ANNALES

DE

L'INSTITUT FOURIER

Giuseppe DELLA SALA

Liouville-type theorems for foliations with complex leaves

Tome 60, nᵒ 2 (2010), p. 711-725.

<http://aif.cedram.org/item?id=AIF_2010__60_2_711_0>

LIOUVILLE-TYPE THEOREMS FOR FOLIATIONS WITH COMPLEX LEAVES

by Giuseppe DELLA SALA

————

ABSTRACT. — In this paper we discuss various problems regarding the structure of the foliation of some foliated submanifolds S of $\mathbb{C}^n$, in particular Levi flat ones. As a general scheme, we suppose that S is bounded along a coordinate (or a subset of coordinates), and prove that the complex leaves of its foliation are planes.

RÉSUMÉ. — Dans cet article nous considérons différentes questions relatives à la structure du feuilletage de certaines sous-variétés $S \subset \mathbb{C}^n$, en particulier les variétés Levi-plates. Comme schéma général, on suppose que S est bornée le long d'une coordonnée (ou d'un sous-ensemble des coordonnées), et on montre que les feuilles complexes de son feuilletage sont des plans.

Introduction

Let S be a foliated submanifold of $\mathbb{C}^{n+d} = \mathbb{C}^n \times \mathbb{C}^d$, $\mathbb{C}^n = \mathbb{C}^n_z$, $\mathbb{C}^d = \mathbb{C}^d_w$. In this paper we address the following general question: If S is bounded in some directions, what can be said about leaves endowed with a structure of regular immersed complex manifold? A particularly important example we considered in this setting is that of Levi-flat manifolds, namely CR manifolds which are foliated by complex leaves. As it is well known Levi-flat hypersurfaces appear in complex analysis as "limit objects" in many extension problems (see, for example, [6]). The study of their geometric properties in the last 15 years matured in a fruitful area of research (see [3], [2], [12], [5], [7] among many others).

We will show that, in some circumstances, it is possible to conclude that such leaves are linear spaces. A first result in this direction states that a

smooth Levi-flat submanifold of codimension $2d - 1$ of $\mathbb{C}^n \times \mathbb{C}^d$, contained in the w-cylinder

$$C = \left\{ (z, w) \in \mathbb{C}^{n+d} \colon \sum_{i=1}^{d} |w_i|^2 < 1 \right\}$$

and closed in $\overline{C}$ is foliated by coordinate hyperplanes $\{w = \text{const}\}$ (Theorem 1.1).

The proof of Theorem 1.1 for codimension one (*i.e.* $d = 1$) is a rather easy consequence of Liouville's Theorem for *analytic multifunctions*. Originally introduced by Oka [8], the analytic multifunctions are set-valued functions $\mathbb{C} \to \mathbf{k}(\mathbb{C})$ (where $\mathbf{k}(\mathbb{C})$ denote the subset of $\mathcal{P}(\mathbb{C})$ formed by the compact subsets of $\mathbb{C}$) which behave in some ways as analytic functions; Namely, according to Oka's definition, the complement of their graph is pseudoconvex. For our scope it is more convenient to use the characterization found by Slodkowski [13] by means of plurisubharmonic functions. The proof for higher codimension requires a slightly less trivial application of the Liouville's Theorem.

In Section 2 we consider the case of a smooth codimension one foliation on the graph of a bounded function on $\mathbb{C}_z^n \times \mathbb{R}_u$, in particular a Levi flat graph. In this case, the methods of analytic multifunctions do not seem sufficient, and we proceed by an analysis of each single complex leaf. The main result is contained in Theorems 2.1, 2.2.

1. Levi-flat manifolds in a w-cylinder

Let $\mathbb{C}^{n+d} = \mathbb{C}^n \times \mathbb{C}^d$, $\mathbb{C}^n = \mathbb{C}_z^n$, $\mathbb{C}^d = \mathbb{C}_w^d$, with complex coordinates $z_1, \ldots, z_n, w_1, \ldots, w_d$. In what follows, we want to consider Levi flat submanifolds which are not necessarily hypersurfaces:

DEFINITION. — *We say that a maximally complex $(2d-1)$-codimensional submanifold S of $\mathbb{C}^{n+d}$ of class at least C^1 is Levi flat if its CR tangent bundle $H(S)$ is integrable: In such a case S admits a foliation by n-dimensional complex leaves.*

The aim of this section is to prove the following

THEOREM 1.1. — *Let $S \subset \mathbb{C}^{n+d}$ be a $(2d-1)$-codimensional Levi flat submanifold of class C^1 contained in the w-cylinder*

$$C = \left\{ (z, w) \in \mathbb{C}^{n+d} : \sum_{i=1}^{d} |w_i|^2 < 1 \right\}$$

and closed in $\overline{C}$. Then S is foliated by complex coordinate n-planes $\{w_1 = c_1, \ldots, w_d = c_d\}$.

Let us recall some results on analytic multifunctions that will be used in the proof.

1.1. Analytic multifunctions and Liouville's Theorem

Consider a function $f \colon \mathbb{C}^n \to \mathcal{P}(\mathbb{C}^k)$, *i.e.* a set-valued function from $\mathbb{C}^n$ to the power set of $\mathbb{C}^k$. Then f is said to be a *multifunction* and its graph is the set $\Gamma(f) \subset \mathbb{C}^{n+k}$ defined as

$$\Gamma(f) = \bigcup_{z \in \mathbb{C}^n} \{z\} \times f(z).$$

We will always suppose that each value $f(z)$ is a compact set and that f satisfies the following upper semicontinuity property: For every $z_0 \in \mathbb{C}^n$ and for every $\varepsilon > 0$ there exist $\delta > 0$ such that $f(z)$ is contained in the ε-neighborhood of $f(z_0)$ for every $z \in B(z_0, \delta)$ (this property holds, for example, whenever f is continuous in the Hausdorff metric); In particular, $\Gamma(f)$ is closed.

DEFINITION. — *Let $f \colon \mathbb{C}^n \to \mathcal{P}(\mathbb{C}^k)$ be an upper semicontinuous multifunction. We say that f is an* analytic multifunction *if, for every continuous plurisubharmonic function ρ defined in a neighborhood of $\Gamma(f)$ in $\mathbb{C}^{n+k}$, the function $\rho' \colon \mathbb{C}^n \to \mathbb{R}$ defined as*

$$\rho'(z) = \max_{w \in f(z)} \rho(w)$$

is plurisubharmonic.

714 Giuseppe DELLA SALA

The concept of analytic multifunction has been introduced by Oka for $k = 1$, by requiring $\mathbb{C}^{n+1} \smallsetminus \Gamma(f)$ to be pseudoconvex. In this way, a holomorphic function $f \in \mathcal{O}(\mathbb{C}^n)$ is clearly an analytic multifunction. The definition we are adopting follows from a result by Slodkowski: In [13] it is proved that the property of ρ' being p.s.h. for any p.s.h. ρ characterizes analytic multifunctions for $k = 1$. In our context, such a property is actually weaker than Oka's definition (as can be seen, for example, by considering a complex line in $\mathbb{C}^3$) and this allows us to consider, in addition to hypersurfaces, higher codimensional submanifolds.

For analytic multifunctions the following Liouville's result holds (see also [10]):

LEMMA 1.2. — Let f be an analytic multifunction $\mathbb{C}^n \to \mathcal{P}(\mathbb{C}^k)$, and suppose that f is bounded in the following sense:

$$\Gamma(f) \subset \big\{|w| < M\big\} \subset \mathbb{C}^{n+k}$$

for some $M > 0$. Let $\widehat{f}$ be the multifunction defined as

$$\widehat{f}(z) = \widehat{f(z)}, \ z \in \mathbb{C}^n$$

where $\widehat{K}$ is the polynomial hull of K. Then $\widehat{f}$ is constant.

Proof. — Let $P(w)$ be a polynomial on $\mathbb{C}^k_w$, and denote again by P the trivial extension to $\mathbb{C}^{n+k}$ $P(z, w) = P(w)$. Then $|P|$ is a plurisubharmonic function on $\mathbb{C}^{n+k}$, therefore by definition

$$P'(z) = \max\big\{|P(w)| : w \in f(z)\big\}$$

is p.s.h. on $\mathbb{C}^n$. But, defining

$$C = \max_{|w| \leqslant M} P(w)$$

we have that $P'(z) \leqslant C$ for all $z \in \mathbb{C}^n$. Then, by Liouville's Theorem for plurisubharmonic functions it follows that P' is constant. We deduce that $\widehat{f}$ also is constant. Indeed, in the opposite case we could find $w_1 \in \mathbb{C}^k$ and $z_1, z_2 \in \mathbb{C}^n$ such that $w_1 \in (\widehat{f}(z_1) \smallsetminus \widehat{f}(z_2))$, i.e. there would exist a polynomial P_1 such that

$$|P_1(w_1)| > \max_{\widehat{f}(z_2)} |P_1|,$$

hence

$$P_1'(z_2) < |P_1(w_1)| \leqslant P_1'(z_1)$$

which is a contradiction. □

ANNALES DE L'INSTITUT FOURIER

Example 1.3. — The hypothesis of Lemma 1.2 does not imply that f is in turn a constant multifunction. A simple example is the following:

$$f(z) = \begin{cases} \{|w| = 1\}, & z \neq 0; \\ \{|w| \leqslant 1\}, & z = 0. \end{cases}$$

Example 1.4. — A modification of the previous example shows that, even if $\Gamma(f)$ is a (disconnected) manifold, f need not be constant if ρ' is subharmonic whenever ρ is p.s.h. and defined on the whole $\mathbb{C}^2$ (this is slightly weaker than our definition, but it is all that is needed to prove Lemma 1.2). Indeed, in this case we may define $f(z)$ to be the union of the unit circle bD and any compact set contained in the unit disc D, as any subharmonic function can "detect" the behaviour of f only in bD. As we show below, anyway, the result holds if $\Gamma(f)$ has the structure of a (even disconnected) Levi flat manifold (which is obviously not the case in the previous example).

1.2. Proof of Theorem 1.1

The proof of Theorem 1.1 can be achieved by choosing a complex projection on $\mathbb{C}^n_z$ and interpreting S as a multifunction f_S with values in $\mathcal{P}(\mathbb{C}^d)$. It is immediate to see that f_S is analytic (according to Oka's definition) when $d = 1$; When $d > 1$, though, the complement of S is no longer pseudoconvex. Lemma 1.5 shows that, nevertheless, f_S satisfies our definition. Afterwards, to prove that f_S is constant whenever $\widehat{f_S}$ is we use the results of [14] (note that, also for this, the case when $d = 1$ is "easier" as it involves only the hull of curves contained in $\mathbb{C}$).

Let L_z, $z \in \mathbb{C}^n$, be the vertical complex d-plane over z i.e.

$$L_z = \left\{(\zeta, w) \in \mathbb{C}^{n+d} \colon \zeta = z\right\}.$$

Consider the set-valued function f_S defined by $f_S(z) = L_z \cap S$: We want to show that f_S is an analytic multifunction.

LEMMA 1.5. — f_S is an analytic multifunction.

Proof. — Let U be a neighborhood of S in $\mathbb{C}^{n+d}$, and let $\rho \colon U \to \mathbb{R}$ be a p.s.h. function; Define ρ' as above. Let $z_0 \in \mathbb{C}^n$, and let $\pi_1 \colon \mathbb{C}^n \to \mathbb{CP}^{n-1}$ be the projection associated to z_0; Moreover, let $\pi_2 \colon \mathbb{C}^{n+d} \to \mathbb{C}^n$. Observe that the Levi foliation of S is still of class C^1 by Barrett and Fornaess' result in [1]. Then, $\pi = \pi_1 \circ \pi_2 \colon S \to \mathbb{CP}^{n-1}$, seen as locally defined in charts $\cong \mathbb{R} \times \mathbb{C}^n$, is a map of class $C^{1 \times \omega}$ in the sense defined by Pugh

in [9]. By the same paper, it follows that Sard's lemma holds for π: In other words, for a generic choice of a complex line $\mathcal{L} \subset \mathbb{C}^n$ passing through z_0, the intersection of S with the complex $(d+1)$-plane

$$\left\{ (z, w) \in \mathbb{C}^{n+d} \colon z \in \mathcal{L} \right\}$$

is transversal, and thus a Levi flat submanifold of $\mathbb{C}^{d+1}$. Therefore, since it is sufficient to show that the restriction of ρ' to a generic $\mathcal{L}$ is subharmonic, we can suppose $n = 1$.

Assume, then, that f_S is a $\mathcal{P}(\mathbb{C}^d)$-valued multifunction defined over $\mathbb{C}$, and fix $z^0 \in \mathbb{C}$. If $w \in f(z^0)$, we denote by Σ_w the leaf of the foliation of S through w. Two cases are possible:

$$T_{(z^0, w)}(\Sigma_w) \nsubseteq \mathbb{C}^d \quad \text{and} \quad T_{(z^0, w)}(\Sigma_w) \subset \mathbb{C}^d.$$

In the former, for a sufficiently small neighborhood $V_w = (\Delta \times U)_w$ of (z^0, w) we have that $\Sigma_w \cap V_w$ can be written as

$$\Sigma_w \cap V_w = \left\{ (z, w) \in \Delta \times U \colon w_1 = g_1^w(z), \ldots, w_d = g_d^w(z) \right\}$$

for some holomorphic function $g_i^w \in \mathcal{O}(\Delta)$. Moreover, observe that for $w' \in f(z^0)$ in a small enough neighborhood $\mathcal{W}_w$ of w, we can choose a Δ which does not depend on w';

In the latter, consider the restriction of the projection $\pi \colon \mathbb{C}^{d+1} \to \mathbb{C}$ to a small neighborhood $\mathcal{V}_w$ of (z^0, w) in Σ_w. We can suppose that $\mathcal{V}_w$ is a local chart such that $(z^0, w) = 0$. Denote by ζ the complex coordinate on $\mathcal{V}_w$. Since $\pi|_{\mathcal{V}_w}$ is a holomorphic function, and its first derivative vanishes in 0, there exists $k \geqslant 1$ such that

$$\frac{\partial^j}{\partial \zeta^j} \pi|_{\mathcal{V}_w} = 0 \ \text{ for } \ j \leqslant k, \quad \frac{\partial^{k+1}}{\partial \zeta^{k+1}} \pi|_{\mathcal{V}_w} \neq 0.$$

Otherwise, we would have $\pi|_{\mathcal{V}_w} \equiv z^0$ and so Σ_w would be a complex line contained in $\mathbb{C}^d$, which is impossible since it must be contained in the w-cylinder C of Theorem 1.1. It follows that $\pi|_{\mathcal{V}_w}$ is a $(k+1)$-sheeted covering over some neighborhood Δ of z_0. Now, the restriction of π to the leaves $\Sigma_{w'}$ passing through the points (z^0, w') of a small neighborhood of (z^0, w) can be interpreted as a smooth one-parameter family of holomorphic functions $\pi_t \colon \mathcal{V}_k \to \mathbb{C}_z$, such that $\pi_0 = \pi$. For $|t| \ll 1$, the argument principle implies that the sum of the orders of the zeroes of $(\partial/\partial \zeta)\pi_t$ is still k. This in turn means that for w' sufficiently close to w the projection $\pi|_{\Sigma_{w'}}$ is still a $(k+1)$-sheeted covering over some neighborhood $\Delta_{w'}$; In a possibly smaller neighborhood $\mathcal{W}_w$ we can assume to have chosen a Δ independent of w'.

Since $f(z^0)$ is a compact set, we can choose finitely many open sets as above, $\mathcal{W}_{w_1}, \ldots, \mathcal{W}_{w_h}$, in such a way that

$$\bigcup_{i=1}^{h} \mathcal{W}_{w_i} = f(z_0).$$

Choose a disc $\Delta \subset \Delta_{w_1} \cap \cdots \cap \Delta_{w_h}$. We claim that ρ' is plurisubharmonic on Δ. In order to prove this, choose $w \in f(z^0)$:

- If $w \in \mathcal{W}_{w_j}$ with w_j of the first kind, then we define

$$\rho_w^j = \rho|_{\Sigma_w \cap \pi^{-1}(\Delta)};$$

- If $w \in \mathcal{W}_{w_j}$ with w_j of the second kind, we define

$$\rho_w^j = \left(\max_{\Sigma_w \cap \pi^{-1}(\Delta_{w_j})} \rho(z, w) \right)\Big|_{\Delta}.$$

In both cases, ρ_w^j is a plurisubharmonic function. Observe that possibly $\rho_w^i \neq \rho_w^j$ when $i \neq j$. Nevertheless, consider

$$(1.1) \qquad \varrho(z) = \max_{1 \leqslant i \leqslant h, w \in f(z^0)} \rho_w^i(z);$$

We have that $\varrho(z) = \rho'(z)$. In fact, the arguments above show that

$$\bigcup_{w \in f(z_0)} \Sigma_w \cap \pi^{-1}(\Delta) = S \cap \pi^{-1}(\Delta)$$

and so the maximum of equation (1.1) is attained exactly on $f(z)$ rather than on a proper subset (as would be the case if leaves of S which accumulate on $f(z^0)$ without intersecting it existed). Since we already know that $\rho'(z)$ is continuous, (1.1) implies that $\rho'(z)$ is plurisubharmonic. $\qquad \square$

By lemma 1.5 and lemma 1.2 we have that $\widehat{f}_S$ is a constant multifunction. We must show that f_S is in turn constant.

Proof of Theorem 1.1. — Observe that, as in the proof of lemma 1.5, the projection $S \to \mathbb{C}^n$ is of class $C^{1 \times \omega}$ (*cf.* [9]), hence, for z belonging to a dense, open subset J of $\mathbb{C}^n$, L_z intersects S transversally. For $z \in J$, $f(z) = L_z \cap S$ is the disjoint union of a finite set $\{\gamma_i(z)\}_{1 \leqslant i \leqslant k(z)}$ of loops in $\mathbb{C}^d$. It is a well-known fact ([14]) that, in this case, the polynomial hull $\widehat{f}(z)$ of $f(z)$ is given by the union of some of the loops γ_i and some complex varieties Λ_j whose boundaries are the others γ_i's. We choose the minimal subsets of loops $\{\alpha_i(z)\}_{1 \leqslant i \leqslant h(z)}$ such that, if $\mathcal{M}(z) = \alpha_1 \cup \cdots \cup \alpha_{h(z)}$, then $\widehat{\mathcal{M}}(z) = \widehat{f}(z)$ (in particular, $\widehat{\mathcal{M}}(z)$ is constant for $z \in J$); Observe that $\mathcal{M}(z)$ is univocally defined. Choose $z_0 \in J$: We want to show that $\mathcal{M}(z) = \mathcal{M}(z_0)$ for $z \in J$ and, moreover, that $\mathcal{M}(z_0)$ consists of a subset of connected components of $f(z)$ for any z. In such a case, $S' = S \subset \cup_z \mathcal{M}(z)$

is a Levi flat submanifold with strictly less connected components than S and the proof can be achieved inductively.

Observe that, by [14], $\widehat{\mathcal{M}}(z)$ identifies $\mathcal{M}(z)$ univocally, so that clearly $\mathcal{M}(z)$ is constant on J. Since J is dense in $\mathbb{C}^n$ and S is closed, it follows that $\mathcal{M}(z_0)$ is contained in $f(z)$ for any z. Then $\mathbb{C}^n \times \mathcal{M}(z_0)$ is a $2n + 1$ dimensional manifold contained in S, hence it consists of a subset of its connected components: This concludes the proof. $\square$

1.3. Density of the projection

Let S be a Levi flat hypersurface.The scheme of the proof of Theorem 1.1 can be employed to show that, in fact, the projection of S along a coordinate w is, in general, dense in $\mathbb{C}_w$.

THEOREM 1.6. — *Let S be a closed Levi flat hypersurface embedded in $\mathbb{C}^{n+1}$ of class C^1. Then S is either foliated by complex hyperplanes of the kind $\{w = c\}$, or its projection to $\mathbb{C}_w$ is dense.*

Proof. — Suppose that the projection is not dense, and let D be a disc in $\mathbb{C}_w$, centered in w_0, which is contained in the complement of the projection of S. For $M \gg 0$, we denote by S_M the set

$$S_M = S \cup \{|w - w_0| \geqslant M\};$$

Observe that the complement of S_M in $\mathbb{C}^{n+1}$ is pseudoconvex. It is sufficient to show that S_M is a union of complex hyperplanes for each $M > 0$. Up to a rational change of coordinates, we can suppose that S_M lies in $\{|w| < 1\}$. As before, we consider the multifunction f_{S_M} whose fiber over $z_0 \in \mathbb{C}^n$ is $S_M \cap \{z = z_0\}$: f_{S_M} is analytic and, generically, $f_{S_M}(z)$ is the union of a fixed closed disc $\overline{D}'$ and finitely many arcs with endpoints in $\overline{D}'$. Then, a proof completely analogous to that of Theorem 1.1 shows that f_{S_M} is a constant multifunction. $\square$

2. Foliation of a graph

Consider in $\mathbb{C}^{n+1} = \mathbb{C}^n \times \mathbb{C}$ coordinates $(z_1, \ldots, z_n, w) = (z, w)$, $z_j = x_j + iy_j$, $w = u + iv$. We denote by π the projection $\pi \colon \mathbb{C}^{n+1} \to \mathbb{C}^n_z$ and by τ the projection $\tau \colon \mathbb{C}^{n+1} \to \mathbb{C}^n_z \times \mathbb{R}_u$. Let $\rho \colon \mathbb{C}^n \times \mathbb{R}_u \to \mathbb{R}_v$ be a function of class C^1, and suppose that its graph

$$S = \{v = \rho(z, u)\}$$

carries a C^1 foliation $\mathcal{F}$ of codimension one; Note that we are not assuming that all the leaves are complex manifolds. We say that a leaf Σ of $\mathcal{F}$ is *properly embedded* if, for (almost) every ball $B \subset \mathbb{C}^{n+1}$, the connected components of $\overline{B} \cap \Sigma$ are compact, embedded submanifolds of $\overline{B} \cap S$ with boundary. We say that $\mathcal{F}$ is *proper* if all the leaves are properly embedded.

The aim of this section is to prove the following

THEOREM 2.1. — *Let $S = \{v = \rho(z, u)\}$ be a properly foliated hypersurface of $\mathbb{C}^{n+1}$ of class C^1, and suppose that ρ is bounded. Then every complex leaf of S is a complex hyperplane.*

In particular, Theorem 2.1 applies to Levi- flat graphs. In this case, though, we can actually prove it when ρ is just continuous: In such a situation, we say that S is Levi flat if it locally separates $\mathbb{C}^{n+1}$ into pseudoconvex domains. Note that, by the results of [11], a Levi flat graph is properly foliated in the sense given above.

THEOREM 2.2. — *Suppose that S is Levi-flat and ρ is C^0 and bounded by some constant M. Then S is foliated by complex hyperplanes, i.e.*

$$\rho(z, u) = \rho(u).$$

Theorem 2.2 can, once again, be proved by means of analytic multifunctions.

Proof of Theorem 2.2. — We may assume $n = 1$. Indeed, let $p_1 = (z^1, u)$ and $p_2 = (z^2, u)$ be two points in $\mathbb{C}^n_z \times \mathbb{R}_u$ with the same u-coordinate, and consider the complex line $L \subset \mathbb{C}^n_z$ such that $z^1, z^2 \in L$. Then the restriction of ρ to $L \times \mathbb{R}_u$ has a Levi-flat graph

$$S_L = S \cap (L \times \mathbb{C}_w) \subset L \times \mathbb{C}_w \cong \mathbb{C}^2.$$

so Theorem 2.2 for $n = 1$ applies to S_L, showing that $\rho|_{L \times \mathbb{R}_u}$ is a function of u and thus that $\rho(p_1) = \rho(p_2)$.

Thus let $n = 1$. By hypothesis there exists a complex line $\{w = c\}$ such that S lies outside the w-cylinder

$$C = \big\{(z, w) \colon |w - c| < \varepsilon\big\}.$$

Then, we can perform a rational change of coordinates (acting only on the w-coordinate) which sends ∞ to 0, and such that the image S' of S is contained in $\mathbb{C}_z \times D_w$, where D_w is the unit disc. The complement of

$$\overline{S}' = S' \cup \big(\mathbb{C}_z \times \{0\}\big)$$

in $\mathbb{C}_z \times D_w$ is pseudoconvex. Indeed, a plurisubharmonic exhaustion function φ for the complement of S in $\mathbb{C}^2$ induces a plurisubharmonic exhaustion function φ' for the complement of S' in $\mathbb{C}_z \times (D_w \smallsetminus \{0\})$; Then

$$\psi = \max\left\{\varphi' \left| \frac{1}{w} \right| \right\}$$

is a plurisubharmonic exhaustion function for the complement of $\overline{S}'$ in $\mathbb{C}_z \times D_w$. Now we can argue as in the proof of Theorem 1.1: In fact, if f is the multifunction representing $\overline{S}'$, $f(z)$ is a simple continuous Jordan curve, univocally determined by its polynomial hull. $\qquad\square$

We are not able to put the method of analytic multifunctions to work for the proof of Theorem 2.1, so we proceed by analyzing the foliation "leaf by leaf" instead. It is sufficient to study the case $n = 1$ (see Theorem 2.2).

2.1. Preliminary facts

First of all, we show the following

LEMMA 2.3. — *Let Σ be any complex leaf of the foliation of S. Then the projection $\pi|_\Sigma$ is a local homeomorphism.*

Proof. — For any $p \in \Sigma$, we have $T_p(\Sigma) = H_p(S) \subset T_p(S)$; Since by hypothesis $\partial/\partial v \notin T_p(S)$, we deduce $\partial/\partial w \notin T_p^{\mathbb{C}}(\Sigma)$ *i.e.* the differential of $\pi|_\Sigma$ is onto. $\qquad\square$

Lemma 2.3 shows that a complex leaf Σ of the foliation is locally a graph over $\mathbb{C}_z$ but, since we do not know whether $\pi \colon \Sigma \to \mathbb{C}_z$ is actually a covering, we cannot conclude immediately that $\pi|_\Sigma^{-1}$ is single-valued. However, if this is the case, it is easy to deduce that the thesis of Theorem 2.2 holds true for Σ, provided that the projection $\pi|_\Sigma$ is onto:

LEMMA 2.4. — *Let Σ be a complex leaf of S, and suppose that*
- $\pi(\Sigma) = \mathbb{C}_z$;
- *For every $z^0 \in \mathbb{C}_z$, $\pi^{-1}(z^0) \cap \Sigma$ is a single point.*

Then there exists $c \in \mathbb{C}$ such that

$$\Sigma = \{w = c\}.$$

Proof. — Indeed, in this case the leaf Σ is biholomorphic to $\mathbb{C}$ as $\pi|_\Sigma$ is one to one; Then, denoting by v the projection on the v-coordinate, $v \circ (\pi|_\Sigma)^{-1}$ is a harmonic, bounded function on $\mathbb{C}_z$, which is constant by Liouville's Theorem. Therefore $v|_\Sigma$ is also constant and so is $u|_\Sigma$, which is conjugate to v in Σ. $\qquad\square$

Remark 2.5. — One may ask whether the latter hypothesis in Lemma 2.4 can be replaced by

- $\pi|_\Sigma$ is a local homeomorphism.

This is not the case: It is not difficult to find examples of surjective (even finite-to-one) local biholomorphisms $D \to \mathbb{C}$ (however, much more is true: Fornaess and Stout [4] showed that any complex manifold is the image of a polydisc by a finite-to-one local biholomorphism).

In order to prove Theorem 2.2 our strategy is to apply Lemma 2.4 and so, from now on, we shall focus on a single complex leaf Σ of the foliation of S and we will prove that its projection over $\mathbb{C}_z$ is a biholomorphism. We set

$$\pi(\Sigma) = \Omega \subset \mathbb{C}_z;$$

Then, since Σ is a complex curve (or also because of Lemma 2.3), Ω is an open subset of $\mathbb{C}_z$.

2.2. Analysis of Ω

Our purpose is now to show that Ω is simply connected. In order to achieve this we employ some lemmas contained in [11], which are part the in-depth analysis which is carried out therein on the leaves of the foliation of the Levi-flat solution for graphs. First of all, we prove that $\pi|_\Sigma^{-1}$ is actually single-valued over $\Omega = \pi(\Sigma)$.

LEMMA 2.6. — *Let Ω and Σ be as above. Then $\pi|_\Sigma^{-1}(z)$ consists of a point for every $z \in \Omega$.*

Proof. — Suppose that, for some $z \in \Omega$, there exist $p, q \in \Sigma$ ($p \neq q$) such that $\pi(p) = \pi(q) = z$. Since, by definition, Σ is connected, there exists an arc $\widetilde{\gamma}$ joining p and q. Denote $\gamma = \pi \circ \widetilde{\gamma}$ be the corresponding loop in Ω and let B be a ball in $\mathbb{C}_z \times \mathbb{R}_u$, centered at z, with a large enough radius such that $\gamma \subset B$, $\tau \circ \widetilde{\gamma} \subset B$. Then

$$S \cap \tau^{-1}(B) = \Gamma(\rho|_B) \subset \mathbb{C}^2$$

is a hypersurface whose boundary is the graph

$$S \cap \tau^{-1}(bB) = \Gamma(\rho|_{bB}).$$

Since, by hypothesis, Σ is properly embedded in $S \cap \tau^{-1}(B)$ $\tau(\Sigma)$ is properly embedded in B. By the choice of B, $\tau(p)$ and $\tau(q)$ belong to the same connected component of $\tau(\Sigma) \cap B$, say Σ'. Lemma 3.2 in [11] states that

a connected surface of $\mathbb{C} \times \mathbb{R}$, properly embedded in a convex domain and which is locally a graph over $\mathbb{C}$, is globally a graph. By Lemma 2.3 we have that Σ' is locally a graph over $\mathbb{C}_z$; Since B is convex, we deduce that Σ' is globally a graph over some subdomain of Ω. Since $\tau(p)$ and $\tau(q)$ have the same projection over Ω, it follows $\tau(p) = \tau(q)$ and consequently $p = q$, a contradiction. $\qquad\square$

By Lemma 2.6, Σ is represented by the graph of a holomorphic function $u + iv$ on Ω. The following is also a direct consequence of a Lemma in [11]:

LEMMA 2.7. — Ω is simply connected.

Proof. — Observe that, if Ω is not simply connected, then $D \cap \Omega$ is not simply connected for some open disc $D \subset \mathbb{C}_z$. Again, $\tau(\Sigma)$ is properly embedded in some subdomain

$$D \times (-R, R) \subset \mathbb{C}_z \times \mathbb{R}_u, \ R \gg 0.$$

We recall that Lemma 3.3 in [11] states that, if a harmonic function defined in a domain of $\mathbb{C}$ has a graph properly embedded (in a convex domain of $\mathbb{C} \times \mathbb{R}$) and admits a (single valued) harmonic conjugate, then its domain of definition is simply connected. In our situation, $\tau(\Sigma)$ is the graph of u over $D \cap \Omega$; The since v is a single-valued harmonic conjugate of $\mathcal{U}$, we obtain that $D \cap \Omega$ is actually simply connected. $\qquad\square$

2.3. Proof of Theorem 2.1

What is left to prove is that the projection $\Sigma \to \mathbb{C}_z$ is onto. Suppose, by contradiction, that $\Omega \subsetneq \mathbb{C}_z$, and let $z_0 \in b\Omega$. The following result shows that in fact z_0 must belong to Ω at least in some special case.

LEMMA 2.8. — *Let $z_0 \in \mathbb{C}_z$ and suppose that there exist p_0 such that $\pi(p_0) = z_0$ and p_0 is a cluster point for Σ. Then $z_0 \in \Omega$.*

Remark 2.9. — Since we do not know, at this stage, whether Σ is a closed submanifold or not, it is a priori possible that $p_0 \notin \Sigma$. Nevertheless, $\pi^{-1}(z_0) \cap \Sigma \neq \emptyset$.

Proof of Lemma 2.8. — Let V be a neighborhood of p_0 on which the foliation of $S \cap V$ is trivial. Then, either $\Sigma \cap V$ has finitely many connected components - in this case one of them must contain p_0 - or the connected components of $\Sigma \cap V$ accumulate to the leaf Σ' of $S \cap V$ containing p_0. Then Σ' must be a complex leaf, too. Thus, from Lemma 2.3 it follows

that, if $V' \Subset V$ $(p_0 \in V')$ is small enough, all the leaves of $S \cap V'$ intersect (possibly in V) $\pi^{-1}(z_0)$. By hypothesis

$$V' \cap \Sigma \neq \emptyset,$$

so Σ contains a leaf of $S \cap V'$ and consequently

$$\pi^{-1}(z_0) \cap \Sigma \neq \emptyset. \qquad \square$$

The previous lemma does not depend on $\pi|_{\Sigma}^{-1}$ being single-valued; However, since we know by Section 2.2 that it is the case, we will denote by $w(z)$ (resp. $u(z), v(z)$) the w-coordinates (resp. the u- and v-coordinate) of $\pi|_{\Sigma}^{-1}(z)$. With these notations, we can state the following straightforward corollary of Lemma 2.8:

COROLLARY 2.10. — *Let $z_0 \in b\Omega$, and let $\{\mathcal{U}_k\}_{k \in \mathbb{N}}$ be a fundamental system of neighborhoods of z_0 in $\mathbb{C}_z$. Then, for any $M > 0$ there exists $K \in \mathbb{N}$ such that $|w(z)| > M$ for all $z \in \Omega \cap \mathcal{U}_k$ with $k \geqslant K$.*

Proof. — Otherwise, there would exist $M > 0$ and a sequence $\{z_n\}_{n \in \mathbb{N}}$ such that

- $z_n \in \Omega$ for every $n \in \mathbb{N}$;
- $z_n \to z_0$;
- For every $n \in \mathbb{N}$ there exists $p_n \in \Sigma$ such that $\pi(p_n) = z_n$ and $|w(p_n)| \leqslant M$.

Then $\{p_n\}_{n \in \mathbb{N}}$ would admit an accumulation point p_0 in $\mathbb{C}^2$ such that $\pi(p_0) = z_0$. By Lemma 2.8 this would imply $z_0 \in \Omega$, a contradiction. $\quad \square$

Since, by the main hypothesis, $v(z)$ is bounded on Ω, it follows immediately

COROLLARY 2.11. — *Let $z_0 \in b\Omega$, and let $\{\mathcal{U}_k\}_{k \in \mathbb{N}}$ be a fundamental system of neighborhoods of z_0 in $\mathbb{C}_z$. Then for any $M > 0$ there exists $K \in \mathbb{N}$ such that $|u(z)| > M$ for all $z \in \Omega \cap \mathcal{U}_k$ with $k \geqslant K$.*

Remark 2.12. — Since $\Omega \cap \mathcal{U}_k$ need not be connected even for large k, u could assume both signs in every neighborhood of z_0. Later on we will prove that it is not the case.

LEMMA 2.13. — *Let C be a connected component of the boundary of Ω. Then there exist a neighborhood $\mathcal{U}$ of C in $\overline{\Omega}$ such that either $u > 0$ on $\mathcal{U}$ or $u < 0$ on $\mathcal{U}$.*

Proof. — Let K be a compact connected subset C, chosen in such a way that $C \smallsetminus K$ does not contain relatively compact connected components;

It is enough to prove that the thesis holds for any such K. Observe that, since Ω is connected, $C \smallsetminus K$ has at most two connected components. By Corollary 2.11, for any $z \in K$ there exists a disc $D(z, \varepsilon)$ such that $|u| > 0$ on $D(z, \varepsilon) \cap \Omega$. Cover K by discs $\{D_1, \ldots, D_k\}$ and take δ so small that

$$\mathcal{U}' = \big\{ z \in \mathbb{C} \colon d(z, K) < \delta \big\} \subset D_1 \cup \cdots \cup D_k.$$

Then the thesis of the Lemma is a consequence of the following fact: There is a connected component of $\mathcal{U}' \cap \Omega$ whose boundary contains K. Indeed, suppose that this is not the case, and choose a connected component $\mathcal{V}$ of $\mathcal{U}' \cap \Omega$ such that $\emptyset \neq E = b\mathcal{V} \cap K \subsetneq K$. Observe that $b\mathcal{V} = E \cup F \cup G$, where

$$F = b\mathcal{V} \cap \big\{ z \in \mathbb{C} \colon d(z, K) = \delta \big\} \quad \text{and} \quad G = b\mathcal{V} \cap C \smallsetminus K;$$

Obviously $E \cap F = \emptyset$ and thus G has at least two connected component. Moreover, E is connected since otherwise $C \smallsetminus K$ would have more than two connected components. But if $E \subsetneq K$ is connected then it can touch at most one connected component of $C \smallsetminus K$ and thus of G; It follows $E = K$. $\square$

COROLLARY 2.14. — *Let C be a connected component of $b\Omega$. Then there is a fundamental system $\{\mathcal{V}_n\}_{n \in \mathbb{N}}$ of neighborhoods of C in $\overline{\Omega}$ such that either*

$$\inf_{\mathcal{V}_n} u \to +\infty \quad or \quad \sup_{\mathcal{V}_n} u \to -\infty \quad as \quad n \to \infty.$$

Proof. — This is a consequence of Corollary 2.11 and Lemma 2.13. $\square$

Now we are in position to prove Theorem 2.1. Choose a point $w \in C$ and observe that, since Ω is simply connected by Section 2.2, there exists a disc $D = D(w, \varepsilon)$ such that $D \smallsetminus C$ is disconnected; Moreover suppose, first, that we can choose w and D in such a way that $D \smallsetminus C$ is not contained in Ω. Let

$$g = \frac{1}{u + iv};$$

Then g is well-defined and holomorphic on $D \cap \Omega$. Define a function $\widetilde{g} \colon D \to \mathbb{C}$ as

$$\widetilde{g}(z) = \begin{cases} g(z), & z \in \overline{\Omega} \cap D; \\ 0, & z \in D \smallsetminus \overline{\Omega}. \end{cases}$$

Then $\widetilde{g}$ is continuous by Corollary 2.14. Moreover, by definition $\widetilde{g}$ is holomorphic outside the set $\{\widetilde{g} = 0\}$; Therefore, by Rado's Theorem, $\widetilde{g} \in \mathcal{O}(D)$. Since the interior part of $\{\widetilde{g} = 0\}$ is nonempty, we have $\widetilde{g} \equiv 0$ on D and consequently $g \equiv 0$ on Ω, which is a contradiction. Suppose, then, that for any choice of w and D we have $D \smallsetminus C \subset \Omega$; By the same argument

as before, we have that nevertheless the function $\widetilde{g}$ defined as g in Ω and 0 in its complement is meromorphic (and not everywhere vanishing) on all $\mathbb{C}$, hence its null set is discrete. Since Ω is simply connected, the only possibility is $\mathbb{C} \smallsetminus \Omega = \emptyset$, against our assumptions.

It follows that u cannot be unbounded on Ω. Then by Corollary 2.11 we have that $\Omega = \mathbb{C}_z$ and so π is onto. Lemma 2.6 implies that π is one to one, therefore we can apply Lemma 2.4 and conclude that $\Sigma = \{w = c\}$ for some $c \in \mathbb{C}$, whence the thesis of Theorem 2.1. □

BIBLIOGRAPHY

[1] D. E. Barrett & J. E. Fornæss, "On the smoothness of Levi-foliations", *Publ. Mat.* **32** (1988), no. 2, p. 171-177.

[2] D. E. Barrett & T. Inaba, "On the topology of compact smooth three-dimensional Levi-flat hypersurfaces", *J. Geom. Anal.* **2** (1992), no. 6, p. 489-497.

[3] R. L. Bryant, "Levi-flat minimal hypersurfaces in two-dimensional complex space forms", in *Lie groups, geometric structures and differential equations—one hundred years after Sophus Lie (Kyoto/Nara, 1999)*, Adv. Stud. Pure Math., vol. 37, Math. Soc. Japan, Tokyo, 2002, p. 1-44.

[4] J. E. Fornaess & E. L. Stout, "Polydiscs in complex manifolds", *Math. Ann.* **227** (1977), no. 2, p. 145-153.

[5] A. Lins Neto, "A note on projective Levi flats and minimal sets of algebraic foliations", *Ann. Inst. Fourier (Grenoble)* **49** (1999), no. 4, p. 1369-1385.

[6] G. Lupacciolu & G. Tomassini, "An extension theorem for CR-functions", *Ann. Mat. Pura Appl. (4)* **137** (1984), p. 257-263.

[7] T. Ohsawa, "On the Levi-flats in complex tori of dimension two", *Publ. Res. Inst. Math. Sci.* **42** (2006), no. 2, p. 361-377.

[8] K. Oka, "Note sur les familles de fonctions analytiques multiformes", *J. Sci. Hiroshima Univ. Ser. A* **4** (1934), p. 93-98.

[9] C. C. Pugh, "The Morse-Sard theorem with mixed differentiability", *Indiana Univ. Math. J.* **40** (1991), no. 4, p. 1387-1396.

[10] T. Ransford, "A new approach to analytic multifunctions", *Set-Valued Anal.* **7** (1999), no. 2, p. 159-194.

[11] N. V. Shcherbina, "On the polynomial hull of a graph", *Indiana Univ. Math. J.* **42** (1993), no. 2, p. 477-503.

[12] Y.-T. Siu, "Nonexistence of smooth Levi-flat hypersurfaces in complex projective spaces of dimension $\geqslant 3$", *Ann. of Math. (2)* **151** (2000), no. 3, p. 1217-1243.

[13] Z. Slodkowski, "Analytic set-valued functions and spectra", *Math. Ann.* **256** (1981), no. 3, p. 363-386.

[14] G. Stolzenberg, "Uniform approximation on smooth curves", *Acta Math.* **115** (1966), p. 185-198.

Giuseppe DELLA SALA
Institut fuer Mathematik
Nordbergstrasse 15
1090 Wien (Autriche)
beppe.dellasala@gmail.com