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SEMI-ÉTALE GROUPOIDS AND APPLICATIONS

by Klaus THOMSEN

ABSTRACT. — We associate a C^*-algebra to a locally compact Hausdorff groupoid with the property that the range map is locally injective. The construction generalizes J. Renault's reduced groupoid C^*-algebra of an étale groupoid and has the advantage that it works for the groupoid arising from a locally injective dynamical system by the method introduced in increasing generality by Renault, Deaconu and Anantharaman-Delaroche. We study the C^*-algebras of such groupoids and give necessary and sufficient conditions for simplicity, and show that many of them contain a Cartan subalgebra as defined by Renault. In particular, this holds when the dynamical system is a shift space, in which case the C^*-algebra coincides with the one introduced by Matsumoto and Carlsen.

RÉSUMÉ. — Nous associons une C^*-algèbre à un groupoïde de Hausdorff localement compact avec la propriété pour l'application but d'être localement injective. La construction généralise celle de la C^*-algèbre réduite de J. Renault pour un groupoïde étale. Elle possède l'avantage de fonctionner pour le groupoïde résultant d'un système dynamique localement injectif par la méthode introduite par Renault, Deaconu et Anantharaman-Delaroche, en augmentant la généralité. Nous étudions les C^*-algèbres de tels groupoïdes et nous donnons des conditions nécessaires et suffisantes pour la simplicité. Nous montrons qu'un grand nombre d'entre elles contiennent une sous-algèbre de Cartan au sens de Renault. En particulier, cela est valable pour un système dynamique symbolique, auquel cas la C^*-algèbre coïncide avec celle introduite par Matsumoto et Carlsen.

1. Introduction

The main purpose of this paper is to develop new tools for the investigation of C^*-algebras which have been constructed from shift spaces in a series of papers by K. Matsumoto and T. Carlsen, *cf.* [18]–[22], [7], [8]. The main results about the structure of these algebras which we obtain here give necessary and sufficient conditions for the algebras to be simple, and show that they all contain a Cartan subalgebra in the sense introduced by

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J. Renault in [28]. Previous results on simplicity of the C^*-algebras defined from subshifts are all due to Matsumoto and give only sufficient conditions under various additional assumptions on the subshift. As a step on the way we show that each of these algebras is a crossed product in the spirit of Paschke [23], arising from a full corner endomorphism of an AF-algebra.

The methods we employ are useful beyond the study of C^*-algebras of subshifts because they extend the applicability of locally compact groupoids to the construction and study of C^*-algebras. The use of groupoids in relation to C^*-algebras was initiated by the pioneering work of J. Renault in [26]. After a relatively slow beginning during the eighties the last two decades have witnessed an increasing recognition of the importance of groupoids as a tool to encode various mathematical structures in a C^*-algebra. Of particular importance in this respect are the so-called étale groupoids which have been used in many different contexts, for example in connection with graph algebras and dynamical systems. In an étale groupoid the range and source maps are local homeomorphisms, and in particular open as they must be if there is a Haar system in the sense of Renault, *cf.* [26]. But in the locally compact groupoid which is naturally associated to a dynamical system by the construction of Renault, Deaconu and Anantharaman-Delaroche, *cf.* [26], [11] and [2], the range and source maps are only open if the map of the dynamical system is also open, and this is a serious limitation which for example prevents the method from being used on subshifts which are not of finite type. For this reason we propose here a construction of a C^*-algebra from a more general class of locally compact Hausdorff groupoids which differ from the étale groupoids in that the range and source maps are locally injective, but not necessarily open. This class of groupoids is not new; it coincides with the locally compact Hausdorff groupoids which were called r-discrete by Renault in [26] and they are equipped with a (continuous) Haar system if and only if they are étale. In many influential places in the literature on the C^*-algebras of groupoids, such [2] or [24] for example, an r-discrete groupoid is assumed to have a continuous Haar system and hence to be étale in the terminology which is now generally accepted. In order to avoid any misunderstanding we therefore propose the name semi-étale for the class of locally compact groupoids where the range map is locally injective, but not necessarily open.

The algebra we associate to a locally compact semi-étale groupoid is the reduced groupoid C^*-algebra of Renault when the groupoid is étale and the construction is a generalization of his. To some extend the only price one has to pay when dealing with groupoids which are not étale, and only semi-étale, is that the continuous and compactly supported functions no longer are invariant under the convolution product and hence do not constitute a $*$-algebra with respect to that product. Nonetheless they still generate a C^*-algebra and we obtain results on its structure which go beyond those known in the étale case, as far as necessary and sufficient conditions for simplicity and the presence of a Cartan subalgebra is concerned.

In the second part of the paper we make a first investigation of the C^*-algebras which arise from the construction of Renault, Deaconu and Anantharaman-Delaroche when the map of the dynamical system is locally injective but not necessarily open. An interesting class of such dynamical systems are the one-sided subshifts since the shift map is locally injective but only open when the shift space is of finite type. We show that the (reduced) C^*-algebra of the semi-étale groupoid constructed from a one-sided subshift is a copy of the Matsumoto-algebra of Carlsen, *cf.* [7], and the results concerning its structure are obtained by specializing results on the groupoid C^*-algebra arising from a general locally injective map.

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2. The C^*-algebra of a semi-étale groupoid

2.1. Definitions and fundamental tools

Let G be a locally compact groupoid, *cf.* [26]. As in [26] we denote the unit space of G by G^0 and use the letters r and s for the range and source maps, respectively. We will say that G is *semi-étale* when $r\colon G \to G^0$ is locally injective, *i.e.* when the topology of G has a base consisting of open sets U such that $r\colon U \to G^0$ and $s\colon U \to G^0$ are injective. An open subset $U \subseteq G$ with this property will be called *a bisection.*

Lemma 2.1. — *G^0 is open in G if and only if r is locally injective.*

Proof. — Assume first that r is locally injective. Let $x \in G^0$ and fix a bisection U containing x. If every open neighborhood of x contained an element from $G \setminus G^0$ the continuity of the groupoid operations would imply the existence of an element $\gamma \in U \setminus G^0$ with $r(\gamma) \in U$. This violates the injectivity of r on U.

Conversely, assume that G^0 is open in G. Let $\gamma \in G$. By continuity of the groupoid operations there is an open neighborhood U of γ in G such that $\mu^{-1}\nu \in G^0$ for all $\mu, \nu \in U$ with $r(\mu) = r(\nu)$. Then r is injective on U. ☐

Assume now that G is semi-étale.

Lemma 2.2 (Lemma 2.7 (i) in Chapter I of [26]). — *Let $x \in G^0$. Then $r^{-1}(x)$ and $s^{-1}(x)$ are discrete sets in the topology inherited from G.*

Proof. — Let U be a bisection containing x. Since $U \cap r^{-1}(x) = \{x\}$ we see that x is isolated in $r^{-1}(x)$. A similar argument shows that y is isolated in $s^{-1}(y)$ for all $y \in G^0$. Let $\gamma \in r^{-1}(x)$ and define $\Phi\colon r^{-1}(x) \to r^{-1}(s(\gamma))$ such that $\Phi(\eta) = \gamma^{-1}\eta$. Then Φ is a homeomorphism with inverse $\eta \mapsto \gamma\eta$. Since $\Phi(\gamma) = s(\gamma)$ and $s(\gamma)$ is isolated in $r^{-1}(s(\gamma))$ it follows that γ is isolated in $r^{-1}(x)$. This proves that $r^{-1}(x)$ is discrete. The argument concerning $s^{-1}(x)$ is identical. ☐

It follows from Lemma 2.2 that $r^{-1}(x)$ and $s^{-1}(x)$ both have a finite intersection with any compact subset of G. Therefore, when $f, g\colon G \to \mathbb{C}$ are compactly supported functions, we can define $f \star g\colon G \to \mathbb{C}$ by the usual formula

$$f \star g(\gamma) = \sum_{\gamma_1\gamma_2=\gamma} f(\gamma_1)g(\gamma_2). \tag{2.1}$$

Then $f \star g$ is again compactly supported, and $f \star g$ is bounded when f and g both are. It follows that the set $B_c(G)$ of bounded compactly supported functions on G is a $*$-algebra with the product $\star$ and the involution $f \mapsto f^*$ defined such that

$$f^*(\gamma) = \overline{f\left(\gamma^{-1}\right)}. \tag{2.2}$$

To obtain a C^*-norm we use the usual representations: For each $x \in G^0$ we define a $*$-representation π_x of $B_c(G)$ on $l^2\left(s^{-1}(x)\right)$ such that

$$\left(\pi_x(f)\psi\right)(\gamma) = \sum_{\gamma_1\gamma_2=\gamma} f\left(\gamma_1\right)\psi\left(\gamma_2\right).$$

We define the C^*-algebra $B^*_r(G)$ to be the completion of $B_c(G)$ in the norm

$$\|f\| = \sup_{x\in G^0} \|\pi_x(f)\|.$$

Let $C_c(G)$ be the subspace of $B_c(G)$ consisting of the functions on G which are compactly supported and continuous. We let $C_r^*(G)$ be the C^*-subalgebra of $B_r^*(G)$ generated by $C_c(G) \subseteq B_c(G)$, *i.e.*

$$C_r^*(G) = \overline{\operatorname{alg}^* G}$$

when $\operatorname{alg}^* G$ denotes the $*$-subalgebra of $B_r^*(G)$ generated by $C_c(G)$. Note that $C_r^*(G)$ is separable when G is second countable while $B_r^*(G)$ essentially never is.

Lemma 2.3 (Proposition 4.1 in Chapter II of [26]). — *Let $f \in B_c(G)$. Then*

$$\sup_{\gamma \in G} |f(\gamma)| \leqslant \|f\| \tag{2.3}$$

and

$$\sum_{\gamma \in s^{-1}(x)} |f(\gamma)|^2 \leqslant \|f\|^2 \tag{2.4}$$

for all $x \in G^0$.

Proof. — Let $\gamma \in G$ and set $x = s(\gamma)$. Let $\delta_x, \delta_\gamma \in l^2\left(s^{-1}(x)\right)$ denote the characteristic functions of $\{x\}$ and $\{\gamma\}$, respectively. Then $\left(\pi_x(f)\delta_x\right)(\gamma) = \sum_{\gamma_1\gamma_2=\gamma} f\left(\gamma_1\right)\delta_x\left(\gamma_2\right) = f\left(\gamma\right)$ which shows that

$$\langle \pi_x(f)\delta_x, \delta_\gamma\rangle = f\left(\gamma\right), \tag{2.5}$$

and

$$\|\pi_x(f)\delta_x\|^2_{l^2(s^{-1}(x))} = \sum_{\gamma \in s^{-1}(x)} |f(\gamma)|^2 . \tag{2.6}$$

(2.3) follows from (2.5) and (2.4) follows from (2.6). □

Lemma 2.4. — *Let $f \in B_c(G)$ be supported in a bisection. Then $\|f\| = \sup_{\gamma \in G} |f(\gamma)|$.*

Proof. — Let U be a bisection containing $\operatorname{supp} f$. Define $\tilde{f} : G \to \mathbb{C}$ such that $\tilde{f}(\gamma) = 0$ when $r(\gamma) \notin r(U)$ and $\tilde{f}(\gamma) = \overline{f(\mu)}$ where $\mu \in U$ is the unique element with $r(\mu) = r(\gamma)$ when $r(\gamma) \in r(U)$. Let $x \in G^0$ and define $V\colon l^2(s^{-1}(x)) \to l^2(s^{-1}(x))$ such that $V\varphi(\gamma) = 0$ when $r(\gamma) \notin r(U)$ and $V\varphi(\gamma) = \varphi(\mu^{-1}\gamma)$ when $r(\gamma) \in r(U)$, where $\mu \in U$ is the element with $r(\mu) = r(\gamma)$. Then $\|V\| \leqslant 1$. Let $\varphi, \psi \in l^2\left(s^{-1}(x)\right)$. Then

$$\begin{aligned}|\langle \pi_x(f)\varphi,\psi\rangle| &= \Big|\sum_{\gamma\in s^{-1}(x)}\sum_{\gamma_1\gamma_2=\gamma} f(\gamma_1)\varphi(\gamma_2)\overline{\psi(\gamma)}\Big| \\ &= \Big|\sum_{\gamma\in s^{-1}(x)} V\varphi(\gamma)\overline{\tilde f(\gamma)\psi(\gamma)}\Big| \\ &\leqslant \|V\varphi\|\,\|\psi\|\sup_{\gamma}\left|\tilde f(\gamma)\right| \\ &\leqslant \|\varphi\|\|\psi\|\sup_{\gamma\in G}|f(\gamma)|\,.\end{aligned}$$

It follows that $\|f\| \leqslant \sup_{\gamma\in G}|f(\gamma)|$. Equality holds by (2.3). □

Let $B_0(G)$ denote the space of bounded functions on G which vanishes at infinity. We consider $B_0(G)$ as a Banach space in the supremum norm $\|\cdot\|_\infty$. It follows from (2.3) that the inclusion $B_c(G) \subseteq B_0(G)$ extends to a linear map $j\colon B_r^*(G) \to B_0(G)$ such that

(2.7) $$\|j(b)\|_\infty \leqslant \|b\|_{B_r^*(G)}\,.$$

Lemma 2.5 (Proposition 4.2 (iii) in Chapter 3 of [26]). — *Let $a,b \in B_r^*(G)$. Then $j(b)|_{s^{-1}(x)} \in l^2\left(s^{-1}(x)\right), j(a)|_{r^{-1}(x)} \in l^2\left(r^{-1}(x)\right)$ for all $x \in G^0$, and*

(2.8) $$j(a^*)(\gamma) = \overline{j(a)(\gamma^{-1})},$$

and

(2.9) $$j(ab)(\gamma) = \sum_{\gamma_1\gamma_2=\gamma} j(a)\left(\gamma_1\right)j(b)\left(\gamma_2\right)$$

for all $\gamma\in G$.

Proof. — Choose sequences $\{f_n\},\{g_n\}\subseteq B_c(G)$ such that $a=\lim_{n\to\infty} f_n$ and $b=\lim_{n\to\infty} g_n$ in $B_r^*(G)$. It follows from (2.3) that

$$j(a^*)(\gamma) = \lim_{n\to\infty} f_n^*(\gamma) = \lim_{n\to\infty}\overline{f_n(\gamma^{-1})} = \overline{j(a)\left(\gamma^{-1}\right)}$$

which gives (2.8). It follows from (2.4) that $j(b)|_{s^{-1}(x)}$ is the limit in $l^2\left(s^{-1}(x)\right)$ of the sequence $\left\{g_n|_{s^{-1}(x)}\right\}$. Inserting f^* for f in (2.4) we obtain the inequality

(2.10) $$\sum_{\gamma\in r^{-1}(x)}|f(\gamma)|^2 \leqslant \|f\|^2$$

when $x\in G^0$ and $f\in B_c(G)$. Then (2.10) implies that $j(a)|_{r^{-1}(x)}$ is the limit in $l^2\left(r^{-1}(x)\right)$ of the sequence $\left\{f_n|_{r^{-1}(x)}\right\}$. In particular, $j(b)|_{s^{-1}(x)}$ and $j(a)|_{r^{-1}(x)}$ are both square-summable functions for all $x\in G^0$ and hence the righthand side of (2.9) makes sense for each $\gamma \in G$. Let $\gamma \in G$ and set $x = r(\gamma), y = s(\gamma)$. We have then the estimate

$$\begin{aligned}
&\Big|\sum_{\gamma_1\gamma_2=\gamma} j(a)(\gamma_1)\, j(b)(\gamma_2) - j(f_n \star g_n)(\gamma)\Big| \\
&\qquad = \Big|\sum_{\gamma_1\gamma_2=\gamma} j(a)(\gamma_1)\, j(b)(\gamma_2) - \sum_{\gamma_1\gamma_2=\gamma} f_n(\gamma_1)\, g_n(\gamma_2)\Big| \\
&\qquad \leqslant \sum_{\gamma_1\gamma_2=\gamma} |j(a)(\gamma_1) - f_n(\gamma_1)|\, |j(b)(\gamma_2)| \\
&\qquad\qquad\qquad + \sum_{\gamma_1\gamma_2=\gamma} |f_n(\gamma_1)|\, |g_n(\gamma_2) - j(b)(\gamma_2)| \\
&\qquad \leqslant \|j(a) - f_n\|_{l^2(r^{-1}(x))}\, \|j(b)\|_{l^2(s^{-1}(y))} \\
&\qquad\qquad\qquad + \|f_n\|_{l^2(r^{-1}(x))}\, \|g_n - j(b)\|_{l^2(s^{-1}(y))}\,.
\end{aligned}$$

The equality (2.9) follows then by letting n tend to infinity. □

Corollary 2.6 (Proposition 4.2 (i) in Chapter 3 of [26]). *$j\colon B_r^*(G) \to B_0(G)$ is injective.*

Proof. — If $j(a) = 0$ it follows from (2.9) that $j(ab) = 0$ for all $b \in B_r^*(G)$. Now note that it follows from (2.3) that the equality (2.5) extends by continuity to the equality

$$\langle \pi_x(d)\delta_x, \delta_\gamma\rangle = j(d)(\gamma)\,,$$

valid for all $d \in B_r^*(G)$, all $x \in G^0$ and all $\gamma \in s^{-1}(x)$. Since $j(ab) = 0$ for all $b \in B_r^*(G)$ this implies that

$$\langle \pi_x(a)\pi_x(b)\delta_x, \delta_\gamma\rangle = \langle \pi_x(ab)\delta_x, \delta_\gamma\rangle = 0$$

for all $x \in G^0, b \in B_r^*(G)$ and all $\gamma \in s^{-1}(x)$. Since δ_x is cyclic for π_x this implies that $\pi_x(a) = 0$ for all x, i.e. $a = 0$. □

Since G^0 is closed in G we have an embedding $B_c(G^0) \subseteq B_c(G)$. Let $B_0(G^0)$ be the C^*-algebra of bounded functions on G^0 which vanish at infinity. Note that $\sup_{x\in G^0} \|\pi_x(f)\| = \sup_{y\in G^0} |f(y)|$ when $f \in B_c(G^0)$ by Lemma 2.4. It follows that the embedding $B_c(G^0) \subseteq B_c(G)$ extends by continuity to an isometric $*$-homomorphism $B_0(G^0) \to B_r^*(G)$. In the following we will consider $B_0(G^0)$ as a C^*-subalgebra of $B_r^*(G)$ via this embedding. It follows from (2.3) and (2.5) that there is a conditional expectation

$$P_G\colon B_r^*(G) \to B_0(G^0)$$

defined such that $P_G(a)(x) = \langle \pi_x(a)\delta_x, \delta_x\rangle$. Then

$$P_G(a)(x) = j(a)(x) \tag{2.11}$$

for all $a \in B_r^*(G)$, $x \in G^0$.

LEMMA 2.7. — *Let $E \subseteq G$ be a subset which is both closed and open in G. Let $f_1, f_2, \dots, f_n \in C_c(G)$, and let $V_\alpha, \alpha \in I$, be a collection of open sets in G such that $E \subseteq \bigcup_{\alpha \in I} V_\alpha$.*

It follows that there are functions $h_1^j, h_2^j, \dots, h_n^j \in C_c(G), j = 1, 2, \dots, m$, such that

a) $\sum_{j=1}^m h_1^j \star h_2^j \star \cdots \star h_n^j(\gamma) = \begin{cases} f_1 \star f_2 \star \cdots \star f_n(\gamma), & \gamma \in E \\ 0, & \gamma \notin E, \end{cases}$

and

b) *for each $j \in \{1, 2, \dots, m\}$ there is an $\alpha_j \in I$ such that*

$$\operatorname{supp} h_1^j \star h_2^j \star \cdots \star h_n^j \ \subseteq V_{\alpha_j}.$$

Proof. — We say that a function $k \colon G^n \to \mathbb{C}$ is of *product type* when there are functions $k_1, k_2, \dots, k_n \in C_c(G)$ such that

$$k(\gamma_1, \gamma_2, \dots, \gamma_n) = k_1(\gamma_1) k_2(\gamma_2) \cdots k_n(\gamma_n)$$

for all $(\gamma_1, \gamma_2, \dots, \gamma_n) \in G^n$. Set

$$G^{(n)} = \left\{ (\gamma_1, \gamma_2, \dots, \gamma_n) \in G^n \colon s(\gamma_i) = r(\gamma_{i+1}),\ i = 1, 2, \dots, n-1 \right\}.$$

For each $\alpha \in I$, set

$$A_\alpha = \left\{ (\gamma_1, \gamma_2, \dots, \gamma_n) \in G^{(n)} \colon \gamma_1 \gamma_2 \cdots \gamma_n \in V_\alpha \cap E \right\}$$

which is an open subset of $G^{(n)}$. Let

$$A = \left\{ (\gamma_1, \gamma_2, \dots, \gamma_n) \in G^{(n)} \colon \gamma_1 \gamma_2 \cdots \gamma_n \in E \right\}$$

and note that A is both open and closed in $G^{(n)}$. Let $\Omega_\alpha \subseteq G^n$ be an open subset such that $\Omega_\alpha \cap G^{(n)} = A_\alpha$. Since $(\operatorname{supp} f_1 \times \operatorname{supp} f_2 \times \cdots \times \operatorname{supp} f_n) \cap A \cap G^{(n)}$ is a compact subset of G^n contained in $\bigcup_{\alpha \in I} \Omega_\alpha$ there is a cover $\Omega'_\beta, \beta \in I'$, of

$$(\operatorname{supp} f_1 \times \operatorname{supp} f_2 \times \cdots \times \operatorname{supp} f_n) \cap A \cap G^{(n)}$$

in G^n such that each Ω'_β is an open rectangle, *i.e.* of the form

$$\Omega'_\beta = U_1 \times U_2 \times \cdots \times U_n,$$

where each U_i is an open subset of G, and such that the closure, $\overline{\Omega'_\beta}$, of each Ω'_β is contained in Ω_α for some α. By compactness there is a finite set $\{\beta_1, \beta_2, \dots, \beta_{m'}\} \subseteq I'$ such that

$$(\operatorname{supp} f_1 \times \operatorname{supp} f_2 \times \cdots \times \operatorname{supp} f_n) \cap A \cap G^{(n)} \subseteq \bigcup_{j=1}^{m'} \Omega'_{\beta_j}.$$

For each $j \in \{1, 2, \ldots, m'\}$ there is a positive function $g_j \in C_c(G^n)$ of product type such that $g_j(\xi) = 1, \xi \in \overline{\Omega'_{\beta_j}}$, and $\operatorname{supp} g_j \subseteq \Omega_{\alpha_j}$ for some $\alpha_j \in I$ with $\overline{\Omega'_{\beta_j}} \subseteq \Omega_{\alpha_j}$. Define $h_j, j = 1, 2, \ldots, m'$, such that $h_1 = g_1$ and $h_{i+1} = (1-g_1)(1-g_2)\cdots(1-g_i)g_{i+1}, 1 \leqslant i \leqslant m'-1$. Then $h_1(\xi)+h_2(\xi)+\cdots+h_{m'}(\xi) = 1$ when $\xi \in (\operatorname{supp} f_1 \times \operatorname{supp} f_2 \times \cdots \times \operatorname{supp} f_n) \cap A \cap G^{(n)}$. Furthermore, each h_j is the sum of functions of product type, each of which has its support contained in some Ω_α. Let $h'_j, j = 1, 2, \ldots, m$, be an enumeration of these functions such that $\sum_{j=1}^m h'_j = \sum_{j=1}^{m'} h_j$. Since h'_j is of product type there are functions $k_1^j, k_2^j, \ldots, k_n^j \in C_c(G)$ such that

$$h'_j(\gamma_1, \gamma_2, \ldots, \gamma_n) = k_1^j(\gamma_1)k_2^j(\gamma_2)\cdots k_n^j(\gamma_n)$$

for all $(\gamma_1, \gamma_2, \ldots, \gamma_n) \in G^n$. Set $h_i^j = k_i^j f_i$. Then $h_1^j, h_2^j, \ldots, h_n^j \in C_c(G)$, $j = 1, 2, \ldots, m$, satisfy a) and b) by construction. □

Lemma 2.8.

i) *P_G is positive, i.e. $a \geqslant 0$ in $B_r^*(G) \Rightarrow P_G(a) \geqslant 0$ in $B_0\left(G^0\right)$.*
ii) *$P_G(b) = b$ when $b \in B_0\left(G^0\right)$.*
iii) *$\|P_G\| = 1$.*
iv) *P_G is faithful, i.e. $a \neq 0 \Rightarrow P_G(a^*a) \neq 0$.*
v) *$P_G\left(C_r^*(G)\right) = C_r^*(G) \cap B_0(G^0) = \overline{\operatorname{alg}^* G \cap B_c(G^0)}$.*

Proof.

i), ii) and iii) hold by construction.

iv): Let $a \in B_r^*(G)$ and assume that $P_G(a^*a) = 0$. It follows then from (2.11) and (2.9) that

$$\sum_{\mu \in s^{-1}(x)} |j(a)(\mu)|^2 = \sum_{\gamma_1\gamma_2 = x} \overline{j(a)\left(\gamma_1^{-1}\right)} j(a)(\gamma_2) = j(a^*a)(x) = 0$$

for all $x \in G^0$. This shows that $j(a) = 0$ and it follows then from Corollary 2.6 that $a = 0$.

v) : The inclusions $\overline{\operatorname{alg}^* G \cap B_c(G^0)} \subseteq C_r^*(G) \cap B_0(G^0) \subseteq P_G\left(C_r^*(G)\right)$ are obvious so it suffices to show that $P_G\left(\operatorname{alg}^* G\right) \subseteq \operatorname{alg}^* G \cap B_c(G^0)$. Since $P_G(a) = j(a)|_{G^0} = a|_{G^0}$ when $a \in \operatorname{alg}^* G$, this follows from Lemma 2.7, applied with $E = G^0$. □

It follows from Lemma 2.8 and a result of Tomiyama that

$$P_G(d_1 a d_2) = d_1 P_G(a) d_2 \tag{2.12}$$

for all $a \in C_r^*(G),\ d_1, d_2 \in C_r^*(G) \cap B_0(G^0)$; a fact which can also easily be established directly.

Lemma 2.9. — *Assume that $n \in \operatorname{alg}^* G$ is supported in a bisection. It follows that*

$$n^* P_G(a) n = P_G(n^* a n) \tag{2.13}$$

for all $a \in C_r^*(G)$.

Proof. — Let U be a bisection containing $\operatorname{supp} n$. It follows from Lemma 2.5 that

$$\begin{aligned} j(n^* P_G(a) n)(\gamma) &= \sum_{\gamma_1\gamma_2\gamma_3=\gamma} \overline{n(\gamma_1^{-1})} j(P_G(a))(\gamma_2) n(\gamma_3) \\ &= \begin{cases} 0, & \text{when } \gamma \notin s(U) \\ \overline{n(\mu)} j(a)(r(\mu)) n(\mu) & \text{where } \mu \in U \cap s^{-1}(\gamma), \text{ when } \gamma \in s(U)\ . \end{cases} \end{aligned}$$

This is the same expression we find for $j(P_G(n^* a n))(\gamma)$ and hence (2.13) follows from Corollary 2.6. □

Lemma 2.10. — *Let $H \subseteq G$ be an open subgroupoid, i.e. H is open, $H^{-1} = H$ and $(\gamma_1, \gamma_2) \in H^2 \cap G^{(2)} \Rightarrow \gamma_1\gamma_2 \in H$. Then the inclusions $C_c(H) \subseteq C_c(G)$ and $B_c(H) \subseteq B_c(G)$ extend to C^*-algebra embeddings $C_r^*(H) \subseteq C_r^*(G)$ and $B_r^*(H) \subseteq B_r^*(G)$, respectively.*

Proof. — Clearly, the inclusion $C_c(H) \subseteq C_c(G)$ extends to an inclusion $\operatorname{alg}^* H \subseteq \operatorname{alg}^* G$ of $*$-algebras so it remains only to show that $\|f\|_{B_r^*(G)} = \|f\|_{B_r^*(H)}$ when $f \in B_c(H)$. To this end, let $x \in G^0$ and let $f \in B_c(H)$. We define an equivalence relation $\sim$ on $Hs^{-1}(x)$ such that $\gamma \sim \gamma'$ if and only if $\gamma' = \mu\gamma$ for some $\mu \in H$. Let $\left[Hs^{-1}(x)\right]$ denote the set of equivalence classes in $Hs^{-1}(x)$. Then

$$l^2\left(s^{-1}(x)\right) = \oplus_{\xi \in [Hs^{-1}(x)]} l^2(\xi) \oplus l^2\left(s^{-1}(x) \setminus Hs^{-1}(x)\right)$$

and $\pi_x(f)$ respects this direct sum decomposition. Since $\pi_x(f) = 0$ on $l^2\left(s^{-1}(x) \setminus Hs^{-1}(x)\right)$ we find that

$$\|\pi_x(f)\| = \sup_{\xi \in [Hs^{-1}(x)]} \left\|\pi_x(f)|_{l^2(\xi)}\right\|. \tag{2.14}$$

Let $\xi \in \left[Hs^{-1}(x)\right]$ and fix a representative $\gamma_0 \in \xi$. We can then define a unitary $V\colon l^2\left(H \cap s^{-1}\left(r(\gamma_0)\right)\right) \to l^2(\xi)$ such that $V\psi(\mu) = \psi\left(\mu\gamma_0^{-1}\right)$. It is then straightforward to verify that

$$V \pi_{r(\gamma_0)}(f)|_{l^2(H \cap s^{-1}(r(\gamma_0)))} V^* = \pi_x(f)|_{l^2(\xi)},$$

and we conclude that

$$\left\|\pi_x(f)|_{l^2(\xi)}\right\| = \left\|\pi_{r(\gamma_0)}(f)|_{l^2(H \cap s^{-1}(r(\gamma_0)))}\right\| \leqslant \|f\|_{B_r^*(H)}\,.$$

Combined with (2.14), and using that $x \in G^0$ was arbitrary, this shows that $\|f\|_{B_r^*(G)} \leqslant \|f\|_{B_r^*(H)}$. Since the reversed inequality is trivial, this completes the proof. □

Lemma 2.11. — *Let $g \in C_c(G)$ and $f \in \operatorname{alg}^* G$. Then the pointwise product*

$$g \cdot f(\gamma) = g(\gamma) f(\gamma), \gamma \in G,$$

is in $\operatorname{alg}^* G$.

Proof. — It follows from Lemma 2.7, applied with $E = G$, that f is a finite sum of elements from $\operatorname{alg}^* G$ whose compact supports are contained in bisections. We may therefore assume that f has support in a bisection. Then an argument from Lemma 4.3 of [14] completes the proof: Define first $u_0 \colon r(\operatorname{supp} f) \to \mathbb{C}$ such that $u_0(x) = g(\mu)$, where $\mu \in \operatorname{supp} f$ is the unique element with $r(\mu) = x$, and let $u \in C_c(G^0)$ be an extension of u_0. Then $g(\gamma) f(\gamma) = u(r(\gamma)) f(\gamma) = u \star f(\gamma)$ for all $\gamma \in G$. □

Lemma 2.12. — *Let $a \in C_r^*(G)$ and let $h \in C_c(G)$ be supported in a bisection. There is then an element $h \cdot a \in C_r^*(G)$ such that $j(h \cdot a)(\gamma) = h(\gamma) j(a)(\gamma)$ for all $\gamma \in G$.*

Proof. — Define a function $\tilde{h} \colon G \to \mathbb{C}$ such that $\tilde{h}(\gamma) = 0$ when $r(\gamma) \notin s(\operatorname{supp} h)$ and $\tilde{h}(\gamma) = h(\gamma')$ where $\gamma' \in \operatorname{supp} h$ is the unique element of $s^{-1}(r(\gamma)) \cap \operatorname{supp} h$ when $r(\gamma) \in s(\operatorname{supp} h)$. Then

$$\sum_{\gamma_1 \gamma_2 = \gamma} h(\gamma_1) f(\gamma_1) \varphi(\gamma_2) = \sum_{\gamma_1 \gamma_2 = \gamma} f(\gamma_1) \tilde{h}(\gamma_2) \varphi(\gamma_2)$$

when $f \in \operatorname{alg}^* G$ and $\varphi \in l^2\left(s^{-1}(x)\right)$. It follows that $\|h \cdot f\| \leqslant \|f\| \, \|h\|_\infty$ for all $f \in \operatorname{alg}^* G$. In particular, it follows that $\{h \cdot f_n\}$ converges in $B_r^*(G)$ when $\{f_n\} \subseteq \operatorname{alg}^* G$ converges to a. It follows from Lemma 2.11 that the limit $h \cdot a = \lim_{n \to \infty} h \cdot f_n$ exists in $C_r^*(G)$. The limit will have the stated property since $j(h \cdot a)(\gamma) = \lim_{n \to \infty} j(h \cdot f_n)(\gamma)$ for all γ. □

2.2. Ideals

Let A be a C^*-algebra and $D \subseteq A$ an abelian C^*-subalgebra. An element $a \in A$ is a *D-normalizer* when $a^* D a \subseteq D$ and $a D a^* \subseteq D$. The set of D-normalizers will be denoted by $N(D)$.

Consider now the case where $A = C_r^*(G)$ and

$$D = D_G = C_r^*(G) \cap B_0(G^0).$$

Let $N_0(D_G)$ denote the set of functions g from $\operatorname{alg}^* G$ that are supported in a bisection. It follows from Lemma 2.9 that $N_0(D_G) \subseteq N(D_G)$. A (closed) ideal $J \subseteq D_G$ is said to be *G-invariant* when $n^* J n \subseteq J$ for all $n \in N_0(D_G)$. Note that $I \cap D_G$ is a G-invariant ideal in D_G when I is a (closed and twosided) ideal in $C_r^*(G)$.

LEMMA 2.13. — *Let $J \subseteq D_G$ be a G-invariant ideal. It follows that*

$$\widehat{J} = \left\{a \in C_r^*(G) \colon P_G(a^*a) \in J\right\}$$

is an ideal in $C_r^*(G)$ such that $J = \widehat{J} \cap D_G$.

Proof. — It follows easily, by using the relations $x^*y^*yx \leqslant \|y\|^2 x^*x$ and $(x+y)^*(x+y) \leqslant 2x^*x + 2y^*y$, that $\widehat{J}$ is a left ideal in $C_r^*(G)$. It follows from Lemma 2.9 that $a \in \widehat{J} \Rightarrow an \in \widehat{J}$ when $n \in N_0(D)$ because J is G-invariant. It follows from Lemma 2.11 that the elements of $N_0(D)$ span a dense subspace in $C_r^*(G)$. We conclude therefore that $\widehat{J}$ is also a right-ideal. This proves the lemma because the identity $J = \widehat{J} \cap D_G$ is obvious. □

Note that it follows from Lemma 2.13 that the lattice of G-invariant ideals in D_G has a copy inside the lattice of ideals in $C_r^*(G)$.

An ideal in a C^*-algebra is said to be *non-trivial* when it is neither $\{0\}$ nor the whole algebra. With this terminology we have

COROLLARY 2.14. — *Assume that D_G contains a non-trivial ideal which is G-invariant. It follows that $C_r^*(G)$ contains a non-trivial ideal.*

For $x \in G^0$ we let $G_x = \{\gamma \in G \colon r(\gamma) = s(\gamma) = x\}$ denote the *isotropy group at x.*

LEMMA 2.15. — *Let $I \subseteq C_r^*(G)$ be an ideal such that $I \cap D_G = \{0\}$. It follows that*

$$j(a)(x) = 0$$

for all $a \in I$ and all $x \in G^0$ with $G_x = \{x\}$.

Proof. — Let $h \in \operatorname{alg}^* G$ and let $x \in G^0$ be a point with trivial isotropy (*i.e.* $G_x = \{x\}$). We assume that $h(x) \neq 0$. Consider a point $\gamma \in G$. If $r(\gamma) = x$ and $\gamma \neq x$, we know that $s(\gamma) \neq x$. There is therefore an open neighborhood U_γ of γ such that $r\left(\overline{U_\gamma}\right) \cap s\left(\overline{U_\gamma}\right) = \emptyset$. If $r(\gamma) \neq x$ there is an open neighborhood U_γ of γ such that $x \notin r\left(\overline{U_\gamma}\right)$. Finally, if $\gamma = x$ there is an open neighborhood U_γ of γ such that $U_\gamma \subseteq G^0$. It follows from Lemma 2.7, applied with $E = G$, that there are elements $h_i \in \operatorname{alg}^* G$ and distinct elements $\gamma_i \in G$ such that $\operatorname{supp} h_i \subseteq U_{\gamma_i}, i = 1, 2, \ldots, N$, and $h = \sum_{i=1}^N h_i$. By construction x is only element of one member from

$U_{\gamma_i}, i = 1, 2, \ldots, N$. For convenience we assume that $x \in U_{\gamma_1}$. Then $\gamma_1 = x$ and $U_{\gamma_1} \subseteq G^0$. For each $j \geqslant 2$, $x \notin r\left(\overline{U_{\gamma_j}}\right)$ or $x \notin s\left(\overline{U_{\gamma_j}}\right)$. There is therefore a function $f \in C_c(G^0)$ such that $0 \leqslant f \leqslant 1, f(x) = 1$ and $f \star h_j = 0$ or $h_j \star f = 0$, $j \geqslant 2$. It follows that $f \star h_i \star f = 0$ when $i \neq 1$. Hence $f \star h \star f = f \star h_1 \star f \in D_G$. Let $q : C_r^*(G) \to C_r^*(G)/I$ be the quotient map. Since q is injective on D_G we find that

$$\begin{aligned} \|q(h)\| \geqslant \|q(f \star h \star f)\| &= \|q\left(f \star h_1 \star f\right)\| \\ &= \|f \star h_1 \star f\| \\ &= \sup_{y \in G^0} |f \star h_1 \star f(y)| \geqslant |h(x)|. \end{aligned}$$

Let $a \in C_r^*(G)$. There is a sequence $\{h_k\} \subseteq \operatorname{alg}^* G$ such that $a = \lim_{k\to\infty} h_k$ in $C_r^*(G)$. It follows that

$$\|q(a)\| = \lim_{k\to\infty} \|q(h_k)\| \geqslant \lim_{k\to\infty} |h_k(x)| = |j(a)(x)|.$$

This proves the lemma. □

Lemma 2.16. — *Assume that $G_x = \{x\}$ for some $x \in G^0$. Let I be a non-trivial ideal in $C_r^*(G)$. It follows that either $I \cap D_G$ or $\overline{P_G(I)}$ is a non-trivial G-invariant ideal in D_G.*

Proof. — Unless the intersection $I \cap D_G$ is zero it will constitute an ideal in D_G which must be non-trivial because D_G contains an approximate unit for $C_r^*(G)$. Since $I \cap D_G$ is G-invariant it suffices to show that $\overline{P_G(I)}$ is a non-trivial G-invariant ideal in D_G when $I \cap D_G = \{0\}$. First observe that it is an ideal because of (2.12). Since P_G is faithful by iv) of Lemma 2.8 we have that $P_G(I) \neq 0$ since $I \neq 0$. By assumption there is a point $x \in G^0$ with trivial isotropy and it follows then from Lemma 2.15 and (2.11) that $g(x) = 0$ for all $g \in \overline{P_G(I)}$. In particular, $\overline{P_G(I)} \neq D_G$. Thus $\overline{P_G(I)}$ is a non-trivial ideal in D_G when $I \cap D_G$ fails to be. It is G-invariant by Lemma 2.9. □

Theorem 2.17. — *Assume that $G_x = \{x\}$ for some $x \in G^0$. Then $C_r^*(G)$ is simple if and only if there are no non-trivial G-invariant ideals in D_G.*

Proof. — Combine Lemma 2.16 and Corollary 2.14. □

For the formulation of the following corollary remember that a subset $V \subseteq G^0$ is G-invariant when $\gamma \in G,\ s(\gamma) \in V \Rightarrow r(\gamma) \in V$.

Corollary 2.18. — *Assume that G is étale and that $G_x = \{x\}$ for some $x \in G^0$. It follows that $C_r^*(G)$ is simple if and only if there are no open non-trivial G-invariant subsets of G^0.*

Proof. — Since G is étale, $D_G = C_0(G^0)$. Let $U \subseteq G^0$ be an open subset. By Theorem 2.17 it suffices to show that the ideal $C_0(U)$ of D_G is G-invariant if and only U is G-invariant. Assume first that $C_0(U)$ is G-invariant and let $\gamma \in G$ be such that $s(\gamma) \in U$. There is then an element $h \in N_0(D_G)$ such that $h(\gamma) = 1$. It follows that $h^*h(s(\gamma)) = |h(\gamma)|^2 = hh^*(r(\gamma)) = 1$. Since $s(\gamma) \in U$ there is an $f \in C_0(U)$ such that $f^*h^*hf \in C_0(U)$ and $f^*h^*hf(s(\gamma)) = 1$. Since $hf \in N_0(D_G)$ we find that $hf\,(f^*h^*hf)\,f^*h^* \in C_0(U)$ and hence that $hff^*h^* \in C_0(U)$. Since $hff^*h^*(r(\gamma)) = f^*h^*hf(s(\gamma)) = 1$ this implies that $r(\gamma) \in U$.

Assume next that U is G-invariant and let $f \in C_0(U), h \in N_0(D_G)$. A term in the sum

$$\sum_{\gamma_1\gamma_2\gamma_3=\gamma} h(\gamma_1)f(\gamma_2)\overline{h(\gamma_3^{-1})}$$

is zero unless $\gamma_2 = s(\gamma_1)$ and $\gamma = r(\gamma_1)$. Since U is G-invariant this shows that $hfh^* \in C_0(U)$. □

In comparison with the condition for simplicity which can be derived from Renault's work, note that although the statement does not appear explicitly in [26] his methods can give the conclusion in Corollary 2.18, that simplicity is equivalent to the absence of any non-trivial open G-invariant subset in G^0, under the assumption that points with trivial isotropy is dense in G^0. So what we do in Corollary 2.18 is to reduce the assumption, and in fact to the least possible. Any discrete group whose reduced group C^*-algebra is not simple is an example which shows that in general the existence of at least one unit with trivial isotropy can not be omitted in Theorem 2.17.

In a weak moment one might hope that there is a bijection between the ideals of $C_r^*(G)$ and the G-invariant ideals of D_G in the setting of Theorem 2.17, but elementary examples such as the product of a discrete group and a locally compact Hausdorff space, shows that this is certainly not the case. Theorem 2.17 is only a result on the presence or absence of ideals in $C_r^*(G)$.

2.3. Discrete abelian isotropy and Cartan subalgebras

In general an abelian C^*-subalgebra D of a given C^*-algebra A is *regular* when A is generated as a C^*-algebra by $N(D)$. Following Renault, *cf.* [28], we say that D is a *Cartan subalgebra* in A when

(i) D contains an approximate unit in A;

(ii) D is maximal abelian;
(iii) D is regular, and
(iv) there exists a faithful conditional expectation $Q\colon A \to D$ of A onto D.

Returning to the case where $A = C_r^*(G)$ and $D = D_G = C_r^*(G) \cap B_0(G^0)$, it follows from Lemma 2.8 that P_G is a faithful conditional expectation of $C_r^*(G)$ onto D_G, and from Lemma 2.9 that every $n \in \operatorname{alg}^* G$ which is a supported in a bisection is a D_G-normalizer. This shows that D_G is regular. It is easy to see that D_G contains an approximate unit for $C_r^*(G)$, *cf.* the proof of Theorem 2.23, and there is therefore only one thing missing in Renault's definition of a Cartan subalgebra from [28]: In general D_G is not maximal abelian. In this section we impose additional conditions on G which hold in many of the applications of the theory to dynamical systems and which ensure that D_G is a subalgebra of a larger abelian C^*-algebra which *is* a Cartan subalgebra in the sense of Renault.

Set

$$\operatorname{Is} G = \left\{\gamma \in G\colon r(\gamma) = s(\gamma)\right\}$$

which is sometimes called *the isotropy bundle* of G. Note that $\operatorname{Is} G$ is a closed sub-groupoid of G. In the following we often assume that $\operatorname{Is}(G)$ is abelian, *i.e.* that $\gamma_1\gamma_2 = \gamma_2\gamma_1$ for all $\gamma_1\gamma_2 \in G^{(2)} \cap (\operatorname{Is} G \times \operatorname{Is} G)$.

Set

$$D_G' = \left\{a \in C_r^*(G)\colon \operatorname{supp} j(a) \subseteq \operatorname{Is}(G)\right\}.$$

Lemma 2.19. — *D_G' is a C^*-subalgebra of $C_r^*(G)$. In fact,*

$$D_G' = \left\{a \in C_r^*(G)\colon ah = ha \,\forall h \in C_c(G^0)\right\}. \tag{2.15}$$

Proof. — It suffices to prove (2.15). Let $h \in C_c(G^0)$, $a \in C_r^*(G)$. Then $j(ah)(\gamma) = j(a)(\gamma)h\left(s(\gamma)\right)$ and $j(ha)(\gamma) = h\left(r(\gamma)\right) j(a)(\gamma)$ for all $\gamma \in G$ by Lemma 2.5. Hence $j(ah) = j(ha)$ when $a \in D_G'$ and by Corollary 2.6 this implies that $ah = ha$.

Assume next that $a \in C_r^*(G)$ commutes with every element of $C_c(G^0)$ and consider an element $\gamma \in G$ with $j(a)(\gamma) \neq 0$. If $\gamma \notin \operatorname{Is} G$ we can pick an element $h \in C_c(G^0)$ such that $h(r(\gamma)) = 0$ while $h(s(\gamma)) = 1$. Then $j(ha)(\gamma) = h(r(\gamma))j(a)(\gamma) = 0$ while $j(ah)(\gamma) = j(a)(\gamma)h(s(\gamma)) = j(a)(\gamma) \neq 0$, proving that $j(ah - ha) \neq 0$. By Corollary 2.6 this implies that $ah \neq ha$, contradicting our assumption on a. It follows that $j(a)(\gamma) = 0$ for all $\gamma \in G \setminus \operatorname{Is} G$, *i.e.* $a \in D_G'$. □

Definition 2.20. — *We say that* $\operatorname{Is} G$ *is* discrete *when* $\operatorname{Is} G \setminus G^0$ *is discrete in the topology inherited from* G.

Lemma 2.21. — *Assume that* $\operatorname{Is} G$ *is discrete. It follows that there is a faithful surjective conditional expectation* $Q\colon C_r^*(G) \to D'_G$.

Proof. — Set $\operatorname{Is}_{\mathrm{ess}} G = \{\gamma \in \operatorname{Is} G\colon j(a)(\gamma) \neq 0 \text{ for some } a \in D'_G\}$. Let $\gamma \in \operatorname{Is}_{\mathrm{ess}} G \setminus G^0$. Since $\operatorname{Is} G$ is discrete in the topology inherited from G there is a bisection U such that $U \cap \operatorname{Is} G = \{\gamma\}$. Since $\gamma \in \operatorname{Is}_{\mathrm{ess}} G$ there is also an element $a \in D'_G$ such that $j(a)(\gamma) = 1$. Let $h \in C_c(G)$ be supported in U such that $h(\gamma) = 1$. By Lemma 2.12 there is an element $a_\gamma = h \cdot a \in C_r^*(G)$ such that $j(a_\gamma) = 1_{\{\gamma\}}$. It follows from Corollary 2.6 that $a_\gamma = 1_{\{\gamma\}}$. *I.e.* we have shown that $1_{\{\gamma\}} \in D'_G$ when $\gamma \in \operatorname{Is}_{\mathrm{ess}} G \setminus G^0$.

When $f \in \operatorname{alg}^* G$ the set $\operatorname{supp} f \cap \left(\operatorname{Is} G \setminus G^0\right)$ is finite and we set

$$Q(f) = f|_{\operatorname{Is} G} = f|_{G^0} + \sum_{\gamma \in \operatorname{Is}_{\mathrm{ess}} G \setminus G^0} f(\gamma) 1_{\{\gamma\}}. \tag{2.16}$$

Then $Q(f) \in D'_G$. To estimate the norm of $Q(f)$ in D'_G, observe that for every $x \in G^0$ we have a direct sum decomposition

$$l^2\left(s^{-1}(x)\right) = \oplus_{y \in G^0} l^2\left(s^{-1}(x) \cap r^{-1}(y)\right)$$

which is respected by $\pi_x(g)$ when $g \in B_c(\operatorname{Is} G)$. It follows that

$$\|\pi_x(g)\| = \sup_{y \in G^0} \left\| \pi_x(g)|_{l^2(s^{-1}(x) \cap r^{-1}(y))} \right\|. \tag{2.17}$$

Consider a $y \in G^0$ such that $r^{-1}(y) \cap s^{-1}(x) \neq \emptyset$ and choose $\gamma_0 \in r^{-1}(y) \cap s^{-1}(x)$. We define a unitary $V\colon l^2\left(r^{-1}(y) \cap s^{-1}(x)\right) \to l^2\left(r^{-1}(y) \cap \operatorname{Is} G\right)$ such that $V\psi(\eta) = \psi(\eta\gamma_0)$. Then

$$V \pi_x(g) V^* = \pi_y(g)|_{l^2(r^{-1}(y) \cap \operatorname{Is} G)}$$

and hence (2.17) implies that $\|\pi_x(g)\| \leqslant \sup_{y \in G^0} \left\|\pi_y(g)|_{l^2(r^{-1}(y) \cap \operatorname{Is} G)}\right\|$. It follows first that $\|\pi_x(g)\| \leqslant \|g\|_{B_r^*(\operatorname{Is} G)}$, and then that $\|g\|_{B_r^*(G)} \leqslant \|g\|_{B_r^*(\operatorname{Is} G)}$. Since the reversed inequality is obvious, we conclude that $\|g\|_{B_r^*(G)} = \|g\|_{B_r^*(\operatorname{Is} G)}$. In particular, $\|Q(f)\|_{D'_G} = \|Q(f)\|_{B_r^*(\operatorname{Is} G)}$.

Let $y \in G^0$ and note that

$$\|\pi_y\left(Q(f)\right)\|_{l^2(s^{-1}(y) \cap \operatorname{Is} G)} = \|P_y \pi_y(f) P_y\|_{l^2(s^{-1}(y))},$$

where $P_y\colon l^2\left(s^{-1}(y)\right) \to l^2\left(s^{-1}(y) \cap \operatorname{Is} G\right)$ is the orthogonal projection. It follows that

$$\|\pi_y\left(Q(f)\right)\|_{l^2(s^{-1}(y) \cap \operatorname{Is} G)} \leqslant \|\pi_y(f)\|_{l^2(s^{-1}(y))}.$$

Since $y \in G^0$ was arbitrary we conclude that

$$\|Q(f)\|_{D'_G} = \|Q(f)\|_{B_r^*(\operatorname{Is} G)} \leqslant \|f\|_{C_r^*(G)}.$$

Hence Q extends by continuity to a linear map $Q\colon C_r^*(G) \to D'_G$ of norm 1.

Let $a \in D'_G$. Choose a sequence $\{f_n\} \subseteq \mathrm{alg}^* G$ such that $\lim_{n\to\infty} f_n = a$ in $C^*_r(G)$. Then $\lim_{n\to\infty} Q(f_n) = Q(a)$. Furthermore,

$$\begin{aligned} j(Q(a))(\gamma) &= \lim_{n\to\infty} Q(f_n)(\gamma) \\ &= \begin{cases} 0, & \text{when } \gamma \notin \mathrm{Is}_{\mathrm{ess}}\, G \\ \lim_{n\to\infty} f_n(\gamma) = j(a)(\gamma), & \text{when } \gamma \in \mathrm{Is}_{\mathrm{ess}}\, G. \end{cases} \end{aligned}$$

This shows that $j(Q(a)) = j(a)$, and it follows then from Corollary 2.6 that $Q(a) = a$. Thus $Q\colon C^*_r(G) \to D'_G$ is a linear surjective idempotent map of norm one. It is easy to check that Q is also positive and hence a conditional expectation. Q is faithful because $P_G \circ Q = P_G$ and P_G is faithful by Lemma 2.8. $\square$

Corollary 2.22. — *Assume that* $\mathrm{Is}\, G$ *is discrete. Then*

$$D'_G = \overline{C^*_r(G) \cap B_c(\mathrm{Is}\, G)}.$$

Proof. — The inclusion $C^*_r(G) \cap B_c(\mathrm{Is}\, G) \subseteq D'_G$ is obvious and it follows from Lemma 2.21 and (2.16) that $C^*_r(G) \cap B_c(\mathrm{Is}\, G)$ is dense in D'_G. $\square$

Theorem 2.23. — *Assume that* $\mathrm{Is}\, G$ *is abelian and discrete. It follows that* D'_G *is a Cartan subalgebra of* $C^*_r(G)$.

Proof. — Let $a, b \in D'_G$. Since $\mathrm{Is}\, G$ is abelian it follows from Lemma 2.5 that $j(ab) = j(ba)$. By Corollary 2.6 this implies that $ab = ba$, proving that D'_G is abelian. We check the conditions (i) through (iv) which were listed at the beginning of this section: To check condition (i), note that $C_c\left(G^0\right) \subseteq D'_G$ by Lemma 2.1. It is elementary to check that a bounded and increasing net of non-negative functions from $C_c(G^0)$ which eventually become constant 1 on every compact subset of G^0 will be an approximate unit relative to elements from $\mathrm{alg}^* G$ and hence on all of $C^*_r(G)$.

(ii) follows from (2.15).

To establish (iii) it suffices to show that an element $f \in C_c(G)$ which is supported in a bisection is a D'_G-normalizer. Let $a \in D'_G$, $\gamma \in G$. By Lemma 2.5

$$j(f^*af)(\gamma) = \sum_{\gamma_1\gamma_2\gamma_3=\gamma} \overline{f\left(\gamma_1^{-1}\right)} j(a)\left(\gamma_2\right) f\left(\gamma_3\right).$$

Since $j(a)$ is supported in $\mathrm{Is}\, G$, $\overline{f\left(\gamma_1^{-1}\right)} j(a)\left(\gamma_2\right) f\left(\gamma_3\right)$ is zero unless $r\left(\gamma_3\right) = s\left(\gamma_2\right) = r\left(\gamma_2\right) = s\left(\gamma_1\right)$. Since r and s are both injective on $\mathrm{supp}\, f$ there is an (injective) map $\theta\colon r(\mathrm{supp}\, f) \to s(\mathrm{supp}\, f)$ such that $f(\mu) = 0$ unless $\theta\left(r(\mu)\right) = s(\mu)$. So if $\overline{f\left(\gamma_1^{-1}\right)} j(a)\left(\gamma_2\right) f\left(\gamma_3\right)$ is not zero we must also have that $\theta\left(r(\gamma_3)\right) = s\left(\gamma_3\right)$ and $\theta\left(s\left(\gamma_1\right)\right) = \theta\left(r\left(\gamma_1^{-1}\right)\right) = s\left(\gamma_1^{-1}\right) = r\left(\gamma_1\right)$.

As observed we must also have that $s(\gamma_1) = r(\gamma_3)$ and it follows that $s(\gamma_3) = r(\gamma_1)$. Since $r(\gamma) = r(\gamma_1)$ and $s(\gamma_3) = s(\gamma)$ this implies that $r(\gamma) = s(\gamma)$. Thus $j(f^* a f)$ is supported in $\operatorname{Is} G$, *i.e.* $f^* a f \in D'_G$.

(iv) follows from Lemma 2.21. □

A semi-étale groupoid G is a *semi-étale equivalence relation* when $\operatorname{Is} G = G^0$. To distinguish these groupoids from the more general ones we shall denote a semi-étale equivalence relation by R.

LEMMA 2.24. — *Let R be a semi-étale equivalence relation and $P_R \colon C_r^*(R) \to D_R$ the corresponding conditional expectation. Let $\epsilon > 0$ and $a \in C_r^*(R)$ be given. It follows that there are positive elements $d_i, i = 1, 2, \ldots, N$, in $C_c\left(R^0\right) \subseteq D_R$ such that*

$$\left\| P_R(a) - \sum_{i=1}^{N} d_i a d_i \right\| \leqslant \epsilon.$$

Proof. — Choose $f \in \operatorname{alg}^* R$ such that $\|a - f\| \leqslant \epsilon$. Set $E = R \setminus R^0$. Since there is only trivial isotropy in R we can cover E by open bisections U such that $r(U) \cap s(U) = \emptyset$. We can therefore apply Lemma 2.7 to obtain a decomposition $f = f|_{R^0} + \sum_{j=1}^{M} h_j$ where $h_j \in \operatorname{alg}^* R$ is supported in a bisection U_j with $r(U_j) \cap s(U_j) = \emptyset$. Set $K = (\operatorname{supp} f|_{R^0}) \cup \bigcup_{j=1}^{M} (r(\operatorname{supp} h_j) \cup s(\operatorname{supp} h_j))$ and cover K by a finite open cover $V_i, i = 1, 2, \ldots, N$, such that $V_i \cap r(\operatorname{supp} h_j) \neq \emptyset \Rightarrow V_i \cap s(\operatorname{supp} h_j) = \emptyset$ for all i, j, and let $k_i \in C_c\left(R^0\right), i = 1, 2, \ldots, N$, be a partition of unity on K subordinate to $V_i, i = 1, 2, \ldots, N$. Set $d_i = \sqrt{k_i}$. Then $P_R(f) = f|_{R^0} = \sum_{i=1}^{N} d_i f d_i$ and $\left\| P_R(a) - \sum_{i=1}^{N} d_i a d_i \right\| \leqslant 2\epsilon$. □

3. The C^*-algebra of a locally injective map

Let X and Y be compact metrizable Hausdorff spaces and $\varphi \colon X \to Y$ a continuous and locally injective map. Set

$$R(\varphi) = \left\{(x, y) \in X \times X \colon \varphi(x) = \varphi(y)\right\}.$$

This is a semi-étale relation in the topology inherited from $X \times X$ and we present the information on the structure of $C_r^*(R(\varphi))$ which will be needed in the subsequent sections.

LEMMA 3.1. — *Set $K = \max_x \# \varphi^{-1}(\varphi(x))$. Then*

$$\|d\| \leqslant K \sup_{(x,y) \in R(\varphi)} |j(d)(x, y)|$$

for all $d \in B_r^*(R(\varphi))$.

Proof. — Let $\{f_n\} \subseteq B_c(R(\varphi))$ be a sequence converging to d in $B^*_r(R(\varphi))$. (The subscript c is redundant in this case because $R(\varphi)$ is compact, but we keep it on for consistency with the notation of Section 2.1.) Then $\{f_n\}$ also converges to $j(d)$, uniformly on $R(\varphi)$, by (2.3). It suffices therefore to prove the desired inequality when $d \in B_c(R(\varphi))$: When $x \in X$ and $\psi \in l^2\left(s^{-1}(x)\right)$ we find that

$$\begin{aligned} \|\pi_x(d)\psi\|^2 &= \sum_{y \in s^{-1}(x)} \left| \sum_z d(y,z)\psi(z) \right|^2 \\ &\leqslant \sum_{y \in s^{-1}(x)} \|\psi\|^2_{l^2(s^{-1}(x))} \sum_z |d(y,z)|^2 \\ &\leqslant K^2 \|\psi\|^2_{l^2(s^{-1}(x))} \left(\sup_{(x,y)\in R(\varphi)} |d(x,y)| \right)^2 . \end{aligned}$$

□

We will consider the elements of $B^*_r(R(\varphi))$ as functions on $R(\varphi)$, as we can by Corollary 2.6. It follows then from Lemma 3.1 that the C^*-norm of $B^*_r(R(\varphi))$ is equivalent to the supremum norm.

Set

$$Y \times_\varphi X = \left\{(a,x) \in Y \times X \colon a = \varphi(x)\right\}$$

which is a closed subset of $\varphi(X) \times X$. Let $B(Y \times_\varphi X)$ denote the vector space of bounded complex functions on $Y \times_\varphi X$. We intend to construct an imprimitivity bimodule, in the sense of Rieffel, out of $B(Y \times_\varphi X)$, and we refer to [25] for a nice exposition of the theory we rely on.

When $h, k \in B(Y \times_\varphi X)$ we define $\langle h, k\rangle : R(\varphi) \to \mathbb{C}$ such that

$$\langle h,k\rangle(x,y) = \overline{h(\varphi(x),x)}\, k(\varphi(y),y),$$

and $(h,k)\colon \varphi(X) \to \mathbb{C}$ such that

$$(h,k)(a) = \sum_{z \in \varphi^{-1}(a)} h(a,z)\overline{k(a,z)}.$$

Note that $\langle h, h\rangle$ is positive in $B^*_r(R(\varphi))$ since $\pi_z(\langle h,h\rangle)$ is positive as an operator on $l^2\left(s^{-1}(z)\right)$ for every $z \in X$.

Let $B(\varphi(X))$ be the C^*-algebra of bounded complex functions on $\varphi(X)$. When $g \in B(\varphi(X))$ and $h \in B(Y \times_\varphi X)$ we define $gh \in B(Y \times_\varphi X)$ such that

$$gh(a,x) = g(a)h(a,x),$$

and when $h \in B(Y \times_\varphi X)$, $f \in B_c(R(\varphi))$ we define $hf \in B(Y \times_\varphi X)$ such that

$$hf(a,x) = \sum_{z\in\varphi^{-1}(a)} h(a,z)f(z,x).$$

It is straightforward to check that with these definitions $B(Y \times_\varphi X)$ has all the properties of a $B(\varphi(X))$-$B_r^*(R(\varphi))$-imprimitivity bimodule, except that the fullness of the Hilbert modules may fail. See Definition 3.1 on page 42 of [25].

Let E_φ be the Hilbert $C_r^*(R(\varphi))$-module

$$E_\varphi = \overline{\operatorname{Span}\{fg\colon\ f \in C(Y \times_\varphi X),\ g \in C_r^*(R(\varphi))\}}.$$

In particular, $\langle h,k\rangle \in C_r^*(R(\varphi))$ when $h,k \in E_\varphi$. Set

$$A_\varphi = \overline{\operatorname{Span}\{(h,k)\colon\ h,\ k \in E_\varphi\}},$$

which is unital C^*-subalgebra of $B(\varphi(X))$. Note that E_φ is then a full left Hilbert A_φ-module.

Theorem 3.2. — *E_φ is a full Hilbert $C_r^*(R(\varphi))$-module and hence an A_φ-$C_r^*(R(\varphi))$-imprimitivity bimodule.*

Proof. — We must show that the closed twosided ideal of $C_r^*(R(\varphi))$ generated by

$$\{\langle h,k\rangle : h,k \in C(Y \times_\varphi X)\} \tag{3.1}$$

is all of $C_r^*(R(\varphi))$. By definition $C_r^*(R(\varphi))$ is generated as a C^*-algebra by $C(R(\varphi))$, and it suffices therefore to show that $C(R(\varphi))$ is contained in the closed linear span of the elements from (3.1). Let $f \in C(R(\varphi))$. By Tietze's extension theorem there is a $g \in C(X \times X)$ such that $g|_{R(\varphi)} = f$ and we can therefore approximate f in the supremum norm, and hence also in the C^*-norm of $C_r^*(R(\varphi))$ by a linear combination of functions of the form $\mu \otimes \kappa$, where $\mu, \kappa \in C(X)$ and $\mu \otimes \kappa(x,y) = \mu(x)\kappa(y)$. Define $h,k\colon Y \times_\varphi X \to \mathbb{C}$ such that $h(a,x) = \overline{\mu(x)}$ and $k(a,x) = \kappa(x)$. Since $\langle h,k\rangle = \mu \otimes \kappa$, this completes the proof. $\square$

Corollary 3.3. — *Let Z_φ denote the Gelfand spectrum of A_φ. It follows that there is an $n \in \mathbb{N}$ and a projection $p \in C(Z_\varphi, M_n(\mathbb{C}))$ such that*

$$C_r^*(R(\varphi)) \simeq pC(Z_\varphi, M_n(\mathbb{C}))\,p.$$

Proof. — It follows from Theorem 3.2 that $C_r^*(R(\varphi))$ is Morita equivalent to $C(Z_\varphi)$ and then from [6] and [5] that $C_r^*(R(\varphi))$ is a corner in $C(Z_\varphi) \otimes \mathbb{K}$, where $\mathbb{K}$ denotes the C^*-algebra of compact operators on an infinite dimensional separable Hilbert space. Such a corner has the form $pC(Z_\varphi, M_n(\mathbb{C}))\,p$ for some n and some $p \in C(Z_\varphi, M_n(\mathbb{C}))$. $\square$

In the terminology from the classification program of C^*-algebras, *cf. e.g.* [13], what Corollary 3.3 says is that $C_r^*(R(\varphi))$ is a direct sum of homogeneous C^*-algebras. In the context of type I C^*-algebras, *cf. e.g.* [25], it says that $C_r^*(R(\varphi))$ is a direct sum of n-homogeneous C^*-algebras with trivial Dixmier-Douady invariant. In particular the primitive ideal space of $C_r^*(R(\varphi))$ is Hausdorff. It is possible, but lengthy to give a complete description of the primitive ideal space. Only the following partial results in that direction will be needed.

Let $a \in \varphi(X)$. We can then define a $*$-homomorphism $\tilde{\psi}_a \colon B_c(R(\varphi)) \to M_{\varphi^{-1}(a)}(\mathbb{C})$ such that $\tilde{\psi}_a(h) = h|_{\varphi^{-1}(a)\times\varphi^{-1}(a)}$. Since

$$\left\|h|_{\varphi^{-1}(a)\times\varphi^{-1}(a)}\right\|_{M_{\varphi^{-1}(a)}(\mathbb{C})} = \|\pi_x(h)\| \text{ for any } x \in \varphi^{-1}(a)$$

we find that

$$\left\|h|_{\varphi^{-1}(a)\times\varphi^{-1}(a)}\right\|_{M_{\varphi^{-1}(a)}(\mathbb{C})} \leqslant \|h\| \text{ for all } h \in B_c(R(\varphi)).$$

Hence $\tilde{\psi}_a$ extends to a $*$-homomorphism $\tilde{\psi}_a \colon B_r^*(R(\varphi)) \to M_{\varphi^{-1}(a)}(\mathbb{C})$ which is clearly surjective. Set $\psi_a = \tilde{\psi}_a|_{C_r^*(R(\varphi))}$ which is also surjective since $C(R(\varphi)) \subseteq C_r^*(R(\varphi))$.

Lemma 3.4. — *$\{\ker\psi_a \colon a \in \varphi(X)\}$ is dense in the primitive ideal space $\operatorname{Prim} C_r^*(R(\varphi))$ of $C_r^*(R(\varphi))$.*

Proof. — Let W be a non-empty open subset of $\operatorname{Prim} C_r^*(R(\varphi))$. Since $\operatorname{Prim} C_r^*(R(\varphi))$ is Hausdorff there is an element $d \in C_r^*(R(\varphi))$ such that $d \neq 0$ and

$$\left\{\ker\pi \in \operatorname{Prim} C_r^*(R(\varphi)) : \pi(d) \neq 0\right\} \subseteq W,$$

cf. e.g. [25]. Since $d \neq 0$ there is an $a \in \varphi(X)$ such that $d|_{\varphi^{-1}(a)\times\varphi^{-1}(a)} \neq 0$. It follows that $\ker\psi_a \in W$. □

Lemma 3.5. — *For each $j = 1, 2, 3, \dots$ there is a (possibly zero) projection p_j in the center of $C_r^*(R(\varphi))$ such that*

$$p_j(x,y) = \begin{cases} 1, & \text{when } x = y \text{ and } \#\varphi^{-1}(\varphi(x)) = j, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. — Let p be the function on $R(\varphi)$ which is constant 1. Then $p \in C(R(\varphi)) \subseteq C_r^*(R(\varphi))$ and hence $p \star p^*|_X \in C_r^*(R(\varphi))$. Since

$$p \star p^*(x,x) = \sum_{j=1}^{\infty} j p_j(x,x),$$

it follows from spectral theory that $p_j \in C_r^*(R(\varphi))$ for all j. It is straightforward to check that p_j is central. □

Note that there are only finitely many $j \in \mathbb{N}$ for which $p_j \neq 0$ and that $\sum_j p_j = 1$.

Lemma 3.6. — *Let $z \in \varphi(X)$ and set $j = \#\varphi^{-1}(z)$. There is then an open neighborhood U of z and open sets $V_i, i = 1, 2, \ldots, j$, in X, such that*

1) $\varphi^{-1}\left(\overline{U}\right) \subseteq V_1 \cup V_2 \cup \cdots \cup V_j$,
2) $\overline{V_i} \cap \overline{V_{i'}} = \emptyset, i \neq i'$, *and*
3) φ *is injective on* $\overline{V_i}$ *for each* i.

Proof. — Since φ is locally injective there are open sets $V_i, i = 1, 2, \ldots, j$, such that 2) and 3) hold and

$$\varphi^{-1}(z) \subseteq \bigcup_{i=1}^{j} V_i. \tag{3.2}$$

If there is no open neighborhood U of z for which 1) holds there is a sequence $\{x_n\} \subseteq X \setminus \bigcup_{i=1}^{j} V_i$ such that $\lim_n \varphi(x_n) = z$. A condensation point x of this sequence gives us an element $x \in X \setminus \bigcup_{i=1}^{j} V_i$ such that $\varphi(x) = z$, contradicting (3.2). $\square$

For each $j \in \mathbb{N}$, set

$$L_j = \left\{(x, y) \in R(\varphi) \colon \#\varphi^{-1}(\varphi(x)) = j\right\}.$$

Lemma 3.7. — *Let $a \in C^*_r(R(\varphi))$. Then $a|_{L_j}$ is continuous on L_j for every j.*

Proof. — Since continuity is preserved under uniform convergence it suffices to prove this when $a \in \mathrm{alg}^* R(\varphi)$, and hence in fact when $a = f_1 \star f_2 \star \cdots \star f_N$ for some $f_1, f_2, \ldots, f_N \in C(R(\varphi))$. Let $(x, y) \in L_j$ and set $z = \varphi(x) = \varphi(y)$. Let U and $V_i, i = 1, 2, \ldots, j$, be as in Lemma 3.6. For every $z' \in U$ with $\#\varphi^{-1}(z') = j$ there are unique elements $\lambda_k(z') \in V_k$ such that $\varphi^{-1}(z') = \{\lambda_1(z'), \lambda_2(z'), \ldots, \lambda_j(z')\}$. Then

$$\begin{aligned} &f_1 \star f_2 \star \cdots \star f_N(x', y') \\ &\quad = \sum_{k_1, k_2, \ldots, k_{N-1}} f_1(x', \lambda_{k_1}(\varphi(x'))) f_2(\lambda_{k_1}(\varphi(x')), \lambda_{k_2}(\varphi(x'))) \\ &\qquad\qquad \cdots f_{N-1}\left(\lambda_{k_{N-2}}(\varphi(x')), \lambda_{k_{N-1}}(\varphi(x'))\right) f_N\left(\lambda_{k_{N-1}}(\varphi(x')), y'\right) \end{aligned}$$

when $(x', y') \in L_j$ and $x' \in \varphi^{-1}(U)$. It suffices therefore to prove that each λ_k is continuous on $U \cap \left\{z \in Y \colon \#\varphi^{-1}(z) = j\right\}$. Let $\{a_n\}$ be a sequence in $U \cap \left\{z \in Y \colon \#\varphi^{-1}(z) = j\right\}$ converging to $a \in U \cap \left\{z \in Y \colon \#\varphi^{-1}(z) = j\right\}$. If $\{\lambda_k(a_n)\}$ does not converge to $\lambda_k(a)$ for some k, the sequence $\{\lambda_k(a_n)\}$ will have a condensation point $x \in \overline{V_k} \setminus \{\lambda_k(a)\}$. Since $\varphi(x) = \varphi(\lambda_k(a)) = a$, this contradicts condition 3) of Lemma 3.6. $\square$

4. Dynamical systems

Let X be a compact metrizable Hausdorff space and $\varphi\colon X \to X$ a continuous map. We assume that φ is locally injective. Set

$$\Gamma_\varphi = \left\{(x,k,y) \in X \times \mathbb{Z} \times X \colon \exists a,b \in \mathbb{N},\ k = a-b,\ \varphi^a(x) = \varphi^b(y)\right\}.$$

This is a groupoid with the set of composable pairs being

$$\Gamma_\varphi^{(2)} = \left\{((x,k,y),(x',k',y')) \in \Gamma_\varphi \times \Gamma_\varphi \colon y = x'\right\}.$$

The multiplication and inversion are given by

$$(x,k,y)(y,k',y') = (x,k+k',y') \text{ and } (x,k,y)^{-1} = (y,-k,x).$$

To turn Γ_φ into a locally compact topological groupoid, fix $k \in \mathbb{Z}$. For each $n \in \mathbb{N}$ such that $n + k \geqslant 0$, set

$$\Gamma_\varphi(k,n) = \left\{(x,l,y) \in X \times \mathbb{Z} \times X \colon l = k, \varphi^{k+i}(x) = \varphi^i(y),\ i \geqslant n\right\}.$$

This is a closed subset of the topological product $X \times \mathbb{Z} \times X$ and hence a locally compact Hausdorff space in the relative topology. Since φ is locally injective $\Gamma_\varphi(k,n)$ is an open subset of $\Gamma_\varphi(k,n+1)$ and hence the union

$$\Gamma_\varphi(k) = \bigcup_{n \geqslant -k} \Gamma_\varphi(k,n)$$

is a locally compact Hausdorff space in the inductive limit topology. The disjoint union

$$\Gamma_\varphi = \bigcup_{k \in \mathbb{Z}} \Gamma_\varphi(k)$$

is then a locally compact Hausdorff space in the topology where each $\Gamma_\varphi(k)$ is an open and closed set. In fact, as is easily verified, Γ_φ is a locally compact groupoid in the sense of [26]. In the following we shall often identify the unit space Γ_φ^0 of Γ_φ with X via the map $x \to (x,0,x)$ which is a homeomorphism. The local injectivity of φ ensures that the range map $r(x,k,y) = x$ is locally injective, *i.e.* Γ_φ is semi-étale. Note that every isotropy group of Γ_φ is a subgroup of $\mathbb{Z}$. In particular, $\operatorname{Is}\Gamma_\varphi$ is abelian.

When φ besides being locally injective is also open, and hence a local homeomorphism, Γ_φ is an étale groupoid, which was introduced in increasing generality in [26], [11], [2] and [27]. However, when φ is not open Γ_φ is no longer étale, merely semi-étale.

Lemma 4.1. — *Assume that $\left\{x \in X \colon \varphi^k(x) = x\right\}$ is discrete in the topology inherited from X for all $k \in \mathbb{N}$. It follows that $\operatorname{Is}\Gamma_\varphi$ is discrete.*

Proof. — Let $\gamma = (x_0, k, x_0) \in \operatorname{Is}\Gamma_\varphi \setminus \Gamma^0_\varphi$. Then $k \neq 0$ and $x_0 \in \Gamma_\varphi(k,n)$ for some $n \geqslant 1$. Note that $\varphi^n(x_0)$ is $|k|$-periodic. By assumption there is an open neighborhood U of x_0 such that x_0 is the only element x of U for which $\varphi^n(x)$ is $|k|$-periodic. Then

$$W = \left\{(x,k,y) \in \Gamma_\varphi(k,n) \colon x, y \in U\right\}$$

is an open subset of Γ_φ such that $W \cap \operatorname{Is}\Gamma_\varphi = \{\gamma\}$. □

Theorem 4.2. — *Assume that $\left\{x \in X \colon \varphi^k(x) = x\right\}$ is discrete in the topology inherited from X for all $k \in \mathbb{N}$. It follows that D'_{Γ_φ} is a Cartan subalgebra of $C^*_r(\Gamma_\varphi)$.*

Proof. — This is now a consequence of Theorem 2.23. □

Lemma 4.3. — *Let $\mathbb{T}$ be the unit circle in $\mathbb{C}$. There is a continuous action $\mathbb{T} \ni z \mapsto \beta_z \in \operatorname{Aut} C^*_r(\Gamma_\varphi)$ such that*

$$\beta_z(f)(x,k,y) = z^k f(x,k,y) \tag{4.1}$$

when $f \in C_c(\Gamma_\varphi)$ and $(x,k,y) \in \Gamma_\varphi$.

Proof. — It is straightforward to check that the formula (4.1) defines an automorphism β_z of $B_c(\Gamma_\varphi)$ such that $\beta_z\left(\operatorname{alg}^* \Gamma_\varphi\right) = \operatorname{alg}^* \Gamma_\varphi$. To see that β_z extends by continuity to $C^*_r(\Gamma_\varphi)$, let $x \in X$ and define a unitary U_z on $l^2\left(s^{-1}(x)\right)$ such that

$$U_z \psi(y,k,x) = z^k \psi(y,k,x).$$

Then $\pi_x\left(\beta_z(a)\right) = U_z \pi_x(a) U_z^*$ and hence

$$\|\pi_x(\beta_z(a))\|_{l^2(s^{-1}(x))} = \|\pi_x(a)\|_{l^2(s^{-1}(x))}.$$

It follows that β_z extends to an automorphism of $C^*_r(\Gamma)$ for each $z \in \mathbb{T}$. To check the continuity of $z \mapsto \beta_z(a)$ for each $a \in C^*_r(\Gamma_\varphi)$ it suffices to check when $a = f \in C_c(\Gamma_\varphi)$ is supported in a bisection inside $\Gamma_\varphi(k)$ for some $k \in \mathbb{Z}$. In this case we have the estimate

$$\|\beta_z(f) - \beta_{z'}(f)\| \leqslant \left|z^k - z'^k\right| \sup_{\gamma \in G} |f(\gamma)|$$

by Lemma 2.4. This proves the continuity of $z \mapsto \beta_z$. □

We will refer to the action β from Lemma 4.3 as the *gauge action*. The fixed point algebra of the gauge action will be denoted by $C^*_r\left(\Gamma_\varphi\right)^{\mathbb{T}}$.

4.1. A crossed product description of $C_r^*(\Gamma_\varphi)$

Note that

$$\Gamma_\varphi(0) = \left\{(x,k,y) \in \Gamma_\varphi \colon k = 0\right\}$$

is an open subgroupoid of Γ_φ and hence a semi-étale groupoid in itself. We identify $\Gamma_\varphi(0)$ with

$$\left\{(x,y) \in X \times X \colon \varphi^i(x) = \varphi^i(y) \ \text{ for some } i \in \mathbb{N}\right\}$$

under the map $(x,y) \mapsto (x,0,y)$. Note that $\Gamma_\varphi(0)$ is a semi-étale equivalence relation which we denote by R_φ in the following. Similarly, $\Gamma_\varphi(0,n)$ is an open subgroupoid of $R_\varphi \subseteq \Gamma_\varphi$ and a semi-étale equivalence relation in itself. In fact, $\Gamma_\varphi(0,n)$ is isomorphic, as a semi-étale equivalence relation, to the semi-étale equivalence relation $R(\varphi^n)$ corresponding to the locally injective map φ^n. The isomorphism is given the map $R(\varphi^n) \ni (x,y) \mapsto (x,0,y)$, and it induces an isomorphism $C_r^*(R(\varphi^n)) \simeq C_r^*(\Gamma_\varphi(0,n))$. In the following we suppress these isomorphisms in the notation and identify $R(\varphi^n)$ with $\Gamma_\varphi(0,n)$ and $C_r^*(R(\varphi^n))$ with $C_r^*(\Gamma_\varphi(0,n))$. Then

$$R_\varphi = \bigcup_{n\in\mathbb{N}} R(\varphi^n).$$

It follows from Lemma 2.10 that there are isometric embeddings

$$C_r^*(R(\varphi^n)) \subseteq C_r^*\left(R\left(\varphi^{n+1}\right)\right) \subseteq C_r^*(R_\varphi) \ \text{ for all } n.$$

Since $C_c(R_\varphi) = \bigcup_n C_c(R(\varphi^n))$ it follows that

$$C_r^*(R_\varphi) = \overline{\bigcup_n C_r^*(R(\varphi^n))}. \tag{4.2}$$

Combined with Corollary 3.3 this shows that $C_r^*(R_\varphi)$ is an AH-algebra in the terminology from the classification program for C^*-algebras, *cf. e.g.* [13].

We assume that φ is surjective. The aim is to show that there is then an endomorphism of $C_r^*(R_\varphi)$ such that $C_r^*(\Gamma_\varphi)$ is the crossed product of $C_r^*(R_\varphi)$ by this endomorphism. In the étale case, where φ is open, this crossed product decomposition was established in [2]. Define $m\colon X \to \mathbb{R}$ such that

$$m(x) = \#\left\{y \in X \colon \varphi(y) = \varphi(x)\right\}.$$

It follows from Lemma 3.5 that m is an element of $D_{R(\varphi)}$ which is central in $C_r^*(R(\varphi))$. Note that m is positive and invertible.

Lemma 4.4. — *For each $k \geqslant 1$ there is a $*$-homomorphism $h_k\colon C_r^*\left(R\left(\varphi^k\right)\right) \to C_r^*\left(R\left(\varphi^{k+1}\right)\right)$ such that*

$$h_k(f)(x,y) = m(x)^{-\frac{1}{2}} m(y)^{-\frac{1}{2}} f(\varphi(x),\varphi(y)). \tag{4.3}$$

Proof. — The formula (4.3) makes sense for all $f \in B_c\left(R\left(\varphi^k\right)\right)$ and defines a linear map $B_c\left(R\left(\varphi^k\right)\right) \to B_c\left(R\left(\varphi^{k+1}\right)\right)$ which is continuous for the supremum norms. When $f \in C\left(R\left(\varphi^k\right)\right)$,

$$h_k(f) = m^{-1/2} \star \left[f \circ (\varphi \times \varphi)\right] \star m^{-1/2}$$

which is in $C_r^*\left(R\left(\varphi^{k+1}\right)\right)$ since m is by Lemma 3.5. It suffices therefore to check that $h_k(f^*) = h_k(f)^*$ and $h_k(f \star g) = h_k(f) \star h_k(g)$ when $f, g \in B_c\left(R\left(\varphi^k\right)\right)$. The first property is obvious. We check the second:

$$\begin{aligned}
&h_k(f) \star h_k(g)(x,y) \\
&\quad = \sum_{\{z \in X :\ \varphi^{k+1}(z) = \varphi^{k+1}(x)\}} m(x)^{-\frac{1}{2}} m(y)^{-\frac{1}{2}} m(z)^{-1} \\
&\qquad\qquad \cdot f\left(\varphi(x), \varphi(z)\right) g\left(\varphi(z), \varphi(y)\right) \\
&\quad = \sum_{\{z \in X :\ \varphi(z) = w\}} \sum_{\{w \in X :\ \varphi^k(w) = \varphi^{k+1}(x)\}} m(x)^{-\frac{1}{2}} m(y)^{-\frac{1}{2}} m(z)^{-1} \\
&\qquad\qquad \cdot f\left(\varphi(x), w\right) g\left(w, \varphi(y)\right) \\
&\quad = \sum_{\{w \in X :\ \varphi^k(w) = \varphi^{k+1}(x)\}} m(x)^{-\frac{1}{2}} m(y)^{-\frac{1}{2}} f\left(\varphi(x), w\right) g\left(w, \varphi(y)\right) \\
&\quad = h_k(f \star g)(x,y),
\end{aligned}$$

where the surjectivity of φ was used for the second equality. □

Corollary 4.5. — *The function $X \ni x \mapsto m(\varphi^k(x))$ is in $C_r^*(R(\varphi^{k+1}))$ for $k = 0, 1, 2, 3, \ldots$.*

Proof. — When h_k is the $*$-homomorphism from Lemma 4.4 we have that $m \circ \varphi^k = P_{R(\varphi^{k+1})}\left(m^{\frac{1}{2}} \star h_k(m \circ \varphi^{k-1}) \star m^{\frac{1}{2}}\right)$. In this way the assertion follows from Lemma 4.4 by induction. □

Note that the diagram

$$\begin{tikzcd}
C_r^*\left(R\left(\varphi^k\right)\right) \arrow[r, hook] \arrow[d, "h_k"'] & C_r^*\left(R\left(\varphi^{k+1}\right)\right) \arrow[d, "h_{k+1}"'] \\
C_r^*\left(R\left(\varphi^{k+1}\right)\right) \arrow[r, hook] & C_r^*\left(R\left(\varphi^{k+2}\right)\right)
\end{tikzcd} \tag{4.4}$$

commutes for each k so that we obtain a $*$-endomorphism $\widehat{\varphi}\colon C_r^*\left(R_\varphi\right) \to C_r^*\left(R_\varphi\right)$ defined such that $\widehat{\varphi}|_{C_r^*(R(\varphi^k))} = h_k$.

We define a function $v\colon \Gamma_\varphi \to \mathbb{C}$ such that

$$v(x,k,y) = \begin{cases} m(x)^{-\frac{1}{2}} & \text{when } k=1 \text{ and } y=\varphi(x) \\ 0 & \text{otherwise.} \end{cases}$$

Then v is the product $v = m^{-\frac{1}{2}} 1_{\Gamma_\varphi(1,0)}$ in $C_r^*(\Gamma_\varphi)$. In particular, $v \in C_r^*(\Gamma_\varphi)$. By checking on $C_r^*\left(R\left(\varphi^k\right)\right)$ one finds that

$$vav^* = \widehat{\varphi}(a) \tag{4.5}$$

for all $a \in C_r^*(R_\varphi)$. Similarly a direct computation shows that v is an isometry, *i.e.* $v^*v = 1$. Unlike the étale case considered in [2], the inclusion $v^* C_r^*(R_\varphi)\, v \subseteq C_r^*(R_\varphi)$ can fail in the semi-étale case.

Theorem 4.6. — *Assume that φ is surjective. It follows that $C_r^*(\Gamma_\varphi)$ is generated, as a C^*-algebra, by the isometry v and $C_r^*(R_\varphi)$. In fact, $C_r^*(\Gamma_\varphi)$ is the crossed product*

$$C_r^*(R_\varphi) \times_{\widehat{\varphi}} \mathbb{N}$$

in the sense of Stacey [29]*, and Boyd, Keswani and Raeburn* [4]*.*

Proof. — By definition $C_r^*(\Gamma_\varphi)$ is generated by

$$\bigcup_{n,l\in\mathbb{N}} C_c\left(\Gamma_\varphi(l,n)\right) \cup C_c\left(\Gamma_\varphi(-l,n)\right)$$

so to prove the first assertion it suffices to show that $C_c\left(\Gamma_\varphi(l,n)\right)$ and $C_c\left(\Gamma_\varphi(-l,n)\right)$ are both subsets of the C^*-algebra generated by v and $C_r^*(R_\varphi)$ for every l,n. Assume that $f \in C_c\left(\Gamma_\varphi(l,n)\right)$. Define the function $g\colon R(\varphi^n) \to \mathbb{C}$ such that

$$g(x,y) = f\left(x,l,\varphi^l(y)\right) m\left(\varphi^{l-1}(y)\right)^{-\frac{1}{2}} m\left(\varphi^{l-2}(y)\right)^{-\frac{1}{2}} \cdots m(y)^{-\frac{1}{2}}.$$

It follows from Corollary 4.5 that $g \in C_r^*\left(R\left(\varphi^{l+n}\right)\right)$. Since $f = f\,(v^*)^l\, v^l$ and $f\,(v^*)^l = g$, this shows that f is in the C^*-algebra generated by v and $C_r^*(R_\varphi)$. When $f \in C_c\left(\Gamma_\varphi(-l,n)\right)$ a similar calculation shows that $v^l f \in C_r^*(R_\varphi)$ and hence $f = (v^*)^l\, v^l f$ is in the C^*-algebra generated by v and $C_r^*(R_\varphi)$.

It follows now from the universal property of $C_r^*(R_\varphi) \times_{\widehat{\varphi}} \mathbb{N}$ that there is a surjective $*$-homomorphism $C_r^*(R_\varphi) \times_{\widehat{\varphi}} \mathbb{N} \to C_r^*(\Gamma_\varphi)$ which is the identity on $C_r^*(R_\varphi)$. To show that it is an isomorphism it suffices, by Proposition 2.1 of [4], to show that

$$\left\| \sum_{i\in F} (v^*)^i\, a_{i,i} v^i \right\| \leqslant \left\| \sum_{i,j\in F} (v^*)^i\, a_{i,j} v^j \right\|$$

in $C_r^*(\Gamma_\varphi)$ when $F \subseteq \mathbb{N}$ is a finite set and $\{a_{i,j}\}_{i,j\in F}$ is any collection of elements from $C_r^*(R_\varphi)$. This inequality follows from the existence of the gauge action β of $\mathbb{T}$ on $C_r^*(\Gamma_\varphi)$ since

$$\sum_{i\in F} (v^*)^i a_{i,i} v^i = \int_{\mathbb{T}} \beta_z \left(\sum_{i,j\in F} (v^*)^i a_{i,j} v^j \right) dz.$$

$\square$

Lemma 4.7. — *The endomorphism $\widehat{\varphi}\colon C_r^*(R_\varphi) \to C_r^*(R_\varphi)$ is a full corner endomorphism in the sense that the projection $\widehat{\varphi}(1) = vv^*$ is full in $C_r^*(R_\varphi)$.*

Proof. — Note that

$$\widehat{\varphi}(1)(x,y) = vv^*(x,y) = m(x)^{-\frac{1}{2}} m(y)^{-\frac{1}{2}} 1_{R(\varphi)}(x,y). \tag{4.6}$$

It follows from Lemma 2.24 that the ideal in $C_r^*(R_\varphi)$ generated by $\widehat{\varphi}(1)$ contains $P_{R_\varphi}(\widehat{\varphi}(1))$. And it follows from (4.6) that $P_{R_\varphi}(\widehat{\varphi}(1))$ is the invertible element m^{-1}. Hence the ideal in $C_r^*(R_\varphi)$ generated by $\widehat{\varphi}(1)$ is all of $C_r^*(R_\varphi)$. $\square$

Note that v is a unitary, *i.e.* $vv^* = 1$, if and only if $m = 1$ if and only if φ is a homeomorphism. In this case $C_r^*(R_\varphi) = C(X)$ and $C_r^*(\Gamma_\varphi) \simeq C(X) \times_\varphi \mathbb{Z}$. Such crossed products have been intensively studied and we shall have nothing to add in this case. We therefore restrict attention to the case where φ is surjective, but not injective.

Assume that φ is surjective and not injective. Let B_φ be the inductive limit of the sequence

$$C_r^*(R_\varphi) \xrightarrow{\widehat{\varphi}} C_r^*(R_\varphi) \xrightarrow{\widehat{\varphi}} C_r^*(R_\varphi) \xrightarrow{\widehat{\varphi}} \cdots \tag{4.7}$$

We can then define an automorphism $\widehat{\varphi}_\infty \in \operatorname{Aut} B_\varphi$ such that $\widehat{\varphi}_\infty \circ \rho_{\infty,n} = \rho_{\infty,n} \circ \widehat{\varphi}$, where $\rho_{\infty,n}\colon C_r^*(R_\varphi) \to B_\varphi$ is the canonical $*$-homomorphism from the n'th level in the sequence (4.7) into the inductive limit algebra. In this notation the inverse of $\widehat{\varphi}_\infty$ is defined such that $\widehat{\varphi}_\infty^{-1} \circ \rho_{\infty,n} = \rho_{\infty,n+1}$.

Theorem 4.8. — *Assume that φ is surjective and not injective. It follows that $p = \rho_{\infty,1}(1) \in B_\varphi \subseteq B_\varphi \times_{\widehat{\varphi}_\infty} \mathbb{Z}$ is a full projection and $C_r^*(\Gamma_\varphi)$ is $*$-isomorphic to the corner $p\big(B_\varphi \times_{\widehat{\varphi}_\infty} \mathbb{Z}\big)p$.*

Proof. — In view of Proposition 3.3. of [29], which was restated in [4], it suffices to check that $\rho_{\infty,1}(1)$ is a full projection in B_φ. To this end it suffices to show that $v^n (v^*)^n = \widehat{\varphi}^n(1)$ a full projection in $C_r^*(R_\varphi)$. By noting that

$$v^n (v^*)^n (x,y) = \Big[m(x)m(\varphi(x)) \cdots m(\varphi^{n-1}(x))m(y)m(\varphi(y)) \\ \cdots m(\varphi^{n-1}(y))\Big]^{-\frac{1}{2}} 1_{R(\varphi^n)}(x,y),$$

this follows from Lemma 2.24 as in the proof of Lemma 4.7. □

Corollary 4.9. — *Assume that φ is surjective and not injective. It follows that there is a (non-unital) AH-algebra A and an automorphism α of A such that $C_r^*(\Gamma_\varphi)$ is stably isomorphic to $A \times_\alpha \mathbb{Z}$.*

Proof. — Note that B_φ is the AH-decomposition of $C_r^*(R_\varphi)$ is compatible with $\widehat{\varphi}$. Hence the assertion follows from Theorem 4.8 by the use of Brown's theorem [5]. □

One virtue of Theorem 4.6 and Theorem 4.8 is that the crossed product description and the Pimsner-Voiculescu exact sequence give us a six-term exact sequence which makes it possible to calculate the K-theory of $C_r^*(\Gamma_\varphi)$ from the K-theory of $C_r^*(R_\varphi)$ and the action of $\widehat{\varphi}$ on K-theory. See *e.g.* [11] and [12] for such K-theory calculations in the étale case.

4.2. Simplicity of $C_r^*(\Gamma_\varphi)$ and $C_r^*(R_\varphi)$

Theorem 4.10. — *Assume that φ is surjective and not injective. Then $C_r^*(\Gamma_\varphi)$ is simple if and only if there is no non-trivial ideal I in $C_r^*(R_\varphi)$ such that $\widehat{\varphi}(I) \subseteq I$.*

Proof. — Assume first that I is a non-trivial ideal in $C_r^*(R_\varphi)$ which is $\widehat{\varphi}$-invariant in the specified way. In the notation established before Theorem 4.8, set $J = \overline{\bigcup_n \rho_{\infty,n}(I)}$. Then J is a non-zero $\widehat{\varphi}_\infty$-invariant ideal in B_φ. To prove that $J \neq B_\varphi$, we show that $\rho_{\infty,1}(1) \notin J$. Indeed, if $\rho_{\infty,1}(1) \in J$ there is, for any $\epsilon > 0$, a $k \in \mathbb{N}$ and an element $a \in I$ such that

$$\left\|\widehat{\varphi}^k(1) - a\right\| = \|\rho_{\infty,1}(1) - \rho_{\infty,k}(a)\| \leqslant \epsilon.$$

Since $\widehat{\varphi}^k(1)$ is a projection and I is an ideal in $C_r^*(R_\varphi)$ this implies, with ϵ appropriately small, that $\widehat{\varphi}^k(1) \in I$. And as argued in the proof of Theorem 4.8 $\widehat{\varphi}^k(1) = v^k {v^*}^k$ is a full projection in $C_r^*(R_\varphi)$ and hence $\widehat{\varphi}^k(1) \in I$ implies that $I = C_r^*(R_\varphi)$, contrary to assumption. Hence J is a non-trivial ideal in B_φ. Being $\widehat{\varphi}_\infty$-invariant it gives rise to a non-trivial ideal in $B_\varphi \times_{\widehat{\varphi}_\infty} \mathbb{Z}$. Since $C_r^*(\Gamma_\varphi)$ is stably isomorphic to $B_\varphi \times_{\widehat{\varphi}_\infty} \mathbb{Z}$ by Theorem 4.8 and [5], this means that $C_r^*(\Gamma_\varphi)$ is not simple.

The reversed implication, that $C_r^*(\Gamma_\varphi)$ is simple when there are no non-trivial $\widehat{\varphi}$-invariant ideals in $C_r^*(R_\varphi)$ follows from [4] (and [1]), in particular, from Corollary 2.7 of [4], because $C_r^*(R_\varphi)$ is AH, and hence also strongly amenable. $\square$

Remark 4.11. — The inclusion $C_r^*(R_\varphi) \subseteq C_r^*(\Gamma_\varphi)^{\mathbb{T}}$ is obvious. When φ is open the two algebras are identical, but there are examples of shift spaces where this is a strict inclusion. Similarly, the inclusion $D_{R_\varphi} \subseteq D_{\Gamma_\varphi}$ is obvious and is an identity when φ is open, but a strict inclusion for certain shift spaces.

Let $P_{R_\varphi} : C_r^*(R_\varphi) \to D_{R_\varphi}$ be the conditional expectation. Note that

$$mP_{R_\varphi} \circ \widehat{\varphi}(f) = f \circ \varphi \tag{4.8}$$

for $f \in D_{R_\varphi}$. Thus $mP_{R_\varphi} \circ \widehat{\varphi}$ is a unital injective endomorphism of D_{R_φ} which we denote by $\overline{\varphi}$.

Lemma 4.12.

a) *Let $I \subseteq C_r^*(R_\varphi)$ be a non-trivial ideal in $C_r^*(R_\varphi)$ such that $\widehat{\varphi}(I) \subseteq I$. It follows that $J = I \cap D_{R_\varphi}$ is a non-trivial R_φ-invariant ideal in D_{R_φ} such that $\overline{\varphi}(J) \subseteq J$.*

b) *Let $J \subseteq D_{R_\varphi}$ be a non-trivial R_φ-invariant ideal such that $\overline{\varphi}(J) \subseteq J$. It follows that*

$$\widehat{J} = \left\{a \in C_r^*(R_\varphi) : P_{R_\varphi}(a^*a) \in J\right\}$$

is a non-trivial ideal in $C_r^*(R_\varphi)$ such that $\widehat{\varphi}(\widehat{J}) \subseteq \widehat{J}$.

Proof.

a) It follows from Lemma 2.24 that $P_{R_\varphi}(I) \subseteq I$ and hence that $I \cap D_{R_\varphi}$ is not zero since I is not and P_{R_φ} is faithful. It is not all of D_{R_φ} because it does not contain the unit. Finally, it follows that $\overline{\varphi}\left(I \cap D_{R_\varphi}\right) \subseteq mP_{R_\varphi}\widehat{\varphi}(I) \subseteq mP_{R_\varphi}(I) \subseteq I \cap D_{R_\varphi}$ since $\widehat{\varphi}(I) \subseteq I$, $P_{R_\varphi}(I) \subseteq I \cap D_{R_\varphi}$ and $m \in D_{R_\varphi}$.

b) Recall that $\widehat{J}$ is a non-trivial ideal by Lemma 2.13. Since $P_{R_\varphi} \circ \widehat{\varphi} \circ P_{R_\varphi} = P_{R_\varphi} \circ \widehat{\varphi}$ it follows that

$$P_{R_\varphi}\left(\widehat{\varphi}(a)^*\widehat{\varphi}(a)\right) = P_{R_\varphi} \circ \widehat{\varphi} \circ P_{R_\varphi}(a^*a) = m^{-1}\overline{\varphi}\left(P_{R_\varphi}(a^*a)\right) \in J$$

when $a \in \widehat{J}$. This shows that $\widehat{\varphi}\left(\widehat{J}\right) \subseteq \widehat{J}$. $\square$

Theorem 4.13. — *Assume that φ is surjective and not injective. Then $C_r^*(\Gamma_\varphi)$ is simple if and only if there is no non-trivial R_φ-invariant ideal J in D_{R_φ} such that $\overline{\varphi}(J) \subseteq J$.*

Proof. — Combine Lemma 4.12 and Theorem 4.10. $\square$

When φ is open, and hence a local homeomorphism, Theorem 4.13 follows from Proposition 4.3 of [12].

For all $k, l \in \mathbb{N}$, set

$$X_{k,l} = \left\{x \in X \colon \#\varphi^{-k}\left(\varphi^k(x)\right) = l\right\}.$$

THEOREM 4.14. — *Assume that φ is surjective and not injective. The following conditions are equivalent:*

1) *$C_r^*(\Gamma_\varphi)$ is simple.*
2) *For every open subset $U \subseteq X$ and $k, l \in \mathbb{N}$ such that $U \cap X_{k,l} \neq \emptyset$ there is an $m \in \mathbb{N}$ such that*

$$\bigcup_{j=0}^{m} \varphi^j\left(U \cap X_{k,l}\right) = X.$$

Proof. — For the organization of the proof it is convenient to observe that condition 2) is equivalent to the following:

2′) For every open subset $U \subseteq X$ and $k, l \in \mathbb{N}$ such that $U \cap X_{k,l} \neq \emptyset$ there is a $m \in \mathbb{N}$ such that

$$\varphi^{m+k}\left(\bigcup_{j=0}^{m-1} \varphi^{-j}\left(U \cap X_{k,l}\right)\right) = X.$$

1) $\Rightarrow$ 2′): Assume first that $C_r^*(\Gamma_\varphi)$ is simple. If condition 2′) fails there is an open subset U in X and a pair $k, l \in \mathbb{N}$ such that $U \cap X_{k,l} \neq \emptyset$ and for each $m \in \mathbb{N}$ there is an element $x_m \in X$ such that

$$\varphi^{-m-k}(x_m) \cap \left(\bigcup_{j=0}^{m-1} \varphi^{-j}\left(U \cap X_{k,l}\right)\right) = \emptyset. \tag{4.9}$$

Let $R\left(\varphi^k\right)|_U$ be the reduction of $R\left(\varphi^k\right)$ to U, *i.e.*

$$R\left(\varphi^k\right)|_U = \left\{(x,y) \in R\left(\varphi^k\right) : x, y \in U\right\},$$

which is an open subgroupoid of $R\left(\varphi^k\right)$. It follows from Lemma 2.10 that there is an isometric inclusion

$$C_r^*\left(R\left(\varphi^k\right)|_U\right) \subseteq C_r^*\left(R\left(\varphi^k\right)\right).$$

In the notation used in Lemma 3.4, let ψ_{x_m} be the irreducible representation of $C_r^*\left(R\left(\varphi^{k+m}\right)\right)$ corresponding to x_m. Let $\{p_j\}$ be the central projections of Lemma 3.5 relative to φ^k. It follows from (4.9) that $\psi_{x_m}\left(\widehat{\varphi}^j\left(C_r^*\left(R\left(\varphi^k\right)|_U\right)p_l\right)\right) = 0$ for all $j \in \{0, 1, 2, \ldots, m-1\}$. By composing the normalized trace of $M_{\varphi^{-m-k}(x_m)}(\mathbb{C})$ with ψ_{x_m} we obtain in this way, for each $m \in \mathbb{N}$, a trace state ω_m on $C_r^*\left(R\left(\varphi^{m+k}\right)\right)$ which annihilates $\widehat{\varphi}^j\left(C_r^*\left(R\left(\varphi^k\right)|_U\right)p_l\right)$ for $j = 0, 1, \ldots, m-1$. For each m we choose a state extension ω'_m of ω_m to $C_r^*(R_\varphi)$. Any weak* condensation point of the sequence $\{\omega'_m\}$ will be a trace state ω on $C_r^*(R_\varphi)$ which annihilates $\widehat{\varphi}^j\left(C_r^*\left(R\left(\varphi^k\right)|_U\right)p_l\right)$ for all $j \in \mathbb{N}$. It follows that the closed two-sided ideal I in $C_r^*(R_\varphi)$ generated by

$$\bigcup_{j=0}^{\infty} \widehat{\varphi}^j\left(C_r^*\left(R\left(\varphi^k\right)|_U\right)p_l\right)$$

is contained in $\{a \in C_r^*(R_\varphi) : \omega(a^*a) = 0\}$. Hence I is a non-trivial ideal in $C_r^*(R_\varphi)$. Since $\widehat{\varphi}(I) \subseteq I$ this contradicts the simplicity of $C_r^*(\Gamma_\varphi)$ by Theorem 4.10.

2′) $\Rightarrow$ 1): Let $I \subseteq C_r^*(R_\varphi)$ be a non-zero closed twosided ideal such that $\widehat{\varphi}(I) \subseteq I$. By Theorem 4.10 it suffices to show that $1 \in I$. Since $I \neq 0$ there is a $k \in \mathbb{N}$ such that $I \cap C_r^*\left(R\left(\varphi^k\right)\right) \neq 0$. Let $d' \in I \cap C_r^*\left(R\left(\varphi^k\right)\right)$ be an element with $\|d'\| = 1$. There is then an l such that $d'p_l \neq 0$. In particular, there is an $a \in X$ such that $\psi_a(d'p_l) \neq 0$. Since $I \cap C_r^*\left(R\left(\varphi^k\right)\right)$ is an ideal in $C_r^*\left(R\left(\varphi^k\right)\right)$, p_l is central in $C_r^*\left(R\left(\varphi^k\right)\right)$ and $\psi_a\left(C_r^*\left(R\left(\varphi^k\right)\right)\right) \simeq M_{\varphi^{-k}(a)}(\mathbb{C})$, there is a positive element $d \in I \cap C_r^*\left(R\left(\varphi^k\right)\right)$ such that $dp_l(\xi,\xi) > 1$ for all $\xi \in \varphi^{-k}(a) \cap X_{k,l}$. It follows from Lemma 3.7 that the map $\xi \mapsto dp_l(\xi,\xi)$ is continuous on $X_{k,l}$. There is therefore an open set W in X such that $W \cap X_{k,l} \supseteq \varphi^{-k}(a) \cap X_{k,l}$ and

$$dp_l(\xi,\xi) \geqslant 1. \tag{4.10}$$

for all $\xi \in W \cap X_{k,l}$. Since condition 2′) holds there is an $m \in \mathbb{N}$ such that

$$\varphi^{m+k}\left(\bigcup_{j=0}^{m-1} \varphi^{-j}\left(W \cap X_{k,l}\right)\right) = X. \tag{4.11}$$

Now, assuming that $1 \notin I$, there is an irreducible ideal in $C_r^*\left(R\left(\varphi^{m+k}\right)\right)$ which contains $I \cap C_r^*\left(R\left(\varphi^{m+k}\right)\right)$. It follows then from Lemma 3.4 and the $\widehat{\varphi}$-invariance of I that there is a point $b \in X$ such that

$$\left\|\psi_b\left(\widehat{\varphi}^i\left(dp_l\right)\right)\right\| < \inf_{y \in X, j \in \{0,1,2,\dots,m-1\}} \left(m(y)m\left(\varphi(y)\right)\cdots m\left(\varphi^{j-1}(y)\right)\right)^{-1} \tag{4.12}$$

for all $i \in \{0,1,2,\dots,m-1\}$. It follows from (4.11) that there is a $j_0 \in \{0,1,2,\dots,m-1\}$ and an element $z \in \varphi^{-m-k}(b) \cap \varphi^{-j_0}\left(W \cap X_{k,l}\right)$. Then

$$\begin{aligned} \left\|\psi_b \circ \widehat{\varphi}^{j_0}\left(dp_l\right)\right\| &\geqslant \widehat{\varphi}^{j_0}\left(dp_l\right)(z,z) \\ &= \big(m(z)m\left(\varphi(z)\right)m\left(\varphi^2(z)\right) \\ &\qquad \cdots m\left(\varphi^{j_0-1}(z)\right)\big)^{-1} dp_l\left(\varphi^{j_0}(z),\varphi^{j_0}(z)\right) \\ &\geqslant \left(m(z)m\left(\varphi(z)\right)m\left(\varphi^2(z)\right)\cdots m\left(\varphi^{j_0-1}(z)\right)\right)^{-1}, \end{aligned} \tag{4.13}$$

where we in the last step used that $\varphi^{j_0}(z) \in W \cap X_{k,l}$ so that (4.10) applies. (4.13) contradicts (4.12). □

It is easy to modify the proof of Theorem 4.14 to obtain the following

Theorem 4.15. — *Assume that φ is surjective and not injective. The following conditions are equivalent:*

1) *$C_r^*(R_\varphi)$ is simple.*
2) *For every open subset $U \subseteq X$ and $k, l \in \mathbb{N}$ such that $U \cap X_{k,l} \neq \emptyset$ there is a $m \in \mathbb{N}$ such that*

$$\varphi^m (U \cap X_{k,l}) = X.$$

By Theorem 2.17 the two conditions, 1) and 2), in Theorem 4.15 are equivalent to the absence of any non-trivial R_φ-invariant ideal in D_{R_φ}.

When φ is open and hence a local homeomorphism it was observed in [2] that the sets $X_{k,l}$ are all open by a result of Eilenberg. So in this case condition 2) of Theorem 4.14 is equivalent to strong transitivity of φ in the sense of [12]: For every non-empty open set U of X there is an $m \in \mathbb{N}$ such that $\bigcup_{j=0}^m \varphi^j(U) = X$. Similarly when φ is open condition 2) of Theorem 4.15 is equivalent to exactness of φ: For every non-empty open set U of X there is an $m \in \mathbb{N}$ such that $\varphi^m(U) = X$. So when φ is a local homeomorphism Theorem 4.14 follows from Proposition 4.3 of [12] and Theorem 4.15 from Proposition 4.1 of [12].

We show next that it is possible to use the methods of this paper to improve the known simplicity criteria in the étale case to handle a non-surjective local homeomorphism of a locally compact space. Let X be a locally compact Hausdorff space and $\varphi\colon X \to X$ a local homeomorphism. We say that φ is *irreducible* when

$$X = \bigcup_{0 \leqslant i,j} \varphi^{-i}\left(\varphi^j(U)\right). \tag{4.14}$$

for every open non-empty set U in X. As observed in [15] a simple argument shows that φ is irreducible if and only if there is no non-trivial open subset $V \subseteq X$ such that $\varphi^{-1}(V) = V$.

Theorem 4.16. — *Let X be a locally compact second countable Hausdorff space and $\varphi\colon X \to X$ a local homeomorphism. The following are equivalent:*

1) *$C_r^*(\Gamma_\varphi)$ is simple.*
2) *$\left\{x \in X : \varphi^k(x) = x\right\}$ has empty interior for each $k \geqslant 1$ and φ is irreducible.*

3) *There is a point in X which is not pre-periodic under φ and φ is irreducible.*

Proof. — 1) $\Rightarrow$ 2): Assume that $\{x \in X : \varphi^k(x) = x\}$ contains a non-empty open set V for some $k \geqslant 1$. Set $W = \bigcup_{j=0}^{k-1} \varphi^j(V)$. Then the reduction

$$\Gamma_\varphi|_W = \{(x,k,y) \in \Gamma_\varphi : x, y \in W\}$$

is an étale groupoid in itself and $C_r^*(\Gamma_\varphi|_W)$ is a C^*-subalgebra of $C_r^*(\Gamma_\varphi)$. It is easy to check that

$$C_r^*(\Gamma_\varphi|_W) = \overline{C_0(W) C_r^*(\Gamma_\varphi) C_0(W)},$$

showing that $C_r^*(\Gamma_\varphi|_W)$ is a hereditary C^*-subalgebra. Note that $C_r^*(\Gamma_\varphi)$ is separable since we assume that X is second countable. Since we assume that $C_r^*(\Gamma_\varphi)$ is simple we can then apply [5] to conclude that $C_r^*(\Gamma_\varphi)$ is stably isomorphic to $C_r^*(\Gamma_\varphi|_W)$. However, since φ is k-periodic on W, every orbit of an element in W is a $\Gamma_\varphi|_W$-invariant closed subset of W. As $C_r^*(\Gamma_\varphi|_W)$ must be simple since $C_r^*(\Gamma_\varphi)$ is, it follows from Corollary 2.14 and (the proof of) Corollary 2.18 that W must be a single orbit. But then

$$C_r^*(\Gamma_\varphi|_W) \simeq C(\mathbb{T}) \otimes M_{k'}(\mathbb{C})$$

where $k' \leqslant k$ is the number of elements in W. This algebra is obviously not simple, contradicting the assumption that $C_r^*(\Gamma_\varphi)$ is. It follows that $\{x \in X : \varphi^k(x) = x\}$ must have empty interior for each $k \geqslant 1$.

It follows from Corollary 2.14 and (the proof of) Corollary 2.18 that X contains no non-trivial open Γ_φ-invariant subset. It is easy to see that this is equivalent to the assertion that (4.14) holds for every non-empty open subset U.

2) $\Rightarrow$ 3): Assume to reach a contradiction that every element of X is pre-periodic under φ. This means that

$$X = \bigcup_{n \geqslant 1, k \geqslant 0} \varphi^{-k}(\mathrm{Per}_n X) \tag{4.15}$$

where $\mathrm{Per}_n X = \{y \in X : \varphi^n(y) = y\}$. It follows from the Baire category theorem that there are $n \geqslant 1$, $k \geqslant 0$ and a non-empty open set $V \subseteq \varphi^{-k}(\mathrm{Per}_n X)$. Since φ is open this implies that $\varphi^k(V)$ is an open subset of $\mathrm{Per}_n X$, contradicting our assumption.

3) $\Rightarrow$ 1): As we observed above irreducibility of φ is equivalent to the absence of any non-trivial Γ_φ-invariant open subset in X. Furthermore, a point x of X which is not pre-periodic under φ must have trivial isotropy group in Γ_φ. Hence the simplicity of $C_r^*(\Gamma_\varphi)$ follows from Theorem 2.17. □

Concerning the simplicity of $C_r^*(R_\varphi)$ when φ is open we get the following conclusion directly from Corollary 2.18. It generalizes Proposition 4.1 of [12].

THEOREM 4.17. — *Let X be a locally compact Hausdorff space and $\varphi\colon X \to X$ a local homeomorphism. It follows that $C_r^*(R_\varphi)$ is simple if and only if $\bigcup_{k=0}^\infty \varphi^{-k}\left(\varphi^k(U)\right) = X$ for every open non-empty subset $U \subseteq X$.*

4.3. Subshifts: Carlsen-Matsumoto algebras

K. Matsumoto was the first to encode structures from general subshifts in a C^*-algebra, [18], generalizing the original construction of Cuntz and Krieger [10]. Later, slightly different constructions were suggested by Carlsen and Matsumoto [8], and by Carlsen [7]. The exact relation between the various constructions is a little obscure. Some of the known connections between them are described in [8] and [9]. As we shall see the approach we take here, based on the groupoids of Renault, Deaconu and Anantharaman-Delaroche, gives rise to the algebras introduced by Carlsen in [7].

Set $A = \{1, 2, \ldots, n\}$ and $A^{\mathbb{N}} = \{(x_1, x_2, x_3, \ldots) : x_i \in A\}$. We consider $A^{\mathbb{N}}$ as a compact metric space with the metric

$$d(x, y) = \sum_{i=1}^{\infty} 2^{-i} |x_i - y_i| .$$

The shift σ acts on $A^{\mathbb{N}}$ in the usual way: $\sigma(x)_i = x_{i+1}$. Let $S \subseteq A^{\mathbb{N}}$ be a subshift, *i.e.* S is closed and $\sigma(S) = S$. Such a subshift defines in a canonical way an abstract language whose words $\mathbb{W}(S)$ are the finite strings of "letters" from the "alphabet" A which occur in an element from S. We refer to [17] for more on subshifts.

Since $\sigma\colon S \to S$ is locally injective we can apply the construction of the previous section to obtain a semi-étale groupoid which we denote by Γ_S. Similarly, the corresponding semi-étale equivalence relation will be denoted by R_S in this setting. Given a word $u \in \mathbb{W}(S)$ of length $|u| = n$, set

$$C(u) = \{x \in S\colon x_1 x_2 \cdots x_n = u\} .$$

These are the *standard cylinder* sets in S and they form a base for the topology. Now set

$$C(u, v) = C(u) \cap \sigma^{-|u|}\left(\sigma^{|v|}\left(C(v)\right)\right) .$$

Thus $C(u,v)$ consists of the elements of $C(u)$ with the property that when the prefix u is replaced by v the infinite row of letters is still an element of S. Since the empty word $\emptyset$ by convention is also a word in $\mathbb{W}(S)$, with cylinder $C_\emptyset = S$, we have that $C(u) = C(u,\emptyset)$. While the cylinder sets are both closed and open, the set $C(u,v)$ is in general only closed. The characteristic functions $1_{C(u,v)}, u,v \in \mathbb{W}(S)$, generate a separable C^*-subalgebra in $l^\infty(S)$ which we denote by D_S. The C^*-algebra $\mathcal{O}_S$ of Carlsen from [7] is generated by partial isometries $s_u, u \in \mathbb{W}(S)$, such that

$$s_u s_v = s_{uv}$$

when $uv \in \mathbb{W}(S)$, $s_u s_v = 0$ when $uv \notin \mathbb{W}(S)$, and such that $s_v s_u^* s_u s_v^*$, $u,v \in \mathbb{W}(S)$, are projections which generate an abelian C^*-subalgebra isomorphic to D_S under a map sending $s_v s_u^* s_u s_v^*$ to $1_{C(v,u)}$, *cf.* [9].

The algebra $\mathcal{O}_S$ is blessed with the following universal property which enhances its applicability:

(A) *When B is a C^*-algebra containing partial isometries $S_u, u \in \mathbb{W}(S)$, such that*

$$S_{uv} = S_u S_v$$

when $uv \in \mathbb{W}(S)$, $S_u S_v = 0$ when $uv \notin \mathbb{W}(S)$, and admitting a $*$-homomorphism $D_S \to B$ sending $1_{C(v,u)}$ to $S_v S_u^* S_u S_v^*$ for all $u,v \in \mathbb{W}(S)$, then there is a $*$-homomorphism $\mathcal{O}_S \to B$ sending s_u to S_u for all $u \in \mathbb{W}(S)$.

In particular, it follows from (A) that there is a continuous action γ of the unit circle $\mathbb{T}$ on $\mathcal{O}_S$ such that $\gamma_z(s_u) = z^{|u|} s_u$ for all $z \in \mathbb{T}$ and all $u \in \mathbb{W}(S)$. This action is called the *gauge action*, *cf.* [7],[9].

The universal property (A) of $\mathcal{O}_S$ is established in Theorem 10 of [9]. As we shall show in the following proof, property (A) will provide us with a $*$-homomorphism $\mathcal{O}_S \to C_r^*(\Gamma_S)$ which we show is surjective. To conclude that it is also injective we shall need that $\mathcal{O}_S$ is a crossed product of a type dealt with by Exel and Vershik in [15]. In the notation of [9] and [15], $\mathcal{O}_S = D_S \rtimes_{\alpha,\mathcal{L}} \mathbb{N}$, where α is the endomorphism of D_S defined such that $\alpha(f) = f \circ \sigma$, and the transfer operator $\mathcal{L}$ is given by

$$\mathcal{L}(f)(x) = \frac{1}{\#\sigma^{-1}(x)} \sum_{y \in \sigma^{-1}(x)} f(y),$$

cf. [9]. Note that both σ and $\mathcal{L}$ are unital and faithful so that the Hypotheses 3.1 of [15] are satisfied. Furthermore, by inspection of the proof of Theorem 10 in [9] one sees that the gauge action γ of $\mathcal{O}_S$ is the same as the gauge action considered in [15]. In this way we can use Theorem 4.2 of [15] to supplement property (A) with the following "gauge invariant uniqueness property":

(B) *Let B be C^*-algebra and $\lambda\colon \mathcal{O}_S \to B$ a $*$-homomorphism which is injective on D_S. Assume that B admits a continuous action of $\mathbb{T}$ by automorphisms such that λ is equivariant with respect to the gauge-action on $\mathcal{O}_S$. It follows that λ is injective.*

Theorem 4.18. — *Let S be a one-sided subshift. Then the Carlsen-Matsumoto algebra $\mathcal{O}_S$ is $*$-isomorphic to the C^*-algebra $C_r^*(\Gamma_S)$ under a $*$-isomorphism which maps D_S onto D_{Γ_S}.*

Proof. — When $u \in \mathbb{W}(S)$ is a word, we let $t_u \in B_c(\Gamma_S)$ be the characteristic function of the set

$$\left\{(x,l,y) \in S \times \mathbb{Z} \times S\colon x \in C(u),\ l = |u|,\ y_i = x_{|u|+i},\ i \geqslant 1\right\}.$$

Note that $\left\{(x,l,y) \in S \times \mathbb{Z} \times S\colon x \in C(u), l = |u|, y_i = x_{|u|+i},\ i \geqslant 1\right\}$ is an open and compact subset of $\Gamma_S\left(|u|,0\right)$ and hence of Γ_S. Therefore $t_u \in C_c\left(\Gamma_S\right)$. Straightforward calculations show that

$$t_v t_u^* t_u t_v^* = 1_{C(v,u)} \tag{4.16}$$

for all $u,v \in \mathbb{W}(S)$ when we identify S with the unit space of Γ_S, and that $t_u t_v = t_{uv}$ when $uv \in \mathbb{W}(S)$, and $t_u t_v = 0$ when $uv \notin \mathbb{W}(S)$. It follows therefore from the universal property (A) of $\mathcal{O}_S$ described above that there is a $*$-homomorphism $\lambda\colon \mathcal{O}_S \to C_r^*(\Gamma_S)$ such that $\lambda\left(s_u\right) = t_u$ for all $u \in \mathbb{W}(S)$.

To see that λ is surjective note first that it follows from (4.16) that $1_{C(u)} = 1_{C(u,u)}$ is in the range of λ for all $u \in \mathbb{W}(S)$. Note next that $t_u t_v^* = 1_{A(u,v)}$ where

$$A(u,v) = \Big\{(x,k,y) \in S \times \mathbb{Z} \times S\colon k = |u| - |v|,\\ x \in C(u),\ y \in C(v),\ x_{|u|+i} = y_{|v|+i}, i \geqslant 1\Big\} \tag{4.17}$$

is a compact and open subset of Γ_S. In fact, sets of this form constitute a base for the topology of Γ_S so in order to show that every element of $C_c\left(\Gamma_S\right)$ is in the range of λ, which implies that λ is surjective, it will be enough to consider an $f \in C_c\left(\Gamma_S\right)$ such that f has support in $A(u,v)$ for some $u,v \in \mathbb{W}(S)$. Let $\epsilon > 0$. Note that the range map r is injective on $A(u,v)$. By combining this fact with the continuity of f it follows that we can find words $u_i, i = 1,2,\ldots,N$, in $\mathbb{W}(S)$ such that $C(u) = \bigcup_{i=1}^N C(u_i)$ and

$$|f(\xi) - f(\xi')| \leqslant \epsilon \tag{4.18}$$

when $\xi, \xi' \in A(u,v) \cap r^{-1}(C(u_i))$, and such that

$$C(u_i) \cap C(u_j) = \emptyset, i \neq j. \tag{4.19}$$

Define a function $h\colon S \to \mathbb{C}$ such that

$$h(x) = \begin{cases} f(x,|u|-|v|,y), & \text{when } (x,|u|-|v|,y) \in A(u,v) \text{ for some } y \in C(v) \\ 0, & \text{otherwise.} \end{cases}$$

Then h is bounded and supported in $C(u)$. Let

$$J = \left\{ i\colon A(u,v) \cap r^{-1}(C(u_i)) \neq \emptyset \right\}.$$

For each $i \in J$ we pick an element $\xi_i \in A(u,v) \cap r^{-1}(C(u_i))$ and define $k\colon S \to \mathbb{C}$ such that

$$k = \sum_{i \in J} h(\xi_i) 1_{C(u_i)}.$$

Then k, h are both bounded and compactly supported in S and k is in the range of λ. Furthermore, it follows from (4.18) and (4.19) that

$$\|h-k\|_\infty \leqslant \epsilon \tag{4.20}$$

in $B_c\left(\Gamma_S^0\right)$. By using the canonical inclusion $B_c\left(\Gamma_S^0\right) \subseteq B_c(\Gamma_S)$ we consider h and k as elements of $B_c(\Gamma_S)$, and find then that $f = h \star t_u t_v^*$. Hence

$$\|f - k \star t_u t_v^*\|_{C_r^*(\Gamma_S)} \leqslant \|h-k\|_\infty \|t_u t_v^*\|_{C_r^*(\Gamma_S)}. \tag{4.21}$$

It follows from (4.16) that $t_u t_v^*$ is a partial isometry and hence that $\|t_u t_v^*\| \leqslant 1$. We can therefore combine (4.21) and (4.20) to get the estimate

$$\|f - k \star t_u t_v^*\| \leqslant \epsilon.$$

Since $\epsilon > 0$ is arbitrary and $k \star t_u t_v^*$ is in the range of λ, it follows that so is f, completing the proof that λ is surjective. Note that $1_{C(u_i)} \star t_u t_v^* = 1_{A(u_i,v_i')}$ for an appropriate word $v_i' \in \mathbb{W}(S)$ so that $k \star t_u t_v^*$ is a linear combination of such characteristic functions.

To see that λ is injective note first that (4.16) implies that λ is injective on D_S. Therefore property (B) above shows that the injectivity of λ will follow if we can exhibit a continuous action $\beta\colon \mathbb{T} \to \operatorname{Aut} C_r^*(\Gamma_S)$ such that $\beta_z(t_u) = z^{|u|} t_u$ for all $u \in \mathbb{W}(S)$. The gauge action from Lemma 4.3 is such an action.

It remains to show that

$$\lambda(D_S) = D_{\Gamma_S}. \tag{4.22}$$

The inclusion $\lambda(D_S) \subseteq D_{\Gamma_S}$ follows from (4.16) and the definition of λ, so it remains only to show that $D_{\Gamma_S} \subseteq \lambda(D_S)$. To this end we let $1_{A(u,v)}$ denote the characteristic function of the set (4.17). Let $u \in \mathbb{W}(S)$, and let

$F \subseteq \mathbb{W}(S)$ be a finite set of words in S, not necessarily of the same length as u. We set

$$(4.23)\quad C'(u;F) = \Big\{x \in C(u) \colon \forall v \in F \exists y^v \in C(v) \text{ such that } y^v_{|v|+i} = x_{|u|+i}, i \geqslant 1\Big\}.$$

Thus $C'(u;F) = \bigcap_{v\in F} C(u,v)$. Let $v_1, v_2, \dots, v_N$ be the elements of F. It is straightforward to check that

$$(4.24)\quad 1_{C'(u;F)} = P_{\Gamma_S}\left(1_{A(u,v_1)} \star 1_{A(v_1,v_2)} \star \cdots \star 1_{A(v_{N-1},v_N)} \star 1_{A(v_N,u)}\right)$$

where $P_{\Gamma_S} \colon C^*_r(\Gamma_S) \to D_{\Gamma_S}$ is the conditional expectation of Lemma 2.8 corresponding to Γ_S. In particular, $1_{C'(u;F)} \in D_{\Gamma_S}$. As we have just shown every element of $C_c(\Gamma_S)$ can be approximated arbitrarily well in $C^*_r(\Gamma_S)$ by a linear combination of functions of the form $1_{A(u,v)}$. It follows that $C^*_r(\Gamma_S)$ is the closed linear span of elements of the form

$$(4.25)\qquad 1_{A(u_1,v_1)} \star 1_{A(u_2,v_2)} \star \cdots \star 1_{A(u_N,v_N)}.$$

By using that $1_{A(u,v)} = \sum_{i=1}^n 1_{A(ui,vi)}$ we can write the convolution product (4.25) as a sum of similar products, with the additional property that $|v_i| = |u_{i+1}|, i = 1, 2, \dots, N-1$. Then

$$P_{\Gamma_S}\left(1_{A(u_1,v_1)} \star 1_{A(u_2,v_2)} \star \cdots \star 1_{A(u_N,v_N)}\right) = 0$$

unless $v_1 = u_2, v_2 = u_3, \dots, v_N = u_1$, in which case

$$P_{\Gamma_S}\left(1_{A(u_1,v_1)} \star 1_{A(u_2,v_2)} \star \cdots \star 1_{A(u_N,v_N)}\right) = 1_{C'(u_1;F)}$$

where $F = \{v_1, v_2, \dots, v_{N-1}\}$. This shows that D_{Γ_S} is the closed linear span of projections of the form $1_{C'(u;F)}$ for some u and F. Since $1_{C'(u;F)} = 1_{C(u,v_1)} 1_{C(u,v_2)} \cdots 1_{C(u,v_k)}$ when $F = \{v_1, v_2, \dots, v_k\}$ and since $\lambda\left(s_u s^*_{v_i} s_{v_i} s^*_u\right) = 1_{C(u,v_i)}$, we obtain (4.22). □

Let $u \in \mathbb{W}(S)$, and let F be a finite set of words in S of the same length as u. We set

$$C(u;F) = \Big\{x \in C(u) \colon \forall v \in F \exists y^v \in C(v) \text{ such that } y^v_{|v|+i} = x_{|u|+i},\ i \geqslant 1\Big\}.$$

Similar sets were used in the proof of Theorem 4.18, but note that we now require the words in F to have the same length as u. We will then call $C(u;F)$ *a generalized cylinder* in S. Following the notation used in Theorem 4.14 we set

$$S_{k,l} = \left\{x \in S \colon \#\sigma^{-k}\left(\sigma^k(x)\right) = l\right\}$$

for all $k, l \in \mathbb{N}$.

Lemma 4.19.

a) When $U \subseteq S$ is an open subset and $k, l \in \mathbb{N}$ are numbers such that $U \cap S_{k,l} \neq \emptyset$ there is a non-empty generalized cylinder $C(u; F)$ such that $C(u; F) \subseteq U \cap S_{k,l}$.

b) When $C(u; F)$ is a non-empty generalized cylinder there is an open subset $U \subseteq S$ and natural numbers $k, l \in \mathbb{N}$ such that $U \cap S_{k,l} \neq \emptyset$ and $U \cap S_{k,l} \subseteq C(u; F)$.

Proof.

a) Let $x \in U \cap S_{k,l}$, and let $\mathbb{W}_k(S)$ denote the set of words in S of length k. There are then exactly l words $v_1, v_2, \ldots, v_l$ in $\mathbb{W}_k(S)$ such that $v_j x_{[k+1,\infty)} = v_j x_{k+1} x_{k+2} x_{k+3} \cdots \in S$. The word $x_1 x_2 \cdots x_k$ is one of them and we arrange that it is v_1. For each $j \in \{1, 2, \ldots, l\}$ and some $m > k$, set $w_j = v_j x_{k+1} x_{k+2} \cdots x_m$. We can then choose m so large that $x \in C(w_1; \{w_2, w_3, \ldots, w_l\}) \subseteq U \cap S_{k,l}$.

b) Let F' be a maximal collection of words from $\mathbb{W}_{|u|}(S)$ with the property that $F \subseteq F'$ and $C(u; F') \neq \emptyset$. Let $x \in C(u; F')$ and set $k = |u|, l = \#F'$. For each word $v \in \mathbb{W}_k(S) \setminus F'$ there is a natural number m_v such that

$$v x_{[k+1,i]} \notin \mathbb{W}(S)$$

when $i \geqslant m_v$. Set $m = \max_v m_v$. Then

$$x \in S_{k,l} \cap C\left(x_{[1,m]}\right) \subseteq C(u; F') \subseteq C(u; F).$$

Let $U = C\left(x_{[1,m]}\right)$. This handles the case when $F' \neq \mathbb{W}_k(S)$. In case $F' = \mathbb{W}_k(S)$ we have that $l = \#\mathbb{W}_k(S)$, and we can then take $U = C\left(x_{[1,k]}\right)$. $\square$

Theorem 4.20. — *Let S be an infinite one-sided subshift. Then $C_r^*(\Gamma_S)$ is simple if and only if the following holds: For every non-empty generalized cylinder $C(u; F)$ there is an $m \in \mathbb{N}$ with the property that for all $x \in S$ there is an element $y \in C(u; F)$ and a $k \in \{0, 1, 2, \ldots, m\}$ such that $x_i = y_{i+k}$ for all $i \in \mathbb{N}$.*

Proof. — Note that the shift is not injective since we assume that $\sigma(S) = S$ and that S is infinite. Combine Theorem 4.18, Lemma 4.19 and Theorem 4.14. $\square$

Similarly, for subshifts Theorem 4.15 can now be re-formulated as follows:

Theorem 4.21. — *Let S be an infinite one-sided subshift. Then $C_r^*(R_S)$ is simple if and only if the following holds: For every non-empty generalized cylinder $C(u; F)$ there is an $m \in \mathbb{N}$ with the property that for all $x \in S$ there is an element $y \in C(u; F)$ such that $x_i = y_{i+m}$ for all $i \in \mathbb{N}$.*

Concerning the existence of a Cartan subalgebra of $\mathcal{O}_S$ note that a subshift only has finitely many periodic points of each period. We can therefore combine Theorem 4.18 and Theorem 4.2 to obtain the following

THEOREM 4.22. — *The Carlsen-Matsumoto algebra $\mathcal{O}_S$ of a subshift S contains a Cartan subalgebra in the sense of Renault* [28].

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