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OBSTRUCTIONS TO DEFORMING CURVES ON A 3-FOLD, II: DEFORMATIONS OF DEGENERATE CURVES ON A DEL PEZZO 3-FOLD

by Hirokazu NASU (*)

Abstract. — We study the Hilbert scheme $\mathrm{Hilb}^{sc}V$ of smooth connected curves on a smooth del Pezzo 3-fold V. We prove that any degenerate curve C, *i.e.* any curve C contained in a smooth hyperplane section S of V, does not deform to a non-degenerate curve if the following two conditions are satisfied: (i) $\chi(V, \mathcal{I}_C(S)) \geqslant 1$ and (ii) for every line ℓ on S such that $\ell \cap C = \emptyset$, the normal bundle $N_{\ell/V}$ is trivial (*i.e.* $N_{\ell/V} \simeq \mathcal{O}_{\mathbb{P}^1}{}^{\oplus 2}$). As a consequence, we prove an analogue (for $\mathrm{Hilb}^{sc}V$) of a conjecture of J. O. Kleppe, which is concerned with non-reduced components of the Hilbert scheme $\mathrm{Hilb}^{sc}\mathbb{P}^3$ of curves in the projective 3-space $\mathbb{P}^3$.

Résumé. — Nous étudions le schéma de Hilbert $\mathrm{Hilb}^{sc}V$ des courbes lisses connexes sur une variété de del Pezzo lisse V de dimension 3. Nous montrons qu'aucune courbe C dégénérée, c'est-à-dire, aucune courbe C contenue dans une section hyperplane S de V, se déforme en une courbe non-dégénérée, si les deux conditions suivantes sont satisfaites : (i) $\chi(V, \mathcal{I}_C(S)) \geqslant 1$ et (ii) pour chaque droite ℓ sur S telle que $\ell \cap C = \emptyset$, le fibré normal $N_{\ell/V}$ de ℓ dans V est trivial. Par conséquent, nous prouvons un analogue (pour $\mathrm{Hilb}^{sc}V$) d'une conjecture de J. O. Kleppe, qui concerne les composantes non-réduites du schéma de Hilbert $\mathrm{Hilb}^{sc}\mathbb{P}^3$ des courbes dans l'espace projectif $\mathbb{P}^3$ de dimension 3.

1. Introduction

This paper is a sequel to a joint work [13] with Shigeru Mukai. In [13] the embedded deformations of smooth curves C on a smooth projective 3-fold V have been studied under the presence of a smooth surface S such that $C \subset S \subset V$, especially when V is a uniruled 3-fold. In this paper, the same subject is studied in detail especially when V is a del Pezzo 3-fold.

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It is known that even if the deformations of C in S and the deformations of S in V behave well, those of C in V behave badly in general. For example, even if $\operatorname{Hilb} V$ and $\operatorname{Hilb} S$ are nonsingular of expected dimension $\chi(N_{S/V})$ and $\chi(N_{C/S})$ at $[S]$ and $[C]$ respectively, $\operatorname{Hilb} V$ can be generically non-reduced along some component passing through $[C]$ (*cf.* Mumford's example in [14]). Such non-reduced components of the Hilbert scheme $\operatorname{Hilb}^{sc} V$ of smooth connected curves on V have been constructed for many uniruled 3-folds V in [13]. The non-reducedness is originated from the non-surjectivity of the restriction map

$$H^0(S, N_{S/V}) \xrightarrow{|_C} H^0(C, N_{S/V}|_C). \tag{1.1}$$

We say that C is *stably degenerate* if every small global deformation of C in V is contained in a deformation S' of S in V (*cf.* Definition 4.1). If (1.1) is surjective, then C is stably degenerate (*cf.* Proposition 4.3). However if it is not surjective, then there exists a first order deformation $\widetilde{C}$ of C in V which is not contained in any first order deformation $\widetilde{S}$ of S. In this paper, we consider the following problem raised by Mukai:

Problem 1.1. — *Suppose that* (1.1) *is not surjective and* $\chi(V, \mathcal{I}_C(S)) > 0$. *Then (1) Is C stably degenerate? (2) Is* $\operatorname{Hilb}^{sc} V$ *singular at* $[C]$*?*

Here $\mathcal{I}_C$ denotes the ideal sheaf of C in V and $\mathcal{I}_C(S) := \mathcal{I}_C \otimes \mathcal{O}_V(S)$. J. O. Kleppe [8] and Ph. Ellia [2] considered Problem 1.1 for the case where V is the projective 3-space $\mathbb{P}^3$, S is a smooth cubic surface in $\mathbb{P}^3$ and C is a smooth connected curve of degree d lying on S. Kleppe gave a conjecture (*cf.* Conjectures 5.1), which can be reformulated as follows:

Conjecture 1.2. — *Let* $C \subset S \subset \mathbb{P}^3$ *be as above and assume that* $\chi(\mathbb{P}^3, \mathcal{I}_C(3)) \geqslant 1$. *Then:*

(1) *If C is linearly normal, then every small global deformation C' of C in $\mathbb{P}^3$ is contained in a cubic surface $S' \subset \mathbb{P}^3$, i.e., C is stably degenerate, and*
(2) *Suppose that C is general and $d > 9$. Then* $\operatorname{Hilb}^{sc} \mathbb{P}^3$ *is nonsingular at $[C]$ if and only if $H^1(\mathbb{P}^3, \mathcal{I}_C(3)) = 0$.*

As a testing ground of his conjecture, we consider Problem 1.1 for the case where V is a smooth del Pezzo 3-fold (*cf.* §2.2), S is a smooth member of the class $|H|$ of the polarization H of V, *i.e.,* a smooth del Pezzo surface in V, and C is a smooth connected curve on S. The following theorem is an analogue of Kleppe's conjecture.

Theorem 1.3. — *Let $C \subset S \subset V$ be as above and assume that $\chi(V, \mathcal{I}_C(S)) \geqslant 1$. If every line ℓ on S such that $C \cap \ell = \emptyset$ is a good line on V (i.e., the normal bundle $N_{\ell/V}$ of ℓ in V is trivial), then:*

(1) *C is stably degenerate, and*
(2) *$\operatorname{Hilb}^{sc} V$ is nonsingular at $[C]$ if and only if $H^1(V, \mathcal{I}_C(S)) = 0$.*

If $\chi(V, \mathcal{I}_C(S)) < 1$, then it follows from a dimension count that C is not stably degenerate (Proposition 4.7). If some ℓ is a *bad line* on V (*i.e.*, $N_{\ell/V} \not\simeq {\mathcal{O}_{\mathbb{P}^1}}^{\oplus 2}$) then C is not necessarily stably degenerate (Proposition 5.4). As a corollary to Theorem 1.3, we give a sufficient condition for a maximal family W of degenerate curves on V to become an irreducible component of the Hilbert scheme $\operatorname{Hilb}^{sc} V$ and determine whether $\operatorname{Hilb}^{sc} V$ is generically non-reduced along W or not (Theorem 4.14).

One of the main tools used in this paper is the infinitesimal analysis of the Hilbert scheme developed in [13]. As is well known, every infinitesimal deformation $\widetilde{C}$ of C in V of the first order (*i.e.*, over $\operatorname{Spec} k[t]/(t^2)$) determines a global section $\alpha \in H^0(N_{C/V})$ and a cohomology class $\operatorname{ob}(\alpha) \in H^1(N_{C/V})$ (called the *obstruction*) such that $\widetilde{C}$ lifts to a deformation over $\operatorname{Spec} k[t]/(t^3)$ if and only if $\operatorname{ob}(\alpha) = 0$ (*cf.* §2.3). Let $\pi_{C/S} : N_{C/V} \to N_{S/V}\big|_C$ be the natural projection. In [13] Mukai and Nasu studied the *exterior component* of α and $\operatorname{ob}(\alpha)$, *i.e.*, the images of α and $\operatorname{ob}(\alpha)$ by the induced maps $H^i(\pi_{C/S}) : H^i(N_{C/V}) \to H^i(N_{S/V}\big|_C)$ $(i = 0, 1)$, respectively. They proved that if there exists a curve E on S such that $(E^2)_S < 0$ (*e.g.* (-1)-$\mathbb{P}^1$ on S) and the exterior component of α lifts to a global section $v \in H^0(N_{S/V}(E)) \setminus H^0(N_{S/V})$, then the exterior component of $\operatorname{ob}(\alpha)$ is nonzero provided that certain additional conditions on E, C and v hold (see [13, Theorem 2.2]). Such a rational section v of $N_{S/V}$ admitting a pole along E is called an *infinitesimal deformation with a pole*. In §3 we see that an infinitesimal deformation with a pole along E induces an obstructed infinitesimal deformation of the open surface $S^\circ := S \setminus E$ in the open 3-fold $V^\circ := V \setminus E$ (Theorem 3.1). By using this fact, we prove Theorem 1.3 in §4. In §5 we give some examples of generically non-reduced components of the Hilbert scheme of curves on a del Pezzo 3-fold as an application.

Acknowledgements. I should like to express my sincere gratitude to Professor Shigeru Mukai. He showed me the example of non-reduced components of the Hilbert scheme of canonical curves in §5.2 as a simplification of Mumford's example of a non-reduced component of $\operatorname{Hilb}^{sc} \mathbb{P}^3$. This motivated me to research the topic of this paper. Throughout this research, he made many suggestions which are useful for obtaining and improving the proofs. According to his suggestion, I studied the deformation theory of an open surface in an open 3-fold and organized §3. I am grateful to Professor Jan Oddvar Kleppe for giving me useful comments on Hilbert-flag schemes and for finding a gap in the proof of Lemma 4.8 in a earlier version. I should like to thank the referee, who showed me a straight proof of Proposition 4.3 and led me to a simplification of §4.1 and §4.2. According to his/her recommendation, I give the classes of non-reduced components of the Hilbert scheme explicitly in Proposition 5.5.

Notation and Conventions. We work over an algebraically closed field k of characteristic 0. Let V be a scheme over k and let X be a closed subscheme of V. Then $\mathcal{I}_X$ denotes the ideal sheaf of X in V and $N_{X/V}$ denotes the normal sheaf $(\mathcal{I}_X/{\mathcal{I}_X}^2)^\vee$ of X in V. For a sheaf $\mathcal{F}$ on V, we denote the restriction map $H^i(V,\mathcal{F}) \to H^i(X,\mathcal{F}\big|_X)$ by $\big|_X$. We denote the Euler-Poincaré characteristic of $\mathcal{F}$ by $\chi(V,\mathcal{F})$ or $\chi(\mathcal{F})$. $\operatorname{Hilb}^{sc} V$ denotes the open subscheme of the Hilbert scheme $\operatorname{Hilb} V$ whose point corresponds to a smooth connected curve on V.

2. Preliminaries

The results in this section will be used in § 4. Proposition 2.4 and Lemma 2.5 are important to our proof of Proposition 4.9 and 4.10, respectively.

2.1. Del Pezzo surfaces

A *del Pezzo surface* is a smooth surface S whose anti-canonical divisor $-K_S$ is ample. Every del Pezzo surface is isomorphic to $\mathbb{P}^2$ blown up at fewer than 9 points or $\mathbb{P}^1 \times \mathbb{P}^1$. We denote the blow-up of $\mathbb{P}^2$ at $(9-n)$-points by S_n. A curve $\ell \simeq \mathbb{P}^1$ on S_n is called a *line*[(1)] if $\ell \cdot (-K_S) = 1$. Every (-1)-$\mathbb{P}^1$ on S_n is a line and every line on S_n is a (-1)-$\mathbb{P}^1$. A curve q on S_n is called a *conic* if $q \cdot (-K_S) = 2$ and $q^2 = 0$.

Lemma 2.1. — *Let D be a divisor on a del Pezzo surface S. If D is nef and $\chi(S,-D) \geqslant 0$, then $H^1(S,-D) = 0$.*

(1) There exists no line on $\mathbb{P}^2$ and on $\mathbb{P}^1 \times \mathbb{P}^1$.

Proof. — If $D^2 > 0$ then the assertion follows the Kawamata-Viehweg vanishing. Since D is a nef divisor on a del Pezzo surface, we have $D^2 \geqslant 0$. Now we assume that $D^2 = 0$. If $S = S_n$, then D is linearly equivalent to a multiple mq ($m \geqslant 0$) of a conic q on S. By the Riemann-Roch theorem, we have

$$\begin{aligned}\chi(S, -D) &= \frac{1}{2}(-mq) \cdot (-mq - K_S) + \chi(\mathcal{O}_S) \\ &= -m + 1.\end{aligned}$$

Thus we have $m = 0$ or 1 by assumption. This implies that $H^1(-mq) = 0$. If $S = \mathbb{P}^1 \times \mathbb{P}^1$, then D is of bidegree $(m, 0)$ or $(0, m)$ with $m \geqslant 0$. Again by the Riemann-Roch theorem, we have $\chi(-D) = -m + 1 \geqslant 0$. Thus $H^1(\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(-D)) = 0$. □

LEMMA 2.2. — *Let D be an effective divisor on a del Pezzo surface S. Then the lines ℓ such that $D \cdot \ell < 0$ are mutually disjoint. The fixed part*[(2)] $\mathrm{Bs}\,|D|$ *of the linear system $|D|$ on S is equal to*

$$-\sum_{D \cdot \ell < 0} (D \cdot \ell)\ell.$$

Proof. — We prove the two assertions at the same time. It is clear that any line ℓ satisfying $D \cdot \ell < 0$ is contained in $\mathrm{Bs}\,|D|$. On the other hand, except for lines on S every irreducible curve C on S can move on S by the linearly equivalence since $\chi(C) \geqslant 2$ and $H^2(C) = 0$. Hence $|D|$ is decomposed into the sum

$$|D| = |D'| + \sum_{i=1}^{k} m_i \ell_i,$$

of a linear system $|D'|$ on S such that $\mathrm{Bs}\,|D'| = \emptyset$ and some lines ℓ_i on S with coefficients $m_i \in \mathbb{Z}_{>0}$ ($1 \leqslant i \leqslant k$). If $\ell_i \cap \ell_j \neq \emptyset$ for some $i \neq j$, then $\ell_i + \ell_j$ is a (reducible) conic on S and can move on S by $\chi(\ell_i + \ell_j) = 2$. Thus ℓ_i's are mutually disjoint. Now we prove that $D \cdot \ell_i < 0$ for any i. Since $m_i = (D' - D) \cdot \ell_i > 0$, it suffices to show that $D' \cdot \ell_i = 0$. Since D' is nef, we have $(D')^2 \geqslant 0$. Since $-K_S$ is ample, so is $D' - K_S$. Hence we have $H^1(D') = H^1((D' - K_S) + K_S) = 0$ by the Kodaira vanishing. If $D' \cdot \ell_i \geqslant 1$, then it follows from the exact sequence

$$0 \longrightarrow \mathcal{O}_S(D') \longrightarrow \mathcal{O}_S(D' + \ell_i) \longrightarrow \mathcal{O}_S(D' + \ell_i)\big|_{\ell_i} \longrightarrow 0$$

that $h^0(D' + \ell_i) > h^0(D')$. Thus we have $D' \cdot \ell_i = 0$. □

(2) the base locus of dimension one

Lemma 2.3. — *Let E be a disjoint union of m lines ($m \geqslant 0$) on a del Pezzo surface S and let $\varepsilon : S \to F$ be the blow-down of E from S. If a divisor D on F satisfies $h^0(F, D) \geqslant m$, then we have the following:*

(1) *$h^0(S, \varepsilon^* D - E) = h^0(F, D) - m$, and*
(2) *If $H^1(S, \varepsilon^* D) = 0$, then $H^1(S, \varepsilon^* D - E) = 0$.*

Proof. — (1) Let ℓ_i ($1 \leqslant i \leqslant m$) be the disjoint lines on S and let $E := \sum_{i=1}^m \ell_i$. We put $D_j := \varepsilon^* D - \sum_{1 \leqslant i \leqslant j} \ell_i$. Since the image of ℓ_i on F is a point, we have $h^0(D_j) \geqslant h^0(D) - j$ for every $1 \leqslant j \leqslant m$. Moreover since $D_{j-1} \cdot \ell_j = 0$, Lemma 2.2 shows that ℓ_j is not contained in $\mathrm{Bs}\,|D_{j-1}|$. Hence $\dim |D_j|$ decreases one by one as j increases. Therefore we have $h^0(\varepsilon^* D - E) = h^0(D_m) = h^0(D) - m$.

(2) An exact sequence $0 \to \mathcal{O}_S(\varepsilon^* D - E) \to \mathcal{O}_S(\varepsilon^* D) \to \mathcal{O}_E \to 0$ on S induces an exact sequence

$$H^0(S, \varepsilon^* D) \xrightarrow{\rho} H^0(E, \mathcal{O}_E) \longrightarrow H^1(S, \varepsilon^* D - E) \longrightarrow H^1(S, \varepsilon^* D)$$

of cohomology groups. Then ρ is surjective by (1) and $H^1(S, \varepsilon^* D) = 0$ by assumption. Hence we have $H^1(S, \varepsilon^* D - E) = 0$. $\square$

Let C be a smooth connected curve on a del Pezzo surface S. We consider the restriction to C of the anti-canonical linear system $|-K_S|$ on S. The restriction map $H^0(-K_S) \to H^0(-K_S|_C)$ is not surjective in general. Let ℓ_i ($1 \leqslant i \leqslant m$) be the lines on S disjoint to C. Let us define an effective divisor E on S by the sum

$$E := \sum_{i=1}^m \ell_i$$

and we put $E := 0$ if there exists no such ℓ_i. If C is neither a line nor a conic, then the ℓ_i's are mutually disjoint: indeed if $\ell_i \cap \ell_j \neq \emptyset$ for some $i \neq j$, then $q := \ell_i + \ell_j$ is a conic on S and hence C intersects q by $C \cdot q > 0$.

Proposition 2.4. — *Assume that C is irrational and $\chi(S, -K_S - C) \geqslant 0$. Then we have $H^1(S, -K_S + E - C) = 0$ and the restriction map*

$$H^0(S, -K_S + E) \xrightarrow{|_C} H^0(C, -K_S|_C) \tag{2.1}$$

is surjective. If C is not elliptic either, then the map (2.1) is an isomorphism.

Proof. — It suffices to show that $H^1(-K_S + E - C) = 0$ by the exact sequence

$$0 \longrightarrow \mathcal{O}_S(-K_S + E - C) \longrightarrow \mathcal{O}_S(-K_S + E) \longrightarrow \mathcal{O}_S(-K_S)|_C \longrightarrow 0. \tag{2.2}$$

Claim. Put $D_1 := C + K_S - E$. Then D_1 is nef.

Since S is regular (*i.e.*, $H^1(K_S)=0$), the restriction map $\big|_C : H^0(C+K_S) \to H^0(K_C)$ is surjective. Since $C \not\simeq \mathbb{P}^1$, the linear system $|C+K_S|$ on S is non-empty. Let l be a line on S. Since C is not a line, we have $C\cdot\ell \geqslant 0$ and hence $(C+K_S)\cdot\ell \geqslant -1$. By Lemma 2.2, ℓ is contained in $\mathrm{Bs}\,|C+K_S|$ if and only if $C\cap\ell=\emptyset$. Thus we have $E=\mathrm{Bs}\,|C+K_S|$ and $|D_1|$ does not have base components. In particular, D_1 is nef.

It follows from the exact sequence

(2.3)

$$0 \longrightarrow \mathcal{O}_S(-K_S-C) \longrightarrow \mathcal{O}_S(-K_S+E-C) \longrightarrow \underbrace{\mathcal{O}_S(-K_S+E)\big|_E}_{\simeq \mathcal{O}_E} \longrightarrow 0$$

that $\chi(-D_1)=\chi(-K_S-C)+\chi(\mathcal{O}_E)\geqslant 0$. Hence we have $H^1(-D_1)=0$ by Lemma 2.1.

Now we assume that C is not elliptic. Then $K_C \not\sim 0$ and hence $C+K_S \not\sim E$ by adjunction. Thus $D_1 \not\sim 0$ and $H^0(-D_1)=0$. Therefore (2.1) is injective. □

Lemma 2.5. — *If C is not rational nor elliptic and $\chi(S,-K_S-C)\geqslant 0$, then the map*

$$H^1(S,-K_S+3E) \xrightarrow{|_C} H^1(C,-K_S\big|_C)$$

induced by (2.2) $\otimes\mathcal{O}_S(2E)$ *is injective.*

Proof. — It suffices to show that $H^1(-K_S+3E-C)=0$. Let $\varepsilon : S\to F$ be the blow-down of E from S. Then there exists a divisor D_2 on F such that $\varepsilon^*D_2 \sim C+2K_S-2E$. By the Serre duality, it suffices to show that $H^1(\varepsilon^*D_2-E)=0$.

Claim. $H^i(S,\varepsilon^*D_2)=0$ for $i=1,2$.

By (2.3)$\otimes\mathcal{O}_S(E)$, there exists an exact sequence

$$H^1(S,-K_S+E-C) \longrightarrow H^1(S,-K_S+2E-C) \longrightarrow H^1(E,(-K_S+2E)\big|_E).$$

Since $H^1((-K_S+2E)\big|_E)\simeq H^1(\mathcal{O}_E(E))=0$ and $H^1(-K_S+E-C)=0$ by Proposition 2.4, we have $H^1(-K_S+2E-C)=0$. By the Serre duality, we have $H^1(\varepsilon^*D_2)=0$. Similarly by the Serre duality, we have $H^2(\varepsilon^*D_2)\simeq H^0(K_S-\varepsilon^*D_2)^\vee$. Since C is not rational nor elliptic, we have $(K_S-\varepsilon^*D_2)\cdot C=(-K_S-C)\cdot C=-\deg K_C<0$. Hence we have $H^2(\varepsilon^*D_2)=0$ because C is nef. Thus the claim has been proved.

By this claim, we have $h^0(F,D_2)=h^0(S,\varepsilon^*D_2)=\chi(S,\varepsilon^*D_2)$. Then an easy calculation shows that $\chi(\varepsilon^*D_2)=\chi(-K_S-C)+\chi(\mathcal{O}_E)$. Since $\chi(-K_S-C)\geqslant 0$, we have $h^0(F,D_2)=\chi(S,\varepsilon^*D_2)\geqslant m$, where m is the number of components of E. Since $H^1(\varepsilon^*D_2)=0$, Lemma 2.3 (2) shows that $H^1(\varepsilon^*D_2-E)=0$. □

Let S be a smooth projective surface and let L be a line bundle on S.

Lemma 2.6. — *Let E be a disjoint union of irreducible curves E_i ($i = 1, \dots, m$) on S such that $E_i^2 < 0$ and let $\iota : S^\circ := S \setminus E \hookrightarrow S$ be the open immersion. If $\deg(L\big|_{E_i}) \leqslant 0$ for every i, then the map*

$$H^1(S, L) \to H^1(S^\circ, L\big|_{S^\circ})$$

induced by the sheaf inclusion $L \hookrightarrow L \otimes \iota_* \mathcal{O}_{S^\circ}$ is injective.

The proof is similar to that of [13, Lemma 2.5] and we omit it here. Lemma 2.6 allows us to identify $H^1(S, L(nE))$ ($n \geqslant 0$) with their images in $H^1(S^\circ, L\big|_{S^\circ})$. As a result, under the identification we obtain a natural filtration

$$H^1(S, L) \subset H^1(S, L(E)) \subset H^1(S, L(2E)) \subset \cdots \subset H^1(S^\circ, L\big|_{S^\circ})$$

on $H^1(S^\circ, L\big|_{S^\circ})$.

2.2. Del Pezzo threefolds

A *del Pezzo threefold* is a pair (V, H) consisting of a (smooth) irreducible projective variety V of dimension 3 and an ample Cartier divisor H on V such that $-K_V = 2H$. Here H is called the *polarization* of V and sometimes omitted. The self-intersection number $n := H^3$ is called the *degree* of V. It is known that the linear system $|H|$ on V determines a double cover $\varphi_{|H|} : V \to \mathbb{P}^3$ if $n = 2$, and an embedding $\varphi_{|H|} : V \hookrightarrow \mathbb{P}^{n+1}$ if $n \geqslant 3$. If S is a smooth member of $|H|$, then the pair $(S, H\big|_S)$ is a del Pezzo surface of degree n. Every smooth del Pezzo 3-fold is one of V_n ($1 \leqslant n \leqslant 8$) or V_6' in Table 2.1, in which $\mathbb{L}^{(i)}$ denotes a linear subspace of dimension i, and n and ρ respectively denote the degree and the Picard number of V_n (and of V_6') (*cf.* [4],[5],[6]). It is known that a smooth 3-fold $V \subset \mathbb{P}^{n+1}$ ($n \geqslant 3$) is a del Pezzo 3-fold of degree n if a linear section $[V \subset \mathbb{P}^{n+1}] \cap H_1 \cap H_2$ with two general hyperplanes $H_1, H_2 \subset \mathbb{P}^{n+1}$ is an elliptic normal curve in $\mathbb{P}^{n-1}$.

We briefly review the basics of the Hilbert scheme of lines on a del Pezzo 3-fold. We refer to Iskovskih ([6],[7]) for the details. Let (V, H) be a smooth del Pezzo 3-fold of degree n. By a *line* on (V, H), we mean a reduced irreducible curve ℓ on V such that $(\ell \cdot H)_V = 1$ and $\ell \simeq \mathbb{P}^1$. If $n \leqslant 7$ then V contains a line ℓ. Then there are only the following possibilities for the

Table 2.1. Del Pezzo 3-folds

del Pezzo 3-folds	n	ρ	
$V_1 = (6) \subset \mathbb{P}(3,2,1,1,1)$	1	1	a weighted hypersurface of degree 6
$V_2 = (4) \subset \mathbb{P}(2,1,1,1,1)$	2	1	a weighted hypersurface of degree 4 [(a)]
$V_3 = (3) \subset \mathbb{P}^4$	3	1	a cubic hypersurface
$V_4 = (2) \cap (2) \subset \mathbb{P}^5$	4	1	a complete intersection of two quadrics
$V_5 = [\mathrm{Gr}(2,5) \overset{\text{Plücker}}{\hookrightarrow} \mathbb{P}^9] \cap \mathbb{L}^{(6)}$	5	1	a linear section of Grassmannian
$V_6 = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \overset{\text{Segre}}{\hookrightarrow} \mathbb{P}^7$	6	3	
$V_6' = [\mathbb{P}^2 \times \mathbb{P}^2 \overset{\text{Segre}}{\hookrightarrow} \mathbb{P}^8] \cap \mathbb{L}^{(7)}$	6	2	
$V_7 = \mathrm{Bl}_{\mathrm{pt}}\, \mathbb{P}^3 \subset \mathbb{P}^8$	7	2	the blow-up of $\mathbb{P}^3$ at a point [(b)]
$V_8 = \mathbb{P}^3 \overset{\text{Veronese}}{\hookrightarrow} \mathbb{P}^9$	8	1	the Veronese image of $\mathbb{P}^3$

[(a)] Another realization of V_2 is a double cover of $\mathbb{P}^3$ branched along a quartic surface.
[(b)] V_7 is realized as the projection of $V_8 \subset \mathbb{P}^9$ from one of its points.

normal bundle $N_{\ell/V}$ of ℓ in V:

$$
\begin{array}{lll}
(0,0): & N_{\ell/V} \simeq \mathcal{O}_{\mathbb{P}^1}{}^{\oplus 2} & (\textit{i.e.}, \text{trivial}), \\
(1,-1): & N_{\ell/V} \simeq \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(1), & \\
(2,-2): & N_{\ell/V} \simeq \mathcal{O}_{\mathbb{P}^1}(-2) \oplus \mathcal{O}_{\mathbb{P}^1}(2) & (\text{only if } n = 1 \text{ or } 2), \\
(3,-3): & N_{\ell/V} \simeq \mathcal{O}_{\mathbb{P}^1}(-3) \oplus \mathcal{O}_{\mathbb{P}^1}(3) & (\text{only if } n = 1).
\end{array}
$$

In this paper, ℓ is called a *good line* if $N_{\ell/V}$ is trivial, and called a *bad line* otherwise. If $n \geqslant 3$, then every line on V is of type $(0,0)$ or $(1,-1)$. The Hilbert scheme Γ of lines on V is called the *Fano surface* of V, and in fact every irreducible (non-embedded) component of Γ is of dimension two. Let $\Gamma_i \subset \Gamma$ be an irreducible component and let S_i be the universal family of lines on V over Γ_i. Then there exists a natural diagram

$$
\begin{array}{ccc}
S_i & \xrightarrow{\;p\;} & V \\
{\scriptstyle\pi}\downarrow & & \\
\Gamma_i . & &
\end{array}
$$

By [7, Chap. III, Proposition 1.3 (iv)], if $n \geqslant 3$ then we have either

(a) p is surjective; in this case a general line in Γ_i is a good line; or
(b) $p(S_i) \simeq \mathbb{P}^2$ is a plane on $V \subset \mathbb{P}^{n+1}$; in this case every line in Γ_i is a bad line.

We have either (a) or (b) also when $n \leqslant 2$. (See the proof[(3)] in [7], which works for $n \leqslant 2$.) If $n \neq 7$ then every irreducible component of Γ is of type (a). If $n = 7$ then Γ consists of two irreducible components $\Gamma_i \simeq \mathbb{P}^2 (i = 0, 1)$, one of which is of type (a), while the other is of type (b). Consequently, we have

[(3)] In the proof, the assumption that char $k = 0$ is used.

Lemma 2.7 (Iskovskih). — *Every smooth del Pezzo 3-fold of degree $n \neq 8$ contains a good line.*

Lemma 2.8. — *Let (V, H) be a smooth del Pezzo 3-fold of degree n and let S be a general member of $|H|$. If $n \neq 7$ then every line on S is good. If $n = 7$ then there exist three lines ℓ_0, ℓ_1, ℓ_2 on S forming the configuration in Figure 2.1. Then ℓ_0 is bad, while ℓ_1 and ℓ_2 are good.*

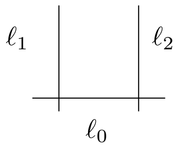

Figure 2.1. (-1)-$\mathbb{P}^1$*'s on* S_7

Proof. — There exists no line on V_8. If $n \neq 7$, then the locus $\mathfrak{B}$ of bad lines in the Fano surface Γ is of dimension one. Let p_i denote the projection of

$$\{(\ell, S) \mid \ell \subset S\} \subset \Gamma \times |H|$$

to the i-th factor. Since the fiber of p_1 is of dimension $n-1$, $p_2(p_1^{-1}(\mathfrak{B}))$ is a proper closed subset of $|H| \simeq \mathbb{P}^{n+1}$. Hence every line on a general member S of $|H|$ is a good line.

Suppose that $V = V_7$, *i.e.*, the blow-up of $\mathbb{P}^3$ at a point. Then S is a del Pezzo surface S_7, *i.e.*, a blow-up of $\mathbb{P}^2$ at two distinct points, and hence there exist three lines ℓ_0, ℓ_1, ℓ_2 on S as in Figure 2.1. Here ℓ_0 is distinguished by the fact that it intersects both of the other lines. Let P be the exceptional divisor of the blow-up $V_7 \to \mathbb{P}^3$. Then $P \simeq \mathbb{P}^2$ is a unique plane on V_7 and ℓ_0 is the intersection of S with P (*cf.* [7, Chap. II, §1.4]). Since $N_{\ell_0/P} \simeq \mathcal{O}_{\mathbb{P}^1}(1)$, ℓ_0 is a bad line on V_7. On the other hand, ℓ_1 and ℓ_2 are good lines on V_7 since S is general. □

2.3. Infinitesimal deformations and obstructions

Let V be a smooth variety and let X be a smooth closed subvariety of V. An *(embedded) first order infinitesimal deformation* of X in V is a closed subscheme $\widetilde{X} \subset V \times \operatorname{Spec} k[t]/(t^2)$ which is flat over $\operatorname{Spec} k[t]/(t^2)$ and whose central fiber is X. It is well known that there exists a one to one correspondence between the group of homomorphisms $\alpha : \mathcal{I}_X \to \mathcal{O}_X$ and the first order infinitesimal deformations $\widetilde{X}$ of X in V. In what follows, we identify $\widetilde{X}$ with α and abuse the notation. The standard exact sequence

$$0 \longrightarrow \mathcal{I}_X \longrightarrow \mathcal{O}_V \longrightarrow \mathcal{O}_X \longrightarrow 0 \tag{2.4}$$

induces $\delta : \mathrm{Hom}(\mathcal{I}_X, \mathcal{O}_X) \to \mathrm{Ext}^1(\mathcal{I}_X, \mathcal{I}_X)$ as a coboundary map. Then $\alpha \in \mathrm{Hom}(\mathcal{I}_X, \mathcal{O}_X)$ (*i.e.*, $\widetilde{X}$) lifts to a deformation over $\mathrm{Spec}\, k[t]/(t^3)$ if and only if

$$\mathrm{ob}(\alpha) := \delta(\alpha) \cup \alpha \in \mathrm{Ext}^1(\mathcal{I}_X, \mathcal{O}_X)$$

is zero, where $\cup$ is the cup product map

$$\mathrm{Ext}^1(\mathcal{I}_X, \mathcal{I}_X) \times \mathrm{Hom}(\mathcal{I}_X, \mathcal{O}_X) \xrightarrow{\cup} \mathrm{Ext}^1(\mathcal{I}_X, \mathcal{O}_X).$$

(We refer to [11, Chap. I §2]. See also [15], [1], [3] and [10].) Then $\mathrm{ob}(\alpha)$ is called the *obstruction* of α (*i.e.*, $\widetilde{X}$). Since both X and V are smooth, $\mathrm{ob}(\alpha)$ is contained in $H^1(X, N_{X/V}) \subset \mathrm{Ext}^1(\mathcal{I}_X, \mathcal{O}_X)$ (*cf.* [11, Chap. I, Prop. 2.14]). Since $\mathrm{Hom}(\mathcal{I}_X, \mathcal{O}_X) \simeq H^0(N_{X/V})$, we regard α as a global section of $N_{X/V}$ from now on.

If X is a hypersurface of V, *i.e.*, of codimension one in V, then $\mathrm{ob}(\alpha)$ becomes a simple cup product. Let $\delta_1 : H^0(X, N_{X/V}) \to H^1(V, \mathcal{O}_V)$ be the coboundary map of the exact sequence $0 \to \mathcal{O}_V \to \mathcal{O}_V(X) \to N_{X/V} \to 0$. Let us define a map[4]

$$d_X : H^0(X, N_{X/V}) \longrightarrow H^1(X, \mathcal{O}_X) \tag{2.5}$$

by the composition of δ_1 and the restriction map $H^1(\mathcal{O}_V) \xrightarrow{|_X} H^1(\mathcal{O}_X)$. Then we have

Lemma 2.9. — *Let X be a smooth hypersurface of V. Then $\mathrm{ob}(\alpha)$ for $\alpha \in H^0(N_{X/V})$ is equal to the cup product $d_X(\alpha) \cup \alpha$, where $\cup$ is the cup product map*

$$H^1(X, \mathcal{O}_X) \times H^0(X, N_{X/V}) \xrightarrow{\cup} H^1(X, N_{X/V}).$$

Proof. — Since $\mathcal{I}_X \simeq \mathcal{O}_V(-X)$ is a line bundle on V, we have $\mathrm{Ext}^i(\mathcal{I}_X, \mathcal{O}_X) \simeq H^i(N_{X/V})$ $(i = 0, 1)$ and $\mathrm{Ext}^1(\mathcal{I}_X, \mathcal{I}_X) \simeq H^1(\mathcal{O}_V)$. Hence the coboundary map δ appearing in the definition of $\mathrm{ob}(\alpha)$ is nothing but the coboundary map δ_1 of $(2.4) \otimes \mathcal{O}_V(X)$. Since α is a cohomology class on X, the cup product map $H^1(\mathcal{O}_V) \to H^1(N_{X/V})$ with α factors through the restriction map $\big|_X$. □

(4) The map d_X is equal to the map $d_{X, \mathcal{O}_V(X)}$ defined in [13, §2.1 (2.3)].

We recall the definition of exterior component introduced in [13]. Let X be a smooth closed subvariety of V and let Y be a smooth hypersurface of V containing X. Then the natural projection $\pi_{X/Y} : N_{X/V} \to N_{Y/V}\big|_X \simeq \mathcal{O}_X(Y)$ of normal bundles induces the maps $H^i(\pi_{X/Y}) : H^i(N_{X/V}) \to H^i(N_{Y/V}\big|_X)$, where $i = 0, 1$, of their cohomology groups. Let α be a global section of $N_{X/V}$.

Definition 2.10. — *$\pi_{X/Y}(\alpha)$ and $\mathrm{ob}_Y(\alpha)$ denote the images of α and $\mathrm{ob}(\alpha)$ by the maps $H^0(\pi_{X/Y})$ and $H^1(\pi_{X/Y})$, respectively. They are called the exterior components of α and $\mathrm{ob}(\alpha)$, respectively.*

Roughly speaking, $\pi_{X/Y}(\alpha)$ is the projection of the normal vector α of X in V onto the normal directions to Y in V. Then $\mathrm{ob}_Y(\alpha)$ represents the obstruction to deforming X into this direction. We recall a basic fact on exterior components.

Lemma 2.11 ([13, Lemma 2.4]). — *Let $\pi_{X/Y}(\alpha)$ and $\mathrm{ob}_Y(\alpha)$ be the exterior components of α and $\mathrm{ob}(\alpha)$, respectively. If there exists a global section v of $N_{Y/V}$ whose restriction $v\big|_X$ to X coincides with $\pi_{X/Y}(\alpha)$, then we have*

$$\mathrm{ob}_Y(\alpha) = \mathrm{ob}(v)\big|_X$$

where $\mathrm{ob}(v)\big|_X \in H^1(X, N_{Y/V}\big|_X)$ is the restriction of $\mathrm{ob}(v) \in H^1(Y, N_{Y/V})$ to X.

Lemma 2.11 together with Lemma 2.9 shows that $\mathrm{ob}_Y(\alpha) = d_Y(v)\big|_X \cup \pi_{X/Y}(\alpha)$, where d_Y is the map (2.5) for Y and $\cup$ is the cup product map

$$H^1(X, \mathcal{O}_X) \times H^0(X, N_{Y/V}\big|_X) \xrightarrow{\cup} H^1(X, N_{Y/V}\big|_X). \tag{2.6}$$

Let E be an effective divisor of Y disjoint to X (*i.e.*, $X \cap E = \emptyset$). Let Y° and V° denote the two complements of E in Y and V, respectively. Every rational section v of $N_{Y/V} \simeq \mathcal{O}_Y(Y)$ having poles only along E determines a global section v° of the normal bundle N_{Y°/V° of Y° in V° and hence the obstruction $\mathrm{ob}(v^\circ) \in H^1(N_{Y^\circ/V^\circ})$ to deforming Y° in V°. Let ι denote the open immersion of $Y^\circ \hookrightarrow Y$. Then a natural homomorphism $\iota_* N_{Y^\circ/V^\circ} \to N_{Y/V}\big|_X$ $(= [\iota_* \mathcal{O}_{Y^\circ} \to \mathcal{O}_X] \otimes N_{Y/V})$ of sheaves on Y induces a map $H^1(N_{Y^\circ/V^\circ}) \xrightarrow{|_X} H^1(N_{Y/V}\big|_X)$. Since $\mathrm{ob}(\alpha)$ is (and hence $\mathrm{ob}_Y(\alpha)$ is) determined by a neighborhood of X, we have the following variant of Lemma 2.11.

Lemma 2.12. — *Let α be a global section of $N_{X/V}$. If there exists a rational section v of $N_{Y/V}$ whose only poles are along E and whose restriction* *to X coincides with $\pi_{X/Y}(\alpha)$, then we have*

$$\mathrm{ob}_Y(\alpha) = \mathrm{ob}(v^\circ)\big|_X,$$

where $\mathrm{ob}(v^\circ)\big|_X$ *is the image of* $\mathrm{ob}(v^\circ)$ *by the map* $H^1(Y^\circ, N_{Y^\circ/V^\circ}) \xrightarrow{|_X} H^1(X, N_{Y/V}\big|_X)$.

3. Infinitesimal deformations with a pole

Let V be a smooth projective 3-fold, S a smooth surface in V, E a smooth connected curve on S. We put $V^\circ := V \setminus E$ and $S^\circ := S \setminus E$, the complementary open subvarieties. In this section, we study the first order infinitesimal deformations of S° in V°, when the self-intersection number of E on S is negative. We are interested in a rational section v of $N_{S/V}$ having a pole only along E and of order one, that is, $v \in H^0(N_{S/V}(E)) \setminus H^0(N_{S/V})$. Let $\iota : S^\circ \hookrightarrow S$ be the open immersion. Then $\iota_* \mathcal{O}_{S^\circ}$ contains $\mathcal{O}_S(nE)$ as a subsheaf for any $n \geqslant 0$. Hence the natural sheaf injection $N_{S/V}(nE) \hookrightarrow \iota_* N_{S^\circ/V^\circ}$ induces $H^0(S, N_{S/V}(nE)) \hookrightarrow H^0(S^\circ, N_{S^\circ/V^\circ})$ for each n. Therefore v determines a first order infinitesimal deformation of S° in V°. The main theorem of this section is the following.

THEOREM 3.1. — *Let v be as above and assume that $E^2 < 0$ and* $\det N_{E/V} := \bigwedge^2 N_{E/V}$ *is trivial. If the exact sequence*

$$0 \longrightarrow N_{E/S} \longrightarrow N_{E/V} \longrightarrow N_{S/V}\big|_E \longrightarrow 0 \tag{3.1}$$

does not split, then the first order infinitesimal deformation of $S^\circ \subset V^\circ$ determined by v does not lift to a deformation over $\operatorname{Spec} k[t]/(t^3)$.

Let n be a non-negative integer. In what follows, we identify $H^0(N_{S/V}(nE))$ with its image in $H^0(N_{S^\circ/V^\circ})$. We shall prove that the obstruction $\mathrm{ob}(v)$ is nonzero in $H^1(N_{S^\circ/V^\circ})$. Let d_{S° denote the map (2.5) for $X = S^\circ$. Then by Lemma 2.9, $\mathrm{ob}(v)$ is equal to the cup product $d_{S^\circ}(v) \cup v$, where $\cup$ is the cup product map

$$H^1(S^\circ, \mathcal{O}_{S^\circ}) \times H^0(S^\circ, N_{S^\circ/V^\circ}) \xrightarrow{\cup} H^1(S^\circ, N_{S^\circ/V^\circ}).$$

The inclusion $\mathcal{O}_S(nE) \hookrightarrow \iota_* \mathcal{O}_{S^\circ}$ of sheaves induces a map $H^1(S, \mathcal{O}_S(nE)) \to H^1(S^\circ, \mathcal{O}_{S^\circ})$ of cohomology groups. Suppose that $E^2 < 0$. Then this map is injective by Lemma 2.6. Hence we identify $H^1(\mathcal{O}_S(nE))$ with its image in $H^1(S^\circ, \mathcal{O}_{S^\circ})$. Under this identification, there exists a natural filtration

$$H^1(S, \mathcal{O}_S) \subset H^1(S, \mathcal{O}_S(E)) \subset H^1(S, \mathcal{O}_S(2E)) \subset \cdots \subset H^1(S^\circ, \mathcal{O}_{S^\circ})$$

on $H^1(S^\circ, \mathcal{O}_{S^\circ})$. Suppose now that $\det N_{E/V}$ is trivial. Then under similar identifications, there exists a natural filtration

$$H^1(S, N_{S/V}(E)) \subset H^1(S, N_{S/V}(2E)) \subset \cdots \subset H^1(S^\circ, N_{S^\circ/V^\circ})$$

on $H^1(S^\circ, N_{S^\circ/V^\circ})$, because we have $\deg N_{S/V}(nE)\big|_E = \deg(\det N_{E/V}) + (n-1)E^2 = (n-1)E^2 \leqslant 0$ for $n \geqslant 1$. Then it follows from [13, Proposition 2.6 (1)] that the image of d_{S° over $H^0(N_{S/V}(E))$ is contained in $H^1(\mathcal{O}_S(2E))$. By the commutative diagram

$$\begin{array}{ccccc} H^1(\mathcal{O}_{S^\circ}) & \times & H^0(N_{S^\circ/V^\circ}) & \stackrel{\cup}{\longrightarrow} & H^1(N_{S^\circ/V^\circ}) \\ \bigcup & & \bigcup & & \bigcup \\ H^1(\mathcal{O}_S(2E)) & \times & H^0(N_{S/V}(E)) & \stackrel{\cup}{\longrightarrow} & H^1(N_{S/V}(3E)), \end{array}$$

the image of the obstruction map ob over $H^0(N_{S/V}(E))$ is contained in $H^1(N_{S/V}(3E))$. The following lemma is essential to the proof of Theorem 3.1. Let d_S denote the restriction of the map d_{S° to $H^0(S, N_{S/V}(E))$.

Lemma 3.2 ([13, Proposition 2.6 (2)]). — *Let* $\partial : H^0(N_{S/V}(E)\big|_E) \to H^1(\mathcal{O}_E(2E)) \simeq H^1(N_{E/S}(E))$ *be the coboundary map of the exact sequence* (3.1)$\otimes\mathcal{O}_S(E)$. *Then the diagram*

$$\begin{array}{ccc} H^0(S, N_{S/V}(E)) & \stackrel{d_S}{\longrightarrow} & H^1(S, \mathcal{O}_S(2E)) \\ {\scriptstyle |_E}\downarrow & & {\scriptstyle |_E}\downarrow \\ H^0(E, N_{S/V}(E)\big|_E) & \stackrel{\partial}{\longrightarrow} & H^1(E, \mathcal{O}_E(2E)) \end{array}$$

is commutative.

Proof of Theorem 3.1. It suffices to show that the image $\mathrm{ob}(v)\big|_E \in H^1(N_{S/V}(3E)\big|_E)$ of $\mathrm{ob}(v) \in H^1(N_{S/V}(3E))$ is nonzero. By the definition of v, we have $v\big|_E \neq 0$ in $H^0(N_{S/V}(E)\big|_E)$. Then the line bundle $N_{S/V}(E)\big|_E \simeq \det N_{E/V}$ on E is trivial. Since (3.1) does not split by assumption, we have $\partial(v\big|_E) \neq 0$. Hence by Lemma 3.2, we conclude that

$$\mathrm{ob}(v)\big|_E = d_{S^\circ}(v)\big|_E \cup v\big|_E = \partial(v\big|_E) \cup v\big|_E \neq 0. \quad \square$$

If E is a (-1)-$\mathbb{P}^1$ on S with $\det N_{E/V} \simeq \mathcal{O}_{\mathbb{P}^1}$, then the exact sequence (3.1) does not split if and only if $N_{E/V}$ is trivial.

Example 3.3. — Let V_n be a smooth del Pezzo 3-fold of degree $n \neq 8$ and let E be a good line on V_n, *i.e.*, N_{E/V_n} is trivial (*cf.* §2.2). If S_n is a smooth hyperplane section of V_n containing E, then there exists an obstructed infinitesimal deformation of $S_n^\circ := S_n \setminus E$ in $V_n^\circ := V_n \setminus E$. Indeed, let $\varepsilon : S_n \to S_{n+1}$ be the blow-down of E from S_n. Since $N_{S_n/V_n} \simeq -K_{S_n}$, $N_{S_n/V_n}(E) \simeq \varepsilon^*(-K_{S_{n+1}})$, and $h^0(-K_{S_{n+1}}) > h^0(-K_{S_n})$, there exists a global section $v \in H^0(N_{S_n/V_n}(E)) \setminus H^0(N_{S_n/V_n})$. Then by Theorem 3.1, the first order infinitesimal deformation of S_n° in V_n° determined by v is obstructed.

In the rest of this section, we discuss a generalization of Theorem 3.1, which will be used in the proof of Theorem 1.3. Let E be a disjoint union of smooth connected curves E_i $(i = 1, \dots, m)$ on S such that $E_i^2 < 0$ and $\det N_{E_i/V}$ is trivial. By the same symbol E we also denote the divisor $\sum_{i=1}^m E_i$ on S. Let us define V° and S° as above and identify $H^0(N_{S/V}(E))$ with its image in $H^0(N_{S^\circ/V^\circ})$. We compute the restriction to $H^0(N_{S/V}(E))$ of the obstruction map $\mathrm{ob} : H^0(N_{S^\circ/V^\circ}) \to H^1(N_{S^\circ/V^\circ})$. Lemma 2.6 allows us to regard $H^1(\mathcal{O}_S(2E))$ and $H^1(N_{S/V}(3E))$ as subgroups of $H^1(\mathcal{O}_{S^\circ})$ and $H^1(N_{S^\circ/V^\circ})$, respectively. Then an argument similar to [13, Proposition 2.6 (1)] shows that the image of $H^0(N_{S/V}(E))$ by d_{S° is contained in $H^1(\mathcal{O}_S(2E))$. Therefore we conclude that

Lemma 3.4. — *The image of $H^0(N_{S/V}(E))$ by* ob *is contained in $H^1(N_{S/V}(3E)) \subset H^1(N_{S^\circ/V^\circ})$.*

Let v and v' be any global sections of $N_{S/V}(E)$ and $N_{S/V}$, respectively. Then we have $\mathrm{ob}(v+v')\big|_E = \mathrm{ob}(v)\big|_E$ in $H^1(N_{S/V}(3E)\big|_E)$. Indeed it follows from the definition of d_{S° (*cf.* (2.5)) that $d_{S^\circ}(v')$ is contained in $H^1(\mathcal{O}_S)$ and hence

$$\begin{aligned}\mathrm{ob}(v+v') &= (d_{S^\circ}(v) + d_{S^\circ}(v')) \cup (v+v') \\ &= \mathrm{ob}(v) + \underbrace{d_{S^\circ}(v) \cup v' + d_{S^\circ}(v') \cup v + d_{S^\circ}(v') \cup v'}_{\text{contained in } H^1(N_{S/V}(2E))}.\end{aligned}$$

Therefore the obstruction map ob induces a map

$$\overline{\mathrm{ob}} : H^0(N_{S/V}(E))/H^0(N_{S/V}) \longrightarrow H^1(N_{S/V}(3E)\big|_E). \tag{3.2}$$

Proposition 3.5. — *If $H^1(N_{S/V}) = 0$ and the exact sequence*

$$0 \longrightarrow N_{E_i/S} \longrightarrow N_{E_i/V} \longrightarrow N_{S/V}\big|_{E_i} \longrightarrow 0 \tag{3.3}$$

does not split for every i, then $\overline{\mathrm{ob}}$ is injective.

This is an immediate consequence of the next lemma.

Lemma 3.6. — *Under the assumption of Proposition 3.5, $\overline{\mathrm{ob}}$ is equivalent to the quadratic map*

$$k^m \longrightarrow k^n, \qquad (a_1, \dots, a_m) \longmapsto (a_1^2, \dots, a_m^2, 0, \dots, 0)$$

of diagonal type, where $n = \dim H^1(N_{S/V}(3E)\big|_E)$.

Proof. — Since $H^1(N_{S/V})=0$, the source of the map $\overline{\mathrm{ob}}$ is isomorphic to $H^0(N_{S/V}(E)\big|_E)$. Moreover there exist global sections v_i of $N_{S/V}(E_i)$ such that $v_i\big|_E \neq 0$ in $H^0(N_{S/V}(E_i)\big|_{E_i})$ for all i. Since E_i's are mutually disjoint, we have $N_{S/V}(E)\big|_E \simeq \bigoplus_{i=1}^m N_{S/V}(E_i)\big|_{E_i} \simeq \bigoplus_{i=1}^m \mathcal{O}_{E_i}$. Then there exists a commutative diagram

$$\begin{array}{ccccccccc} 0 \to & H^0(N_{S/V}) & \to & H^0(N_{S/V}(E)) & \to & H^0(N_{S/V}(E)\big|_E) & \to 0 \\ & \uparrow{\scriptstyle a_1} & & \uparrow{\scriptstyle a_2} & & \uparrow{\scriptstyle a_3} & \\ 0 \to & \bigoplus_i H^0(N_{S/V}) & \to & \bigoplus_i H^0(N_{S/V}(E_i)) & \to & \bigoplus_i H^0(N_{S/V}(E_i)\big|_{E_i}) & \to 0, \end{array}$$

where the two horizontal sequences are exact and a_i ($1 \leqslant i \leqslant 3$) are defined by addition. Since a_1 and a_3 are surjective, so is a_2. Hence every element $v \in H^0(N_{S/V}(E))$ is written as a k-linear combination $\sum_{i=1}^m c_i v_i$ of $v_i \in H^0(N_{S/V}(E_i))$ and the expression is unique modulo $H^0(N_{S/V})$. By the commutative diagram

$$\begin{array}{ccccc} H^1(\mathcal{O}_S(2E)) & \times & H^0(N_{S/V}(E)) & \xrightarrow{\cup} & H^1(N_{S/V}(3E)) \\ {\scriptstyle |_E}\downarrow & & {\scriptstyle |_E}\downarrow & & {\scriptstyle |_E}\downarrow \\ \bigoplus_i H^1(\mathcal{O}_{E_i}(2E_i)) & \times & \bigoplus_i H^0(N_{S/V}(E_i)\big|_{E_i}) & \xrightarrow{\cup} & \bigoplus_i H^1(N_{S/V}(3E_i)\big|_{E_i}), \end{array}$$

we have

$$\mathrm{ob}(v)\big|_E = (d_{S^\circ}(v) \cup v)\big|_E = d_{S^\circ}(v)\big|_E \cup v\big|_E = \sum_i c_i^2 d_{S^\circ}(v_i)\big|_{E_i} \cup v_i\big|_{E_i}.$$

By Lemma 3.2, $d_{S^\circ}(v_i)\big|_{E_i}$ is equal to $\partial_i(v\big|_{E_i})$ in $H^1(\mathcal{O}_{E_i}(2E_i))$, where ∂_i is the coboundary map of (3.3). Since (3.3) does not split by assumption, we have $\partial_i(v\big|_{E_i}) \neq 0$ and hence $d_{S^\circ}(v_i)\big|_{E_i} \neq 0$ for any i. As a result, $d_{S^\circ}(v_i)\big|_{E_i} \cup v_i\big|_{E_i}$ ($1 \leqslant i \leqslant m$) form a sub-basis of $H^1(N_{S/V}(3E)\big|_E)$. □

Suppose now that E_i is a (-1)-$\mathbb{P}^1$ on S and $N_{E_i/V}$ is trivial for every $1 \leqslant i \leqslant m$. Then Proposition 3.5 shows that

Corollary 3.7. — *Let $v \in H^0(N_{S/V}(E)) \setminus H^0(N_{S/V})$ be a global section. If $H^1(N_{S/V}) = 0$, then we have $\mathrm{ob}(v) \neq 0$ in $H^1(N_{S/V}(3E))$ ($\subset H^1(N_{S^\circ/V^\circ})$).*

4. Obstructions to deforming curves

Let V be a smooth projective 3-fold. In this section we study the deformation of smooth curves C on V under the presence of smooth surface S such that $C \subset S \subset V$.

4.1. S-normal curves and S-maximal families

In what follows, we assume that $\operatorname{Hilb} V$ is nonsingular at $[S]$. Then there exists a unique irreducible component U_S of $\operatorname{Hilb} V$ passing through $[S]$. We use the following convention.

Definition 4.1.

(1) *C is said to be stably degenerate if there exists an (Zariski) open neighborhood $U \subset \operatorname{Hilb} V$ of $[C]$ such that for any member $[C'] \in U$, there exists a deformation S' of S in V such that $C' \subset S'$ and $[S'] \in U_S$.*

(2) *C is said to be S-normal if the restriction map* (1.1) *is surjective.*

Let

$$V \times U_S \supset \mathcal{S} \xrightarrow{p_2} U_S$$

be the universal family of U_S. Let us denote the Hilbert scheme of smooth connected curves in $\mathcal{S}$ by $\operatorname{Hilb}^{sc} \mathcal{S}$, which is the relative Hilbert scheme of $\mathcal{S}/U_S$. $\operatorname{Hilb}^{sc} \mathcal{S}$ is regarded as an open subscheme of the Hilbert-flag scheme of V (see [8] for the definition), which parametrizes all flat families of pairs (C, S) of a curve C and a surface S in V such that $C \subset S$. The projection $p_1 : \mathcal{S} \to V$ induces a natural morphism

$$pr_1 : \operatorname{Hilb}^{sc} \mathcal{S} \longrightarrow \operatorname{Hilb}^{sc} V, \tag{4.1}$$

which is the forgetful morphism $(C, S) \mapsto C$. Then by definition C is stably degenerate if and only if pr_1 is surjective in a neighborhood of $[C] \in \operatorname{Hilb}^{sc} V$.

The next lemma plays an important role in our proof of Theorem 1.3 later (*cf.* § 4.3).

Lemma 4.2. — *Assume that $H^1(C, N_{C/S}) = 0$. Then:*

(1) *The kernel and the cokernel of the tangential map*

$$\kappa_{C,S} : H^0(C, N_{C/\mathcal{S}}) \longrightarrow H^0(C, N_{C/V}). \tag{4.2}$$

of pr_1 at (C, S) are isomorphic to those of the restriction map (1.1)*, respectively.*

(2) $\operatorname{Hilb}^{sc} \mathcal{S}$ *is nonsingular at (C, S).*

For the proof we refer to [13, Lemma 3.1] for (1) and [9, Lemma 1.10] for (2). We can also prove (1) by using the "fundamental exact sequence relating $A^i(C \subset S)$ and $H^{i-1}(N_{C/V})$" in [9].

In what follows, we assume that $H^1(C, N_{C/S}) = 0$. If C is S-normal, then $\kappa_{C,S}$ is surjective by Lemma 4.2 (1). Then by (2) of the same lemma,

$\mathrm{Hilb}^{sc} V$ is nonsingular at $[C]$ and furthermore pr_1 is surjective in a neighborhood of $[C]$. In fact, if C is S-normal, then the morphism pr_1 is smooth at (C, S) (*cf.* [8, Lemma A10]). Thus we conclude that

PROPOSITION 4.3 (*cf.* [8],[9]). — *If C is S-normal, then C is stably degenerate and $\mathrm{Hilb}^{sc} V$ is nonsingular at $[C]$.*

We recall the S-maximal family introduced in [13, §3.2]. By the smoothness of $\mathrm{Hilb}^{sc} \mathcal{S}$, there exists a unique irreducible component $\mathcal{W}_{S,C}$ of $\mathrm{Hilb}^{sc} \mathcal{S}$ containing (C, S).

DEFINITION 4.4. — *We define the S-maximal family of curves containing C to be the image of $\mathcal{W}_{S,C}$ in $\mathrm{Hilb}^{sc} V$ and denote it by $W_{S,C}$.*

By Proposition 4.3, if C is S-normal then $W_{S,C}$ is an irreducible component of $\mathrm{Hilb}^{sc} V$ and $\mathrm{Hilb}^{sc} V$ is generically smooth along $W_{S,C}$.

4.2. Deformation of curves on a del Pezzo 3-fold

Let V be a smooth del Pezzo 3-fold with the polarization H, S a smooth member of $|H|$, and C a smooth connected curve on S. Let n denote the degree of V and let d and g denote the degree $(:= (C \cdot H)_V)$ and the genus of C, respectively.

By adjunction we have $N_{S/V} \simeq -K_V\big|_S + K_S$ and $N_{C/S} \simeq -K_S\big|_C + K_C$. Since $-K_V$ and $-K_S$ are ample, we have $H^1(N_{S/V}) = H^1(N_{C/S}) = 0$. Hence $\mathrm{Hilb}\, V$ and $\mathrm{Hilb}\, S$ are nonsingular at $[S]$ and $[C]$, respectively. Thus if C is S-normal, then by Proposition 4.3, C is stably degenerate and $\mathrm{Hilb}^{sc} V$ is nonsingular at $[C]$. Because $H^1(N_{S/V}) = 0$, it follows from the exact sequence

$$0 \longrightarrow N_{S/V}(-C) \longrightarrow N_{S/V} \longrightarrow N_{S/V}\big|_C \longrightarrow 0 \tag{4.3}$$

that C is S-normal if and only if $H^1(N_{S/V}(-C)) = 0$. There exists a natural exact sequence

$$0 \longrightarrow N_{C/S} \longrightarrow N_{C/V} \xrightarrow{\pi_{C/S}} N_{S/V}\big|_C \longrightarrow 0. \tag{4.4}$$

Since $H^1(N_{C/S}) = 0$, we have $H^1(N_{C/V}) \simeq H^1(N_{S/V}\big|_C)$. Thus every obstruction to deforming C is contained in the cohomology group $H^1(N_{S/V}\big|_C)$. Since $\chi(N_{C/V}) = (-K_V \cdot C)_V = 2d$, we also have

LEMMA 4.5. — *If $H^1(N_{S/V}\big|_C) = 0$, then $\mathrm{Hilb}^{sc} V$ is nonsingular of expected dimension $2d$ at $[C]$.*

In particular, if C is rational ($g = 0$) or elliptic ($g = 1$), then the $\mathrm{Hilb}^{sc} V$ is nonsingular at $[C]$ because $H^1(N_{S/V}\big|_C) \simeq H^1(-K_S\big|_C) = 0$.

Let $W_{S,C}$ be the S-maximal family $W_{S,C}$ of curves containing C. We compute the dimension of $W_{S,C}$. Let $pr_1 : \mathrm{Hilb}^{sc} \mathcal{S} \to \mathrm{Hilb}^{sc} V$ be the morphism (4.1).

Lemma 4.6.

(1) $\mathrm{Hilb}^{sc} \mathcal{S}$ *is nonsingular of dimension* $d + g + n$ *at* (C, S).

(2) *If* $g \geqslant 2$ *or* $d \geqslant n + 1$, *then* pr_1 *is a closed embedding in a neighborhood of* (C, S) *and* $\dim W_{S,C} = d + g + n$.

Proof. — (1) Let $\mathcal{W}_{S,C}$ be the irreducible component of $\mathrm{Hilb}^{sc} \mathcal{S}$ containing (C, S). By the Riemann-Roch theorem on S, we have $\dim |\mathcal{O}_S(C)| = d + g - 1$. Then $\mathcal{W}_{S,C}$ is birationally equivalent to $\mathbb{P}^{d+g-1}$-bundle over an open subset of $|H| \simeq \mathbb{P}^{n+1}$. Hence we have $\dim \mathcal{W}_{S,C} = d + g + n$.

(2) By assumption, we have $(-K_S - C) \cdot C = 2 - 2g < 0$ or $(-K_S - C) \cdot (-K_S) = n - d < 0$. Since both C and $-K_S$ are nef, we have $H^0(N_{S/V}(-C)) \simeq H^0(-K_S - C) = 0$. By Lemma 4.2 (1), pr_1 is a closed embedding near (C, S). Hence we have $\dim W_{S,C} = \dim \mathcal{W}_{S,C}$. □

We denote by $\mathrm{Hilb}^{sc}_{d,g} V$ the open and closed subscheme of $\mathrm{Hilb}^{sc} V$ of curves of degree d and genus g. It is known that the dimension of every irreducible component of $\mathrm{Hilb}^{sc}_{d,g} V$ is greater than or equal to the expected dimension $\chi(N_{C/V}) = 2d$ (*cf.* [11, Chap. I, Theorem 2.8]).

Proposition 4.7. — *If* $\chi(V, \mathcal{I}_C(S)) < 1$, *then* C *is not stably degenerate, i.e., there exists a deformation* C' *of* C *in* V *which is not contained in any deformation* S' *of* S *in* V.

Proof. — There exists an exact sequence $[0 \to \mathcal{I}_C \to \mathcal{O}_V \to \mathcal{O}_C \to 0] \otimes \mathcal{O}_V(S)$ on V. We see that $\chi(\mathcal{O}_C(S)) = d+1-g$ and $\chi(\mathcal{O}_V(S)) = n+2$. Hence $\chi(\mathcal{I}_C(S)) < 1$ is equivalent to $g < d - n$. Then we have $\dim W_{S,C} \leqslant \dim \mathcal{W}_{S,C} = d + g + n < 2d$. Hence there exists an irreducible component $W' \supset W_{S,C}$ of $\mathrm{Hilb}^{sc} V$ such that $\dim W' > \dim W_{S,C}$. A general member C' of $W' \setminus W_{S,C}$ is such a deformation of C in V. □

4.3. Stably degenerate curves

We devote this subsection to the proof of Theorem 1.3. Notation is the same as in the previous subsection. The following are equivalent: (i) $\chi(V, \mathcal{I}_C(S)) \geqslant 1$, (ii) $\chi(S, N_{S/V}(-C)) \geqslant 0$ and (iii) $g \geqslant d - n$. Indeed we have already seen in the proof of Proposition 4.7 that (i) and (iii) are equivalent. Also (i) and (ii) are equivalent because we have $\chi(N_{S/V}(-C)) = \chi(\mathcal{I}_C(S)) - 1$ by the exact sequence

$$(4.5) \qquad [0 \longrightarrow \mathcal{I}_S \longrightarrow \mathcal{I}_C \longrightarrow \mathcal{O}_S(-C) \longrightarrow 0] \otimes \mathcal{O}_V(S).$$

Throughout this subsection, we assume one of (i),(ii) and (iii) (and hence all).

Lemma 4.8. — *If $H^1(C, N_{S/V}\big|_C) = 0$ then C is S-normal.*

Proof. — It suffices to show that $H^1(N_{S/V}(-C)) = 0$. Since $H^2(N_{S/V}) \simeq H^2(-K_S) = 0$ and $H^1(N_{S/V}\big|_C) = 0$, we obtain $H^2(N_{S/V}(-C)) = 0$ by (4.3). Then by assumption, we have $0 \leqslant \chi(N_{S/V}(-C)) = h^0(N_{S/V}(-C)) - h^1(N_{S/V}(-C))$. Therefore if $H^0(N_{S/V}(-C)) = 0$, then we have $H^1(N_{S/V}(-C)) = 0$. Suppose that $H^0(N_{S/V}(-C)) \neq 0$. There exists an effective divisor D on S such that $N_{S/V}(-C) \simeq \mathcal{O}_S(D)$. If $D = 0$, then $H^1(N_{S/V}(-C)) = 0$. Suppose that $D \neq 0$. Let h be a general member of $|-K_S|$. Then h is a smooth elliptic curve on S. Since $-K_S$ is ample, we have $\deg \mathcal{O}_S(D)\big|_h = D \cdot (-K_S) > 0$ and hence $H^1(\mathcal{O}_S(D)\big|_h) = 0$. Since C is connected, we obtain $H^1(D-h) \simeq H^1(-C) = 0$ from the exact sequence $0 \to \mathcal{O}_S(-C) \to \mathcal{O}_S \to \mathcal{O}_C \to 0$. Therefore it follows from the exact sequence

$$0 \longrightarrow \mathcal{O}_S(D-h) \longrightarrow \mathcal{O}_S(D) \longrightarrow \mathcal{O}_S(D)\big|_h \longrightarrow 0$$

that $H^1(N_{S/V}(-C)) \simeq H^1(D) = 0$. □

Let $E_1, \ldots, E_m$ be lines on S disjoint to C. We define an effective divisor E on S by $E := \sum_{i=1}^m E_i$. If C is not S-normal, then E is responsible for the abnormality.

Proposition 4.9. — *Suppose that C is not rational nor elliptic.*

1. *The restriction map $H^0(S, N_{S/V}(E)) \xrightarrow{|_C} H^0(C, N_{S/V}\big|_C)$ is an isomorphism.*
2. *C is S-normal if and only if there exists no line ℓ such that $C \cap \ell = \emptyset$ (i.e., $E = 0$).*

Proof. — (1) Since $N_{S/V} \simeq -K_S$, we have the assertion by Proposition 2.4.

(2) The "if" part follows from (1). We prove the "only if" part. Suppose that there exist such lines on S. Let $\varepsilon : S \to F$ be the blow-down of E from S. Then F is also a del Pezzo surface and $\varepsilon^*(-K_F) = -K_S + E$. Since $\deg F > \deg S$, we have $h^0(-K_F) > h^0(-K_S)$. Hence it follows from $N_{S/V} \simeq -K_S$ that $N_{S/V}(E)$ has more global sections than $N_{S/V}$. Hence

we have $h^0(N_{S/V}\big|_C) = h^0(N_{S/V}(E)) > H^0(N_{S/V})$ by (1). Therefore C is not S-normal. □

Let $\kappa_{C,S} : H^0(N_{C/S}) \to H^0(N_{C/V})$ denote the tangential map (4.2).

Proposition 4.10. — *Suppose that C is not S-normal. If every E_i is a good line on V, then the obstruction* $\mathrm{ob}(\alpha)$ *is nonzero for any* $\alpha \in H^0(C, N_{C/V}) \setminus \mathrm{im}\,\kappa_{C,S}$.

Proof. — Let $\pi_{C/S}(\alpha) \in H^0(N_{S/V}\big|_C)$ and $\mathrm{ob}_S(\alpha) \in H^1(N_{S/V}\big|_C)$ be the exterior component of α and $\mathrm{ob}(\alpha)$, respectively (*cf.* Definition 2.10). We compute $\mathrm{ob}_S(\alpha)$ instead of $\mathrm{ob}(\alpha)$ itself. Since C is not S-normal, by Lemma 4.8, we have $H^1(N_{S/V}\big|_C) \neq 0$. In particular, C is not rational nor elliptic. By Proposition 4.9 (1), there exists a global section v of $N_{S/V}(E)$ whose restriction $v\big|_C \in H^0(N_{S/V}\big|_C)$ to C coincides with $\pi_{C/S}(\alpha)$. Since α is not contained in the image of $\kappa_{C,S}$, $\pi_{C/S}(\alpha)$ is not contained in the image of (1.1) by Lemma 4.2 (1). Hence v is not a global section of $N_{S/V}$, in other words, an infinitesimal deformation with a pole (*cf.* §3).

Let S° and V° respectively denote the two complements $S\setminus E$ and $V\setminus E$ of E. There exists a natural injection $H^0(S, N_{S/V}(E)) \hookrightarrow H^0(S^\circ, N_{S^\circ/V^\circ})$ of cohomology groups. In what follows, we identify $v \in H^0(S, N_{S/V}(E))$ with its image $v^\circ \in H^0(S^\circ, N_{S^\circ/V^\circ})$, *i.e.*, a first order infinitesimal deformation of S° in V°. Now we prove that the obstruction $\mathrm{ob}(v) \in H^1(S^\circ, N_{S^\circ/V^\circ})$ is nonzero. By Lemma 3.4, $\mathrm{ob}(v)$ is contained in the subgroup $H^1(S,N_{S/V}(3E))$ of $H^1(S^\circ, N_{S^\circ/V^\circ})$. Every component E_i of E is a (-1)-$\mathbb{P}^1$ on S and its normal bundle $N_{E_i/V}$ in V is trivial by assumption. Therefore by virtue of Corollary 3.7, we have $\mathrm{ob}(v) \neq 0$ in $H^1(S, N_{S/V}(3E))$.

Finally we show that $\mathrm{ob}_S(\alpha) \neq 0$ in $H^1(C, N_{S/V}\big|_C)$. There exists an exact sequence

$$0 \longrightarrow N_{S/V}(3E - C) \longrightarrow N_{S/V}(3E) \xrightarrow{|_C} N_{S/V}\big|_C \longrightarrow 0.$$

Since $N_{S/V} \simeq -K_S$, the restriction map $H^1(S, N_{S/V}(3E)) \to H^1(C, N_{S/V}\big|_C)$ is injective by Lemma 2.5. Therefore we have $\mathrm{ob}_S(\alpha) = \mathrm{ob}(v)\big|_C \neq 0$ by Lemma 2.12. □

Now we prove Theorem 1.3. Let C be as in the theorem. Then we have

Lemma 4.11. — *Every small global deformation of C in V is contained in the S-maximal family $W_{S,C}$ of curves containing C.*

Proof. — Let $C_T \subset V \times T$ be a small global deformation of C, *i.e.*, a flat family C_T over a small open variety T, having a point $0 \in T$ with $C_0 = C$. Given an element of the Zariski tangent space of T at 0, we obtain a morphism $\operatorname{Spec} k[t]/(t^2) \to T$ and a first order infinitesimal deformation $\widetilde{C} \to \operatorname{Spec} k[t]/(t^2)$ of C by base extension. Then by Proposition 4.10 $\widetilde{C}$ is contained in the image of the map $\kappa_{(C,S)}$. Hence there exists a first order infinitesimal deformation $\widetilde{S}$ of S such that $\widetilde{S} \supset \widetilde{C}$. Since $\operatorname{Hilb}^{sc} \mathcal{S}$ is nonsingular at (C, S), the first order infinitesimal deformation $(\widetilde{C}, \widetilde{S})$ of (C, S) lifts to a global deformation (C_T, S_T) over T. □

Therefore C is stably degenerate. The rest of the proof is as follows. If C is S-normal, then $\operatorname{Hilb}^{sc} V$ is nonsingular at $[C]$ by Proposition 4.3. Otherwise, there exists a first order infinitesimal deformation $\widetilde{C}$ of C not contained in the image of $\kappa_{(C,S)}$. Then $\operatorname{Hilb}^{sc} V$ is singular at $[C]$ by Proposition 4.10. We have an isomorphism $H^1(S, N_{S/V}(-C)) \simeq H^1(V, \mathcal{I}_C(S))$ by the exact sequence (4.5) together with that $H^i(V, \mathcal{I}_S(S)) = H^i(V, \mathcal{O}_V) = 0$ for $i = 1, 2$. Hence C is S-normal if and only if $H^1(V, \mathcal{I}_C(S)) = 0$. The proof of Theorem 1.3 has been completed.

Remark 4.12. — We give two remarks on Theorem 1.3.

(1) Suppose that V is not isomorphic to a blow-up V_7 of $\mathbb{P}^3$ at a point. If $S \in |H|$ is general, then by Lemma 2.8, every line on S is a good line on V. Hence every curve C on S is stably degenerate by the theorem. Meanwhile there exists a non-stably degenerate curve C on V_7 which is contained in a general member S of $|H|$ (*cf.* Proposition 5.4).

(2) There exists no line on a del Pezzo 3-fold $V_8 \simeq \mathbb{P}^3$. Hence if $V = V_8$, then the assumption of the theorem concerning lines ℓ on S such that $C \cap \ell = \emptyset$ is empty. In fact, if $g \geqslant d - 8$ then every curve C on V_8 is S-normal and hence stably degenerate. This coincides with the previous result [15, Appendix, Proposition 4.11], which proved that every curve of degree e and genus $p \geqslant 2e - 8$ in $\mathbb{P}^3$ lying on a smooth quadric surface $Q_2 \simeq \mathbb{P}^1 \times \mathbb{P}^1$ is stably degenerate.

The following proposition is more practical than Proposition 4.10 in showing that $\operatorname{Hilb}^{sc} V$ is singular at $[C]$.

Proposition 4.13. — *Suppose that C is not rational nor elliptic. If there exists a good line ℓ on V such that $\ell \subset S$ and $C \cap \ell = \emptyset$, then $\operatorname{Hilb}^{sc} V$ is singular at $[C]$.*

The proofs of Proposition 4.10 and Proposition 4.13 are very similar. Take a global section $v \in H^0(N_{S/V}(\ell)) \setminus H^0(N_{S/V})$ and put $\alpha \in H^0(N_{C/V})$ as a lift of $v\big|_C \in H^0(N_{S/V})$ by the surjective map $\pi_{C/S} : H^0(N_{C/V}) \twoheadrightarrow H^0(N_{S/V}\big|_C)$. Then it is enough to show that $\operatorname{ob}_S(\alpha) \neq 0$ in $H^1(N_{S/V}\big|_C)$

by reducing it to $\mathrm{ob}(v)\big|_\ell \neq 0$ as in the proof of Proposition 4.10. We omit the details.

The following is an analogue of Conjecture 5.1 due to Kleppe and Ellia.

THEOREM 4.14. — *Let C be the curve in Theorem 1.3. Then:*

(1) *The S-maximal family $W_{S,C} \subset \mathrm{Hilb}^{sc} V$ of curves containing C is an irreducible component of $(\mathrm{Hilb}^{sc} V)_{\mathrm{red}}$.*

(2) *$\mathrm{Hilb}^{sc} V$ is generically smooth along $W_{S,C}$ if $H^1(V, \mathcal{I}_C(S)) = 0$, and generically non-reduced along $W_{S,C}$ otherwise.*

Proof. — (1) By definition $W_{S,C}$ is an irreducible closed subset of $\mathrm{Hilb}^{sc}V$. By Lemma 4.11, $W_{S,C}$ is maximal among all such subsets.

(2) Let C' be a general member of $W_{S,C}$. Then C' is contained in a smooth surface $S' \sim S$ in V. Since C' is general, so is S' in $|S|$. Suppose that $H^1(\mathcal{I}_C(S)) = 0$. Then since (C', S') is a generalization of (C, S), we have $H^1(\mathcal{I}_{C'}(S')) = H^1(\mathcal{I}_C(S)) = 0$ by the upper semicontinuity. Hence $\mathrm{Hilb}^{sc} V$ is nonsingular at $[C']$ and hence generically smooth along $W_{S,C}$. Suppose that $H^1(\mathcal{I}_C(S)) \neq 0$, *i.e.*, C is not S-normal. Then Lemma 4.8 shows that $H^1(N_{S/V}\big|_C) \neq 0$ and hence $g \geqslant 2$. By Proposition 4.9 (2), there exists a line ℓ on S such that $C \cap \ell = \emptyset$. Since $H^1(\mathcal{O}_S) = 0$, the Picard group of S does not change under the smooth deformation of S and hence $\mathrm{Pic}\, S \simeq \mathrm{Pic}\, S'$. Since $H^1(\mathcal{O}_S(\ell)) = 0$, the line ℓ is deformed to a line ℓ' on S'. Then we have $C' \cap \ell' = \emptyset$. Moreover since ℓ is a good line, so is ℓ'. Hence $\mathrm{Hilb}^{sc} V$ is singular at $[C']$ by Proposition 4.13. Since C' is a general member of $W_{S,C}$, $\mathrm{Hilb}^{sc} V$ is everywhere singular along $W_{S,C}$ and hence generically non-reduced along $W_{S,C}$. □

5. Original motivation and examples

5.1. Kleppe's conjecture

The original motivation of the present work was to show the following conjecture due to Kleppe. We denote by $\mathrm{Hilb}^{sc}_{d,g} \mathbb{P}^3$ the open and closed subscheme of $\mathrm{Hilb}^{sc} \mathbb{P}^3$ consisting of curves of degree d and genus g.

CONJECTURE 5.1 (Kleppe, Ellia). — *Let W be a maximal irreducible closed subset of $\mathrm{Hilb}^{sc}_{d,g} \mathbb{P}^3$ whose general member C is contained in a smooth cubic surface. If*

$$d \geqslant 14, \quad g \geqslant 3d - 18, \quad H^1(\mathbb{P}^3, \mathcal{I}_C(3)) \neq 0 \quad \text{and} \quad H^1(\mathbb{P}^3, \mathcal{I}_C(1)) = 0,$$

then W is a component of $(\mathrm{Hilb}^{sc} \mathbb{P}^3)_{\mathrm{red}}$ and $\mathrm{Hilb}^{sc} \mathbb{P}^3$ is generically non-reduced along W.

In the original conjecture [8, Conjecture 4] of Kleppe, the assumption of the linearly normality of C (*i.e.*, $H^1(\mathbb{P}^3, \mathcal{I}_C(1)) = 0$) was missing. However Ellia [2] pointed out that the conjecture does not hold for linearly non-normal curves C by a counterexample, and suggested restricting the conjecture to linearly normal ones. The most crucial part to prove this conjecture is the proof of the maximality of W in $(\operatorname{Hilb}^{sc} \mathbb{P}^3)_{\mathrm{red}}$. Once we prove that W is a component of $(\operatorname{Hilb}^{sc} \mathbb{P}^3)_{\mathrm{red}}$, then the non-reducedness of $\operatorname{Hilb}^{sc} \mathbb{P}^3$ along W naturally follows. Therefore Conjecture 5.1 follows from Conjecture 1.2 (1), where the condition $\chi(\mathbb{P}^3, \mathcal{I}_C(3)) \geqslant 1$ is equivalent to $g \geqslant 3d - 18$. Recently it has been proved in [15] that Conjecture 5.1 is true if $h^1(\mathbb{P}^3, \mathcal{I}_C(3)) = 1$. Kleppe and Ellia gave a proof for the conjecture under some other conditions, however the whole conjecture is still open.

5.2. Hilbert scheme of canonical curves

Let (V, H) be a polarized variety. We say that a curve $C \subset V$ is *canonical* if $f^*H = K_C$, where $f : C \hookrightarrow V$ is the embedding, or equivalently C is embedded into V by a linear subsystem of $|K_C|$. We apply Theorem 4.14 to prove the following:

Proposition 5.2 (*cf.* [13]). — *Let V be a smooth del Pezzo 3-fold of degree n. If $n \leqslant 7$ then the Hilbert scheme $\operatorname{Hilb}^{sc} V$ of smooth connected curves on V has a generically non-reduced component W, whose general member is a canonical curve on V.*

Proof. — Since $n \leqslant 7$, there exists a good line ℓ on V by Lemma 2.7. Let S_n be a smooth member of $|H|$ containing ℓ. We consider the complete linear system $\Lambda := |-2K_{S_n} + 2\ell|$ on S_n. Let S_{n+1} be the the blow-down of ℓ from S_n, which is a del Pezzo surface of degree $n + 1$. Then Λ is the pull-back of $|-2K_{S_{n+1}}| \simeq \mathbb{P}^{3n+3}$ on S_{n+1}. Since Λ is base point free, every general member C of Λ is a smooth connected curve of degree $d = 2n + 2$ and genus $g = n + 2$. Therefore we have $g = d - n$ and hence $\chi(V, \mathcal{I}_C(S)) = 1$. Then ℓ does not intersect C by $(-2K_{S_n} + 2\ell)\cdot\ell = 2-2 = 0$. Moreover ℓ is the only such line on S_n. By Theorem 4.14 (1), $W_{S_n,C}$ is an irreducible component of $(\operatorname{Hilb}^{sc} V)_{\mathrm{red}}$. Since $C \cap \ell = \emptyset$, C is not S_n-normal by Proposition 4.9 (2). Therefore $\operatorname{Hilb}^{sc} V$ is generically non-reduced along $W_{S_n,C}$ by Theorem 4.14 (2). By construction, C is the image of a canonical curve $C' \sim -2K_{S_{n+1}}$ on S_{n+1} by the projection $S_{n+1} \cdots \to S_n$ from a point $p \in S_{n+1} \setminus C'$. □

Remark 5.3. — The dimension of the irreducible component $W_{S_n,C}$ is equal to $d+g+n = 4n+4$ by Lemma 4.6 (2). The tangential dimension of $\mathrm{Hilb}^{sc} V$ at a general point $[C]$ of $W_{S_n,C}$ is equal to $h^0(N_{C/V}) = 4n+5$. Indeed the exact sequence (4.4) is

$$0 \longrightarrow \mathcal{O}_C(2K_C) \longrightarrow N_{C/V} \longrightarrow \mathcal{O}_C(K_C) \longrightarrow 0,$$

since $N_{S/V}\big|_C \simeq -K_S\big|_C \simeq K_C$. Hence we have

$$h^0(N_{C/V}) = h^0(2K_C) + h^0(K_C) = (3n+3) + (n+2) = 4n+5.$$

The next example shows that the curve C in Theorem 1.3 is not necessarily stably degenerate if there exists a bad line ℓ on S such that $C \cap \ell = \emptyset$.

Let $V_7 \subset \mathbb{P}^8$ be a smooth del Pezzo 3-fold of degree 7, S_7 a smooth hyperplane section of V_7. Let ℓ_0, ℓ_1, ℓ_2 be the three lines on S_7 explained in Lemma 2.8, *i.e.*, ℓ_0 is bad and ℓ_1 and ℓ_2 are good. Consider a general member C of $\Lambda := |-2K_{S_7} + 2\ell_0|$. Then C is a smooth connected curve of degree 16 and genus $9 = 16-7$ and not S_7-normal by $C \cap \ell_0 = \emptyset$.

PROPOSITION 5.4. — *Let C be as above. Then there exists a smooth deformation $C' \subset V_7$ of C not contained in any hyperplane section. In other words, C is not stably degenerate.*

Proof. — Recall that V_7 is isomorphic to the blow-up of $\mathbb{P}^3$ at a point p. It is realized as the projection of the Veronese image $V_8 \subset \mathbb{P}^9$ of $\mathbb{P}^3$ from $p \in V_8$ (*cf.* §2.2). Then S_7 is the image by the projection of a hyperplane section $Q_2 \simeq \mathbb{P}^1 \times \mathbb{P}^1$ of V_8 containing p. Hence we have a diagram

$$\begin{array}{ccccc} S_7 \simeq \mathrm{Bl}_{2\mathrm{pts}}\, \mathbb{P}^2 & \subset & V_7 \simeq \mathrm{Bl}_p\, \mathbb{P}^3 & \subset & \mathbb{P}^8 \\ \downarrow\uparrow & & {\scriptstyle \pi_p}\downarrow\uparrow{\scriptstyle \Pi_p} & & \uparrow \\ Q_2 \simeq \mathbb{P}^1 \times \mathbb{P}^1 & \subset & V_8 \simeq \mathbb{P}^3 & \subset & \mathbb{P}^9, \end{array} \tag{5.1}$$

where the down arrows (resp. the up arrows) are the blow-up morphisms at (resp. the projections from) $p \in Q_2 \subset V_8 \subset \mathbb{P}^9$. Let $P \simeq \mathbb{P}^2$ denote the exceptional divisor of π_p. Then its intersection with S_7 is equal to the bad line ℓ_0.

Since $C \cap \ell_0 = \emptyset$ and $C \cdot \ell_i = 4$ for each $i = 1, 2$, π_p maps C isomorphically onto a curve of bidegree $(4,4)$ on Q_2. Let Q_2' be a general hyperplane section of V_8. Then $Q_2' \simeq \mathbb{P}^1 \times \mathbb{P}^1$ is mapped isomorphically onto a surface Q_2'' on V_7 by Π_p. Here Q_2'' is linearly equivalent to $S_7 + P$ as a divisor of V_7 and contains a smooth deformation C' of C. Then there exists no hyperplane section of V_7 containing C'. Suppose that there exists such a hyperplane section S_7'. Then the image $\pi_p(C')$ is contained in the intersection of two hyperplane sections $\pi_p(S_7')$ and Q_2' of V_8. Hence the pull-back of $\pi_p(C')$

in $\mathbb{P}^3$ by the Veronese embedding is contained in a complete intersection of two quadrics. This is impossible since the degree of the inverse image is equal to $8 > 4$. □

5.3. Non-reduced components of the Hilbert scheme

In this subsection, we give the classes of irreducible components of the Hilbert scheme of del Pezzo 3-folds which are non-reduced by Theorem 4.14 more precisely (*cf.* Proposition 5.5).

Let V_n be a smooth del Pezzo 3-fold of degree $n \leqslant 7$ and let $S \subset V_n$ be a smooth member of the class $|H|$ of the polarization of V_n, *i.e.*, a del Pezzo surface. Put $r := 9 - n$ ($\geqslant 2$). Then S is isomorphic to a $\mathbb{P}^2$ blown up at r points in general position, *i.e.*, no three are on a line, no six are on a conic and any cubic containing eight points is smooth at each of them. The class of the pullback l of a line in $\mathbb{P}^2$ and the r exceptional curves e_i ($1 \leqslant i \leqslant r$) form a free $\mathbb{Z}$-basis of the Picard group $\operatorname{Pic} S \simeq \mathbb{Z}^{r+1}$ of S. Thus given a divisor D on S, we obtain a $(r+1)$-tuple $(a; b_1, \ldots, b_r)$ of integers as the coefficients of the divisor class $D = al - \sum_{i=1}^{r} b_i e_i$. On the other hand, for each $r \geqslant 2$ there exists a Weyl group $W_r \subset \operatorname{Aut}(\operatorname{Pic} S)$. Here W_r is the subgroup generated by the permutations of e_i ($1 \leqslant i \leqslant r$) and (for $r \geqslant 3$) by the additional Cremona element σ given by $\sigma(l) = 2l - e_1 - e_2 - e_3$, $\sigma(e_1) = l - e_2 - e_3$, $\sigma(e_2) = l - e_1 - e_3$, $\sigma(e_3) = l - e_1 - e_2$ and $\sigma(e_i) = e_i$ for $i \notin \{1, 2, 3\}$. The root systems corresponding to the Weyl group W_r ($r = 2, 3, 4, 5, 6, 7, 8$) are $A_1, A_1 \times A_2, A_4, D_5, E_6, E_7, E_8$, respectively (See [12] for the details). Every element of W_r induces a base change of $\operatorname{Pic} S$. By virtue of the Weyl groups and this base change, given a divisor D on S there exists a suitable blow-up $S \to \mathbb{P}^2$ (in other words, a suitable choice of r exceptional curves on S) such that we have

$$b_1 \geqslant \cdots \geqslant b_r \quad \text{and} \quad a \geqslant b_1 + b_2 + b_3 \quad (\text{only for } r \geqslant 3). \tag{5.2}$$

When (5.2) holds, we say the basis $\{l, e_1, \ldots, e_r\}$ of $\operatorname{Pic} S$ is *standard* for D. For the standard basis of $\operatorname{Pic} S$ for D, the linear system $|D|$ on S contains a smooth connected curve C of degree > 2 if and only if $a > b_1$ and $b_r \geqslant 0$ (and $a \geqslant b_1 + b_2$ for $r = 2$). The degree d and genus g of C is computed as

$$d = 3a - \sum_{i=1}^{r} b_i \quad \text{and} \quad g = \binom{a-1}{2} - \sum_{i=1}^{r} \binom{b_i}{2}. \tag{5.3}$$

Let (d, g) be a pair of integers with $d > 2$ and let $(a; b_1, \ldots, b_r)$ be a $(r+1)$-tuple of integers satisfying (5.2), (5.3), $a > b_r$ and $b_r \geqslant 0$ (and

$a \geqslant b_1 + b_2$ as well for $r = 2$). Then the linear system $|al - \sum_{i=1}^{r} b_i e_i|$ on S contains a smooth connected member C of degree d and genus g. Then we denote by $W_{(a;b_1,\dots,b_r)}$ the S-maximal family $W_{S,C} \subset \mathrm{Hilb}^{sc}_{d,g} V_n$ of curves containing C (*cf.* Definition 4.4). By definition, $W_{(a;b_1,\dots,b_r)}$ contains every smooth connected curve C' on V_n such that C' is contained in a smooth member $S' \in |H|$ and such that $C' \sim al' - \sum_{i=1}^{r} b_i e'_i$ on S' for a standard basis $\{l', e'_1, \dots, e'_r\}$ of $\mathrm{Pic}\, S'$ for C'.

Proposition 5.5. — *Suppose that $g \geqslant 2$ and $g \geqslant d - n$. If $b_r = 0$, then $W_{(a;b_1,\dots,b_r)}$ is an irreducible component of $(\mathrm{Hilb}^{sc}_{d,g} V_n)_{\mathrm{red}}$ of dimension $d + g + n$ and $\mathrm{Hilb}^{sc}_{d,g} V_n$ is generically non-reduced along $W_{(a;b_1,\dots,b_r)}$.*

Proof. — Let C denote a general member of $W_{(a:b_1,\dots,b_r)}$. Then C is contained in a smooth member $S \in |H|$. Since C is general, so is S in $|H|$. By Lemma 2.8 every line on S is good except for the bad line ℓ_0 on V_7. If $n = 7$ then ℓ_0 is linearly equivalent to $l - e_1 - e_2$. Since $b_2 = 0$, C intersects ℓ_0 by $C \cdot \ell_0 = (al - b_1 e_1) \cdot \ell_0 = a - b_1 > 0$. We recall that $g \geqslant d - n$ is equivalent to $\chi(V, \mathcal{I}_C(S)) \geqslant 1$. Therefore $W_{(a;b_1,\dots,b_r)}$ is an irreducible component of $(\mathrm{Hilb}^{sc}_{d,g} V_n)_{\mathrm{red}}$ by Theorem 4.14 (1), and of dimension $d+g+n$ by Lemma 4.6.

Since $b_r = 0$, the line e_r on S does not intersect C. Since $g \geqslant 2$, C is not S-normal by Proposition 4.9 (2), and hence we have $H^1(V, \mathcal{I}_C(S)) \neq 0$. Thus $\mathrm{Hilb}^{sc}_{d,g} V_n$ is generically non-reduced along $W_{(a;b_1,\dots,b_r)}$ by Theorem 4.14 (2). □

The next example shows that for every integer $d \geqslant 12$ the Hilbert scheme of smooth connected curves of degree d on a smooth cubic 3-fold V_3 has a generically non-reduced component.

Example 5.6. — Let $\lambda \in \mathbb{Z}_{\geqslant 0}$ and let W be one of the S-maximal families

$$W_{(\lambda+6;\lambda+1,1,1,1,1,0)} \subset \mathrm{Hilb}^{sc}_{d,2d-16} V_3 \quad (d = 2\lambda + 13) \quad \text{and}$$
$$W_{(\lambda+6;\lambda+2,1,1,1,1,0)} \subset \mathrm{Hilb}^{sc}_{d,\frac{3}{2}d-9} V_3 \quad (d = 2\lambda + 12).$$

Then W is an irreducible component of $(\mathrm{Hilb}^{sc} V_3)_{\mathrm{red}}$ and $\mathrm{Hilb}^{sc} V_3$ is generically non-reduced along W.

It was shown in [13, Theorem 1.4] that for many uniruled 3-folds V the Hilbert scheme $\mathrm{Hilb}^{sc} V$ has infinitely many generically non-reduced components.

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