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# THE MONODROMY CONJECTURE FOR ZETA FUNCTIONS ASSOCIATED TO IDEALS IN DIMENSION TWO

**by Lise VAN PROEYEN & Willem VEYS (*)**

Abstract. — The monodromy conjecture states that every pole of the topological (or related) zeta function induces an eigenvalue of monodromy. This conjecture has already been studied a lot. However in full generality it is proven only for zeta functions associated to polynomials in two variables.

In this article we work with zeta functions associated to an ideal. First we work in arbitrary dimension and obtain a formula (like the one of A'Campo) to compute the "Verdier monodromy" eigenvalues associated to an ideal. Afterwards we prove a generalized monodromy conjecture for arbitrary ideals in two variables.

Résumé. — La conjecture de la monodromie prédit que chaque pôle de la fonction zêta topologique (ou analogue) induit une valeur propre de la monodromie. Cette conjecture a déjà beaucoup été étudiée ; toutefois elle est prouvée en général seulement pour des fonctions zêta associées à un polynôme en deux variables. Dans cet article nous traitons des fonctions zêta associées à un idéal. En dimension quelconque nous obtenons une formule (semblable à celle d'A'Campo) qui calcule les valeurs propres de la "monodromie de Verdier". Pour des idéaux en deux variables, nous prouvons ensuite une conjecture généralisée de la monodromie.

## 1. Introduction

Classically the invariants called topological, motivic and $p$-adic Igusa zeta function are associated to *one* polynomial $f$ over $\mathbb{C}$, over an arbitrary field of characteristic zero and over a $p$-adic field, respectively. There are fascinating conjectures relating their poles with the roots of the Bernstein-Sato polynomial (also called $b$-function) of $f$ and with the eigenvalues of

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the local Milnor monodromy of $f$ in points of $\{f = 0\}$, up to now proven in full generality only for polynomials in two variables.

One associates similarly in a natural way all these functions to several polynomials or to an ideal. Not so obvious is the notion of Bernstein-Sato polynomials associated to several polynomials or an ideal. There is a construction of Sabbah [24] and more recently also of Budur, Mustaţă and Saito [5]. Concerning monodromy, the classical construction of local Milnor fibre does not generalize to arbitrary maps $(f_1, \ldots, f_r)$. However, there is a notion of "Verdier monodromy" in this general context [26].

In this paper we prove a relation between the poles of these zeta functions associated to an arbitrary ideal in two variables and the "Verdier monodromy eigenvalues" of this ideal, generalizing the result for one polynomial.

We now provide more details, focussing on the topological zeta function. Let $\mathcal{I} = (f_1, \ldots, f_r)$ be a nontrivial ideal in $\mathbb{C}[x_1, \ldots, x_n]$ and $Y = \mathbf{V}(\mathcal{I})$ its associated subscheme of $\mathbb{A}^n_{\mathbb{C}}$. We assume that $Y$ contains the origin.

We first fix notation to define the topological zeta function. Take a principalization $\psi : \tilde{X} \to \mathbb{A}^n$ of $\mathcal{I}$. By this we mean that $\psi$ is a proper birational map from a nonsingular variety $\tilde{X}$ such that the total transform $\psi^*\mathcal{I}$ is a principal ideal with support a simple normal crossings divisor, and moreover that the exceptional locus of $\psi$ is contained in the support of $\psi^*\mathcal{I}$. If $\mathcal{I}$ has components of codimension one, we can write this total transform as a product of two (principal) ideals: the support of the first one is the exceptional locus, where the support of the second one is formed by the irreducible components of the total transform that are not contained in the exceptional locus. This second ideal is the "weak transform" of $\mathcal{I}$.

Note that we use the word "principalization", where other authors may also use log-principalization, log-resolution or monomialization.

Let $\tilde{E} = \sum_{i \in J} N_i E_i$ denote the divisor of $\psi^*\mathcal{I}$, *i.e.,* its irreducible components are the $E_i$, $i \in J$, occurring with multiplicity $N_i$. (Alternatively, one can say that $\psi^{-1}Y = \tilde{E}$.) Let the relative canonical divisor of $\psi$ be $\sum_{i \in J}(\nu_i - 1)E_i$, *i.e.,* $\nu_i - 1$ is the multiplicity of $E_i$ in the divisor of $\psi^*(dx_1 \wedge \ldots \wedge dx_n)$. Finally put $E_I^\circ := (\cap_{i \in I} E_i) \backslash (\cup_{l \notin I} E_l)$ for $I \subset J$; these $E_I^\circ$ form a natural locally closed stratification of $\tilde{X}$. (Note that $E_\emptyset^\circ = \tilde{X} \backslash \cup_{l \in J} E_l$.) The (local) *topological zeta function of* $\mathcal{I}$ *at* 0 is

$$Z_{\mathrm{top},\mathcal{I}}(s) := \sum_{I \subset J} \chi(E_I^\circ \cap \psi^{-1}\{0\}) \prod_{i \in I} \frac{1}{\nu_i + sN_i} \in \mathbb{Q}(s).$$

There is a global version replacing $E_I^\circ \cap \psi^{-1}\{0\}$ by $E_I^\circ$.

When $r = 1$, Denef and Loeser showed in [8] that the expression above does not depend on the chosen principalization (which for $r = 1$ is just an embedded resolution) by writing it as a limit of $p$-adic Igusa zeta functions. Alternatively, they obtained it later in [9] as a specialization of the motivic zeta function. This can be generalized to arbitrary $r$, see *e.g.* [29, (2.4)]. Still another possibility for arbitrary $r$ is to use the Weak Factorization Theorem of Włodarczyk et al. [1] to compare two principalizations. At any rate, observe that a complete list of possible poles of $Z_{\mathrm{top},\mathcal{I}}(s)$ is given by the $-\frac{\nu_i}{N_i}, i \in J$.

For $r = 1$, say $\mathcal{I} = (f)$, there are the following intriguing conjectures [8]. (They were originally formulated for the $p$-adic Igusa zeta function, which is a certain $p$-adic integral, and partially motivated by analogous statements that are true for a similar complex integral. See [16], in particular section 5.4.)

CONJECTURE 1.1. — *If $s_0$ is a pole of $Z_{\mathrm{top},f}(s)$, then $s_0$ is a root of the (local) Bernstein-Sato polynomial $b_{f,0}(s)$ of $f$.*

CONJECTURE 1.2 (Monodromy Conjecture). — *If $s_0$ is a pole of $Z_{\mathrm{top},f}(s)$, then $e^{2\pi i s_0}$ is an eigenvalue of the local monodromy action on some cohomology group of the Milnor fibre of $f$ at some point of $\{f = 0\}$ close to 0.*

Note that Conjecture 1.1 implies Conjecture 1.2 since for any root $s_0$ of $b_{f,0}(s)$ we have that $e^{2\pi i s_0}$ is such a monodromy eigenvalue [21]. For $n = 2$ Conjecture 1.1 was proved by Loeser [19]. He also verified it for non-degenerate polynomials satisfying extra assumptions [20]. (There is a more elementary proof of Conjecture 1.2 for $n = 2$ by Rodrigues [22].) Concerning Conjecture 1.2, there are various partial results, mainly for $n = 3$, by Artal, Cassou-Noguès, Luengo and Melle [3], [4], and Lemahieu, Rodrigues and the second author [27], [23], [28], [17], [18].

Budur, Mustaţă and Saito introduced a Bernstein-Sato polynomial associated to an arbitrary ideal $\mathcal{I} = (f_1, \ldots, f_r) \subset \mathbb{C}[x_1, \ldots, x_n]$. (Their polynomial coincides with a polynomial that appears in [13].)

Still for arbitrary $r$, the notion of local Milnor fibre is in general not well-defined. There is however the following construction of Verdier. To any constructible complex of sheaves $\mathcal{F}^\bullet$ on $\mathbb{A}^n$, he associates a similar complex on $C_Y\mathbb{A}^n$, the normal cone of $Y = \mathbf{V}(\mathcal{I})$ in $\mathbb{A}^n$. This complex is called the *specialization* of $\mathcal{F}^\bullet$. It is moreover equipped with a "canonical monodromy operator". In particular when $\mathcal{I} = (f)$ it turns out that this specialization of $\mathbb{C}^\bullet$ is in some sense equivalent to the usual complex of nearby cycles on $Y = \mathbf{V}(f)$, and that the two monodromy notions essentially correspond.

A nice feature of the Bernstein-Sato polynomial of [5] is that for any of its roots $s_0$, we have that $e^{2\pi i s_0}$ is a "Verdier monodromy eigenvalue", see [5, Corollary 2.8]. This thus generalizes the implication for $r = 1$ mentioned above.

It is natural to ask also for arbitrary $\mathcal{I}$ if poles of the topological zeta function of $\mathcal{I}$ are always roots of its Bernstein-Sato polynomial, remembering of course that this question turned out to be very difficult already for $r = 1$. In the special case of a monomial ideal $\mathcal{I}$ this was verified in [15] by Howald, Mustaţă and Yuen (for the $p$-adic Igusa zeta function). Maybe more accessible, do the poles of the topological zeta function of an arbitrary ideal $\mathcal{I}$ induce monodromy eigenvalues in the sense of Verdier? The main result of this paper is to provide an affirmative answer to the last question for arbitrary ideals in two variables.

The plan of the paper is as follows. We work over the base field of complex numbers. In §2 we explain the construction of the specialization functor of Verdier. In arbitrary dimension we show in §3 a formula for the "Verdier monodromy eigenvalues" of an ideal $\mathcal{I}$ in terms of a principalization of the ideal, in the same spirit as A'Campo's formula [2] for the eigenvalues of one function $f$ in terms of an embedded resolution of $Y = \mathbf{V}(f)$. Note that this is a priori not obvious; for $r = 1$ the complex of nearby cycles lives on $Y$ but for $r > 1$ the specialized complex of Verdier lives not on $Y$ but on $C_Y\mathbb{A}^n$. In §4 we prove the "generalized Monodromy Conjecture" for ideals in two variables, and finally in §5 we provide some examples.

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## 2. The specialization functor of Verdier

Let $\mathcal{I}$ be a coherent ideal sheaf on a variety $X$. Consider the associated subscheme $Y = \mathbf{V}(\mathcal{I})$ of $X$. We construct the blow-up $\pi : B = Bl_Y X \to X$ of $X$ in $Y$ and denote by $E$ the inverse image $\pi^{-1}(Y)$.

For every $e \in E$ we want to study the zeta function of monodromy of the ideal $\mathcal{I}$. To define what monodromy is in this context, we will need notions as the normal cone of $Y$ in $X$ and the specialization functor of Verdier.

The normal cone of $Y$ in $X$ is a cone over $Y$ defined as

$$C_Y X = \mathrm{Spec}_{\mathcal{O}_Y}(\oplus_{n\geqslant 0}\mathcal{I}^n/\mathcal{I}^{n+1}),$$

see *e.g.* [12, B.6]. Interesting to notice is that the projectivization $P(C_Y X)$ of the normal cone is exactly the exceptional variety $E$ of the blowing-up of

$X$ in $Y$ (with its non-reduced scheme structure). We can identify the "locus of vertices" of this cone with $Y$. So we have an embedding $j : Y \to C_Y X$ and a projection $p : C_Y X \to Y$. These maps satisfy $p \circ j = \mathrm{Id}_Y$ . Moreover, we have an action of $\mathbb{C}^*$ on the normal cone, coming from the graduation on $\oplus_{n \geqslant 0} \mathcal{I}^n / \mathcal{I}^{n+1}$. The locus of vertices is the scheme of the fixed points of the action and the morphisms $j$ and $p$ commute with it. Starting with a point $e \in (C_Y X) \backslash Y$, we define the *ruler* through $e$ as the orbit of $e$ under the action of $\mathbb{C}^*$ (with its reduced scheme structure), so every ruler can be identified with $\mathbb{C}^*$.

The construction of the normal cone is functorial in the following sense. Suppose we have a map $f : X' \to X$ of schemes and two subschemes $Y \subset X$ and $Y' \subset X'$ such that

$$\begin{array}{ccc} Y' & \hookrightarrow & X' \\ \downarrow f|_{Y'} & & \downarrow f \\ Y & \hookrightarrow & X \end{array}$$

is a cartesian diagram, then we can associate a map $C(f) : C_{Y'} X' \to C_Y X$ to $f$. In particular, the map $\pi : B \to X$ has an associated map $C(\pi) : C_E B \to C_Y X$. This last one has the following interesting properties which we will use further on. If we restrict the map $C(\pi)$ to the "punctured" cones, $C'(\pi) : (C_E B) \backslash E \to (C_Y X) \backslash Y$, this is an isomorphism. Likewise, the map of projectivized cones $C''(\pi) : P(C_E B) \to P(C_Y X)$ is well-defined and is an isomorphism. Moreover, the following diagram is commutative.

$$\begin{array}{ccccc} E & \xleftarrow{\sim} & P(C_E B) & \longleftarrow & (C_E B) \backslash E \\ \big\downarrow \pi & \circ & \wr \big\downarrow C''(\pi) & \circ & \wr \big\downarrow C'(\pi) \\ Y & \longleftarrow & P(C_Y X) & \longleftarrow & (C_Y X) \backslash Y \end{array} \tag{2.1}$$

The (canonical) deformation to the normal cone of $Y$ in $X$ is a scheme $\hat{X}$ with a morphism $\hat{\pi} : \hat{X} \to \mathbb{A}^1$ such that $\hat{\pi}^{-1}(\mathbb{A}^1 \setminus \{0\}) \cong X \times (\mathbb{A}^1 \setminus \{0\})$ and the fibre over $0 \in \mathbb{A}^1$ is the normal cone $C_Y X$. This construction can be found in [12, Chapter 5]. So we obtain the following commutative diagram.

$$\begin{array}{ccccc} C_Y X & \longrightarrow & \hat{X} & \overset{j}{\hookleftarrow} & X \times (\mathbb{A}^1 \setminus \{0\}) \\ \big\downarrow & \circ & \big\downarrow \hat{\pi} & \circ & \big\downarrow \\ 0 & \longrightarrow & \mathbb{A}^1 & \overset{j_0}{\hookleftarrow} & \mathbb{A}^1 \setminus \{0\} \end{array}$$

We introduce now the specialization functor of Verdier that associates (by means of the previous diagram) to a constructible complex of sheaves $\mathcal{F}^\bullet$ on $X$ a constructible complex of sheaves on $C_Y X$.

*Note.* By a constructible complex of sheaves (of $\mathbb{C}$-vector spaces) $\mathcal{G}^\bullet$ on a scheme $Z$ we mean an object in the full subcategory $D^b_c(Z)$ of the derived category $D(Z)$; its cohomology sheaves $H^n(\mathcal{G}^\bullet)$ are thus constructible and moreover zero for $|n| >> 0$. (A sheaf on $Z$ is called constructible if there is a Zariski-locally closed stratification of $Z_{red}$ such that the restriction of the sheaf to each stratum is locally constant for the complex topology.)

In [26] the specialization of $\mathcal{F}^\bullet$ is defined as

$$Sp_{Y\setminus X}(\mathcal{F}^\bullet) := \psi_{\hat{\pi}}(j_! \, pr_1^* \, \mathcal{F}^\bullet),$$

where $\hat{\pi}$ and $j$ are as before, $pr_1$ is the projection of $X \times (\mathbb{A}^1 \setminus \{0\})$ on the first factor and $\psi_{\hat{\pi}}$ is the nearby cycle functor of Deligne [6] (see also [10, section 4.2]). In [26, p.356-357], Verdier defines a canonical transformation of monodromy on the complex of sheaves $Sp_{Y\setminus X}(\mathcal{F}^\bullet)$.

This specialization functor has a number of important properties, from which we will give the ones that we need in this article. These properties are stated in [26, sections 8 and 10].

(SP1) *Monodromy.* For every constructible complex of sheaves $\mathcal{F}^\bullet$, the complex $Sp_{Y\setminus X}(\mathcal{F}^\bullet)$ is monodromic. This means that $Sp_{Y\setminus X}(\mathcal{F}^\bullet)$ is locally constant (with respect to the usual complex topology) on every ruler of $C_Y X$.

(SP2) *Proper direct image.* Suppose we have a cartesian diagram

$$\begin{array}{ccc} Y' & \hookrightarrow & X' \\ \downarrow & & \downarrow f \\ Y & \hookrightarrow & X \end{array}$$

where $f$ is proper. Let $C(f) : C_{Y'} X' \to C_Y X$ be the morphism associated to $f$. Then $C(f)$ is proper and for each constructible complex $\mathcal{F}^\bullet$ on $X'$, the natural morphism

$$Sp_{Y\setminus X}(Rf_* \mathcal{F}^\bullet) \to RC(f)_*(Sp_{Y'\setminus X'}(\mathcal{F}^\bullet))$$

is an isomorphism.

(SP6) *Normalization.* Suppose that $Y$ is a principal divisor with equation $f = 0$. Then the morphism $p \times C(f) : C_Y X \to Y \times \mathbb{A}^1$ is an isomorphism, where $p : C_Y X \to Y$ is the projection. Notice that $C_{\{0\}}\mathbb{A}^1 = \mathbb{A}^1$, so $C(f)$ is indeed a map $C_Y X \to \mathbb{A}^1$. With this

isomorphism, we can define a section for $z \in \mathbb{C}\backslash\{0\}$ :

$$\begin{array}{rccl} s_z & : & Y & \to & C_Y X \\ & & y & \mapsto & (p \times C(f))^{-1}(y, z). \end{array}$$

Then we have for each constructible complex of sheaves $\mathcal{F}^\bullet$ on $X$ an isomorphism

$$s_z^*(Sp_{Y\backslash X}(\mathcal{F}^\bullet)) \to \psi_f(\mathcal{F}^\bullet).$$

This isomorphism is compatible with the two monodromy operations defined on both sheaves on $Y$, in the sense that they are each other's opposite. (We refer to [6] or [10, section 4.2] for a description of the 'classical' monodromy on $\psi_f(\mathcal{F}^\bullet)$.)

(SP7) *Perversity.* The specialization functor transforms perverse sheaves into perverse sheaves.

## 3. The zeta function of monodromy

As we said before, Verdier defined a canonical transformation of monodromy on the complex of sheaves $Sp_{Y\backslash X}(\mathbb{C}_X^\bullet)$, which we will denote by $M$. For each $m \in \mathbb{Z}$ and $y \in (C_Y X)\backslash Y$, we have an automorphism $M_y^m$ on the stalk $H^m(Sp_{Y\backslash X}(\mathbb{C}_X^\bullet)_y)$. *Eigenvalues of monodromy* are eigenvalues of these vector space automorphisms.

We now define the zeta function of monodromy $Z_{\mathcal{I},e}(t)$ of $\mathcal{I}$ for a point $e \in E$. Choose an arbitrary point $e'$ on $(C_Y X)\backslash Y$ that is mapped to $e$ by the projectivization. Then $Z_{\mathcal{I},e}(t)$ is the (finite) product

$$Z(Sp_{Y\backslash X}(\mathbb{C}_X^\bullet))(e') := \prod_{m\in\mathbb{Z}} \det(\mathrm{Id} - t\, M_{e'}^m)^{(-1)^m}.$$

Note that this is the usual notion of a zeta function for $M$ and $e'$, and that, by (SP1), this definition is independent of the choice of $e'$.

In the next section, we use the zeta function of monodromy to prove the monodromy conjecture in dimension two. We will prove that a pole $s_0$ of the topological zeta function induces a zero or a pole $e^{2\pi i s_0}$ of the monodromy zeta function for some point $e \in E$. This implies that this number $e^{2\pi i s_0}$ is an eigenvalue of monodromy.

*Remark 3.1.* — Note that it is also true that each eigenvalue of monodromy is a zero or a pole of the monodromy zeta function for some point $e \in E$. Since we know from (SP7) that $Sp_{Y\backslash X}(\mathbb{C}_X^\bullet)$ is a (shifted) perverse sheaf, we can copy the proof of [7, Lemma 4.6] to obtain this result.

Now we prove a generalization of the formula of A'Campo [2, Theorem 3] to the case of ideals.

THEOREM 3.2. — *Let $\mathcal{I}$ be a sheaf of ideals on a variety $X$. Let $Y = \mathbf{V}(\mathcal{I})$ be the associated subscheme of $X$ and suppose that $Sing(X) \subset Y$. Let $\pi : B = Bl_Y X \to X$ be the blow-up of $X$ in $Y$ and $\psi : \tilde{X} \to X$ a principalization of $\mathcal{I}$. Define $\varphi : \tilde{X} \to B$ as the unique morphism such that $\psi = \pi \circ \varphi$. We denote by $E$ the inverse image $\pi^{-1}(Y)$ and $\tilde{E} = \psi^{-1}(Y)$. We use $E_i$, $i \in J$, for the irreducible components of $\tilde{E}$ and $N_i$ for the according multiplicities in $\tilde{E}$. Put $E_i^\circ = E_i \setminus \cup_{j \in J,\, j \neq i} E_j$. For a point $e \in E$, the zeta function of monodromy is equal to*

$$Z_{\mathcal{I},e}(t) = \prod_{j \in J} (1 - t^{N_j})^{\chi(E_j^\circ \cap \varphi^{-1}(e))}.$$

*Remark 3.3.* — (1) Note that the principalization $\psi$ factorizes through the blow-up $\pi$ since the inverse image $\psi^{-1}(Y)$ is a Cartier subscheme in $\tilde{X}$.

(2) When $\mathcal{I}$ is principal we can consider $\pi$ as the identity and thus $\psi = \varphi$. Then $Z_{\mathcal{I},e}(t)$ is the usual hypersurface zeta function of monodromy for $e \in E = Y$ and we recover A'Campo's formula.

*Proof.* — Fix $e \in E$ and choose an element $e' \in (C_Y X) \backslash Y$ that is mapped to $e$ by the projectivization map.

The restriction of $\pi$ is an isomorphism $B \setminus E \to X \setminus Y$, which implies that $R\pi_* \mathbb{C}_B^\bullet|_{X \setminus Y} = \mathbb{C}_X^\bullet|_{X \setminus Y}$. Since $Sp_{Y \setminus X}(\mathcal{F}^\bullet)|_{(C_Y X) \setminus Y}$ only depends on $\mathcal{F}|_{X \setminus Y}$ (see [26, p. 354]), we can deduce that

$$Sp_{Y \setminus X}(\mathbb{C}_X^\bullet)|_{(C_Y X) \setminus Y} = Sp_{Y \setminus X}(R\pi_* \mathbb{C}_B^\bullet)|_{(C_Y X) \setminus Y}$$

and

$$Z(Sp_{Y \setminus X}(\mathbb{C}_X^\bullet))(e') = Z(Sp_{Y \setminus X}(R\pi_* \mathbb{C}_B^\bullet))(e').$$

Because we have the cartesian diagram

$$\begin{array}{ccc} E & \hookrightarrow & B \\ \downarrow & & \downarrow \pi \\ Y & \hookrightarrow & X \end{array}$$

we can use (SP2) to write that

$$Sp_{Y \setminus X}(R\pi_* \mathbb{C}_B^\bullet) \xrightarrow{\sim} RC(\pi)_*(Sp_{E \setminus B}(\mathbb{C}_B^\bullet))$$

is an isomorphism, or that

$$Z(Sp_{Y \setminus X}(R\pi_* \mathbb{C}_B^\bullet))(e') = Z(Sp_{E \setminus B}(\mathbb{C}_B^\bullet))(C(\pi)^{-1}(e')).$$

(Note that the restriction of $C(\pi)$ to $(C_E B)\backslash E$ is an isomorphism, as said in §2.)

There exists an open $V_e$ around $e$ such that $E \cap V_e$ is the zero locus of one nonzerodivisor $f$. So we can use (SP6) to see that locally, we have an isomorphism $p \times C(f) : C_E B \to E \times \mathbb{A}^1$. Take $z \in \mathbb{C}\backslash\{0\}$ such that $(p \times C(f))(C(\pi)^{-1}(e')) = (e, z)$. This is possible, using the commutative diagram (2.1). We can conclude that we have an isomorphism

$$s_z^*(Sp_{E\backslash B}(\mathbb{C}_B^\bullet)) \longrightarrow \psi_f(\mathbb{C}_B^\bullet).$$

This isomorphism is "compatible" with the monodromy actions, so

$$Z(Sp_{E\backslash B}(\mathbb{C}_B^\bullet))(s_z(e)) = Z(\psi_f(\mathbb{C}_B^\bullet))(e).$$

The right hand side is the monodromy zeta function of the map $f : V_e \subset B \to \mathbb{C}$ in the point $e$. Because we already have an embedded resolution of $E$ in $B$, namely the map $\varphi : \tilde{X} \to B$, we can use the formula of A'Campo [2, Theorem 3]. (This formula was originally proven for functions on a smooth variety. But one can check that for instance the proof that Dimca gives in [10, Corollary 6.1.15] is still valid in this more general context where the hypersurface contains the singular locus of the ambient variety.) We obtain

$$Z(\psi_f(\mathbb{C}_B^\bullet))(e) = \prod_{j \in J} (1 - t^{N_j})^{\chi(E_j^\circ \cap \varphi^{-1}(e))}.$$

For this last equality, we need that the multiplicities of the irreducible components $E_i$ of $\tilde{E}$ in the divisor $\varphi^{-1}(E)$ are precisely the $N_i$. Indeed:

$$\tilde{E} = \sum_{i \in J} N_i E_i = (\pi \circ \varphi)^{-1}(Y) = \varphi^{-1}(\pi^{-1}Y) = \varphi^{-1}(E).$$

Putting all these equalities of zeta functions together, proves our theorem. □

## 4. The Monodromy Conjecture

From now on, we will work in dimension 2, but we will use the same notation as before. So let $\mathcal{I} \subset \mathbb{C}[x, y]$ be an ideal satisfying $\{0\} \subset \operatorname{Supp} \mathcal{I}$. Put $Y = V(\mathcal{I})$ the subscheme of $X = \mathbb{C}^2$ defined by $\mathcal{I}$. Let $\pi : B = Bl_Y X \to X$ be the blowing-up of $X$ in $Y$ and $\psi : \tilde{X} \to X$ be the minimal principalization of $\mathcal{I}$. The map $\varphi : \tilde{X} \to B$ is defined such that $\psi = \pi \circ \varphi$. We denote $E = \pi^{-1}(Y)$ as divisor on $B$, and $\tilde{E} = \psi^{-1}(Y) = \sum_{i \in J} N_i E_i$ as divisor on $\tilde{X}$. Here the $E_i(N_i, \nu_i)$ are the irreducible components of $\tilde{E}$, together with their numerical data.

THEOREM 4.1 (Generalized Monodromy Conjecture). — *If $-\frac{\nu}{N}$ is a pole of the local topological zeta function of an ideal $\mathcal{I} \subset \mathbb{C}[x,y]$, then there exists a point $y \in E$ such that $e^{-2\pi i \frac{\nu}{N}}$ is an eigenvalue of monodromy in $y$.*

We first mention some results that will be useful in the proof.

For an ideal $\mathcal{I} = (f_1, \ldots, f_r) \subset \mathbb{C}[x,y]$ we can look at the linear system $\{\lambda_1 f_1 + \ldots + \lambda_r f_r = 0 \,|\, \lambda_i \in \mathbb{C} \text{ for } i = 1, \ldots, r\}$. A *total generic curve* of $\mathcal{I}$ is a general element of this linear system. Now we determine whether there are common components among the $f_i$ and put them together. So we can write $\mathcal{I} = (h)(f_1', \ldots, f_r')$, where $(f_1', \ldots, f_r')$ is a finitely supported ideal. A *generic curve* of the ideal $\mathcal{I}$ is a general element of the linear system $\{\lambda_1 f_1' + \ldots + \lambda_r f_r' = 0 \,|\, \lambda_i \in \mathbb{C} \text{ for } i = 1, \ldots, r\}$. Notice that the definition of the (total) generic curve depends on the choice of generators we use to represent the ideal.

LEMMA 4.2. — *If an irreducible component $E_i$ of $\tilde{E}$ is contracted by $\varphi$, then there is no intersection between $E_i$ and the strict transform of a generic curve of the ideal $\mathcal{I}$ in $\tilde{X}$.*

*Proof.* — If there is an intersection, the strict transform of a generic curve of $\mathcal{I}$ in $B$ contains the point $\varphi(E_i)$. But if we denote $\widetilde{f}_i$ for the strict transform of $f_i'$ in $B$, the linear system $\{\lambda_1 \widetilde{f}_1 + \ldots + \lambda_r \widetilde{f}_r = 0 \,|\, \lambda_i \in \mathbb{C} \text{ for } i = 1, \ldots, r\}$ should be base point free (see *e.g.* [14, Example 7.17.3]). □

LEMMA 4.3. — *Let $(f_1, \ldots, f_l)$ be an ideal in $\mathbb{C}[x,y]$ and $E_0(N,\nu)$ an exceptional curve of the principalization (together with the numerical data). Suppose $E_0$ intersects $n$ times the strict transform of a generic curve and $m$ times other components $E_1(N_1,\nu_1), \ldots, E_m(N_m,\nu_m)$ of the principalization. Put $\alpha_i = \nu_i - \frac{\nu}{N} N_i$ for $i = 1, \ldots, m$. Then we have*

(1) $\sum_{i=1}^{m} \alpha_i = m - 2 + \frac{\nu n}{N}$, *where the left hand side is zero when $m = 0$, and*

(2) $-1 \leqslant \alpha_i < 1$ *for $i \in \{1, \ldots, m\}$. Moreover, $\alpha_i = -1$ only occurs when $m = 1$.*

The first equality can be found in [25, §3]. It is a reformulation of the relation between the numerical data proved by Loeser in [19]. The second statement is [25, Proposition 3.1].

In [25], we gave a complete list of five conditions in which a candidate pole is indeed a pole of the local topological zeta function of an ideal in dimension two. Using Lemma 4.3, it is not difficult to see that the conditions given in numbers 2, 3 and 4 of [25, Theorem 4.2] are equivalent to the condition of being intersected by the strict transform of a generic curve. So we can reformulate this theorem as follows.

THEOREM 4.4. — *Let $\mathcal{I} \subset \mathbb{C}[x,y]$ be an ideal satisfying $0 \in Supp(\mathcal{I})$ and $\psi : \tilde{X} \to \mathbb{C}^2$ the minimal principalization of $\mathcal{I}$ in a neighbourhood of 0. Let $E_\bullet(N_\bullet, \nu_\bullet)$ be the components of the support of the total transform $\psi^*\mathcal{I}$ with their associated numerical data.*

*The rational number $s_0$ is a pole of the local topological zeta function of $\mathcal{I}$ if and only if at least one of the following conditions is satisfied:*

(1) *$s_0 = -\frac{1}{N}$ for a component $E_0(N,\nu)$ of the support of the weak transform of $\mathcal{I}$;*
(2) *$s_0 = -\frac{\nu}{N}$ for $E_0(N,\nu)$ an exceptional curve that has non-empty intersection with the strict transform of a generic curve of the ideal $\mathcal{I}$;*
(3) *$s_0 = -\frac{\nu}{N}$ for $E_0(N,\nu)$ an exceptional curve that intersects at least three times other components.*

Now we are ready to prove the monodromy conjecture for ideals in dimension 2. We were inspired by [22], where Rodrigues gives an elementary proof of the monodromy conjecture for curves on normal surfaces.

*Proof of Theorem 4.1.* — Choose a pole $s_0$ of the local topological zeta function of the ideal $\mathcal{I}$ and take $a, d \in \mathbb{Z}$ satisfying $s_0 = -\frac{a}{d}$ and $\gcd(a,d) = 1$.

Suppose that there exists an irreducible component $E_m(N_m, \nu_m)$ with $d \mid N_m$ that is not contracted to a point by $\varphi : \tilde{X} \to B$. Choose a point $y \in \varphi(E_m^\circ)$ such that $\varphi^{-1}(y)$ is a finite set of points. Then there exists for every $j \in J$ a nonnegative integer $k_j$ (with $k_m \neq 0$) such that

$$Z_{\mathcal{I},y}(t) = \prod_{j \in J} (1 - t^{N_j})^{k_j},$$

from which we see that $e^{2\pi i s_0}$ is a zero of the zeta function of monodromy in $y$, so it is an eigenvalue of monodromy.

On the other hand, suppose that every $E_i(N_i, \nu_i)$ that satisfies $d \mid N_i$ is contracted to a point by $\varphi$. This implies (by using Lemma 4.2 and Theorem 4.4) that there exists an index $m \in J$, with $s_0 = -\frac{\nu_m}{N_m}$, such that $E_m$ is exceptional and $E_m$ intersects at least three times other components of $\tilde{E}$. Fix such a $m$ and take $y = \varphi(E_m)$. Define $T \subset J$ such that $\varphi^{-1}\{y\} = \cup_{i \in T} E_i$. To prove that $e^{2\pi i s_0}$ is a zero or a pole of the zeta function of monodromy in $y$, it is enough to show that

$$\sum_{i \in T,\, d \mid N_i} \chi(E_i^\circ) < 0.$$

First notice that the strict transform of a generic curve intersects at least one exceptional component, so not every exceptional curve can be

contracted and it is impossible that $d$ divides $N_i$ for every $i \in J$. We define

$$\tilde{M} := \bigcup_{i \in T,\, d | N_i} E_i.$$

Every connected component of $\tilde{M}$ contains at least one irreducible component that intersects a component $E_k(N_k, \nu_k)$ outside $\tilde{M}$. If $d \mid N_k$, then $E_k$ is contracted by $\varphi$ and since $E_k$ has a non-empty intersection with $\tilde{M}$, this also implies that $k \in T$. So $d \nmid N_k$. Moreover, the irreducible component of $\tilde{M}$ that intersects $E_k$ needs to intersect a second component $E_{k'}(N_{k'}, \nu_{k'})$ with $d \nmid N_{k'}$. We can deduce this from the formula $\kappa N_l = \sum_{i=1}^{r} N_i$, where $E_l$ is an exceptional curve intersecting $r$ times other components $E_i$, for $i = 1, \ldots, r$, and where $-\kappa$ denotes the self-intersection number of $E_l$.

Every connected component $\bigcup_{i \in I_0} E_i$ of $\tilde{M}$ satisfies

$$\sum_{i \in I_0} \chi(E_i^\circ) \leqslant 0.$$

Because such a component has at least two external intersections, this is a direct consequence of the following fact: if $\bigcup_{i=1}^{r} E_i$ is a tree consisting of rational curves, then $\sum_{i=1}^{r} \chi(E_i \backslash (\cup_{j=1,\ldots,r,\, j \neq i} E_j)) = 2$. See *e.g.* [22, Lemma 2.2].

Now fix the connected component $M = \bigcup_{i \in I_M} E_i$ of $\tilde{M}$ that contains $E_m$. For this one, we prove the strict inequality

$$\sum_{i \in I_M} \chi(E_i^\circ) < 0.$$

If this sum would equal 0, then $M$ is a tree of rational curves that intersects precisely two times with components $E_j$ outside $\tilde{M}$. From a previous argument, we know that these intersections are on one curve $E_s$ with $s \in I_M$. There are two possibilities.

- $E_s = E_m$**:** The component $E_m$ will intersect at least one other curve $E_j(N_j, \nu_j)$. This curve belongs to $M$, so $d \mid N_j$ and the number $\nu_j - \frac{a}{d} N_j$ is an integer. The only possibility is 0 (see Lemma 4.3), so this implies that $-\frac{\nu_j}{N_j} = -\frac{a}{d}$. We know that $E_j$ is contracted to a point, so it is not intersected by the strict transform of a generic curve. From the relations between the numerical data (see also Lemma 4.3), we can deduce that $E_j$ intersects at least one other component of $M$ and we can repeat this process infinitely many times. This leads to a contradiction.
- $E_s \neq E_m$**:** Now we can start for each of the (at least three) components that intersect $E_m$ the same procedure of constructing a series of

curves in $M$. Only one of these can eventually stop, because there is only one of these series that has intersections outside of $M$.

This ends the proof. □

*Remark 4.5.* — Theorem 4.1 is also true for the motivic and the Hodge zeta function of $\mathcal{I}$, and for $p$-adic Igusa zeta functions associated to several polynomials in two variables, because the necessary condition of [25, Theorem 4.2] to be a pole is still valid for these zeta functions. We refer to [25, §6] for definitions and more explanation.

## 5. Some examples

(1) $\mathcal{I} = (x^3y, x^6 + y^4) \subset \mathbb{C}[x, y]$. The intersection diagram of the principalization $\tilde{X}$ is as in Figure 5.1, where the dashed line denotes the strict transform of a generic curve.

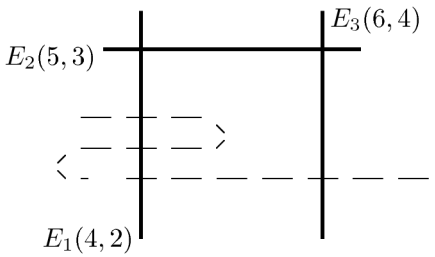


*Figure 5.1. Intersection diagram of the principalization of* $(x^3y, x^6 + y^4)$.

We see that the poles of the local topological zeta function are $-\frac{1}{2}$ and $-\frac{2}{3}$.

The blowing-up $B = Bl_Y\mathbb{C}^2$ is given by

$$\operatorname{Proj} \frac{\mathbb{C}[x, y][A, B]}{(Ax^3y - B(x^6 + y^4))},$$

see *e.g.* [11, Section IV.2.1]. The exceptional variety consists of only one irreducible curve, say $E$. The map $\varphi : \tilde{X} \to B$ contracts $E_2$ to a point $a$ on $E$. It maps $E_1$ and $E_3$ surjectively on $E$. The restriction to $E_3$ is one-to-one, but the restriction to $E_1$ is three-to-one.

For a point $e$ on $E$ different from $a$, we compute that the monodromy zeta function is

$$Z_{\mathcal{I},e}(t) = (1 - t^4)^3(1 - t^6).$$

We can immediately check that $e^{-\pi i}$ and $e^{-\frac{4\pi i}{3}}$ are zeroes of this function. Since $-\frac{1}{2}$ and $-\frac{2}{3}$ were the only poles of the local topological zeta function, this illustrates the monodromy conjecture.

(2) $\mathcal{I} = (x^4, xy^2, y^3) \subset \mathbb{C}[x, y]$. The intersection diagram of the principalization (with numerical data) is given in Figure 5.2. We can easily deduce that the poles of $Z_{\mathrm{top},\mathcal{I}}(s)$ are $-\frac{2}{3}$ and $-\frac{5}{8}$.

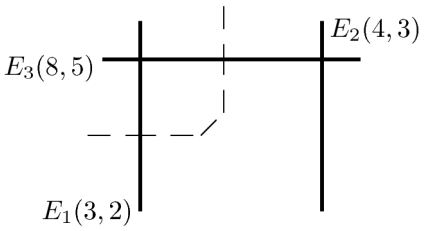


*Figure 5.2. Intersection diagram of the principalization of* $(x^4, xy^2, y^3)$.

The blowing-up $B = Bl_Y \mathbb{A}^2$ is given by

$$B = \mathrm{Proj}\, \frac{\mathbb{C}[x, y][A, B, C]}{(y^2A + x^3B, xC + yB, B^3x + AC^2)}$$

and the exceptional variety $E$ consists of two projective lines $E'$ and $E''$. The map $\varphi : \tilde{X} \to B$ maps $E_1$ and $E_3$ one-to-one onto $E'$ and $E''$, respectively. It contracts $E_2$ to a point $a$ on $E''$.

We choose a point $x \in E'^{\circ}$ and a point $y \in E''^{\circ} \backslash \{a\}$. For these points the monodromy zeta functions are

$$Z_{\mathcal{I},x}(t) = 1 - t^3, \qquad Z_{\mathcal{I},y}(t) = 1 - t^8.$$

So we conclude that the two poles of the topological zeta function induce two eigenvalues of monodromy $e^{-\frac{4\pi i}{3}}$ and $e^{-\frac{5\pi i}{4}}$.

*Remark 5.1.* — Note that this example is a monomial ideal. As mentioned in the introduction, the monodromy conjecture had already implicitly been verified in this case.

(3) $\mathcal{I} = (x^3y, x^3 - y^2) \subset \mathbb{C}[x, y]$. In Figure 5.3 we can see that $Z_{\mathrm{top},\mathcal{I}}(s)$ has poles in $-\frac{5}{6}$ and $-\frac{8}{9}$.

We know that

$$B = Bl_Y X = \mathrm{Proj}\, \frac{\mathbb{C}[x, y][A, B]}{(A(x^3 - y^2) - Bx^3y)}.$$

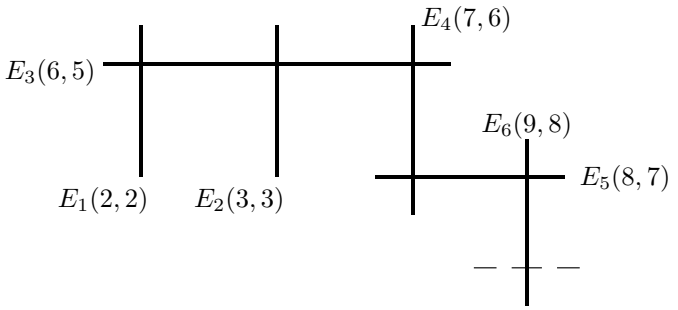


*Figure 5.3. Intersection diagram of the principalization of* $(x^3y, x^3 - y^2)$.

The blowing-up has one irreducible exceptional component $E$. The map $\varphi : \tilde{X} \to B$ maps $E_6$ surjectively on $E$ and all the other exceptional components of the principalization are contracted to a point $a$ on $E$. We fix a point $x \in E^\circ, x \neq a$, and look at the following two monodromy zeta functions:

$$Z_{\mathcal{I},a}(t) = \frac{(1-t^2)(1-t^3)}{1-t^6}, \qquad Z_{\mathcal{I},x}(t) = 1 - t^9.$$

We can see that the two poles of the topological zeta function $-\frac{5}{6}$ and $-\frac{8}{9}$ give rise to eigenvalues of monodromy $e^{-\frac{5\pi i}{3}}$ and $e^{-\frac{16\pi i}{9}}$.

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