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CODIMENSION TWO TRANSCENDENTAL SUBMANIFOLDS OF PROJECTIVE SPACE

by Wojciech KUCHARZ & Santiago R. SIMANCA

ABSTRACT. — We provide a simple characterization of codimension two submanifolds of $\mathbb{P}^n(\mathbb{R})$ that are of algebraic type, and use this criterion to provide examples of transcendental submanifolds when $n \geqslant 6$. If the codimension two submanifold is a nonsingular algebraic subset of $\mathbb{P}^n(\mathbb{R})$ whose Zariski closure in $\mathbb{P}^n(\mathbb{C})$ is a nonsingular complex algebraic set, then it must be an algebraic complete intersection in $\mathbb{P}^n(\mathbb{R})$.

RÉSUMÉ. — Nous fournissons une caractérisation simple des variétés de codimension deux de $\mathbb{P}^n(\mathbb{R})$ qui sont de type algébrique, et employons ce critère pour fournir des exemples des sous-variétés transcendantales quand $n \geqslant 6$. Si la sous-variété de codimension deux est un sous-ensemble algébrique non singulier de $\mathbb{P}^n(\mathbb{R})$ dont la fermeture de Zariski dans $\mathbb{P}^n(\mathbb{C})$ est un ensemble algébrique complexe non singulier, alors ce doit être une intersection algébrique complète dans $\mathbb{P}^n(\mathbb{R})$.

1. Introduction

Let us denote by $\mathbb{P}^n(\mathbb{R})$ and by $\mathbb{P}^n(\mathbb{C})$ the real and complex projective n-spaces, respectively. We regard the first of these as a subset of the second. A compact smooth (of class C^∞) submanifold M of $\mathbb{P}^n(\mathbb{R})$ is said to be of *algebraic type* if it is isotopic, in $\mathbb{P}^n(\mathbb{R})$, to the set of real points of a nonsingular complex algebraic subset of $\mathbb{P}^n(\mathbb{C})$ defined over $\mathbb{R}$. A submanifold M of $\mathbb{P}^n(\mathbb{R})$ that is not of algebraic type is said to be *transcendental*. Any $M \subset \mathbb{P}^n(\mathbb{R})$ such that either $\operatorname{codim} M = 1$, or $2 \dim M + 1 \leqslant n$, is of algebraic type in $\mathbb{P}^n(\mathbb{R})$. Precisely, there exists a smooth embedding $e\colon M \to \mathbb{P}^n(\mathbb{R})$, arbitrarily close in the C^∞ topology to the inclusion map $M \hookrightarrow \mathbb{P}^n(\mathbb{R})$, such that $e(M)$ is the set of real points of a nonsingular complex algebraic subset of $\mathbb{P}^n(\mathbb{C})$ defined over $\mathbb{R}$ [11, Remark 1.2] (in the

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case when $2 \dim M + 1 \leqslant n$, this assertion is highly nontrivial and relies on a projective version of the Nash-Tognoli theorem [14, 16] proven in [9], and on Hironaka's resolution of singularities [7]). In particular, any M such that $\dim M \leqslant 1$ is always of algebraic type in $\mathbb{P}^n(\mathbb{R})$.

It is known, but in no way obvious, that transcendental submanifolds exist. The first of their examples were obtained by Akbulut and King [2] (see also [1] for the preliminary results required in [2]). A substantially simpler and explicit construction is contained in [11].

In the present article, we characterize the codimension 2 compact smooth submanifolds of algebraic type in $\mathbb{P}^n(\mathbb{R})$ for $n \geqslant 6$; the cases $n = 4$ and $n = 5$ remain mysterious.

Let us recall that a codimension 2 compact smooth submanifold M of $\mathbb{P}^n(\mathbb{R})$ is said to be a *complete intersection* if it can be expressed as $M = M_1 \cap M_2$, where M_1 and M_2 are compact smooth hypersurfaces in $\mathbb{P}^n(\mathbb{R})$ that meet transversally. As usual, we denote the first Stiefel-Whitney class of a smooth manifold N by $w_1(N)$. It is a standard fact that N is orientable if, and only if, $w_1(N) = 0$.

THEOREM 1.1. — *Let M be a codimension 2 compact smooth submanifold of $\mathbb{P}^n(\mathbb{R})$. Then the following two conditions are equivalent:*

(a) *$w_1(M)$ is in the image of the restriction homomorphism*

$$H^1(\mathbb{P}^n(\mathbb{R}); \mathbb{Z}/2) \to H^1(M; \mathbb{Z}/2).$$

(b) *M is a complete intersection in $\mathbb{P}^n(\mathbb{R})$.*

These conditions imply

(c) *M is of algebraic type in $\mathbb{P}^n(\mathbb{R})$.*

If $n \geqslant 6$, all three conditions (a), (b), *and* (c) *are equivalent.*

We postpone the proof until §2. An attractive feature of Theorem 1.1 is that conditions (b) and (c) have an obvious geometric flavor, while the algebraic condition (a) is verifiable directly in many cases.

COROLLARY 1.2. — *Each codimension 2 compact orientable smooth submanifold of $\mathbb{P}^n(\mathbb{R})$ is a complete intersection, and hence, of algebraic type in $\mathbb{P}^n(\mathbb{R})$.*

Proof. — The assertion follows by Theorem 1.1 since condition (a) is satisfied. □

Remark 1.3. — It is plausible that each codimension 2 compact smooth submanifold of either $\mathbb{P}^4(\mathbb{R})$ or $\mathbb{P}^5(\mathbb{R})$ is of algebraic type. However, in these

dimensions we are able to show only that condition (c) in Theorem 1.1 is not equivalent to conditions (a) and (b):

(1) Suppose that $n = 4$. The mapping

$$\varepsilon\colon \mathbb{P}^2(\mathbb{C}) \to \mathbb{P}^4(\mathbb{C})$$

defined by

$$\varepsilon((y_1\colon y_2\colon y_3)) = \left(y_1^2 + y_2^2 + y_3^2\colon y_1y_2\colon y_1y_3\colon y_2y_3\colon y_1^2 + 2y_2^2 + 3y_3^2\right),$$

is an algebraic embedding (see [11, Lemma 3.2] if desired). Hence $\varepsilon(\mathbb{P}^2(\mathbb{C}))$ is a nonsingular complex algebraic subset of $\mathbb{P}^4(\mathbb{C})$ defined over $\mathbb{R}$. The set $M = \varepsilon(\mathbb{P}^2(\mathbb{R}))$ of real points of $\varepsilon(\mathbb{P}^2(\mathbb{C}))$ is a smooth submanifold of $\mathbb{P}^4(\mathbb{R})$ contained in the affine part $\mathbb{R}^4$ of $\mathbb{P}^4(\mathbb{R})$. Since M is nonorientable, condition (a) in Theorem 1.1 is not satisfied, and hence neither is condition (b). Obviously, M is of algebraic type in $\mathbb{P}^4(\mathbb{R})$.

(2) Suppose that $n = 5$. Let

$$\varphi\colon \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^2(\mathbb{C}) \to \mathbb{P}^5(\mathbb{C})$$

be the Segre embedding

$$\varphi\left((x_1\colon x_2), (y_1\colon y_2\colon y_3)\right) = (x_1y_1\colon x_1y_2\colon x_1y_3\colon x_2y_1\colon x_2y_2\colon x_2y_3).$$

The set $\varphi(\mathbb{P}^1(\mathbb{R}) \times \mathbb{P}^2(\mathbb{R}))$ of real points of $\varphi(\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^2(\mathbb{C}))$ is a smooth submanifold M of $\mathbb{P}^5(\mathbb{R})$. Note that condition (a) in Theorem 1.1 is not satisfied, and so neither is condition (b). Indeed, given a positive integer k, let us denote by v_k the unique generator of the cohomology group $H^1(\mathbb{P}^k(\mathbb{R}); \mathbb{Z}/2) \cong \mathbb{Z}/2$. We define the mappings $j_1\colon \mathbb{P}^1(\mathbb{R}) \to \mathbb{P}^1(\mathbb{R}) \times \mathbb{P}^2(\mathbb{R})$ and $j_2\colon \mathbb{P}^2(\mathbb{R}) \to \mathbb{P}^1(\mathbb{R}) \times \mathbb{P}^2(\mathbb{R})$ by

$$j_1\left((x_1\colon x_2)\right) = \left((x_1\colon x_2), (1\colon 0\colon 0)\right),$$
$$j_2\left((y_1\colon y_2\colon y_3)\right) = \left((1\colon 0), (y_1\colon y_2\colon y_3)\right).$$

If $\psi\colon \mathbb{P}^1(\mathbb{R}) \times \mathbb{P}^2(\mathbb{R}) \to \mathbb{P}^5(\mathbb{R})$ is the restriction of φ, then

$$j_1^*(\psi^*(v_5)) = (\psi \circ j_1)^*(v_5) = v_1,$$

and

$$j_2^*(\psi^*(v_5)) = (\psi \circ j_2)^*(v_5) = v_2,$$

and, therefore, $\psi^*(v_5) = v_1 \times 1 + 1 \times v_2$, where 1 denotes the unit element in $H^0(-; \mathbb{Z}/2)$, and $\times$ stands for the cross product in cohomology. Thus, $\psi^*(v_5)$ is not equal to $w_1(\mathbb{P}^1(\mathbb{R}) \times \mathbb{P}^2(\mathbb{R})) = 1 \times v_2$. Consequently, condition (a) in Theorem 1.1 is not satisfied. By construction, M is of algebraic type in $\mathbb{P}^5(\mathbb{R})$.

Clearly, Theorem 1.1 with $n \geqslant 6$ gives a characterization of codimension 2 transcendental submanifolds of $\mathbb{P}^n(\mathbb{R})$. The following observation, which for emphasis we state as a corollary, provides numerous examples of such submanifolds.

COROLLARY 1.4. — *Let N be a compact nonorientable smooth surface, and let $\mathbb{S}^d$ denote the unit d-sphere in $\mathbb{R}^{d+1}$. Assume that $n \geqslant 6$. Then the manifold $N \times \mathbb{S}^{n-4}$ embeds in $\mathbb{R}^n \subset \mathbb{P}^n(\mathbb{R})$ and is transcendental in $\mathbb{P}^n(\mathbb{R})$.*

Proof. — Let us identify $\mathbb{R}^n$ with a subset of $\mathbb{P}^n(\mathbb{R})$ via the mapping $(x_1, \ldots, x_n) \to (1 \colon x_1 \colon \cdots \colon x_n)$. If M is any codimension 2 compact nonorientable smooth submanifold of $\mathbb{P}^n(\mathbb{R})$ that is contained in $\mathbb{R}^n$, then condition (a) in Theorem 1.1 is not satisfied. Consequently, if $n \geqslant 6$, any such M is a transcendental submanifold of $\mathbb{P}^n(\mathbb{R})$. If N is a compact nonorientable smooth surface, then the manifold $N \times \mathbb{S}^{n-4}$ can be smoothly embedded in $\mathbb{R}^n$. This is so since N can be embedded in $\mathbb{R}^4$, and $\mathbb{S}^{n-4} \times \mathbb{R}^4$ can be identified with the total space of the normal bundle of $\mathbb{S}^{n-4}$ in $\mathbb{R}^n$, where we regard $\mathbb{R}^{n-3}$ as a subset of $\mathbb{R}^{n-3} \times \mathbb{R}^3 = \mathbb{R}^n$. □

We now examine a different aspect of the problem addressed by Theorem 1.1. A codimension 2 nonsingular algebraic subset X of $\mathbb{P}^n(\mathbb{R})$ is said to be an *algebraic complete intersection* if it can be expressed as $X = X_1 \cap X_2$, where X_1 and X_2 are nonsingular algebraic hypersurfaces in $\mathbb{P}^n(\mathbb{R})$ that meet transversally. Clearly, if X is an algebraic complete intersection in $\mathbb{P}^n(\mathbb{R})$ then, when it is regarded as a smooth submanifold of $\mathbb{P}^n(\mathbb{R})$, X is a complete intersection. It turns out that the converse is also true.

THEOREM 1.5. — *Let X be a codimension 2 nonsingular algebraic subset of $\mathbb{P}^n(\mathbb{R})$. Then the following conditions are equivalent:*

(a) *X is an algebraic complete intersection in $\mathbb{P}^n(\mathbb{R})$.*
(b) *X, regarded as a smooth submanifold of $\mathbb{P}^n(\mathbb{R})$, is a complete intersection.*
(c) *$w_1(X)$ is in the image of the restriction homomorphism*

$$H^1(\mathbb{P}^n(\mathbb{R}); \mathbb{Z}/2) \to H^1(X; \mathbb{Z}/2).$$

The proof of Theorem 1.5 is given in §2.

COROLLARY 1.6. — *Let X be a codimension 2 nonsingular algebraic subset of $\mathbb{P}^n(\mathbb{R})$. If $n \geqslant 6$ and the Zariski closure of X in $\mathbb{P}^n(\mathbb{C})$ is a nonsingular complex algebraic set, then X is an algebraic complete intersection in $\mathbb{P}^n(\mathbb{R})$.*

Proof. — Obviously X, regarded as a smooth submanifold of $\mathbb{P}^n(\mathbb{R})$, is of algebraic type. The result then follows by applying Theorems 1.1 and 1.5. □

As the reader may have noticed, Corollary 1.6 resembles Hartshorne's conjecture [6] on complete intersections in $\mathbb{P}^n(\mathbb{C})$, a conjecture that remains open to this date.

2. Submanifolds and vector bundles

Throughout this section, we assume that submanifolds are closed subsets of the ambient manifold.

Let P be a smooth manifold and let M be a smooth submanifold of P of codimension r. In general, it is hard to decide whether or not there exist a smooth real vector bundle E on P and a smooth section $s\colon P \to E$ such that $\operatorname{rank} E = r$, s is transversal to the zero section, and M is equal to the zero locus $Z(s) = \{x \in P \mid s(x) = 0\}$ of s. As it is well known, the case $r = 1$ is exceptional: E and s always exist (this is, of course, a standard fact, see [4, Remark 12.4.3] for example). For $r = 2$, we have the following result.

PROPOSITION 2.1. — *Let P be a smooth manifold and let M be a codimension 2 smooth submanifold of P. Then the following conditions are equivalent:*

(a) *$w_1(M)$ is in the image of the restriction homomorphism*

$$H^1(P;\mathbb{Z}/2) \to H^1(M;\mathbb{Z}/2).$$

(b) *There exist a rank 2 smooth real vector bundle E on P and a smooth section $s\colon P \to E$, with s transversal to the zero section of E and such that $Z(s) = M$.*

If the normal bundle of M in P is assumed to be trivial, then the vector bundle E in (b) *can be chosen to be orientable.*

Proof. — The equivalence of (a) and (b) is proved in [10, Lemma 2.3]. Suppose now that the normal bundle of M in P is trivial. By the classical Pontryagin-Thom construction, there exist a smooth mapping $f\colon P \to \mathbb{S}^2$ and a regular value $y_0 \in \mathbb{S}^2$ of f with $f^{-1}(y_0) = M$ (see [12] for details). We regard $\mathbb{S}^2$ as $\mathbb{P}^1(\mathbb{C})$, and readily find a rank 2 smooth real vector bundle F over $\mathbb{P}^1(\mathbb{C})$ and a smooth section $u\colon \mathbb{P}^1(\mathbb{C}) \to F$ such that u is transversal to the zero section, and $Z(u) = \{y_0\}$. The pullback vector bundle $E = f^*F$

on P and the pullback section $s = f^*u$ of E satisfy condition (b). By construction, the vector bundle E is orientable. □

Our next goal is the classification of rank 2 real vector bundles on $\mathbb{P}^n(\mathbb{R})$ for $n \geqslant 2$.

We denote by Γ the universal real line bundle on $\mathbb{P}^n(\mathbb{R})$, and by R the trivial real line bundle on $\mathbb{P}^n(\mathbb{R})$ with total space $\mathbb{P}^n(\mathbb{R}) \times \mathbb{R}$. We recall that each real line bundle on $\mathbb{P}^n(\mathbb{R})$ is isomorphic to either R or Γ. Below, $w_i(E)$ denotes the i^{th} Stiefel-Whitney class of the bundle E.

We denote by v_n the unique generator of the cohomology group $H^1(\mathbb{P}^n(\mathbb{R}); \mathbb{Z}/2) \cong \mathbb{Z}/2$. Thus, $v_n = w_1(\Gamma)$. The cohomology group $H^2(\mathbb{P}^n(\mathbb{R}); \mathbb{Z}/2) \cong \mathbb{Z}/2$ is generated by the cup product $v_n \cup v_n$.

PROPOSITION 2.2. — *Let E be a rank 2 real vector bundle on $\mathbb{P}^n(\mathbb{R})$.*

(1) *If E is orientable and $n \geqslant 2$, then E is isomorphic to either $R \oplus R$ or $\Gamma \oplus \Gamma$.*
(2) *If E is nonorientable and $n \geqslant 3$, then E is isomorphic to $\Gamma \oplus R$.*

Proof. — We prove the two cases separately:

(1) Since the vector bundle E is orientable, its structure group can be reduced to $\mathbb{SO}(2) \cong \mathbb{U}(1)$. Hence, there exists a complex line bundle L on $\mathbb{P}^n(\mathbb{R})$ with $L_{\mathbb{R}}$ isomorphic to E, where the notation $L_{\mathbb{R}}$ is used to indicate that L is regarded as a real vector bundle by restricting the scalars from $\mathbb{C}$ to $\mathbb{R}$. As it is well known [8], L is determined up to isomorphism by its first Chern class $c_1(L) \in H^2(\mathbb{P}^n(\mathbb{R}); \mathbb{Z}) \cong \mathbb{Z}/2$. The homomorphism $\rho\colon H^2(\mathbb{P}^n(\mathbb{R}); \mathbb{Z}) \to H^2(\mathbb{P}^n(\mathbb{R}); \mathbb{Z}/2)$ induced by the epimorphism $\mathbb{Z} \to \mathbb{Z}/2$ is an isomorphism. Since $\rho(c_1(L)) = w_2(L_{\mathbb{R}})$ [13], it follows that E is determined up to isomorphism by its second Stiefel-Whitney class $w_2(E) \in H^2(\mathbb{P}^n(\mathbb{R}); \mathbb{Z}/2)$. Clearly, $w_2(R \oplus R) = 0$ and $w_2(\Gamma \oplus \Gamma) = v_n \cup v_n$. This completes the proof of this case.

(2) Let M be a closed manifold. Then the groups of isomorphism classes of real line bundles over M and $H^1(M; \mathbb{Z}/2)$ are isomorphic. Thus, each real line bundle on $\mathbb{P}^n(\mathbb{R})$ is isomorphic to one of either R or Γ, and in order to prove the stated result, it suffices to show that the nontrivial vector bundle E has a nowhere zero section. Equivalently, we have to prove that the $\mathbb{S}^1$-bundle $\mathbb{S}(E)$ determined by E has a section.

Obstruction theory can be applied to decide whether or not $\mathbb{S}(E)$ has a section. Accordingly, we are required to use cohomology of $\mathbb{P}^n(\mathbb{R})$ with coefficients in the local system $\{\pi_k(S(E)_x)\}$ consisting of the kth homotopy group $\pi_k(S(E)_x)$ for each fiber $\mathbb{S}(E)_x$ of $\mathbb{S}(E)$ over x, as x varies in $\mathbb{P}^n(\mathbb{R})$. Since $\mathbb{S}(E)_x \cong \mathbb{S}^1$, we have that $\pi_1(\mathbb{S}(E)_x) \cong \mathbb{Z}$ and $\pi_k(\mathbb{S}(E)_x) = 0$ for

$k \geqslant 2$. Hence, the only nonzero local system is $\{\pi_1(S(E)_x)\}$, which is the local system $\mathbb{Z}_{v_n}$ of integer coefficients $\mathbb{Z}$ twisted by $v_n = w_1(E)$. We consider the usual CW decomposition of $\mathbb{P}^n(\mathbb{R})$ with i-skeleton $\mathbb{P}^i(\mathbb{R}) \subset \mathbb{P}^n(\mathbb{R})$ for $0 \leqslant i \leqslant n$. By a basic result in obstruction theory [15, Theorem 34.2], it follows that if the $\mathbb{S}^1$-bundle $\mathbb{S}(E)$ has a section over the 2-skeleton of $\mathbb{P}^n(\mathbb{R})$, then it has a section over $\mathbb{P}^n(\mathbb{R})$, which is what we need. By [15, Theorem 35.4], there is only one obstruction to the existence of a section of $\mathbb{S}(E)$ over the 2-skeleton, and this obstruction is given by a cohomology class in $H^2(\mathbb{P}^n(\mathbb{R}); \mathbb{Z}_{v_n})$. We assert that $H^2(\mathbb{P}^n(\mathbb{R}); \mathbb{Z}_{v_n}) = 0$, so the obstruction vanishes, the section exists over the 2-skeleton, and the desired result follows.

The group $H^2(\mathbb{P}^n(\mathbb{R}); \mathbb{Z}_{v_n})$ depends on the 3-skeleton of $\mathbb{P}^n(\mathbb{R})$ only. Hence, by naturality, the groups $H^2(\mathbb{P}^k(\mathbb{R}); \mathbb{Z}_{v_k})$, $k \geqslant 3$, are all isomorphic. Since $v_4 = w_1(\mathbb{P}^4(\mathbb{R}))$, by Poincaré duality (see for example [5, Theorem 5.7]), we have that $H^2(\mathbb{P}^4(\mathbb{R}); \mathbb{Z}_{v_4}) \cong H_2(\mathbb{P}^4(\mathbb{R}); \mathbb{Z})$, and the latter group is trivial. The assertion follows. □

Proof of Theorem 1.1. — We first show that (a) is equivalent to (b). Suppose that (a) holds. By Proposition 2.1, there exist a rank 2 smooth real vector bundle E on $\mathbb{P}^n(\mathbb{R})$ and a section $s\colon \mathbb{P}^n(\mathbb{R}) \to E$ that is transversal to the zero section of E and is such that $Z(s) = M$. If $n = 2$, we may assume that E is orientable. Notice that in such a case, the manifold M is a finite set.

By Proposition 2.2, we may assume that $E = E_1 \oplus E_2$ is the direct sum of two smooth line bundles E_1 and E_2, and hence, $s = s_1 \oplus s_2$, where $s_i\colon \mathbb{P}^n(\mathbb{R}) \to E_i$ is a smooth section of E_i, $i = 1, 2$. Notice that s_i is transversal to the zero section of E_i at each point of M. Consequently, there exists a smooth section $u_i\colon \mathbb{P}^n(\mathbb{R}) \to E_i$, arbitrarily close in the C^∞ topology to s_i, such that $u_i = s_i$ on M and u_i is transversal to the zero section of E_i. We get $M = Z(u_1) \cap Z(u_2)$, and the smooth hypersurfaces $Z(u_1)$ and $Z(u_2)$ in $\mathbb{P}^n(\mathbb{R})$ intersect transversally. Therefore, (b) is satisfied.

Conversely, suppose that (b) holds. Then M can be expressed as $M = M_1 \cap M_2$, where M_1 and M_2 are compact smooth hypersurfaces in $\mathbb{P}^n(\mathbb{R})$ that intersect transversally. Let L_i be a smooth line bundle on $\mathbb{P}^n(\mathbb{R})$, and let $\sigma_i\colon \mathbb{P}^n(\mathbb{R}) \to L_i$ be a smooth section of L_i such that σ_i is transversal to the zero section, and $Z(\sigma_i) = M_i$ for $i = 1, 2$. The section $\sigma_1 \oplus \sigma_2\colon \mathbb{P}^n(\mathbb{R}) \to L_1 \oplus L_2$ is transversal to the zero section and $Z(\sigma_1 \oplus \sigma_2) = M$. By Proposition 2.1, (a) is satisfied. The proof of the equivalence of conditions (a) and (b) is complete.

It follows by [3, Theorem 7.1] that (b) implies (c).

Now suppose that (c) holds and that $n \geqslant 6$. It remains to prove that (a) is satisfied. This can be done as follows. There exists a nonsingular complex algebraic subset V of $\mathbb{P}^n(\mathbb{C})$, defined over $\mathbb{R}$, such that M is isotopic in $\mathbb{P}^n(\mathbb{R})$ to the set $V(\mathbb{R})$ of real points of V. Clearly, (a) holds if, and only if, $w_1(V(\mathbb{R}))$ is in the image of the restriction homomorphism $r\colon H^1(\mathbb{P}^n(\mathbb{R});\mathbb{Z}/2) \to H^1(V(\mathbb{R});\mathbb{Z}/2)$. Since V is defined over $\mathbb{R}$, the canonical line bundle κ on V is also defined over $\mathbb{R}$. Let us denote by $\kappa(\mathbb{R})$ the real line bundle on $V(\mathbb{R})$ determined by κ. According to [11, Lemma 2.1], $w_1(\kappa(\mathbb{R}))$ is in the image of r (the assumption $n \geqslant 6$ is used here). The proof is complete since $w_1(\kappa(\mathbb{R})) = w_1(V(\mathbb{R}))$. □

In order to treat algebraic complete intersections in $\mathbb{P}^n(\mathbb{R})$ we make use of algebraic vector bundles. Following the terminology used in [4], by an algebraic vector bundle on an algebraic subset Y of $\mathbb{R}^p$ we mean an algebraic subbundle of the trivial bundle on Y with total space $Y \times \mathbb{R}^q$ for some q.

LEMMA 2.3. — *Let Y be a compact nonsingular algebraic subset of $\mathbb{R}^p$ and let X be a nonsingular algebraic subset of Y. Let E be an algebraic vector bundle on Y and let $s\colon Y \to E$ be a smooth section with $X \subset Z(s)$. Then there exists an algebraic section $u\colon Y \to E$, arbitrarily close in the C^∞ topology to s, such that $X \subset Z(u)$.*

Proof. — By [4, Theorem 12.1.7], there exists an algebraic vector bundle F on Y such that the direct sum $E \oplus F$ is an algebraically trivial vector bundle. If $0\colon Y \to F$ is the zero section, then the smooth section $\sigma = s \oplus 0\colon Y \to E \oplus F$ satisfies $X \subset Z(\sigma)$. The section σ corresponds to a smooth mapping $f\colon Y \to \mathbb{R}^q$ with $X \subset f^{-1}(0)$, where q is the rank of $E \oplus F$. By the Weierstrass relative approximation theorem [4, Lemma 12.5.5], there exists a regular mapping $\overline{f}\colon Y \to \mathbb{R}^q$ that is arbitrarily close in the C^∞ topology to f and satisfies $X \subset \overline{f}^{-1}(0)$. The algebraic section $\overline{\sigma}\colon Y \to E \oplus F$ determined by $\overline{f}$ is close in the C^∞ topology to σ, and $X \subset Z(\overline{\sigma})$. Hence the algebraic section $u\colon Y \to E$ that is given by the composition of $\overline{\sigma}$ and the canonical projection $E \oplus F \to E$ has the required properties. □

It is well known that $\mathbb{P}^n(\mathbb{R})$ can be regarded as an algebraic subset of $\mathbb{R}^p$ for some p [4, Theorem 3.4.4], and hence Lemma 2.3 is applicable to sections of algebraic vector bundles on $\mathbb{P}^n(\mathbb{R})$.

Proof of Theorem 1.5. — By Theorem 1.1, it suffices to prove that (b) implies (a). Suppose that (b) holds and that $X = M_1 \cap M_2$, where M_1 and M_2 are compact smooth hypersurfaces in $\mathbb{P}^n(\mathbb{R})$ that intersect transversally. Let L_i be a smooth real line bundle on $\mathbb{P}^n(\mathbb{R})$ and let $s_i\colon \mathbb{P}^n(\mathbb{R}) \to L_i$

be a smooth section such that s_i is transversal to the zero section and $M_i = Z(s_i)$ for $i = 1, 2$. We know that L_i is isomorphic to either R or Γ, and hence we may assume that L_i is an algebraic line bundle on $\mathbb{P}^n(\mathbb{R})$, $i = 1, 2$. By Lemma 2.3, there exists an algebraic section $u_i \colon \mathbb{P}^n(\mathbb{R}) \to L_i$, arbitrarily close in the C^∞ topology to s_i, with $X \subset Z(u_i)$ for $i = 1, 2$. Since s_i is transversal to the zero section, then so is u_i. Hence $Z(u_1)$ and $Z(u_2)$ are nonsingular algebraic hypersurfaces in $\mathbb{P}^n(\mathbb{R})$ that intersect transversally, and $X = Z(u_1) \cap Z(u_2)$. This means that condition (a) is satisfied. □

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