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EFFECTIVE LOCAL FINITE GENERATION OF MULTIPLIER IDEAL SHEAVES

by Dan POPOVICI

Abstract. — Let φ be a psh function on a bounded pseudoconvex open set $\Omega \subset \mathbb{C}^n$, and let $\mathcal{I}(m\varphi)$ be the associated multiplier ideal sheaves, $m \in \mathbb{N}^\star$. Motivated by global geometric issues, we establish an effective version of the coherence property of $\mathcal{I}(m\varphi)$ as $m \to +\infty$. Namely, given any $B \Subset \Omega$, we estimate the asymptotic growth rate in m of the number of generators of $\mathcal{I}(m\varphi)_{|B}$ over $\mathcal{O}_\Omega$, as well as the growth of the coefficients of sections in $\Gamma(B, \mathcal{I}(m\varphi))$ with respect to finitely many generators globally defined on Ω. Our approach relies on proving asymptotic integral estimates for Bergman kernels associated with singular weights. These estimates extend to the singular case previous estimates obtained by Lindholm and Berndtsson for Bergman kernels with smooth weights and are of independent interest. In the final section, we estimate asymptotically the additivity defect of multiplier ideal sheaves. As $m \to +\infty$, the decay rate of $\mathcal{I}(m\varphi)$ is proved to be almost linear if the singularities of φ are analytic.

Résumé. — Soit φ une fonction psh sur un ouvert pseudo-convexe borné $\Omega \subset \mathbb{C}^n$ et soit $\mathcal{I}(m\varphi)$ les faisceaux d'idéaux multiplicateurs associés, $m \in \mathbb{N}^\star$. Motivé par des considérations de géométrie globale, nous donnons une version effective de la propriété de cohérence de $\mathcal{I}(m\varphi)$ lorsque $m \to +\infty$. Étant donné $B \Subset \Omega$, nous estimons la croissance asymptotique en m du nombre de générateurs du $\mathcal{O}_\Omega$-module $\mathcal{I}(m\varphi)_{|B}$, ainsi que la croissance des coefficients des sections de $\Gamma(B, \mathcal{I}(m\varphi))$ par rapport à un nombre fini de générateurs globalement définis sur Ω. Notre approche consiste à démontrer des estimations intégrales asymptotiques pour des noyaux de Bergman associés à des poids singuliers. Ces estimations généralisent au cas singulier des estimations obtenues antérieurement par Lindholm et Berndtsson pour des noyaux de Bergman à poids lisses et présentent un intérêt propre. Nous donnons également des estimations asymptotiques pour le défaut d'additivité des faisceaux d'idéaux multiplicateurs. Nous montrons que lorsque $m \to +\infty$ le taux de décroissance de $\mathcal{I}(m\varphi)$ est presque linéaire si les singularités de φ sont analytiques.

Keywords: Bergman kernel, closed positive current, L^2 estimates, multiplier ideal sheaf, psh function, singular Hermitian metric, Stein manifold.
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1. Introduction

Let L be a holomorphic line bundle over a complex manifold X of complex dimension n. Suppose L is equipped with a possibly singular Hermitian metric h. This means that on each open subset of X on which L is trivial, the metric is defined as $h = e^{-\varphi}$ for some local weight function φ which is only assumed to be L^1_{loc} with respect to the Lebesgue measure. The induced metrics h^m on the tensor power line bundles L^m are then defined by the local weight functions $m\varphi$, $m \in \mathbb{N}^\star$.

In all that follows, the curvature current $i\Theta_h(L)$ of bidegree $(1,1)$ associated with (L, h) is assumed to be $\geqslant 0$ on X. Equivalently, the local weight functions φ are assumed to be plurisubharmonic (psh). Following Nadel ([14]), one can associate with h a multiplier ideal sheaf $\mathcal{I}(h) \subset \mathcal{O}_X$ defined as follows. If $h = e^{-\varphi}$ on a trivialising open set $\Omega \subset X$ for L, we set

$$\mathcal{I}(h)_{|\Omega} = \mathcal{I}(\varphi),$$

where the stalk $\mathcal{I}(\varphi)_x$ at any point $x \in \Omega$ is defined as the set of germs of holomorphic functions $f \in \mathcal{O}_{\Omega,\,x}$ such that $|f|^2\, e^{-2\varphi}$ is integrable with respect to the Lebesgue measure in some local coordinates in a neighbourhood of x. The prime objective of the present paper is to study the variation of $\mathcal{I}(h^m)$ (or equivalently of $\mathcal{I}(m\varphi)$ on each trivialising open set Ω) as $m \to +\infty$.

The sequence of multiplier ideal sheaves $\mathcal{I}(h^m)$ is easily seen to be non-increasing as $m \to +\infty$. Indeed, to offset the possible non-integrability of $e^{-2m\varphi}$ near points $x \in \Omega$ where $\varphi(x) = -\infty$ (singularities of the metric h), holomorphic germs $f \in \mathcal{I}(m\varphi)_x$ may need to vanish to increasingly high orders at x as m increases. On the other hand, the Demailly-Ein-Lazarsfeld subadditivity property of multiplier ideal sheaves ([8]) implies that

$$\mathcal{I}(h^m) \subset \mathcal{I}(h)^m, \quad (resp.\, \mathcal{I}(m\varphi) \subset \mathcal{I}(\varphi)^m), \quad m \in \mathbb{N}^\star,$$

while the inclusion is strict in general. In other words, $\mathcal{I}(h^m)$ may decrease more quickly than linearly as $m \to +\infty$. The main thrust of the ensuing development is to obtain an effective control of the decay rate of $\mathcal{I}(h^m)$ as $m \to +\infty$.

As the problems being dealt with are local in nature, we shall focus our attention on psh functions φ defined on a bounded pseudoconvex open set $\Omega \Subset \mathbb{C}^n$. Examples of such functions are provided by the so-called psh functions with **analytic singularities**, namely those psh functions that can be locally written as

$$\varphi = \frac{c}{2}\log(|g_1|^2 + \cdots + |g_N|^2) + C^\infty, \tag{$\star$}$$

for some holomorphic functions $g_1, \dots, g_N$ on Ω, and some constant $c > 0$. The $-\infty$-poles (or singularities) of φ are precisely the common zeroes of $g_1, \dots, g_N$ and they only depend, up to equivalence of singularities, on the ideal sheaf generated by $g_1, \dots, g_N$. More generally, Ω may be any Stein manifold of any complex dimension n. According to whether Ω is a bounded pseudoconvex open subset of $\mathbb{C}^n$ or an arbitrary Stein manifold, ω will denote the standard Kähler metric of $\mathbb{C}^n$ or a fixed Kähler metric on Ω.

The issue of the decay rate of $\mathcal{I}(m\varphi)$ is addressed in two ways.

First, let $\mathcal{H}_\Omega(m\varphi)$ be the Hilbert space of holomorphic functions f on Ω such that $|f|^2 e^{-2m\varphi}$ is integrable with respect to the Lebesgue measure on Ω. It is well-known that the ideal sheaf $\mathcal{I}(m\varphi)$ is coherent and generated as an $\mathcal{O}_\Omega$-module by an arbitrary orthonormal basis $(\sigma_{m,j})_{j\in\mathbb{N}^\star}$ of $\mathcal{H}_\Omega(m\varphi)$ ([14], see also [[7], Lemma 4.4]). By the strong Noetherian property of coherent sheaves, it is then generated, on every relatively compact open subset $\Omega' \Subset \Omega$, by only finitely many $\sigma_{m,j}$'s. Our first goal is to make this local finite generation property effective; in other words, to estimate the number N_m of generators needed, as well as the growth rate of the (holomorphic function) coefficients appearing in the decomposition of an arbitrary section of $\mathcal{I}(m\varphi)$ on Ω' as a finite linear combination of $\sigma_{m,j}$'s, as $m \to +\infty$. The first set of results can be summed up as follows.

Theorem 1.1. — *Let φ be a strictly psh function on a Stein manifold Ω such that $i\partial\bar\partial\varphi \geqslant C_0\,\omega$ for some constant $C_0 > 0$. Let $B' \Subset B \Subset \Omega$ be any pair of relatively compact pseudoconvex domains in Ω. Then there exists $m_0 = m_0(C_0) \in \mathbb{N}$ and every point $x \in B'$ has a neighbourhood $B_0 \subset B$ such that for every $m \geqslant m_0$ the following property holds. Every $g \in \mathcal{H}_B(m\varphi)$ admits, with respect to some suitable finitely many elements $\sigma_{m,1}, \dots \sigma_{m,N_m}$ in a suitable orthonormal basis $(\sigma_{m,j})_{j\in\mathbb{N}^\star}$ of $\mathcal{H}_\Omega(m\varphi)$, a decomposition:*

$$g(z) = \sum_{j=1}^{N_m} b_{m,j}(z)\,\sigma_{m,j}(z), \quad z \in B_0,$$

with holomorphic functions $b_{m,j}$ on B_0 satisfying:

$$\sup_{B_0} \sum_{j=1}^{N_m} |b_{m,j}|^2 \leqslant C \int_B |g|^2 e^{-2m\varphi} < +\infty,$$

where the constant $C > 0$ and the size of the neighbourhood B_0 depend only on $n = \dim_{\mathbb{C}}\Omega$, Ω, B and B', while m_0 depends only on C_0, n, Ω, B and B' (so they are all independent of φ and m); when $\Omega \Subset \mathbb{C}^n$, they even *depend only on* n *(and* C_0 *for* m_0*), the diameters of* Ω*,* B *and the distance* $d(B', \Omega \setminus B)$*.*

Moreover, if φ *has analytic singularities, then* $N_m \leqslant C_\varphi m^n$ *for* $m \gg 1$*, where* $C_\varphi > 0$ *is a constant depending only on* φ*,* B*, and* n*.*

This can be seen as a local counterpart to Siu's effective version of the global generation of multiplier ideal sheaves (cf. [17, Theorem 2.1]) which was one of the steps in his proof of the invariance of the plurigenera under projective deformations. Siu dealt with spaces of global sections of coherent analytic sheaves over compact complex manifolds; these are always finite dimensional. However, in the present case, the corresponding spaces $\mathcal{H}_\Omega(m\varphi) \subset \Gamma(\Omega, \mathcal{I}(m\varphi))$ of sections of $\mathcal{I}(m\varphi)$ over a Stein manifold are infinite dimensional. The thrust of our results is to reduce an infinite dimensional situation to a finite dimensional one. To this end, the main idea is to concentrate the L^2 norms of holomorphic functions on a relatively compact open subset of Ω by means of compact operators, the so-called Toeplitz concentration operators. The procedure involves appropriate choices of generators for $\mathcal{H}_\Omega(m\varphi)$ and produces estimates derived from an asymptotic study of Bergman kernels associated with singular weights $m\varphi$ as $m \to +\infty$.

Second, to complement the Demailly-Ein-Lazarsfeld subadditivity property [8], we obtain a *superadditivity* result showing that the variation of $\mathcal{I}(m\varphi)$, though not necessarily linear in m, is almost linear if the singularities of φ are analytic. Namely, the following qualitative statement holds.

Theorem 1.2. — *Let* $\varphi = \frac{c}{2}\log(|g_1|^2 + \cdots + |g_N|^2)$ *be a psh function with analytic singularities on* $\Omega \subset \mathbb{C}^n$ *(cf. (*$\star$*)). Then, for every* $0 < \delta < 1$*, we have:*

$$\mathcal{I}(m\varphi)^p \subset \mathcal{I}(mp(1-\delta)\varphi), \quad \textit{for all} \quad m \geqslant \frac{n+1}{c\,\delta} \quad \textit{and} \quad p \in \mathbb{N}^\star.$$

Recall that subadditivity means that $\mathcal{I}(mp\varphi) \subset \mathcal{I}(m\varphi)^p$ (cf. [8]). Theorem 1.2 is only the qualitative version of a much stronger effective result obtained as Theorem 4.1. Specifically, given any sections $f_1, \dots, f_p \in \Gamma(\Omega, \mathcal{I}(m\varphi))$ with $e^{-2m\varphi}$-weighted L^2-norms equal to 1, the $e^{-2mp(1-\delta)\varphi}$-weighted L^2-norm of the product $f_1 \cdots f_p$ on any $\Omega' \Subset \Omega$ is estimated and shown to be finite. Hence $f_1 \cdots f_p \in \Gamma(\Omega', \mathcal{I}(mp(1-\delta)\varphi))$. This effective result is partly motivated by the work [15] on singular Morse inequalities where precise estimates of products $f_1 \cdots f_p$ and their derivatives on fixed-size interior open subsets play a key role (cf. [15, Proposition 7.2]).

Here is an outline of our approach. In section 2, we recall the definition of Toeplitz concentration operators and observe some basic properties. We then go on to use Hörmander's L^2 estimates ([10]) and Skoda's L^2 division theorem ([18]) to prove an effective local finite generation property of $\mathcal{I}(m\varphi)$ which amounts to proving the first part of Theorem 1.1 regarding generation.

In section 3, we complete the proof of Theorem 1.1 by estimating the number N_m of generators of $\mathcal{I}(m\varphi)$ needed on an interior ball $B(x_0, r_0) \Subset \Omega$ as $m \to +\infty$. To this purpose, we extend to the case of analytic singularities the asymptotic integral estimates for Bergman kernels previously obtained for smooth weight functions by Lindholm ([13]) and Berndtsson ([2]). Namely, if $(\sigma_{m,j})_{j\in\mathbb{N}^\star}$ is any orthonormal basis of the singularly weighted Bergman space $\mathcal{H}_\Omega(m\varphi)$, the trace of the associated Bergman kernel on the diagonal of $\Omega \times \Omega$ is defined as

$$B_{m\varphi} := \sum_{j=0}^{+\infty} |\sigma_{m,j}|^2 \quad \text{on } \Omega, \tag{1.1}$$

the definition being independent of the orthonormal basis chosen. We obtain, among other things, the following result.

Theorem 1.3. — *Let φ be a psh function with analytic singularities on $\Omega \Subset \mathbb{C}^n$ such that $i\partial\bar\partial\varphi \geqslant C_0\omega$ for some constant $C_0 > 0$. Then, for any relatively compact open subset $B \Subset \Omega$, we have:*

$$\lim_{m\to+\infty} \frac{n!}{m^n} \int_B B_{m\varphi}\, e^{-2m\varphi}\, dV_n = \frac{2^n}{\pi^n} \int_B (i\partial\bar\partial\varphi)^n_{ac} < +\infty, \tag{1.2}$$

where dV_n denotes the Lebesgue measure in $\mathbb{C}^n$ and $(i\partial\bar\partial\varphi)_{ac}$ stands for the absolutely continuous part in the Lebesgue decomposition of the (complex measure) coefficients of the current $i\partial\bar\partial\varphi$ with respect to the Lebesgue measure.

This can be seen as the first step (giving the integral of the leading term) towards a possible asymptotic expansion of the Bergman kernel associated with singular weights that would parallel the Tian-Yau-Zelditch asymptotic expansion obtained for smooth weights (cf. [19], [20], [21]).

The search for asymptotic integral estimates for singularly weighted Bergman kernels leads naturally to defining an invariant for functions $\varphi \in PSH(\Omega)$ that we find convenient to call the *volume* of φ on $B \Subset \Omega$ (see Definition 3.1 below). After noticing that a psh function φ with arbitrary singularities may have infinite volume, section 3 goes on to prove the finiteness of the volume for any φ with analytic singularities. We can actually manage rather more by proving that the volume of φ on B equals the Monge-Ampère mass of $i\partial\bar\partial\varphi$ on B in the case of analytic singularities

(Theorem 1.3 above). The Monge-Ampère mass is easily seen to be finite in this case. The finiteness of the volume for φ with analytic singulatities translates to the estimate $N_m = O(m^n)$ as $m \to +\infty$.

Section 4 deals with the different but related issue of estimating the subadditivity defect of multiplier ideal sheaves. Demailly's local approximation of arbitrary psh functions by psh functions with analytic singularities ([6], Proposition 3.1) is revisited to get improved effective estimates from above on arbitrary fixed-sized interior subsets (Theorem 4.1). The key ingredient is again Skoda's L^2 division theorem. Theorem 1.2 then follows as a corollary.

We now put these results in context. In [15], some of them are applied to the global geometry of compact (not necessarily Kähler) complex manifolds X to get singular Morse inequalities which extend Demailly's holomorphic Morse inequalities (cf. [5]) to arbitrary singular metrics. This leads to a complete characterisation of the volume of a line bundle L in terms of all (possibly singular) Hermitian metrics with positive curvature current that one can define on L (see [15, Theorem 1.3]). A consequence of this is a new metric characterisation of big line bundles, and implicitly of Moishezon manifolds, that generalises previous results arisen from Siu's resolution of the Grauert-Riemenschneider conjecture. Since in the non-Kähler context of [15] no background Hermitian metric ω of X is closed, one has to work locally on coordinate patches. This strategy motivated in part some of the local problems treated in the present paper.

2. Effective local finite generation

In this section we prove the first part of Theorem 1.1 regarding generation. What is at stake is to find appropriate choices of an orthonormal basis of $\mathcal{H}_\Omega(m\varphi)$ and of finitely many local generators of $\mathcal{I}(m\varphi)$ that satisfy effective growth estimates as $m \to +\infty$.

Let (Ω, ω) be a bounded pseudoconvex open subset of $\mathbb{C}^n$ or, more generally, a Stein manifold of complex dimension n with Kähler metric ω. Fix a psh function φ on Ω. Let $B \Subset \Omega$ be any relatively compact pseudoconvex domain. Following [13] and [2] (themselves inspired by [12]), we consider the following Toeplitz concentration operators for B:

$$T_{B,m} : \mathcal{H}_\Omega(m\varphi) \to \mathcal{H}_\Omega(m\varphi), \quad T_{B,m}(f) = P_m(\chi_B\, f), \quad m \in \mathbb{N}^\star,$$

where χ_B is the characteristic function of B and $P_m : L^2(\Omega,\, e^{-2m\varphi}) \to \mathcal{H}_\Omega(m\varphi)$ is the orthogonal projection from the Hilbert space of (equivalence classes of) measurable functions f for which $|f|^2\, e^{-2m\varphi}$ is Lebesgue integrable on Ω, onto the closed subspace of holomorphic such functions. It is easy to see that

$$T_{B,\,m}(f) = \chi_B\, f - u,$$

where u is the solution of the equation $\bar\partial u = \bar\partial(\chi_B\, f)$ of minimal $e^{-2m\varphi}$-weighted L^2-norm. Alternatively, if we consider the Bergman kernel:

$$K_{m\varphi} : \Omega\times\Omega \to \mathbb{C}, \quad K_{m\varphi}(z,\,\zeta) = \sum_{j=1}^{+\infty}\sigma_{m,\,j}(z)\,\overline{\sigma_{m,\,j}(\zeta)},$$

its reproducing property shows the concentration operator to be also given by

$$T_{B,\,m}(f)(z) = \int_B K_{m\varphi}(z,\,\zeta)\, f(\zeta)\, e^{-2m\varphi(\zeta)}\, d\,V_n(\zeta), \quad z\in\Omega,$$

where $d\,V_n$ is the Lebesgue measure of Ω. Being defined by a square integrable kernel, $T_{B,\,m}$ is a compact operator. Its eigenvalues $\lambda_{m,\,1}\geqslant\lambda_{m,\,2}\geqslant\dots$ lie in the open interval $(0,\,1)$. If $f\in\mathcal{H}_\Omega(m\varphi)$ is an eigenvector of $T_{B,\,m}$ corresponding to some eigenvalue λ, we see that $\langle\langle T_{B,\,m}(f),\, f\rangle\rangle = \lambda\,||f||^2$, and implicitly:

$$\lambda = \frac{\int_B |f|^2\, e^{-2m\varphi}}{\int_\Omega |f|^2\, e^{-2m\varphi}}.$$

Therefore, $f\in\mathcal{H}_\Omega(m\varphi)$ is an eigenvector of $T_{B,\,m}$ if and only if

$$f(z) = \frac{\int_\Omega |f|^2\, e^{-2m\varphi}}{\int_B |f|^2\, e^{-2m\varphi}}\int_B K_{m\varphi}(z,\,\zeta)\, f(\zeta)\, e^{-2m\varphi(\zeta)}\, d\,V_n(\zeta), \quad z\in\Omega,$$

which amounts to having:

$$f(z) = \frac{\int_\Omega |f|^2\, e^{-2m\varphi}}{\int_B |f|^2\, e^{-2m\varphi}}\sum_{j=1}^{+\infty}\sigma_{m,\,j}(z)\int_B f(\zeta)\,\overline{\sigma_{m,\,j}(\zeta)}\, e^{-2m\varphi(\zeta)}\, d\,V_n(\zeta),\ z\in\Omega.$$

On the other hand, every $f\in\mathcal{H}_\Omega(m\varphi)$ has a Hilbert space decomposition with Fourier coefficients:

$$f(z) = \sum_{j=1}^{+\infty}\left(\int_\Omega f(\zeta)\,\overline{\sigma_{m,\,j}(\zeta)}\, e^{-2m\varphi(\zeta)}\, d\,V_n(\zeta)\right)\sigma_{m,\,j}(z),\ z\in\Omega.$$

The uniqueness of the decomposition into a linear combination of elements in an orthonormal basis implies the following simple observation.

Lemma 2.1. — *A function $f\in\mathcal{H}_\Omega(m\varphi)$ is an eigenvector of $T_{B,\,m}$ if and only if*

$$\frac{1}{\int_\Omega |f|^2\, e^{-2m\varphi}}\int_\Omega f\,\bar\sigma_{m,\,j}\, e^{-2m\varphi} = \frac{1}{\int_B |f|^2\, e^{-2m\varphi}}\int_B f\,\bar\sigma_{m,\,j}\, e^{-2m\varphi},$$

for every $j \geqslant 1$.

We shall now be studying the behaviour of the eigenvalues of $T_{B,\,m}$ as $m \to +\infty$. Let us fix an orthonormal basis $(\sigma_{m,\,j})_{j\in\mathbb{N}^\star}$ of $\mathcal{H}_\Omega(m\varphi)$ made up of eigenvectors of $T_{B,\,m}$ corresponding respectively to its eigenvalues $\lambda_{m,\,1} \geqslant \lambda_{m,\,2} \geqslant \dots$ listed non-increasingly. Let $0 < \varepsilon < 1$ be a constant whose choice will be made explicit later on. Since $T_{B,\,m}$ is a compact operator, there are (if any) at most finitely many eigenvalues $\lambda_{m,\,1} \geqslant \lambda_{m,\,2} \geqslant \dots \geqslant \lambda_{m,\,N_m} \geqslant 1-\varepsilon$. In other words,

$$\int_B |\sigma_{m,\,1}|^2\, e^{-2m\varphi} \geqslant \dots \geqslant \int_B |\sigma_{m,\,N_m}|^2\, e^{-2m\varphi} \geqslant 1-\varepsilon > \int_B |\sigma_{m,\,k}|^2\, e^{-2m\varphi},$$

for every $k \geqslant N_m + 1$.

Lemma 2.1 above shows, in particular, that the restrictions to B of the $\sigma_{m,\,j}$'s are still orthogonal to one another. If we let

$$(2.1) \quad B_{m\varphi}(z) := K_{m\varphi}(z,\, z) = \sum_{j=1}^{+\infty} |\sigma_{m,\,j}(z)|^2 = \sup_{f\in B(1)} |f(z)|^2, \quad z \in \Omega,$$

where $B(1)$ is the unit ball in $\mathcal{H}_\Omega(m\varphi)$, the traces of $T_{B,\,m}$ and $T^2_{B,\,m}$ are easily computed as:

$$\mathrm{Tr}\,(T_{B,\,m}) = \int_B B_{m\varphi}(z)\, e^{-2m\varphi(z)}\, dV_n(z), \quad \text{and}$$

$$\mathrm{Tr}\,(T^2_{B,\,m}) = \int_{B\times B} |K_{m\varphi}(z,\,\zeta)|^2\, e^{-2m\varphi(z)}\, e^{-2m\varphi(\zeta)}\, dV_n(z)\, dV_n(\zeta).$$

We now proceed to proving the first part of Theorem 1.1 by constructing finitely many generators for the ideal sheaf $\mathcal{I}(m\varphi)$ restricted to a smaller domain allowing asymptotic growth estimates.

Definition 2.2. — *Given $0 < \delta < 1$, a nonzero function $f \in \mathcal{H}_\Omega(m\varphi)$ is said to be δ-**concentrated** on B if* $\dfrac{\int_B |f|^2\, e^{-2m\varphi}}{\int_\Omega |f|^2\, e^{-2m\varphi}} \geqslant 1-\delta$.

It is clear that $\sigma_{m,\,1}, \dots, \sigma_{m,\,N_m}$ are the only elements of the family $(\sigma_{m,\,j})_{j\in\mathbb{N}^\star}$ that are ε-concentrated on B. Since the restrictions to B of the $\sigma_{m,\,j}$'s are still orthogonal to one another (cf. Lemma 2.1), any element f belonging to the subspace of $\mathcal{H}_\Omega(m\varphi)$ which is generated by $\sigma_{m,\,1}, \dots, \sigma_{m,\,N_m}$ is ε-concentrated on B. The following simple observation shows that the converse is not far from being true.

Lemma 2.3. — *Let $f \in \mathcal{H}_\Omega(m\varphi)$ be ε^2-concentrated on B. Then, there is an element g in the subspace of $\mathcal{H}_\Omega(m\varphi)$ generated by the ε-concentrated $\sigma_{m,\,1}, \dots, \sigma_{m,\,N_m}$ such that $\int_\Omega |f-g|^2\, e^{-2m\varphi} < \varepsilon \int_\Omega |f|^2\, e^{-2m\varphi}$.*

Proof. — We may assume that $\int_\Omega |f|^2 e^{-2m\varphi} = 1$. Let $f = \sum_{j=1}^{N_m} a_j\,\sigma_{m,\,j} + \sum_{k=N_m+1}^{+\infty} a_k\,\sigma_{m,\,k}$ be the decomposition of f with respect to the chosen orthonormal basis of $\mathcal{H}_\Omega(m\varphi)$, where $a_j,\, a_k \in \mathbb{C}$. Then $\sum_{j=1}^{+\infty} |a_j|^2 = 1$ and

$$\int_B |f|^2 e^{-2m\varphi} = \sum_{j=1}^{+\infty} |a_j|^2 \int_B |\sigma_{m,\,j}|^2 e^{-2m\varphi}.$$

We then get:

$$\int_{\Omega\setminus B} |f|^2 e^{-2m\varphi} = \sum_{j=1}^{+\infty} |a_j|^2 \int_{\Omega\setminus B} |\sigma_{m,\,j}|^2 e^{-2m\varphi} \leqslant \varepsilon^2.$$

Since $\int_{\Omega\setminus B} |\sigma_{m,\,k}|^2 e^{-2m\varphi} > \varepsilon$, for every $k \geqslant N_m + 1$, we get $\sum_{k=N_m+1}^{+\infty} |a_k|^2 < \varepsilon$. If we set $g := \sum_{j=1}^{N_m} a_j\,\sigma_{m,\,j}$, the lemma is proved. □

This strongly suggests where to turn for the most likely choice of finitely many local generators for $\mathcal{I}(m\varphi)$. Indeed, we will now show that, if ε is well chosen and m is large enough, the ideal sheaf $\mathcal{I}(m\varphi)$ is generated, on a relatively compact open subset, by $\sigma_{m,\,1}, \dots, \sigma_{m,\,N_m}$, with an effective control of the coefficients. We shall proceed inductively following an idea of [17, Theorem 2.1] which uses Skoda's L^2 division theorem to get an effective version of Nakayama's Lemma. The crux is the following approximation to order one of a local section of $\mathcal{I}(m\varphi)$ by a finite linear combination of $\sigma_{m,\,j}$'s. As compared to [17], the new feature now is that only finitely many, suitably chosen, elements in an orthonormal basis are used rather than the whole basis.

Lemma 2.4. — *Let φ be a psh function on Ω such that $i\partial\bar\partial\varphi \geqslant C_0\,\omega$ for some constant $C_0 > 0$. Let $B \Subset \Omega$ be any pseudoconvex domain and $x \in B' \Subset B$ any point in an arbitrary fixed relatively compact domain $B' \Subset B$. Then there exist $\varepsilon > 0$ and $m_0 = m_0(C_0) \in \mathbb{N}$ such that, for every $m \geqslant m_0$, the following property holds. Every $g \in \mathcal{H}_B(m\varphi)$ admits, with respect to the (on B) ε-concentrated $\sigma_{m,\,j}$'s and the standard holomorphic coordinates $z_1, \dots, z_n$ of $\mathbb{C}^n$ (or global holomorphic functions $f_1, \dots, f_n$ on Ω that define local coordinates about x if Ω is an arbitrary Stein manifold),* *a decomposition:*

$$g(z)=\sum_{j=1}^{N_m} c_j\,\sigma_{m,\,j}(z)+\sum_{l=1}^{n}(z_l-x_l)\,h_l(z),\quad z\in B,$$

with some $c_j\in\mathbb{C}$ *satisfying* $\sum_{j=1}^{N_m}|c_j|^2\leqslant C\int_B|g|^2\,e^{-2m\varphi}$ *and some holomorphic functions* h_l *on* B *satisfying:*

$$\sum_{l=1}^{n}\int_B|h_l|^2\,e^{-2m\varphi}\leqslant C\int_B|g|^2\,e^{-2m\varphi},$$

where the constant $C>0$ *depends only on* n*,* Ω*,* B *and* B'*. When* $B=B(x,\,r)\Subset\Omega\Subset\mathbb{C}^n$*,* C *depends only on* n*,* r *and the diameter* d *of* Ω*; when* $B'\Subset B\Subset\Omega\Subset\mathbb{C}^n$ *are arbitrary pseudoconvex domains,* C *depends only on* n*, the diameters of* Ω*,* B *and on the distance* $d(B',\,\Omega\setminus B)$*.*

Proof. — We may assume that $x=0$ and $B=B(0,\,r)\Subset\Omega\Subset\mathbb{C}^n$. The minor changes to the arguments needed when B is not a ball or Ω is an arbitrary Stein manifold will be specified at the end of the proof. Let $\theta\in C^\infty(\Omega), 0\leqslant\theta\leqslant1$, be a cut-off function such that $\mathrm{Supp}\,\theta\subset B$, $\theta\equiv1$ on some arbitrary ball $B(0,\,r')\Subset B$, and $|\bar\partial\theta|\leqslant\frac{3}{r}$. Fix $g\in\mathcal{H}_B(m\varphi)$ such that $C_g:=\int_B|g|^2\,e^{-2m\varphi}<+\infty$. We use Hörmander's L^2 estimates ([10]) to solve the equation

$$\bar\partial u=\bar\partial(\theta g)\quad\text{on }\Omega\text{ with weight }\ m\varphi(z)+(n+1)\,\log|z|,$$

where $|z|^2=|z_1|^2+\cdots+|z_n|^2$. (If Ω is arbitrary Stein, $|z|$ is replaced with $|\tilde f|$ defined similarly from $\tilde f_j\in C^\infty(\Omega)$ such that $\tilde f_j=f_j$ on B and $\mathrm{Supp}\tilde f_j\Subset\Omega$; if the f_j's are bounded on Ω, simply replace $|z|^2$ with $|f|^2=|f_1|^2+\cdots+|f_n|^2$.) We get a solution $u\in C^\infty(\Omega)$ satisfying the estimate:

$$\begin{aligned}\int_\Omega\frac{|u|^2}{|z|^{2(n+1)}}\,e^{-2m\varphi}&\leqslant\frac{1}{C_0\,m}\int_\Omega\frac{|\bar\partial\theta|^2\,|g|^2}{|z|^{2(n+1)}}\,e^{-2m\varphi}\\&\leqslant\frac{9}{r^2}\,\frac{1}{C_0\,m}\int_{B\setminus B(0,r')}\frac{|g|^2}{|z|^{2(n+1)}}\,e^{-2m\varphi}.\end{aligned}\tag{2.2}$$

Put $F_m := \theta g - u \in \mathcal{H}_\Omega(m\varphi)$ and get the decomposition $g = F_m + (g - F_m)$ on B. The above estimate for u implies:

$$\begin{aligned}\int_B \frac{|g - F_m|^2}{|z|^{2(n+1)}}\, e^{-2m\varphi} \leqslant 2 \int_{B\setminus B(0,r')} & \frac{|1-\theta|^2|g|^2}{|z|^{2(n+1)}}\, e^{-2m\varphi} \\ & + 2\int_B \frac{|u|^2}{|z|^{2(n+1)}}\, e^{-2m\varphi} \\ \leqslant C(r,\, r',\, d)\, C_g, &\end{aligned} \tag{2.3}$$

with a constant $C(r, r', d) > 0$ depending only on r, r', and the diameter d of Ω, if m is chosen so large that $\frac{1}{C_0\, m} < 1$.

We now apply Skoda's L^2 division theorem (cf. [18]) to obtain:

$$g(z) - F_m(z) = \sum_{l=1}^{n} z_l\, v_l(z), \quad z \in B, \tag{2.4}$$

for some holomorphic functions v_l on B satisfying $\sum_{l=1}^{n} \int_B \frac{|v_l|^2}{|z|^{2n}}\, e^{-2m\varphi} \leqslant 2\, C(r, r', d)\, C_g$. Since $|z|^{2n} \leqslant r^{2n}$ for $z \in B$, we get:

$$\sum_{l=1}^{n} \int_B |v_l|^2\, e^{-2m\varphi} \leqslant C_1(r,\, r',\, d)\, C_g, \tag{2.5}$$

where $C_1(r, r', d) = 2\, r^{2n}\, C(r, r', d)$. As for F_m, the obvious pointwise inequality $|F_m|^2 \leqslant 2(|\theta\, g|^2 + |u|^2)$, combined with estimate (2.2) for u, gives:

$$\int_B |F_m|^2\, e^{-2m\varphi} \leqslant C(r,\, r',\, d)\, C_g,$$

after absorbing an extra $d^{2(n+1)}$ in the constant $C(r, r', d)$. On the other hand, the factor $\frac{1}{C_0\, m}$ in estimate (2.2) for u shows that if m is chosen large enough, the L^2 norm of u on Ω is very small compared to the L^2 norm of g on B. This is where the strict psh assumption on φ comes in. Since $F_m = g - u$ on $B(0, r')$ and $F_m = -u$ on $\Omega \setminus B$, we get, for m large enough and some constant $C_1 = C_1(C_0) > 0$ independent of m:

$$1 - \frac{C_1}{m} \leqslant \frac{\int_B |F_m|^2\, e^{-2m\varphi}}{\int_\Omega |F_m|^2\, e^{-2m\varphi}} < 1.$$

In other words, F_m is $\frac{C_1}{m}$-concentrated on B. Fix some small $\varepsilon > 0$ whose choice will be specified later. If m_ε is chosen such that $\frac{C_1}{m} < \varepsilon^2$ for $m \geqslant m_\varepsilon$, then F_m is, in particular, ε^2-concentrated on B. Then, Lemma 2.3 shows that in the decomposition:

$$F_m(z) = \sum_{j=1}^{N_m} a_j\, \sigma_{m,\,j}(z) + \sum_{k=N_m+1}^{+\infty} a_k\, \sigma_{m,\,k}(z), \quad z \in \Omega,$$

of F_m with respect to the chosen orthonormal basis of $\mathcal{H}_\Omega(m\varphi)$, we have:

$$\sum_{k=N_m+1}^{+\infty} |a_k|^2 \leqslant C(r,\, r',\, d)\, \varepsilon\, C_g, \qquad \sum_{j=1}^{N_m} |a_j|^2 \leqslant C(r,\, r',\, d)\, C_g. \tag{2.6}$$

If we set $g_1(z) := \sum_{k=N_m+1}^{+\infty} a_k\, \sigma_{m,\,k}(z)$, (2.4) gives the decomposition:

$$g(z) = \sum_{j=1}^{N_m} a_j\, \sigma_{m,\,j}(z) + \sum_{l=1}^{n} z_l\, v_l(z) + g_1(z), \quad z \in B, \tag{2.7}$$

with an effective control of the a_j's and v_l's (see (2.6) and (2.5)), and such that:

$$\int_B |g_1|^2\, e^{-2m\varphi} \leqslant \int_\Omega |g_1|^2\, e^{-2m\varphi} = \sum_{k=N_m+1}^{+\infty} |a_k|^2 \leqslant C(r,\, r',\, d)\, \varepsilon\, C_g. \tag{2.8}$$

Thus, for every $g \in \mathcal{H}_B(m\varphi)$, formula (2.7) gives the required decomposition of g up to an error term g_1. Setting $g_1 = Q(g)$, we get a linear operator $Q : \mathcal{H}_B(m\varphi) \to \mathcal{H}_B(m\varphi)$ which, by the estimate (2.8), has norm satisfying

$$||Q||^2 \leqslant C(r,\, r',\, d)\, \varepsilon. \tag{2.9}$$

Now, if $\varepsilon > 0$ is chosen so small that $C(r,\, r',\, d)\, \varepsilon < 1$, then $I - Q$ is invertible. Moreover, formula (2.7) gives for $(I-Q)(g)$ the kind of decomposition that we expect for g. Bringing these together, we see that if we start off with an arbitrary $g \in \mathcal{H}_B(m\varphi)$, we get a $\tilde{g} \in \mathcal{H}_B(m\varphi)$ such that $g = (I-Q)(\tilde{g})$, by invertibility of $I - Q$. Furthermore, applying to $\tilde{g}$ the arguments that have previously been applied to g, we get a decomposition analogous to (2.7):

$$g(z) = (I-Q)(\tilde{g})(z) = \sum_{j=1}^{N_m} c_j\, \sigma_{m,j}(z) + \sum_{l=1}^{n} z_l\, h_l(z), \quad z \in B, \tag{2.10}$$

for some $c_j \in \mathbb{C}$ and some holomorphic functions h_l on B satisfying estimates analogous to (2.6) and (2.5):

$$\sum_{j=1}^{N_m} |c_j|^2 \leqslant C(r, r', d)\, C_{\tilde{g}} \ ; \quad \sum_{l=1}^{n} \int_B |h_l|^2\, e^{-2m\varphi} \leqslant C_1(r, r', d)\, C_{\tilde{g}}, \tag{2.11}$$

where $C_{\tilde{g}} = \int\limits_B |\tilde{g}|^2 e^{-2m\varphi} = ||\tilde{g}||^2$ denotes the square-norm of $\tilde{g}$ in the Hilbert space $\mathcal{H}_B(m\varphi)$. Furthermore, using (2.9), we can estimate $C_{\tilde{g}}$ in terms of C_g:

$$C_{\tilde{g}} = ||(I-Q)^{-1}g||^2 = ||\sum_{j=0}^{+\infty} Q^j(g)||^2 \leqslant \left(\sum_{j=0}^{+\infty} ||Q^j(g)||\right)^2 \leqslant C_2\, C_g, \tag{2.12}$$

where $C_2 := \left(\sum\limits_{j=0}^{+\infty} (C(r,r',d)\,\varepsilon)^{j/2}\right)^2 < +\infty$. Thus decomposition (2.10) alongside estimates (2.11) and (2.12) completes the proof of Lemma 2.4 in the case when $\Omega \Subset \mathbb{C}^n$ and B is a ball. When B is not a ball, r can be taken to be the radius of the largest ball about x that is contained in B. (This radius is of course larger than the distance $d(B', \Omega \setminus B)$). The case of a general Stein manifold Ω follows in a similar way if global functions $f_1, \dots, f_n \in \mathcal{O}(\Omega)$ defining coordinates at x replace $z_1, \dots, z_n$ as Hörmander's and Skoda's L^2 estimates apply on any Stein manifold Ω, resp. any pseudoconvex $B \Subset \Omega$. □

As in [17], we can now run an induction argument using Lemma 2.4 repeatedly to get, at every step p, an approximation to order p of the original local section g of $\mathcal{I}(m\varphi)$ by a finite linear combination of $\sigma_{m,\,j}$'s. The following is a slightly more precise rewording of the first part of Theorem 1.1.

Theorem 2.5. — *Given a psh function φ on Ω such that $i\partial\bar{\partial}\varphi \geqslant C_0\,\omega$ and having fixed a pair of pseudoconvex domains $B' \Subset B \Subset \Omega$, there exist $\varepsilon > 0$, $m_0 = m_0(C_0) \in \mathbb{N}$ and every point $x \in B'$ has a neighbourhood $B_0 \subset B$ such that for every $m \geqslant m_0$ the following property holds. Every $g \in \mathcal{H}_B(m\varphi)$ admits, with respect to the ε-concentrated $\sigma_{m,\,j}$'s in an orthonormal basis of $\mathcal{H}_\Omega(m\varphi)$ consisting of eigenvectors of $T_{B,\,m}$, a decomposition:*

$$g(z) = \sum_{j=1}^{N_m} b_{m,\,j}(z)\,\sigma_{m,\,j}(z), \quad z \in B_0,$$

with some holomorphic functions $b_{m,\,j}$ on B_0 satisfying:

$$\sup_{B_0} \sum_{j=1}^{N_m} |b_{m,\,j}|^2 \leqslant C \int_B |g|^2\, e^{-2m\varphi} < +\infty,$$

where the constant $C > 0$, the size of the neighbourhood B_0 and ε depend only on n, Ω, B and B', while m_0 depends only on C_0, n, Ω, B and B' (so C, the size of B_0, ε and m_0 are all independent of φ and m). When $\Omega \Subset \mathbb{C}^n$, all the above dependencies on Ω, B and B' reduce to dependencies only on the diameters of Ω, B and on the distance $d(B', \Omega \setminus B)$.

Proof. — As in Lemma 2.4, we may assume that $B = B(0, r) \Subset \Omega \Subset \mathbb{C}^n$. Fix $\varepsilon > 0$ as in Lemma 2.4 associated with an arbitrary fixed open ball $B(0, r') \Subset B(0, r)$. Let $g \in \mathcal{H}_B(m\varphi)$ with $C_g := \int_B |g|^2 e^{-2m\varphi} < +\infty$. Lemma 2.4 gives a decomposition $g(z) = \sum_{j=1}^{N_m} a_j \sigma_{m,j}(z) + \sum_{l_1=1}^{n} z_{l_1} h_{l_1}(z)$ for every $z \in B(0, r)$, with coefficients under control. We now apply Lemma 2.4 again to every function h_{l_1}, $l_1 = 1, \ldots, n$, and get:

$$h_{l_1}(z) = \sum_{j=1}^{N_m} a_{j,l_1} \sigma_{m,j}(z) + \sum_{l_2=1}^{n} z_{l_2} h_{l_1,l_2}(z), \quad z \in B(0, r),$$

with constant coefficients $a_{j,l_1} \in \mathbb{C}$, satisfying:

$$\sum_{l_1=1}^{n} \sum_{j=1}^{N_m} |a_{j,l_1}|^2 \leqslant C \sum_{l_1=1}^{n} \int_B |h_{l_1}|^2 e^{-2m\varphi} \leqslant C^2 C_g,$$

and holomorphic functions h_{l_1,l_2} on B, satisfying:

$$\sum_{l_1=1}^{n} \sum_{l_2=1}^{n} \int_B |h_{l_1,l_2}|^2 e^{-2m\varphi} \leqslant C \sum_{l_1=1}^{n} \int_B |h_{l_1}|^2 e^{-2m\varphi} \leqslant C^2 C_g.$$

We thus obtain, after p applications of Lemma 2.4, the decomposition:

$$g(z) = \sum_{j=1}^{N_m} \left(a_j + \sum_{\nu=1}^{p-1} \sum_{l_1,\ldots,l_\nu=1}^{n} a_{j,l_1,\ldots,l_\nu} z_{l_1} \ldots z_{l_\nu} \right) \sigma_{m,j}(z) + \sum_{l_1,\ldots,l_p=1}^{n} z_{l_1} \ldots z_{l_p} v_{l_1,\ldots,l_p}(z),$$

for every $z \in B(0, r)$, with coefficients $a_{j,l_1,\ldots,l_\nu=1} \in \mathbb{C}$ and $v_{l_1,\ldots,l_p} \in \mathcal{O}(B(0, r))$, satisfying the estimates:

$$\sum_{l_1,\ldots,l_\nu=1}^{n} \sum_{j=1}^{N_m} |a_{j,l_1,\ldots,l_\nu}|^2 \leqslant C^{\nu+1} C_g, \quad \nu = 1, \ldots, p-1,$$

$$\sum_{l_1,\ldots,l_p=1}^{n} \int_B |v_{l_1,\ldots,l_p}|^2 e^{-2m\varphi} \leqslant C^p C_g.$$

We now set $b_{m,j}(z) := a_j + \sum_{\nu=1}^{+\infty} \sum_{l_1,\ldots,l_\nu=1}^{n} a_{j,l_1,\ldots,l_\nu} z_{l_1} \ldots z_{l_\nu}$, for $j = 1, \ldots, N_m$, and will prove that the series defining $b_{m,j}$ converges to a holomorphic function on some smaller ball $B_0 := B(0,\, r_0)$ and that, with an appropriate choice of r_0 depending only on r and the diameter d of Ω, we have:

$$\sup_{B(0,\, r_0)} \sum_{j=1}^{N_m} |b_{m,\, j}|^2 \leqslant \frac{1}{(1-r/d)^2}\, C\, C_g.$$

Since $\sup\limits_{B(0,\, r_0)} |z_{l_1} \dots z_{l_\nu}|^2 \leqslant r_0^{2\nu}$, we get, for every $1 \leqslant \nu < +\infty$ and every $z \in B(0,\, r_0)$, the estimate:

$$\sum_{j=1}^{N_m} \Bigg| \sum_{l_1,\dots,\, l_\nu=1}^{n} a_{j,\, l_1,\dots,\, l_\nu}\, z_{l_1} \dots z_{l_\nu} \Bigg|^2 \leqslant r_0^{2\nu} \sum_{j=1}^{N_m} \Bigg(\sum_{l_1,\dots,\, l_\nu=1}^{n} |a_{j,\, l_1,\dots,\, l_\nu}| \Bigg)^2$$

$$\leqslant r_0^{2\nu}\, n^\nu \sum_{j=1}^{N_m} \sum_{l_1,\dots,\, l_\nu=1}^{n} |a_{j,\, l_1,\dots,\, l_\nu}|^2 \leqslant \Big(n\, C\, r_0^2 \Big)^\nu C\, C_g = \Big(\frac{r}{d} \Big)^{2\nu} C\, C_g,$$

if we choose $r_0 = \dfrac{r}{d} \dfrac{1}{\sqrt{n\, C}}$. This choice of r_0 is merely an example, the convergence of the series defining $b_{m,\, j}$ is guaranteed whenever $n\, C\, r_0^2 < 1$, so any $r_0 > 0$ such that $r_0 < 1/\sqrt{n\, C}$ would do. As $C > 0$ is the constant of Lemma 2.4, it only depends on n, r and d, hence so does r_0. Thus B_0 is independent of φ and m. To get convergence of the series defining $b_{m,\, j}$, set:

$$F_{\nu,\, j} := \sum_{l_1,\dots,\, l_\nu=1}^{n} a_{j,\, l_1,\dots,\, l_\nu}\, z_{l_1} \dots z_{l_\nu}, \quad \text{for } \nu \geqslant 1 \text{ and } j = 1, \dots,\, N_m,$$

$F_{0,\, j} = a_j, \quad$ for $j = 1, \dots,\, N_m$.

Thus $b_{m,\, j} = \sum\limits_{\nu=0}^{+\infty} F_{\nu,\, j}$, for $j = 1, \dots,\, N_m$, so we have:

$$\sum_{j=1}^{N_m} |b_{m,\, j}|^2 \leqslant \sum_{j=1}^{N_m} \Big(\sum_{\nu=0}^{+\infty} |F_{\nu,j}| \Big)^2 = \sum_{\nu_1,\nu_2} \sum_{j=1}^{N_m} |F_{\nu_1,j}|\, |F_{\nu_2,j}|,$$

and, as $\sum\limits_{j=1}^{N_m} |F_{\nu_1,j}|\, |F_{\nu_2,j}| \leqslant \Big(\sum\limits_{j=1}^{N_m} |F_{\nu_1,j}|^2 \Big)^{1/2} \Big(\sum\limits_{j=1}^{N_m} |F_{\nu_2,j}|^2 \Big)^{1/2}$, we finally get:

$$\Big(\sum_{j=1}^{N_m} |b_{m,\, j}|^2 \Big)^{\frac{1}{2}} \leqslant \sum_{\nu=0}^{+\infty} \Big(\sum_{j=1}^{N_m} |F_{\nu,\, j}|^2 \Big)^{\frac{1}{2}} \leqslant \sum_{\nu=0}^{+\infty} \Big(\frac{r}{d} \Big)^\nu \sqrt{C\, C_g} = \frac{\sqrt{C\, C_g}}{1 - r/d},$$

at every point in $B(0,\, r_0)$. If we absorb the denominator in the constant $C > 0$, the proof is complete. □

The first part of Theorem 1.1 stated in the Introduction is thus proved.

Corollary 2.6. — *Under the hypotheses of Theorem 2.5, the following estimate holds:*

$$\sum_{j=1}^{+\infty} |\sigma_{m,\,j}(z)|^2 \leqslant C \sum_{j=1}^{N_m} |\sigma_{m,\,j}(z)|^2, \quad z \in B_0,$$

where $C > 0$ is a constant independent of φ and m as in Theorem 2.5.

Proof. — As $\sum_{j=1}^{+\infty} |\sigma_{m,\,j}(z)|^2 = \sup_{g \in \bar{B}_m(1)} |g(z)|^2$, where $\bar{B}_m(1)$ is the closed unit ball in $\mathcal{H}_\Omega(m\varphi)$, and as every $g \in \bar{B}_m(1)$ has a decomposition on B_0 as in Theorem 2.5, the estimate follows from the Cauchy-Schwarz inequality. □

3. Volume of psh functions

In this section we will obtain the estimate $N_m = O(m^n)$ for the asymptotic growth rate, as $m \to +\infty$, of the number N_m of local generators of $\mathcal{I}(m\varphi)$ when φ is assumed to have analytic singularities. This will complete the proof of Theorem 1.1. The method will be to prove asymptotic estimates of independent interest for the Bergman kernel associated with singular psh weight functions φ with analytic singularities.

Fix $B \Subset \Omega \Subset \mathbb{C}^n$ pseudoconvex domains and φ a psh function on Ω. The arguments of this section also apply to the case of an arbitrary Stein manifold Ω. Recall that the generators $\sigma_{m,\,1}, \dots, \sigma_{m,\,N_m}$ of $\mathcal{I}(m\varphi)$ on the smaller domain B were chosen in the previous section as eigenvectors of $T_{B,\,m}$ corresponding to the (finitely many) eigenvalues $\lambda_{m,\,1}, \dots, \lambda_{m,\,N_m}$ that are $\geqslant 1-\varepsilon$ for a suitable $0 < \varepsilon < 1$. They form part of an orthonormal basis $(\sigma_{m,\,j})_{j\in\mathbb{N}^\star}$ of $\mathcal{H}_\Omega(m\varphi)$ and $\int_B |\sigma_{m,\,j}|^2\, e^{-2m\varphi}$ equals the corresponding eigenvalue $\lambda_{m,\,j}$ for all $j \in \mathbb{N}^\star$. The obvious relations

$$\int_B B_{m\varphi}\, e^{-2m\varphi}\, d\,V_n = \sum_{j=1}^{+\infty} \lambda_{m,\,j} \geqslant \sum_{j=1}^{N_m} \lambda_{m,\,j} \geqslant (1-\varepsilon)\, N_m \tag{3.1}$$

reduce the problem of estimating N_m to that of estimating $\int_B B_{m\varphi}\, e^{-2m\varphi}\, d\,V_n$ as $m \to +\infty$.

Bearing in mind the analogy with the L^2 volume of a line bundle defined on a compact complex manifold in terms of global sections and characterised in terms of curvature currents (see e.g. [3], [15]), we are naturally led to set the following.

Definition 3.1. — *The volume on $B \Subset \Omega$ of a psh function φ on $\Omega \Subset \mathbb{C}^n$ is defined as*

$$v_B(\varphi) := \limsup_{m\to+\infty} \frac{n!}{m^n} \int_B B_{m\varphi}\, e^{-2m\varphi}\, dV_n,$$

where dV_n is the Lebesgue measure of $\mathbb{C}^n$.

By the strong Noetherian property, the coherent sheaf $\mathcal{I}(m\varphi)$ is generated on $B \Subset \Omega$ by finitely many $\sigma_{m,j}$'s. Thus $B_m\, e^{-2m\varphi}$ is integrable on $B \Subset \Omega$ since it is dominated there by a constant multiple of a finite sum $\sum |\sigma_{m,j}|^2\, e^{-2m\varphi}$. The volume $v_B(\varphi)$ compares the growth rate of the finite quantities $\int_B B_{m\varphi}\, e^{-2m\varphi}\, dV_n$ with that of m^n as $m \to +\infty$.

While $v_B(\varphi)$ is always non-negative, the volume of a psh function with arbitrary singularities may be infinite. The following example is an apt illustration. Let φ be a psh function on Ω such that φ is C^∞ in the complement of an analytic subset $V \subset \Omega$. As the discussion of the Lindholm-Berndtsson results in the smooth setting (see Theorem 3.4 below) and their extension to the singular setting (see beginning of the proof of Theorem 3.5 below) will show, there is a pointwise convergence on $\Omega \setminus V$:

$$\frac{n!}{m^n} B_{m\varphi}\, e^{-2m\varphi} \longrightarrow \frac{2^n}{\pi^n} (i\partial\bar{\partial}\varphi)^n, \quad \text{as } m \to +\infty.$$

Hence, using Fatou's Lemma, we get

$$\liminf_{m\to+\infty} \frac{n!}{m^n} \int_{B\setminus V} B_{m\varphi}\, e^{-2m\varphi}\, dV_n \geqslant \frac{2^n}{\pi^n} \int_{B\setminus V} (i\partial\bar{\partial}\varphi)^n.$$

Now, the examples of Kiselman in [11, p.141-143] and that of Shiffman and Taylor in [16, p.451-453] produce psh functions φ on a domain $\Omega \subset \mathbb{C}^n$ which are smooth outside an analytic subset $V \subset \Omega$ and for which the Monge-Ampère mass in the right-hand side above is infinite even if B is indefinitely shrunk about the singular set V. As $B_m\, e^{-2m\varphi}$ is integrable on $B \Subset \Omega$, its integral on B equals the integral on $B \setminus V$ since V is Lebesgue negligible. The volume $v_B(\varphi)$ of such a function φ is clearly infinite.

However, this section will focus on proving that the volume $v_B(\varphi)$ of a psh function with analytic singularities is always finite.

Let us now make the very simple observation that the addition of a pluriharmonic function or of a psh function with divisorial singularities of integral coefficients does not affect the volume.

Remark 3.2. — If φ is any psh function, ψ is any pluriharmonic function, and g is any holomorphic function on a simply connected domain $\Omega \Subset \mathbb{C}^n$, then:

$$B_{m\varphi}\, e^{-2m\varphi} = B_{m(\varphi+\psi)}\, e^{-2m(\varphi+\psi)} = B_{m(\varphi+p\log|g|)}\, e^{-2m(\varphi+p\log|g|)},$$

at every point of Ω for every $m, p \in \mathbb{N}$. Implicitly,

$$v_B(\varphi) = v_B(\varphi + \psi) = v_B(\varphi + p\log|g|), \quad p \in \mathbb{N}.$$

Proof. — Any $f \in \mathcal{O}(\Omega)$ satisfies

$$1 = \int_\Omega |f|^2 \, e^{-2m(\varphi + p\log|g|)} = \int_\Omega \frac{|f|^2}{|g^{mp}|^2} \, e^{-2m\varphi}$$

if and only if $f = g^{mp}\, h$ for some $h \in \mathcal{O}(\Omega)$ which satisfies $\int_\Omega |h|^2 \, e^{-2m\varphi} = 1$. Taking supremum over all such f and h in $|f|^2 = |g^{mp}|^2 \, |h|^2$ and using (2.1), we get $B_{m(\varphi+p\log|g|)} = |g^{mp}|^2 \, B_{m\varphi} = e^{2mp\log|g|} \, B_{m\varphi}$.

As ψ is pluriharmonic, there exists a holomorphic function u on Ω such that $\psi = \operatorname{Re} u$. Hence $e^{-2m\psi} = |e^{-2m\,u}|$. The same argument as above shows that any f satisfying $\int_\Omega |f|^2 \, e^{-2m(\varphi+\psi)} = 1$ must be divisible by $e^{m\,u}$. Hence $B_{m(\varphi+\psi)} = |e^{m\,u}|^2 \, B_{m\varphi} = e^{2m\psi} \, B_{m\varphi}$. The proof is complete. □

There is a global-to-local issue in that $B_{m\varphi}$ is defined on the whole Ω while $v_B(\varphi)$ only involves integration on the smaller $B \Subset \Omega$. In passing from Bergman kernels defined on larger sets to Bergman kernels defined on smaller sets, we shall need the following comparison lemma.

Lemma 3.3. — *If $U \Subset \Omega$ is a pseudoconvex open subset, the Bergman kernels associated with the weight $m\varphi$ on Ω and respectively U:*

$$B_{m\varphi,\,\Omega} := \sum_{k=0}^{+\infty} |\sigma_{m,\,k}|^2, \quad B_{m\varphi,\,U} := \sum_{k=0}^{+\infty} |\mu_{m,\,k}|^2,$$

defined by orthonormal bases $(\sigma_{m,\,k})_{k\in\mathbb{N}}$ and $(\mu_{m,\,k})_{k\in\mathbb{N}}$ of the Hilbert spaces $\mathcal{H}_\Omega(m\varphi)$ and respectively $\mathcal{H}_U(m\varphi_{|U})$, can be compared, for every $m \in \mathbb{N}$, as:

$$B_{m\varphi,\,\Omega} \leqslant B_{m\varphi,\,U} \leqslant C_{n,\,d,\,r} \, B_{m\varphi,\,\Omega} \quad \text{on any } \; U_0 \Subset U \Subset \Omega,$$

where $C_{n,\,d,\,r} > 0$ is a constant depending only on n, the diameter d of Ω, and the distance $r > 0$ between the boundaries of U_0 and U.

Proof. — As the restriction to U defines an injection of the unit ball of $\mathcal{H}_\Omega(m\varphi)$ into the unit ball of $\mathcal{H}_B(m\varphi)$, the former inequality follows from (2.1). For the latter inequality, fix $x \in U_0$ and let $f \in \mathcal{O}(U)$ such that $\int_U |f|^2 \, e^{-2m\varphi} = 1$ be an arbitrary element in the unit sphere of $\mathcal{H}_U(m\varphi)$. We use Hörmander's L^2 estimates ([10]) to construct a holomorphic function $F \in \mathcal{H}_\Omega(m\varphi)$ such that $F(x) = f(x)$ and

$$\int\limits_\Omega |F|^2 \, e^{-2m\varphi} \leqslant 2\left(1 + C_n \, \frac{d^{2n} \, e^{2d^2}}{r^{2(n+1)}}\right) \int\limits_U |f|^2 \, e^{-2m\varphi} = C_{n,\,d,\,r}.$$

This is done by solving the equation:

$$\bar\partial u = \bar\partial(\theta\, f) \quad \text{on } \Omega$$

with a cut-off function $\theta \in C^\infty(\mathbb{C}^n)$, $\operatorname{Supp}\theta \Subset U$, $\theta \equiv 1$ on $U_0 \Subset U$, and with the strictly psh weight $m\varphi + n\log|z-x| + |z-x|^2$. There exists a C^∞ solution u satisfying the estimate:

$$\int_\Omega \frac{|u|^2}{|z-x|^{2n}}\, e^{-2m\varphi}\, e^{-2|z-x|^2} \leqslant 2\int_\Omega \frac{|\bar\partial\theta|^2\,|f|^2}{|z-x|^{2n}}\, e^{-2m\varphi}\, e^{-2|z-x|^2}.$$

Due to the non-integrability of $|z-x|^{-2n}$ near x, we have $u(x) = 0$. Thus F is obtained as:

$$F := \theta\, f - u \in \mathcal{O}(\Omega).$$

Now $F/\sqrt{C_{n,\,d,\,r}}$ belongs to the unit ball of $\mathcal{H}_\Omega(m\varphi)$, and we get:

$$|f(x)|^2 = |F(x)|^2 \leqslant C_{n,\,d,\,r}\, B_{m\varphi,\,\Omega}(x),$$

which proves the latter inequality by taking the supremum over all f in the unit sphere of $\mathcal{H}_U(m\varphi)$. □

We now begin the study of the finiteness of the volume $v_B(\varphi)$ of psh functions with analytic singularities. The outcome will be an asymptotic control of the number N_m of generators of $\mathcal{I}(m\varphi)$ on B. We take our cue from the following asymptotic estimates of the Bergman kernel associated with a smooth φ, due to N. Lindholm (cf. Theorems 10, 11, and 13 in [13]), and subsequently rewritten in a slightly different setting by B.Berndtsson (cf. Theorems 2.3, 2.4, and 3.1 in [2]). Compare also [1]. The standard Kähler form on $\mathbb{C}^n$ will be denoted throughout by ω. Clearly, $d\,V_n = \omega^n/n!$.

Theorem 3.4. — *(Lindholm, Berndtsson) Let φ be a C^∞ psh function such that $i\partial\bar\partial\varphi \geqslant C_0\,\omega$ on $\Omega \Subset \mathbb{C}^n$ for some constant $C_0 > 0$. Then, for any $B \Subset \Omega$:*

(a) $v_B(\varphi) = \lim\limits_{m\to+\infty} \dfrac{n!}{m^n} \displaystyle\int_B B_{m\varphi}(z)\, e^{-2m\varphi(z)}\, d\,V_n(z) = \dfrac{2^n}{\pi^n} \displaystyle\int_B (i\partial\bar\partial\varphi)^n$;

(b) *the sequence of measures on $\Omega\times\Omega$ defined as*

$$\frac{n!}{m^n}\,|K_{m\varphi}(z,\,\zeta)|^2\, e^{-2m\varphi(z)}\, e^{-2m\varphi(\zeta)}\, d\,V_n(z)\, d\,V_n(\zeta)$$

converges to $\dfrac{2^n}{\pi^n}\,(i\partial\bar\partial\varphi)^n \wedge [\Delta]$ *in the weak topology of measures, as $m\to+\infty$, where $[\Delta]$ is the current of integration on the diagonal of $\Omega\times\Omega$;*

(c) *finally,* $\lim\limits_{m\to+\infty} \dfrac{n!}{m^n} \displaystyle\sum_{j=1}^{+\infty} \lambda_{m,\,j} = \lim\limits_{m\to+\infty} \dfrac{n!}{m^n} \displaystyle\sum_{j=1}^{+\infty} \lambda_{m,\,j}^2 = \dfrac{2^n}{\pi^n} \displaystyle\int_B (i\partial\bar\partial\varphi)^n,$

and since $\lambda_{m,1} \geqslant \cdots \geqslant \lambda_{m,N_m} \geqslant 1-\varepsilon > \lambda_{m,k}$*, for* $k \geqslant N_m + 1$*, we get:*

$$\lim_{m\to+\infty} \frac{n!}{m^n}\, N_m = \frac{2^n}{\pi^n}\int_B (i\partial\bar\partial\varphi)^n \quad \textit{and} \quad \lim_{m\to+\infty} \frac{n!}{m^n} \sum_{k=N_m+1}^{+\infty} \lambda_{m,k} = 0.$$

We now single out the major steps in the proof given to this theorem in [2, p. 4-6], as they will be subsequently adapted to a more general context. The proof of (a) in [2] hinges on two facts. First, if $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of $i\partial\bar\partial\varphi$ with respect to ω, the following pointwise convergence is established:

$$\lim_{m\to+\infty} \frac{n!}{m^n}\, B_{m\varphi}(z)\, e^{-2m\varphi(z)} = \frac{2^n}{\pi^n}\, \lambda_1(z)\ldots\lambda_n(z), \quad z \in \Omega. \tag{3.2}$$

The mean value inequality gives the upper estimate of the left-hand side by the right-hand side. The converse estimate is obtained by means of Hörmander's L^2 estimates. These are local procedures carried out on small balls about x, and the desired estimates are obtained in the limit while shrinking the balls to x.

Second, the mean value inequality further gives the following uniform estimate on a relatively compact open subset:

$$0 < \frac{n!}{m^n}\, B_{m\varphi}(z)\, e^{-2m\varphi(z)} \leqslant C\, \frac{2^n}{\pi^n}\, \lambda_1(z)\ldots\lambda_n(z), \quad z \in B, \tag{3.3}$$

if m is large enough, for a constant $C > 0$ independent of m. Uniformity can be achieved as φ is C^∞ and B is relatively compact. One can then conclude by dominated convergence.

We will now prove that Theorem 3.4 is still essentially valid if we allow analytic singularities for φ (see definition $(\star)$ in the Introduction), provided the current $i\partial\bar\partial\varphi$ is replaced throughout by its absolutely continuous part $(i\partial\bar\partial\varphi)_{ac}$ in the Lebesgue decomposition of its measure coefficients (into an absolutely continuous part and a singular part w.r.t. Lebesgue measure). We clearly have $(i\partial\bar\partial\varphi)_{ac} = i\partial\bar\partial\varphi$ if φ is C^∞. Hence $(i\partial\bar\partial\varphi)^n_{ac} = \lambda_1 \ldots \lambda_n\, \omega^n$ in the complement of the singular set of φ.

We will split the analysis of the analytic singularity case into two steps according to whether the coefficient of these singularities is an integer or not.

(a) Analytic singularities with an integral coefficient

In this case, we have a complete analogue of Theorem 3.4.

Theorem 3.5. — *Let* $\varphi = \frac{p}{2}\log(|g_1|^2 + \cdots + |g_N|^2) + u$ *for some holomorphic functions* $g_1, \ldots, g_N$*, some* $p \in \mathbb{N}$*, and some* C^∞ *function* u *on*

$\Omega \Subset \mathbb{C}^n$. Assume $i\partial\bar\partial\varphi \geqslant C_0\,\omega$ for some constant $C_0 > 0$. Then, for $B \Subset \Omega$, we have:

(a) $v_B(\varphi) = \lim\limits_{m\to+\infty} \dfrac{n!}{m^n} \displaystyle\int\limits_B B_{m\varphi}\, e^{-2m\varphi}\, dV_n = \dfrac{2^n}{\pi^n} \int\limits_B (i\partial\bar\partial\varphi)^n_{ac} < +\infty.$

(b) *the analogue of* (*b*) *in Theorem 3.4 holds, with convergence to*

$$\frac{2^n}{\pi^n}\,(i\partial\bar\partial\varphi)^n_{ac} \wedge [\Delta]$$

in the weak topology of measures;

(c) *the analogue of* (*c*) *in Theorem 3.4 holds, with*

$$\lim_{m\to+\infty} \frac{n!}{m^n}\, N_m = \frac{2^n}{\pi^n} \int_B (i\partial\bar\partial\varphi)^n_{ac} < +\infty.$$

A key standard observation in this new setting (see, e.g. [3]) is that $(i\partial\bar\partial\varphi)^n_{ac}$ is of locally finite mass, and thus the integrals involving $(i\partial\bar\partial\varphi)^n_{ac}$ above are finite. Indeed, we can resolve the analytic singularities of φ by blowing up the ideal sheaf $\mathcal{I}$ generated as an $\mathcal{O}_\Omega$-module by $g_1, \ldots, g_N$. According to Hironaka, there exists a proper modification $\mu : \tilde\Omega \to \Omega$, arising as a locally finite sequence of smooth-centred blow-ups, such that $\mu^\star\mathcal{I} = \mathcal{O}(-E)$ for an effective divisor E on $\tilde\Omega$. We thus get the following Zariski decomposition (the same as Siu's decomposition in this case) of the pull-back current:

$$\mu^\star(i\partial\bar\partial\varphi) = \alpha + c\,[E], \quad on\tilde\Omega,$$

with a C^∞ closed (1, 1)-form $\alpha \geqslant 0$, where $[E]$ stands for the E-supported current of integration on E. If

$$V := \{g_1 = \cdots = g_N = 0\}$$

is the singular set of φ, we clearly have $\int_B (i\partial\bar\partial\varphi)^n_{ac} = \int_{\mu^{-1}(B)} \alpha^n$, and this quantity is finite since the smooth volume form α^n has locally finite mass. If $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of $i\partial\bar\partial\varphi$ with respect to ω on $\Omega \setminus V$, this means that the product $\lambda_1 \ldots \lambda_n$ is integrable on $B \setminus V$.

Proof of Theorem 3.5. — The overall idea is to run Berndtsson's proof of the smooth case (cf. [2], p. 4-6) on $\Omega \setminus V$ where φ is C^∞. The pointwise convergence (3.2) still holds at points $x \in \Omega \setminus V$. Indeed, the mean value inequality still applies on small balls about x that are contained in $\Omega \setminus V$, while Hörmander's L^2 estimates apply even to singular weights φ on Ω under the strict psh assumption that has been made.

The main difficulty stems from the uniform estimate (3.3) not being immediately clear near the singular set V. We claim, however, that the analogous uniform upper estimate does hold outside the singular set, namely:

$$0 < \frac{n!}{m^n}\, B_{m\varphi}(z)\, e^{-2m\varphi(z)} \leqslant C\, \frac{2^n}{\pi^n}\, \lambda_1(z)\dots\lambda_n(z), \quad z \in B\setminus V, \tag{3.4}$$

if m is large enough, for a constant $C>0$ independent of m.

Once this is proved, as $\lambda_1(z)\dots\lambda_n(z)$ is integrable on $B\setminus V$ by the key standard observation above, (a) follows by dominated convergence as in the smooth case.

The proof of (b) in [2] (p. 6-7) can be repeated on $\Omega\setminus V$ and extended across V in a similar way. Explicitly, what we have to prove is that for every compactly supported continuous function g on $\Omega\times\Omega$, we have:

$$\lim_{m\to+\infty}\frac{n!}{m^n}\int\limits_{\Omega\times\Omega} g(z,\,\zeta)\,|K_{m\varphi}(z,\,\zeta)|^2\, e^{-2m\varphi(z)}\, e^{-2m\varphi(\zeta)}\, dV_n(z)\, dV_n(\zeta)$$
$$= \frac{2^n}{\pi^n}\int\limits_{\Omega\setminus V} g(z,\,z)\,(i\partial\bar\partial\varphi)^n(z).$$

Again, $(i\partial\bar\partial\varphi)^n_{ac}$ having locally finite mass on Ω implies the well-definedness of $(i\partial\bar\partial\varphi)^n_{ac}\wedge[\Delta]$ as a complex measure on $\Omega\times\Omega$. If $M:=\sup|g|$, we notice that:

$$\left|\frac{n!}{m^n}\int\limits_{\Omega} g(z,\,\zeta)\,|K_{m\varphi}(z,\,\zeta)|^2\, e^{-2m\varphi(z)}\, e^{-2m\varphi(\zeta)}\, dV_n(z)\right|$$
$$\leqslant M\,\frac{n!}{m^n}\int\limits_{\Omega} |K_{m\varphi}(z,\,\zeta)|^2\, e^{-2m\varphi(z)}\, e^{-2m\varphi(\zeta)}\, dV_n(z)$$
$$= M\,\frac{n!}{m^n}\, B_{m\varphi}(\zeta)\, e^{-2m\varphi(\zeta)} \leqslant 2\,M\,\frac{2^n}{\pi^n}\,\lambda_1(\zeta)\dots\lambda_n(\zeta), \quad \zeta\in\Omega\setminus V,$$

where the equality above follows from the reproducing property of the Bergman kernel, and the last expression is locally integrable on $\Omega\setminus V$ by the key standard observation above. By dominated convergence, it is then enough to prove that:

$$\lim_{m\to+\infty}\frac{n!}{m^n}\int\limits_{\Omega} g(z,\,\zeta)\,|K_{m\varphi}(z,\,\zeta)|^2\, e^{-2m\varphi(z)}\, e^{-2m\varphi(\zeta)}\, dV_n(z)$$
$$= \frac{2^n}{\pi^n}\, g(\zeta,\,\zeta)\,\lambda_1(\zeta)\dots\lambda_n(\zeta), \quad \zeta\in\Omega\setminus V.$$

To this end, we can repeat Berndtsson's arguments showing the Bergman kernel to decay rapidly off the diagonal, namely that for every $\varepsilon>0$ and every $\zeta \in \Omega \setminus V$, we have:

$$\lim_{m\to+\infty} \frac{1}{m^n} \int\limits_{|z-\zeta|>\varepsilon} |K_{m\varphi}(z,\,\zeta)|^2\, e^{-2m\varphi(z)}\, e^{-2m\varphi(\zeta)}\, dV_n(z) = 0.$$

The reproducing property of the Bergman kernel then leads to (b). Point (c) is an easy consequence of (a) and (b).

The whole proof of Theorem 3.5 thus boils down to establishing the uniform upper estimate (3.4). We will proceed in several steps.

Step 1. Assume $\varphi = \psi + \log|g|$ on Ω, for some $g \in \mathcal{O}(\Omega)$ such that $\operatorname{div} g$ is a normal crossing divisor, and for some smooth and strictly psh ψ on Ω. Then $V = \{g = 0\}$ and $i\partial\bar\partial\varphi = i\partial\bar\partial\psi$ on $\Omega \setminus V$. By Remark 3.2 we have:

$$\frac{n!}{m^n}\, B_{m\varphi}\, e^{-2m\varphi} = \frac{n!}{m^n}\, B_{m\psi}\, e^{-2m\psi}, \quad \text{on } \Omega.$$

This last expression satisfies the uniform upper estimate claimed in (3.4) on $B \Subset \Omega$ thanks to the Berndtsson-Lindholm inequality (3.3) applied to the smooth function ψ.

Step 2. Assume φ has locally divisorial singularities; namely, every point in Ω has a neighbourhood on which $\varphi = \psi + \log|g|$ for some holomorphic function g such that $\operatorname{div} g$ has normal crossings, and for some smooth strictly psh function ψ.

Let $U \Subset \Omega$ be a pseudoconvex such neighbourhood. By Lemma 3.3 above, we get:

$$\frac{n!}{m^n}\, B_{m\varphi}\, e^{-2m\varphi} \leqslant \frac{n!}{m^n}\, B_{m\varphi,\,U}\, e^{-2m\varphi} = \frac{n!}{m^n}\, B_{m\psi,\,U}\, e^{-2m\psi}, \quad \text{on } U.$$

The last term satisfies the uniform upper estimate claimed in (3.4) on $U \setminus V$ thanks again to the smooth case applied to $\psi_{|U}$ and its associated Bergman kernels $B_{m\psi,\,U}$. The resulting constant $C > 0$ depends on U, but we get a constant independent of m for the estimate on $B \setminus V$ after taking a finite covering of $\bar B$ by such open sets U.

Step 3. Assume $\varphi = \frac{p}{2}\log(|g_1|^2 + \cdots + |g_N|^2) + u$ on Ω, for some holomorphic functions $g_1, \ldots, g_N$ and some smooth function u.

Let $J := (g_1, \ldots, g_N) \subset \mathcal{O}_\Omega$ be the ideal sheaf generated by $g_1, \ldots, g_N$, and let $\mu : \tilde\Omega \to \Omega$ be a proper modification such that

$$\mu^\star J = \mathcal{O}(-E),$$

$\tilde\Omega$ is a smooth manifold, and E is an effective normal crossing divisor on $\tilde\Omega$. We still denote by $\mathcal{H}_{\tilde\Omega}(m\varphi \circ \mu)$ the Hilbert space of global holomorphic functions f on $\tilde{\Omega}$ such that $|f|^2\, e^{-2m\varphi\circ\mu}$ is integrable with respect to the Lebesgue measure on $\tilde{\Omega}$. The (global) Bergman kernel $B_{m\varphi\circ\mu}$ is defined as in (1.1) by any orthonormal basis of $\mathcal{H}_{\tilde{\Omega}}(m\varphi\circ\mu)$, $m\in\mathbb{N}^\star$.

We may assume, without loss of generality, that the Jacobian J_μ of μ is globally defined on $\tilde{\Omega}$. Otherwise we can cover $\tilde{\Omega}$ by a fixed finite collection $(\tilde{U}_j)_{j=1,\dots,N}$, with N independent of m, of open subsets contained in coordinate patches and we can work separately on each $\tilde{U}_j$. Lemma 3.3 above shows that the Bergman kernels $B_{m\varphi\circ\mu}$ defined by global holomorphic functions in $\mathcal{H}_{\tilde{\Omega}}(m\varphi\circ\mu)$ are only distorted from those $B_{m\varphi\circ\mu}$ defined by local holomorphic functions in each $\mathcal{H}_{\tilde{U}_j}(m\varphi\circ\mu)$ by insignificant constants $C_{n,\,d,\,r}$ independent of m. The fact that, in general, J_μ is not a global function on $\tilde{\Omega}$ but a section of the line bundle (associated with) $\mathcal{O}(E)$ plays no role here. Remark 3.2 shows that the expressions $B_{m\varphi\circ\mu}\, e^{-2m\varphi\circ\mu}$ are unaffected by the addition to the weight $\varphi\circ\mu$ of terms such as $p\log|h|$, where h is a locally defined holomorphic function representing J_μ in a local trivialisation of $\mathcal{O}(E)$. Thus, switching from a $\tilde{U}_j$ to another $\tilde{U}_k$ introduces only distortions by constants independent of m. With these reductions in place, a change of variable shows that, for each $\sigma_{m,\,j}$, we have:

$$\begin{aligned}1 &= \int_\Omega |\sigma_{m,\,j}|^2\, e^{-2m\varphi}\, dV_n = \int_{\Omega\setminus V} |\sigma_{m,\,j}|^2\, e^{-2m\varphi}\, dV_n \\ &= \int_{\tilde{\Omega}\setminus \mathrm{Supp}\, E} |\sigma_{m,\,j}\circ\mu|^2\, |J_\mu|^2\, e^{-2m\varphi\circ\mu}\, d\tilde{V}_n \\ &= \int_{\tilde{\Omega}} |\sigma_{m,\,j}\circ\mu|^2\, |J_\mu|^2\, e^{-2m\varphi\circ\mu}\, d\tilde{V}_n,\end{aligned}$$

for a suitable volume form $d\tilde{V}_n$ on $\tilde{\Omega}$. The actual choice of a smooth volume form on $\tilde{\Omega}$ is immaterial since smooth volume forms are comparable on local coordinate patches and $\tilde{\Omega}$ can be covered by finitely many such patches independent of m. A change of volume form would only distort the Bergman kernels by constants independent of m, hence irrelevant as $m\to+\infty$. We get

$$B_{m\varphi\circ\mu} = |J_\mu|^2\, B_{m\varphi}\circ\mu.$$

Thus, proving the upper estimate claimed in (3.4) amounts to proving:

$$\frac{n!}{m^n}\, B_{m\varphi\circ\mu}\, e^{-2m\varphi\circ\mu} \leqslant C\, \frac{2^n}{\pi^n}\, |J_\mu|^2\, \lambda_1\circ\mu\dots\lambda_n\circ\mu, \tag{3.5}$$

on $\mu^{-1}(B)\setminus \mathrm{Supp}\, E$, since μ defines an isomorphism between $\tilde{\Omega}\setminus\mathrm{Supp}\, E$ and $\Omega\setminus V$, and J_μ does not vanish on $\tilde{\Omega}\setminus\mathrm{Supp}\, E$. If $\tilde{\lambda}_1,\dots,\tilde{\lambda}_n$ are the eigenvalues of $i\partial\bar\partial(\varphi\circ\mu)$ (calculated on each coordinate open subset $\tilde{U}_j \subset \tilde\Omega$ with respect to the standard Kähler form of $\tilde{U}_j$ defined by the coordinates), it can easily be seen that

$$|J_\mu|^2\, \lambda_1\circ\mu \dots \lambda_n\circ\mu = \tilde{\lambda_1}\dots\tilde{\lambda_n}.$$

The divisor E can be locally represented as $E = \operatorname{div} g$, for some locally defined holomorphic function g. We then get, locally on $\tilde\Omega$:

$$\varphi\circ\mu = \frac{p}{2}\,\log\left(\left|\frac{g_1\circ\mu}{g}\right|^2 + \dots + \left|\frac{g_N\circ\mu}{g}\right|^2\right) + u\circ\mu + \log|g^p| = \psi + \log|g^p|,$$

where ψ is a smooth psh function. Thus $\varphi\circ\mu$ has locally divisorial singularities like the weight functions considered in Step 2. Although $\varphi\circ\mu$ is psh on $\tilde\Omega$, it is strictly psh only in the complement of the support of E as the Jacobian J_μ vanishes along $\operatorname{Supp} E$. Nonetheless, this causes no trouble owing to the assumption that $i\partial\bar\partial\varphi \geqslant C_0\,\omega$ on the whole set Ω for a constant $C_0 > 0$. Indeed, this assumption implies that $\lambda_j(x) \geqslant C_0 > 0$ for every x in the complement in Ω of the singular set $V = \{g_1 = \dots = g_N = 0\}$ of φ. Having fixed an arbitrary point $x \in \Omega\setminus V$, the mean value inequality method described in [2, p. 3-4], can be applied on a small ball of radius $c_m/\sqrt{m}$ about x to yield:

$$\frac{n!}{m^n}\, B_{m\varphi}(x)\, e^{-2m\varphi(x)} \leqslant \frac{1+\varepsilon_m}{\pi^n \prod_{j=1}^{n}(1-e^{-c_m^2\,\lambda_j(x)})}\,\lambda_1(x)\dots\lambda_n(x), \tag{3.6}$$

where the constants $c_m > 0$ have been chosen such that $c_m \to +\infty$ and $c_m^3/\sqrt{m} \to 0$ when $m \to +\infty$, while $\varepsilon_m := e^{c_m^3/\sqrt{m}} - 1 \to 0$. (In [2], m is labelled as k). The fact that φ is not smooth on the whole of Ω is not a problem here since the change of variable procedure described above reduces integrals on balls in Ω involving φ to integrals on open subsets of $\tilde\Omega$ involving ψ, the latter function being smooth. Remark 3.2 allows one to replace $m\varphi\circ\mu$ with $m\psi$.

Now, since $\lambda_j(x) \geqslant C_0 > 0$ for every $x \in \Omega\setminus V$, we see that $c_m^2\,\lambda_j(x) \geqslant c_m^2\, C_0 > C' > 0$ for some constant $C' > 0$, if m is large enough. Thus, the coefficient of $\lambda_1(x)\dots\lambda_n(x)$ in the above formula (3.6) is uniformly bounded above by some constant $C'' > 0$, proving the uniform estimate claimed in (3.4). This completes the proof of Theorem 3.5. □

(b) Analytic singularities with arbitrary coefficients

Let us first consider the case of divisorial singularities with non-integral coefficients. For the sake of simplicity, we assume that our domains are polydiscs, $\Omega = D^n$ and $B = D^n_{1-\varepsilon}$, where D (respectively $D_{1-\varepsilon}$) is the unit disc (respectively the disc of radius $1-\varepsilon$) in $\mathbb{C}$. We begin by considering the following toric situation.

Proposition 3.6. — *Assume that $\varphi(z) = \psi(z) + c_1 \log|z_1| + \cdots + c_n \log|z_n|$, with $\psi(z) = \psi_1(z_1) + \cdots + \psi_n(z_n)$ for some C^∞ functions ψ_j on $\mathbb{C}$ depending only on $|z_j|$ respectively, and some constants $c_j > 0$, $j = 1, \ldots, n$. If $i\partial\bar\partial\varphi \geqslant C_0\,\omega$ for some $C_0 > 0$, then:*

$$v_B(\psi + \sum_{j=1}^{n} c_j \log|z_j|) = v_B(\psi) = \frac{2^n}{\pi^n}\int_B (i\partial\bar\partial\psi)^n < +\infty.$$

Proof. — By Remark 3.2, we have:

$$B_{m\varphi}\, e^{-2m\varphi} = B_{m\psi+\Sigma\{mc_j\}\log|z_j|}\, e^{-2(m\psi+\Sigma\{mc_j\}\log|z_j|)}, \tag{3.7}$$

where $\{\cdot\}$ stands for the fractional part. The same identity holds with $m\{c_j\}$ in place of $\{mc_j\}$. Thus we may assume, without loss of generality, that $0 \leqslant c_j < 1$ for $j = 1, \ldots, n$.

Of the two equalities in the statement of the Proposition, only the former needs a proof. The latter equality follows from Theorem 3.4 for smooth functions. To begin with, we shall prove that:

$$\int_B B_{m\varphi}\, e^{-2m\varphi}\, dV_n \geqslant \int_B B_{m\psi}\, e^{-2m\psi}\, dV_n, \quad \text{for every} \quad m \in \mathbb{N}^\star, \tag{3.8}$$

which clearly implies that

$$v_B(\psi + \sum_{j=1}^{n} c_j \log|z_j|) \geqslant v_B(\psi). \tag{3.9}$$

Fix $m \in \mathbb{N}^\star$, and let $a_j := \{m\,c_j\}$ for $j = 1, \ldots, n$. Since $a_j = \{mc_j\} < 1$ for every $j = 1, \ldots, n$, the exponential

$$e^{-2m\psi - 2\{mc_1\}\log|z_1| - \cdots - 2\{mc_n\}\log|z_n|}$$

is easily seen to be locally integrable. Indeed, by Fubini's theorem, the integral is a product of integrals depending each on one complex variable. Thus all the monomials $z^\alpha = z_1^{\alpha_1}\ldots z_n^{\alpha_n}$, $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, form a complete orthogonal set of $\mathcal{H}_\Omega(m\psi + \{mc_1\}\log|z_1| + \cdots + \{mc_n\}\log|z_n|)$. This set becomes an orthonormal basis after each monomial is normalised to have norm 1. As $a_j = \{mc_j\}$, in view of (3.7) we get:

$$B_{m\varphi}\, e^{-2m\varphi} = \sum_{\alpha_1,\ldots,\alpha_n=0}^{+\infty} \prod_{j=1}^{n} \frac{|z_j|^{2(\alpha_j-a_j)}\, e^{-2m\psi_j}}{\int_D |z_j|^{2(\alpha_j-a_j)}\, e^{-2m\psi_j(z_j)}\, dV_1(z_j)},$$

where $d V_1$ denotes the Lebesgue measure on $\mathbb{C}$. As ψ is smooth, there is an analogous formula for $m\psi$:

$$B_{m\psi}\, e^{-2m\psi} = \sum_{\alpha_1,\dots,\,\alpha_n=0}^{+\infty} \prod_{j=1}^{n} \frac{|z_j|^{2\alpha_j}\, e^{-2m\psi_j}}{\displaystyle\int_D |z_j|^{2\alpha_j}\, e^{-2m\psi_j(z_j)}\, d V_1(z_j)}.$$

Inequality (3.8) will follow from Fubini's theorem if we can prove that, for every $j = 1, \dots, n$, we have:

$$\frac{\displaystyle\int_{D_{1-\varepsilon}} |z_j|^{2(\alpha_j - a_j)}\, e^{-2m\psi_j}\, d V_1(z_j)}{\displaystyle\int_D |z_j|^{2(\alpha_j - a_j)}\, e^{-2m\psi_j}\, d V_1(z_j)} \geqslant \frac{\displaystyle\int_{D_{1-\varepsilon}} |z_j|^{2\alpha_j}\, e^{-2m\psi_j}\, d V_1(z_j)}{\displaystyle\int_D |z_j|^{2\alpha_j}\, e^{-2m\psi_j}\, d V_1(z_j)}. \tag{3.10}$$

Taking polar coordinates $z_j = r_j\, \rho_j$, $0 \leqslant r_j \leqslant 1$, $\rho_j \in S^1$, we are then reduced to proving that, given a C^∞ function $u \geqslant 0$ and a constant $0 \leqslant c < 1$, we have:

$$\frac{\displaystyle\int_0^{1-\varepsilon} x^{2k+1}\, u(x)\, \frac{1}{x^{2c}}\, dx}{\displaystyle\int_0^1 x^{2k+1}\, u(x)\, \frac{1}{x^{2c}}\, dx} \geqslant \frac{\displaystyle\int_0^{1-\varepsilon} x^{2k+1}\, u(x)\, dx}{\displaystyle\int_0^1 x^{2k+1}\, u(x)\, dx}.$$

Writing $\int_0^1 = \int_0^{1-\varepsilon} + \int_{1-\varepsilon}^1$, this is seen to be equivalent to:

$$\frac{\displaystyle\int_0^{1-\varepsilon} x^{2k+1}\, u(x)\, \frac{1}{x^{2c}}\, dx}{\displaystyle\int_0^{1-\varepsilon} x^{2k+1}\, u(x)\, dx} \geqslant \frac{\displaystyle\int_{1-\varepsilon}^1 x^{2k+1}\, u(x)\, \frac{1}{x^{2c}}\, dx}{\displaystyle\int_{1-\varepsilon}^1 x^{2k+1}\, u(x)\, dx},$$

which is clear since the left-hand side is $\geqslant \frac{1}{(1-\varepsilon)^{2c}}$, and the right-hand side is $\leqslant \frac{1}{(1-\varepsilon)^{2c}}$. Inequalities (3.8) and (3.9) are thus proved.

We shall now prove that:

$$v_B(\psi + \sum_{j=1}^{n} c_j\, \log |z_j|) \leqslant v_B(\psi + \sum_{j=1}^{n} \log |z_j|). \tag{3.11}$$

Let $v(z) := \psi(z) + \log |z_1| + \cdots + \log |z_n|$. Again thanks to Fubini's theorem, the monomials $z^\alpha = z_1^{\alpha_1} \dots z_n^{\alpha_n}$, with $\alpha_1, \dots, \alpha_n \geqslant m$, form a complete orthogonal system of $\mathcal{H}_\Omega(mv)$. Therefore,

$$\begin{aligned} B_{mv}(z)\,e^{-2mv(z)} &= \sum_{\alpha_1,\ldots,\alpha_n=m}^{+\infty} \frac{|z_1|^{2(\alpha_1-m)}\ldots|z_n|^{2(\alpha_n-m)}}{\int_\Omega |z_1|^{2(\alpha_1-m)}\ldots|z_n|^{2(\alpha_n-m)}\,e^{-2m\psi(z)}}\,e^{-2m\psi(z)} \\ &= \sum_{\alpha_1,\ldots,\alpha_n=1}^{+\infty} \frac{|z_1|^{2(\alpha_1-1)}\ldots|z_n|^{2(\alpha_n-1)}}{\int_\Omega |z_1|^{2(\alpha_1-1)}\ldots|z_n|^{2(\alpha_n-1)}\,e^{-2m\psi(z)}}\,e^{-2m\psi(z)} \\ &= \sum_{\alpha_1,\ldots,\alpha_n=1}^{+\infty} \prod_{j=1}^{n} \frac{|z_j|^{2(\alpha_j-1)}\,e^{-2m\psi_j(z_j)}}{\int_D |z_j|^{2(\alpha_j-1)}\,e^{-2m\psi_j(z_j)}\,dV_1(z_j)}, \end{aligned}$$

which entails:

$$\begin{aligned} &\frac{1}{m^n}\int_B B_{mv}(z)\,e^{-2mv(z)}\,dV_n(z) \\ &= \frac{1}{m^n}\prod_{j=1}^{n}\sum_{\alpha_j=1}^{+\infty} \frac{\int_{D_{1-\varepsilon}} |z_j|^{2(\alpha_j-1)}\,e^{-2m\psi_j(z_j)}\,dV_1(z_j)}{\int_D |z_j|^{2(\alpha_j-1)}\,e^{-2m\psi_j(z_j)}\,dV_1(z_j)} \\ &\geqslant \frac{1}{m^n}\prod_{j=1}^{n}\left(\sum_{\alpha_j=0}^{+\infty} \frac{\int_{D_{1-\varepsilon}} |z_j|^{2(\alpha_j-c_j)}\,e^{-2m\psi_j(z_j)}\,dV_1(z_j)}{\int_D |z_j|^{2(\alpha_j-c_j)}\,e^{-2m\psi_j(z_j)}\,dV_1(z_j)}\right. \\ &\qquad \left. - \frac{\int_{D_{1-\varepsilon}} |z_j|^{-2c_j}\,e^{-2m\psi_j(z_j)}\,dV_1(z_j)}{\int_D |z_j|^{-2c_j}\,e^{-2m\psi_j(z_j)}\,dV_1(z_j)}\right) \\ &\geqslant \frac{1}{m^n}\prod_{j=1}^{n}\left(\sum_{\alpha_j=0}^{+\infty} \frac{\int_{D_{1-\varepsilon}} |z_j|^{2(\alpha_j-c_j)}\,e^{-2m\psi_j(z_j)}\,dV_1(z_j)}{\int_D |z_j|^{2(\alpha_j-c_j)}\,e^{-2m\psi_j(z_j)}\,dV_1(z_j)} - 1\right) \\ &\geqslant \frac{1}{m^n}\int_B B_{m\varphi}\,e^{-2m\varphi}\,dV_n - \frac{1}{m}\sum_{k=1}^{n}\frac{1}{m^{n-1}}\int_{D_{1-\varepsilon}^{n-1}} B_{mu_k}\,e^{-2mu_k}\,dV_{n-1} \\ &+ \frac{1}{m^2}\sum_{k_1,\,k_2}\frac{1}{m^{n-2}}\int_{D_{1-\varepsilon}^{n-2}} B_{mu_{k_1,\,k_2}}\,e^{-2mu_{k,\,k_2}}\,dV_{n-2} - \cdots + \frac{(-1)^n}{m^n}, \end{aligned}$$

where we have denoted $u_k(z_1,\ldots,\hat{z_k},\ldots z_n) := \varphi(z) - \psi_k(z_k) - c_k \log|z_k|$, and the analogous expressions when several indices are missing. The k-dimensional Lebesgue measure has been denoted dV_k. Note that as we assume $0 \leqslant c_j < 1$, the first inequality above is implied by estimate (3.10) with α_j replaced with $\alpha_j - c_j$, and c_j replaced with $1 - c_j$.

We can thus run an induction on the dimension n. If we assume the finiteness of the volume for psh functions of the form under consideration defined in less than n variables, all the terms appearing on the right-hand side, except the first one, tend to 0 as $m \to +\infty$. Inequality (3.11) is then what we get in the limit.

Now, by Remark 3.2

$$v_B(\psi) = v_B(\psi + \sum_{j=1}^{n} \log|z_j|),$$

which, alongside inequalities (3.9) and (3.11), completes the proof. □

We can now round off the study of the finiteness of the volume of a psh function in the case of general analytic singularities with arbitrary (not necessarily integral) coefficients.

Proposition 3.7. — *Let $\varphi = \frac{c}{2}\log(|g_1|^2 + \cdots + |g_N|^2) + u$, for some holomorphic functions $g_1, \ldots, g_N$, some $c > 0$, and some C^∞ function u on $\Omega \Subset \mathbb{C}^n$. Assume $i\partial\bar\partial\varphi \geqslant C_0\,\omega$ for some constant $C_0 > 0$. Then:*

$$v_B(\varphi) := \limsup_{m\to+\infty} \frac{n!}{m^n} \int\limits_B B_{m\varphi}\, e^{-2m\varphi}\, dV_n < +\infty.$$

Proof. — Let $J = (g_1, \ldots, g_N) \subset \mathcal{O}_\Omega$, and let $\mu : \tilde\Omega \to \Omega$ be a proper modification such that $\tilde\Omega$ is smooth and $\mu^\star J = \mathcal{O}(-E)$ for an effective normal crossing divisor E on $\tilde\Omega$. The change of variable formula shows, as at Step 3 in the proof of Theorem 3.5, that

$$v_B(\varphi) = v_{\mu^{-1}(B)}(\varphi\circ\mu).$$

If we cover $\overline{\mu^{-1}(B)}$ by finitely many open polydiscs U_k such that $E = \mathrm{div} g_k$ on U_k and use the comparison Lemma 3.3 on the behaviour of Bergman kernels under restrictions, we see that:

$$v_{\mu^{-1}(B)}(\varphi\circ\mu) \leqslant \sum_k v_{U_k}(\varphi\circ\mu_{|U_k}).$$

Now, the singularities of $\varphi\circ\mu_{|U_k}$ are concentrated along the divisor E. As E has normal crossings, the singular part of $\varphi\circ\mu_{|U_k}$ can be written in the form

$$c_1\,\log|w_1| + \cdots + c_n\,\log|w_n|,$$

for constants $c_1, \ldots, c_n \geqslant 0$ and local holomorphic coordinates $w_1, \ldots, w_n$ on $U_k \subset \tilde\Omega$. The smooth part ψ of $\varphi\circ\mu_{|U_k}$ can be handled as follows. Fix any point $x \in U_k$ and choose local coordinates w about x. After possibly adding a harmonic function to ψ, we can write $\psi = \psi_0 + o(|w|^2)$ near x, where $\psi_0 = \sum c_{jk}\, w_j \bar{w}_k$ with some constant coefficients c_{jk}. By Remark 3.2, $B_{m\psi}\, e^{-2m\psi}$ is unaffected by the addition of a harmonic function. So is, obviously, $i\partial\bar{\partial}\psi$. After possibly changing coordinates, ψ_0 can be diagonalised as $\psi_0 = \sum \lambda_j\, |w_j|^2$, where $\lambda_1, \dots, \lambda_n$ are the eigenvalues of $i\partial\bar{\partial}\psi$ at $x = w(0)$. As the distance between ψ and ψ_0 is under control, the previous Proposition 3.6 can be applied to $\sum \lambda_j\, |w_j|^2 + c_1\, \log|w_1| + \cdots + c_n\, \log|w_n|$ to give $v_{U_k}(\varphi \circ \mu_{|U_k}) < +\infty$ for every k. This completes the proof. $\square$

Summing up, we have proved that the volume of a psh function with analytic singularities is finite. As already explained, this implies the following estimate for the number N_m of local generators of $\mathcal{I}(m\varphi)$:

$$\limsup_{m\to+\infty} \frac{n!}{m^n}\, N_m \leqslant \frac{v_B(\varphi)}{1-\varepsilon} < +\infty, \quad \text{if } \varphi \text{ has analytic singularities.} \tag{3.12}$$

The last statement of Theorem 1.1 in the Introduction is thus proved.

4. Approximations of psh functions and multiplier ideal sheaves

We now turn to the second part of the paper which discusses the independent but related question of how far multiplier ideal sheaves $\mathcal{I}(h^m) = \mathcal{I}(m\varphi)$ are from behaving additively when $m \to +\infty$. As pointed out in the Introduction, we will show that for metrics $h = e^{-\varphi}$ with analytic singularities, $\mathcal{I}(h^m)$ comes arbitrarily close to an additive decay rate provided that m is chosen large enough. The main concern is to obtain an effective asymptotic control of the subadditivity defect. We will cast this result in the language of approximations of plurisubharmonic functions.

A by now classical result of Demailly's ([6], *Proposition 3.1.*) states that a psh function φ with arbitrary singularities can be approximated by psh functions φ_m with analytic singularities (cf. $(\star)$) constructed as

$$\varphi_m(z) = \frac{1}{2m}\, \log\sum_{j=1}^{+\infty} |\sigma_{m,\,j}(z)|^2 = \sup_{f\in \bar{B}_m(1)} \frac{1}{m}\, \log|f(z)|, \qquad z \in \Omega, \tag{4.1}$$

where $(\sigma_{m,\,j})_{j\in\mathbb{N}^\star}$ is an arbitrary orthonormal basis and $\bar{B}_m(1)$ is the unit ball of the Hilbert space $\mathcal{H}_\Omega(m\varphi)$ considered in the previous sections. We even have

$$\varphi(z) - \frac{C_1}{m} \leqslant \varphi_m(z) \leqslant \sup_{|\zeta - z|<r} \varphi(\zeta) + \frac{1}{m}\, \log(\frac{C_2}{r^n}), \tag{4.2}$$

for every $z \in \Omega$ and every $r < d(z, \partial\Omega)$. The lower estimate is a consequence of the Ohsawa-Takegoshi L^2 extension theorem. The upper estimate is far easier, coming from an application of the submean value inequality satisfied by squares of absolute values of holomorphic functions. Our aim is to improve the upper estimate by replacing the supremum by a pointwise upper bound which affords a much better understanding of singularities. This is not possible for an arbitrary φ, but the following proposition shows it to be possible if φ is assumed to have analytic singularities.

THEOREM 4.1. — *Let $\Omega \Subset \mathbb{C}^n$ be a bounded pseudoconvex open set and let $\varphi = \frac{c}{2}\log(|g_1|^2 + \cdots + |g_N|^2)$ be a plurisubharmonic function with analytic singularities on Ω. If $\Omega' \Subset \Omega'' \Subset \Omega$ are any relatively compact open subsets, then for every $0 < \delta < 1$ and every $m \geqslant \frac{n+1}{c\,\delta}$ we have:*

$$\varphi_m(z) \leqslant (1-\delta)\,\varphi(z) + c\,\delta\,\log A + \frac{\log(C_n\,C_m)}{2m}, \quad z \in \Omega',$$

where $C_m = m\,c - n$ if $m\,c$ is an integer and $C_m = \frac{m\,c-n}{m\,c-[m\,c]}$ if $m\,c$ is not an integer. We also denote $A := \max\{\sup_{\Omega''} \sqrt{|g_1|^2 + \cdots + |g_N|^2},\, 1\}$ and $C_n > 0$ is a constant depending only on Ω', Ω'' and n (while [ ] is the round-down).

Remark 4.2. — (a) The upper bound given in Theorem 4.1, combined with Demailly's lower bound of (4.2), implies that φ and its approximations φ_m have the same $-\infty$ poles on Ω' if m is large enough. Only the Lelong numbers may be slightly different up to an arbitrarily small $\delta > 0$ when $m \gg 1$. Since the set of poles of an arbitrary psh function φ is not necessarily analytic, whereas the polar set of φ_m is always analytic, we see that some analyticity assumption on φ is necessary. We may ask for the weakest assumption on φ under which Theorem 4.1 holds.

(b) A similar estimate was noticed in [9, 2.2]. While that estimate applies to the larger class of psh functions φ for which e^{φ} is Hölder continuous, it is non-effective as it only holds on a small ball of uncontrollable size and with an uncontrollable constant. Our main concern here is effectiveness. A crucial application of a variant of Theorem 4.1 above in [15] makes an essential use of effective estimates on fixed-size open balls to obtain a new regularisation theorem for $(1,1)$-currents with controlled Monge-Ampère masses. Moreover, the present approach seems rather flexible as, besides effectiveness, it can also yield estimates for the derivatives of the $\sigma_{m,j}$'s up to any pregiven order (see [15, Proposition 7.2.]).

Proof. — Fix $0 < \delta < 1$. Let $m \geqslant \frac{n+1}{c\,\delta}$ and let f be an arbitrary holomorphic function on Ω such that

$$\int_\Omega |f|^2\, e^{-2m\varphi} = 1 \quad \Leftrightarrow \quad \int_\Omega \frac{|f|^2}{(|g_1|^2 + \cdots + |g_N|^2)^{mc}} = 1. \tag{4.3}$$

We clearly have $mc > n+1$. If $m\,c \notin \mathbb{N}$, set $q := [mc] - n \geqslant 1$ and $\varepsilon := m\,c - [m\,c] = \{m\,c\} > 0$. If $m\,c \in \mathbb{N}$, then $m\,c \geqslant n+2$; set $q := m\,c - n - 1 \geqslant 1$ and $\varepsilon = 1$. In both cases we have

$$m\,c = n + q + \varepsilon, \qquad 0 < \varepsilon \leqslant 1. \tag{4.4}$$

Notice that the optimal choice of the integer q (= number of g_i's used in products dividing f below) is the largest possible, so choosing $q = mc - n$ and $\varepsilon = 0$ when $m\,c \in \mathbb{N}$ would appear preferable, but the present form of Skoda's theorem (see below) requires $\varepsilon > 0$. If the L^2 estimates in Skoda's division theorem can be improved to allow the choice $\varepsilon = 0$, then our result would hold for $m \geqslant \frac{n}{c\,\delta}$ when $m\,c \in \mathbb{N}$. However, the requirement $m \geqslant \frac{n+1}{c\,\delta}$ is optimal when $m\,c \notin \mathbb{N}$.

By Skoda's L^2 division theorem ([18]) applied in the form given in [6, Corollary A.5.], there exist holomorphic functions $h_{i_1,\dots,i_q}$ on Ω for all multi-indices $(i_1, \dots, i_q) \in \{1, \dots, N\}^q$ such that:

$$f(z) = \sum_{i_1,\dots,i_q=1}^{N} h_{i_1,\dots,i_q}(z)\, g_{i_1}(z) \dots g_{i_q}(z), \quad z \in \Omega,$$

and

$$\begin{aligned}\sum_{i_1,\dots,i_q=1}^{N} \int_\Omega \frac{|h_{i_1,\dots,i_q}|^2}{(|g_1|^2 + \cdots + |g_N|^2)^{mc-q}} &\leqslant \frac{q+\varepsilon}{\varepsilon} \int_\Omega \frac{|f|^2}{(|g_1|^2 + \cdots + |g_N|^2)^{mc}} \\ &= \frac{q+\varepsilon}{\varepsilon} := C_m,\end{aligned}$$

where $C_m > 0$ is the constant specified in the statement. Set $|g|^2 := |g_1|^2 + \cdots + |g_N|^2$. As Ω'' satisfies $\Omega' \Subset \Omega'' \Subset \Omega$, we get, for every $z \in \Omega'$:

$$\begin{aligned}|f(z)|^2 &\leqslant |g(z)|^{2q} \sum_{i_1,\dots,i_q=1}^{N} |h_{i_1,\dots,i_q}(z)|^2 \\ &\leqslant C_n\, |g(z)|^{2q} \sum_{i_1,\dots,i_q=1}^{N} \int_{\Omega''} \frac{|h_{i_1,\dots,i_q}|^2}{(|g_1|^2 + \cdots + |g_N|^2)^{mc-q}}\, |g|^{2(mc-q)} \\ &\leqslant C_n\, C_m\, (\sup_{\Omega''} |g|)^{2(mc-q)}\, |g(z)|^{2q} \leqslant C_n\, C_m\, A^{2(mc-q)}\, |g(z)|^{2q},\end{aligned}$$

where $C_n > 0$ is the constant involved in the submean value inequality applied to every $|h_{i_1,\dots,i_q}|^2$ at z (thus depending only on Ω', Ω'' and n, but independent of m) and recall that $A = \max\{\sup_{\Omega''}|g|, 1\}$. Hence we get:

$$\begin{aligned}|f(z)|^2\, e^{-2m(1-\delta)\varphi(z)} &\leqslant C_n\, C_m\, A^{2(mc-q)}\, \frac{|g(z)|^{2q}}{|g(z)|^{2mc(1-\delta)}}\\ &= C_n\, C_m\, A^{2(mc-q)}\, |g(z)|^{2(q-mc+mc\delta)}, \quad z\in\Omega'.\end{aligned}$$

Now m has been chosen such that $mc\delta \geqslant n+1$ and the choice of q guarantees that $n+1 \geqslant mc-q = n+\varepsilon$ (the inequality is strict if $m\,c \notin \mathbb{N}$, while equality holds otherwise). Thus $mc\delta \geqslant mc-q$, hence the exponent $2(q-mc+mc\delta)$ above is nonnegative and we finally get:

$$\begin{aligned}|f(z)|^2\, e^{-2m(1-\delta)\varphi(z)} &\leqslant C_n\, C_m\, A^{2(mc-q)}\, A^{2(q-mc+mc\delta)}\\ &= C_n\, C_m\, A^{2mc\delta}, \quad z\in\Omega'. \qquad (4.5)\end{aligned}$$

In view of (4.1), taking log and dividing by $2m$ on both sides and then taking supremum over all holomorphic functions f on Ω satisfying (4.3), we get:

$$\varphi_m(z) \leqslant (1-\delta)\,\varphi(z) + c\delta\, \log A + \frac{\log(C_n\, C_m)}{2m}, \quad z\in\Omega'.$$

The proof is complete. □

Now Theorem 1.2 in the Introduction follows from Theorem 4.1.

Proof of Theorem 1.2. — As repeatedly pointed out above, the ideal sheaf $\mathcal{I}(m\varphi)$ is generated as an $\mathcal{O}_\Omega$-module by the Hilbert space $\mathcal{H}_\Omega(m\varphi)$. Now, using (4.1), the estimate for φ_m obtained in Theorem 4.1 implies estimate (4.5) for every $f\in\mathcal{H}_\Omega(m\varphi)$ with norm 1 (i.e. satisfying (4.3)):

$$|f(z)|^2 \leqslant C_n\, C_m\, A^{2\,m\,c\,\delta}\, e^{2m(1-\delta)\varphi(z)}, \quad z\in\Omega', \quad m\geqslant \frac{n+1}{c\,\delta}.$$

If $f_1,\dots, f_p \in \mathcal{H}_\Omega(m\varphi)$ have norm 1, then $|f_1\dots f_p|^2\, e^{-2mp(1-\delta)\varphi}$ is bounded and therefore integrable on Ω', proving that $f_1,\dots, f_p$ is a section of $\mathcal{I}(mp(1-\delta)\varphi)$ on Ω'. The proof is complete. □

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