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ON A GENERALIZED CALABI-YAU EQUATION

by Hongyu WANG & Peng ZHU (*)

Abstract. — Dealing with the generalized Calabi-Yau equation proposed by Gromov on closed almost-Kähler manifolds, we extend to arbitrary dimension a non-existence result proved in complex dimension 2.

Résumé. — En travaillant sur l'équation de Calabi-Yau généralisée proposée par Gromov pour des variétés presque-Kalhériennes fermées, nous étendons le résultat de la non-existence prouvé en dimension complexe 2, à des dimensions arbitraires.

1. Introduction

The Calabi conjecture [2] asserts that any representative of the first Chern class of a closed Kähler manifold (M, ω) of real dimension $2n$ can be written as the Ricci curvature of a Kähler metric ω' cohomologous to ω. This conjecture was proved by S. T. Yau [14]. Yau's result is equivalent to finding a Kähler metric in a given Kähler class with a prescribed volume form. More precisely, by $\partial\bar{\partial}$-lemma on a Kähler manifold, this amounts to solve the complex Monge-Ampère equation:

$$(\omega + \sqrt{-1}\partial\bar{\partial}\phi)^n = e^F \omega^n, \tag{1.1}$$

for some real function ϕ with

$$\omega + \sqrt{-1}\partial\bar{\partial}\phi > 0,$$

where $F \in C^\infty(M; \mathbb{R})$ satisfies that

$$\int_M e^F \omega^n = \int_M \omega^n.$$

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We call Equation (1.1) Calabi-Yau equation. Yau solved this equation by virtue of the continuity method. Consider the system of equations obtained by replacing F by $tF+c_t$, where c_t is a constant for $t \in [0,1]$ ($c_0 = c_1 = 0$). An openness argument is obtained by the implicit function theorem and a closeness is derived by a priori estimate. Similar questions are proposed in symplectic manifolds in different cases and studied by many authors [3, 4, 10, 12].

Suppose that (M, ω) is a closed $2n$-dimensional symplectic manifold with a volume form $\sigma \in [\omega^n]$. In [9], Moser proved that there exists a symplectic form $\omega' \in [\omega]$ which is symplectomorphic to ω satisfying that

$$\omega'^n = \sigma. \tag{1.2}$$

That is, there exists a diffeomorphism,

$$f : M \to M,$$

isotopic to the identity such that

$$f^* \omega^n = \sigma.$$

It is easy to see that $\omega' = f^*\omega$ is cohomologous to ω and $f^*\omega$ satisfies Equation (1.2). This is independent of the almost complex structures. It is well known that there are many almost complex structures which are compatible with the symplectic form ω and they form a contractible space [1, 8]. Here ω is compatible with an almost complex structure J, that is, at every point $p \in M$, $\omega_p(v, Jv) > 0$ for every nonzero vector $v \in T_pM$ and $\omega(JY, JZ) = \omega(Y, Z)$ for all vector fields Y and Z.

Now we suppose that (M, g, J, ω) is an almost Kähler manifold, that is, the symplectic form ω is compatible with the almost complex structure J and $g(X, Y) = \omega(X, JY)$. Obviously, g is a Riemannian metric. Consider the existence of the solution of Equation (1.2) in the following form,

$$\omega' = \omega(\phi) \equiv \omega + d(Jd\phi), \tag{1.3}$$

for

$$\phi \in C^\infty(M; \mathbb{R}).$$

Here

$$(Jd\phi)(X) = d\phi(JX),$$

and ω' tames J, that is, at every point $p \in M$, $\omega'_p(v, Jv) > 0$ for every nonzero vector $v \in T_pM$. More precisely, this question can be expressed as follows:

Does there exist a smooth function,

$$\phi \in C^\infty(M; \mathbb{R}),$$

satisfying the following conditions?

$$
\begin{cases} \omega'^n = \sigma, \\ \omega' = \omega(\phi) \equiv \omega + dJd\phi \text{ tames } J, \end{cases} \tag{1.4}
$$

where σ is a given volume form in $[\omega^n]$. Following a suggestion of M. Gromov, P. Delanoë studied this problem in [3].

We call Equation (1.4) generalized Calabi-Yau equation. In particular, if (M, g, J, ω) is Kähler (that is, J is integrable), then,

$$
\begin{aligned}
d(Jd\phi) &= d(J(\partial + \bar{\partial})\phi) \\
&= \sqrt{-1}(\partial + \bar{\partial})(\partial - \bar{\partial})\phi \\
&= 2\sqrt{-1}\bar{\partial}\partial\phi.
\end{aligned}
$$

This implies that the form ω' is a Kähler form. So Equation (1.4) reduces to the Calabi-Yau equation on Kähler manifolds, which was solved by S. T. Yau [14].

We define an operator F from $C^\infty(M;\mathbb{R})$ to $C^\infty(M;\mathbb{R})$ as follows:

$$
\phi \mapsto F(\phi),
$$

where

$$
F(\phi)\omega^n = (\omega(\phi))^n. \tag{1.5}
$$

Therefore, Equation (1.4) is equivalent to the following problem:

Suppose that a positive function $f \in C^\infty(M;\mathbb{R})$ satisfies the following equality,

$$
\int_M \omega^n = \int_M f\omega^n.
$$

Does there exist a solution of $\phi \in C^\infty(M;\mathbb{R})$ which satisfies the following equation?

$$
\begin{cases} F(\phi) = f, \\ \omega(\phi) \equiv \omega + dJd\phi \text{ tames } J. \end{cases} \tag{1.6}
$$

We need some notations in [3]:

Definition 1.1. — *Suppose that (M, g, J, ω) is an almost Kähler manifold of real dimension $2n$. The sets A, B, A_+ and B_+ are defined as follows:*

$$\begin{aligned} A &:= \{\phi \in C^\infty(M;\mathbb{R}) \mid \int_M \phi\, \omega^n = 0\}; \\ B &:= \{f \in C^\infty(M;\mathbb{R}) \mid \int_M f\omega^n = \int_M \omega^n\}; \\ A_+ &:= A \cap \{\phi \in C^\infty(M;\mathbb{R}) \mid \omega(\phi) \text{ tames } J\}; \\ B_+ &:= B \cap \{f \in C^\infty(M;\mathbb{R}) \mid f > 0\}. \end{aligned}$$

Note that A_+ can be regarded as a convex open set of symplectic potential functions (analogue of Kähler potential functions). Restricting the operator F to A_+, we get

$$F(A_+) \subset B_+.$$

Thus, the existence of a solution to Equation (1.6) is equivalent to that the restricted operator

$$F|_{A_+} : A_+ \to B_+, \tag{1.7}$$

is surjective.

Suppose that (M, g, J, ω) is a closed almost Kähler manifold of real dimension $2n$. If J is integrable, then $F : A_+ \to B_+$ is surjective. Conversely, if $F : A_+ \to B_+$ is bijective and $n = 2$, then J is integrable. Delanoë [3] proved this result by constructing a suitable smooth function ϕ_0 on the boundary of A_+ and he conjectured [3, Conjecture p.837] that the same result holds when $n \geqslant 2$.

In this paper, we prove Delanoë's conjecture. Since an oriented surface is Kähler, we always consider the case $n \geqslant 2$.

Theorem 1.2. — *Suppose that (M, g, J, ω) is a closed almost Kähler manifold of real dimension $2n$. Then the restricted operator $F|_{A_+} : A_+ \to F(A_+)$ is a diffeomorphism. Moreover, the restricted operator $F|_{A_+} : A_+ \to B_+$ is a surjectivity map if and only if J is integrable.*

Remark 1.3. — 1) S. K. Donaldson gave a conjecture in [4]. Suppose that J is an almost complex structure on a closed symplectic manifold (M, ω) of dimension 4, and is tamed by the symplectic form ω. Let σ be a smooth volume form on M with

$$\int_M \sigma = \int_M \omega^2.$$

Let $\tilde{\omega}$ be an almost Kähler form corresponding to the almost complex structure J (that is, $\tilde{\omega}$ is a symplectic form compatible with J) with $[\tilde{\omega}] = [\omega]$, and solve the Calabi-Yau equation

$$\tilde{\omega}^2 = \sigma.$$

Donaldson conjectured that there are C^∞ a prior bounds on $\tilde{\omega}$ depending only on ω, J and σ. This is related to his broader program [4].

2) V. Tosatti, B. Weinkove and S. Y. Yau [10] proved that Donaldson's conjecture on estimates for Calabi-Yau equation in terms of a taming symplectic form can be reduced to an integral estimate of a scalar potential function. We will study Donaldson's conjecture in a future paper [11].

This article is organized as follows. Section 2 contains a local theory of the operator F (Equality (1.5)) and the local expression of $\omega(\phi)$ (Equality (1.3)) by choosing the second canonical connection. Section 3 gives the proof of Theorem 1.2. In the following, we simply call Equation (1.4) the Calabi-Yau equation.

2. Local theory of Calabi-Yau equation

This section is devoted to a local theory of Calabi-Yau equation. Let (M, g, J, ω) be a closed almost Kähler manifold of real dimension $2n$. The $\mathbb{C}$-extension of the almost complex structure J on the complexified tangent bundle $TM \otimes \mathbb{C}$ is defined by

$$J(X + iY) = JX + iJY$$

for $X, Y \in TM$. Obviously,

$$J^2 = -Id$$

on $TM \otimes \mathbb{C}$. Therefore,

$$TM \otimes \mathbb{C} = T^{1,0} \oplus T^{0,1},$$

where

$$T^{1,0} = \{X \in TM \otimes \mathbb{C} : JX = iX\}.$$

J induces an almost complex structure on $\wedge^k T^*M \otimes \mathbb{C}$. Hence

$$\wedge^2 T^*M \otimes \mathbb{C} = T_{2,0} + T_{1,1} + T_{0,2},$$

where $T_{p,q}$ is the space of (p,q)-forms. Let $P_{2,0}$, $P_{1,1}$ and $P_{0,2}$ be the projections to $T_{2,0}$, $T_{1,1}$ and $T_{0,2}$, respectively. For each $\phi \in C^\infty(M; \mathbb{R})$, set

$$\tau(\phi) = P_{2,0}(\omega(\phi)), \tag{2.1}$$

$$H(\phi) = P_{1,1}(\omega(\phi)). \tag{2.2}$$

Proposition 2.1. — *Suppose that (M, g, J, ω) is a closed almost Kähler manifold of real dimension $2n$. Then $\omega(\phi)$ tames J if and only if $H(\phi)$ tames J.*

Proof. — From the definition of $\omega(\phi)$ and $H(\phi)$, we have

$$\begin{aligned}\omega(\phi)(X, JX) &= (\tau(\phi) + H(\phi) + \overline{\tau(\phi)})(X, JX)\\ &= H(\phi)(X, JX).\end{aligned}$$

The last equality holds since $\tau(\phi)(X, JX) = 0$, for every vector field $X \in TM$. □

Remark 2.2. — Observe that the taming $(1,1)$-form $H(\phi)$ in Proposition 2.1 is not necessarily closed; it is, when J is integrable (Kähler case) in which case it coincides with $\omega(\phi)$.

To compute $\tau(\phi)$ and $H(\phi)$, we choose a local coordinate system and the second canonical connection on an almost Hermitian manifold, (M, g, J, ω) of real dimension $2n$, (that is, a Riemannian metric g is J-invariant and $\omega(X, Y) = g(JX, Y)$.).

There exists a local orthonormal basis $\{\epsilon_1, J\epsilon_1, \cdots, \epsilon_n, J\epsilon_n\}$ of M. Set $e_j = \frac{\epsilon_j - \sqrt{-1}J\epsilon_j}{\sqrt{2}}$. Then $\{e_1, \cdots, e_n, \bar{e}_1, \cdots, \bar{e}_n\}$ is a local basis of $TM \otimes \mathbb{C}$ and $g(e_i, e_j) = g(\bar{e}_i, \bar{e}_j) = 0$, $g(e_i, \bar{e}_j) = \delta_{ij}$. Obviously, $\{e_1, \cdots, e_n\}$ is a local basis of $T^{1,0}$. Let $\{\theta^1, \cdots, \theta^n\}$ be its dual basis. Obviously, after complexification, J, g, and ω can be expressed locally as follows,

$$J = \sqrt{-1}(\theta^\alpha \otimes e_\alpha - \bar{\theta}^\alpha \otimes \bar{e}_\alpha),$$

$$g = \sum_{i=1}^{n} (\theta^\alpha \otimes \bar{\theta}^\alpha + \bar{\theta}^\alpha \otimes \theta^\alpha),$$

$$\omega = \sqrt{-1} \sum_{\alpha=1}^{n} \theta^\alpha \wedge \bar{\theta}^\alpha. \tag{2.3}$$

Now, although this choice is not essential for the proof, we choose a different connection than the one used in [3], namely we choose the second canonical connection ∇^1 (an almost Hermitian connection) on an almost Kähler manifold [5, 6, 10, 15]. Note that the second canonical connection on the almost Hermitian manifold (M, g, J, ω) is an affine connection ∇^1 satisfying

$$\nabla^1 g = 0 = \nabla^1 J, \tag{2.4}$$

and the $(1,1)$-component of its torsion vanishes [6, 10, 15]. We must redo with ∇^1 calculations analogous to those presented in [3, Appendix 1]. From Equality (2.4), we obtain that

$$\begin{gathered}\nabla^1 : T^{1,0} \to (T^*M \otimes \mathbb{C}) \otimes T^{1,0},\\ e_\alpha \mapsto \omega_\alpha^\beta \otimes e_\beta.\end{gathered}$$

The second canonical connection, ∇^1, induces

$$\begin{gathered}\nabla^1 : T_{1,0} \to (T^*M \otimes \mathbb{C}) \otimes T_{1,0},\\ \theta^\alpha \mapsto -\omega_\beta^\alpha \otimes \theta^\beta.\end{gathered}$$

Let $\phi \in C^\infty(M;\mathbb{R})$. Then

$$d\phi = \phi_\alpha \theta^\alpha + \bar{\phi}_\alpha \bar{\theta}^\alpha. \tag{2.5}$$

Thus,

$$Jd\phi = \sqrt{-1}(\phi_\alpha \theta^\alpha - \bar{\phi}_\alpha \bar{\theta}^\alpha). \tag{2.6}$$

Let

$$d\phi_\alpha - \phi_\beta \omega_\alpha^\beta = \phi_{\alpha\beta}\theta^\beta + \phi_{\alpha\bar{\beta}}\bar{\theta}^\beta. \tag{2.7}$$

Then, equality $d^2\phi = 0$ and Equation (2.5) imply that

$$d(\phi_\alpha \theta^\alpha) + d(\bar{\phi}_\alpha \bar{\theta}^\alpha) = 0. \tag{2.8}$$

Let

$$\Theta^\alpha = d\theta^\alpha + \omega_\beta^\alpha \wedge \theta^\beta \tag{2.9}$$

be the torsion of the second canonical connection ∇^1. Thus, Θ^α contains $(2,0)$ and $(0,2)$ components only. Therefore,

$$\Theta^\alpha = T_{\beta\gamma}^\alpha \theta^\beta \wedge \theta^\gamma + N_{\bar{\beta}\bar{\gamma}}^\alpha \bar{\theta}^\beta \wedge \bar{\theta}^\gamma, \tag{2.10}$$

with $T_{\beta\gamma}^\alpha = -T_{\gamma\beta}^\alpha$ and $N_{\bar{\beta}\bar{\gamma}}^\alpha = -N_{\bar{\gamma}\bar{\beta}}^\alpha$. Indeed, the $(0,2)$ component of the torsion is independent of the choice of a metric and can be regarded as the Nijenhuis tensor, N, of the almost complex structure J [10, 15]. From

Equations (2.3) and (2.9), we obtain that

$$\begin{aligned}
d\omega &= d(\sqrt{-1}\sum_{\alpha=1}^{n}\theta^\alpha\wedge\bar\theta^\alpha)\\
&= \sqrt{-1}\sum_{\alpha=1}^{n}(d\theta^\alpha\wedge\bar\theta^\alpha-\theta^\alpha\wedge d\bar\theta^\alpha)\\
&= \sqrt{-1}\sum_{\alpha=1}^{n}(\Theta^\alpha\wedge\bar\theta^\alpha-\theta^\alpha\wedge\bar\Theta^\alpha)\\
&= \sqrt{-1}\sum_{\alpha,\beta,\gamma=1}^{n}(T^\alpha_{\beta\gamma}\bar\theta^\alpha\wedge\theta^\beta\wedge\theta^\gamma-\bar T^\alpha_{\beta\gamma}\theta^\alpha\wedge\bar\theta^\beta\wedge\bar\theta^\gamma\\
&\quad+N^\alpha_{\bar\beta\bar\gamma}\bar\theta^\alpha\wedge\bar\theta^\beta\wedge\bar\theta^\gamma-\bar N^\alpha_{\bar\beta\bar\gamma}\theta^\alpha\wedge\theta^\beta\wedge\theta^\gamma).
\end{aligned}\tag{2.11}$$

Suppose that (M,g,J,ω) is an almost Kähler manifold. By Equation (2.11), we have

$$T^\alpha_{\beta\gamma}=0,$$

and

$$N^\gamma_{\bar\alpha\bar\beta}+N^\alpha_{\bar\beta\bar\gamma}+N^\beta_{\bar\gamma\bar\alpha}=0.$$

Combining with Equations (2.6)–(2.10), we obtain that

$$\begin{aligned}
dJd\phi &= 2\sqrt{-1}d(\phi_\alpha\theta^\alpha)\\
&= 2\sqrt{-1}(d\phi_\alpha\wedge\theta^\alpha+\phi_\alpha d\theta^\alpha)\\
&= 2\sqrt{-1}(\phi_{\alpha\beta}\theta^\beta\wedge\theta^\alpha+\phi_{\alpha\bar\beta}\bar\theta^\beta\wedge\theta^\alpha+\phi_\alpha\Theta^\alpha)\\
&= 2\sqrt{-1}(\phi_{\alpha\beta}\theta^\beta\wedge\theta^\alpha+\phi_{\alpha\bar\beta}\bar\theta^\beta\wedge\theta^\alpha+\phi_\alpha N^\alpha_{\bar\beta\bar\gamma}\bar\theta^\beta\wedge\bar\theta^\gamma).
\end{aligned}$$

The reality of $dJd\phi$ implies that

$$\phi_{\alpha\bar\beta}=\overline{\phi_{\beta\bar\alpha}},$$

and

$$\phi_{\alpha\beta}\theta^\beta\wedge\theta^\alpha=-\overline{\phi_\alpha N^\alpha_{\bar\beta\bar\gamma}\bar\theta^\beta\wedge\bar\theta^\gamma}.$$

Set

$$h(\phi)(X,Y)=\frac{1}{2}(H(\phi)(X,JY)+H(\phi)(Y,JX)).\tag{2.12}$$

Then,

$$\tau(\phi)=-2\sqrt{-1}\bar\phi_\alpha\bar N^\alpha_{\bar\beta\bar\gamma}\theta^\beta\wedge\theta^\gamma,\tag{2.13}$$

$$H(\phi)=\sqrt{-1}(\delta_{\alpha\bar\beta}-2\phi_{\alpha\bar\beta})\theta^\alpha\wedge\bar\theta^\beta,\tag{2.14}$$

$$h(\phi) = (\delta_{\alpha\bar{\beta}} - 2\phi_{\alpha\bar{\beta}})(\theta^{\alpha} \otimes \bar{\theta}^{\beta} + \bar{\theta}^{\beta} \otimes \theta^{\alpha}), \tag{2.15}$$

$$dJd\phi = -2\sqrt{-1}(\bar{\phi}_{\alpha}\bar{N}^{\alpha}_{\bar{\beta}\bar{\gamma}}\theta^{\beta} \wedge \theta^{\gamma} + \phi_{\alpha\bar{\beta}}\theta^{\alpha} \wedge \bar{\theta}^{\beta} - \phi_{\alpha}N^{\alpha}_{\bar{\beta}\bar{\gamma}}\bar{\theta}^{\beta} \wedge \bar{\theta}^{\gamma}).$$

Where $\delta_{\alpha\bar{\beta}} = 1$ if $\alpha = \beta$; otherwise, $\delta_{\alpha\bar{\beta}} = 0$. Hence, we have the following lemma:

LEMMA 2.3. — *Suppose that (M, g, J, ω) is a $2n$-dimensional almost Kähler manifold, and ϕ is a smooth real function on M. Then,*

$$\omega(\phi) = \tau(\phi) + H(\phi) + \bar{\tau}(\phi),$$

where $\tau(\phi)$ and $H(\phi)$ are given by Equations (2.13) *and* (2.14), *respectively.*

Let $[\frac{n}{2}]$ denote the integral part of a positive number $\frac{n}{2}$. Following [3], let us define operators F_j $(j = 0, 1, \cdots, [\frac{n}{2}])$ as follows:

$$F_j : C^{\infty}(M; \mathbb{R}) \to C^{\infty}(M; \mathbb{R}),$$
$$\phi \mapsto F_j(\phi),$$

where

$$F_j(\phi)\omega^n = \frac{n!}{j!^2(n-2j)!}(\tau(\phi) \wedge \overline{\tau(\phi)})^j \wedge H(\phi)^{n-2j}. \tag{2.16}$$

Then

$$\sum_{j=0}^{[\frac{n}{2}]} F_j(\phi)\omega^n = (\tau(\phi) + H(\phi) + \bar{\tau}(\phi))^n = F(\phi)\omega^n.$$

The following result was proved in [3, Proposition 5]:

PROPOSITION 2.4. — *Suppose that (M, g, J, ω) is a closed almost Kähler manifold of real dimension $2n$. Then*

$$F(\phi) = F_0(\phi) + \cdots + F_{[\frac{n}{2}]}(\phi).$$

If $\phi \in A_+$, then $F_0(\phi) > 0$ and $F_j(\phi) \geqslant 0$, for $j = 1, 2, \cdots, [\frac{n}{2}]$.

For any $\phi \in A_+$, it is easy to see that the tangent space at ϕ, $T_{\phi}A_+$, is A. For $u \in T_{\phi}A_+ = A$, define $L(\phi)(u)$ by,

$$L(\phi)(u) = \frac{d}{dt}F(\phi + tu)|_{t=0}.$$

For a local theory of Calabi-Yau equation, we should study the tangent map of the restricted operator $F|_{A_+}$. The following result was obtained in

[3, Proposition 1]:

Lemma 2.5. — *Suppose that* (M, g, J, ω) *is a closed almost Kähler manifold of dimension* $2n$. *Then the restricted operator*

$$F : A_+ \to B_+$$

is elliptic type on A_+. *Moreover, the tangent map,*

$$dF_\phi = L(\phi)$$

of F *at* $\phi \in A_+$ *is a linear elliptic differential operator of second order without zero-th term.*

The following result was proved in [3, Theorem 2]:

Proposition 2.6. — *Suppose that* (M, g, J, ω) *is a closed almost Kähler manifold of dimension* $2n$. *Then the restricted operator*

$$F : A_+ \to F(A_+)$$

is a diffeomorphic map.

Remark 2.7. — From the definition of $F(\phi)$ (Equality (1.5)), we see that

$$F(0)\omega^n = (\omega(0))^n = \omega^n.$$

Thus, $F(0) = 1$. Therefore, Proposition 2.6 implies that the solution to Equation (1.4) exists and is unique if σ is a small perturbation of ω^n.

3. Global theory of Calabi-Yau equation

In this section, we give a proof of the main theorem. Let (M, g, J, ω) be a closed almost Kähler manifold of real dimension $2n$. If the almost complex structure J is integrable, then the surjectivity of the restricted operator (1.7) is equivalent to the existence of solutions of Calabi-Yau equation on Kähler manifolds which was solved by S. T. Yau [14]. Suppose that J is not integrable. We will prove that the restricted operator (1.7) is not surjective. For the case $n = 2$, it was proved by P. Delanoë [3]. Now we give the proof for the case $n \geqslant 2$. More precisely, as in [3, p.835], we want to construct a function ϕ_0 belonging to the boundary of the convex open set A_+ such that $F(\phi_0) \in B_+$. First, let us give the definition of a pseudo holomorphic function [1, 7].

Definition 3.1. — *A smooth complex-valued function* f *on a manifold* M *is a pseudo holomorphic function at some point* $p \in M$ *if* $df \circ J = i df$ *at* p.

To construct ϕ_0 belonging to the boundary of A_+ and satisfying inequality $F(\phi_0) > 0$, we need the following lemmas:

LEMMA 3.2. — *Let (M, J) be an almost complex manifold. Suppose that f is a pseudo holomorphic function at some point $p \in M$. Then, for all vector fields X, Y,*

$$df(N(X, Y)) = 0,$$

at p, where N is the Nijenhuis tensor.

Proof. — If $X, Y \in T^{1,0}$, then

$$\begin{aligned} N(X, Y) &= [JX, JY] - J[X, JY] - J[JX, Y] - [X, Y] \\ &= -2[X, Y] - 2iJ[X, Y]. \end{aligned}$$

Thus

$$\begin{aligned} JN(X, Y) &= -2J[X, Y] - 2iJ^2[X, Y] \\ &= -iN(X, Y). \end{aligned} \tag{3.1}$$

If

$$df \circ J = idf,$$

then

$$df \circ J(N(X, Y)) = idf(N(X, Y)).$$

Combining with Equality (3.1), we get

$$df(N(X, Y)) = 0,$$

at $p \in M$. If $X, Y \in T^{0,1}$, then

$$df(N(X, Y)) = \overline{df(N(\bar{X}, \bar{Y}))} = 0,$$

at $p \in M$. If $X \in T^{0,1}$ and $Y \in T^{1,0}$, then $N(X, Y) = 0$. Hence,

$$df(N(X, Y)) = 0,$$

at $p \in M$. The proof is completed. □

Remark 3.3. — Note that if there exist n pseudo holomorphic functions on a real $2n$-dimensional almost Hermitian manifold (M, g, J) which are independent at some point $p \in M$, then the Nijenhuis tensor N identically vanishes at p. This means that an integrable complex structure is one with many (pseudo)-holomorphic functions. It is a hard theorem (Newlander-Nirenberg integrability theorem for almost complex structures) that the converse is also true. In general, an almost complex manifold has no holomorphic functions at all. On the other hand, it has a lot of pseudo-holomorphic curves (i.e., maps $f : \mathbb{C} \to (M^{2n}, g, J)$ such that $df \circ i = J \circ df$) [7].

Let (M, g, J, ω) be a closed almost Kähler manifold of real dimension $2n$, where the almost complex structure J is not integrable. Hence there exists a neighborhood U_0 of some point $p_0 \in M$ where the Nijenhuis tensor N does not vanish. Pick $f = \phi_1 + i\phi_2$, with ϕ_1 and ϕ_2 real-valued, a complex function on M which is not holomorphic on U_0. By Lemma 3.2, it satisfies

$$df(N(X,Y)) \neq 0,$$

on the neighborhood U_0. Without loss generality, we can suppose that for some $1 \leqslant \alpha < \beta \leqslant n$,

$$d\phi_1(N(e_\alpha, e_\beta)) = d\phi_1(e_{\bar\gamma} N^{\bar\gamma}_{\alpha\beta}) \neq 0,$$

on the neighborhood U_0 of p_0. By Equality (2.15), there exists a suitable (non-zero) constant c such that $h(c\phi_1)$, defined in Section 2 (Equalities (2.12) and (2.15)), is a positive definite Hermitian matrix on M. Set $\phi = c\phi_1$. Then $\tau(\phi)$, defined in Section 2 (Equalities (2.1) and (2.13)) is non-zero on the neighborhood U_0 of p_0. So we obtain the following lemma:

Lemma 3.4. — *Suppose that (M, g, J, ω) is a closed almost Kähler manifold of real dimension $2n$, where the almost complex structure J is not integrable. There exists a function $\phi \in C^\infty(M; \mathbb{R})$ satisfying the following conditions:* (1) *$h(\phi)$ is a positive definite Hermitian matrix on M;* (2) *there is a neighborhood U_0 of a point $p_0 \in M$ such that $\tau(\phi)$ is non-zero on the neighborhood U_0.*

We can suppose that

$$H(\phi) = \sqrt{-1}\lambda_\alpha(0)\theta^\alpha \wedge \bar\theta^\alpha \tag{3.2}$$

at p_0 [13]. By Lemma 3.4, we have that $\lambda_\alpha(0) > 0$, for $\alpha = 1, \cdots, n$. There exists a local coordinate system $\varphi^{-1} : U_0 \to \mathbb{C}^n$ such that $\varphi(0) = p_0$ and

$$\frac{\partial}{\partial z_\alpha} = e_\alpha \tag{3.3}$$

at p_0, for $\alpha = 1, \cdots, n$. Without loss generality, we can suppose that there is a polydisk $\triangle$ with center p_0:

$$\triangle = \{(z_1, \cdots, z_n) \in \mathbb{C}^n : |z_i| \leqslant 1, 1 \leqslant i \leqslant n\} \subset \varphi^{-1}(U_0)$$

and there is $\epsilon_1 > 0$ such that, on $\varphi(\triangle)$,

$$|\tau_{12}(\phi)| \geqslant \epsilon_1 > 0. \tag{3.4}$$

Let $\tilde{\eta}(r)$ be a C^2 cut-off function such that

$$\tilde{\eta}(r) = \begin{cases} 1 & r \leqslant \frac{1}{2}, \\ 0 & r \geqslant 1. \end{cases} \tag{3.5}$$

Given a real number $R > 2$, let

$$\Phi_R(z) = \sum_{i=1}^{2} \frac{\lambda_i(0)}{2} |z_i|^2 \cdot \tilde{\eta}(|R^2 z_i|) \tag{3.6}$$

and

$$\eta_R(z) = \tilde{\eta}(|Rz_1|) \cdots \tilde{\eta}(|Rz_n|). \tag{3.7}$$

Define a function ψ_R as follows,

$$\psi_R(z) = \Phi_R(z)\eta_R(z), \tag{3.8}$$

where $z \in \triangle$. In the sequel, for short, we will abusively denote by z_i, η_R, Φ_R, ψ_R, σ, the corresponding functions on $U_0 \subset M$ obtained by composition with φ^{-1}. With this abuse of notation, recalling (2.7), we obtain by a direct calculation that

$$P_{1,1}(dJd(\psi_R)) = -2\sqrt{-1} \sum_{1 \leqslant \alpha,\beta \leqslant n} (\psi_R)_{\alpha\bar{\beta}} \theta^\alpha \wedge \bar{\theta}^\beta, \tag{3.9}$$

with

$$(\psi_R)_{\alpha\bar{\beta}} = \bar{e}_\beta e_\alpha \psi_R - (\psi_R)_\gamma \Gamma^\gamma_{\alpha\bar{\beta}}, \tag{3.10}$$

where $\Gamma^\gamma_{\alpha\bar{\beta}} = \omega^\gamma_\alpha(\bar{e}_\beta)$ are connection coefficients of the second canonical connection bounded on $\triangle$. Set

$$\triangle_R = \{(z_1, \cdots, z_n) : |z_i| \leqslant \frac{1}{R}, 1 \leqslant i \leqslant n\}.$$

Obviously, $\triangle_R \subset \triangle$. Henceforth, we will freely denote by the same letter C various positive constants independent of $R > 2$.

Lemma 3.5. — *Suppose that (M, g, J, ω) is a closed almost Kähler manifold of real dimension $2n$, and $\tilde{\eta}$, ψ_R, $\triangle$ and U_0 are defined as before. There exists a constant C (depending only on $\lambda_1(0)$, $\lambda_2(0)$, $\tilde{\eta}$, ω, g and J) such that,* (1) *on $\triangle$, for $1 \leqslant \alpha \leqslant n$,*

$$|(\psi_R)_\alpha| \leqslant \frac{C}{R^2};$$

(2) *on $\triangle$, for $1 \leqslant \alpha = \beta \leqslant 2$,*

$$|(\psi_R)_{\alpha\bar{\beta}}| \leqslant C;$$

otherwise,

$$|(\psi_R)_{\alpha\bar{\beta}}| \leqslant C(\sigma + \frac{1}{R}),$$

where σ is a continuous function (independent of R) and $\sigma(0) = 0$. Moreover, at the center of $\triangle$, $|(\psi_R)_{\alpha\bar{\beta}}| = \frac{\lambda_\alpha(0)}{2}$, for $1 \leqslant \alpha = \beta \leqslant 2$; otherwise, $|(\psi_R)_{\alpha\bar{\beta}}| = 0$.

Proof. — From (3.5), we can see that ψ_R is zero on $\triangle \setminus \triangle_R$. Hence it suffices to prove that the desired estimate holds on $\triangle_R$. Note that, for $1 \leqslant \alpha, \beta, i \leqslant n$, the functions $|e_\alpha z_i|$, $|e_\alpha \bar{z}_i|$, $|\bar{e}_\beta z_i|$, $|\bar{e}_\beta \bar{z}_i|$, $|\bar{e}_\beta e_\alpha z_i|$ and $|\bar{e}_\beta e_\alpha \bar{z}_i|$ are continuous and bounded on Δ. From (3.6) and (3.7), we have on $\triangle_R$

$$|\eta_R| \leqslant 1, \quad |\Phi_R| \leqslant CR^{-4}. \tag{3.11}$$

A direct computation shows that

$$\begin{aligned} e_\alpha \eta_R(z) = \sum_{i=1}^{n} \big\{ & \tilde{\eta}(|Rz_1|) \cdots \widehat{\tilde{\eta}(|Rz_i|)} \cdots \tilde{\eta}(|Rz_n|) \\ & \cdot \tilde{\eta}'(|Rz_i|) \cdot [R(2|z_i|)^{-1}((e_\alpha z_i)\bar{z}_i + (e_\alpha \bar{z}_i) z_i)] \big\}, \end{aligned}$$

where the hat means that the term is omitted. It is easy to see that

$$|e_\alpha \eta_R| \leqslant CR, \quad |\bar{e}_\alpha \eta_R| \leqslant CR. \tag{3.12}$$

Since

$$\begin{aligned} e_\alpha \Phi_R(z) = 4^{-1} \sum_{i=1}^{2} \big\{ & \lambda_i(0)[(e_\alpha z_i)\bar{z}_i + (e_\alpha \bar{z}_i) z_i] \\ & \cdot [2\tilde{\eta}(|R^2 z_i|) + R^2 |z_i| \tilde{\eta}'(|R^2 z_i|)] \big\}, \end{aligned}$$

we obtain that

$$|e_\alpha \Phi_R| \leqslant CR^{-2}, \quad |\bar{e}_\alpha \Phi_R| \leqslant CR^{-2}. \tag{3.13}$$

Thus

$$|(\psi_R)_\alpha| \leqslant |(e_\alpha \Phi_R)\eta_R| + |\Phi_R(e_\alpha \eta_R)| \leqslant CR^{-2}. \tag{3.14}$$

Now we consider

$$\begin{aligned} \bar{e}_\beta e_\alpha \psi_R = & \Phi_R(\bar{e}_\beta e_\alpha \eta_R) + (e_\alpha \Phi_R)(\bar{e}_\beta \eta_R) \\ & + (\bar{e}_\beta \Phi_R)(e_\alpha \eta_R) + (\bar{e}_\beta e_\alpha \Phi_R)\eta_R. \end{aligned} \tag{3.15}$$

A direct computation shows that

$$\begin{aligned}\bar{e}_\beta e_\alpha \eta_R(z) = \sum_{1\leqslant i\neq j\leqslant n} &\big\{\tilde{\eta}(|Rz_1|)\cdots\widehat{\tilde{\eta}(|Rz_i|)}\cdots\widehat{\tilde{\eta}(|Rz_j|)}\cdots\tilde{\eta}(|Rz_n|)\\ &\cdot\tilde{\eta}'(|Rz_i|)\cdot[R(2|z_i|)^{-1}((e_\alpha z_i)\bar{z}_i+(e_\alpha\bar{z}_i)z_i)]\\ &\cdot\tilde{\eta}'(|Rz_j|)\cdot[R(2|z_j|)^{-1}(\bar{e}_\beta z_j)\bar{z}_j+(\bar{e}_\beta\bar{z}_j)z_j)]\big\}\\ &+\sum_{1\leqslant i\leqslant n}\big\{\tilde{\eta}(|Rz_1|)\cdots\widehat{\tilde{\eta}(|Rz_i|)}\cdots\tilde{\eta}(|Rz_n|)\\ &\cdot\big[\tilde{\eta}''(|Rz_i|)R^2(2|z_i|)^{-2}\cdot((e_\alpha z_i)\bar{z}_i+(e_\alpha\bar{z}_i)z_i)\\ &\cdot((\bar{e}_\beta z_i)\bar{z}_i+(\bar{e}_\beta\bar{z}_i)z_i)+4^{-1}R\tilde{\eta}'(|Rz_i|)\\ &\cdot\big(2(\bar{e}_\beta e_\alpha z_i)|z_i|^{-1}\bar{z}_i+(e_\alpha z_i)(\bar{e}_\beta\bar{z}_i|z_i|^{-1}-(\bar{e}_\beta z_i)|z_i|^{-3}\bar{z}_i^2)\\ &+2(e_{\bar{\beta}}e_\alpha\bar{z}_i)|z_i|^{-1}z_i+(e_\alpha\bar{z}_i)(\bar{e}_\beta z_i|z_i|^{-1}-(\bar{e}_\beta\bar{z}_i)|z_i|^{-3}{z_i}^2)\big)\big]\big\}.\end{aligned}$$

Thus,

$$|\bar{e}_\beta e_\alpha \eta_R| \leqslant CR^2. \tag{3.16}$$

A direct computation shows that

$$\begin{aligned}\bar{e}_\beta e_\alpha\Phi_R = 4^{-1}\sum_{i=1}^{2}\lambda_i(0)\big\{&\big[(\bar{e}_\beta e_\alpha z_i)\bar{z}_i+(e_\alpha z_i)(\bar{e}_\beta\bar{z}_i)+(\bar{e}_\beta e_\alpha\bar{z}_i)z_i\\ &+(e_\alpha\bar{z}_i)(\bar{e}_\beta z_i)\big]\cdot\big[2\tilde{\eta}(|R^2z_i|)+R^2|z_i|\tilde{\eta}'(|R^2z_i|)\big]\\ &+2^{-1}\big[(e_\alpha z_i)\bar{z}_i+(e_\alpha\bar{z}_i)z_i\big]\cdot\big[(\bar{e}_\beta z_i)\bar{z}_i+(\bar{e}_\beta\bar{z}_i)z_i\big]\\ &\cdot\big[3R^2|z_i|^{-1}\tilde{\eta}'(|R^2z_i|)+R^4\tilde{\eta}''(|R^2z_i|)\big]\big\}.\end{aligned}\tag{3.17}$$

If $\alpha=\beta=1$ or $\alpha=\beta=2$, then

$$|\bar{e}_\beta e_\alpha\Phi_R|\leqslant C; \tag{3.18}$$

If $3\leqslant\alpha=\beta\leqslant n$ or $1\leqslant\alpha\neq\beta\leqslant n$, then

$$|\bar{e}_\beta e_\alpha\Phi_R|\leqslant C(\sigma+R^{-2})\leqslant C(\sigma+R^{-1}), \tag{3.19}$$

where σ is the continuous function on $\triangle$ such that $\sigma(0)=0$, defined by:

$$\begin{aligned}\sigma(z)=\max\{&|(e_\alpha z_i)(\bar{e}_\beta z_i)|(z),|(e_\alpha z_i)(\bar{e}_\beta\bar{z}_i)|(z),|(e_\alpha\bar{z}_i)(\bar{e}_\beta z_i)|(z),\\ &|(e_\alpha\bar{z}_i)(\bar{e}_\beta\bar{z}_i)|(z): i=1,2, 1\leqslant\alpha\neq\beta\leqslant n\ \text{ or }\ 3\leqslant\alpha=\beta\leqslant n\}.\end{aligned}$$

From (3.11)-(3.19), we obtain that

$$|(\psi_R)_{\alpha\bar{\beta}}|\leqslant\begin{cases}C, & 1\leqslant\alpha=\beta\leqslant 2;\\ C(\sigma+\frac{1}{R}), & \text{otherwise,}\end{cases}$$

where σ is a continuous function and $\sigma(0)=0$. Besides, by (3.3) combined with (3.5) and (3.17), we obtain at $z=0$ that $|(\psi_R)_{\alpha\bar{\alpha}}|(0)=\frac{\lambda_\alpha(0)}{2}$, for $1\leqslant\alpha\leqslant 2$; otherwise, $|(\psi_R)_{\alpha\bar{\beta}}|(0)=0$. □

Lemma 3.6. — *Suppose that (M,g,J,ω) is a closed almost Kähler manifold of real dimension $2n$, where the almost complex structure J is not integrable. Suppose ϕ, ψ_R, φ, $\tilde{\eta}$, $\triangle$ and U_0 are defined as before. Let $s\in[0,1]$. Then there exists $R_1>2$ independent of $s\in[0,1]$ such that, for each $R\geqslant R_1$, on $\varphi(\triangle_R)$,*

$$|\tau_{12}(\phi+s\psi_R\circ\varphi^{-1})|\geqslant\frac{\epsilon_1}{2}>0,$$

where $\epsilon_1>0$ is the constant occuring in (3.4).

Proof. — Combining Lemma 3.5 with

$$\tau_{\alpha\beta}(\psi_R\circ\varphi^{-1})=-2\sqrt{-1}(\psi_R\circ\varphi^{-1})_\gamma\bar{N}^\gamma_{\bar{\alpha}\bar{\beta}},$$

on $\varphi(\triangle_R)$, we have the following inequality,

$$|\tau_{\alpha\beta}(\psi_R\circ\varphi^{-1})|\leqslant\frac{C}{R^2}\leqslant\frac{C}{R}.$$

By Lemma 3.4 and Inequality (3.4), we obtain,

$$\begin{aligned}|\tau_{12}(\phi+s\psi_R\circ\varphi^{-1})|&\geqslant|\tau_{12}(\phi)|-s|\tau_{12}(\psi_R\circ\varphi^{-1})|\\&\geqslant|\tau_{12}(\phi)|-|\tau_{12}(\psi_R\circ\varphi^{-1})|\\&\geqslant\epsilon_1-\frac{C}{R}.\end{aligned}$$

We can choose $R_1>2$ such that $\epsilon_1-\frac{C}{R_1}\geqslant\frac{\epsilon_1}{2}$. Thus, for each $R\geqslant R_1$, the desired estimate follows. □

For $s\in[0,1]$, we consider the function $\phi+s\psi_R\circ\varphi^{-1}$. If $s=0$, then

$$h(\phi+0\cdot\psi_R\circ\varphi^{-1})=h(\phi)\tag{3.20}$$

is a positive definite Hermitian metric on M. As observed above, we have

$$P_{1,1}(dJd(\psi_R\circ\varphi^{-1}))(p_0)=-\sqrt{-1}\sum_{1\leqslant\alpha\leqslant2}\lambda_\alpha(0)\theta^\alpha\wedge\bar{\theta}^\alpha,\tag{3.21}$$

which, recalling (3.2), yields:

$$h(\phi+1\cdot\psi_R\circ\varphi^{-1})\circ\varphi(0)=\sum_{3\leqslant\alpha\leqslant n}\lambda_\alpha(0)(\theta^\alpha\otimes\bar{\theta}^\alpha+\bar{\theta}^\alpha\otimes\theta^\alpha).\tag{3.22}$$

The latter is a semi-positive definite Hermitian matrix at the center of $\triangle_R$ (i.e., $p_0=\varphi(0)\in M$). Following [3, Definition 3], let us recall the notion of positivity amplitude:

Definition 3.7. — *Suppose that* (M, g, J, ω) *is a closed almost Hermitian manifold of real dimension* $2n$*. For each non-constant function* $\phi \in C^\infty(M;\mathbb{R})$*, the positivity amplitude of* ϕ *is the real number*

$$a_\omega(\phi) = \sup\{s \in (0,\infty) : \ h(s\phi) \text{ is a positive Hermitian matrix function on M}\},$$

where $h(\phi)$ *is defined in Section 2 (Equalities* (2.12) *and* (2.15)*).*

Since M is closed and the support of ψ_R is compact, the positivity amplitude is finite. Hence, by Lemmas 3.4–3.6 and Equalities (3.20), (3.22), we obtain the following proposition:

Proposition 3.8. — *Suppose that* (M, g, J, ω) *is a closed almost Kähler manifold of real dimension* $2n$*, where* J *is not integrable. Suppose* $\triangle_R$*,* ψ_R*,* R_1*,* ϕ*,* φ*,* U_0 *,*$\triangle$ *and* $a_\omega(\phi)$ *are defined as before and set*

$$\phi_R = \phi + a_{\omega(\phi)}(\psi_R \circ \varphi^{-1}) \cdot \psi_R \circ \varphi^{-1}.$$

Then, for each $R \geqslant R_1$*, we have: (1)* $0 < a_{\omega(\phi)}(\psi_R \circ \varphi^{-1}) \leqslant 1$*; (2)* ϕ_R *lies on the boundary of* A_+*; (3)* $h(\phi + a_{\omega(\phi)}(\psi_R \circ \varphi^{-1}) \cdot \psi_R \circ \varphi^{-1})$ *is a semi-positive definite Hermitian matrix function on* $\varphi(\triangle_R)$ *and a positive definite Hermitian matrix function on* $M \setminus \varphi(\triangle_R)$*.*

Now we will prove the following key lemma, sticking to the assumptions of Proposition 3.8:

Lemma 3.9. — *There exists* $R_0 \geqslant R_1$ *large enough such that the function* $\phi_0 = \phi_{R_0}$ *satisfies* $F(\phi_0) > 0$ *on* M*.*

Proof. — Note that

$$H(\phi_R) = H(\phi) + a_{\omega(\phi)}(\psi_R \circ \varphi^{-1}) \cdot P_{1,1}\big(dJd(\psi_R \circ \varphi^{-1})\big). \tag{3.23}$$

From (2.16), we have at p_0

$$F_1(\phi_R) = \sum_{1 \leqslant k \neq l \leqslant n} |\tau_{kl}(\phi_R)|^2 \cdot \det\big((h(\phi_R)_{\alpha\bar\beta})_{\alpha,\beta \in \{1,\ldots,\hat k,\ldots,\hat l,\ldots,n\}}\big).$$

From Equalities (3.2), (3.21), Lemma 3.6 and Proposition 3.8, we have, at p_0, that

$$F_1(\phi_R) \geqslant |\tau_{12}(\phi_R)|^2 \cdot \det((h(\phi_R)_{\alpha\bar\beta})_{3 \leqslant \alpha,\beta \leqslant n}) \geqslant \big(\frac{\epsilon_1}{2}\big)^2 \prod_{3 \leqslant \alpha \leqslant n} \lambda_\alpha(0) > 0.$$

Let

$$H(\phi_R) = H' + H'',$$

where

$$H' = \sqrt{-1} \sum_{1 \leqslant \alpha \neq \beta \leqslant n} H(\phi)_{\alpha\bar{\beta}} \theta^{\alpha} \wedge \bar{\theta}^{\beta} + \sqrt{-1} a_{\omega(\phi)} (\psi_R \circ \varphi^{-1})$$
$$\cdot \Big[\sum_{1 \leqslant \alpha \neq \beta \leqslant n} (\psi_R)_{\alpha\bar{\beta}} \theta^{\alpha} \wedge \bar{\theta}^{\beta} + \sum_{3 \leqslant \alpha \leqslant n,} (\psi_R)_{\alpha\bar{\alpha}} \theta^{\alpha} \wedge \bar{\theta}^{\alpha} \Big]$$

and

$$H'' = \sqrt{-1} \sum_{1 \leqslant \alpha \leqslant 2} H(\phi_R)_{\alpha\bar{\alpha}} \theta^{\alpha} \wedge \bar{\theta}^{\alpha} + \sqrt{-1} \sum_{3 \leqslant \alpha \leqslant n} H(\phi)_{\alpha\bar{\alpha}} \theta^{\alpha} \wedge \bar{\theta}^{\alpha}.$$

Note that $H(\phi)_{\alpha\bar{\alpha}}(p_0) = \lambda_{\alpha}(0)$, for $1 \leqslant \alpha \leqslant n$. $H(\phi)_{\alpha\bar{\beta}}(p_0) = 0$, for $\alpha \neq \beta$. Let us define a neighborhood of $0 \in \triangle$ by:

$$V_{\delta} = \{ z \in \triangle : |\sigma|(z) < \delta, |H(\phi)_{\alpha\bar{\alpha}}|(\varphi(z)) > \frac{\lambda_{\alpha}(0)}{2},$$
$$|H(\phi)_{\beta\bar{\gamma}}|(\varphi(z)) < \delta, \beta \neq \gamma \},$$

for some $\delta \in (0, \frac{1}{2})$. Recalling (2.16), we have

$$\begin{aligned} F_1(\phi_R) &= n(n-1) \tau(\phi_R) \wedge \overline{\tau(\phi_R)} \wedge (H' + H'')^{n-2} / \omega^n \\ &= n(n-1) \{ \tau(\phi_R) \wedge \overline{\tau(\phi_R)} \wedge (H'')^{n-2} \\ &\quad + \tau(\phi_R) \wedge \overline{\tau(\phi_R)} \wedge \sum_{k=1}^{n-2} C_{n-2}^{k} (H')^k \wedge (H'')^{n-2-k} \} / \omega^n. \end{aligned}$$

By Lemma 3.5, we obtain that, on $\triangle_R \bigcap V_{\delta}$, each component of H' is less than $C(\delta + \frac{1}{R})$ and $|(\tau(\phi_R))_{\alpha\beta}| \leqslant C$, where C is independent of R. By Lemma 3.4, we have that, on $\triangle_R \cap V_{\delta}$,

$$0 < H(\phi)_{\alpha\bar{\alpha}} \leqslant C,$$

where C is independent of R and $\alpha = 3, \cdots, n$. By Lemma 3.5 and Proposition 3.8, we have that, on $\triangle_R \cap V_{\delta}$,

$$0 \leqslant H(\phi_R)_{\alpha\bar{\alpha}} \leqslant H(\phi)_{\alpha\bar{\alpha}} + |(\psi_R)_{\alpha\bar{\alpha}}| \leqslant C,$$

where $\alpha = 1, 2$ and C is independent of R. Therefore, on $\triangle_R \cap V_{\delta}$, recalling the positivity lemma of [3, Appendix 3], we have

$$\begin{aligned} F_1(\phi_R) &\geqslant |(\tau(\phi_R))_{12}|^2 \Pi_{\alpha=3}^{n} (H(\phi)_{\alpha\bar{\alpha}}) - C(\delta + \frac{1}{R}) \\ &\geqslant 2^{-n} \epsilon_1^2 \Pi_{\alpha=3}^{n} \lambda_{\alpha}(0) - C(\delta + \frac{1}{R}) \\ &\equiv \epsilon_0 - C(\delta + \frac{1}{R}), \end{aligned}$$

where C and ϵ_0 is independent of R. Choose

$$\delta_1 < \min\{\frac{\epsilon_0}{4C}, \frac{1}{2}\}, \qquad R_2 > \max\{R_1, \frac{4C}{\epsilon_0}\}.$$

Thus, for each $R > R_2$,

$$F_1(\phi_R) \geqslant \frac{\epsilon_0}{2} > 0,$$

on $V_{\delta_1} \bigcap \triangle_R$. Moreover, we can choose $R_0 > R_2$ such that $\triangle_{R_0} \subset V_{\delta_1}$. Define

$$\phi_0 = \phi + a_{\omega(\phi)}(\psi_{R_0} \circ \varphi^{-1}) \cdot \psi_{R_0} \circ \varphi^{-1}. \tag{3.24}$$

By construction, we have $\phi_0 = \phi$ on $M \setminus \varphi(\triangle_{R_0})$ and, on $\varphi(\triangle_{R_0})$:

$$F_1(\phi_0) \geqslant \frac{\epsilon_0}{2} > 0, \qquad F_0(\phi_0) \geqslant 0,$$

and also [3, Appendix 3]: $\forall j \geqslant 2, F_j(\phi_0) \geqslant 0$. Therefore, by Proposition 2.4, we conclude that $F(\phi_0) > 0$ on M, or else $F(\phi_0) \in B_+$. □

Proof of Theorem 1.2. — For completeness, let us redo the argument of [3]. First, Proposition 2.6 has shown that $F|_{A_+} : A_+ \to F(A_+)$ is a diffeomorphism. If J is integrable, then (M, g, J, ω) is a closed Kähler manifold. The restricted operator $F|_{A_+}$ is a surjectivity map since there always exists a solution of Calabi-Yau equation on Kähler manifold [14]. Suppose that J is not integrable. From Lemma 3.9, we have constructed a function ϕ_0 belonging to the boundary of A_+ and $F(\phi_0) > 0$ on M. We claim that $F(\phi_0)$ does not belong to the image of F on A_+. Otherwise, we may assume that there exists a function $\phi_1 \in A_+$ such that

$$F(\phi_0) = F(\phi_1).$$

For $t \in (0, 1]$, let

$$\phi_t = t\phi_1 + (1 - t)\phi_0.$$

Obviously, for all $t \in (0, 1]$, $\phi_t \in A_+$. Hence

$$\int_0^1 \frac{d}{dt}[F(\phi_t)]dt = 0.$$

But

$$\int_0^1 \frac{d}{dt}[F(\phi_t)]dt\omega^n = \int_0^1 n(\omega(\phi_t))^{n-1} \wedge (d(Jd(\phi_1 - \phi_0))dt.$$

Then, by Lemma 2.5, we have that

$$L(\phi_0)(f)\omega^n = \int_0^1 n(\omega(\phi_t))^{n-1}dt \wedge d(Jdf)$$

is a linear elliptic operator of second order without zero-th term. Making use of maximal principle of Hopf, we obtain that the kernel of $L(\phi_0)$ consists of constant functions. Therefore,

$$\phi_0 = \phi_1$$

(note that ϕ_0, ϕ_1 belong to $\bar{A}_+$) which implies that

$$\phi_0 \in A_+.$$

This contradicts the fact that ϕ_0 belongs to the boundary of A_+. $\square$

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