



ANNALES

DE

L'INSTITUT FOURIER

Masayuki ASAOKA

Regular projectively Anosov flows on three-dimensional manifolds

Tome 60, n° 5 (2010), p. 1649-1684.

http://aif.cedram.org/item?id=AIF_2010__60_5_1649_0

© Association des Annales de l'institut Fourier, 2010, tous droits réservés.

L'accès aux articles de la revue « Annales de l'institut Fourier » (<http://aif.cedram.org/>), implique l'accord avec les conditions générales d'utilisation (<http://aif.cedram.org/legal/>). Toute reproduction en tout ou partie cet article sous quelque forme que ce soit pour tout usage autre que l'utilisation à fin strictement personnelle du copiste est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

cedram

Article mis en ligne dans le cadre du
Centre de diffusion des revues académiques de mathématiques
<http://www.cedram.org/>

REGULAR PROJECTIVELY ANOSOV FLOWS ON THREE-DIMENSIONAL MANIFOLDS

by Masayuki ASAOKA (*)

ABSTRACT. — We give the complete classification of regular projectively Anosov flows on closed three-dimensional manifolds. More precisely, we show that such a flow must be either an Anosov flow or decomposed into a finite union of $T^2 \times I$ -models. We also apply our method to rigidity problems of some group actions.

RÉSUMÉ. — Nous classifions complètement les flots projectivement Anosov réguliers en dimension trois. Plus précisément, nous prouvons qu'un tel flot est un flot d'Anosov ou se décompose en une union finie de $T^2 \times I$ -modèles. Nous appliquons aussi notre méthode au problème de rigidité de certaines actions de groupes.

1. Introduction

1.1. Regular projectively Anosov flows

In [19], Mitsumatsu introduced a *bi-contact structure* on a three-dimensional manifold, *i.e.*, a pair of mutually transverse positive and negative contact structures. He observed that a three-dimensional Anosov flow naturally induces a bi-contact structure whose intersection as a pair of plane fields is tangent to the flow. In general, the intersection of a bi-contact structure does not define an Anosov flow. In fact, he showed that a bi-contact structure corresponds to a *projectively Anosov flow*, which is a generalization of an Anosov flow. In [13], Eliashberg and Thurston also studied bi-contact structures and projectively Anosov flows (*conformally Anosov*

Keywords: Projectively Anosov flows, bi-contact structures.

Math. classification: 37D30, 57R30.

(*) Partially supported by Grant-in-Aid for Encouragement of Young Scientists (B) and JSPS Postdoctoral Fellowships for Research Abroad.

flows in their book) from the viewpoint of confoliation theory. They observed that a bi-contact structure naturally appears in a linear deformation of a foliation into contact structures.

A flow $\Phi = \{\Phi^t\}_{t \in \mathbb{R}}$ on a three-dimensional manifold M is called a *projectively Anosov flow* (or a $\mathbb{P}A$ flow) if it has no stationary points and admits a decomposition $TM = E^u + E^s$ by continuous plane fields such that

- $E^u(z) \cap E^s(z) = T\Phi(z)$ for any $z \in M$, where $T\Phi$ is the line field tangent to the orbits of Φ ,
- $D\Phi^t(E^\sigma(z)) = E^\sigma(\Phi^t(z))$ for any $\sigma \in \{u, s\}$, $z \in M$, and $t \in \mathbb{R}$, and
- there exist two constants $C > 0$ and $\lambda > 1$ such that

$$\|N\Phi^t|_{(E^s/T\Phi)(z)}\| \cdot \|(N\Phi^t|_{(E^u/T\Phi)(z)})^{-1}\| \leq C\lambda^{-t}$$

for any $z \in M$ and $t \geq 0$, where $N\Phi = \{N\Phi^t\}_{t \in \mathbb{R}}$ is the flow on $TM/T\Phi$ induced by Φ .

We call the decomposition $TM = E^u + E^s$ a $\mathbb{P}A$ *splitting*. If it satisfies stronger inequalities

$$\|N\Phi^t|_{(E^s/T\Phi)(z)}\| \leq C\lambda^{-t}, \quad \|(N\Phi^t|_{(E^u/T\Phi)(z)})^{-1}\| \leq C\lambda^{-t}$$

for any $z \in M$ and $t \geq 0$, then the flow is called an *Anosov flow* and the splitting is called a *weak Anosov splitting* ⁽¹⁾. We remark that a $\mathbb{P}A$ splitting is a *dominated splitting* on the whole manifold. Such a splitting plays important roles in the modern theory of dynamical systems. See [7] for example.

It is known that a $\mathbb{P}A$ splitting is always integrable. However, the splitting is not smooth in general ⁽²⁾. In fact, any orientable closed three-dimensional manifold admits a smooth $\mathbb{P}A$ flow, but no $\mathbb{P}A$ flow on the three-dimensional sphere admits a C^1 $\mathbb{P}A$ splitting. See Theorems 4.2.6 and 4.3.1 in [20]. From the viewpoint of confoliation theory, a $\mathbb{P}A$ flow with a smooth $\mathbb{P}A$ splitting corresponds to a linear deformation of a smooth foliation into contact structures whose derivative generates another smooth foliation (see [13, Proposition 2.2.3]).

In [15], Ghys classified three-dimensional Anosov flows with a smooth weak Anosov splitting. We say an Anosov flow Φ is *algebraic* if there exists a Lie group G , its cocompact lattice Γ , and a one-parameter subgroup

⁽¹⁾ It is different from but equivalent to the common definition of an Anosov flow as pointed out by Doering [12, Proposition 1.1].

⁽²⁾ A $\mathbb{P}A$ flow with a smooth $\mathbb{P}A$ splitting is called *regular*. However, we do not use the term 'regular' in this sense since we use this term in other context below.

$\{a^t\}_{t \in \mathbb{R}}$ of G such that Φ is a flow on $\Gamma \backslash G$ given by $\Phi^t(\Gamma g) = \Gamma(g \cdot a^t)$. It is known that there are only two choices of G up to covering:

- (1) The universal covering group $\widetilde{\text{PSL}}(2, \mathbb{R})$ of the special linear group of \mathbb{R}^2 .
- (2) The semi-direct product $\mathbb{R} \ltimes \mathbb{R}^2$ associated to a homomorphism $H : \mathbb{R} \rightarrow \text{GL}(2, \mathbb{R})$ given by $H(t)(x, y) = (e^t x, e^{-t} y)$.

In the former case, the algebraic Anosov flow can be identified with the geodesic flow on a closed surface with a hyperbolic metric up to finite cover. In the latter case, the algebraic Anosov flow can be identified with the suspension flow of a hyperbolic toral automorphism.

THEOREM 1.1 ([15]). — *If an Anosov flow on a closed three-dimensional manifold admits a C^2 PA splitting, it is smoothly equivalent to an algebraic Anosov flow.*

It is natural to ask whether any PA flow with a smooth PA splitting is equivalent to an algebraic model or not. In [23], Noda showed that if a PA flow on a \mathbb{T}^2 -bundle over S^1 admits a smooth PA splitting and has an invariant torus, then it must be represented as a finite union of so-called $\mathbb{T}^2 \times I$ -models. Roughly speaking, a $\mathbb{T}^2 \times I$ -model is a flow on $\mathbb{T}^2 \times [0, 1]$ which is transverse to $\mathbb{T}^2 \times \{z\}$ for any $z \in (0, 1)$ and is equivalent to a linear flow on each boundary. See [23] for the precise definition. In a series of papers, he and Tsuboi gave a classification for certain manifolds, which is summarized as follows.

THEOREM 1.2 ([23, 24, 25, 28]). — *If a PA flow on a Seifert manifold or a \mathbb{T}^2 -bundle over S^1 admits a smooth PA splitting, then it is either an Anosov flow or represented as a finite union of $\mathbb{T}^2 \times I$ -models.*

The author of this paper also approached the classification from another direction. In [4], he showed that if a PA flow on any closed three-dimensional manifold admits a smooth PA splitting and all periodic orbits are hyperbolic, then it is equivalent to one of the above.

In [24], Noda conjectured that the above is the complete list of three-dimensional PA flows with a smooth PA splitting. The goal of this paper is an affirmative solution to the conjecture.

THEOREM 1.3. — *If a PA flow on a closed, connected, and three-dimensional manifold admits a C^2 PA splitting, then it is either an Anosov flow or represented as a finite union of $\mathbb{T}^2 \times I$ models.*

The theorem gives a solution to a conjecture posed by Mitsumatsu (Conjecture 4.3.3 in [20]) immediately.

COROLLARY 1.4. — Any bi-contact structure associated to a $\mathbb{P}A$ flow with a smooth $\mathbb{P}A$ splitting consists of tight contact structures.

We give the proof of Theorem 1.3 in Sections 2 and 3. In Section 2, we show a dichotomy on dynamics of a $\mathbb{P}A$ flow with a C^2 $\mathbb{P}A$ splitting. Namely, either the flow is topologically transitive or the non-wandering set is the union of invariant tori with rotational dynamics. It is not so hard to see that the latter implies that the flow is represented by $\mathbb{T}^2 \times I$ -models. In Section 3, we show the former implies that the flow is Anosov. It is done by proving the hyperbolicity of all periodic orbits.

1.2. Foliations with a tangentially contracting flow

Let \mathcal{F} be a codimension-one foliation on a three-dimensional manifold M . We say a flow Φ is *tangentially contracting* with respect to \mathcal{F} if there exist $C > 0$ and $\lambda > 1$ such that $\|N\Phi^t|_{T\mathcal{F}/T\Phi(z)}\| \leq C\lambda^{-t}$ for any $z \in M$ and $t \geq 0$. We apply the method developed in this paper to a classification of foliations which admit a tangentially contracting flow.

THEOREM 1.5. — Let M be a closed three-dimensional manifold and \mathcal{F} a C^r codimension-one foliation on M with $r \geq 2$. Suppose that \mathcal{F} admits a C^r tangentially contracting flow Φ . Then, Φ is Anosov and \mathcal{F} is C^r -diffeomorphic to the weak stable foliation of an algebraic Anosov flow.

We give two examples of group actions which induce a foliation with a tangentially contracting flow naturally. The above theorem implies the rigidity of such actions.

Locally free actions of the affine group. Let GA be the group of orientation preserving affine transformations of the real line \mathbb{R} . It is generated by two one-parameter subgroups $\{a^t\}_{t \in \mathbb{R}}$ and $\{b^x\}_{x \in \mathbb{R}}$ with a relation $b^x \cdot a^t = a^t \cdot b^{\exp(-t)x}$. We say an action $\rho: M \times \text{GA} \rightarrow M$ on a manifold M is *locally free* if the isotropy subgroup $\{g \in \text{GA} \mid \rho(p, g) = p\}$ is discrete for any $p \in M$. By $\mathcal{O}(p, \rho)$, we denote the ρ -orbit $\{\rho(p, g) \mid g \in \text{GA}\}$ of $p \in M$. If ρ is of class C^1 and M is closed, then the partition $\mathcal{O}_\rho = \{\mathcal{O}(p, \rho) \mid p \in M\}$ is a foliation. The flow $\{\rho(\cdot, a^t)\}_{t \in \mathbb{R}}$ is tangentially contracting with respect to \mathcal{O}_ρ .

In [14], Ghys classified C^r locally free action of GA on closed three-dimensional manifolds for $r \geq 2$ up to C^r conjugacy assuming the existence of an invariant volume. Applying Theorem 1.5 to \mathcal{O}_ρ , we obtain a classification of the orbit foliation of actions without the assumption on an invariant volume.

THEOREM 1.6. — *Let ρ be a C^r locally free action of GA on a closed three-dimensional manifold with $r \geq 2$. Then, the orbit foliation of ρ is C^r diffeomorphic to the weak stable foliation of an algebraic Anosov flow.*

In the forthcoming paper [2], we will give a classification of the actions of GA up to smooth conjugacy.

Actions of Fuchsian groups on the circle. Let Γ_g be the fundamental group of the oriented closed surface of genus $g \geq 2$. We identify the circle S^1 with the real projective line. It induces a projective structure to the circle. We call an action Φ of Γ_g on the circle *projective* if it preserves the projective structure. We say two actions Φ_1 and Φ_2 of Γ_g are C^r -conjugate if there exists a C^r diffeomorphism H (or a homeomorphism if $r = 0$) of S^1 such that $H(\Phi_1(\gamma, p)) = \Phi_2(\gamma, H(p))$ for any $\gamma \in \Gamma_g$ and $p \in S^1$. In [15], Ghys proved the rigidity of projective actions.

THEOREM 1.7 ([15]). — *Let $\Phi : \Gamma_g \times S^1 \rightarrow S^1$ be a C^r action with $r \geq 3$. Suppose that Φ is C^0 -conjugate to a projective action. Then, it is C^r -conjugate to a projective action.*

His proof can be divided into two steps. The first step is to show that the suspension foliation of the action admits a tangentially contracting flow. The second step is to construct a transverse projective structure of the foliation using the flow obtained in the first step. The first step can be done even for $r = 2$. The second step also can be done even for $r = 2$ if the flow obtained in the first step is Anosov, as Ghys mentioned in Section 5 of [15]. Hence, Theorem 1.5 implies the following improvement of the above theorem.

THEOREM 1.8. — *Theorem 1.7 holds even for $r = 2$.*

It is known that the theorem does not hold for $r = 1$. See [15].

Acknowledgments. The author would like to thank Professor Takashi Inaba for letting the author know the stability theory and the level theory of Cantwell and Conlon, which are necessary to prove Lemma 2.14. This paper was partially written when the author stayed at Unité de Mathématiques Pures et Appliquées, École Normale Supérieure de Lyon. He is grateful to the members of UMPA, especially to Professor Étienne Ghys for their warm hospitality. He would also like to thank an anonymous referee for many comments to improve the paper.

2. A dichotomy on dynamics

In the rest of the article, we fix an orientable, closed, connected, and three-dimensional manifold M . Let Φ be a C^2 PA flow on M with a continuous PA splitting $TM = E^u + E^s$. For a compact Φ -invariant set Λ , we define *the stable set* $W^s(\Lambda)$ and *the unstable set* $W^u(\Lambda)$ by

$$W^s(\Lambda) = \left\{ z \in M \mid \lim_{t \rightarrow +\infty} d(\Phi^t(z), \Lambda) = 0 \right\}$$

and $W^u(\Lambda) = W^s(\Lambda; \Phi^{-1})$, where Φ^{-1} is the time-reverse of Φ .

We call a Φ -invariant torus T is *normally attracting* if there exists $C > 0$ and $\lambda > 1$ such that $\|N\Phi^t|_{E^s/T\Phi(z)}\| \leq C\lambda^{-t}$ for any $z \in T$ and $t \geq 0$. By the existence of a PA splitting, our definition coincides with the usual definition. It is known that if T is a normally attracting invariant torus, then $W^s(T)$ is an open neighborhood of T and is diffeomorphic to $\mathbb{T}^2 \times \mathbb{R}$. We say an invariant torus T' is *normally repelling* if T' is normally attracting with respect to the time-reverse Φ^{-1} .

Let Ω_* be the union of invariant embedded tori to which the restriction of Φ are topologically equivalent to a linear flow. For $\rho \in \{u, s\}$, let Ω_*^ρ be the union of tori in Ω_* tangent to E^ρ . By the linearity of the flow on tori and the domination property of the splitting, we have $\Omega_* = \Omega_*^u \cup \Omega_*^s$, Ω_*^u is a union of normally attracting Φ -invariant tori, and Ω_*^s is a union of normally repelling Φ -invariant tori. The exactly same argument as Proposition 3.9 of [1] shows that Ω_* consists of finite number of tori.

The aim of this section is to show the following dichotomy.

PROPOSITION 2.1. — *If Φ admits a C^2 PA splitting, then either*

- (1) Φ is topologically transitive, or
- (2) $M = W^s(\Omega_*^u) \cup \Omega_*^s = W^u(\Omega_*^s) \cup \Omega_*^u$.

The latter implies that Φ is equivalent to one of the known models.

PROPOSITION 2.2. — *In the latter case of Proposition 2.1, Φ is represented by a finite union of $\mathbb{T}^2 \times I$ -models.*

Proof. — Fix connected components T_0 of Ω_*^s and U of $W^u(T_0) \setminus T_0$. Take an embedding $\psi : \mathbb{T}^2 \times [0, 1] \rightarrow M$ such that $\psi(\mathbb{T}^2 \times 0) = T_0$, $\text{Im } \psi \subset U \cup T_0$, and $\psi(\mathbb{T}^2 \times 1)$ is transverse to the flow. Put $T' = \psi(\mathbb{T}^2 \times 1)$. Since $W^s(\Omega_*^u)$ is a disjoint union of the stable sets of connected components of Ω_*^u , we have $T' \subset W^s(T_1)$ for some connected component T_1 of Ω_*^u .

Let U_1 be the connected component of $W^s(T_1) \setminus T_1$ which contains T' . Since T_1 is normally attracting there exists an embedding $\psi_1 : \mathbb{T}^2 \times$

$[0, 1] \rightarrow M$ such that $\text{Im } \psi_1 \subset T_1 \cup (U_1 \setminus \text{Im } \psi)$, $\psi_1(\mathbb{T}^2 \times 0) = T_1$, $\psi_1(\mathbb{T}^2 \times 1)$ is transverse to the flow, and each Φ -orbit contained in U_1 intersects with $\psi_1(\mathbb{T}^2 \times 1)$ exactly once. Then, we can take a smooth positive function τ on T' such that $\Phi^{\tau(z)}(z) \in \psi_1(\mathbb{T}^2 \times 1)$ for any $z \in T'$. It implies that

$$\overline{U} = \overline{U_1} = \text{Im } \psi \cup \text{Im } \psi_1 \cup \{\Phi^t(z) \mid z \in T', t \in [0, \tau(z)]\}$$

is diffeomorphic to $\mathbb{T}^2 \times [0, 1]$ and its boundary is $T_0 \cup T_1$.

Inductively, we obtain sequences $(T_n)_{n \geq 0}$ and $(B_n)_{n \geq 0}$ of subsets of M such that T_n is a connected component of Ω_* , B_n is diffeomorphic to $\mathbb{T}^2 \times [0, 1]$, $\partial B_n = T_n \cup T_{n+1}$, and $B_n \cap B_{n+1} = T_{n+1}$ for any n . Since Ω_* contains only finitely many tori, we have $T_n = T_m$ for some $n \neq m$. It implies that M is a \mathbb{T}^2 -bundle over S^1 . By Noda's classification [23], Φ is represented by a finite union of $\mathbb{T}^2 \times I$ -models. □

The rest of this section is devoted to the proof of Proposition 2.1. In Subsection 2.1, we show the existence of the stable and unstable manifolds of invariant sets without irregular periodic points. In Section 2.2, we prove the dichotomy when all periodic points outside Ω_* are regular. In the both of subsections, we assume only a weaker condition on the regularity of the $\mathbb{P}A$ splitting since the C^2 regularity of the splitting is too strong when we apply the results to a foliation with a tangentially contracting flow. At last, in Subsection 2.3, we prove that all periodic points are regular for a $\mathbb{P}A$ flow with a C^2 $\mathbb{P}A$ splitting.

2.1. Hyperbolic-like behavior

Let Φ be a C^2 $\mathbb{P}A$ flow and $TM = E^u + E^s$ its $\mathbb{P}A$ splitting. In this subsection, we do not assume the the splitting is of class C^2 .

For $z \in M$, we define *the orbit* $\mathcal{O}(z)$, *the α -limit set* $\alpha(z)$, and *the ω -limit set* $\omega(z)$ by

$$\begin{aligned} \mathcal{O}(z) &= \{\Phi^t(z) \mid z \in \mathbb{R}\}, \\ \alpha(z) &= \bigcap_{T > 0} \overline{\{\Phi^t(z) \mid t \leq -T\}}, \\ \omega(z) &= \bigcap_{T > 0} \overline{\{\Phi^t(z) \mid t \geq T\}}. \end{aligned}$$

We say a point $z \in M$ is *periodic* if there exists $T > 0$ such that $\Phi^T(z) = z$. The minimum of $\{t > 0 \mid \Phi^t(z) = z\}$ is called *the period* of z . We denote the set of periodic points of Φ by $\text{Per}(\Phi)$, and the non-wandering set of Φ by $\Omega(\Phi)$.

We say a periodic point z_0 is *s-regular* when there exists an embedded closed annulus A tangent to E^s such that $\Phi^t(A) \subset \text{Int } A$ for any $t > 0$ and $\bigcap_{t>0} \Phi^t(A) = \mathcal{O}(z_0)$. Similarly, we say a periodic point z_0 is *u-regular* when there exists an embedded closed annulus A tangent to E^u such that $\Phi^{-t}(A) \subset \text{Int } A$ for any $t > 0$ and $\bigcap_{t>0} \Phi^{-t}(A) = \mathcal{O}(z_0)$. We also say z_0 is ρ -irregular for $\rho \in \{u, s\}$ if z_0 is not ρ -regular. Let $\text{Per}_{\text{irr}}^\rho(\Phi)$ be the set of ρ -irregular periodic points. Put $\text{Per}_{\text{irr}}(\Phi) = \text{Per}_{\text{irr}}^s(\Phi) \cup \text{Per}_{\text{irr}}^u(\Phi)$. The aim of this subsection is to show the existence of the unstable manifolds for a compact invariant set which does not intersect with $\Omega_* \cup \overline{\text{Per}_{\text{irr}}(\Phi)}$.

Fix a continuous family $\{\phi_z\}_{z \in M}$ of C^2 embeddings of $[-1, 1]^2$ into M such that $\text{Im } \phi_z$ is transverse to $T\Phi$ and $\phi_z(0, 0) = z$ for any $z \in M$. We call $\{\phi_z\}_{z \in M}$ a *family of local cross sections*. Let r_z^t be the holonomy map of the orbit foliation of Φ between $\text{Im } \phi_z$ and $\text{Im } \phi_{\Phi^t(z)}$ along the path $\{\Phi^{t'}(z) \mid t' \in [0, t]\}$. We call $\{r_z^t\}_{(z,t) \in M \times \mathbb{R}}$ the *family of local returns* associated to $\{\phi_z\}_{z \in M}$. For $\Delta > 0$, put $D_\Delta(z) = \{z' \in \text{Im } \phi_z \mid d(z, z') \leq \Delta\}$, where $d(z, z')$ is the distance of $z, z' \in M$. By the continuity of the family $\{\phi_z^t\}_{z \in M}$, there exists $\Delta_\phi > 0$ such that r_z^t is well-defined on $D_{\Delta_\phi}(z)$ for any $z \in M$ and $t \in [-1, 1]$.

The splitting $TM/T\Phi = (E^s/T\Phi) \oplus (E^u/T\Phi)$ defines projections π^s and π^u from TM to $E^s/T\Phi$ and $E^u/T\Phi$ respectively. For $\alpha > 0$, we say an embedded interval I in M is an (E^s, α) -transversal if $\|\pi^s(v)\| \leq \alpha \|\pi^u(v)\|$ for any $z \in I$ and $v \in T_z I$. Similarly, we say an embedded interval I in M is an (E^u, α) -transversal if $\|\pi^u(v)\| \leq \alpha \|\pi^s(v)\|$ for any $z \in I$ and $v \in T_z I$. For $\Delta > 0$, an interval I is called a (Δ, E^s) -interval if it is an $(E^s, 1)$ -transversal and $r_z^t(I) \subset D_\Delta(\Phi^t(z))$ for any $t \geq 0$. Similarly, an interval I is called a (Δ, E^u) -interval if it is an $(E^u, 1)$ -transversal and $r_z^{-t}(I) \subset D_\Delta(\Phi^{-t}(z))$ for any $t \geq 0$.

The next lemma is a variant of “the Denjoy property”, which was proved by Arroyo and Rodrigues-Hertz in [1] for flows without non-hyperbolic periodic points.

LEMMA 2.3. — *There exists $\Delta_0 > 0$ such that*

- (1) *the interior of any (Δ_0, E^s) -interval contains a point z such that $\omega(z)$ is a periodic orbit in $\text{Per}_{\text{irr}}^u(\Phi)$ or is a torus in Ω_*^u , and*
- (2) *the interior of any (Δ_0, E^u) -interval contains a point z such that $\alpha(z)$ is a periodic orbit in $\text{Per}_{\text{irr}}^s(\Phi)$ or is a torus in Ω_*^s .*

Proof. — We will show the former assertion since that the latter can be obtained in the same way. In the proof of Proposition 4.2 in [1], Arroyo and Rodrigues-Hertz used the hyperbolicity of all periodic points only in the proof of Lemma 4.3 and 4.4 and the other part of the proof works even if

there are non-hyperbolic periodic points. Hence, it is sufficient to see how to recover the proof of Lemmas 4.3 and 4.4 for our case.

Fix a (δ, E^s) -interval I which contains z . Put $I_t = r_z^t(I)$ for $t \geq 0$. Let $\{J_s\}_{s \geq 0}$ be the family of (Δ, E^s) -intervals in the proof of Proposition 4.2 of [1], i.e., the maximal one among families of (Δ, E^s) -intervals satisfying $I_s \subset J_s$ and $r_z^{t-s}(J_s) \subset J_t$ for any $t \geq s \geq 0$. As shown in Lemma 4.1 of [1] (its proof does not require the hyperbolicity of periodic orbits), there is a uniform bound from below of the length of the local stable manifold of each point of J_s . Let J_s^ϵ be the union of the local stable manifold of each point of J_s . Lemma 4.3 of [1] deals with the case that $r_z^t(J_s^\epsilon) \subset J_s^\epsilon$ for some $z \in M$ and $t > 0$. In this case, the exactly same argument as Lemma 4.3 of [1] shows that the forward orbit of some point in $\text{Int } J_0$ converges to a u -irregular periodic orbit.

Lemma 4.4 of [1] deals with the case that $\limsup |J_s| > 0$ and $r_z^t(J_s^\epsilon) \cap \text{Per}(\Phi) \neq \emptyset$ for some s and $t > 0$. In this case, we need to show the inclination lemma for an (Δ, E^s) -interval J_s and a periodic point which is sufficiently close to J_s . However, it is an easy consequence of the existence of the local stable manifolds with uniform length (Lemma 4.1 in [1]). \square

Let Λ be a compact Φ -invariant set such that $\Lambda \cap (\Omega_* \cup \overline{\text{Per}_{\text{irr}}(\Phi)}) = \emptyset$. In the rest of the subsection, we will show that the stable and unstable manifolds are well-defined for any point of Λ .

For a subset S of M and $\delta > 0$, we denote the δ -neighborhood $\{p \in M \mid \inf_{q \in S} d(p, q) \leq \delta\}$ by $\mathcal{N}_\delta(S)$. Fix $0 < \Delta_1 < \Delta_0$ such that $\mathcal{N}_{\Delta_1}(\Lambda) \cap (\Omega_* \cup \overline{\text{Per}_{\text{irr}}(\Phi)}) = \emptyset$. By the center-unstable manifold theorem, there exist constants $0 < \delta_1 < \delta_2 < \Delta_1$ and a continuous family $\{W_{\text{loc}}^{cu}(z)\}_{z \in M}$ of $C^2(E^s, 1)$ -transversals such that

- $z \in W_{\text{loc}}^{cu}(z) \subset D_{\delta_2}(z)$ and $\partial W_{\text{loc}}^{cu}(z) \subset \partial D_{\delta_2}(z)$ for any $z \in M$,
- $W_\delta^{cu}(z) = W_{\text{loc}}^{cu}(z) \cap D_\delta(z)$ is an interval for any $0 < \delta < \delta_2$ and $z \in M$, and
- $r_z^{-t}(W_{\delta_1}^{cu}(z)) \subset W_{\delta_2}^{cu}(\Phi^t(z))$ for any $0 \leq t \leq 1$ and $z \in M$.

PROPOSITION 2.4. — For any given $\delta > 0$, there exists $\epsilon_1 > 0$ such that

- (1) $r_z^{-t}(W_{\epsilon_1}^{cu}(z)) \subset W_\delta^{cu}(\Phi^t(z))$ for any $z \in \Lambda$ and $t \geq 0$, and
- (2) $\lim_{t \rightarrow \infty} (\sup_{z \in M} |r_z^{-t}(W_{\epsilon_1}^{cu}(z))|) = 0$.

Proof. — Without loss of generality, we may assume $\delta < \delta_1$. If the lemma does not hold, then there exist sequences $\{\epsilon_k > 0\}_{k \geq 1}$, $\{t_k > 0\}_{k \geq 1}$, and $\{z_k \in \Lambda\}_{k=0}^\infty$ such that

- $\lim_{k \rightarrow \infty} \epsilon_k = 0$,
- $r_{z_k}^{-t}(W_{\epsilon_k}^{cu}(z_k)) \subset W_\delta^{cu}(\Phi^{-t}(z_k))$ for any $k \geq 0$ and $0 \leq t \leq t_k$, and

- $\limsup_{k \rightarrow \infty} |r_{z_k}^{-t_k}(W_{\epsilon_k}^{cu}(z_k))| > 0.$

By the first and the last items, we have $\lim_{k \rightarrow \infty} t_k = \infty.$ By taking subsequences, we may assume that $\Phi^{-t_k}(z_k)$ converges to a point z_* of Λ and $r_{z_k}^{-t_k}(W_{\epsilon_k}^{cu}(z_k))$ converges to an interval $I \subset W_{\delta}^{cu}(z_*)$ with positive length. Then, I is a (δ, E^s) -interval. Since z_* is a point of $I \cap \Lambda,$ we have $\omega(z) \subset \mathcal{N}_{\delta}(\Lambda)$ for any $z \in I,$ and hence, $\omega(z) \cap (\Omega_*(\Phi) \cup \overline{\text{Per}_{\text{irr}}(\Phi)}) = \emptyset.$ However, it contradicts Lemma 2.3. □

COROLLARY 2.5. — *For any sufficiently small $\epsilon > 0,$ $W_{\epsilon}^{cu}(z)$ is tangent to E^u for any $z \in \Lambda.$*

Proof. — Fix $\delta > 0$ and take $\epsilon_1 > 0$ in Proposition 2.4. For $z \in \Lambda, z' \in W_{\epsilon_1}^{cu}(z),$ and $t \geq 0,$ let $\alpha(z, z', t)$ be the angle between $T_{r_z^{-t}(z')}r_z^{-t}(W_{\epsilon_1}^{cu}(z))$ and $E^u(r_z^{-t}(z')).$ By the domination property of the $\mathbb{P}A$ splitting, there exist $C > 0$ and $\lambda > 1$ such that $\alpha(z, z', t) \geq C\lambda^t\alpha(z, z', 0).$ On the other hand, the continuity of the family $\{W_{\epsilon_1}^{cu}(z)\}_{z \in M}$ and the proposition implies that α is bounded as a function of z, z' and $t.$ Hence, we have $\alpha(z, z', 0) = 0$ for any z and $z'.$ □

LEMMA 2.6. — *There exists $\Delta_2 \in (0, \Delta_1/2)$ which satisfies the following property: If an $(E^u, 1)$ -transversal I is contained in $D_{\Delta_2}(z)$ for some $z \in M$ and satisfies $r_z^{-t}(\partial I) \subset D_{\Delta_2}(\Phi^{-t}(z))$ for any $t \geq 0,$ then it is a $(\Delta_1/2, E^u)$ -interval.*

Proof. — Since $TM = E^s + E^u$ is a $\mathbb{P}A$ splitting, there exists $\alpha > 0$ such that if I is an $(E^u, 1)$ -transversal and $r_z^{-t}(I)$ is well-defined for some $z \in M$ and $t \geq 0,$ then $r_z^{-t}(I)$ is an (E^u, α) -transversal. By the uniform transversality of (E^u, α) -transversals to $E^u,$ we can take $\delta \in (0, \Delta_1/2)$ and $\beta > 1$ such that $|J| \leq \beta \cdot \text{diam}(\partial I)$ for any $z \in M$ and any (E^u, α) -transversal with $J \subset D_{\delta}(z).$ Put $\Delta_2 = \delta/4\beta.$ We remark that $\Delta_2 < \delta/4 < \Delta_1/8.$

Let I be an $(E^u, 1)$ -transversal contained in $D_{\Delta_2}(z)$ for some $z \in M$ such that $r_z^{-t}(\partial I) \subset D_{\Delta_2}(\Phi^{-t}(z))$ for any $t \geq 0.$ It is sufficient to show that

$$t_0 = \sup\{t_1 \geq 0 \mid r_z^{-t}(I) \subset D_{\delta}(\Phi^{-t}(z)), |r_z^{-t}(I)| \leq \delta \text{ for any } t \in [0, t_1]\}$$

is infinite. Suppose that t_0 is a finite number. Since $r_z^{-t}(I)$ is an (E^u, α) -transversal and $r_z^{-t}(\partial I) \subset D_{\Delta_2}(\Phi^{-t}(z))$ for $0 \leq t \leq t_0,$ we have

$$(2.1) \quad |r_z^{-t_0}(I)| \leq \beta \cdot \text{diam}(r_z^{-t_0}(\partial I)) \leq 2\beta\Delta_2 = \delta/2.$$

It implies $|r_z^t(I)| < \delta$ for any t sufficiently close to $t_0.$ By the inclusion $r_z^{-t_0}(\partial I) \subset D_{\Delta_2}(\Phi^{-t_0}(z))$ again, the inequality (2.1) implies

$$(2.2) \quad r_z^{-t_0}(I) \subset D_{(\delta/2)+\Delta_2}(\Phi^{-t_0}(z)) \subset D_{(3/4)\delta}(\Phi^{-t_0}(z)).$$

Hence, $r_z^{-t}(I) \subset D_\delta(\Phi^{-t}(z))$ for any t sufficiently close to t_0 . It contradicts the choice of t_0 . □

PROPOSITION 2.7. — *There exists $\epsilon_2 > 0$ such that*

$$\bigcap_{t \geq 0} r_z^t(D_\epsilon(\Phi^{-t}(z))) \subset W_\epsilon^{cu}(z)$$

for any $z \in \Lambda$ and $0 < \epsilon < \epsilon_2$.

Proof. — Let $\epsilon_1 > 0$ be the constant obtained by applying Proposition 2.4 for $\delta = \Delta_2$. By the uniform transversality of (Δ_1, E^s) -interval to E^s , we can take a constant $\epsilon_2 \in (0, \epsilon_1)$ which satisfies the following property: For any $z \in \Lambda$ and $z' \in D_{\epsilon_2}(z) \setminus W_{\epsilon_1}^{cu}(z)$, there exists an $(E^u, 1)$ -transversal J in $D_{\Delta_2}(z)$ and $z_J \in J \cap W_{\epsilon_1}^{cu}(z)$ such that $\partial J = \{z', z_J\}$.

Suppose that the proposition does not hold. Then, there exists $z \in \Lambda$ and $z' \in \bigcap_{t \geq 0} r_z^t(D_{\epsilon_2}(\Phi^{-t}(z))) \setminus W_{\epsilon_2}^{cu}(z)$. Take an $(E^u, 1)$ -transversal J in $D_{\Delta_2}(z)$ and $z_J \in J \cap W_{\epsilon_1}^{cu}(z)$ such that $\partial J = \{z', z_J\}$. Since both $r_z^{-t}(z')$ and $r_z^{-t}(z_J)$ are contained in $D_{\Delta_2}(\Phi^{-t}(z))$ for any $t \geq 0$, J is an $(\Delta_1/2, E^u)$ -interval by Lemma 2.6. Hence, we have

$$r_z^{-t}(J) \subset D_{(\Delta_1/2)+\Delta_2}(\Phi^{-t}(z)) \subset D_{\Delta_1}(\Phi^{-t}(z)).$$

By Lemma 2.3, the set $\Sigma = \overline{\bigcup_{t \geq 0} r_z^{-t}(J)}$ intersects with $\Omega_*^s \cup \text{Per}_{\text{irr}}^s(\Phi)$. However, Σ is contained in the Δ_1 -neighborhood of Λ . It contradicts the choice of Δ_1 . □

We define a family $\{V^u(z)\}_{z \in \Lambda}$ of subsets of M by

$$V^u(z) = \bigcup_{t > 0} \bigcup_{z' \in \mathcal{O}(z)} \Phi^t(W_\epsilon^{cu}(z')).$$

It is a continuous family of C^2 open immersed surfaces tangent to E^u by Corollary 2.5. By Proposition 2.4, $V^u(z)$ does not depend on the choice of sufficiently small $\epsilon > 0$. It is easy to see that

- $V^u(z_0)$ is diffeomorphic to $S^1 \times \mathbb{R}$ for any periodic point $z_0 \in \Lambda$, and
- $V^u(z_1) \cap V^u(z_2) \neq \emptyset$ for $z_1, z_2 \in \Lambda$ implies $V^u(z_1) = V^u(z_2)$.

Similar to $\{W_\delta^{cu}(z)\}_{z \in \Lambda}$, we can take a family $\{W_\delta^{cs}(z)\}_{z \in \Lambda}$ of $(E^u, 1)$ -transversals. We define a family $\{V^s(z)\}_{z \in \Lambda}$ by

$$V^s(z) = \bigcup_{t > 0} \bigcup_{z' \in \mathcal{O}(z)} \Phi^{-t}(W_\epsilon^{cs}(z'))$$

for any small $\epsilon > 0$. It has analogous properties to $\{V^u(z)\}_{z \in \Lambda}$.

For Φ -invariant compact subsets Λ_1 and Λ_2 of Λ , we write $\Lambda_1 \preceq \Lambda_2$ if $W^s(\Lambda_1) \cap W^u(\Lambda_2) \neq \emptyset$.

PROPOSITION 2.8. — *Suppose that Λ is locally maximal, i.e., there exists a neighborhood U of Λ such that $\Lambda = \bigcap_{t \in \mathbb{R}} \Phi^t(U)$. Then,*

- $W^s(\Lambda) = \bigcup_{z \in \Lambda \cap \Omega(\Phi)} V^s(z)$,
- *there exists a decomposition $\Lambda \cap \Omega(\Phi) = \bigcup_{i=1}^m \Lambda_i$ into mutually disjoint topologically transitive compact invariant subsets, and*
- \preceq *is a partial order on $\{\Lambda_1, \dots, \Lambda_m\}$.*

Proof. — By Propositions 2.4 and 2.7, Λ has the shadowing property (see e.g. [27]). We can show the required properties by the same argument as the case of locally maximal hyperbolic sets. □

The partially ordered set $(\{\Lambda_1, \dots, \Lambda_m\}, \preceq)$ is called *the spectral decomposition* of $\Lambda \cap \Omega(\Phi)$.

We say a point z of a topological space X is *accessible* from a subset A of X if there exists a continuous map $l : [0, 1] \rightarrow X$ such that $l(1) = z$ and $l(t) \in A$ for any $t \in [0, 1)$.

LEMMA 2.9. — *Let Λ' be a topologically transitive compact invariant subset of Λ such that $W^s(\Lambda') \cap W^u(\Lambda') = \Lambda'$. If $z \in \Lambda'$ is accessible from $V^s(z) \setminus \Lambda'$, then $V^u(z)$ contains a periodic point $z_* \in \Lambda'$ which is accessible from $V^s(z_*) \setminus \Lambda'$. Similarly, if $z \in \Lambda'$ is accessible from $V^u(z) \setminus \Lambda'$, then $V^s(z)$ contains a periodic orbit z_* which is accessible from $V^u(z_*) \setminus \Lambda'$.*

Proof. — The same argument as the proof of Proposition 1 of [22] shows that $V^u(z)$ contains a periodic point z_* . Since the α -limit set of z coincides with the orbit of z_* , the invariance of Λ' implies that $z_* \in \Lambda'$. By the accessibility of z from $V^s(z) \setminus \Lambda'$, there exists a curve $I \subset V^s(z)$ transverse to E^s such that $I \cap \Lambda' = \{z\}$. Suppose that z_* is not accessible from $V^s(z_*) \setminus \Lambda'$. By the continuity of $V^u(z')$ with respect to $z' \in \Lambda$ there exists $z_1 \in \Lambda' \cap V^u(z_*)$ such that $V^u(z_1) \cap (I \setminus \{z\}) \neq \emptyset$. Since $W^s(\Lambda') \cap W^u(\Lambda') = \Lambda'$ and $V^\sigma(z') \subset W^\sigma(\Lambda')$ for any $z' \in \Lambda'$ and $\sigma = s, u$, we have $I \cap V^u(z_1) \subset \Lambda'$. However, it contradicts the choice of I . Therefore, z_* is accessible from $V^s(z_*) \setminus \Lambda'$.

We obtain the latter from the former by reversing the time. □

2.2. Dichotomy under regularity of periodic orbits

The aim of this subsection is to show Proposition 2.1 under some additional assumptions.

PROPOSITION 2.10. — *Let Φ be a C^2 $\mathbb{P}A$ flow and $TM = E^s + E^u$ be its $\mathbb{P}A$ splitting. Suppose that $\text{Per}_{\text{irr}}(\Phi) \subset \Omega_*$ and E^s generates a C^2 foliation. Then, either*

- (1) $\Omega_* = \text{Per}_{\text{irr}}(\Phi) = \emptyset$ and Φ is topologically transitive, or
- (2) $M = W^u(\Omega_*^s) \cup \Omega_*^u = W^s(\Omega_*^u) \cup \Omega_*^s$.

Remark that we do not assume the C^2 -regularity of E^u .
 Put $\Omega_h = M \setminus (W^u(\Omega_*^s) \cup W^s(\Omega_*^u))$.

LEMMA 2.11. — Ω_h is a locally maximal closed invariant set.

Proof. — Since Ω_*^s is normally repelling, there exists a compact neighborhood K^s of Ω_*^s such that $\Phi^{-t}(K^s) \subset K^s$ for any $t > 0$, $\bigcap_{t \geq 0} \Phi^{-t}(K^s) = \Omega_*^s$, and $\bigcup_{t \geq 0} \Phi^t(K^s) = W^u(\Omega_*^s)$. Similarly, there exists a compact neighborhood K^u of Ω_*^u such that $\Phi^t(K^u) \subset K^u$ for any $t > 0$, $\bigcap_{t \geq 0} \Phi^t(K^u) = \Omega_*^u$, and $\bigcup_{t \geq 0} \Phi^{-t}(K^u) = W^s(\Omega_*^u)$. Then, $U = M \setminus (K^u \cup K^s)$ is a neighborhood of Ω_h such that $\bigcap_{t \in \mathbb{R}} \Phi^t(U) = \Omega_h$. □

It is easy to see that $\alpha(z) \cup \omega(z) \subset \Omega_*^s \cup \Omega_*^u \cup \Omega_h$ for any $z \in M$. Since Ω_*^u is normally attracting and Ω_*^s is normally repelling, we have

$$M = W^u(\Omega_h) \cup W^u(\Omega_*^s) \cup \Omega_*^u = W^s(\Omega_h) \cup W^s(\Omega_*^u) \cup \Omega_*^s.$$

We assume $\Omega_h \neq \emptyset$ and show that $M = \Omega_h$ and Φ is topologically transitive. It implies that $\Omega_*^u = \Omega_*^s = \emptyset$, and hence, $\text{Per}_{\text{irr}}(\Phi) = \emptyset$ by the assumption.

By $\mathcal{G}(z)$, we denote the leaf of a foliation \mathcal{G} that contains a point z . Let \mathcal{F}^s be the C^2 foliation generated by E^s .

LEMMA 2.12. — $\mathcal{F}^s(z) = V^s(z)$ for any $z \in \Omega_h$.

Proof. — Since $V^s(z')$ is tangent to E^s for any $z' \in \Omega_h$, it is a connected open subset of $\mathcal{F}^s(z')$. Since $\mathcal{F}^s(z) \subset M \setminus \Omega_*^s = W^s(\Omega_h) \cup W^s(\Omega_*^u)$, we have a decomposition

$$\mathcal{F}^s(z) = (\mathcal{F}^s(z) \cap W^s(\Omega_*^u)) \cup \bigcup_{z' \in \Omega_h \cap \mathcal{F}^s(z)} V^s(z')$$

of $\mathcal{F}^s(z)$ into mutually disjoint open subsets. It implies that $V^s(z)$ coincides with $\mathcal{F}^s(z)$. □

By Proposition 2.8 and Lemma 2.11, the invariant set $\Omega_h \cap \Omega(\Phi)$ admits a spectral decomposition $(\{\Lambda_1, \dots, \Lambda_m\}, \preceq)$. Take a maximal element Λ_+ with respect to \preceq . By the same argument as the hyperbolic case, we have $W^s(\Lambda_+) \cap W^u(\Lambda_+) \subset \Omega(\Phi)$. The maximality implies that $W^s(\Lambda_+) \subset \Lambda_+ \cup W^u(\Omega_*^s)$. Since $W^u(\Lambda_+) \cap W^u(\Omega_*^s) = \emptyset$, we have $W^s(\Lambda_+) \cap W^u(\Lambda_+) = \Lambda_+$.

Recall that a subset Λ of M is called a saturated set of \mathcal{F}^s if $\mathcal{F}^s(z) \subset \Lambda$ for any $z \in \Lambda$.

LEMMA 2.13. — Λ_+ is a closed saturated set of \mathcal{F}^s .

Proof. — We will show $W^s(\Lambda_+) \subset \Lambda_+$. It completes the proof of the lemma since $\mathcal{F}^s(z) = V^s(z) \subset W^s(\Lambda_+)$ for any $z \in \Lambda_+$.

Suppose that $W^s(\Lambda_+) \not\subset \Lambda_+$. Then, there exists $z_* \in \Lambda_+$ which is accessible from $V^s(z_*) \setminus \Lambda_+$. Since $W^s(\Lambda_+) \cap W^u(\Lambda_+) = \Lambda_+$, we can apply Lemma 2.9 to Λ_+ . Hence, we may assume that z_* is a periodic point. Accessibility implies that a connected component L of $V^s(z_*) \setminus \mathcal{O}(z_*)$ is a subset of $W^s(\Lambda_+) \setminus \Lambda_+$, and hence, is contained in $W^u(\Omega_*^s)$.

Take a simple closed curve $\gamma \subset L$ which is homotopic to $\mathcal{O}(z_*)$ in $\mathcal{F}^s(z_*)$. Since z_* is a u -regular periodic point, the holonomy of \mathcal{F}^s along γ is non-trivial.

Since $W^u(T)$ is a connected open subset of M for any torus T in Ω_*^s , there exists a torus T_* in Ω_*^s such that $L \subset W^u(T_*)$. Take an embedding $\psi : \mathbb{T}^2 \times [-1, 1] \rightarrow W^u(T_*)$ such that $\psi(\mathbb{T}^2 \times 0) = T_*$ and $\psi(\mathbb{T}^2 \times \{-1, 1\})$ is transverse to the flow. There exists $t > 0$ such that $\Phi^{-t}(\gamma) \subset \psi(\mathbb{T}^2 \times (-1, 1))$. Let \mathcal{G} be the restriction of \mathcal{F} to $\text{Im } \psi$. Since T_* is the unique compact leaf of \mathcal{G} , a classification theorem of C^2 foliation on $\mathbb{T}^2 \times [0, 1]$ due to Moussu and Roussarie [21] implies that T_* is the only leaf of \mathcal{G} that has non-trivial holonomy. It contradicts that the holonomy of \mathcal{F}^s along $\Phi^{-t}(\gamma)$ is non-trivial but $\Phi^{-t}(\gamma)$ is not contained in T_* . □

Recall that a leaf of a codimension-one foliation is called *semi-proper* when it accumulates to itself from at most one side. We also say a leaf is *proper* when it does not accumulate to itself from either sides.

LEMMA 2.14. — *Let \mathcal{G} be a C^2 codimension-one foliation of a closed three-dimensional manifold. Then, any semi-proper leaf of \mathcal{G} diffeomorphic to $S^1 \times \mathbb{R}$ is proper and it has trivial holonomy.*

Proof. — Let L be a leaf of \mathcal{G} which is diffeomorphic to $S^1 \times \mathbb{R}$.

By the level theory of Cantwell and Conlon [9], L is either proper or contained in an exceptional local minimal set. Duminy’s theorem [10] implies that the end set of a semi-proper leaf in an exceptional local minimal set must be a Cantor set. Since the end set of L consists of two points, the leaf L is proper. By a stability theorem of proper leaves with finite ends due to Cantwell and Conlon [8, Theorem 1], L has trivial holonomy. □

Now, we prove Proposition 2.10. Let Φ be a $\mathbb{P}A$ flow satisfying the assumptions of the proposition. Then, \mathcal{F}^s is a C^2 foliation and all periodic points in Ω_h are u -regular. Suppose that the proposition does not hold. Let Λ_+ be the maximal set in the spectral decomposition of $\Omega_h \cap \Omega(\Phi)$. By Lemma 2.13, it is a closed saturated set of \mathcal{F}^s . Since the restriction of Φ to Λ_+ is topologically transitive, the assumption implies $\Lambda_+ \neq M$.

In particular, Λ_+ contains a semi-proper leaf L of \mathcal{F}^s . By Lemma 2.9, L contains a periodic point q in Λ_+ , and hence, it is diffeomorphic to $S^1 \times \mathbb{R}$. Lemma 2.14 implies that the holonomy of \mathcal{F}^s along the orbit of q is trivial. In particular, q is a u -irregular periodic point. However, it contradicts that all periodic points in Ω_h are u -regular.

2.3. Local dynamics at periodic points

Let Φ be a C^2 PA flow with a PA splitting $TM = E^u + E^s$. In this subsection, we suppose that E^u and E^s generate C^2 foliations \mathcal{F}^u and \mathcal{F}^s , respectively. Remark that Ω_*^ρ is a union of closed leaves of \mathcal{F}^ρ for $\rho = s, u$.

The main aim of this subsection is to show the following proposition, which completes the proof of Proposition 2.1 by combining with Proposition 2.10.

PROPOSITION 2.15. — $\text{Per}_{\text{irr}}^u(\Phi) \subset \Omega_*^u$ and $\text{Per}_{\text{irr}}^s(\Phi) \subset \Omega_*^s$.

Fix a family $\{\phi_z : [-1, 1]^2 \rightarrow M\}_{z \in M}$ of C^2 local cross sections so that $\phi_z(0, 0) = z$, $\phi_z([-1, 1] \times y)$ is tangent to E^s , and $\phi_z(x \times [-1, 1])$ is tangent to E^u for any $(x, y) \in [-1, 1]^2$. Let $\{r_z^t\}$ be the family of local returns associated to $\{\phi_z\}_{z \in M}$.

Recall that $D_\delta(z)$ be the δ -ball in $\text{Im } \phi_z$ centered at z . Let $\Delta > 0$ be the constant obtained in Lemma 2.3. For $0 < \delta < \Delta$, put

$$I_\delta^s(z) = D_\delta(z) \cap \phi_z([-1, 1] \times 0),$$

$$I_\delta^u(z) = D_\delta(z) \cap \phi_z(0 \times [-1, 1]).$$

By replacing Δ with a smaller one, we may assume that $I_\delta^u(z)$ and $I_\delta^s(z)$ are intervals for any $z \in M$ and $0 < \delta < \Delta$.

LEMMA 2.16. — *Suppose that sequences $(z_n \in M)_{n \geq 1}$, $(\delta_n > 0)_{n \geq 1}$, and $(t_n > 0)_{n \geq 1}$ satisfy the following properties:*

- $\lim_{n \rightarrow \infty} \delta_n = 0$.
- $r_{z_n}^t(I_{\delta_n}^s(z_n))$ is well-defined for any $n \geq 1$ and $0 \leq t \leq t_n$.
- $\limsup_{n \rightarrow \infty} |r_{z_n}^{t_n}(I_{\delta_n}^s(z_n))| > 0$.

Then, any accumulation point of $\{z_n\}_{n \geq 1}$ is contained in $\text{Per}_{\text{irr}}^s(\Phi) \cup \Omega_*^s$.

Proof. — Take an accumulation point z_* of $(z_n)_{n \geq 1}$. By taking subsequences if it is necessary, we may assume that z_n converges to z_* , $\Phi^{t_n}(z_n)$ converges to a point z_∞ , and $r_{z_n}^{t_n}(I_{\delta_n}^s(z_n))$ converges to an interval $I_\infty \subset I_\Delta^s(z_\infty)$. Remark that t_n goes to infinity. In fact, $|r_{z_n}^T(I_{\delta_n}^s(z_n))|$ converges to zero for any given $T > 0$ since δ_n goes to zero.

The interval I_∞ is a $C^2(\Delta, E^u)$ -interval, By Lemma 2.3, there exists $z' \in \text{Int } I_\infty$ such that its α -limit set $\alpha(z')$ is a periodic orbit in $\text{Per}_{\text{irr}}^s(\Phi)$ or an embedded torus in Ω_*^s . In each case, $N\Phi^{-t}|_{E^u/T\Phi}$ is uniformly contracting on $\alpha(z')$. Hence, there exists a compact neighborhood V of z' in $\mathcal{F}^u(z')$ such that $\bigcap_{t>0} \overline{\bigcup_{t'>t} \Phi^{-t'}(V)} = \alpha(z')$. For any sufficiently large $n \geq 1$, the interval $r_{z_n}^{t_n}(I_{\delta_n}^s(z_n))$ contains a point z'_n of V . Then, $z_* = \lim_{n \rightarrow \infty} (r_{z_n}^{t_n})^{-1}(z'_n)$ is contained in $\alpha(z')$. Hence, z_* is contained in $\text{Per}_{\text{irr}}^s(\Phi)$ or Ω_*^s . \square

LEMMA 2.17. — *Let z_0 be an s -regular periodic point. Then, the following holds:*

- *There exists $\delta > 0$ and $\tau : W^s(\mathcal{O}(z_0)) \rightarrow \mathbb{R}$ such that $I_\delta^s(\Phi^{\tau(z)}(z)) \subset W^s(\mathcal{O}(z_0))$ for any $z \in W^s(\mathcal{O}(z_0))$.*
- *For any $z \in W^s(\mathcal{O}(z_0))$, $\mathcal{F}^s(z) \cap W^s(\mathcal{O}(z_0))$ is an open subset of $\mathcal{F}^s(z)$ with respect to the leafwise topology.*
- *If A is any open annulus such that $\mathcal{O}(z_0) \subset A \subset \mathcal{F}^s(z_0) \cap W^s(\mathcal{O}(z_0))$, then $\bigcup_{t \geq 0} \Phi^{-t}(A)$ is a connected component of $\mathcal{F}^s(\mathcal{O}(z_0)) \cap W^s(\mathcal{O}(z_0))$.*

Proof. — Let T be the period of z_0 . There exists a closed interval $I \subset [-1, 1]$ and C^2 maps $f, g : I \rightarrow [-1, 1]$ such that $0 \in f(I) \subset \text{Int } I$, $r_{z_0}^T \circ \phi_{z_0}(x, y) = \phi_{z_0}(f(x), g(y))$, and $\bigcap_{n \geq 0} f^n(I) = \{0\}$. Put $\Lambda^u = \bigcap_{n \geq 0} g^{-n}(I)$ and $\Lambda_0^u = \{y \in \Lambda^u \mid \lim_{n \rightarrow \infty} g^n(y) = 0\}$. For any $(x, y) \in \text{Int } I \times \Lambda^u$, we have $r_{z_0}^{nT} \circ \phi_{z_0}(x, y) = \phi_{z_0}(f^n(x), g^n(y))$. By the compactness of Λ^u , there exists $\delta > 0$ such that

$$(2.3) \quad I_\delta^s(\phi_{z_0}(x, y)) \subset \phi_{z_0}(\text{Int } I \times y)$$

for any $(x, y) \in f(I) \times \Lambda^u$.

If $(x, y) \in \text{Int } I \times \Lambda_0^u$, then $(f^n(x), g^n(y))$ converges to $(0, 0)$ as n goes to infinity. For any $z \in W^s(\mathcal{O}(z_0))$, its positive orbit $\{\Phi^t(z) \mid t \geq 0\}$ intersects with $\phi_{z_0}(f(I) \times \Lambda_0^u)$. Hence, we have

$$W^s(\mathcal{O}(z_0)) = \bigcup_{t \geq 0} \Phi^{-t} \circ \phi_{z_0}(f(I) \times \Lambda_0^u) = \bigcup_{t \geq 0} \Phi^{-t} \circ \phi_{z_0}(\text{Int } I \times \Lambda_0^u).$$

It completes the proof of the first assertion of the lemma with the inclusion (2.3). The second assertion is an immediate consequence of the first.

Put

$$V_1 = \bigcup_{t \geq 0} \Phi^{-t} \circ \phi_{z_0}(\text{Int } I \times 0),$$

$$V_2 = \bigcup_{t \geq 0} \Phi^{-t} \circ \phi_{z_0}(\text{Int } I \times (\Lambda_0^u \setminus \{0\})).$$

Since $f(I) \subset \text{Int } I$, $g(0) = 0$, and $g(\Lambda_0^u \setminus \{0\}) \subset \Lambda_0^u \setminus \{0\}$, the set $\mathcal{F}^s(z_0) \cap W^s(\mathcal{O}(z_0))$ is a disjoint union of its open subsets V_1 and $V_2 \cap \mathcal{F}^s(z_0)$. Hence, V_1 is a connected component of $\mathcal{F}^s(z_0) \cap W^s(\mathcal{O}(z_0))$. Let A be the annulus in the third assertion of the lemma. Then, $\phi_{z_0}(f^m(I)) \subset A$ for some large $m \geq 1$. It implies

$$V_1 = \bigcup_{t \geq 0} \Phi^{-t} \circ \phi_{z_0}(f^m(I) \times 0) \subset \bigcup_{t \geq 0} \Phi^{-t}(A) \subset V_1.$$

□

LEMMA 2.18. — *Let z_0 and z_1 be periodic points of Φ .*

- (1) *If z_0 is s -regular and z_1 is accessible from a connected component V of $W^s(\mathcal{O}(z_0)) \cap \mathcal{F}^s(z_1)$. then $\mathcal{F}^s(z_0) = \mathcal{F}^s(z_1)$, the orbits of z_0 and z_1 are homotopic in $\mathcal{F}^s(z_0)$ as unoriented curves, and V contains $\mathcal{O}(z_0)$.*
- (2) *If z_0 attracting and z_1 is accessible from $W^s(\mathcal{O}(z_0)) \cap \mathcal{F}^u(z_1)$, then $\mathcal{F}^u(z_0) = \mathcal{F}^u(z_1)$ and the orbits of z_0 and z_1 are homotopic in $\mathcal{F}^u(z_0)$ as unoriented curves.*

Proof. — First, we show the former assertion of the lemma. Suppose that z_0 is s -regular. Let T be the period of z_0 . There exists a closed interval I and C^2 maps $f, g : I \rightarrow [-1, 1]$ such that $0 \in \text{Int } I \subset f(I)$, $\bigcap_{n \geq 0} f^n(I) = \{0\}$, $\phi_{z_0}(I \times I) \cap \mathcal{O}(z_0) = \{z_0\}$, and $r_{z_0}^T \circ \phi_{z_0}(x, y) = \phi_{z_0}(f(x), g(y))$ for any $(x, y) \in I \times I$. Put $U = \bigcup_{t=0}^T r_{z_0}^t(I \times I)$ and let $\mathcal{G}(y)$ be the connected component of $\mathcal{F}^s(\phi_{z_0}(0, y)) \cap U$ which contains $\phi_{z_0}(0, y)$. Then, we can see the following properties of \mathcal{G} and U :

- $\mathcal{G}(y)$ is not contractible if and only if $g(y) = y$.
- If $z = \phi_{z_0}(x, y)$ is a point of $W^s(\mathcal{O}(z_0))$ and $\Phi^t(z) \in U$ for any $t \geq 0$, then $g^n(y)$ converges to 0 as n tends to infinity.

Suppose that a periodic point z_1 is accessible from a connected component V of $\mathcal{F}^s(z_1) \cap W^s(\mathcal{O}(z_0))$. Then, there exists a simple closed curve γ in V which is homotopic to $\mathcal{O}(z_1)$. By the Poincaré-Bendixon theorem, $\mathcal{O}(z_1)$ and γ are not null-homotopic in $\mathcal{F}^s(z_1)$. We can take $t_1 > 0$ such that $\Phi^t(\gamma) \subset U$ for any $t \geq t_1$. It implies that $\Phi^{t_1}(\gamma) \subset \mathcal{G}(y)$ for some $y \in \bigcap_{n \geq 0} g^{-n}(I)$ with $\lim_{n \rightarrow \infty} g^n(y) = 0$. Since the closed curve $\Phi^{t_1}(\gamma)$ is not null-homotopic in $\mathcal{G}(y) \subset \mathcal{F}^s(z_1)$, we have $y = 0$. Hence, $\Phi^{t_1}(\gamma)$ is homotopic to $\mathcal{O}(z_0)$ in $\mathcal{F}^s(z_0)$ as an unoriented curve and the set V intersects the connected component of $\mathcal{F}^s(z_0) \cap W^s(\mathcal{O}(z_0))$ which contains $\mathcal{O}(z_0)$. Therefore, we have $\mathcal{F}^s(z_1) = \mathcal{F}^s(z_0)$, $\mathcal{O}(z_0)$ and $\mathcal{O}(z_1)$ are homotopic in $\mathcal{F}^s(z_0)$ as unoriented curves, and $\mathcal{O}(z_0) \subset V$.

Next, we show the latter assertion of the lemma. Suppose that z_0 is attracting. Let T be the period of z_0 . There exists a closed interval I and C^2 maps $f_1, g_1 : I' \rightarrow [-1, 1]$ such that $0 \in \text{Int } I' \subset g_1(I')$, $\bigcap_{n \geq 0} g_1^n(I') = \{0\}$, 0 is the unique fixed point of $f_1|_{I'}$, $\phi_{z_0}(I \times I') \cap \mathcal{O}(z_0) = \{z_0\}$, and $r_{z_0}^T \circ \phi_{z_0}(x, y) = \phi_{z_0}(f_1(x), g_1(y))$ for any $(x, y) \in I' \times I'$. Put $U' = \bigcup_{t=0}^T r_{z_0}^t(I \times I)$ and let $\mathcal{G}'(x)$ be the connected component of $\mathcal{F}^u(\phi_{z_0}(x, 0)) \cap U'$ which contains $\phi_{z_0}(x, 0)$. Then, we can see the following properties of \mathcal{G}' and U' :

- $\mathcal{G}'(x)$ is not contractible if and only if $x = 0$.
- If $z = \phi_{z_0}(x, y)$ is a point of $W^s(\mathcal{O}(z_0))$ and $\Phi^t(z) \in U'$ for any $t \geq 0$, then $f_1^n(x)$ converges to 0 as n tends to infinity.

Now, the same argument as above, where we replace g, \mathcal{G} , and \mathcal{F}^s with f_1, \mathcal{G}' and \mathcal{F}^u , respectively, shows the latter assertion of the lemma. □

LEMMA 2.19. — *The following holds for any s -regular periodic point z_0 :*

- (1) $\mathcal{F}^s(z) \subset W^s(\mathcal{O}(z_0))$ for any $z \in W^s(\mathcal{O}(z_0)) \setminus \mathcal{F}^s(z_0)$.
- (2) $\mathcal{F}^s(z_0) \cap W^s(\mathcal{O}(z_0))$ is homeomorphic to $S^1 \times \mathbb{R}$.
- (3) If $\mathcal{F}^s(z_0) \not\subset W^s(\mathcal{O}(z_0))$, then $\mathcal{F}^s(z_0)$ contains an s -irregular periodic point such that the orbits of z_0 and z_1 are homotopic as unoriented closed curves in $\mathcal{F}^s(z_0)$.

Proof. — Since z_0 is s -regular, there exists an embedded closed annulus $A_0 \subset \mathcal{F}^s(z_0)$ such that $\Phi^t(A_0) \subset \text{Int } A_0$ for any $t > 0$ and $\bigcap_{t > 0} \Phi^t(A_0) = \mathcal{O}(z_0)$. Put $V_0 = \bigcup_{t \geq 0} \Phi^{-t}(A_0)$. It is diffeomorphic to $S^1 \times \mathbb{R}$ and is a connected component of $W^s(\mathcal{O}(z_0)) \cap \mathcal{F}^s(z_0)$ by the third item of Lemma 2.17.

Fix a leaf L of \mathcal{F}^s and a connected component V of $L \cap W^s(\mathcal{O}(z_0))$. We suppose that $V \neq L$ and show $V = V_0$. Take $z_1 \in L \setminus V$ which is accessible from V . There exist sequences $(z'_n \in V)_{n \geq 1}$ and $(\delta_n > 0)_{n \geq 1}$ such that $z_1 \in I_{\delta_n}^s(z'_n)$ for any $n \geq 1$ and δ_n converges to zero. By Lemma 2.17, we can choose $\delta > 0$ and $(T_n > 0)_{n \geq 1}$ such that $I_{\delta}^s(\Phi^{T_n}(z'_n)) \subset W^s(\mathcal{O}(z_0))$ for any $n \geq 1$. Since $\mathcal{O}(z_1) \cap W^s(\mathcal{O}(z_0)) = \emptyset$, there exists a sequence $(t_n \in (0, T_n))_{n \geq 1}$ such that $r_{z_1}^{t_n}(I_{\delta_n}^s(z'_n))$ is well-defined for any $t \in [0, t_n]$ and $|r_{z_1}^{t_n}(I_{\delta_n}^s(z'_n))| = \delta$. By Lemma 2.16, z_1 is a point of $\text{Per}_{\text{irr}}^s(\Phi) \cup \Omega_*^s$. Since $\mathcal{F}^s(z_1)$ contains a periodic point z_1 and some non-periodic points in V , we have $\mathcal{F}^s(z_1) \not\subset \Omega_*^s$. Therefore, z_1 is an s -irregular periodic point. Since z_1 is accessible from $V \subset W^s(\mathcal{O}(z_0))$, we can apply the former part of Lemma 2.18. It implies that $\mathcal{F}^s(z_1) = \mathcal{F}^s(z_0)$, $V = V_0$, and the orbits of z_0 and z_1 are homotopic as unoriented closed curves in $\mathcal{F}^s(z_0)$. □

LEMMA 2.20. — $\text{Per}_{\text{irr}}^s(\Phi) \cap \mathcal{F}^s(z)$ is a closed subset of $\mathcal{F}^s(z)$ for any $z \in M$. Similarly, $\text{Per}_{\text{irr}}^u(\Phi) \cap \mathcal{F}^u(z)$ is a closed subset of $\mathcal{F}^u(z)$ for any $z \in M$.

Proof. — We show the former assertion. The proof of the latter is similar.

Suppose that a sequence $(z_n)_{n \geq 1}$ in $\mathcal{F}^s(z) \cap \text{Per}_{\text{irr}}^s(\Phi)$ converges to z_* with respect to the leafwise topology of $\mathcal{F}^s(z)$. By replacing z_n in its orbit, we may assume that there exists a sequence $(\delta_n > 0)_{n \geq 1}$ such that $\lim_{n \rightarrow \infty} \delta_n = 0$ and $I_{\delta_n}^s(z_n)$ contains z_* for any $n \geq 1$. Let T_n be the period of z_n and $\Delta > 0$ be a constant such that $D_\Delta(z) \subset \text{Im } \phi_z$ for any $z \in M$.

First, we suppose that there exists $n_0 \geq 1$ such that $|r_{z_{n_0}}^t(I_{\delta_{n_0}}^s(z_{n_0}))| \leq \Delta$ for any $t \in [0, T_{n_0}]$. Then, $r_{z_{n_0}}^{T_{n_0}}(I_{\delta_{n_0}}^s(z_{n_0}))$ is well-defined. Since z_* is a limit point of the sequence $(z_n)_{n \geq 1}$ of periodic points with respect to the leafwise topology, we can choose a sequence $(z'_n)_{n \geq 1}$ of periodic points of Φ such that $\lim_{n \rightarrow \infty} z'_n = z_*$ and $z'_n \in I_{\delta_{n_0}}^s(z_{n_0})$ for any n . Since $r_{z_{n_0}}^{T_{n_0}}(z_{n_0}) = z_{n_0}$ and $r_{z_{n_0}}^{T_{n_0}}$ is an orientation preserving homeomorphism on an interval, we have $r_{z_{n_0}}^{T_{n_0}}(z'_n) = z'_n$ for any n . It implies that $r_{z_{n_0}}^{T_{n_0}}(z_*) = z_*$. Therefore, z_* is an s -irregular periodic point of Φ .

Now, it is sufficient to consider the case that there exists a sequence $(t_n \in [0, T_n])_{n \geq 1}$ such that $|r_{z_n}^{t_n}(I_{\delta_n}^s(z_n))| \leq \Delta$ for any $t \in [0, t_n]$ and $|r_{z_n}^{t_n}(I_{\delta_n}^s(z_n))| = \Delta$. By Lemma 2.16, $z_* = \lim_{n \rightarrow \infty} z_n$ is a point of $\text{Per}_{\text{irr}}^s(\Phi) \cup \Omega_*^s$. If $z_* \in \Omega_*^s$, then $\mathcal{F}^s(z_*)$ is an embedded torus contained in $\text{Per}_{\text{irr}}^s(\Phi)$ since $\{z_n\} \subset \text{Per}_{\text{irr}}^s(\Phi) \cap \mathcal{F}^s(z_*)$. □

LEMMA 2.21. — *For any $z_0 \in \text{Per}_{\text{irr}}^u(\Phi)$, the leaf $\mathcal{F}^s(z_0)$ is proper, has trivial holonomy, and is contained in $W^s(\mathcal{O}(z_0))$.*

Proof. — The periodic point z_0 is s -regular. Hence, there exists a closed annulus A^s in $\mathcal{F}^s(z_0) \cap W^s(\mathcal{O}(z_0))$ such that $\Phi^t(A^s) \subset \text{Int } A^s$ for any $t > 0$ and $\bigcap_{t \geq 0} \Phi^t(A^s) = \mathcal{O}(z_0)$. Put $V = \bigcup_{t \geq 0} \Phi^{-t}(A^s)$. Lemmas 2.17 and 2.19 imply that $V = \mathcal{F}^s(z_0) \cap W^s(\mathcal{O}(z_0))$ and $\mathcal{F}^s(z) \subset W^s(\mathcal{O}(z_0))$ for any $z \in W^s(\mathcal{O}(z_0)) \setminus V$. Since z_0 is u -irregular, there exists a closed annulus A_0^u in $\mathcal{F}^u(z_0)$ such that $\mathcal{O}(z_0) \subset \partial A_0^u$, $A_0^u \cap A^s = \mathcal{O}(z_0)$, and $\Phi^t(A_0^u) \subset A_0^u$ for any $t \geq 0$. It is easy to see that $V \cap A_0^u = \mathcal{O}(z_0)$.

First, we show $V = \mathcal{F}^s(z_0)$. Suppose that it does not hold. By Lemma 2.19, $\mathcal{F}^s(z_0)$ contains an s -irregular periodic point z_1 such that the orbits of z_0 and z_1 are homotopic as unoriented closed curves in $\mathcal{F}^s(z_0)$. For $i = 0, 1$, let T_i be the period of z_i . Put $\lambda_i^u = \|D\Phi^{T_i}|_{E^u(z_i)}\|$ and $\lambda_i^s = \|D\Phi^{T_i}|_{E^s(z_i)}\|$. Since z_0 is u -irregular, z_1 is s -irregular, and $TM = E^s + E^u$ is a dominated splitting, we have

$$(2.4) \quad \lambda_0^s < \lambda_0^u \leq 1 \leq \lambda_1^s < \lambda_1^u.$$

Since λ_i^u is the absolute value of the linear holonomy of \mathcal{F}^s along the orbit of z_i for each i and the orbits of z_0 and z_1 are homotopic in $\mathcal{F}^s(z_0)$ as

unoriented curves in $\mathcal{F}^s(z_0)$, we have $\lambda_0^u = (\lambda_1^u)^{\pm 1}$. Hence, the inequality (2.4) implies

$$(2.5) \quad \lambda_0^u = (\lambda_1^u)^{-1} < 1.$$

In particular, $\mathcal{O}(z_0)$ is an attracting periodic orbit. Since $W^s(\mathcal{O}(z_0))$ is an open subset of M , there exists a closed annulus $A_1^u \subset \mathcal{F}^u(z_1)$ such that $\mathcal{O}(z_1) \subset \partial A_1^u$ and $\mathcal{F}^s(z)$ intersects with $(A_0^u \setminus \mathcal{O}(z_0)) \cap W^s(\mathcal{O}(z_0))$ for any $z \in A_1^u \setminus \mathcal{O}(z_1)$. Now, we recall that $V \cap A_0^u = \mathcal{O}(z_0)$ and $\mathcal{F}^s(z) \subset W^s(\mathcal{O}(z_0))$ for any $z \in W^s(\mathcal{O}(z_0)) \setminus V$. They imply that $A_1^u \setminus \mathcal{O}(z_1)$ is contained in $W^s(\mathcal{O}(z_0))$. In particular, z_1 is accessible from $\mathcal{F}^u(z_1) \cap W^s(\mathcal{O}(z_0))$. By Lemma 2.18, we have $\mathcal{F}^u(z_1) = \mathcal{F}^u(z_0)$ and the orbits of z_0 and z_1 are homotopic in $\mathcal{F}^u(z_0)$ as unoriented closed curves. Since λ_i^s is the absolute value of the linear holonomy of \mathcal{F}^u along the orbit of z_i , we have $\lambda_0^s = (\lambda_1^s)^{\pm 1}$. Hence, the inequality (2.4) implies

$$\lambda_0^s = (\lambda_1^s)^{-1}.$$

However, it contradicts with the inequalities (2.4) and (2.5). Therefore, we have $V = \mathcal{F}^s(z_0)$.

Since $V \cap A_0^u = \mathcal{O}(z_0)$, the leaf $\mathcal{F}^s(z_0) = V$ is semi-proper. By Lemma 2.14, $\mathcal{F}^s(z_0)$ is a proper leaf with trivial holonomy. □

Now, we prove Proposition 2.15. We claim that $\text{Per}_{\text{irr}}^u(\Phi) \subset \Omega_*^u$. Once it is done, we can show that $\text{Per}_{\text{irr}}^s(\Phi) \subset \Omega_*^s$ by applying the claim to the time-reverse of Φ .

Take a leaf L^u of \mathcal{F}^u which intersects with $\text{Per}_{\text{irr}}^u(\Phi)$. By Lemmas 2.20 and 2.21, $\text{Per}_{\text{irr}}^u(\Phi) \cap L^u$ is a closed and open subset of L^u . Hence, L^u is a subset of $\text{Per}_{\text{irr}}^u(\Phi)$. It is sufficient to show that L^u is a closed leaf. If it is not, then there exists a transversal J of \mathcal{F}^u such that J is contained in a leaf L^s of \mathcal{F}^s and $J \cap L^u$ is not a relatively compact subset of L^u . Take a point $z_* \in L^u \cap J$. By Lemma 2.21, $J \subset L^s$ is contained in $W^s(\mathcal{O}(z_*))$. Since $L^u \subset \text{Per}(\Phi)$, it implies that $L^u \cap J$ is contained in $\mathcal{O}(z_*)$. However, it contradicts that $L^u \cap J$ is not a relatively compact subset of L^u .

3. Topologically transitive regular $\mathbb{P}A$ flows

The aim of this section is the following proposition, which completes the proof of the main theorem by combining with Propositions 2.1 and 2.2.

PROPOSITION 3.1. — *If a C^2 topologically transitive $\mathbb{P}A$ flow on a closed three-dimensional manifold admits a C^2 $\mathbb{P}A$ splitting, then it is an Anosov flow.*

When the $\mathbb{P}A$ flow Φ admits a global cross section, the proof of Proposition 3.1 is reduced to an observation that the distortion of a holonomy map of a one-dimensional foliation on a surface can be estimated by the area of rectangle swept out by the holonomy. We refer the reader to [3] for the detail. If a $\mathbb{P}A$ flow admits invariant one-dimensional subbundles of E^s and E^u which are transverse to the flow, then we can apply the proof in [3] with a small modification. However, a $\mathbb{P}A$ flow admits no invariant one-dimensional subbundles transverse to the flow in general. This is the main technical difficulty in the proof.

The structure of the section is as follows: In Subsections 3.1 and 3.2, we show the (possibly non-uniform) contraction along the direction transverse to E^u by the standard argument using a Markov partition and a theorem due to Mañé. In Subsection 3.3, in order to overcome the above technical difficulty, we show that the diameter of $\Phi^t(D^s)$ is uniformly bounded for any small disk D^s tangent to E^s and $t \geq 0$ after we replace the original flow Φ by a suitable time-change. This condition makes us enable to apply the method in [3]. It is done in Subsection 3.4.

3.1. One-dimensional topological Markov maps

In this section, we prove some results about piecewise C^2 Markov maps on a finite union of compact intervals, which we use later.

Let I_* be a finite union of compact intervals in \mathbb{R} and Λ_* be a finite subset of $\text{Int } I_*$. We say a map $F : I_* \rightarrow I_*$ is a C^2 pre-Markov map with the set of discontinuity Λ_* if $F(\Lambda_*) \subset \partial I_*$, $\overline{F(I_* \setminus \Lambda_*)} = I_*$, and for each connected component J of $I_* \setminus \Lambda_*$, there exists a connected component I of I_* such that $F|_J$ extends to a C^2 diffeomorphism from \overline{J} to I . Put $I_*^n = I_* \setminus \bigcup_{n'=0}^n F^{-n'}(\Lambda_*)$ and $I_*^\infty = \bigcap_{n \geq 0} I_*^n$. For $x \in I_*^\infty$ and $n \geq 0$, let $I_*^n(x)$ be the connected component of I_*^n that contains x . Then, the restriction of F^m to $\overline{I_*^n(x)}$ extends to a C^2 diffeomorphism onto $\overline{I_*^{n-m}(F^m(x))}$ for any $x \in I_*^n$ and $n \geq m \geq 0$.

For $n \geq 1$, let $\text{Per}_n(F)$ be the set of all periodic points of period n . Since $F(\Lambda_* \cup \partial I_*) \subset \partial I_*$ and $\partial I_* \cap \Lambda_* = \emptyset$, the set $\text{Per}_n(F)$ is a subset of I_*^∞ for any $n \geq 1$.

We say a pre-Markov map F is a C^2 topologically Markov map if $\bigcap_{n \geq 1} \overline{I_*^n(x)}$ consists of exactly one point for any $x \in I_*^\infty$.

LEMMA 3.2. — *Let F be a C^2 topologically Markov map on I_* . Then, $\text{Per}_n(F)$ is finite for any $n \geq 1$ and any periodic point $x \in \text{Per}_n(F)$ satisfies $|DF^n(x)| \geq 1$.*

Proof. — It is an immediate consequence of the equations $\bigcap_{k \geq 0} \overline{I_*^{kn}(x)} = \{x\}$ and

$$F^n \left(\overline{I_*^{(k+1)n}(x)} \right) = \overline{I_*^{kn}(x)} \supset \overline{I_*^{(k+1)n}(x)}$$

for any $x \in \text{Per}_n(F)$. □

Recall the definition and some properties of the distortion of a one-dimensional map. For a C^2 map $h : I \rightarrow I'$ between intervals I and I' , we define *the distortion* $\text{dist}(h, I)$ by

$$\text{dist}(h, I) = \sup_{x, y \in I} (\log |Dh(x)| - \log |Dh(y)|).$$

It is easy to verify

$$(3.1) \quad \text{dist}(h_n \circ \dots \circ h_0, I) \leq \sum_{m=0}^n \text{dist}(h_m, h_{m-1} \circ \dots \circ h_0(I)).$$

By the Mean Value Theorem for h and $\log |Dh|$, we also have

$$(3.2) \quad \text{dist}(h, I) \geq \sup_{x \in I} \left| \log |Dh(x)| - \log \frac{|h(I)|}{|I|} \right|,$$

$$(3.3) \quad \text{dist}(h, I) \leq |I| \cdot \sup_{x \in I} |D(\log |Dh|)(x)|.$$

Let $\text{Per}_*(F)$ be the set of non-hyperbolic periodic points. The following proposition is a variant of Mané’s theorem([18]).

PROPOSITION 3.3. — *Let F be a C^2 topologically Markov map on I_* and Λ_* be the set of discontinuity of F . Then,*

$$(3.4) \quad \lim_{n \rightarrow \infty} (\inf\{|DF^n(x)| \mid x \in \text{Per}_n(F)\}) = \infty$$

and $\text{Per}_*(F)$ consists of only finitely many points. Moreover, for any given neighborhood U of $\text{Per}_*(F)$,

$$(3.5) \quad \lim_{n \rightarrow \infty} (\inf\{|DF^n(x)| \mid x \in I_*^n \setminus F^{-n}(U)\}) = +\infty.$$

Proof. — The equation (3.4) is a consequence of Theorem 5.1 of [17], which is a version of Mañé’s theorem for piecewise C^2 maps. By Lemma 3.2, it implies the finiteness of non-hyperbolic periodic points.

Since I_*^n consists of finitely many connected components for any n and $\bigcap_{n \geq 0} \overline{I_*^n(x)} = \{x\}$ for any $x \in I_*^\infty$, there exist sequences $(K_n)_{n=0}^\infty$ and $(K'_n)_{n=0}^\infty$ such that $\lim_{n \rightarrow \infty} K_n = \lim_{n \rightarrow \infty} K'_n = 0$ and $K_n \leq |I_*^n(x)| \leq K'_n$ for any $n \geq 0$ and $x \in I_*^n$.

Fix a neighborhood U of $\text{Per}_*(F)$. By the finiteness of $\text{Per}_*(F)$, there exists $N \geq 1$ such that $I_*^N(x_*) \subset U$ for any $x_* \in \text{Per}_*(F)$. Remark that

$I_*^N(x) \cap \text{Per}_*(F) = \emptyset$ if $x \notin U$. Put

$$c_1 = |I_*| \cdot \sup\{|D(\log |DF|)|(x) \mid x \in I_* \setminus \Lambda_*\},$$

$$c_2 = \inf\{|DF(x)| \mid x \in I_* \setminus \Lambda_*\}.$$

We say that an interval $J \subset I_*$ is (λ, m) -compatible for $\lambda > 1$ and $m \geq 1$ if $J \subset I_*^m$ and $\sum_{i=0}^m |F^i(J)| < \frac{\lambda}{\lambda-1} |I_*|$. The properties of distortion imply

$$\inf_{x \in J} |DF^m(x)| \geq e^{-\frac{c_1 \lambda}{\lambda-1}} \cdot \frac{|F^m(J)|}{|J|}$$

for any (λ, m) -compatible interval J .

By the argument in Section III.5 of [11], it can be shown that there exists $\lambda > 1$ and $n_0 \geq 1$ such that $I_*^{n+N}(x)$ is a $(\lambda, n - n_0)$ -compatible interval if $x \in I_*^{n+N}$ satisfies $I_*^N(F^n(x)) \cap \text{Per}_*(F) = \emptyset$. It implies that

$$|DF^n(x)| \geq |DF^{n_0}(F^{n-n_0}(x))| \cdot e^{-\frac{c_1 \lambda}{\lambda-1}} \frac{|I_*^{n+n_0}(F^{n-n_0}(x))|}{|I_*^{n+N}(x)|}$$

$$\geq \frac{c_2^{n_0} e^{-\frac{c_1 \lambda}{\lambda-1}} K_{N+n_0}}{K'_n}$$

for any $n \geq 1$ and $x \in I_*$ with $I_*^N(F^n(x)) \cap \text{Per}_*(F) = \emptyset$. Since $\lim_{n \rightarrow \infty} K'_n = 0$, it completes the proof. □

3.2. Markov partitions

Fix a closed three-dimensional Riemannian manifold M . Let $\{(\phi_k, \tau_k)\}_{k=1}^m$ be a family of pairs of a continuous embedding of $[0, 1]^2$ into M and a continuous positive-valued function on $[0, 1]^2$ such that the map $(w, t) \mapsto \Phi^t \circ \phi_k(w)$ is an embedding of $\{(w, t) \mid w \in [0, 1]^2, t \in [0, \tau_k(w)]\}$ into M for each k . We define a family $\{\phi'_k\}_{k=1}^m$ of embeddings of $[0, 1]^2$ into M by $\phi'_k(w) = \Phi^{\tau_k(w)} \circ \phi_k(w)$. Put $R_k = \text{Im } \phi_k$, $R'_k = \text{Im } \phi'_k$, and $P_k = \{\Phi^t \circ \phi_k(w) \mid w \in [0, 1]^2, t \in [0, \tau(w)]\}$.

We say the family $\{(\phi_k, \tau_k)\}$ determines a *Markov partition* $\{P_k\}_{k=1}^m$ associated to a flow Φ if it satisfies the following properties:

- (1) $\phi_k([0, 1] \times y)$ and $\phi_k(x \times [0, 1])$ are intervals tangent to E^s and E^u respectively, for any $(x, y) \in [0, 1]^2$ and $k = 1, \dots, m$.
- (2) $M = \bigcup_{k=1}^m P_k$.
- (3) For each pair (k, l) , $\partial(P_k \cap P_l) = \partial P_k \cap \partial P_l$ and there exist subintervals $I_{k,l}$ and $J_{k,l}$ of $[0, 1]$, which may be empty sets, such that

$$R'_k \cap R_l = \phi'_k([0, 1] \times I_{k,l}) = \phi_l(J_{k,l} \times [0, 1]).$$

Remark that $\{\text{Int } I_{k,l'}\}_{l'=1}^m$ and $\{\text{Int } J_{k',l}\}_{k'=1}^m$ are partition of $[0, 1]$ up to finite set for any fixed (k, l) . Put $I_* = [0, 1] \times \{1, \dots, m\}$. The family $\{(\phi_k, \tau_k)\}_{k=1}^m$ induces a piecewise continuous map $F : I_* \rightarrow I_*$ by $F(y, k) = (y', l)$ if $\phi_l(J_{k,l} \times y') = \phi'_k([0, 1] \times y)$. We call the map F *the reduced return map*.

We say a $\mathbb{P}A$ flow Φ is E^s -fine if

- (1) both Ω_* and $\text{Per}_{\text{irr}}(\Phi)$ are empty, and
- (2) Φ admits a $\mathbb{P}A$ splitting $TM = E^u + E^s$ such that E^s is a C^2 subbundle.

Remark that Φ is topologically transitive by Proposition 2.10.

LEMMA 3.4. — *Let Φ be a C^2 E^s -fine $\mathbb{P}A$ flow. For any given $\epsilon > 0$, Φ admits a Markov partition $\{P_k\}_{k=1}^m$ such that the reduced return map is a C^2 topological Markov map and the diameter of each P_l is smaller than ϵ .*

Proof. — As we see in Section 2, the flow Φ has the shadowing property on M . Hence, we can obtain a Markov partition $\{P_k\}_{k=1}^m$ associated with Φ such that the diameter of each P_k is smaller than a given constant $\epsilon > 0$ by the same argument as the hyperbolic case (see e.g. [26] for the hyperbolic case). By the C^2 -regularity of E^s , we can choose $\{\phi_k\}$ so that $\pi_y \circ \phi_k^{-1}$ is of class C^2 , where $\pi_y(x, y) = y$. It implies that F is a piecewise C^2 map. It is easy to check that F is a pre-Markov map with the set of discontinuity $(\bigcup_{k,l=1}^m \partial J_{k,l}) \setminus \partial I_*$. By Proposition 2.4, F is a topologically Markov map. □

For $\sigma = s, u$, let $\text{Per}_*^\sigma(\Phi)$ be the set of periodic point z_* such that $\|N\Phi^{t_*}|_{(E^\sigma/T\Phi)(z_*)}\| = 1$, where t_* is the period of z_* . The following is an immediate consequence of the above lemma and Proposition 3.3.

PROPOSITION 3.5. — *If a C^2 $\mathbb{P}A$ flow Φ is E^s -fine, $\text{Per}_*^u(\Phi)$ consists of finitely many orbits. Moreover, for any given neighborhood U of $\text{Per}_*^u(\Phi)$, there exists $T = T(U) > 0$ such that*

$$\sup \{ \|N\Phi^{-t}|_{(E^u/T\Phi)(z)}\| \mid t \geq T, z \in M \setminus U \} \leq \frac{1}{2}.$$

COROLLARY 3.6. — *If a C^2 $\mathbb{P}A$ flow Φ is E^s -fine and $\text{Per}_*^u(\Phi)$ is empty, then there exists $C > 0$ and $\lambda > 1$ such that $\|N\Phi^{-t}|_{E^u/T\Phi(z)}\| < C\lambda^{-t}$ for any $z \in M$ and $t > 0$.*

3.3. Non-expansion property

Let M be a closed three-dimensional Riemannian manifold and Φ be a C^2 E^s -fine $\mathbb{P}A$ flow. As we saw in Subsection 2.1, there exists a family

$\{V^u(z)\}_{z \in M}$ of C^2 immersed submanifolds of M which are tangent to E^u . For $z \in M$ and $\delta > 0$, let $B(z, \delta)$ be the closed δ -ball centered at z and $D^u(z, \delta)$ the connected component of $V^u(z) \cap B(z, \delta)$ that contains z . By the definition of $V^u(z)$, we can see that

- $V^u(\Phi^t(z)) = \Phi^t(V^u(z))$ for any $z \in M$ and $t \in \mathbb{R}$,
- $D^u(z, \delta)$ is a C^2 embedded disk which varies continuously with respect to z if δ is sufficiently small.

We say that Φ is *u-bounded* if there exists a positive-valued function $\bar{\delta}$ on \mathbb{R} such that

$$\Phi^{-t}(D^u(z, \bar{\delta}(\epsilon))) \subset D^u(\Phi^{-t}(z), \epsilon)$$

for any $z \in M, t \geq 0$, and $\epsilon > 0$. Similarly, we can define a C^2 disk $D^s(z, \delta)$ which is tangent to E^s for any $z \in M$ and any sufficiently small $\delta > 0$. We say that Φ is *s-bounded* if there exists a positive-valued function $\bar{\delta}'$ on \mathbb{R} such that

$$\Phi^t(D^s(z, \bar{\delta}'(\epsilon))) \subset D^s(\Phi^t(z), \epsilon)$$

for any $z \in M, t \geq 0$, and $\epsilon > 0$.

If E^u admits a continuous $D\Phi$ -invariant splitting $E^u = T\Phi \oplus E^{uu}$, then we can show Φ is *u-bounded* by Proposition 2.4. However, a $\mathbb{P}A$ flow is not *u-bounded* in general. In fact, if there exists a Φ invariant embedded annulus tangent to E^u on which Φ^t is conjugate to the map $(x, y) \mapsto (x + (1 + y)t, y)$ on $S^1 \times [0, 1]$, then Φ is not *u-bounded*.

We say that Φ admits a local invariant foliation \mathcal{G} transverse to a compact Φ -invariant set Λ if \mathcal{G} is a C^2 two dimensional foliation on an open neighborhood U_* of Λ which is transverse to the orbits of Φ and satisfies $D\Phi^t(T\mathcal{G}(z)) = T\mathcal{G}(\Phi^t(z))$ for any $t \geq 0$ and $z \in \bigcap_{t' \in [0, t]} \Phi^{-t'}(U_*)$. Remark that a suitable time change of Φ admits a local invariant foliation transverse to $\text{Per}_*^u(\Phi)$ if $\text{Per}_*^u(\Phi)$ consists of finite number of orbits.

The aim of this subsection is to show the *u-boundedness* of Φ under a mild assumption.

PROPOSITION 3.7. — *If a C^2 E^s -fine $\mathbb{P}A$ flow Φ admits a local invariant foliation transverse to $\text{Per}_*^u(\Phi)$, then Φ is *u-bounded*.*

Fix a local invariant foliation \mathcal{G} which is transverse to $\text{Per}_*^u(\Phi)$ on a neighborhood U_* of $\text{Per}_*^u(\Phi)$.

LEMMA 3.8. — *For any given neighborhood U of $\text{Per}_*^u(\Phi)$,*

$$\sup \{ \|D\Phi^{-t}|_{E^u(z)}\| \mid t \geq 0, z \in M \setminus U \} < \infty.$$

Proof. — By taking a finite covering, we may assume that E^u is orientable without loss of generality. By X , we denote the vector field generating Φ . Fix a continuous vector field Y^u such that $\{X(z), Y^u(z)\}$ spans $E^u(z)$ for any $z \in M$ and $Y^u(z)$ is tangent to $\mathcal{G}(z)$ if $z \in U_*$. We define functions η and λ on $M \times \mathbb{R}$ by

$$D\Phi^{-t}(Y^u(z)) = \eta(z, t)X(\Phi^{-t}(z)) + \lambda(z, t)Y^u(\Phi^{-t}(z)).$$

Since $D\Phi^{-t}(X(z)) = X(\Phi^{-t}(z))$, the following identities hold:

$$\begin{aligned} \eta(z, t + t') &= \eta(z, t) + \lambda(z, t) \cdot \eta(\Phi^{-t}(z), t'), \\ \lambda(z, t + t') &= \lambda(z, t) \cdot \lambda(\Phi^{-t}(z), t'). \end{aligned}$$

By the local invariance of \mathcal{G} , if $\Phi^{-t}(z) \in U_*$ for any $0 < t < t_0$ then $\eta(z, t_0) = 0$.

Without loss of generality, we may assume that U is a subset of U_* . By Proposition 3.5, there exists $T > 0$ such that $|\lambda(z, t)| < 1/2$ for any $t \geq T$ and $z \in M \setminus U$. Take a constant $K_0 > 1$ such that $|\lambda(z, t)| + |\eta(z, t)| \leq K_0 - 1$ for any $z \in M$ and any $0 \leq t \leq T$. It is sufficient to show that $|\lambda(z, t)| + |\eta(z, t)| \leq 4K_0$ for any $z \in M \setminus U$ and $t > 0$.

We define a function $\tau : M \setminus U \rightarrow [T, \infty]$ by

$$\tau(z) = \begin{cases} \inf\{t \geq T \mid \Phi^{-t}(z) \notin U\} & \text{if } z \notin \bigcap_{t \geq T} \Phi^t(U) \\ \infty & \text{otherwise.} \end{cases}$$

We claim that $|\lambda(z, t)| + |\eta(z, t)| \leq K_0$ for any $z \in M \setminus U$ and $0 \leq t \leq \tau(z)$. It is trivial if $t \leq T$. Suppose that $T < t \leq \tau(z)$. Then, $\Phi^{-t}(z)$ is contained in U for any $T \leq t' < t$, and hence, $\eta(\Phi^{-T}(z), t - T) = 0$. Since $\lambda(z, t) < 1/2$, we have

$$\begin{aligned} |\lambda(z, t)| + |\eta(z, t)| &= |\lambda(z, t)| + |\eta(z, T) + \lambda(z, T) \cdot \eta(\Phi^{-T}(z), t - T)| \\ &< \frac{1}{2} + |\eta(z, T)| \leq K_0. \end{aligned}$$

It completes the proof of the claim.

Fix $z \in M \setminus U$. Take a sequence $(t_i)_{i \geq 0}$ in $[0, \infty]$ such that $t_0 = 0$, $t_{i+1} = t_i + \tau(\Phi^{-t_i}(z))$ for any $i \geq 0$. We claim

$$(3.6) \quad \lambda(z, t_i) \leq 2^{-i}, \quad \eta(z, t_i) \leq 2K_0(1 - 2^{-i})$$

for any $i \geq 0$. The proof is by induction. The inequalities for $i = 0$ is trivial. Suppose that they hold for i . Since $T \leq t_{i+1} - t_i = \tau(\Phi^{-t_i}(z))$ and $\tau(\Phi^{-t_i}(z)) \notin U$, we have $|\lambda(\Phi^{-t_i}(z), t_{i+1} - t_i)| \leq 1/2$. The first claim also

implies $|\eta(\Phi^{-t_i}(z), t_{i+1} - t_i)| \leq K_0$. By the assumption of induction,

$$\begin{aligned} |\lambda(z, t_{i+1})| &= |\lambda(z, t_i)| \cdot |\lambda(\Phi^{-t_i}(z), t_{i+1} - t_i)| \leq 2^{-i} \cdot 2^{-1} = 2^{-(i+1)}, \\ |\eta(z, t_{i+1})| &\leq |\eta(z, t_i)| + |\lambda(z, t_i)| \cdot |\eta(\Phi^{t_i}(z), t_{i+1} - t_i)| \\ &\leq 2K_0(1 - 2^{-i}) + 2^{-i}K_0 = 2K_0(1 - 2^{-(i+1)}). \end{aligned}$$

Therefore, the inequalities (3.6) hold for $i + 1$. The claim is proved.

Take $t > 0$. There exists $i \geq 0$ such that $t_i \leq t < t_{i+1}$. Since $0 \leq t - t_i < \tau(\Phi^{-t_i}(z))$, the above claims imply

$$\begin{aligned} |\lambda(z, t)| &= |\lambda(z, t_i)| \cdot |\lambda(\Phi^{-t_i}(z), t - t_i)| \leq 1 \cdot K_0 = K_0, \\ |\eta(z, t)| &\leq |\eta(z, t_i)| + |\lambda(z, t_i)| \cdot |\eta(\Phi^{-t_i}(z), t - t_i)| \\ &\leq 2K_0 + 1 \cdot K_0 = 3K_0. \end{aligned}$$

□

LEMMA 3.9. — *There exists a neighborhood U_1 of $\text{Per}_*^u(\Phi)$ and a positive-valued function $\bar{\delta}_1$ on \mathbb{R} such that*

$$\Phi^{-t}(D^u(z, \bar{\delta}_1(\epsilon))) \subset D^u(\Phi^{-t}(z), \epsilon)$$

for any $\epsilon > 0$, $t > 0$, and $z \in \bigcap_{t' \in [0, t]} \Phi^{t'}(U_1)$.

Proof. — For an interval $J \subset \mathbb{R}$ and a subset S of M , we denote the set $\{\Phi^t(z) \mid z \in S, t \in J\}$ by $\Phi^J(S)$.

Fix a neighborhood U_1 of $\text{Per}_*^u(\Phi)$ and a constant $0 < \epsilon' < \epsilon/2$ such that $\mathcal{N}_{\epsilon'}(U_1) \cap \mathcal{N}_{\epsilon'}(M \setminus U_*) = \emptyset$. Take a family $\{\phi_z\}_{z \in M}$ of local cross-sections so that $\text{Im } \phi_z \subset \mathcal{G}(z)$ for any $z \in U_1$. Let $\{r_z^t\}$ be the family of returns and put $I_\delta^u(z) = D^u(z, \delta) \cap \text{Im } \phi_z$. Since $\{D^u(z, \Delta)\}$ is a continuous family of C^2 -disks, $\{I_\delta^s(z)\}$ is a continuous family of C^2 -intervals. By Proposition 2.4, there exists $\delta' > 0$ such that $r_z^{-t}(I_{\delta'}^u(z)) \subset I_{\epsilon'}^u(\Phi^{-t}(z))$ for any $t \geq 0$ and $z \in M$. Take an open interval $J \subset \mathbb{R}$ containing 0 such that $\Phi^J(I_{\epsilon'}^u(z)) \subset D^u(z, 2\epsilon')$ for any $z \in M$. By the invariance of \mathcal{G} , we have $r_z^{-t}(I_{\delta'}^u(z)) = \Phi^{-t}(I_{\delta'}^u(z))$ for any $z \in \bigcap_{t' \in [0, t]} \Phi^{t'}(U_1)$. We also take $\delta > 0$ so that $D^u(z, \delta) \subset \Phi^J(I_{\delta'}^u(z))$ for any $z \in M$. Then, we have

$$\begin{aligned} \Phi^{-t}(D^u(z, \delta)) &\subset \Phi^{-t}(\Phi^J(I_{\delta'}^u(z))) = \Phi^J(\Phi^{-t}(I_{\delta'}^u(z))) = \Phi^J(r_z^{-t}(I_{\delta'}^u(z))) \\ &\subset \Phi^J(I_{\epsilon'}^u(\Phi^{-t}(z))) \subset D^u(\Phi^{-t}(z), 2\epsilon') \subset D^u(\Phi^{-t}(z), \epsilon). \end{aligned}$$

for any $t > 0$ and $z \in \bigcap_{t' \in [0, t]} \Phi^{t'}(U_1)$. □

Now, we prove Proposition 3.7. Fix $\epsilon > 0$. Let U_1 and $\bar{\delta}_1$ be the neighborhood of $\text{Per}_*^u(\Phi)$ and the function that are given by Lemma 3.9. There

exists a neighborhood U_2 of $\text{Per}_*^u(\Phi)$ and a constant $0 < \epsilon_2 < \epsilon$ such that $\mathcal{N}_{\epsilon_2}(U_2) \cap \mathcal{N}_{\epsilon_2}(M \setminus U_1) = \emptyset$. By Lemma 3.8,

$$K = 1 + \sup \{ \|D\Phi^{-t}|_{E^u(z)}\| \mid t \geq 0, z \in M \setminus U_2 \}$$

is finite. Put $\delta_2 = \bar{\delta}_1(\epsilon_2 K^{-1})$ and $\delta = \min\{\delta_2, \epsilon_2 K^{-1}\}$. It is sufficient to show the inclusion

$$(3.7) \quad \Phi^{-t}(D^u(z, \delta)) \subset D^u(\Phi^{-t}(z), \epsilon)$$

for any $z \in M$ and $t \geq 0$.

For $z \notin U_1$, we have $D^u(z, \epsilon_2 K^{-1}) \cap U_2 = \emptyset$, and hence,

$$(3.8) \quad \Phi^{-t}(D^u(z, \epsilon_2 K^{-1})) \subset D^u(\Phi^{-t}(z), \epsilon_2) \subset D^u(\Phi^{-t}(z), \epsilon)$$

for any $t \geq 0$. It implies the inclusion (3.7) for $z \notin U_1$. If $z \in U_1$, put $T = \inf\{t > 0 \mid \Phi^{-t}(z) \notin U_1\} \in (0, \infty]$. For $0 \leq t < T$, we have

$$\Phi^{-t}(D^u(z, \delta_2)) \subset D^u(\Phi^{-t}(z), \epsilon_2 K^{-1}) \subset D^u(\Phi^{-t}(z), \epsilon).$$

It implies the inclusion (3.7) for the case $T = \infty$ or $0 < t < T$. If T is finite, then $\Phi^{-T}(D^u(z, \delta_2)) \subset D^u(\Phi^{-T}(z), \epsilon_2 K^{-1})$. Since $\Phi^{-T}(z) \notin U_1$, we have $\Phi^{-(T+t')}(D^u(z, \delta_2)) \subset D^u(\Phi^{-(T+t')}(z), \epsilon)$ for any $t' \geq 0$ by (3.8). It implies the inclusion (3.7) for the case $z \in U_1$ and $t \geq T$.

3.4. Hyperbolicity of periodic orbits

The following proposition is the last piece of the proof of Proposition 3.1.

PROPOSITION 3.10. — *If a C^2 E^s -fine $\mathbb{P}A$ flow Φ is s - and u -bounded, then $\text{Per}_*^u(\Phi)$ is empty.*

Proof of Proposition 3.1. — Let Φ be a topologically transitive $\mathbb{P}A$ flow with a C^2 $\mathbb{P}A$ splitting. Topological transitivity implies that Ω_* is empty. By Proposition 2.15, all periodic points are s - and u -regular. In particular, Φ and Φ^{-1} are E^s -fine.

By Proposition 3.5 for Φ and Φ^{-1} , we see that $\text{Per}_*(\Phi)$ consists of finite number of orbits. Take a time-change Φ_1 of Φ which admits a local invariant foliation transverse to $\text{Per}_*(\Phi)$. Remark that both Φ_1 and Φ_1^{-1} are E^s -fine. We apply Propositions 3.7 and 3.10 to Φ_1 and Φ_1^{-1} . The former implies Φ_1 is s - and u -bounded. The latter for Φ_1 implies that $\text{Per}_*^u(\Phi_1)$ is empty and the same for Φ_1^{-1} implies that $\text{Per}_*^s(\Phi_1)$ is empty. Hence, Φ_1 is an Anosov flow by Corollary 3.6. Since Φ is a time-change of Φ_1 , also Φ is. □

The rest of the subsection is devoted to the proof of Proposition 3.10. Fix a C^2 E^s -fine $\mathbb{P}A$ flow Φ on a closed three-dimensional manifold M which is s - and u - bounded. Let $TM = E^u + E^s$ be a $\mathbb{P}A$ splitting of Φ such that E^s generates a C^2 foliation \mathcal{F}^s . Remark that $D^s(z, \delta) \subset \mathcal{F}^s(z)$ for any $z \in M$ and $\delta > 0$.

Take a family $\{\psi_z\}_{z \in M}$ of C^2 embeddings from $[-1, 1]^3$ to M such that $\psi_z(0, 0, 0) = z$, $\psi_z([-1, 1]^2 \times y) \subset \mathcal{F}^s(\psi_z(0, 0, y))$ for any $y \in [-1, 1]$, and the map $(z, w) \mapsto \psi_z(w)$ from $M \times [-1, 1]^3$ to M is of class C^2 . By $B(z, \delta)$, we denote the closed ball of radius δ which is centered at z . There exists $\epsilon_0 > 0$ such that $B(z, 8\epsilon_0) \subset \text{Im } \psi_z$ for any $z \in M$. Since Φ is s - and u -bounded, we can take $\delta_0 > 0$ so that $\Phi^t(D^s(z, \delta_0)) \subset D^s(\Phi^t(z), \epsilon_0)$ and $\Phi^{-t}(D^u(z, \delta_0)) \subset D^u(\Phi^{-t}(z), \epsilon_0)$ for any $z \in M$ and $t \geq 0$.

Suppose that $\text{Per}_*^u(\Phi)$ is non-empty and contains a point p . There exists a continuous injective map $H : (-1, 1)^2 \rightarrow M$ such that

- (1) $H(0, 0) = p$,
- (2) $\text{Im } H \subset \text{Im } \psi_p$,
- (3) $H(x, \cdot)$ is of class C^2 and $H(x \times (-1, 1)) \subset D^u(H(x, 0), \delta_0)$ for any $x \in (-1, 1)$, and
- (4) $H((0, 1) \times y) \subset D^s(H(0, y), \delta_0)$ for any $y \in (-1, 1)$.

We put $V = \bigcup_{x \in (-1, 1)} D^u(H(x, 0), \delta_0)$.

By Proposition 3.5, $\text{Per}_*^u(\Phi)$ consists of finitely many orbits. Since Φ is topologically transitive, the union of periodic orbits is a dense subset of M . Hence, $H((-1, 1)^2)$ contains a hyperbolic periodic point q . Put $(x_*, y_*) = H^{-1}(q)$.

For $x \in [0, x_*]$ and $t \geq 0$, we put $J^u(x, t) = \Phi^{-t}(H(x \times [0, y_*]))$. Let $V(x, t)$ be the arcwise connected component of $\Phi^{-t}(V) \cap B(\Phi^{-t}(H(x, 0)), 3\epsilon_0)$ which contains $\Phi^{-t}(H(x, 0))$. Since

$$J^u(x, t) \subset \Phi^{-t}(D^u(H(x, 0), \delta_0) \subset D^u(\Phi^{-t}(H(x, 0)), \epsilon_0),$$

we have $J^u(x, t) \subset V(x, t)$.

Let $\pi_y : \mathbb{R}^3 \rightarrow \mathbb{R}$ be the map defined by $\pi_y(w, x, y) = y$. Put $I(x, t) = \pi_y \circ \psi_{\Phi^{-t}(H(x, 0))}^{-1}(J^u(x, t))$. We define a map $h_{x,t} : [0, y_*] \rightarrow I(x, t)$ by

$$h_{x,t}(y) = \pi_y \circ \psi_{\Phi^{-t}(H(x, 0))}^{-1}(\Phi^{-t}(H(x, y))).$$

Remark that it is a C^2 diffeomorphism.

The map $h_{x_*,t} \circ h_{0,0}^{-1}$ can be decomposed in two ways;

$$\begin{aligned} (3.9) \quad h_{x_*,t} \circ h_{0,0}^{-1} &= (h_{x_*,t} \circ h_{x_*,0}^{-1}) \circ (h_{x_*,0} \circ h_{0,0}^{-1}) \\ &= (h_{x_*,t} \circ h_{0,t}^{-1}) \circ (h_{0,t} \circ h_{0,0}^{-1}). \end{aligned}$$

We estimate the distortion of each decomposition. It will lead us to a contradiction.

LEMMA 3.11. — $\{\text{dist}(h_{x_*,t} \circ h_{x_*,0}^{-1}, I(x_*, 0)) \mid t \geq 0\}$ is bounded.

Proof. — Let T be the period of q and put $h = h_{x_*,T} \circ h_{x_*,0}^{-1}$. Then, $I(x_*, T) \subset I(x_*, 0)$ and the map h is C^2 conjugate to the local return map of Φ^{-1} on $J^u(x_*, 0)$. Since q is a hyperbolic periodic point, there exist $C > 0$ and $\lambda \in (0, 1)$ such that $|h^n(I(x_*, T))| < C\lambda^n$ for any $n \geq 0$. Take $K > 0$ so that $|D(\log |Dh|)(y)| < K$ for any $y \in I_{x_*,0}$. Then, we have

$$\begin{aligned} \text{dist}(h_{x_*,nT} \circ h_{x_*,0}^{-1}, I(x_*, 0)) &= \text{dist}(h^n, I(x_*, 0)) \\ &= \sum_{m=0}^{n-1} \text{dist}(h, h^m(I(x_*, 0))) \\ &\leq \sum_{m=0}^{n-1} K \cdot C\lambda^m < KC(1 - \lambda)^{-1}. \end{aligned}$$

Since $\text{dist}(h_{x_*,t} \circ h_{x_*,0}^{-1}, (I(x_*, 0)))$ is continuous with respect to t , it is bounded on $[0, T]$. Hence, the lemma follows from the formula (3.1). \square

We will show that the distortion of the last term of (3.9) is unbounded. Once it is shown, it contradicts Lemma 3.11, and hence, $\text{Per}_*^u(\text{Per}(\Phi))$ is empty.

LEMMA 3.12. — $\{\text{dist}(h_{0,t} \circ h_{0,0}^{-1}, I(0, 0)) \mid t \geq 0\}$ is unbounded.

Proof. — Let T be the period of $p = h_{0,0}^{-1}(0)$. Put $h = h_{0,T} \circ h_{0,0}^{-1}$. Since h is C^2 conjugate to the return map of Φ^{-1} on $J^u(0, 0)$, we have $I(0, T) = h(I(0, 0)) \subset I(0, 0)$, $\bigcap_{n \geq 1} h^n(I(0, 0)) = \{0\}$, $h(0) = 0$, and $|Dh(0)| = 1$. By the formula (3.2),

$$\begin{aligned} \text{dist}(h_{0,nT} \circ h_{0,0}^{-1}, I(0, 0)) &= \text{dist}(h^n, I(0, 0)) \\ &\geq |\log |Dh^n(0)| - \log |h^n(I(0, 0))| + \log |I(0, 0)|| \\ &= |-\log |h^n(I(0, 0))| + \log |I(0, 0)||. \end{aligned}$$

The last term goes to infinity as n tends to infinity since $\lim_{n \rightarrow \infty} |h^n(I(0, 0))| = 0$. \square

To estimate the distortion of $h_{x_*,t} \circ h_{0,t}^{-1}$, we need some preparations.

LEMMA 3.13. — If $V(x_1, t) \cap V(x_2, t) \neq \emptyset$ for $0 \leq x_1 < x_2 \leq x_*$ and $t \geq 0$, then

$$\bigcup_{x \in [x_1, x_2]} J^u(x, t) \subset \text{Im } \psi_{\Phi^{-t}(H(x_1, t))}$$

Proof. — Since $V(x_1, t) \cup V(x_2, t)$ is arcwise connected, we can take a continuous map $L: [0, 1] \rightarrow V(x_1, t) \cup V(x_2, t)$ such that $L(0) = \Phi^{-t}(H(x_1, 0))$ and $L(1) = \Phi^{-t}(H(x_2, 0))$. It defines a continuous map $l: [0, 1] \rightarrow (-1, 1)$ such that $L(\xi) \in \Phi^{-t}(D^u(H(l(\xi)), 0), \delta_0)$ for any $\xi \in [0, 1]$. The diameter of $V(x_1, t) \cup V(x_2, t)$ is not greater than $6\epsilon_0$ and

$$J^u(l(\xi), t) \cup \{L(\xi)\} \subset \Phi^{-t}(D^u(H(l(\xi)), 0), \delta_0) \subset D^u(\Phi^{-t}(H(l(\xi)), 0), \epsilon_0).$$

Hence, $J^u(l(\xi), t)$ is contained in $B(\Phi^{-t}(H(x_1, 0)), 8\epsilon_0)$ for any $\xi \in [0, 1]$. Since $[x_1, x_2] \subset \text{Im } l$ and $B(\Phi^{-t}(H(x_1, 0)), 8\epsilon_0) \subset \text{Im } \psi_{\Phi^{-t}(H(x_1, 0))}$, the proof is complete. \square

Let $\text{Vol}(\cdot)$ be the volume on M associated with the fixed Riemannian metric of M .

LEMMA 3.14. — *There exists a constant $K_* > 0$ such that*

$$|I(x, t)| \leq K_* \text{Vol}(V(x, t))$$

for any $x \in [0, x_*]$ and $t \geq 0$.

Proof. — Since $H([0, x_*] \times [0, y_*])$ is contained in $\text{Int } V$, there exists $\epsilon_1 > 0$ such that $D^s(z, \epsilon_1) \subset V$ for any $z \in H([0, x_*] \times [0, y_*])$. Since Φ is s - and u -bounded, we can take $\delta_1 > 0$ such that $\Phi^t(D^s(z, \delta_1)) \subset D^s(\Phi^t(z), \epsilon_1)$ and $\Phi^{-t}(D^u(z, \delta_1)) \subset D^u(\Phi^{-t}(z), \epsilon_1)$ for any $z \in M$.

Put $C(x, t) = \bigcup_{z \in J^u(x, t)} D^s(z, \delta_1)$. For $z = \Phi^{-t}(H(x, y)) \in J^u(x, t)$, $\Phi^t(D^s(z, \delta_1))$ is contained in $D^s(\Phi^t(z), \epsilon_1) = D^s(H(x, y), \epsilon_1)$, and hence, in V . Since

$$J^u(x, t) \subset \Phi^{-t}(D^u(H(x, 0), \delta_0)) \subset D^u(\Phi^{-t}(H(x, 0)), \epsilon_0),$$

we have

$$C(x, t) \subset \Phi^{-t}(V) \cap B(\Phi^{-t}(H(x, 0)), \epsilon_0 + \delta_1) \subset V(x, t).$$

By the C^2 smoothness of the map $(z, w, x, y) \mapsto \psi_z(w, x, y)$, there exists $K_0 > 0$ such that $K_0^{-1}\|v\| \leq \|D\psi_z^{-1}(v)\| \leq K_0\|v\|$ for any $z \in M$, $z' \in \text{Im } \psi_z$, and $v \in T_{z'}M$. Let Leb_n be the Lebesgue measure on \mathbb{R}^n . Since

$$\text{Leb}_2 \left(\psi_{\Phi^{-t}(H(x, 0))}^{-1} (D^s(z, \delta_1)) \right) \geq \pi \delta_1^2 K_0^{-2}$$

for any $z \in J^u(x, t)$, we have

$$\begin{aligned} (3.10) \quad |I(x, t)| &= |\pi_y \circ \psi_{\Phi^{-t}(H(x, 0))}^{-1} (J^u(x, t))| \\ &\leq \frac{K_0^2}{\pi \cdot \delta_1^2} \cdot \text{Leb}_3 \left(\psi_{\Phi^{-t}(H(x, 0))}^{-1} (C(x, t)) \right) \\ &\leq \frac{K_0^5}{\pi \cdot \delta_1^2} \cdot \text{Vol} (C(x, t)) \leq \frac{K_0^5}{\pi \cdot \delta_1^2} \cdot \text{Vol} (V(x, t)). \end{aligned}$$

□

Now, we estimate the distortion of $h_{x_*,t} \circ h_{0,t}^{-1}$.

LEMMA 3.15. — $\{\text{dist}(h_{x_*,t} \circ h_{0,t}^{-1}, I(0, t)) \mid t \geq 0\}$ is bounded.

Proof. — For $z, z' \in M$, the map $\psi_z^{-1} \circ \psi_{z'}$ can be written as

$$\psi_z^{-1} \circ \psi_{z'}^{-1}(w, x, y) = (f_{z,z'}(w, x, y), g_{z,z'}(y))$$

by a map $f_{z,z'}$ valued in \mathbb{R}^2 and a function $g_{z,z'}$. Take $K_1 > 0$ so that $|D(\log |Dg_{z,z'}|)(y)| \leq K_1$ for any $z, z' \in M$ and any y in the domain of $g_{z,z'}$.

By Lemma 3.13, if $V(x_1, t) \cap V(x_2, t) \neq \emptyset$ for $0 \leq x_1 < x_2 \leq x_*$, then $\pi_y \circ \psi_{\Phi^{-t}(H(x_1, 0))}(J^u(x, t)) = I(x_1, t)$ for any $x \in [x_1, x_2]$. It implies that $h_{x_2,t} \circ h_{x_1,t}^{-1} = g_{z_2(t), z_1(t)}$, where $z_i(t) = \Phi^{-t}(H(x_i, 0))$. Hence, we have

$$\text{dist}(h_{x_2,t} \circ h_{x_1,t}^{-1}, I(x_1, t)) \leq K_1.$$

Fix $t \geq 0$. Let \mathcal{S} be the set of sequences $(x_i)_{i=0}^m$ that satisfy $x_0 = 0$, $x_m = x_*$, and $V(x_{i+1}, t) \cap V(x_i, t) \neq \emptyset$ for any $i = 0, \dots, m - 1$. It is non-empty by the compactness of $[0, 1]$. Take $(x_i)_{i=0}^m \in \mathcal{S}$ such that m is minimal in \mathcal{S} . For any $z \in M$, the minimality of m implies that the number of $V(x_i, t)$ containing z is at most two. Let K_* be the constant given by Lemma 3.14. Then, we have

$$\begin{aligned} \text{dist}(h_{x_*,t} \circ h_{0,t}^{-1}, I(0, t)) &\leq \sum_{i=0}^{m-1} \text{dist}(h_{x_{i+1},t} \circ h_{x_i,t}^{-1}, I(x_i, t)) \\ &\leq K_1 \sum_{i=0}^{m-1} |I(x_i, t)| \\ &\leq K_* K_1 \sum_{i=0}^{m-1} \text{Vol}(V(x_i, t)) \\ &\leq 2K_* K_1 \text{Vol}(M). \end{aligned}$$

Since K_1 and K_* does not depend on t , the lemma is proved. □

Since the second component of the middle term of (3.9) does not depend on t , Lemma 3.11 implies that the distortion of the middle term is bounded with respect to t . It contradicts Lemmas 3.12 and 3.15, which imply that the distortion of the last term of (3.9) is unbounded with respect to t . Therefore, $\text{Per}_*^u(\Phi)$ is empty. Now, the proof of Proposition 3.10 is finished.

4. Foliations with tangentially contracting flows

In this section, we prove Theorem 1.5.

Let \mathcal{F} be a C^2 foliation on a closed three-dimensional manifold M . Suppose that \mathcal{F} admits a C^2 tangentially contracting flow Φ . Let $C > 0$ and $\lambda > 1$ be constants such that $\|N\Phi^t|_{(T\mathcal{F}/T\Phi)(z)}\| \leq C\lambda^{-t}$ for any $z \in M$ and $t \geq 0$.

LEMMA 4.1. — *There exists a continuous subbundle E^u of TM such that Φ is a $\mathbb{P}A$ flow with a $\mathbb{P}A$ splitting $TM = T\mathcal{F} + E^u$.*

Proof. — The proof is almost identical to Lemme IV.1.1 in [14].

The differential of the flow Φ induces a flow $N_{\mathcal{F}}\Phi$ on $TM/T\mathcal{F}$. Let S_* be the set of points $z \in M$ that satisfies

$$(4.1) \quad \limsup_{t \rightarrow \infty} \frac{1}{t} \log \|N_{\mathcal{F}}\Phi_z^t\| \leq -\frac{\log \lambda}{3}.$$

We will show that S_* must be empty. Once it is shown, we have

$$\liminf_{t \rightarrow \infty} \frac{\|N\Phi^t_{(T\mathcal{F}/T\Phi)(z)}\|}{\|N_{\mathcal{F}}\Phi_z^t\|} = 0$$

for any $z \in M$. By a standard argument (see e.g. Proposition 2.3 of [5]), we can show that there exists a continuous subbundle E^u of TM such that $TM = T\mathcal{F} + E^u$ is a $\mathbb{P}A$ splitting for Φ .

Suppose that S_* is not empty. Take a point z_0 in S_* . First, we claim that Φ admits an attracting periodic point. As an accumulation point of the uniform measures on $\{\Phi^t(z_0) \mid t \in [0, T]\}$ with $T \rightarrow \infty$, we obtain a Φ -invariant Borel probability measure m_* such that

$$\int_M \left(\frac{d}{dt} \log \|N_{\mathcal{F}}\Phi_z^t\| \Big|_{t=0} \right) dm_*(z) \leq -\frac{\log \lambda}{3}.$$

It implies that there exists a Borel subset U_* of M such that $m_*(U_*) > 0$ and all Lyapunov exponents of Φ are negative on U_* . By Pesin theory, the ω -limit set of any point of U_* is an attracting periodic point. Therefore, the claim is proved.

Suppose that z_a is an attracting periodic point of Φ . Since Φ is tangentially contracting, there exists a compact embedded annulus A in $\mathcal{F}(z_a)$ such that $\Phi^t(A) \subset \text{Int } A$ for any $t > 0$, $\bigcap_{t \geq 0} \Phi^t(A) = \mathcal{O}(z_a)$, and $\bigcup_{t \geq 0} \Phi^{-t}(A) = \mathcal{F}(z_a)$. In particular, the leaf $\mathcal{F}^s(z_a)$ is diffeomorphic to $S^1 \times \mathbb{R}$. Since z_a is attracting, we can take a compact neighborhood U of $\mathcal{O}(z_a)$ in M such that $\partial A \cap U = \emptyset$ and $\Phi^t(U) \subset \text{Int } U$ for any $t > 0$. By the choice of A , we have $U \cap \mathcal{F}(z_a) = U \cap A$. It implies that $\mathcal{F}(z_a)$ is a proper leaf.

By Lemma 2.14, $\mathcal{F}^s(z_a)$ has trivial holonomy. However, it contradicts that $\mathcal{O}(z_a)$ is an attracting periodic orbit. □

LEMMA 4.2. — *The $\mathbb{P}A$ flow Φ is E^s -fine.*

Proof. — By the strong stable manifold theorem, each leaf of \mathcal{F} is diffeomorphic to \mathbb{R}^2 or $S^1 \times \mathbb{R}$. In particular, \mathcal{F} has no closed leaves. By Duminy’s theorem, there exists no exceptional minimal set of \mathcal{F} . Hence, each leaf of \mathcal{F} is dense in M . By the same argument as the above lemma, if z_0 is a u -irregular periodic point, then $\mathcal{F}(z_0)$ is semi-proper. However, it contradicts that each leaf of \mathcal{F} is dense in M . Therefore, any periodic point is u -regular. Since Φ is tangentially contracting, any periodic point is s -regular.

By Proposition 2.10, either Ω_* is empty or $M = W^u(\Omega_*^s) \cap \Omega_*^u$. Since \mathcal{F} has no closed leaves, Ω_*^s is empty. It implies that the latter case can not occur. Therefore, Ω_* is empty. □

PROPOSITION 4.3. — *The flow Φ is an Anosov flow.*

Proof. — Let $TM = T\mathcal{F} + E^u$ be a $\mathbb{P}A$ splitting for Φ . Since Φ is tangentially contracting with respect to \mathcal{F} , we have $\text{Per}_*^s(\Phi) = \emptyset$. In particular, $\text{Per}_*(\Phi) = \text{Per}_*^u(\Phi)$.

By Proposition 3.5, $\text{Per}_*^u(\Phi)$ consists of finitely many non-hyperbolic periodic orbits if it is not empty. Any time-change of Φ preserves each leaf of \mathcal{F} and is tangentially contracting with respect to \mathcal{F} . Hence, we may assume that Φ admits a local invariant foliation transverse to $\text{Per}_*(\Phi) = \text{Per}_*^u(\Phi)$ (see the beginning of Subsection 3.4). By Proposition 3.7, Φ is u -bounded.

By the same argument as the hyperbolic case in [12], there exists a continuous $D\Phi$ -invariant splitting $T\mathcal{F} = T\Phi \oplus E^{ss}$ and constants $C > 0$ and $\lambda > 0$ such that $\|D\Phi^t|_{E^{ss}(z)}\| \leq C\lambda^{-t}$ for any $z \in M$ and $t \geq 0$. It implies that $\{\|D\Phi^t|_{T\mathcal{F}(z)}\| \mid z \in M, t \geq 0\}$ is bounded. Hence, Φ is s -bounded.

Since Φ is a C^2 E^s -fine $\mathbb{P}A$ flow, Proposition 3.10 implies $\text{Per}_*(\Phi) = \text{Per}_*^u(\Phi) = \emptyset$. By Corollary 3.6, Φ is an Anosov flow. □

Now, we prove Theorem 1.5. By Théorème 4.1 of [15], \mathcal{F} admits a C^r transverse projective structure. By Théorème 5.1 of [6] (cf. Théorème 4.7 of [15]) Φ is topologically equivalent to an algebraic Anosov flow. Therefore, \mathcal{F} is homeomorphic to the weak stable foliation of an algebraic Anosov flow. Such a C^r foliation with $r \geq 2$ is classified completely by Ghys [15] and Ghys and Sergiescu [16]. Their result implies that \mathcal{F} is C^r -diffeomorphic to the weak stable foliation of an algebraic Anosov flow.

BIBLIOGRAPHY

- [1] A. ARROYO & F. RODRIGUEZ HERTZ, “Homoclinic bifurcations and uniform hyperbolicity for three-dimensional flows”, *Ann. Inst. H. Poincaré Anal. Non Linéaire* **20** (2003), p. 805-841.
- [2] M. ASAOKA, “Non-homogeneous locally free actions of the affine group”, preprint. arXiv:math.0702833.
- [3] ———, “A classification of three dimensional regular projectively Anosov flows”, *Proc. Japan Acad., Ser. A* **80** (2004), no. 10, p. 194-197.
- [4] ———, “Classification of regular and non-degenerate projectively Anosov diffeomorphisms on three dimensional manifolds”, *J. Math. Kyoto Univ.* **46** (2006), no. 2, p. 349-356.
- [5] ———, “Codimension-one foliations with a transversely contracting flow”, in *Foliations 2005*, World Sci. Publ., Hackensack, NJ, 2006, p. 21-36.
- [6] T. BARBOT, “Caractérisation des flots d’Anosov en dimension 3 par leurs feuilletages faibles”, *Ergodic Theory Dynam. Systems* **15** (1995), no. 2, p. 247-270.
- [7] C. BONATTI, L. J. DÍAZ & M. VIANA, *Dynamics beyond uniform hyperbolicity*, Encyclopaedia of Mathematical Sciences, vol. 102, Springer-Verlag, Berlin, 2005, A global geometric and probabilistic perspective, Mathematical Physics, III, xviii+384 pages.
- [8] J. CANTWELL & L. CONLON, “Reeb stability for noncompact leaves in foliated 3-manifolds”, *Proc. Amer. Math. Soc.* **33** (1981), no. 2, p. 408-410.
- [9] ———, “The theory of levels”, in *Index theory of elliptic operators, foliations, and operator algebras (New Orleans, LA/Indianapolis, IN, 1986)*, Contemp. Math., vol. 70, Amer. Math. Soc., Providence, RI, 1988, p. 1-10.
- [10] ———, “Endsets of exceptional leaves; a theorem of G. Duminy”, in *Foliations: geometry and dynamics (Warsaw, 2000)*, World Sci. Publ., River Edge, NJ, 2002, p. 225-261.
- [11] W. DE MELO & S. VAN STRIEN, *One-dimensional dynamics*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], vol. 25, Springer-Verlag, Berlin, 1993, xiv+605 pages.
- [12] C. I. DOERING, “Persistently transitive vector fields on three-dimensional manifolds”, in *Dynamical systems and bifurcation theory (Rio de Janeiro, 1985)*, Pitman Res. Notes Math. Ser., vol. 160, Longman Sci. Tech., Harlow, 1987, p. 59-89.
- [13] Y. M. ELIASHBERG & W. P. THURSTON, *Confoliations*, University Lecture Series, vol. 13, American Mathematical Society, Providence, RI, 1998, x+66 pages.
- [14] E. GHYS, “Actions localement libres du groupe affine”, *Invent. Math.* **82** (1985), no. 3, p. 479-526.
- [15] ———, “Rigidité différentiable des groupes fuchiens”, *Inst. Hautes Études Sci. Publ. Math.* **78** (1993), p. 163-185.
- [16] E. GHYS & V. SERGIESCU, “Stabilité et conjugaison différentiable pour certains feuilletages”, *Topology* **19** (1980), no. 2, p. 179-197.
- [17] A. J. HOMBURG, “Piecewise smooth interval maps with non-vanishing derivative”, *Ergodic Theory Dynam. Systems* **20** (2000), no. 3, p. 749-773.
- [18] R. MAÑÉ, “Hyperbolicity, sinks and measure in one-dimensional dynamics”, *Comm. Math. Phys.* **100** (1985), no. 4, p. 495-524, Erratum. *Comm. Math. Phys.* **112** (1987), no. 4, 721-724.
- [19] Y. MITSUMATSU, “Anosov flows and non-stein symplectic manifolds”, *Ann. Inst. Fourier* **45** (1995), no. 5, p. 1407-1421.
- [20] ———, “Foliations and contact structures on 3-manifolds”, in *Foliations: geometry and dynamics (Warsaw, 2000)*, World Sci. Publ., River Edge, NJ, 2002, p. 75-125.

- [21] R. MOUSSU & R. ROUSSARIE, “Relations de conjugaison et de cobordisme entre certains feuilletages”, *Inst. Hautes Études Sci. Publ. Math.* **43** (1974), p. 142-168.
- [22] S. NEWHOUSE & J. PALIS, “Hyperbolic nonwandering sets on two-dimensional manifolds”, in *Dynamical systems (Proc. Sympos., Univ. Bahia, Salvador, 1971)*, Academic Press, New York, 1973, p. 293-301.
- [23] T. NODA, “Projectively Anosov flows with differentiable (un)stable foliations”, *Ann. Inst. Fourier* **50** (2000), no. 5, p. 1617-1647.
- [24] ———, “Regular projectively Anosov flows with compact leaves”, *Ann. Inst. Fourier* **54** (2004), no. 2, p. 353-363.
- [25] T. NODA & T. TSUBOI, “Regular projectively Anosov flows without compact leaves”, in *Foliations: geometry and dynamics (Warsaw, 2000)*, World Sci. Publ., River Edge, NJ, 2002, p. 403-419.
- [26] M. E. RATNER, “Markov decomposition for an U-flow on a three-dimensional manifold”, *Mat. Zametki* **6** (1969), p. 693-704 (Translation to English) *Math. Notes* **6** 1969, 880–886.
- [27] M. SHUB, *Global stability of dynamical systems*, Springer-Verlag, New York, 1987, With the collaboration of Albert Fathi and Rémi Langevin, Translated from the French by Joseph Christy, xii+150 pages.
- [28] T. TSUBOI, “Regular projectively Anosov flows on the Seifert fibered 3-manifolds”, *J. Math. Soc. Japan* **56** (2004), no. 4, p. 1233-1253.

Manuscrit reçu le 24 avril 2007,
révisé le 14 août 2009,
accepté le 8 septembre 2009.

Masayuki ASAOKA
Kyoto University
Department of Mathematics
Kitashirakawa Oiwakecho, Sakyo-ku
606-8502 Kyoto (Japan)
asaoka@math.kyoto-u.ac.jp