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UNIFORM MINIMALITY, UNCONDITIONALITY AND INTERPOLATION IN BACKWARD SHIFT INVARIANT SUBSPACES

by Eric AMAR & Andreas HARTMANN

ABSTRACT. — We discuss relations between uniform minimality, unconditionality and interpolation for families of reproducing kernels in backward shift invariant subspaces. This class of spaces contains as prominent examples the Paley-Wiener spaces for which it is known that uniform minimality does in general neither imply interpolation nor unconditionality. Hence, contrarily to the situation of standard Hardy spaces (and of other scales of spaces), changing the size of the space seems necessary to deduce unconditionality or interpolation from uniform minimality. Such a change can take two directions: lowering the power of integration, or “increasing” the defining inner function (e.g. increasing the type in the case of Paley-Wiener space). Khinchin’s inequalities play a substantial role in the proofs of our main results.

RÉSUMÉ. — Nous étudions des relations entre l’uniforme minimalité, l’inconditionnalité et l’interpolation pour des familles de noyaux reproduisants dans des espaces invariants par l’adjoint du shift. Cette classe d’espaces contient en particulier les espaces de Paley-Wiener pour lesquels il est connu que l’uniforme minimalité n’entraîne en général pas l’inconditionnalité. Par conséquent, et contrairement à la situation dans les espaces de Hardy habituels (et dans d’autres échelles d’espaces), il semble nécessaire de changer la taille de l’espace afin de déduire l’inconditionnalité (ou l’interpolation) de l’uniforme minimalité. Un tel changement de la taille de l’espace peut être opéré de deux façons différentes : en diminuant l’exposant d’intégration, ou en “augmentant” la fonction définissante de l’espace (ce qui revient à augmenter le type dans le cas des espaces de Paley-Wiener). Les inégalités de Khinchin jouent un rôle central dans les preuves de nos résultats principaux.

1. Introduction

A famous result by Carleson states that a sequence of points $S = \{a_k\}$ in the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ is an interpolating sequence for

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the space H^∞ of bounded analytic functions on \mathbb{D} , meaning that every bounded sequence indexed by S can be interpolated by a function f in H^∞ on S , i.e., $H^\infty|_S \supset l^\infty$, if and only if the sequence S satisfies the Carleson condition:

$$\inf_{a \in S} |B_a(a)| = \delta > 0,$$

where $B_a = \prod_{u \neq a} b_u$ is the Blaschke product vanishing exactly on $S \setminus \{a\}$, and $b_a(z) = \frac{|a|}{a} \frac{a-z}{1-\bar{a}z}$ (see [6]). We will write $S \in (C)$ for short when S satisfies (1). Obviously in this situation we also have the embedding $H^\infty|_S \subset l^\infty$, so that $S \in (C)$ is equivalent to $H^\infty|_S = l^\infty$. Subsequently it was shown by Shapiro and Shields [26] that for $p \in [1, \infty)$ a similar result holds:

$$H^p|_S \supset l^p(1 - |a|^2) = \left\{ (v_a)_{a \in S} : \sum_{a \in S} (1 - |a|^2) |v_a|^p < \infty \right\}$$

if and only if $S \in (C)$. Here H^p is the Hardy space of holomorphic functions f on \mathbb{D} for which $\|f\|_p^p := \sup_{0 < r < 1} \int_{-\pi}^{\pi} |f(re^{it})|^p dt < +\infty$.

Again, it turns out that we also have $H^p|_S \subset l^p(1 - |a|^2)$ (the measure $\sum_{a \in S} (1 - |a|^2) \delta_a$ is a so-called Carleson measure), so that $S \in (C)$ is equivalent to $H^p|_S = l^p(1 - |a|^2)$. Considering reproducing kernels $k_a(z) = (1 - \bar{a}z)^{-1}$ the interpolation condition and the Carleson condition can be restated in terms of geometric properties of the sequence $(k_a)_{a \in S}$. More precisely, letting p' the conjugate exponent of p , the Carleson condition is equivalent to $(k_a / \|k_a\|_{p'})_{a \in S}$ being uniformly minimal in $H^{p'}$, and the condition $H^p|_S = l^p(1 - |a|^2)$, which is a priori stronger than the sole interpolation condition, to $(k_a / \|k_a\|_{p'})_{a \in S}$ being an unconditional sequence in $H^{p'}$ (precise definitions will be given below; note that unconditionality is more naturally related with so-called *free* interpolation). Hence, another way of stating the interpolation result in Hardy spaces is to say that a sequence of normalized reproducing kernels in $H^{p'}$ is uniformly minimal if and only if it is an unconditional basis in its span (since interpolation in the scale of Hardy spaces does not depend on p , the distinction between p and p' is rather artificial here). This special situation is not isolated. It turns out to be true in the Bergman space (see [23]), and in Fock spaces and Paley-Wiener spaces for certain indices of p (see [24]).

More recently, in [3] the first named author has given a method allowing to deduce interpolation from uniform minimality when the size of the space is increased by lowering the power of integration. This result requires that the underlying space is the L^p -closure of a uniform algebra, and applies in particular to Hardy spaces on the ball.

We would like to use some of the methods discussed in [3] and based on Khinchin's inequalities to show that uniform minimality implies interpolation in a bigger space for certain backward shift invariant subspaces K_I^p . Bigger means here lowering p and/or "increasing" I (replacing I by IE where E is another inner function). This is done in Theorem 4.5. However, in order to obtain the stronger condition of unconditionality of the normalized reproducing kernels in the bigger space, a Carleson type embedding is required (this follows from the interpolation part in H^p , $1 < p < +\infty$, see Shapiro-Shields [26]). We will discuss this situation in Corollary 4.6 appealing to some results by Volberg and Treil [28] on Carleson measures in backward shift invariant subspaces.

Note that the Paley-Wiener spaces are a particular instance of backward shift invariant subspaces K_I^p (we discuss this in more details in Section 3). Recall that for an inner function I , $K_I^p = H^p \cap \overline{IH}_0^p$ (when considered as a space of functions on \mathbb{T}), where $H_0^p := e^{i\theta}H^p = \{f \in H^p : \hat{f}(0) = 0\}$. In particular when $p = 2$, then K_I^2 is the orthogonal complement of IH^2 . Note also that these spaces are projected subspaces of H^p ($1 < p < \infty$), and the projection — orthogonal when $p = 2$ — is given by $P_I = IP_- \bar{I}$, where $P_- = Id - P_+$ and P_+ is the Riesz projection of $f(e^{it}) = \sum_{n \in \mathbb{Z}} a_n e^{int} \in L^p(\mathbb{T})$ onto the analytic part $\sum_{n \geq 0} a_n e^{int}$. The Paley-Wiener spaces PW_τ^p appear in the special situation when $I(z) = I_\tau(z) := \exp(2\tau(z+1)/(z-1))$. Then, K_I^p is isomorphic to PW_τ^p which is the space of entire functions of exponential type at most τ and p -th power integrable on the real line (see Section 3). By the Paley-Wiener theorem, PW_τ^2 is isometrically isomorphic to $L^2(-\tau, \tau)$. Already in this "simple" case no reasonable description of interpolating sequences is known (see more comments after Theorem 1.1 below). There exist sufficient density conditions for unconditionality (and thus interpolation) when $p = 2$. They allow to check that a certain uniform minimal sequence, which is not unconditional (not interpolating), becomes unconditional (interpolating) when we "increase" the inner function meaning that we replace I by $I^{1+\varepsilon}$, $\varepsilon > 0$. (It is well known that $K_I^2 \subset K_{I^{1+\varepsilon}}^2$ and even $K_{I^{1+\varepsilon}}^2 = K_I^2 \oplus IK_{I^\varepsilon}^2$.) The density conditions for $p = 2$ do not seem to generalize to $p \neq 2$ (see Proposition 3.2 and comments at the end of Section 3), so that there is no easy argument that could show that lowering the integration power alone without changing I is sufficient to deduce interpolation (or unconditionality) from uniform minimality. This makes the problem very delicate.

As a consequence of our discussions to come we state here a sample result (see Corollaries 4.7 and 4.8). Note that the reproducing kernel in K_I^p

is given by

$$k_a^I(z) = (P_I k_a)(z) = \frac{1 - \overline{I(a)}I(z)}{1 - \overline{a}z}, \quad z \in \mathbb{D},$$

for $a \in \mathbb{D}$.

THEOREM 1.1. — *Let I be a singular inner function, $S \subset \mathbb{D}$ a sequence, and $1 < p \leq 2$. Suppose that $\sup_{a \in S} |I(a)| < 1$. If $(k_a^I / \|k_a^I\|_{p'})_{a \in S}$ is uniformly minimal in $K_I^{p'}$, where $1/p + 1/p' = 1$, then for every $\varepsilon > 0$ and for every $s < p$, S is an interpolating sequence for $K_{I^{1+\varepsilon}}^s$ and $(k_a^{I^{1+\varepsilon}} / \|k_a^{I^{1+\varepsilon}}\|_{s'})_{a \in S}$ is an unconditional sequence in $K_{I^{1+\varepsilon}}^{s'}$, $1/s + 1/s' = 1$.*

It can be noted that our results also give the existence of an interpolation operator so that, as a byproduct, we obtain sufficient conditions for a space $N := \{f \in K_J^s : f|_S = 0\}$ to be complemented in K_J^s where $J = IE$ is the “increased” inner function (see comments after Lemma 4.3).

As already pointed out, a characterization of interpolating sequences for Paley-Wiener spaces is unknown for general p (when $p = \infty$ Beurling gives a characterization, see [4], and for $0 < p \leq 1$, see [9]; a crucial difference between these cases and $1 < p < \infty$ is the boundedness of the Hilbert transform on L^p). For the case of complete interpolating sequences in PW_τ^p , i.e., interpolating sequences for which the interpolating functions are unique, these are characterized in [15] appealing to the Carleson condition and the Muckenhoupt (A_p) -condition for some function associated with the generating function of S . Sufficient conditions are pointed out in [24] using a kind of uniform non-uniqueness condition in the spirit of Beurling. Such a condition cannot be necessary since there are complete interpolating sequences in the Paley-Wiener spaces (which are in particular uniqueness sets). Another approach is based on invertibility properties of $P_I|K_B^p$, where $B = \prod_{a \in S} b_a$, and discussed in the seminal paper [11] (see also [19]). Let us also mention the paper by Minkin [16] who improves the result of Hruscev-Nikolski-Pavlov in removing the restriction on the zeros to be contained in a half-plane (which corresponds to $\sup_{a \in S} |I(a)| < 1$). In this approach, once having observed that the Carleson condition for S is necessary (under the condition $\sup_{a \in S} |I(a)| < 1$; note that for $p = 2$, one does not need the latter assumption, see [5, Corollary 3.4]), and so $(k_a / \|k_a\|_p)_{a \in S}$ is an unconditional basis for K_B^p , the left invertibility of $P_I|K_B^p$ guarantees that $(k_a^I / \|k_a^I\|_p)_{a \in S}$ is still an unconditional sequence (and invertibility of $P_I|K_B^p$ gives an unconditional basis). The invertibility properties of $P_I|K_B^p$ can be reduced to invertibility properties of the Toeplitz operator $T_{I\overline{B}}$. Again, and also in this approach, one can feel an

essential difference between complete interpolating sequences and not necessarily complete interpolating sequences. On the one hand, the case of complete interpolating sequences corresponds to invertibility of $T_{I\bar{B}}$, and a criterion of invertibility of Toeplitz operators is known. This is the theorem of Devinatz and Widom (see e.g. [19, Theorem B4.3.1]) for $p = 2$ and of Rochberg (see [21]) for $1 < p < \infty$, and again it is based on the Muckenhoupt (A_p) condition (or the Helson-Szegő condition in case $p = 2$), this time for some function $h \in H^p$ such that $I\bar{B} = \bar{h}/h$. On the other hand, a useful description of left-invertibility of Toeplitz operators, the situation corresponding to general not necessarily complete interpolating sequences, is not available. For the case $p = 2$ an implicit condition is given in [11]: $\text{dist}(I\bar{B}, H^\infty) < 1$, and a condition based on the extremal function of the kernel of the adjoint $T_{I\bar{B}}$ can be found in [10].

Resuming the discussions of this introduction we can say that in K_I^p , when $I = B$ is a Blaschke product then uniform minimality, unconditionality (and interpolation when considered in the right space) are equivalent (see Subsection 2.1), whereas in the other extremal case when I is a singular inner function with associated measure supported in one point then the situation becomes much more complicated: uniform minimality is strictly weaker than interpolation (when considered in the right space) and unconditionality (see Section 3).

The paper is organized as follows. In the next section we introduce the necessary material on uniform minimality, dual boundedness and unconditionality. A general characterization of unconditional bases will be given in terms of two embeddings. This will be applied to characterize unconditional bases of point evaluations (or reproducing kernels) where one of the embeddings is replaced by an interpolation condition. We will also discuss some Carleson-type conditions which are naturally connected with embedding problems. Section 3 is devoted to a longer discussion of the situation in the Paley-Wiener spaces. We essentially put the known material in the perspective of our work. This should convince the reader that it seems difficult to improve the results in the general situation. In the last section we give our main results Theorem 4.5 and Corollaries 4.7 and 4.8 on interpolation which as a special case contain Theorem 1.1.

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2. Preliminaries

2.1. Geometric properties of families of vectors of Banach spaces

We begin with some observations in classical H^p spaces concerning the relation between uniform minimality and unconditionality. Recall that the reproducing kernel of H^p in $a \in \mathbb{D}$ is given by $k_a(z) = (1 - \bar{a}z)^{-1}$. The Carleson condition $\inf_{a \in S} |B_a(a)| \geq \delta > 0$ can then be restated as $(k_a / \|k_a\|_{p'})_{a \in S}$ being a uniformly minimal sequence in $H^{p'}$ (which is equivalent here to $(k_a / \|k_a\|_p)_{a \in S}$ being uniformly minimal in H^p). Let us explain this a little bit more. By definition, a sequence of normalized vectors $(x_n)_n$ in a Banach space X is called uniformly minimal if

$$(2.1) \quad \inf_n \text{dist} \left(x_n, \bigvee_{k \neq n} x_k \right) = \delta > 0.$$

(Here $\bigvee_i x_i$ denotes the closed linear span of the vectors x_i .) By the Hahn-Banach theorem this is equivalent to the existence of a sequence of functionals $(\varphi_n)_n$ in X^* such that $\varphi_n(x_k) = \delta_{n,k}$, where $\delta_{n,k}$ is the usual Kronecker symbol, and $\sup_n \|\varphi_n\|_{X^*} < \infty$. In our situation, setting

$$(2.2) \quad \varphi_a = \frac{B_a}{B_a(a)} \frac{k_a}{k_a(a)} \|k_a\|_{p'},$$

we get

$$\left\langle \varphi_a, \frac{k_b}{\|k_b\|_{p'}} \right\rangle = \delta_{a,b}.$$

Recall that $\|k_a\|_s \simeq (1 - |a|^2)^{-(1-1/s)}$, so that we moreover have from (2.2) that $\sup_{a \in S} \|\varphi_a\|_p < \infty$ if and only if $\inf_{a \in S} |B_a(a)| = \delta > 0$, so that $(k_a / \|k_a\|_{p'})_{a \in S}$ is uniformly minimal in $K_I^{p'}$ if and only if S satisfies the Carleson condition. Another way of viewing the uniform minimality condition when $p = 2$ is given in terms of angles: a sequence $(x_n)_n$ of vectors in a Hilbert space is uniformly minimal if the angles between x_n and $\bigvee_{k \neq n} x_k$ are uniformly bounded away from zero.

A notion closely related with uniform minimality is that of dual boundedness (see [3]). Let us give a formal definition

DEFINITION 2.1. — *Let $X \subset \text{Hol}(\Omega)$ be a Banach space of holomorphic functions on a domain Ω . Suppose that the point evaluations E_z are continuous for every $z \in \Omega$. A sequence $S \subset \Omega$ is called dual-bounded if there is a sequence $(\rho_a)_{a \in S}$ of elements in X such that $\langle \rho_b, E_a / \|E_a\|_{X^*} \rangle = \delta_{a,b}$ and $\sup_{a \in S} \|\rho_a\|_X < \infty$.*

When X is reflexive, S is dual bounded if and only if the sequence $(E_a/\|E_a\|_{X^*})_{a \in S}$ of normalized point evaluations (or reproducing kernels) is uniformly minimal in the dual space.

This condition is termed weak interpolation in [24] because it means that it is possible to interpolate with a uniform control of the norm of the interpolating function the sequences $(\lambda_b)_{b \in S}$ which are zero everywhere except in one point $a \in S$ where they take the value $\|E_a\|_{X^*}$.

Let us discuss the unconditionality. Recall that a basis $(x_n)_n$ of vectors in a Banach space X is an unconditional basis if for every $x \in X$, there exists a numerical sequence (α_n) such that the sum $\sum_n \alpha_n x_n$ converges to x , and for every sequence of signs $\varepsilon = (\varepsilon_n)$, the sum $\sum_n \varepsilon_n \alpha_n x_n$ converges in X to a vector x_ε with norm comparable to $\|x\|$. We will discuss the interpolation condition $H^p|S \supset l^p(1 - |a|^2)$ in the light of this definition using reproducing kernels. First recall from [26] that we have $H^p|S \supset l^p(1 - |a|^2)$ if and only if $H^p|S = l^p(1 - |a|^2)$ and this is again equivalent to the Carleson condition $\inf_{a \in S} |B_a(a)| = \delta > 0$. Let $B = B_S$ be the Blaschke product vanishing on S . Set $K_B^p = H^p \cap BH_0^p$. The space K_B^p is a backward shift invariant subspace. Also, $K_B^p = \bigvee_{a \in S} k_a$, and $H^p = K_B^p + BH^p$ ($K_B^p = P_B H^p$ is a projected space). So the interpolation condition is equivalent to $K_B^p|S = l^p(1 - |a|^2)$, and since the interpolation problem has unique solution in K_B^p , we have for every $f \in K_B^p$, $\|f\|_p^p \simeq \sum_{a \in S} (1 - |a|^2)|f(a)|^p$. Clearly under this condition the functions φ_a introduced above exist and are in K_B^p . Then for every finite sequence (v_a) and every sequence of signs (ε_n) we have

$$\begin{aligned} \left\| \sum_{a \in S} \varepsilon_a v_a \varphi_a \right\|_p^p &\simeq \sum_{a \in S} (1 - |a|^2) |\varepsilon_a|^p |v_a|^p = \sum_{a \in S} (1 - |a|^2) |v_a|^p \\ &\simeq \left\| \sum_{a \in S} v_a \varphi_a \right\|_p^p \end{aligned}$$

which shows that $(\varphi_a)_a$ is an unconditional basis in K_B^p . Then $(k_a/\|k_a\|_{p'})$ is also an unconditional basis in $K_B^{p'}$ (this is a general fact, but the reader might also refer to Proposition 2.2 and references given there). Conversely, it is not clear why the unconditionality of $(k_a/\|k_a\|_{p'})$ in $K_B^{p'}$ or of $(\varphi_a)_{a \in S}$ in H^p should imply $H^p|S = l^p(1 - |a|^2)$. According to Corollary 2.4 to be discussed in the next subsection, the unconditionality of $(k_a/\|k_a\|_{p'})$ in $K_B^{p'}$ is equivalent to the existence of a sequence space l in which the canonical basis is unconditional such that $H^p|S$ is equal to a weighted version of l (see the next subsection). It turns out that in H^p , l is necessarily equal to l^p (see e.g. [18, Corollary, p.188]). So $(k_a/\|k_a\|_{p'})_{a \in S}$ is an unconditional basis

(in fact an $l^{p'}$ -basis) in $K_B^{p'}$ (or an unconditional sequence (an $l^{p'}$ -sequence) in $H^{p'}$) if and only if $\inf_{a \in S} |B_a(a)| = \delta > 0$.

Again, the unconditionality can be expressed in terms of angles when $p = 2$: a sequence $(x_n)_n$ of vectors in a Hilbert space is unconditional if the angles between $\bigvee_{k \in \sigma} x_k$ and $\bigvee_{k \in \mathbb{N} \setminus \sigma} x_k$ is uniformly bounded away from zero for every $\sigma \subset \mathbb{N}$.

So the interpolation results tell us that in H^p a sequence of reproducing kernels is uniformly minimal if and only if it is an unconditional sequence (note that the Carleson condition does not depend on p , and so we can replace p by p' if we wish). Such results also hold in other spaces like e.g. Bergman spaces (see [23]) and in Fock and Paley-Wiener spaces for certain values of p (see [24]).

We will be interested in the situation in more general backward shift invariant subspaces K_I^p .

2.2. Unconditional bases and interpolation

A very surprising fact is that when $(x_n)_n$ is an unconditional basis in (a subspace of) $X = L^p$, $1 < p \neq 2$, then there is no general result ensuring that $(x_n)_n$ generates a space isomorphic to l^p . Pelczynski actually constructed unconditional bases in l^p which are not equivalent to the canonical basis in l^p , [20]. In fact a more general result is true. Lindenstrauss and Zippin [13] showed that if in a Banach space X every two normalized unconditional bases are isomorphic to each other, then X is necessarily isomorphic to one of the following spaces: c_0 , l^1 or l^2 . This motivates the general discussion that follows.

We will establish a general link between unconditional basis on the one hand and interpolation with an additional embedding property on the other hand. It turns out that this link can be reformulated, in the spirit of [17, Theorem 1.2], in abstract terms without appealing to the notion of interpolation. We start with this general result before coming back to the special context of interpolation.

Suppose that X is a reflexive Banach space, and let $(y_n)_n$ be a sequence of normalized elements in X^* that we suppose at least minimal: $\text{dist}(y_n, \bigvee_{k \neq n} y_k) > 0$ for every $n \in \mathbb{N}$. We set $Y = \bigvee y_n \subset X^*$ and $N := Y^\perp \subset (X^*)^* = X$. By the minimality condition there exists a sequence $(x_n)_n$ of elements in $X^{**} = X$ such that $\langle x_n, y_k \rangle_{X, X^*} = \delta_{n,k}$, $n, k \in \mathbb{N}$.

For a sequence space l , we consider the canonical system $\{e_n\}_n$ where $e_n = (\delta_{n,k})_k$. The space l will be called ideal if whenever $(a_n)_n = \sum_k a_k e_k \in l$ and $|b_n| \leq |a_n|, n \in \mathbb{N}$, then $(b_n)_n \in l$. Recall also that a family of vectors in a Banach space is called fundamental if it generates a dense set in the Banach space. Observe that the canonical system is an unconditional basis in l if and only if it is fundamental in l and l is ideal. More generally, with these definitions we can give the following characterization of an unconditional basis: a fundamental system $\{z_n\}_n$ of vectors in a Banach space Z forms an unconditional basis in Z if there exists an ideal space l in which the canonical system is fundamental (hence unconditional) and such that $T : Z \rightarrow l, \sum_n \mu_n z_n \rightarrow (\mu_n)$ is an isomorphism. Moreover $l = T(Z)$ can be normed by $\|\mu\|_T = \|Tz\|_T := \|z\|_Z$. (This is in the spirit of [17, Theorem 1.1 and Lemma 1.2]; see also [14, Proposition 1.c.6].)

We obtain the following result.

PROPOSITION 2.2. — *Let X be a reflexive Banach space. With the above notation, the following assertions are equivalent.*

- (1) *The sequence $(y_n)_n$ is an unconditional basis in $Y = \bigvee_n y_n \subset X^*$.*
- (2) *The sequence $(x_n + N)_n$ is an unconditional basis in X/N .*
- (3) *There exists two reflexive Banach sequence spaces l_1, l_2 , in which the respective canonical systems are unconditional bases and such that*

(i) *The set of generalized Fourier coefficients of X contains l_1 :*

$$\{(\langle x, y_n \rangle_{X, X^*})_n : x \in X\} \supset l_1,$$

(ii) *for every $\mu = (\mu_n)_n \in l_2$,*

$$\left\| \sum_n \mu_n y_n \right\|_{X^*} \lesssim \|\mu\|_{l_2};$$

moreover $l_2 \simeq l_1^$ and the duality of l_1 and $l_1^* \simeq l_2$ is given by $\langle (\alpha_n)_n, (\mu_n)_n \rangle_{l_1, l_2} = \sum_n \alpha_n \mu_n$.*

- (4) *There exists a reflexive Banach sequence space l_1 , in which the canonical system is an unconditional basis and such that*

$$\{(\langle x, y_n \rangle_{X, X^*})_n : x \in X\} = l_1.$$

The spaces l_1 appearing in (3)(i) and (4) turn out to be the same, so that we use the same symbol to design them.

This theorem is in the spirit of [17, Theorem 1.2]. In particular the last item (4) is characteristic for $(x_n + N)_n$ to be an l_1 -basis of X/N (in the sense that $\mathcal{J}(x + N) = (\langle x, y_n \rangle_{x_n + N})_n$ defines an isomorphism of X/N onto $l_1(X_n)$ where X_n is the subspace of X/N generated by $x_n + N$, and

$l_1(X_n) := \{(F_n)_n : F_n \in X_n \text{ for every } n, \text{ and } (\|F_n\|_{X_n})_n \in l_1\}$. However, in Nikolski's theorem there does not really appear the condition (i) together with an embedding of type (ii). The condition (i) will later on play the rôle of the interpolation part.

Remark 2.3. — Let us make another remark concerning condition (3)(i) of Proposition 2.2. Suppose the assumptions of that proposition fulfilled. Then we will say that $(y_n)_n$ is of generalized interpolation if we have condition (3)(i), or, in other words, if the operator $R : X \rightarrow l_1$, $Rx = (\langle x, y_n \rangle)_n$ is onto. This implies that the operator $T_N : l_1 \rightarrow X/N$, where $N := \{x \in X : \langle x, y_n \rangle = 0, n \in \mathbb{N}\}$ is well-defined. Moreover, since $x \mapsto \langle x, y_n \rangle$ is continuous on X and since l_1 has bounded coordinate projections (l_1 is supposed to be an ideal Banach space in which the canonical system is fundamental), T_N has a closed graph and is thus bounded (this is the same argument as in the H^p -interpolation, see [26]). Now, in order that a bounded linear “interpolation” operator $T : l_1 \rightarrow X$, $(\alpha_n)_n \mapsto x$ such that $\langle x, y_n \rangle = \alpha_n$, $n \in \mathbb{N}$, exists, it is necessary and sufficient that N is complemented. Indeed, if a bounded linear “interpolation” operator T exists, then it is easily verified that $P = TR$ is a projection from X onto a complement of N in X . Conversely, if a projection $P : X \rightarrow X$ onto a complement K of N in X exists, then $P_N(f + N) = P(f)$ is a well defined bounded operator and $T = P_N T_N$ is a bounded linear “interpolation” operator.

We mention that Rosenthal [22] showed that there exists $X \subset L^{p'}$ which is equivalent to $l^{p'}$ but not complemented in $L^{p'}$. In particular, there exists an $l^{p'}$ -basis $(\varphi_n)_n$ such that $X = \{\sum \lambda_n \varphi_n : (\lambda_n)_n \in l^{p'}\}$. Now, $X^\perp \subset L^p$ is a space of the above type: $X^\perp = N := \{f \in L^p : \langle f, \varphi_n \rangle = 0, n \in \mathbb{N}\}$ which is not complemented since X is not.

Proof. — We begin with observing first that $Y^* = (X^*)^*/Y^\perp = X/N$. Moreover, for every $u \in N = Y^\perp$, $\langle x_n + u, y_k \rangle = \langle x_n, y_k \rangle = \delta_{n,k}$, and hence $((x_n + N)_n, (y_n)_n)$ is a biorthogonal system in $(X/N, Y) = (Y^*, Y)$. Note also that this justifies that $\langle x + N, y \rangle_{X/N, Y} := \langle x, y \rangle_{X, Y}$ is well defined for $y \in Y$. By the general theory (see for instance [27, Corollary I.12.2 and Theorem II.17.7]) we obtain the equivalence of (1) and (2).

By the remark above (*i.e.*, by [17, Theorem 1.2]) we also have the equivalence of (2) with (4).

Let us now prove that (1) and (2) imply (3). Suppose that $(y_n)_n$ is an unconditional basis in Y . By the above comments we can define the

operator

$$\begin{aligned} \tilde{T} : Y &\longrightarrow l_2 \\ \sum_n \mu_n y_n &\longmapsto (\mu_n)_n, \end{aligned}$$

where $l_2 = \tilde{T}Y$ is an ideal space in which the canonical system is an unconditional basis, and l_2 can be equipped with the norm $\|\cdot\|_{\tilde{T}}$. Hence l_2 is isometrically isomorphic to Y . For convenience we will rather consider the inverse mapping

$$T := \tilde{T}^{-1}|_{l_2}.$$

Note that Y is reflexive as a closed subspace of the reflexive Banach space X^* , and so is l_2 . Moreover, the continuity of T implies the estimate in (ii) of (3).

For exactly the same reasons, by (2) there exists a sequence space l_1 with the required properties such that

$$\begin{aligned} S : l_1 &\longrightarrow X/N \\ (\alpha_n)_n &\longmapsto \sum_n \alpha_n x_n + N =: x_\alpha + N \end{aligned}$$

is an (isometric) isomorphism. Note that X/N is reflexive as a quotient space of the reflexive Banach space X , and so is l_1 . Take $(\alpha_n)_n \in l_1$, then $S((\alpha_n)_n) = x_\alpha + N \in X/N$ for a suitable $x_\alpha \in X$. Now $(\langle x_\alpha, y_n \rangle)_n = (\langle \sum_k \alpha_k x_k + N, y_n \rangle_{X/N, Y})_n = (\sum_k \alpha_k \langle x_k + N, y_n \rangle_{X/N, Y})_n = (\alpha_n)_n$ (note that $\sum_k \alpha_k x_k + N$ converges in X/N). So $(\alpha_n)_n \in \{(\langle x, y_n \rangle) : x \in X\}$, and hence (i) of (3).

Finally, since $l_1 \simeq X/N$, $l_2 \simeq Y$ and $Y^* = X/N$ we have $l_2^* \simeq l_1$ and by reflexivity $l_2 \simeq l_1^*$. Moreover, by the identification maps we can write for $(\alpha_n)_n \in l_1$ and $(\mu_n)_n \in l_2 \simeq l_1^*$:

$$\begin{aligned} \langle (\alpha_n)_n, (\mu_n)_n \rangle_{l_1, l_2} &= \left\langle \sum_n \alpha_n x_n + N, \sum_k \mu_k y_k \right\rangle_{X/N, Y} \\ &= \sum_{n,k} \alpha_n \mu_k \langle x_n, y_k \rangle_{X, Y} \\ &= \sum_n \alpha_n \mu_n. \end{aligned}$$

We finish by showing that (3) implies (1). By (ii), the operator $T : l_2 \longrightarrow Y$, $(\mu_n) \longmapsto \sum_n \mu_n y_n$, already introduced above, is bounded and has dense range since by construction the canonical system is dense in l_2 and $(y_n)_n$ is dense in Y . So we are done if we can show that T is left invertible: $\|\mu\|_{l_2} \lesssim \|T\mu\|_Y$ (since then the range of T is closed and the density of

the range implies that T is onto). Now by (i) for $(\alpha_n)_n \in l_1$, there exists $x_\alpha \in X$ such that $\alpha_n = \langle x_\alpha, y_n \rangle$. Let us introduce the operator

$$\begin{aligned} A : l_1 &\longrightarrow X/N \\ (\alpha_n)_n &\longmapsto x_\alpha + N. \end{aligned}$$

This operator is well defined (if we choose x'_α with $\langle x'_\alpha, y_n \rangle = \alpha_n$, then $\langle x'_\alpha - x_\alpha, y_n \rangle = 0$ for every n and $x'_\alpha - x_\alpha \in N$). It is also linear. Let us check that its graph is closed. For this consider a sequence $(\alpha_n^j)_n$ converging to $(\alpha_n)_n$ in l_1 . Since the canonical basis is an unconditional basis in l_1 , we obtain coordinate-wise convergence: $\alpha_n^j \rightarrow \alpha_n$ when $j \rightarrow \infty$. We assume that $A((\alpha_n^j)_n) = x_{\alpha^j} + N \rightarrow x + N$. Note that $A((\alpha_n)_n) = x_\alpha + N$. Then for every n we have

$$\begin{aligned} \langle x, y_n \rangle_{X,Y} &= \langle x + N, y_n \rangle_{X/N,Y} = \lim_{j \rightarrow \infty} \langle x_{\alpha^j} + N, y_n \rangle_{X/N,Y} \\ &= \lim_{j \rightarrow \infty} \alpha_n^j = \alpha_n \\ &= \langle x_\alpha, y_n \rangle. \end{aligned}$$

So $x - x_\alpha \in N$ and $x + N = A((\alpha_n)_n)$. By the closed graph theorem A is bounded.

Let us show that $A^* : (X/N)^* = Y \rightarrow l_1^*$ is the left inverse to T (modulo the isomorphism from l_1^* to l_2). Equivalently it is sufficient to show that $T^*A : l_1 \rightarrow l_2^*$ is an isomorphism. Note that for $(\alpha_n)_n \in l_1$ and $(\mu_n)_n \in l_2$, we have

$$\begin{aligned} \langle T^*A(\alpha_n)_n, (\mu_n)_n \rangle_{l_2^*, l_2} &= \langle A(\alpha_n)_n, T(\mu_n)_n \rangle_{X/N, Y} \\ &= \langle x_\alpha + N, \sum_n \mu_n y_n \rangle_{X/N, Y} \\ &= \sum_n \mu_n \langle x_\alpha, y_n \rangle_{X, X^*} \\ &= \sum_n \mu_n \alpha_n \end{aligned}$$

By assumption this is equal to $\langle (\alpha_n)_n, (\mu_n)_n \rangle_{l_1, l_2}$ so that for every $(\alpha_n)_n \in l_1$ and $(\mu_n)_n \in l_2$, we have

$$\langle T^*A(\alpha_n)_n, (\mu_n)_n \rangle_{l_2^*, l_2} = \langle (\alpha_n)_n, (\mu_n)_n \rangle_{l_1, l_2}.$$

Hence T^*A is the identity (modulo the identification between l_1 and l_2^*), so that T is finally left-invertible and hence onto in view of what has been said before. We conclude that $T^{-1} : Y \rightarrow l_2, \sum_n \mu_n y_n \mapsto (\mu_n)$ is an isomorphism from where we deduce (1). \square

Let us now come to the context of interpolation. Let $X \subset \text{Hol}(\mathbb{D})$ be a reflexive space for which the point evaluations E_a , $a \in \mathbb{D}$, are continuous in X . Pick S a sequence in \mathbb{D} . If the family $\{E_a/\|E_a\|_{X^*}\}_{a \in S}$ is at least minimal, then there exists $\rho_a \in X^{**} = X$ (X being reflexive) such that $\langle \rho_a, E_b/\|E_b\|_{X^*} \rangle = \delta_{a,b}$. A sequence $S \subset \mathbb{D}$ is called l -interpolating for a sequence space l (defined on S) if for every sequence $v = (v_a)_{a \in S}$ with $(v_a/\|E_a\|_{X^*})_{a \in S} \in l$ there is a function $f \in X$ with $f(a) = v_a$, i.e.,

$$X|S \supset l[1/\|E_a\|_{X^*}]$$

where

$$l[1/\|E_a\|_{X^*}] := \{v = (v_a)_{a \in S} : (v_a/\|E_a\|_{X^*})_{a \in S} \in l\}.$$

Since $\|E_a\|_{(H^p)^*} \simeq \|k_a\|_{p'} \simeq (1 - |a|^2)^{-1/p}$ ($1 < p < \infty$), this definition is consistent with the definitions we gave before for H^p , in which case we had chosen $l = l^p$ (the careful reader has observed that $l^p(1 - |a|^2) = l^p[1/\|E_a\|_{(H^p)^*}] = l^p[(1 - |a|^2)^{1/p}]$). Observe now that replacing y_n in the previous proposition by $E_a/\|E_a\|_{X^*}$, condition (i) of item (3) now becomes

$$\{(f(a)/\|E_a\|_{X^*})_{a \in S} : f \in X\} = \{(\langle f, E_a/\|E_a\|_{X^*} \rangle)_{a \in S} : f \in X\} \supset l_1$$

which is exactly the interpolation condition with $l = l_1$:

$$X|S = \{(f(a))_{a \in S} : f \in X\} = \{(\langle f, E_a \rangle)_{a \in S} : f \in X\} \supset l_1[1/\|E_a\|_{X^*}].$$

The reader should note that in the previous subsection we have repeatedly used the fact that interpolation in H^p , i.e., $H^p|S \supset l^p(1 - |a|^2)$ (we will not consider the cases $p = 1, \infty$ here) implies in fact the equality $H^p|S = l^p(1 - |a|^2)$ (this is Shapiro and Shields' result, [26]). By Proposition 2.2 this is equivalent to $(k_a/\|k_a\|_q)_a$ being an unconditional sequence in H^q (unconditional basis in K_B^q , where B is the Blaschke product vanishing exactly on S). In the general case, without any further information, the sole interpolation condition does not guarantee that the sequence of point evaluations forms an unconditional sequence, and an additional embedding is required: this will be given by the condition (ii) appearing in (3) of Proposition 2.2.

The following corollary is now an immediate consequence of the previous discussions.

COROLLARY 2.4. — *Suppose $X \subset \text{Hol}(\mathbb{D})$ is reflexive and S is a sequence in \mathbb{D} . The following assertions are equivalent.*

- (1) $(E_a/\|E_a\|_{X^*})_{a \in S}$ is an unconditional sequence in X^* .
- (2) There exists a reflexive sequence space l in which the canonical system is an unconditional basis such that
 - (i) $X|S \supset l[1/\|E_a\|_{X^*}]$.

(ii) *There is a constant C such that for every finitely supported sequence $\mu = (\mu_a)_{a \in S}$, we have*

$$\left\| \sum_{a \in S} \mu_a \frac{E_a}{\|E_a\|_{X^*}} \right\|_{X^*} \leq C \|\mu\|_{l^*},$$

(3) *There exists a reflexive sequence space l in which the canonical system is an unconditional basis such that*

$$X|S = l[1/\|E_a\|_{X^*}].$$

We should recall here the definition of *free interpolation*. A sequence $S \subset \mathbb{D}$ is of free interpolation for X if for every $f \in X$, and for every sequence $\alpha := (\alpha_a)_{a \in S}$ with $|\alpha_a| \leq |f(a)|$, $a \in S$, there exists $g \in X$ such that $g|S = \alpha$ (this is another way of saying that $X|S$ is ideal).

COROLLARY 2.5. — *Suppose $X \subset \text{Hol}(\mathbb{D})$ is reflexive and S is a sequence in \mathbb{D} . Then each of the assertions (1)-(3) of the preceding corollary is equivalent to*

(4) *S is of free interpolation for X .*

Proof. — If one (and hence all) of the conditions of Corollary 2.4 is satisfied, then S is free interpolating.

Conversely, if S is free interpolating in a space X , then $\tilde{l} := X|S$ is an ideal space. It is also clear that the point evaluations E_a are continuous, and hence, defining a sequence space l by $\tilde{l} = l[1/\|E_a\|_{X^*}]$ (which means that $(\lambda_a)_{a \in S} \in l$ if and only if $(\lambda_a \|E_a\|_{X^*})_{a \in S} \in \tilde{l}$), we get $X|S = l[1/\|E_a\|_{X^*}]$, which is condition (3) of Corollary 2.4. \square

In particular, if the canonical system is moreover fundamental in l , then (provided X is reflexive) the corollary implies that $(E_a/\|E_a\|_{X^*})_{a \in S}$ is an unconditional sequence in X^* .

A sequence S satisfying condition (ii) will be called l^* -Carleson or q -Carleson when $l^* = l^q$ (a Carleson embedding for X^* with respect to the sequence space l^*).

Note that another way of writing (ii) of (2) is

$$\forall f \in X, \forall \mu \in l^*, \quad \left| \sum_{a \in S} \mu_a \frac{f(a)}{\|E_a\|_{X^*}} \right| \leq C \|f\|_X \|\mu\|_{l^*}.$$

It is now clear from Proposition 2.2 that the duality between l and l^* is expressed in terms of the sum. Hence for every $f \in X$, the sequence $(f(a)/\|E_a\|_{X^*})_{a \in S}$ is in $(l^*)^* = l$, so that (ii) is equivalent to the following generalized Carleson measure type condition

$$(2.3) \quad \|(f(a)/\|E_a\|_{X^*})_{a \in S}\|_l \leq C \|f\|_X,$$

which means $X|S \subset l[1/\|E_a\|_{X^*}]$. This again shows that the reverse inclusion in (3) follows from (ii). (There will be more discussions on Carleson measures in Subsection 2.3.)

We have already mentioned that the general theory does not allow to deduce that $l = l^p$ when we consider unconditional bases in (subspaces of) L^p (or H^p). Since $l^{p'}$ is reflexive and the canonical system is an unconditional basis in $l^{p'}$ the following is true

COROLLARY 2.6. — *Let $1 < p < +\infty$. If S is l^p -interpolating for K_I^p and if there is a constant C such that for every finitely supported sequence $\mu = (\mu)_{a \in S}$, we have $\|\sum_{a \in S} \mu_a k_a^I / \|k_a^I\|_{p'}\|_{p'} \leq C \|\mu\|_{l^{p'}}$, then $(k_a^I / \|k_a^I\|_{p'})_{a \in S}$ is an unconditional sequence in $K_I^{p'}$.*

More precisely the conclusion would be that $(k_a^I)_{a \in S}$ is an $l^{p'}$ -basis in its span. This conclusion can in general not be deduced only from the condition of unconditionality as explained above. However, in the special situation $\sup_{a \in S} |I(a)| < 1$, which is a case we will be interested in, we have the following result.

THEOREM 2.7 (Theorem 6.3, Part II [HNP81]). — *Suppose $\sup_{a \in S} |I(a)| < 1$ and $1 < r < \infty$. If the reproducing kernels form an unconditional sequence in K_I^r then they automatically form an l^r -basis in their span.*

2.3. Carleson measures

Let us fix the framework of this subsection. S is a sequence in \mathbb{D} , I an inner function and $1 \leq q < \infty$. For $a \in S$ we denote by $k_{q,a}^I = k_a^I / \|k_a^I\|_q$ the normalized reproducing kernel in K_I^q .

Let $1 \leq q < \infty$. Recall from the preceding subsection that a sequence S is called q -Carleson for K_I^q if

$$\exists D_q > 0, \forall \mu \in l^q, \left\| \sum_{a \in S} \mu_a k_{q,a}^I \right\|_q \leq D_q \|\mu\|_q.$$

We will also use the notion of weak q -Carleson sequences:

DEFINITION 2.8. — *Let $2 \leq q < \infty$. The sequence S is called weakly q -Carleson for K_I^q if*

$$\exists D_q > 0, \forall \mu \in l^q, \left\| \sum_{a \in S} |\mu_a|^2 |k_{q,a}^I|^2 \right\|_{q/2} \leq D_q \|\mu\|_q^2.$$

Note that by [3, Lemma 3.2], the q -Carleson property implies the weak q -Carleson property.

Observe also that $(l^{p'})^* = l^p$, $1/p + 1/p' = 1$, that the dual of $K_I^{p'}$ can be identified with K_I^p , and that the functional of point evaluation E_a can then be identified with k_a^I . Now, using the notation from the preceding subsection, by (2.3), S is p' -Carleson if and only if for every $f \in K_I^p$,

$$\sum_{a \in S} \frac{|f(a)|^p}{\|k_a^I\|_{p'}^p} \leq c \|f\|_p^p,$$

which means that $\nu := \sum_{a \in S} \delta_a / \|k_a^I\|_{p'}^p$ is a K_I^p -Carleson measure: $K_I^p \subset L^p(\nu)$.

In the special situation when I is one-component, which means that the level set $L(I, \varepsilon) = \{z \in \mathbb{D} : |I(z)| < \varepsilon\}$ of I is connected for some $\varepsilon > 0$, then Aleksandrov shows the following estimate (see [1])

$$\|k_a^I\|_{p'} \simeq \left(\frac{1 - |I(a)|^2}{1 - |a|^2} \right)^{1/p},$$

and so, if S is p' -Carleson and I is one-component, then the measure

$$d\nu = \sum_{a \in S} \frac{1 - |a|^2}{1 - |I(a)|^2} \delta_a$$

is K_I^p -Carleson.

Geometric Carleson conditions

In [28], the following geometric condition for Carleson measures appears. In the notation of [1], let $\mathcal{C}(I)$ be the set of measures for which there exists $C > 0$ such that

$$(2.4) \quad |\mu|(S(\zeta, h)) \leq Ch$$

for every Carleson window $S(\zeta = e^{it}, h) := \{z = re^{i\theta} \in \mathbb{D} : 1 - h < r < 1, |t - \theta| < h\}$ meeting $L(I, 1/2)$ (this is of course a weaker notion than the usual one requiring (2.4) on all Carleson windows as necessary in H^p , see [7]; the value $\varepsilon = 1/2$ is of no particular relevance). Let also $\mathcal{C}_p(I)$ be the set of measures for which $K_I^p \subset L^p(\mu)$. Strengthening some of the results of [28], Aleksandrov proved in [1, Theorem 1.4] that for one component inner functions $\mathcal{C}(I) = \mathcal{C}_p(I)$. In other words, the geometric Carleson condition (2.4) on Carleson windows meeting the level set $L(I, 1/2)$ characterizes the K_I^p -Carleson measures for one component inner functions.

Combining these observations, we get the following characterization.

Fact. Let I be a one-component inner function. Then the following assertions are equivalent.

- (i) S is p' -Carleson for $K_I^{p'}$,
- (ii) $\nu = \sum_{a \in S} \frac{1 - |a|^2}{1 - |I(a)|^2} \delta_a$ is K_I^p -Carleson,
- (iii) ν (as defined in point (ii)) satisfies the geometric Carleson condition (2.4) on Carleson windows meeting the level set $L(I, 1/2)$.

Observe that when I is one-component, under the purely geometric condition (iii) (not appealing to the L^p -norm of the underlying space) the equivalence with (i) or (ii) shows that the l -space appearing for this embedding is automatically l^p (for every p).

Note that without requiring that I is one-component, condition (iii) for the measure $\sum_{a \in S} \frac{\delta_a}{\|k_a^I\|_{p'}^p}$ is still sufficient for it to be a Carleson measure (see [28, Theorem 2]).

Question. Does there exist, in a backward shift invariant subspace K_I^p , an interpolating sequence S that is not p' -Carleson?

3. Paley-Wiener spaces

We will discuss a special class of backward shift invariant subspaces. Let $I(z) = e^{i2\pi z}$ be the singular inner function in the upper half plane with sole singularity at ∞ (to fix the ideas, we have chosen the mass of the associated singular measure to be 2π). Recall (see [19, B.1]) that the transformation

$$U_p : H^p(\mathbb{D}) \longrightarrow H^p(\mathbb{C}^+) \\ f \longmapsto \left\{ z \rightarrow (U_p f)(z) = \left(\frac{1}{\pi(z+i)^2} \right)^{1/p} f \left(\frac{z-i}{z+i} \right) \right\}$$

is an isomorphism of the Hardy space on the disk $H^p(\mathbb{D})$ onto the Hardy space $H^p(\mathbb{C}^+)$ of the upper half plane $\mathbb{C}^+ = \{z \in \mathbb{C} : \Im z > 0\}$ for $p < +\infty$. Note that the inner function $I_0(z) = \exp(2\pi(z+1)/(z-1))$ on \mathbb{D} corresponds to I on \mathbb{C}^+ (via the conformal map $\omega : \mathbb{C}^+ \rightarrow \mathbb{D}$, $\omega(z) = \frac{z-i}{z+i}$; $I = I_0 \circ \omega$).

Let PW_π^p be the Paley-Wiener space of entire functions of type at most π which are p -th power integrable on the real line. Pick $f \in PW_\pi^p$. By a theorem by Plancherel and Pólya (see [12, Lecture 7, Theorem 4]) we get

$$(3.1) \quad \int_{\mathbb{R}} |f(x+ia)|^p dx \leq e^{p\pi|a|} \|f\|_p^p$$

for every $a \in \mathbb{R}$. Setting $F(z) = e^{i\pi z} f(z)$ (which means that in a sense we compensate the type in the positive imaginary direction) yields

$$\int_{\mathbb{R}} |F(x+iy)|^p dx = \int_{\mathbb{R}} |f(x+iy)|^p e^{-p\pi y} dx \leq \|f\|_p^p$$

in particular for every $y > 0$ which means that $F \in H^p(\mathbb{C}^+)$. Dividing F by I we obtain an analytic function in the lower halfplane \mathbb{C}_- and for every $y < 0$,

$$\int_{\mathbb{R}} |F(x + iy)e^{-i2\pi(x+iy)}|^p dx = \int_{\mathbb{R}} |f(x + iy)|^p e^{p\pi y} dx \leq \|f\|_p^p$$

so that F/I is in the Hardy space of the lower halfplane $H^p(\mathbb{C}_-)$. Hence $F \in H^p(\mathbb{C}^+) \cap IH^p(\mathbb{C}_-) =: K_{\mathbb{R},I}^p$ (now considered as a space of functions on \mathbb{R} , the elements of which can of course be continued analytically to the whole plane). Conversely, if $F \in K_{\mathbb{R},I}^p$, then f defined by $f(z) = F(z)e^{-i\pi z}$ is in PW_{π}^p . It is clear that $K_{\mathbb{R},I}^p$ can be identified via U_p with $K_{I_0}^p$ on \mathbb{D} (or \mathbb{T}). Hence there is a natural identification between Paley-Wiener spaces and backward invariant subspaces (on \mathbb{T} or \mathbb{R}): $PW_{\pi}^p = e^{-i\pi z}U_p K_{I_0}^p$. It should also be pointed out that the inner function I_0 (or I) occurring here is actually one-component.

It is well known that in the particular case $p = 2$, PW_{π}^p is nothing but $\mathcal{FL}^2(-\pi, \pi)$ (this comes from the Paley-Wiener theorem).

Let us make another observation concerning imaginary translations. For $a \in \mathbb{R}$, let

$$\begin{aligned} \Phi_a : PW_{\pi}^p &\longrightarrow PW_{\pi}^p \\ f &\longmapsto \{\Phi_a f : z \longmapsto f(z - ia)\}. \end{aligned}$$

Using again the Plancherel-Pólya theorem (see (3.1)), we see that Φ_a is well-defined and bounded (it is clearly linear). It is also invertible with inverse $\Phi_a^{-1} = \Phi_{-a}$. So Φ_a is an isomorphism of PW_{π}^p onto itself (the type that we fixed to π here does not really matter).

So the Paley-Wiener spaces are special candidates of our spaces K_I^p , which motivates the following important observations. In general it is not true that uniform minimality implies interpolation or unconditionality which we will explain now following [24].

By definition, a sequence $\Gamma = \{x_k + iy_k\}_k$ is interpolating for PW_{τ}^p if for every numerical sequence $(v_k)_k$ with

$$(3.2) \quad \sum_k |v_k|^p e^{-p\tau|y_k|} (1 + |y_k|) < \infty,$$

there exists $f \in PW_{\tau}^p$ with $f(\gamma_k) = a_k$ (this corresponds to condition (3)(i) of Corollary 2.4).

THEOREM 3.1 (Schuster-Seip, 2000). — *Let $2 \leq p < \infty$. There exists a dual bounded sequence Γ which is not interpolating in PW_{π}^p .*

We would like to recall here the construction of Schuster and Seip which will serve later on.

Proof. — Define a sequence $\Gamma = \{\gamma_k\}_{k \in \mathbb{Z}}$ by $\gamma_0 = 0$ and $\gamma_k(p) = k + \delta_k(p)$, $k \in \mathbb{Z} \setminus \{0\}$, where $\delta_k(p) = \text{sign}(k)/(2p_0)$ and $p_0 = \max(p, p')$, $1/p + 1/p' = 1$.

Now let $G(z) = z \prod_{k \neq 0} (1 - \frac{z}{\gamma_k})$ which defines an entire function of exponential type π with $|G(x)| \simeq d(x, \Gamma)(1 + |x|)^{-1/p_0}$. Note that the p -th power integrability of $|G|$ on \mathbb{R} is determined by $(1 + |x|)^{-1/p_0}$, and the latter function is never p -th power integrable on \mathbb{R} (one could distinguish the case $p > 2$ and $p < 2$). Hence, Γ is a uniqueness set and thus interpolating if and only if it is completely interpolating.

We will use the same type of computations as in the proof of [15, Theorem 2] to check that Γ is not (completely) interpolating when $p \geq 2$. According to [15, Theorem 1], it suffices to check that F^p , where $F(x) = |G(x)/d(x, \Gamma)| \simeq (1 + |x|)^{-1/p_0}$, is not (A_p) , i.e.

$$\frac{1}{|I|} \int_I F^p dt \left(\frac{1}{|I|} \int_I F^{-p'} dt \right)^{p-1}$$

is not uniformly bounded in the intervals I . For $p \geq 2$, we have $p_0 = p$ and hence we have to consider

$$\frac{1}{|I|} \int_I (1 + |t|)^{-1} dt \left(\frac{1}{|I|} \int_I (1 + |t|)^{p'/p} dt \right)^{p-1}.$$

This expression behaves like $\log(1 + |x|)$ when $I = [0, x]$, which is incompatible with the (A_p) -condition. So the sequence Γ is not interpolating. It seems that for $p < 2$, F^p is an (A_p) weight so that we are not completely sure whether the counterexample in [23] works for $p \in (1, 2)$.

On the other hand, $g_k(z) = G(z)/(z - \gamma_k)$ vanishes on $\Gamma \setminus \{\gamma_k\}$ and satisfies

$$(3.3) \quad |g_k(\gamma_k)| \simeq \|g_k\|_{L^p(\mathbb{R})}.$$

We claim that this implies that the sequence is dual bounded. Indeed, note that the reproducing kernel of the Paley-Wiener space PW_π^p in $x \in \mathbb{R}$ is given by $k_x(z) = \text{sinc}(\pi(z - x)) = \sin(\pi(z - x))/(\pi(z - x))$, the norm of which in $L^{p'}(\mathbb{R})$ can be easily estimated to be comparable to a constant independant of x . Hence (3.3) implies that $\tilde{g}_k := g_k/\|g_k\|_p$ is of norm 1, $\tilde{g}_k(\gamma_l) = 0$ for $k \neq l$, and $|\tilde{g}_k(\gamma_k)| \simeq 1 \simeq \|k_{\gamma_k}\|_{L^{p'}(\mathbb{R})}$. Suitably renormed, $(\tilde{g}_k)_k$ thus furnishes the family $(\rho_{\gamma_k})_k$ mentioned after Definition 2.1. \square

As a consequence, in PW_π^p there exists a sequence $\Gamma \subset \mathbb{C}$ such that $\{k_{\gamma_l}/\|k_{\gamma_l}\|_{PW_\pi^{p'}}\}_l$ is uniformly minimal in $PW_\pi^{p'}$ but not unconditional.

Still, it can be observed that Γ is uniformly separated in the euclidean distance and hence by the classical Plancherel-Pólya inequality we have for every $f \in PW_\pi^p$

$$(3.4) \quad \sum_k |f(\gamma_k)|^p \leq C \|f\|_p^p,$$

so that the restriction operator $f \mapsto f|_\Gamma$ is continuous from PW_π^p to l^p (onto when Γ is interpolating), in other words the measure $\sum_{\gamma \in \Gamma} \delta_\gamma$ is PW_π^p -Carleson. The weight 1 appearing here is consistent with (3.2) since the sequence is real.

More can be said. The following result, which we will prove below, is nothing but a re-interpretation of [15].

PROPOSITION 3.2. — *Let $1 < p \leq 2$. Then for every $1 < s < p$ there exists a sequence Γ that is (completely) interpolating for PW_π^p without being interpolating for PW_π^s .*

So, in the scale of Paley-Wiener spaces — which represents a subclass of backward shift invariant subspaces — an interpolating sequence is not necessarily interpolating in an arbitrary bigger space, and so a fortiori a dual bounded sequence for a given p is not necessarily interpolating for a bigger space K_f^s , $s < p$. This should motivate why in our main result discussed in the next section we increase the space in two directions to get interpolation from dual boundedness: we increase the space by multiplying factors to the defining inner function and by decreasing p .

Again we translate the result to the language of unconditionality. The sequence constructed in this proposition is again a real sequence which is uniformly separated in the euclidean metric so that (3.4) holds for p (and s) and hence the measure $\sum_{k \in \mathbb{Z}} \delta_{\gamma_k}$ is a Carleson measure. This implies that if Γ is interpolating for PW_π^p then we do not only have $PW_\pi^p|_\Gamma \supset l^p$ (recall that the reproducing kernel is given by the sinc-function in $\gamma_k \in \mathbb{R}$ the norm of which is comparable to a constant) but $PW_\pi^p|_\Gamma = l^p$. By Corollary 2.4 this means that $(k_\gamma / \|k_\gamma\|_{p'})_{\gamma \in \Gamma}$ is unconditional in $PW_\pi^{p'}$. Clearly, since Γ is not interpolating for PW_π^s , the sequence $(k_\gamma / \|k_\gamma\|_{s'})_{\gamma \in \Gamma}$ cannot be unconditional in $PW_\pi^{s'}$. We recapitulate these observations in the following result.

COROLLARY 3.3. — *Let $1 < p \leq 2$. Then for every $1 < s < p$ there exists a sequence Γ such that $(k_\gamma / \|k_\gamma\|_{p'})_{\gamma \in \Gamma}$ is unconditional in $PW_\pi^{p'}$ and $(k_\gamma / \|k_\gamma\|_{s'})_{\gamma \in \Gamma}$ is not unconditional for $PW_\pi^{s'}$.*

It can be noted that $s' > p'$ so that $PW_\pi^{s'}$ is a smaller space than $PW_\pi^{p'}$.

Proof of Proposition 3.2. — Since $1 < p \leq 2$ we have $p_0 := \max(p, p') = p'$ (recall $1/p + 1/p' = 1$). In contrast to the example in Schuster and Seip's Theorem 3.1, where we have 'spread out' slightly the integers (by adding a constant to the positive integers and subtracting the same constant from the negative integers) to obtain a dual bounded sequence which is not interpolating ($p \geq 2$) we will now narrow the integers: let $\delta_k = -\text{sign}(k)/2s'$. We have in particular $s_0 = \max(s, s') = s' > p'$. Define $\Gamma = (\gamma_k)_{k \in \mathbb{Z}}$ by $\gamma_k = k + \delta_k$, $k \in \mathbb{Z} \setminus \{0\}$, $\gamma_0 = 0$. Then as the example in [15, Theorem 2], the sequence Γ is not interpolating for PW_π^s . On the other hand, since $|\delta_k| = 1/(2s') < 1/(2p')$ we deduce from the sufficiency part of [15, Theorem 2] that Γ is complete interpolating for PW_π^p . \square

Remark 3.4. — We have mentioned the translations Φ_a , $a \in \mathbb{R}$. These allow to translate the above example Γ to any line parallel to the real axis: $\Phi_a \Gamma$. By the properties of Φ_a , we keep the properties of uniform minimality and (non)-interpolation.

We now discuss the effect of increasing the size of the space in the Paley-Wiener case "in the direction of the inner function". More precisely we will consider the situation when we replace I by $I^{1+\varepsilon}$ on the K_I^p -side, which means on the Paley-Wiener side that we replace the type π by $\pi(1 + \varepsilon) =: \pi + \eta$ for some $\eta > 0$. And for $p = 2$, on the Fourier side this means that we replace the supporting interval $[-\pi, \pi]$ by $[-(\pi + \eta), \pi + \eta]$.

We will use [25, Theorem 2.4] to prove the following result.

PROPOSITION 3.5. — *Let $\Gamma = \{\gamma_k\}_{k \in \mathbb{Z}}$ be defined by $\gamma_0 = 0$, $\gamma_k = k + \text{sign}(k)/4$. Then $(k_\gamma)_{\gamma \in \Gamma}$ is uniformly minimal and not unconditional in PW_π^2 , and for every $\eta > 0$, $(k_\gamma)_{\gamma \in \Gamma}$ is an unconditional sequence in $PW_{\pi+\eta}^2$.*

(It is not necessary to normalize k_γ in this proposition since $\Gamma \subset \mathbb{R}$.)

Proof of Proposition 3.5. — The first part of the claim is established by Theorem 3.1.

We use [25, Theorem 2.4] for interpolation in the bigger space. Seip's theorem furnishes a sufficient density condition for unconditional sequences in Paley-Wiener spaces when $p = 2$ which makes this proof very easy. Recall that $n^+(r)$ denotes the largest number of points from a sequence of real numbers Γ to be found in an interval of length r . The upper uniform density is then defined as

$$D^+(\Gamma) := \lim_{r \rightarrow \infty} \frac{n^+(r)}{r}$$

(the limit exists by standard arguments on subadditivity of $n^+(r)$). [25, Theorem 2.4] states that when a sequence Γ , which is uniformly separated

in the euclidean distance, satisfies $D^+(\Gamma) < \frac{\tau}{\pi}$, then $(k_\gamma / \|k_\gamma\|_{PW_\tau^2})_{\gamma \in \Gamma}$ is an unconditional sequence in PW_τ^2 (strictly speaking Seip's theorem yields the unconditionality for exponentials in $L^2([-\tau, \tau])$, but via the Fourier transform this is of course the same as for reproducing kernels). Our sequence Γ clearly satisfies $D^+(\Gamma) = 1$, and hence $D^+(\Gamma) < \frac{\tau}{\pi}$ whenever $\tau > \pi$, so that Γ is interpolating in PW_τ^2 . \square

The proposition can also be shown by appealing to [24, Theorem 3] which gives a kind of uniform non-uniqueness condition as sufficient condition for interpolation in Paley-Wiener spaces. It can in fact be shown using a perturbation result by Redheffer that the weak limits (in the sense of Beurling) of our sequence Γ have the same completeness radius (in the sense of Beurling-Malliavin) as Γ , *i.e.*, π . So increasing the size of the interval makes these weak limits non-uniqueness in the bigger space (this is the most difficult condition of Schuster and Seip's result to be checked; concerning the other conditions appearing in their theorem, *i.e.*, uniform separation and the two-sided Carleson condition, these are immediate).

Question. A natural question arises in the context of these results. Is it possible that the sequence Γ of Proposition 3.5 — which is dual bounded but not interpolating in PW_π^2 — is interpolating in PW_π^p for some $p = 2 - \varepsilon$ (or every p in some interval $(2 - \varepsilon, 2)$) for suitable small ε ?

This means that we increase the size of the space in the direction p . Proposition 3.2 indicates that ε cannot be chosen arbitrarily big. That proposition also motivates another important remark. A sufficient condition for interpolation in terms of a suitable density and depending on the value of p , as encountered *e.g.* in the context of Bergman spaces where a sequence satisfying the critical density is automatically interpolating in the bigger spaces, seems not expectable. This makes the question very delicate (note that the sequence Γ of Proposition 3.5 has the critical density for PW_π^2).

4. The main result

We will now discuss the principal results that lead to Theorem 1.1.

Let I be an inner function. We increase K_I^p , when p is fixed, by multiplying a factor to the inner function I . More precisely let $J = IE$ where E is another inner function. Recall that $K_I^p + IK_E^p = K_J^p$ (which gives an idea on the increase of the space; note that $K_I^p = P_I K_J^p$).

We first discuss when dual boundedness for $p > 1$ implies interpolation for $q = 1$.

LEMMA 4.1. — *Let $S \subset \mathbb{D}$ be dual bounded in K_I^p , $p > 1$, and let E be another inner function. If*

$$(4.1) \quad \|k_a^J\|_\infty \simeq \frac{\|k_a^I\|_{p'} \|k_a^E\|_2^2}{\|k_a^E\|_{p'}}$$

then S is interpolating in K_J^1 with $J = IE$.

Proof. — Let first $c_a = \frac{\|k_a^E\|_{p'} \|k_a^J\|_\infty}{\|k_a^I\|_{p'} \|k_a^E\|_2^2}$ which is comparable to a uniform constant.

Since S is dual bounded in K_I^p , the sequence $(k_a^I / \|k_a^I\|_{p'})_{a \in S}$ is uniformly minimal in $K_I^{p'}$, and there exists a dual sequence $(\rho_{p,a})_{a \in S}$ in K_I^p : $\langle \rho_{p,a}, k_{p',b}^I \rangle = \delta_{a,b}$, i.e., $\rho_{p,a}(b) = \delta_{a,b} \|k_b^I\|_{p'}$, and $\sup_{a \in S} \|\rho_{p,a}\|_p < \infty$. As in [3] the idea is now to take

$$\forall \lambda \in l^1(S), \quad T(\lambda) := \sum_{a \in S} \lambda_a c_a \rho_{p,a} \frac{k_a^E}{\|k_a^E\|_{p'}}$$

The sum defining T converges clearly under the assumption of the theorem since λ is summable. Note also that $\rho_{p,a} \in K_I^p$ implies that $\rho_{p,a} = I\bar{\psi}$ with $\psi \in H_0^p$ and $k_a^E \in K_E^\infty$ that $k_a^E = E\bar{\varphi}$, $\varphi \in H_0^\infty$. Hence $\rho_{p,a} k_a^E = IE\bar{\varphi\psi}$, with $\varphi\psi \in H_0^p \subset H_0^1$ so that $\rho_{p,a} k_a^E \in K_J^1$. Moreover $k_a^E(a) = \|k_a^E\|_2^2$, and hence

$$T(\lambda)(a) = \lambda_a c_a \rho_{p,a}(a) \frac{k_a^E(a)}{\|k_a^E\|_{p'}} = \lambda_a c_a \frac{\|k_a^I\|_{p'} \|k_a^E\|_2^2}{\|k_a^E\|_{p'}} = \lambda_a \|k_a^J\|_\infty.$$

So, S is interpolating in K_J^1 ($l^1[1/\|k_a^J\|_\infty] \subset K_J^1$). □

Remark 4.2. — If $p = 1$ then dual boundedness of S in K_I^1 implies that S is interpolating in K_I^1 (take the interpolation operator constructed in the proof of Lemma 4.1).

We shall now discuss the general situation.

LEMMA 4.3. — *Suppose that I and E are inner functions, and set $J = IE$. Let $S \subset \mathbb{D}$ be a dual bounded sequence in K_I^p ; let $1 \leq s < p$ and q be such that $\frac{1}{s} = \frac{1}{p} + \frac{1}{q}$; suppose that the following conditions are satisfied:*

$$(i) \quad \frac{\|k_a^J\|_{s'}}{\|k_a^I\|_{p'} k_{q,a}^E(a)} \simeq 1;$$

$$(ii) \quad \forall \lambda \in l^p(S) = \{(\lambda_a)_{a \in S} : \|\lambda\|_{l^p}^p := \sum_{a \in S} |\lambda_a|^p < +\infty\},$$

$$\mathbb{E} \left[\left\| \sum_{a \in S} \lambda_a \epsilon_a \rho_{p,a} \right\|_p^p \right] \lesssim \|\lambda\|_{l^p}^p, \text{ where } (\epsilon_a)_{a \in S} \text{ is a sequence of independent Bernoulli variables,}$$

(iii) if $q > 2$, S is weakly q -Carleson in K_E^q ,

Then S is K_j^s interpolating and moreover there exists a bounded linear interpolation operator $T : l^s(S) \rightarrow K_j^s$, $T(\nu)(a) = \nu_a \|k_a^J\|_{s'}$.

We should point out that it is in general not clear whether interpolation implies the existence of a linear extension operator. Suppose that X is a Banach space of holomorphic functions with bounded point evaluations and l a Banach sequence space with bounded projections on the coordinates. Then we are in the conditions of Remark 2.3 with $y_n = E_{a_n}$. So, in view of that remark, the existence of an interpolation operator is equivalent to the existence of a bounded projection from K_j^s onto $N := \{f \in K_j^s : \langle f, E_a \rangle = 0, a \in S\} = \{f \in K_j^s : f|_S = 0\}$, or, in other words, to the fact that N is complemented in K_j^s . This is a very delicate question. Incidentally, as a consequence of our theorem, we get sufficient conditions for N to be complemented in K_j^s (a trivial case occurs of course when N is reduced to $\{0\}$, which can happen for instance when S is complete interpolating for K_j^s).

Observe that in (iii) we do not require the weak q -Carleson condition on S when $q \leq 2$.

Remark 4.4. — Before proving the result, we discuss some special cases where the condition (i) is satisfied.

Recall that an inner function is called one-component when there exists an $\varepsilon \in (0, 1)$ such that $L(I, \varepsilon) = \{z \in \mathbb{D} : |I(z)| < \varepsilon\}$ is connected. Simple examples of such functions are $I(z) = \exp((z + 1)/(z - 1))$ or Blaschke products with zeros not “too far from each other” such as B_S associated with $S = \{1 - 1/n^2\}_n$ or even associated with the interpolating sequence $S = \{1 - 1/2^n\}_n$. One-component inner functions appear for instance in the context of embeddings for star invariant subspaces. We mention that Treil and Volberg [28] discuss the embedding $K_I^p \subset L^p(\mu)$ when I is one-component, see also Cohn [8] who initiated these questions in the case $p = 2$. For one component inner functions we have already pointed out the following estimate by Aleksandrov (see [1])

$$(4.2) \quad \|k_a^I\|_{s'} \simeq \left(\frac{1 - |I(a)|^2}{1 - |a|^2} \right)^{1/s}.$$

Hence when I, E and J are one-component (it is not clear to us whether I and E being one-component implies that J is one-component), we get

$$\frac{\|k_a^J\|_{s'}}{\|k_a^I\|_{p'}k_{q,a}^E(a)} = \frac{\|k_a^J\|_{s'}\|k_a^E\|_q}{\|k_a^I\|_{p'}k_{q,a}^E(a)} \simeq \frac{(1 - |J(a)|^2)^{1/s}(1 - |E(a)|^2)^{1/q'}}{(1 - |I(a)|^2)^{1/p}(1 - |E(a)|^2)}(1 - |a|^2)^{1/p+1-1/s-1/q'}.$$

By hypothesis, $1/p + 1 - 1/s - 1/q' = 0$, and hence

$$(4.3) \quad \frac{\|k_a^J\|_{s'}}{\|k_a^I\|_{p'}k_{q,a}^E(a)} \simeq \frac{(1 - |J(a)|^2)^{1/s}(1 - |E(a)|^2)^{1/q'}}{(1 - |I(a)|^2)^{1/p}(1 - |E(a)|^2)} = \frac{(1 - |J(a)|^2)^{1/s}}{(1 - |I(a)|^2)^{1/p}(1 - |E(a)|^2)^{1/q}}$$

From this we can deduce that (i) holds in the following two cases.

- (1) $E = I$ and I is one-component, then $J = I^2$ (note that it is clear that when $L(I, \varepsilon)$ is connected then so is $L(I^2, \varepsilon^2)$); hence

$$(1 - |J(a)|^2)^{1/s} = (1 - |I(a)|^4)^{1/s} \simeq (1 - |I(a)|^2)^{1/q} (1 - |I(a)|^2)^{1/p},$$

which by (4.2) and (4.3) yields (i);

- (2) I a one component singular inner function and $E = I^\alpha$ for some $\alpha > 0$, then $J = I^{1+\alpha}$, and (4.3) holds, which yields (i).

Another situation where condition (i) is easily fulfilled is when $\sup_{a \in S} |I(a)| < 1$. Note that under this condition, clearly

$$\left| \frac{1 - \overline{I(a)}I(\zeta)}{1 - \overline{a}\zeta} \right| \simeq \frac{1}{|1 - \overline{a}\zeta|}, \quad \zeta \in \mathbb{T},$$

uniformly in $a \in S$, and hence

$$(4.4) \quad \|k_a^I\|_r \simeq \|k_a\|_r \simeq \frac{1}{(1 - |a|^2)^{1-1/r}}$$

Let us distinguish two cases

- (3) General case : If moreover $\sup_{a \in S} |E(a)| < 1$. Then also $\sup_{a \in S} |J(a)| < 1$, and (i) follows as above from

$$\frac{\|k_a^J\|_{s'}}{\|k_a^I\|_{p'}k_{q,a}^E(a)} = \frac{\|k_a^J\|_{s'}\|k_a^E\|_q}{\|k_a^I\|_{p'}k_{q,a}^E(a)} \simeq (1 - |a|^2)^{1/p+1-1/s-1/q'}.$$

- (4) A particular incidence of the former general case is when I is any singular inner function and $E = I^\alpha$ for some $\alpha > 0$.

Proof of the Lemma. — In view of Lemma 4.1 we can suppose $1 < s < p$.

In order to prove the lemma we will construct a function f interpolating a sequence $\nu \in l^s$ weighted by the norm of the reproducing kernels. To do this, we will consider finitely supported sequences ν , say with only the first N components possibly different from zero, and check that the constants do not depend on $N \in \mathbb{N}$. So, for $1 < s < p$ and $\nu \in l_N^s$ we shall build a function $h \in K_j^s$ such that:

$$\forall j = 0, \dots, N - 1, h(a_j) = \nu_j \|k_{a_j}^J\|_{s'} \text{ and } \|h\|_{K_j^s} \leq C \|\nu\|_{l_N^s}.$$

where the constant C is independent of N . The conclusion then follows from a normal families argument (see also [2]).

We choose q such that $\frac{1}{s} = \frac{1}{p} + \frac{1}{q}$; then $q \in]p', \infty[$ with p' the conjugate exponent of p , and we set $\nu_j = \lambda_j \mu_j$ with $\mu_j := |\nu_j|^{s/q} \in l^q$, $\lambda_j := \frac{\nu_j}{|\nu_j|} |\nu_j|^{s/p} \in l^p$ ($\lambda_j = 0$ when $\nu_j = 0$) so that $\|\nu\|_{l^s} = \|\lambda\|_{l^p} \|\mu\|_{l^q}$.

Let now

$$c_a := \frac{\|k_a^J\|_{s'}}{\|k_a^I\|_{p'} k_{q,a}^E(a)}$$

which by (i) is comparable to a constant independent of a .

Next, since S is dual bounded in K_I^p , there exists $\rho_{p,a} \in K_I^p$ such that $\rho_{p,a}(b) = \delta_{a,b} \|k_a^I\|_{p'}$ and $\sup_{a \in S} \|\rho_{p,a}\|_p < \infty$. Set $h(z) := T(\nu)(z) := \sum_{a \in S} \nu_a c_a \rho_{p,a}(z) k_{q,a}^E(z)$. As in Lemma 4.1, we obtain $\rho_{p,a} k_{q,a}^E = IE\overline{\varphi\psi}$, with $\varphi \in H_0^p$ and $\psi \in H_0^\infty \subset H_0^q$ so that $\varphi\psi \in H_0^s$. Hence every summand is in K_{IE}^s , and

$$\forall a \in S, h(a) = \nu_a c_a \|k_a^I\|_{p'} k_{q,a}^E(a).$$

Recall that $k_{q,a}^E(a) = k_a^E(a) / \|k_a^E\|_q$. Hence

$$h(a) = \nu_a c_a \|k_a^I\|_{p'} k_{q,a}^E(a) = \nu_a \|k_a^J\|_{s'}$$

and h satisfies the interpolation condition.

Let us now come to the estimate of the K_j^s norm of h .

For a sequence S of points in \mathbb{D} , we introduce the related sequence $\{\epsilon_a\}_{a \in S}$ of independent Bernoulli variables.

Set

$$f(\epsilon, z) := \sum_{a \in S} \lambda_a c_a \epsilon_a \rho_{p,a}(z), \quad \text{and} \quad g(\epsilon, z) := \sum_{a \in S} \mu_a \epsilon_a k_{q,a}^E(z).$$

Then $h(z) = \mathbb{E}(f(\epsilon, z)g(\epsilon, z))$ because $\mathbb{E}(\epsilon_j \epsilon_k) = \delta_{j,k}$.

So we get

$$|h(z)|^s = |\mathbb{E}(fg)|^s \leq (\mathbb{E}(|fg|))^s \leq \mathbb{E}(|fg|^s),$$

and hence

$$\|h\|_s = \left(\int_{\mathbb{T}} |h(z)|^s d\sigma(z) \right)^{1/s} \leq \left(\int_{\mathbb{T}} \mathbb{E}(|fg|^s) d\sigma(z) \right)^{1/s}.$$

By Fubini's theorem and Hölder's inequality, we get

$$\begin{aligned} \int_{\mathbb{T}} \mathbb{E}(|fg|^s) d\sigma(z) &= \mathbb{E} \left[\int_{\mathbb{T}} |fg|^s d\sigma(z) \right] \\ (4.5) \qquad \qquad \qquad &\leq \left(\mathbb{E} \left[\int_{\mathbb{T}} |f|^p d\sigma \right] \right)^{s/p} \left(\mathbb{E} \left[\int_{\mathbb{T}} |g|^q d\sigma \right] \right)^{s/q}. \end{aligned}$$

Now for $a \in S$, set $\tilde{\lambda}_a := c_a \lambda_a$. Then $\|\tilde{\lambda}\|_p \simeq C \|\lambda\|_p$ and the first factor on the right hand side in (4.5) is controlled by (ii) of the hypotheses of the Lemma:

$$(4.6) \quad \mathbb{E} \left[\int_{\mathbb{T}} |f|^p d\sigma \right] = \mathbb{E} \left[\left\| \sum_{a \in S} \lambda_a c_a \epsilon_a \rho_{p,a} \right\|_p^p \right] \lesssim \|\tilde{\lambda}\|_p^p \lesssim \|\lambda\|_p^p,$$

and the constants appearing here do not depend on N .

Consider the second factor in (4.5). Fubini's theorem gives:

$$\mathbb{E} \left[\int_{\mathbb{T}} |g|^q d\sigma \right] = \int_{\mathbb{T}} \mathbb{E} [|g|^q] d\sigma.$$

We apply Khinchin's inequalities to $\mathbb{E} [|g|^q]$:

$$\mathbb{E} [|g|^q] \simeq \left(\sum_{a \in S} |\mu_a|^2 |k_{q,a}^E|^2 \right)^{q/2}.$$

If $q > 2$, then S weakly q -Carleson in K_E^q implies

$$(4.7) \quad \int_{\mathbb{T}} \mathbb{E} [|g|^q] d\sigma \lesssim \int_{\mathbb{T}} \left(\sum_{a \in S} |\mu_a|^2 |k_{q,a}^E|^2 \right)^{q/2} d\sigma \lesssim \|\mu\|_{l^q}^q,$$

where, again, the constants do not depend on N .

If $q \leq 2$ then $\left(\sum_{a \in S} |\mu_a|^2 |k_{q,a}^E|^2 \right)^{q/2} \leq \sum_{a \in S} |\mu_a|^q |k_{q,a}^E|^q$, and integrating

over \mathbb{T} we get:

$$(4.8) \quad \int_{\mathbb{T}} \mathbb{E} [|g|^q] d\sigma \leq \int_{\mathbb{T}} \left(\sum_{a \in S} |\mu_a|^q |k_{q,a}^E|^q \right) d\sigma \leq \sum_{a \in S} |\mu_a|^q \int_{\mathbb{T}} |k_{q,a}^E|^q d\sigma = \|\mu\|_{l^q}^q.$$

So putting (4.6) and (4.7) or (4.8) in (4.5) we get that S is an interpolating sequence for K_S^j . Clearly the operator T is a bounded linear interpolation operator. □

We are now in a position to state our main result (the proof of which will be postponed to the end of the paper).

THEOREM 4.5. — *Let $1 < p \leq 2$, $1 \leq s < p$ and q such that $\frac{1}{s} = \frac{1}{p} + \frac{1}{q}$. Suppose that I and E are inner functions, and set $J = IE$. If*

- (i) S is dual bounded in K_I^p ,
- (ii) $\frac{\|k_a^J\|_{s'}}{\|k_a^I\|_{p'} k_{q,a}^E(a)} \simeq 1$;
- (iii) S is weakly q -Carleson in K_E^q .

Then S is K_J^s -interpolating and there exists a bounded linear interpolation operator.

Before discussing special cases we mention a first consequence for the case of unconditionality.

COROLLARY 4.6. — *Suppose the conditions of the preceding theorem fulfilled and $1 < s < p$. Assume that the measure $\nu = \sum_{a \in S} \frac{\delta_a}{\|k_a^J\|_{s'}}$ satisfies*

$$(4.9) \quad |\nu|(S(\zeta, h)) \leq Ch$$

for every Carleson window $S(\zeta = e^{it}, h)$ meeting the level set $L(J, 1/2)$. Then $(k_a^J / \|k_a^J\|_{s'})_{a \in S}$ is an unconditional sequence in $K_J^{s'}$ (in fact an $l^{s'}$ -sequence).

Note that when J is one-component, then from Aleksandrov’s result (4.2) we know that $\|k_a^J\|_{s'} \simeq \frac{1 - |I(a)|^2}{1 - |a|^2}$ which can help checking the Carleson measure condition (4.9) in concrete situations. Another instance is given when $\sup_{a \in S} |J(a)| < 1$. Then by (4.4) the weight $1/\|k_a^J\|_{s'}$ simplifies to $(1 - |a|^2)$.

Proof of Corollary 4.6. — In view of Corollary 2.6 it remains to prove that ν is s' -Carleson. But this is a consequence of [28, Theorem 2] (see also the comments at the end of Subsection 2.3) and (4.9). □

We now obtain the first part of Theorem 1.1.

COROLLARY 4.7. — *Let $1 < p \leq 2$. Let I be a singular inner function and $S \subset \mathbb{D}$. Suppose that $\sup_{a \in S} |I(a)| < 1$. If $(k_a^I / \|k_a^I\|_{p'})_{a \in S}$ is uniformly minimal in $K_I^{p'}$, where $1/p + 1/p' = 1$ then for every $\varepsilon > 0$ and for every $1 \leq s < p$, S is an interpolating sequence in $K_{I^{1+\varepsilon}}^s$.*

Proof of Corollary 4.7. — Observe first that the case $s = 1$ corresponds to Lemma 4.1, so that, in what follows, we can suppose $1 < s < p$.

Condition (ii) of the theorem follows from the case (4) of Remark 4.4. The condition (i) of the theorem is fulfilled by the fact that $(k_a^I/\|k_a^I\|_{p'})_{a \in S}$ is uniformly minimal in $K_I^{p'}$. Let $(\rho_{p,a})_{a \in S}$ be the corresponding dual family in K_I^p . It remains to check the weak q -Carleson condition. In fact more is true: from (4.4), we get, up to some constants $c_a, a \in S$, whose moduli are uniformly bounded above and below,

$$\begin{aligned} \delta_{a,b} &= \langle \rho_{p,a}, k_b^I/\|k_b^I\|_{p'} \rangle = c_a \langle \rho_{p,a}, k_b^I/\|k_b^I\|_{p'} \rangle = c_a \langle \rho_{p,a}, P_I(k_b/\|k_b\|_{p'}) \rangle \\ &= c_a \langle P_I \rho_{p,a}, k_b/\|k_b\|_{p'} \rangle \\ &= c_a \langle \rho_{p,a}, k_b/\|k_b\|_{p'} \rangle. \end{aligned}$$

Hence $(k_a/\|k_a\|_{p'})_{a \in S}$ is a uniform minimal sequence in H^p which by the interpolation result is equivalent to $S \in (C)$. (We could also have shown this by using directly (2.1).) In particular, $(k_a/\|k_a\|_r)_{a \in S}$ is an unconditional sequence in any $H^r, 1 < r < \infty$.

From this we can deduce that S is r -Carleson for every $1 < r < \infty$ (and hence weakly q -Carleson): indeed, let $(\mu_a)_{a \in S} \in l^r$, then

$$\begin{aligned} \left\| \sum_{a \in S} \mu_a k_{a,r}^I \right\|_r^r &= \left\| P_I \sum_{a \in S} \mu_a \frac{k_a}{\|k_a^I\|_r} \right\|_r^r \leq c \left\| \sum_{a \in S} \mu_a \frac{k_a}{\|k_a^I\|_r} \right\|_r^r \\ &= c \left\| \sum_{a \in S} \mu_a \frac{\|k_a\|_r}{\|k_a^I\|_r} \frac{k_a}{\|k_a\|_r} \right\|_r^r \simeq \sum_{a \in S} |\mu_a|^r \left(\frac{\|k_a\|_r}{\|k_a^I\|_r} \right)^r \\ (4.10) \quad &\simeq \sum_{a \in S} |\mu_a|^r, \end{aligned}$$

where we have used that $\|k_a\|_r \simeq \|k_a^I\|_r$. This holds in particular for $r = q$, where $1/s = 1/p + 1/q$. □

We are now in a position to deduce also the second part of Theorem 1.1.

COROLLARY 4.8. — *Let $1 < p \leq 2$. Let I be a singular inner function and $S \subset \mathbb{D}$. Suppose that $\sup_{a \in S} |I(a)| < 1$. If $(k_a^I/\|k_a^I\|_{p'})_{a \in S}$ is uniformly minimal in $K_I^{p'}$, where $1/p + 1/p' = 1$ then for every $\varepsilon > 0$ and for every $1 < s < p, (k_a^{I^{1+\varepsilon}}/\|k_a^{I^{1+\varepsilon}}\|_{s'})_{a \in S}$ is an unconditional sequence in $K_{I^{1+\varepsilon}}^{s'}$.*

So, in the present situation, we increase the space in the direction of the inner function and we decrease the space by increasing the power of integration ($s' > p'$) to deduce unconditionality from uniform minimality.

Note that under the conditions of this corollary, we have $\|k_a^{I^{1+\varepsilon}}\|_{s'} \simeq \|k_a\|_{s'} \simeq \|k_a^I\|_{s'}$. Hence, since $(k_a^{I^{1+\varepsilon}}/\|k_a^{I^{1+\varepsilon}}\|_{s'})_{a \in S}$ is an unconditional sequence, by Corollary 2.4, $K_{I^{1+\varepsilon}}^s|_S = l^s[1/\|k_a^{I^{1+\varepsilon}}\|_{s'}] = l^s[1/\|k_a^I\|_{s'}]$. However, $k_a^I \in K_I^{s'}$ does not reproduce all the functions in $K_{I^{1+\varepsilon}}^s$ so that we cannot replace $k_a^{I^{1+\varepsilon}}$ in the corollary by k_a^I .

Let us make another observation. In [19, D4.4.9(5)] it is stated (in conjunction with [19, Lemma D4.4.3]) that under the Carleson condition $S \in (C)$ the condition $\sup_{a \in S} |I(a)| < 1$ is equivalent to the existence of $N \in \mathbb{N}$ such that $(k_a^{I^N}/\|k_a^{I^N}\|_2)_{a \in S}$ is an unconditional sequence in $K_{I^N}^2$. In the situation of Corollary 4.8, when $(k_a^I/\|k_a^I\|_{p'})_{a \in S}$, $p' \geq 2$, is supposed uniformly minimal (which itself implies the Carleson condition under the assumptions on I and S ; we do not know whether the Carleson condition could imply the uniform minimality in our context) then instead of taking I^N we can choose $I^{1+\varepsilon}$ for any $\varepsilon > 0$, and more generally, in the context of Corollary 4.6 (where a Carleson condition is required) we can still choose I^2 instead of I^N (paying in both cases the price of replacing p' by $q' > p'$).

Proof of Corollary 4.8. — In view of the preceding corollary and Corollary 2.6, it remains to check that S is $(l^s)^* = l^{s'}$ -Carleson, which follows at once from (4.10) by taking $r = s'$. □

Proof of Theorem 4.5. — It remains to prove that the hypotheses of the theorem imply those of Lemma 4.3. We thus have to prove that

$$\mathbb{E} \left[\left\| \sum_{a \in S} \lambda_a \varepsilon_a \rho_{p,a} \right\|_p^p \right] \lesssim \|\lambda\|_{l^p}^p .$$

under the assumption that the dual sequence $\{\rho_{p,a}\}_{a \in S}$ is uniformly bounded in K_I^p : $\sup_{a \in S} \|\rho_{p,a}\|_p \leq C$.

By Fubini's theorem

$$\mathbb{E} \left[\left\| \sum_{a \in S} \lambda_a \varepsilon_a \rho_{p,a} \right\|_p^p \right] = \int_{\mathbb{T}} \mathbb{E} \left[\left| \sum_{a \in S} \lambda_a \varepsilon_a \rho_{p,a} \right|^p \right] d\sigma ,$$

and by Khinchin's inequalities we have

$$\mathbb{E} \left[\left| \sum_{a \in S} \lambda_a \varepsilon_a \rho_{p,a} \right|^p \right] \simeq \left(\sum_{a \in S} |\lambda_a|^2 |\rho_{p,a}|^2 \right)^{p/2} .$$

Now, since $p \leq 2$, we have

$$\left(\sum_{a \in S} |\lambda_a|^2 |\rho_{p,a}|^2 \right)^{1/2} \leq \left(\sum_{a \in S} |\lambda_a|^p |\rho_{p,a}|^p \right)^{1/p} ,$$

and hence

$$\int_{\mathbb{T}} \mathbb{E} \left[\left| \sum_{a \in S} \lambda_a \epsilon_a \rho_{p,a} \right|^p \right] d\sigma \leq \int_{\mathbb{T}} \left(\sum_{a \in S} |\lambda_a|^p |\rho_{p,a}|^p \right) d\sigma = \sum_{a \in S} |\lambda_a|^p \|\rho_{p,a}\|_p^p.$$

So, finally

$$\mathbb{E} \left[\left\| \sum_{a \in S} \lambda_a \epsilon_a \rho_{p,a} \right\|_p^p \right] \lesssim \sup_{a \in S} \|\rho_{p,a}\|_p^p \|\lambda\|_p^p,$$

and consequently the theorem holds. \square

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