ANNALES

DE

L'INSTITUT FOURIER

ON THE FOURIER TRANSFORM OF THE
SYMMETRIC DECREASING REARRANGEMENTS

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ABSTRACT. — Inspired by work of Montgomery on Fourier series and Donoho-Strak in signal processing, we investigate two families of rearrangement inequalities for the Fourier transform. More precisely, we show that the L^2 behavior of a Fourier transform of a function over a small set is controlled by the L^2 behavior of the Fourier transform of its symmetric decreasing rearrangement. In the L^1 case, the same is true if we further assume that the function has a support of finite measure.

As a byproduct, we also give a simple proof and an extension of a result of Lieb about the smoothness of a rearrangement. Finally, a straightforward application to solutions of the free Shrödinger equation is given.

RÉSUMÉ. — Le but de cet article est d'approfondir des travaux de Montgomery sur les séries de Fourier et de Donoho et Stark en traitement du signal sur la transformée de Fourier de la réarrangée d'une fonction. Plus précisément, nous montrons que le comportement L^2 sur un petit ensemble de la transformée de Fourier d'une fonction est contrôlé par le comportement L^2 de la transformée de Fourier de sa réarrangée symétrique. Dans le cas L^1 un résultat similaire est démontré pour les fonctions à support de mesure finie.

Par ailleurs, nous donnons une démonstration simple et une extension d'un résultat de Lieb sur la régularité d'une réarrangée. Finalement, nous donnons une application directe aux solutions de l'équation de Shrödinger.

1. Introduction

The use of rearrangement techniques is a major tool for proving functional inequalities. For instance, it has been used extensively for proving the boundedness of the Fourier transform between weighted Lebesgue spaces (see *e.g.* [15, 16, 2] and the references therein). Let us mention that a weighted inequality for the Fourier transform was proved in [2] with the help of a result of Jodeit-Torchinsky [14] showing that an operator that is of type $(1, \infty)$ and of type $(2, 2)$ satisfies some rearrangement inequalities.

Results mentioned so far deal with the rearrangement of Fourier transforms and not with Fourier transforms of rearrangements. The rearrangements we consider here are the spherical rearrangement *i.e.*, we rearrange a function φ on $\mathbb{R}^d$ into an equi-measurable function $|\varphi|^*$ that is radially decreasing. Of course, rearranging and taking Fourier transforms are two operations that are far from commuting, it is thus not possible to deduce anything about the behavior of the Fourier transform of the rearrangement of a function from the results mentioned so far. In that direction, a remarkable theorem is due to Lieb [18, Lemma 4.1] that shows that the decreasing rearrangement $|\varphi|^*$ of a function φ preserves smoothness:

THEOREM 1.1 (Lieb for $s = 1$ [18], Donoho-Stark for $0 < s < 1$ [7]). — *Let $d \geqslant 1$ be an integer and $0 \leqslant s \leqslant 1$. Then there exists a constant C_s such that, for every φ in the Sobolev space $H^s(\mathbb{R}^d)$ i.e., for which*

$$\|\varphi\|_{H^s} := \left(\int_{\mathbb{R}^d} |\widehat{\varphi}(\xi)|^2 (1 + |\xi|^2)^s \, \mathrm{d}\xi \right)^{1/2} < +\infty,$$

$\left\| |\varphi|^* \right\|_{H^s} \leqslant C_s \|\varphi\|_{H^s}.$

Further results relating smoothness and rearrangements may be found *e.g.* in papers by A. Burchard [3], B. Kawohl [17]. The best possible estimate for second order derivatives was obtained by Cianchi [4]. We will also show some sort of dual version of it, namely that $\|\varphi\|_{H^s} \leqslant C_s \||\varphi|^*\|_{H^s}$ if $s < 0$.

In this paper, we are mainly dealing with a slightly different type of estimates. Namely, we will show that the frequency content of the rearrangement of a function controls the frequency content of the function. We first show an "L^1-result" which may be stated as follows:

THEOREM 1.2. — *Let $S, \Omega > 0$. Then there exists a constant $C = C(S, \Omega)$ such that, for every $\varphi \in L^1(\mathbb{R}^d)$ with support of finite measure S, and for every $a, x \in \mathbb{R}^d$,*

$$(1.1) \qquad \left| \int_{|\xi - a| \leqslant \Omega} \widehat{\varphi}(\xi) e^{2i\pi\langle x, \xi \rangle} \, \mathrm{d}\xi \right| \leqslant C \int_{B(0, \Omega)} \widehat{|\varphi|^*}(\xi) \, \mathrm{d}\xi.$$

This theorem is inspired by and generalizes a result of Donoho and Stark [8] which states that, for $d = 1$, $C(S, \Omega) = 1$ provided ΩS is small enough, a result that we will generalize to higher dimension. One does not expect $C(S, \Omega)$ to be bounded when $S \to +\infty$ so that the hypothesis on the support of φ may not be lifted. Nevertheless, we show that the result can be somewhat improved by taking Bochner-Riesz means.

We also show that in the "L^2-case" the situation is better:

THEOREM 1.3. — *Let d be an integer. Then there exists a constant κ_d such that, for every set $\Sigma \subset \mathbb{R}^d$ of finite positive measure and every $\varphi \in L^2(\mathbb{R}^d)$,*

$$\int_\Sigma |\widehat{\varphi}(\xi)|^2 \, \mathrm{d}\xi \leqslant \kappa_d \int_{B(0,\tau)} \left| \widehat{|\varphi|^*}(\xi) \right|^2 \mathrm{d}\xi,$$

where τ is such that $|B(0,\tau)| = |\Sigma|$.

A similar estimate for Fourier series was previously obtained by Montgomery [22]. The result of Montgomery has also been generalized to Fourier transforms in a different way in [15, 16] where rearrangements of Fourier transforms are considered rather than Fourier transforms of rearrangements as in Theorem 1.3. Also, those papers deal with the one-dimensional (non symmetric) decreasing rearrangement.

Theorem 1.3 should be compared to Theorem 1.1. Both theorems state that one can control one of the Fourier transforms of φ or of $|\varphi|^*$ by the other one in a weighted L^2-sense. Our results show that if the weight is "small" (*e.g.* the characteristic function of a set of finite measure), then $\widehat{|\varphi|^*}$ controls $\widehat{\varphi}$. In Lieb's result, the weight is "big" and the control goes the opposite way. We conjecture that this is also the case when the weight is the characteristic function of the complementary of a set of finite measure, at least when φ has a support of finite measure. We will show how this would imply an optimal generalization to higher dimension of an uncertainty principle proved by F. Nazarov in the one-dimensional case.

Further, Theorem 1.3 may be interpreted in the following way: assume that φ has a big "high-frequency component", in the sense that $\int_\Sigma |\widehat{\varphi}(\xi)|^2 \, \mathrm{d}\xi$ stays large for some Σ in a set of balls of radius 1 and centered away from 0, then $|\varphi|^*$ must be big near 0. In other words, high-frequency oscillations are "pushed" to low-frequency oscillations by symmetrization (see Figure 1.1 for an illustration).

The paper is organized as follows. In the next section, we introduce the necessary notation and a few simple preliminary lemmas. In Section 3, we give a new proof of Theorem 1.1 and prove its "dual" version. The following section is devoted to the proof of Theorem 1.2. We pursue in Section 5 with the proof of Theorem 1.1 and present a conjecture related to Nazarov's Uncertainty Principle. We then give a direct illustration of the result in terms of solutions of the free Shrödinger equation. Finally, we explain how to extend our results to other symmetrizations.
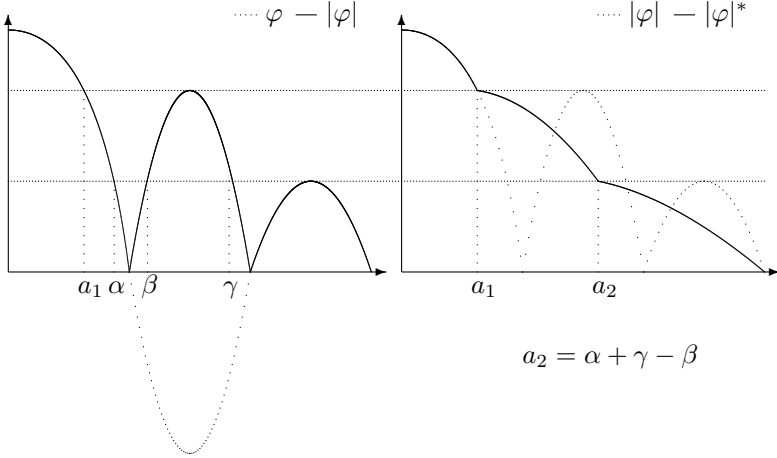
Figure 1.1. *The modulus of a smooth (even) function $|\varphi|$ and its re-arrangement $|\varphi|^*$. Note that $|\varphi|^*$ keeps some of the smoothness of φ but has a second derivative that is not in L^2: $\partial^2|\varphi|^*$ is of the form $\psi + c_1\delta_{a_1} + c_2\delta_{a_2}$. Also $|\varphi|^*$ "oscillates" less than φ which means that its Fourier transform must be more concentrated near 0.*

2. Preliminaries and Notations

2.1. Generalities

Throughout this paper, d will be an integer, $d \geqslant 1$. On $\mathbb{R}^d$, we denote by $\langle .,. \rangle$ and $|.|$ the standard scalar product and the associated norm. For $a \in \mathbb{R}^d$ and $r > 0$, we denote by $B(a,r)$ the open ball centered at a of radius r: $B(a,r) = \{x \in \mathbb{R}^d : |x-a| < r\}$.

The Lebesgue measure on $\mathbb{R}^d$ is denoted $\mathrm{d}x$ and we write $|E|$ for the Lebesgue measure of a Borel set E. The various meanings of $|A|$ should be clear from the context. We will denote by $\omega_d = |B(0,1)| = \dfrac{\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d+2}{2}\right)}$. The characteristic function of a set E will be denoted by χ_E, so that $\chi_E(x) = \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{otherwise} \end{cases}$. The Fourier transform $\mathcal{F}[\varphi] = \widehat{\varphi}$ of $\varphi \in L^1(\mathbb{R}^d)$ is defined by

$$\widehat{\varphi}(\xi) = \int_{\mathbb{R}^d} \varphi(t) e^{-2i\pi\langle t,\xi\rangle}\,\mathrm{d}t.$$

This definition is extended from $L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ to $L^2(\mathbb{R}^d)$ in the standard way. The inverse Fourier transform is denoted $\mathcal{F}^{-1}\varphi = \check{\varphi}$.

2.2. Bessel functions and Fourier transforms

Results in this section can be found in most books on Fourier analysis, for instance [11, Appendix B] and of course in Watson's treatise on the subject [28].

Let λ be a real number with $\lambda > -1/2$. We define the Bessel function J_λ of order λ on $(0, +\infty)$ by its *Poisson representation formula*

$$J_\lambda(x) = \frac{x^\lambda}{2^\lambda \Gamma\left(\lambda + \frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)} \int_{-1}^1 (1-s^2)^\lambda \cos sx \frac{\mathrm{d}s}{\sqrt{1-s^2}}.$$

Let us define $\mathcal{J}_{-1/2}(x) = \cos x$ and for $\lambda > -1/2$, $\mathcal{J}_\lambda(x) := \frac{J_\lambda(x)}{x^\lambda}$. Then $\mathcal{J}_\lambda$ satisfies $|\mathcal{J}_\lambda(x)| \leqslant C_\lambda (1+|x|)^{-\lambda-1/2}$. It is also known that $\mathcal{J}_\lambda$, $\lambda > -1/2$, has only positive real simple zeroes $(j_{\lambda,k})_{k \geqslant 1}$.

We will need the following well-known result:

$$(2.1) \qquad \widehat{\chi_{B(0,1)}}(\xi) = \frac{J_{\frac{d}{2}}(2\pi|\xi|)}{|\xi|^{\frac{d}{2}}}$$

from which we deduce that $|\widehat{\chi_{B(0,1)}}(\xi)| \leqslant C(1+|\xi|)^{-\frac{d+1}{2}}$ thus $\widehat{\chi_{B(0,1)}} \in L^p(\mathbb{R}^d)$ for all $p > \frac{2d}{d+1}$.

More generally, if we denote by $m_\alpha(x) = (1-|x|^2)_+^\alpha$, then

$$\widehat{m_\alpha}(\xi) = \frac{\Gamma(\alpha+1)}{\pi^\alpha} \frac{J_{\frac{d}{2}+\alpha}(2\pi|\xi|)}{|\xi|^{\frac{d}{2}+\alpha}} = 2^{\frac{d}{2}+\alpha}\pi^{\frac{d}{2}}\Gamma(\alpha+1)\mathcal{J}_{\frac{d}{2}+\alpha}(2\pi|\xi|).$$

2.3. Rearrangements

For a Borel subset E of $\mathbb{R}^d$ of finite measure, we define its d-dimensional symmetric rearrangement E^* to be the open ball of $\mathbb{R}^d$ centered at the origin whose volume is that of E. Thus

$$E^* = B(0,r) \quad \text{with} \quad |B(0,r)| = \omega_d r^d = |E|.$$

Let φ be a measurable function on $\mathbb{R}^d$. We will say that φ vanishes at infinity if its level sets have finite measure: *i.e.*, the *distribution function* of φ, $d_\varphi(\lambda) = |\{x \in \mathbb{R}^d : |\varphi(x)| > \lambda\}|$, is finite for all $\lambda > 0$. This is of course the case if $\varphi \in L^p$ for some $p \geqslant 1$. We define the symmetric rearrangement $|\varphi|^*$ via the *layer-cake* representation:

$$|\varphi|^*(x) = \int_0^{+\infty} \chi_{\{y \in \mathbb{R}^d : |\varphi(y)| > \lambda\}^*}(x)\,\mathrm{d}\lambda$$

which should be compared with the layer-cake representation of $|\varphi|$:

$$|\varphi(x)| = \int_0^{+\infty} \chi_{\{y \in \mathbb{R}^d \,:\, |\varphi(y)| > \lambda\}}(x)\,\mathrm{d}\lambda.$$

Rearrangements satisfy many useful properties:

- $|\alpha\varphi|^* = |\alpha||\varphi|^*$.
- For a set E of finite measure and for $\alpha > 0$, $(\alpha E)^* = \alpha E^*$. Therefore, if we write $\varphi_\alpha(x) = \varphi(x/\alpha)$, then $|\varphi_\alpha|^*(x) = |\varphi|^*(x/\alpha)$.
- If ψ is a non-negative radially decreasing function that vanishes at infinity, then $|\psi|^* = \psi$ and if $|\varphi| \leqslant \psi$ then $|\varphi|^* \leqslant \psi$.
- φ and $|\varphi|^*$ are equimeasurable, that is, for all $\lambda > 0$ $d_\varphi(\lambda) = d_{|\varphi|^*}(\lambda)$. In particular, φ and $|\varphi|^*$ have the same L^p norm for $1 \leqslant p \leqslant \infty$.
- The following theorems of Hardy, Littlewood and Riesz will be useful (see *e.g.* [19, Chapter 3] for a proof):

LEMMA 2.1 (Hardy-Littlewood). — *Let* φ, ψ *be non-negative functions vanishing at infinity, then*

$$\int_{\mathbb{R}^d} \varphi(x)\psi(x)\,\mathrm{d}x \leqslant \int_{\mathbb{R}^d} |\varphi|^*(x)|\psi|^*(x)\,\mathrm{d}x.$$

LEMMA 2.2 (Riesz). — *Let* φ, ψ, ζ *be non-negative functions that vanish at infinity. Then*

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \varphi(s)\psi(t)\zeta(s-t)\,\mathrm{d}s\,\mathrm{d}t \leqslant \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |\varphi|^*(s)|\psi|^*(t)|\zeta|^*(s-t)\,\mathrm{d}s\,\mathrm{d}t.$$

3. Lieb's Theorem and its "dual" version

Assume that either $\psi \in L^2(\mathbb{R}^d)$ and $\varphi \in L^2(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$ or vice versa $\varphi \in L^2(\mathbb{R}^d)$ and $\psi \in L^2(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$. Then,

$$\int_{\mathbb{R}^d} \widehat{\psi}(\xi)|\widehat{\varphi}(\xi)|^2\,\mathrm{d}\xi = \int_{\mathbb{R}^d} \widehat{\psi}(\xi)\widehat{\varphi}(\xi)\overline{\widehat{\varphi}(\xi)}\,\mathrm{d}\xi$$

$$= \left\langle \widehat{\psi\varphi}, \widehat{\varphi} \right\rangle = \left\langle \widehat{\psi * \varphi}, \widehat{\varphi} \right\rangle = \langle \psi * \varphi, \varphi \rangle$$

by Parseval. The computations are justified by the fact that, as $\psi \in L^2(\mathbb{R}^d)$ and $\varphi \in L^1(\mathbb{R}^d)$ (or vice versa), $\psi * \varphi \in L^2(\mathbb{R}^d)$. As $\varphi \in L^2(\mathbb{R}^d)$, we may apply Fubini to get

$$\int_{\mathbb{R}^d} \widehat{\psi}(\xi)|\widehat{\varphi}(\xi)|^2\,\mathrm{d}\xi = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \psi(x-y)\varphi(x)\overline{\varphi(y)}\,\mathrm{d}x\,\mathrm{d}y$$

$$\leqslant \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |\psi|^*(x-y)|\varphi|^*(x)|\varphi|^*(y)\,\mathrm{d}x\,\mathrm{d}y$$

with Riesz's Rearrangement Inequality (Lemma 2.2). Unwinding the previous computations, we thus obtain

$$(3.1) \qquad \int_{\mathbb{R}^d} \widehat{\psi}(\xi) |\widehat{\varphi}(\xi)|^2 \, d\xi d\xi. \leqslant \int_{\mathbb{R}^d} |\widehat{\psi}|^*(\xi) \big| \widehat{|\varphi|}^*(\xi) \big|^2 \, d\xi.$$

Remark 3.1. — The hypothesis on φ and ψ can not be totally lifted. For instance, one can not have the inequality with $\widehat{\psi} = \chi_S$ for large enough S (see [8]). However, Theorem 1.3 gives a good substitute in that case.

As an application of (3.1), let us prove the following:

PROPOSITION 3.2. — *Let $s > 0$, and let $\varphi \in L^2$. Then*

$$\int_{\mathbb{R}^d} (1 + |\xi|^2)^{-s} |\widehat{\varphi}(\xi)|^2 \, d\xi \leqslant \int_{\mathbb{R}^d} (1 + |\xi|^2)^{-s} |\widehat{\varphi^*}(\xi)|^2 \, d\xi.$$

Proof. — Let us recall that, for each $s > 0$, there exists a number c_s such that, for every $\xi \in \mathbb{R}^d$,

$$(1 + |\xi|^2)^{-s} = c_s \int_{\mathbb{R}^d} |u|^{2s-d} e^{-\pi(1+|\xi|^2)|u|^2} \, du.$$

Let us define $\psi_u(x) = c_s |u|^{2s-2d} e^{-\pi|u|^2} e^{-\pi|x|^2/|u|^2}$. A simple computation then shows that $\widehat{\psi_u}(\xi) = c_s |u|^{2s-d} e^{-\pi(1+|\xi|^2)|u|^2}$. Applying (3.1) to $\psi = \psi_u$, we obtain

$$\int_{\mathbb{R}^d} c_s |u|^{2s-d} e^{-\pi(1+|\xi|^2)|u|^2} |\widehat{\varphi}(\xi)|^2 d\xi \leqslant \int_{\mathbb{R}^d} c_s |u|^{2s-d} e^{-\pi(1+|\xi|^2)|u|^2} |\widehat{\varphi^*}(\xi)|^2 d\xi.$$

Integrating over $u \in \mathbb{R}^d$ gives the result. $\qquad \square$

As a second consequence of (3.1), we give a new proof of Lieb's result and Donoho-Stark's extension of it, Theorem 1.1. The proof is directly inspired by Lieb's proof.

THEOREM 3.3. — *Let $d \geqslant 1$ be an integer and $0 \leqslant s \leqslant 1$. Then there exists a constant C_s such that, for φ in the Sobolev space $H^s(\mathbb{R}^d)$, $\|\varphi|^*\|_{H^s} \leqslant C_s \|\varphi\|_{H^s}$.*

Proof. — The case $s = 0$ is trivial:

$$(3.2) \qquad \| \widehat{|\varphi|^*} \|_2 = \| |\varphi|^* \|_2 = \| \varphi \|_2 = \| \widehat{\varphi} \|_2.$$

If $0 < s \leqslant 1$, define $g_s(x) = e^{-\pi|x|^s}$ for $x \geqslant 0$. Then, g_s is completely monotonic, that is, for every integer k, $(-1)^k \partial^k g_s(x) \geqslant 0$ for $x > 0$. According to a celebrated theorem of Bernstein (see *e.g.* [9, Chapter 18, Section 4]

or [29, page 161]), g_s is the Laplace transform of a positive measure μ_s on $(0, +\infty)$: $g_s(x) = \int_0^{+\infty} e^{-\pi t x} \, d\mu_s(t)$. In particular, for $\xi \in \mathbb{R}^d$,

$$e^{-\pi|\xi|^{2s}} = \int_0^{+\infty} e^{-\pi t|\xi|^2} \, d\mu_s(t).$$

From (3.1) applied to ψ defined by $\psi(t) = t^{-d/2} e^{-\pi|x|^2/t}$, we obtain

$$\int_{\mathbb{R}^d} e^{-\pi t|\xi|^2} |\widehat{\varphi}(\xi)|^2 \, d\xi \leqslant \int_{\mathbb{R}^d} e^{-\pi t|\xi|^2} |\widehat{|\varphi|^*}(\xi)|^2 \, d\xi.$$

Integrating with respect to $\mu_s(t)$, we thus obtain

$$\int_{\mathbb{R}^d} e^{-\pi t|\xi|^{2s}} |\widehat{\varphi}(\xi)|^2 \, d\xi \leqslant \int_{\mathbb{R}^d} e^{-\pi t|\xi|^{2s}} |\widehat{|\varphi|^*}(\xi)|^2 \, d\xi.$$

With (3.2), it follows that

$$\int_{\mathbb{R}^d} \frac{1 - e^{-\pi t|\xi|^{2s}}}{t} |\widehat{|\varphi|^*}(\xi)|^2 \, d\xi \leqslant \int_{\mathbb{R}^d} \frac{1 - e^{-\pi t|\xi|^{2s}}}{t} |\widehat{\varphi}(\xi)|^2 \, d\xi.$$

Finally, $\dfrac{1 - e^{-\pi t|\xi|^{2s}}}{t} \to \pi|\xi|^{2s}$ as $t \to 0$ and $\dfrac{1 - e^{-\pi t|\xi|^{2s}}}{t} \leqslant \pi|\xi|^{2s}$ so that Lebesgue's Dominated Convergence Theorem implies that

$$(3.3) \qquad \int_{\mathbb{R}^d} |\xi|^{2s} |\widehat{|\varphi|^*}(\xi)|^2 \, d\xi \leqslant \int_{\mathbb{R}^d} |\xi|^{2s} |\widehat{\varphi}(\xi)|^2 \, d\xi.$$

A new appeal to (3.2) and to the fact that $1 + |\xi|^{2s} \simeq (1 + |\xi|^2)^s$ gives the result. □

Note that for $s = 1$, $1 + |\xi|^{2s} = (1 + |\xi|^2)^s$ so that $C_1 = 1$. Note also that the proof does not extend to $s > 1$ as g_s is no longer completely monotonic in that case.

4. An extension of a result of Donoho and Stark

4.1. Higher dimensional generalization of Donoho and Stark's theorem

We will start this section by giving a simple generalization of Donoho and Stark's result. The main purpose of this is to explain the idea of the proof of the following theorem (Theorem 4.4) which may appear a bit obscure and technical otherwise.

PROPOSITION 4.1. — *Let $d \geqslant 1$ and $\alpha > -1/2$. Then there exists a constant $\vartheta = \vartheta(d, \alpha)$ such that, for every $\Omega, S > 0$ with $\Omega S^{1/d} \leqslant \vartheta$ and for every $\varphi \in L^1(\mathbb{R}^d)$ with support of finite measure at most S, we have*

$$\left| \int_{\mathbb{R}^d} m_\alpha(\xi/\Omega) \widehat{\varphi}(\xi) \, d\xi \right| \leqslant \int_{\mathbb{R}^d} m_\alpha(\xi/\Omega) \widehat{|\varphi|^*}(\xi) \, d\xi$$

where $m_\alpha(x) = (1 - |x|^2)_+^\alpha$.

Proof. — Let $R = (S/\omega_d)^{1/d}$, so that the ball of radius R has measure S, $|B(0, R)| = S$. Plancherel's Formula shows that

$$\int_{\mathbb{R}^d} m_\alpha(\xi/\Omega) \widehat{\varphi}(\xi) \, d\xi = 2^{d/2+\alpha} \pi^{d/2} \Gamma(\alpha + 1) \Omega^d \int_{\mathbb{R}^d} \mathcal{J}_{d/2+\alpha}(2\pi\Omega|x|) \varphi(x) \, dx.$$

It follows from Lemma 2.1 that

$$\left| \int_{\mathbb{R}^d} m_\alpha(\xi/\Omega) \widehat{\varphi}(\xi) \, d\xi \right|$$

$$\leqslant \quad 2^{d/2+\alpha} \pi^{d/2} \Gamma(\alpha + 1) \Omega^d \int_{\mathbb{R}^d} |\mathcal{J}_{d/2+\alpha}(2\pi\Omega|x|)| |\varphi(x)| \, dx$$

$$\leqslant \quad 2^{d/2+\alpha} \pi^{d/2} \Gamma(\alpha + 1) \Omega^d \int_{\mathbb{R}^d} |\mathcal{J}_{d/2+\alpha}|^*(2\pi\Omega|x|) |\varphi|^*(x) \, dx$$

$$= \quad 2^{d/2+\alpha} \pi^{d/2} \Gamma(\alpha + 1) \Omega^d \int_{B(0,R)} |\mathcal{J}_{d/2+\alpha}|^*(2\pi\Omega|x|) |\varphi|^*(x) \, dx.$$

In order to complete the proof, we will need the following fact:

Fact. *There exists $\varepsilon_0 > 0$ such that $|\mathcal{J}_{d/2+\alpha}|^* = \mathcal{J}_{d/2+\alpha}$ on $[0, \varepsilon_0)$.*

We will postpone the proof of this fact to the end of this section. An immediate consequence of this fact is that if $2\pi\Omega|x| < \varepsilon_0$ on $B(0, R)$ i.e., $\Omega S^{1/d} < \varepsilon_0 \omega_d^{1/d}/2\pi$, then

$$\left| \int_{\mathbb{R}^d} m_\alpha(\xi/\Omega) \widehat{\varphi}(\xi) \, d\xi \right|$$

$$(4.1) \qquad \leqslant \quad 2^{d/2+\alpha} \pi^{d/2} \Gamma(\alpha + 1) \Omega^d \int_{B(0,R)} \mathcal{J}_{d/2+\alpha}(2\pi\Omega|x|) |\varphi|^*(x) \, dx$$

$$= \quad 2^{d/2+\alpha} \pi^{d/2} \Gamma(\alpha + 1) \Omega^d \int_{\mathbb{R}^d} \mathcal{J}_{d/2+\alpha}(2\pi\Omega|x|) |\varphi|^*(x) \, dx$$

$$= \quad \int_{\mathbb{R}^d} m_\alpha(\xi/\Omega) \widehat{|\varphi|^*}(\xi) \, d\xi$$

which completes the proof once we have proved the fact. $\qquad \square$

Let us now give the proof of the fact (note that this is obvious for the sinc function). Recall that $\mathcal{J}_{d/2+\alpha}(2\pi|x|) = \left(2^{d/2+\alpha} \pi^{d/2} \Gamma(\alpha + 1) \right)^{-1} \widehat{m_\alpha}(x)$. As m_α is non-negative we obtain that $\mathcal{J}_{d/2+\alpha}$ is positive definite. On the other

hand, $\mathcal{J}_{d/2+\alpha}$ is smooth and $\mathcal{J}_{d/2+\alpha}(t) \to 0$ when $t \to +\infty$. The claim then follows once we know that $|\mathcal{J}_{d/2+\alpha}(t)|$ has a global maximum at 0 which is the only place where this maximum is reached. The following lemma shows that this is the case for any non-periodic positive definite function ($\mathbb{R}^d$ may be replaced by any LCA group for that purpose):

LEMMA 4.2. — *Let Φ be a positive definite function on $\mathbb{R}^d$. Then Φ has a global maximum at 0 and, if there is $t_0 \in \mathbb{R}^d \setminus \{0\}$ such that $|\Phi(t_0)| = |\Phi(0)|$, then $|\Phi|$ is t_0-periodic.*

Proof. — Recall that Φ is positive definite if

$$\sum_{i,j=1,\ldots,n} \Phi(t_i - t_j)\xi_i\overline{\xi_j} \geqslant 0$$

for every integer $n \geqslant 2$, every $t_1, \ldots, t_n \in \mathbb{R}^d$ and every $\xi_1, \ldots, \xi_n \in \mathbb{C}$ i.e., if the matrix $[\Phi(t_i - t_j)]_{1 \leqslant i,j \leqslant n}$ is positive definite. In particular, taking $n = 2$ $t_1 = 0$, $t_2 = t$ and $\xi_1 = 1$, $\xi_2 = 1$ and $\xi_2 = i$ shows that $\Phi(-t) = \overline{\Phi(t)}$. Then, taking ξ_2 such that $\xi_2\Phi(t) = -|\Phi(t)|$ shows that $|\Phi|$ has a maximum at 0.

Now note that if Φ is positive definite then, for every $t > 0$, the matrix

$$\begin{pmatrix} \Phi(0) & \Phi(t_0) & \Phi(t - t_0) \\ \Phi(t_0) & \Phi(0) & \Phi(t) \\ \Phi(t - t_0) & \Phi(t) & \Phi(0) \end{pmatrix}$$

is positive definite and thus has non-negative determinant. In particular, if $\Phi(t_0) = \pm\Phi(0)$, $\big(\Phi(t - t_0) \mp \Phi(t)\big)^2 \leqslant 0$. It follows that, for all $t > 0$, $\Phi(t - t_0) = \pm\Phi(t)$ as claimed.   □

Finally, note that if, as in [8] we use Riesz's Inequality (Lemma 2.2) instead of Hardy-Littlewood's Iniequality (Lemma 2.1), we obtain the following:

PROPOSITION 4.3. — *Let $d \geqslant 1$ and $\alpha > -1/2$ and $\vartheta(d,\alpha)$ be the constant of Proposition 4.1. Let $\varphi \in L^2(\mathbb{R}^d)$ with support of finite measure S. If $\Omega S^{1/d} \leqslant \vartheta(d,\alpha)/2$, then*

$$\left| \int_{\mathbb{R}^d} m_\alpha(\xi/\Omega)|\widehat{\varphi}(\xi)|^2 \,\mathrm{d}\xi \right| \leqslant \int_{\mathbb{R}^d} m_\alpha(\xi/\Omega)|\widehat{|\varphi|^*}(\xi)|^2 \,\mathrm{d}\xi.$$

4.2. The main theorem

We are now in position to extend this result in the following way:

THEOREM 4.4. — *Let d be an integer and let $\beta \geqslant \frac{d}{2} - 1$ and $\alpha > -\frac{1}{2}$. Let $S, \Omega > 0$. Set*

$$\zeta_\alpha(s) = \begin{cases} (1+s)^{\frac{1}{d}\left(\frac{d-1}{2} - \alpha\right)} & \text{if } \alpha < \frac{d-1}{2} \\ \ln(2+s) & \text{if } \alpha = \frac{d-1}{2} \\ 1 & \text{if } \alpha > \frac{d-1}{2} \end{cases}.$$

Then there exists a constant $C = C(d, \alpha, \beta)$ such that, for every $\varphi \in L^1(\mathbb{R}^d)$ with support of finite measure S, for every $x, a \in \mathbb{R}^d$,

$$\left| \int_{\mathbb{R}^d} \widehat{\varphi}(\xi)(1 - |\xi - a|^2/\Omega^2)_+^\alpha e^{2i\pi\langle x, \xi\rangle} \, d\xi \right|$$

(4.2)
$$\leqslant C\zeta_\alpha(\Omega^d S) \int_{\mathbb{R}^d} (1 - |\xi|^2/\Omega^2)_+^\beta \widehat{|\varphi|^*}(\xi) \, d\xi.$$

Remark 4.5. — The condition $\alpha > \frac{d-1}{2}$ that appears here for $\zeta_\alpha(s)$ to be constant is the same as the trivial bound for convergence of Bochner-Riesz means (see *e.g.* [11]). Actually both bounds only depend on bounds of appropriate Bessel functions.

Proof. — Throughout this proof, C will be a constant that depends only on α, β and d and, as is usual, may change from line to line.

It is enough to prove (4.2) for $a = x = 0$ and then apply it to $\varphi^{(a,x)}(t) = \varphi(t-x)e^{2i\pi at}$. Note that $|\varphi^{(a,x)}|^* = |\varphi|^*$ so that the right hand side of (1.1) is unaffected by the change of φ to $\varphi^{(a,b)}$.

We may further replace φ by its dilate $\varphi(x/\Omega)$ so that, without loss of generality, we may assume that $\Omega = 1$. More precisely, assume we are able to prove

(4.3)
$$\int_{\mathbb{R}^d} (1 - |\xi|^2)_+^\alpha \widehat{\varphi}(\xi) \, d\xi \leqslant C\zeta_\alpha(S) \int_{\mathbb{R}^d} (1 - |\xi|^2)_+^\beta \widehat{|\varphi|^*}(\xi) \, d\xi$$

for every $S > 0$ and every functions φ with support of finite measure S. We will then apply this to $\varphi_\Omega(x) = \varphi(x/\Omega)$ which has support of measure $\Omega^d S$. Note that

$$\int_{\mathbb{R}^d} (1 - |\xi|^2)_+^\alpha \widehat{\varphi_\Omega}(\xi) \, d\xi = \int_{\mathbb{R}^d} \Omega^d (1 - |\xi|^2)_+^\alpha \widehat{\varphi}(\Omega\xi) \, d\xi$$

$$= \int_{\mathbb{R}^d} (1 - |\xi|^2/\Omega^2)_+^\alpha \widehat{\varphi}(\xi) \, d\xi.$$

On the other hand, $|\varphi_\Omega|^*(x) = |\varphi|^*(x/\Omega)$ therefore $\widehat{|\varphi_\Omega|^*}(\xi) = \Omega^d \widehat{|\varphi|^*}(\Omega\xi)$ so that

$$\int_{\mathbb{R}^d} (1 - |\xi|^2)_+^\beta \widehat{|\varphi_\Omega|^*}(\xi) \, d\xi = \int_{\mathbb{R}^d} (1 - |\xi|^2/\Omega^2)_+^\beta \widehat{|\varphi|^*}(\xi) \, d\xi.$$

It is thus enough to prove (4.3), *i.e.*, to assume that $\Omega = 1$ in Theorem 4.4. The beginning of the proof follows the lines of the proof of Proposition 4.1.

Using Parseval and Hardy-Littlewood's Symmetrization Lemma 2.1, we obtain

$$\left| \int_{\mathbb{R}^d} (1 - |\xi|^2)_+^\alpha \widehat{\varphi}(\xi) \, d\xi \right| = \left| \int_{\mathbb{R}^d} \widehat{m_\alpha}(t) \varphi(t) \, dt \right|$$

$$\leqslant \int_{\mathbb{R}^d} |\widehat{m_\alpha}|^*(t) |\varphi|^*(t) \, dt.$$

Recall that $\widehat{m_\alpha}(t) = C_{d,\alpha} |t|^{-d/2-\alpha} J_{d/2+\alpha}(2\pi|t|)$. We will therefore write $|\widehat{m_\alpha}|^*(t) = \mathcal{J}_{d/2+\alpha}^*(|t|)$.

On the other hand

$$\int_{\mathbb{R}^d} (1 - |\xi|^2)_+^\beta \widehat{|\varphi|^*}(\xi) \, d\xi = \int_{\mathbb{R}^d} C_{d,\beta} |t|^{-d/2-\beta} J_{d/2+\beta}(2\pi|t|) |\varphi|^*(t) \, dt.$$

But, as $|\varphi|^*$ is a radial function that is "radially decreasing", we may write

$$|\varphi|^*(t) = \int_0^{s_S} \chi_{B(0,s)}(t) \, d\mu(s)$$

where μ is a non-negative measure, $\chi_{B(0,s)}$ is the characteristic function of the ball of center 0 and radius s and s_S is defined by $|B(0, s_S)| = S$ *i.e.*, $s_S = (S/\omega_d)^{1/d}$. It is thus enough to prove that, for $s \leqslant s_S$,

$$\int_{\mathbb{R}^d} \mathcal{J}_{d/2+\alpha}^*(|t|) \chi_{B(0,s)}(t) \, dt$$

$$\leqslant \quad C\zeta_\alpha(S) \int_{\mathbb{R}^d} |t|^{-d/2-\beta} J_{d/2+\beta}(2\pi|t|) \chi_{B(0,s)}(t) \, dt.$$

Switching to polar coordinates, this is equivalent to:

$$(4.4) \qquad \int_0^s \mathcal{J}_{d/2+\alpha}^*(r) r^{d-1} \, dr \leqslant C\zeta_\alpha(S) \int_0^s r^{d/2-\beta-1} J_{d/2+\beta}(2\pi r) \, dr.$$

If $2\pi s < j_{d/2+\beta,1}$, the first zero of $J_{d/2+\beta}$, then this inequality follows immediately from the fact that $J_{d/2+\beta}(2\pi r) \sim r^{d/2+\beta}$ and the fact that $\mathcal{J}_{d/2+\alpha}^*(0) > 0$. We will now prove (4.4) when $j_{d/2+\beta,1} \leqslant 2\pi s \leqslant \frac{2\pi}{\omega_d^{1/d}} S^{1/d}$.

Let us first estimate the left hand side. From $|J_\nu(t)| \leqslant C_\nu(1 + |t|)^{-1/2}$ we immediately get $\mathcal{J}_{d/2+\alpha}^*(t) \leqslant C_{d,\alpha}(1 + |t|)^{-\frac{1}{2}(d+1+2\alpha)}$. It follows that,

$$\int_0^s \mathcal{J}_{d/2+\alpha}^*(r) r^{d-1} \, dr \leqslant \begin{cases} C(1 + s)^{\frac{d-1}{2} - \alpha} & \text{if } \alpha < \frac{d-1}{2} \\ C \ln(1 + s) & \text{if } \alpha = \frac{d-1}{2} \\ C & \text{if } \alpha > \frac{d-1}{2}. \end{cases}$$

As $s_S \sim S^{1/d}$, it follows from standard computations that

$$\int_0^s \mathcal{J}_{d/2+\alpha}^*(r) r^{d-1} \, dr \leqslant C\zeta_\alpha(S)$$

for $s \leqslant s_S$. It is thus enough to prove that the integral in the left hand side of (4.4), $\int_0^s r^{d/2-\beta-1} J_{d/2+\beta}(2\pi r) \, dr$, is bounded below.

But, it is well known (see [28] or the remark below for more precise bibliographic details) that the graph of a Bessel function J_ν ($\nu > -1$) consists of "waves" that are alternately positive and negative, the first one being positive. Moreover, the areas of these waves is strictly deceasing. That is, if we denote by $j_{\nu,k}$ the k-th zero of J_ν and $j_{\nu,0} = 0$, then

$$(4.5) \qquad \int_{j_{\nu,k}}^{j_{\nu,k+1}} |J_\nu(r)| \, dr$$

is strictly decreasing. As a consequence, we obtain that, for $\gamma \geqslant 0$,

$$\int_{j_{\nu,k}}^{j_{\nu,k+1}} r^{-\gamma} |J_\nu(r)| \, dr > \int_{j_{\nu,k}}^{j_{\nu,k+1}} j_{\nu,k+1}^{-\gamma} |J_\nu(r)| \, dr$$

$$> \int_{j_{\nu,k+1}}^{j_{\nu,k+2}} j_{\nu,k+1}^{-\gamma} |J_\nu(r)| \, dr$$

$$> \int_{j_{\nu,k+1}}^{j_{\nu,k+2}} r^{-\gamma} |J_\nu(r)| \, dr.$$

It follows that, for $x > j_{\nu,1}$

$$\int_0^x r^{-\gamma} J_\nu(r) \, dr \geqslant C_{\nu,\gamma} := \int_0^{j_{\nu,2}} r^{-\gamma} J_\nu(r) \, dr > 0.$$

Inequality (4.4) follows immediately. $\qquad\qquad\qquad\qquad\qquad\qquad\square$

Remark 4.6. — The behavior of Bessel functions used here is a classical result that was proved originally by Cooke [5, 6] using delicate estimates involving the Lommel functions and several properties of Bessel functions. This is linked to the Gibbs Overshooting Phenomenon of Fourier series which is best seen for $\nu = 1/2$ (the case $d = 1$, $\alpha = 0$) for which $J_{1/2}(t) = \dfrac{\sin t}{t}$. The behaviour of the *sinc* function used here can then be found in text books devoting a chapter on Gibbs Phenomenon like [27, Chapter 4.7] or [30, Chapter 8].

In [20], Makai proved (4.5) for $\nu > -1$ in a simpler way using a differential equation approach of Sturm-Liouville type. A particularly simple proof of Cooke's Theorem has been devised by Steinig in [25].

The range of γ's can be extended to some negative values $\gamma > -\gamma(\nu)$ where $\gamma(\nu)$ is defined in an implicit form for $-1 < \nu < -1/2$, (Askey and Steinig [1]) and $\gamma(\nu) = -1/2$ for $\nu \geqslant -1/2$ (Gasper [10]). Further results may be found in [21].

The argument in the proof may be slightly modified to obtain the following:

COROLLARY 4.7. — *Let d be an integer and let $\beta \geqslant \frac{d}{2}$ and $\alpha > -1/2$. Let $S, \Omega > 0$. Then there exists a constant $C = C(d, \alpha, \beta)$ such that, for $\varphi \in L^1(\mathbb{R}^d)$ with support of finite measure S, for every $a \in \mathbb{R}^d$,*

$$\int_{\mathbb{R}^d} |\widehat{\varphi}(\xi)|(1 - |\xi - a|^2/\Omega^2)^\alpha_+ \, \mathrm{d}\xi \leqslant C(1 + \Omega^d S)^{1/2} \int_{\mathbb{R}^d} (1 - |\xi|^2/\Omega^2)^\beta_+ \widehat{|\varphi|^*}(\xi) \, \mathrm{d}\xi.$$

More generally, there is a constant $\Upsilon = \Upsilon(d)$ such that, if $\psi \in L^2(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$ then, for every $\varphi \in L^1(\mathbb{R}^d)$ with support of finite measure S,
(4.6)
$$\int_{\mathbb{R}^d} |\widehat{\varphi}(\xi)||\psi(\xi/\Omega)| \, \mathrm{d}\xi \leqslant \Upsilon \|\psi\|_2 (1 + \Omega^d S)^{1/2} \int_{\mathbb{R}^d} (1 - |\xi|^2/\Omega^2)^\beta_+ \widehat{|\varphi|^*}(\xi) \, \mathrm{d}\xi$$

while if $\psi \in L^1(\mathbb{R}^d)$ then, for every $\varphi \in L^1(\mathbb{R}^d)$ with support of finite measure S,

$$\int_{\mathbb{R}^d} |\widehat{\varphi}(\xi)||\psi(\xi/\Omega)| \, \mathrm{d}\xi \leqslant \Upsilon \|\psi\|_1 (1 + \Omega^d S) \int_{\mathbb{R}^d} (1 - |\xi|^2/\Omega^2)^\beta_+ \widehat{|\varphi|^*}(\xi) \, \mathrm{d}\xi.$$

Sketch of proof. — The first statement is a consequence of the second one. Let us however only prove the first one as the comparison with the previous proof is easier in that case, the general case follows by replacing m_α by ψ in the argument.

Again, we may assume that $a = 0$. Let us write $|\widehat{\varphi}| = \widehat{\varphi} e^{i\phi}$ then

$$\int_{\mathbb{R}^d} (1 - |\xi|^2)^\alpha_+ |\widehat{\varphi}(\xi)| \, \mathrm{d}\xi = \int_{\mathbb{R}^d} e^{i\phi(\xi)} m_\alpha(\xi) \widehat{\varphi}(\xi) \, \mathrm{d}\xi$$

$$= \left| \int_{\mathbb{R}^d} [e^{i\phi(\cdot)} m_\alpha(\cdot)]\widehat{}(t) \varphi(t) \, \mathrm{d}t \right|$$

$$\leqslant \int_{\mathbb{R}^d} |\vartheta_\alpha|^*(t) |\varphi|^*(t) \, \mathrm{d}t$$

where ϑ_α is the Fourier transform of $e^{i\phi}m_\alpha$. One can not expect any better behavior of this function then to be in $L^2 \cap L^\infty$. In this case, Cauchy-Schwarz gives (with the obvious abuse of notation)

$$\int_0^s |\vartheta_\alpha|^*(r)r^{d-1}\,\mathrm{d}r \leqslant \left(\int_0^s |\vartheta_\alpha|^*(r)^2 r^{d-1}\,\mathrm{d}r\right)^{1/2} \frac{s^{d/2}}{\sqrt{d}}$$

$$= C_d \|\,|\vartheta_\alpha|^*\|_2 s^{d/2} = C_d \|\vartheta_\alpha\|_2 s^{d/2}$$

where C_d is a constant that depends only on the dimension. It follows from Parseval's Identity that

$$\int_0^s |\vartheta_\alpha|^*(r)r^{d-1}\,\mathrm{d}r \leqslant C_d \|m_\alpha\|_2 s^{d/2}.$$

The rest of the proof follows the line of the previous one.

The general result is obtained in the same way by replacing m_α by any $L^1 \cap L^2$-function.

Note that, if one only uses the fact that $m_\alpha \in L^1$, then a worse bound is obtained:

$$\int_0^s |\vartheta_\alpha|^*(r)r^{d-1}\,\mathrm{d}r \leqslant \|\,|\vartheta_\alpha|^*\|_\infty \frac{s^d}{d} = \|\vartheta_\alpha\|_\infty \frac{s^d}{d} \leqslant \|m_\alpha\|_1 \frac{s^d}{d}.$$

One may again replace m_α by any L^1-function. $\qquad\square$

Finally, assume that $\varphi \in L^2$ with support of finite measure S. Then $\varphi \in L^1$ and $|\widehat{\varphi}(\xi)| \leqslant \|\varphi\|_1 \leqslant S^{1/2}\|\varphi\|_2$. Thus

$$\int_{B(0,\Omega)} |\widehat{\varphi}(\xi)|^2\,\mathrm{d}\xi \leqslant S^{1/2}\|\varphi\|_2 \int_{B(0,\Omega)} |\widehat{\varphi}(\xi)|\,\mathrm{d}\xi$$

$$\leqslant \Upsilon\omega_d^{1/2}\Omega^{d/2}(1+S)^{\frac{1}{2}} S^{1/2}\|\varphi\|_2 \int_{B(0,\Omega)} \widehat{|\varphi|}^*(\xi)\,\mathrm{d}\xi$$

where we have used Corollary 4.7 with $\psi = \chi_{B(0,\Omega)}$. Using Cauchy-Schwarz, we thus get

$$\int_{B(0,\Omega)} |\widehat{\varphi}(\xi)|^2\,\mathrm{d}\xi \leqslant \kappa(1+\Omega^d S)\|\varphi\|_2 \left(\int_{B(0,\Omega)} \left|\widehat{|\varphi|}^*(\xi)\right|^2\,\mathrm{d}\xi\right)^{1/2},$$

where κ is a constant that depends only on d. It turns out that this estimate may be improved. This is done in Section 5 by adapting a result originally proved for Fourier series in [22].

5. The L^2 Theorem

Let us now prove Theorem 1.3 from the introduction. For convenience, let us recall the statement here:

THEOREM 5.1. — *Let $d \geqslant 1$ be an integer. Then there exists a constant κ_d such that, for every set $\Sigma \subset \mathbb{R}^d$ of finite positive measure and every $\varphi \in L^2(\mathbb{R}^d)$,*

$$(5.1) \qquad \int_\Sigma |\widehat{\varphi}(\xi)|^2 \, \mathrm{d}\xi \leqslant \kappa_d \int_{\Sigma^*} \left| \widehat{|\varphi|^*}(\xi) \right|^2 \mathrm{d}\xi$$

where Σ^* is the symmetric rearrangement of Σ.

Proof. — Let τ be defined by $|B(0,\tau)| = |\Sigma|$, that is $B(0,\tau) = \Sigma^*$. For $\lambda > 0$, let $\mathbb{D}_\varphi(\lambda)$ be the level set

$$\mathbb{D}_\varphi(\lambda) = \{x \in \mathbb{R}^d : |\varphi(x)| > \lambda\}.$$

Let us first notice that it is enough to prove the theorem for $\varphi \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$. Indeed, once this is done, fix $\varphi \in L^2(\mathbb{R}^d)$ and, for $\lambda > 0$, define $\varphi_\lambda = \varphi \chi_{\mathbb{D}_\varphi(\lambda)}$ so that $|\varphi_\lambda|^* = |\varphi|^* \chi_{\mathbb{D}_\varphi(\lambda)}$. Then, as $\mathbb{D}_\varphi(\lambda)$ has finite measure, $\varphi_\lambda \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ and we may apply (5.1) to it:

$$(5.2) \qquad \int_\Sigma |\widehat{\varphi_\lambda}(\xi)|^2 \, \mathrm{d}\xi \leqslant \kappa_d \int_{\Sigma^*} \left| \widehat{|\varphi_\lambda|^*}(\xi) \right|^2 \mathrm{d}\xi.$$

Moreover, when $\lambda \to 0$, $\varphi_\lambda \to \varphi$ (resp. $|\varphi_\lambda|^* \to |\varphi|^*$) almost everywhere and, with Lebesgue's Dominated Convergence Theorem, in $L^2(\mathbb{R}^d)$. By continuity of the Fourier transform, we may thus go to the limit when $\lambda \to 0$ in (5.2) to obtain the theorem.

Now, let us further choose $\bar{\lambda}$ to be the smallest non-negative real number λ such that $|\mathbb{D}_\varphi(\lambda)| \leqslant |B(0,\tau^{-1})|$ so $\lambda = \lim\limits_{|x| \to \tau^{-1}+0} |\varphi|^*(x)$. For simplicity of notation, we will write $\mathbb{D} = \mathbb{D}_\varphi(\bar{\lambda})$. Let us further write $\widehat{\varphi} = f + g$ where

$$f(\xi) = \int_{\mathbb{D}} \varphi(x) e^{-2i\pi \langle x, \xi \rangle} \, \mathrm{d}x \quad \text{and} \quad g(\xi) = \int_{\mathbb{R}^d \setminus \mathbb{D}} \varphi(x) e^{-2i\pi \langle x, \xi \rangle} \, \mathrm{d}x.$$

First

$$\int_\Sigma |f(\xi)|^2 \, \mathrm{d}\xi \leqslant \|f\|_\infty^2 \int_\Sigma 1 \, \mathrm{d}\xi \leqslant \left(\int_{\mathbb{D}} |\varphi(x)| \, \mathrm{d}x \right)^2 |\Sigma|$$

$$\leqslant |B(0,\tau)| \left(\int_{B(0,\tau^{-1})} |\varphi|^*(x) \, \mathrm{d}x \right)^2.$$

Further, using Parseval's Identity

$$\int_\Sigma |g(\xi)|^2 \,\mathrm{d}\xi \leqslant \int_{\mathbb{R}^d} |g(\xi)|^2 \,\mathrm{d}\xi = \int_{\mathbb{R}^d \setminus \mathbb{D}} |\varphi(x)|^2 \,\mathrm{d}x = \int_{|x| \geqslant \tau^{-1}} |\varphi|^*(x)^2 \,\mathrm{d}x.$$

As $|\widehat{\varphi}|^2 \leqslant 2|f|^2 + 2|g|^2$, we get
(5.3)

$$\int_\Sigma |\widehat{\varphi}(\xi)|^2 \,\mathrm{d}\xi \leqslant 2|B(0,\tau)| \left(\int_{B(0,\tau^{-1})} |\varphi|^*(x) \,\mathrm{d}x \right)^2 + 2 \int_{|x| \geqslant \tau^{-1}} |\varphi|^*(x)^2 \,\mathrm{d}x.$$

On the other hand, let $K = \chi_{B(0,1)} * \chi_{B(0,1)}$ and note that

(i) $0 \leqslant K \leqslant K(0) = \omega_d := |B(0,1)| = \frac{\pi^{d/2}}{\Gamma(d/2+1)}$,

(ii) K is supported in $B(0,2)$,

(iii) $K = \widehat{k^2}$ where $k(x) = \widehat{\chi_{B(0,1)}}(x) = \dfrac{J_{d/2}(2\pi|x|)}{|x|^{d/2}}$.

Further, let $K_\tau(\xi) = \frac{1}{\omega_d} K(2\xi/\tau)$ and $k_\tau(x) = \left(\dfrac{\tau^d}{\omega_d 2^d} \right)^{1/2} k(x\tau/2)$. A simple computation then shows that

$$\int_{B(0,\tau)} \left| \widehat{|\varphi|^*}(\xi) \right|^2 \,\mathrm{d}\xi \quad \geqslant \int_{\mathbb{R}^d} K_\tau(\xi) \left| \widehat{|\varphi|^*}(\xi) \right|^2 \,\mathrm{d}\xi$$

(5.4)
$$= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |\varphi|^*(x)|\varphi|^*(y) \int_{\mathbb{R}^d} K_\tau(\xi) e^{2i\pi\xi(x-y)} \,\mathrm{d}\xi \,\mathrm{d}x \,\mathrm{d}y$$

$$= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |\varphi|^*(x)|\varphi|^*(y) k_\tau^2(x - y) \,\mathrm{d}x \,\mathrm{d}y := I.$$

Now, using the Poisson representation of Bessel functions,

$$k_\tau(u) = \left(\frac{\tau^d}{\omega_d 2^d} \right)^{1/2} k(u\tau/2) = \frac{J_{d/2}(\pi\tau|u|)}{\sqrt{\omega_d}|u|^{d/2}}$$

$$= \gamma_d \sqrt{\omega_d} \tau^{d/2} \int_0^1 (1 - t^2)^{\frac{d-1}{2}} \cos(\pi|u|\tau t) \,\mathrm{d}t$$

where γ_d is a constant that depends only on d. It is then obvious that, for $0 \leqslant |u| \leqslant 1/\tau$,

$$k_\tau(u) \geqslant k_\tau(1/\tau) = \gamma_d \sqrt{\omega_d} \tau^{d/2} \int_0^1 (1 - t^2)^{\frac{d-1}{2}} \cos(\pi t) \,\mathrm{d}t$$

$$= (\upsilon_d |B(0,\tau)|)^{1/2}$$

where

$$\upsilon_d = \left(\gamma_d \int_0^1 (1 - t^2)^{\frac{d-1}{2}} \cos(\pi t) \,\mathrm{d}t \right)^2.$$

We will now write $I \geqslant I_1 + I_2$. For I_1, we restrict the integration in (5.5) to $x, y \in B(0, 1/2\tau)$ and for I_2, the integration is restricted over $|x| \geqslant 1/\tau$ and $|y - x| \leqslant 1/\tau$.

Let us first estimate I_1:

$$I_1 \geqslant v_d |B(0,\tau)| \iint_{x,y \in B(0,1/2\tau)} |\varphi|^*(x) |\varphi|^*(y) \, \mathrm{d}x \, \mathrm{d}y$$

$$\geqslant v_d |B(0,\tau)| \left(\int_{|x| \leqslant \frac{1}{2\tau}} |\varphi|^*(x) \, \mathrm{d}x \right)^2.$$

From

$$\int_{|x| \leqslant \frac{1}{2\tau}} |\varphi|^*(x) \, \mathrm{d}x = \int_{|x| \leqslant \frac{1}{\tau}} |\varphi|^*(x) \, \mathrm{d}x - \int_{\frac{1}{2\tau} \leqslant |x| \leqslant \frac{1}{\tau}} |\varphi|^*(x) \, \mathrm{d}x$$

we deduce that

$$\int_{|x| \leqslant \frac{1}{2\tau}} |\varphi|^*(x) \, \mathrm{d}x \geqslant \frac{1}{2^d} \int_{|x| \leqslant \frac{1}{\tau}} |\varphi|^*(x) \, \mathrm{d}x.$$

We thus obtain the etimate

$$(5.5) \qquad I_1 \geqslant \frac{v_d}{2^d} |B(0,\tau)| \left(\int_{B(0,1/\tau)} |\varphi|^*(x) \, \mathrm{d}x \right)^2.$$

On the other hand, if $|y| < |x|$, then $|\varphi|^*(y) \geqslant |\varphi|^*(x)$, so that

$$\int_{\{y \in \mathbb{R}^d \, : \, |y| < |x|, |x-y| \leqslant \frac{1}{\tau}\}} |\varphi|^*(y) k_\tau^2(x-y) \, \mathrm{d}y$$

$$\geqslant \quad |\varphi|^*(x) \int_{\{y \in \mathbb{R}^d \, : \, |y| < |x|, |x-y| \leqslant \frac{1}{\tau}\}} k_\tau^2(x-y) \, \mathrm{d}y.$$

Using the properties of k_τ we then obtain

$$\int_{\{y \in \mathbb{R}^d \, : \, |y| < |x|, |x-y| \leqslant \frac{1}{\tau}\}} |\varphi|^*(y) k_\tau^2(x-y) \, \mathrm{d}y$$

$$\geqslant \quad |\varphi|^*(x) \int_{|t| \leqslant \frac{1}{\tau}, |x+t| < |x|} k_\tau^2(t) \, \mathrm{d}t$$

$$\geqslant \quad |\varphi|^*(x) k_\tau^2(1/\tau) |B(0,1/\tau) \cap B(x,|x|)|$$

$$\geqslant \quad |\varphi|^*(x) v_d |B(0,\tau)| |B(0,1/2\tau)|$$

$$= \quad \frac{v_d \omega_d^2}{2^d} |\varphi|^*(x)$$

provided $|x| \geqslant \frac{1}{\tau}$, since then

$$\{|t| \leqslant \tau^{-1}, |x+t| < |x|\} \supset B\left(\frac{x}{2\tau|x|}, \frac{1}{2\tau} \right).$$

Therefore, as for I_2 we have restricted the integration in (5.5) to (x,y)'s such that $|x| \geqslant \frac{1}{\tau}$, $|y| < |x|$ and $|x - y| \leqslant \frac{1}{\tau}$, we obtain

$$(5.6) \qquad I_2 \geqslant \frac{v_d \omega_d^2}{2^d} \int_{|x| \geqslant \frac{1}{\tau}} |\varphi|^*(x)^2 \,\mathrm{d}x.$$

From (5.5) and (5.6) we get that

$$\int_{B(0,\tau)} \left| \widehat{|\varphi|^*}(\xi) \right|^2 \mathrm{d}\xi \;\geqslant\; \frac{v_d \min\left(1, \omega_d^2\right)}{2^{d+1}}$$

$$\times \left(2|B(0,\tau)| \left(\int_{B(0,1/\tau)} |\varphi|^*(x) \,\mathrm{d}x \right)^2 + 2 \int_{|x| \geqslant \frac{1}{\tau}} |\varphi|^*(x)^2 \,\mathrm{d}x \right)$$

$$\geqslant \frac{v_d \min\left(1, |B(0,1)|^2\right)}{2^{d+1}} \int_{\Sigma} |\widehat{\varphi}(\xi)|^2 \,\mathrm{d}\xi$$

in view of (5.3), as claimed. $\qquad\qquad\square$

COROLLARY 5.2. — *Let $d \geqslant 1$ be an integer and let κ_d be the constant given by the previous theorem. Let ψ be a non-negative function on $\mathbb{R}^d$ that vanishes at infinity. Then for every $\varphi \in L^2(\mathbb{R}^d)$,*

$$\int_{\mathbb{R}^d} \psi(\xi) |\widehat{\varphi}(\xi)|^2 \,\mathrm{d}\xi \leqslant \kappa_d \int_{\mathbb{R}^d} |\psi|^*(\xi) \left| \widehat{|\varphi|^*}(\xi) \right|^2 \mathrm{d}\xi.$$

Proof. — For $\lambda > 0$, Theorem 5.1 implies that

$$\int_{\mathbb{R}^d} \chi_{\{\xi \,:\, \psi(\xi) > \lambda\}} |\widehat{\varphi}(\xi)|^2 \,\mathrm{d}\xi \leqslant \kappa_d \int_{\mathbb{R}^d} \chi_{\{\xi \,:\, \psi(\xi) > \lambda\}^*} \left| \widehat{|\varphi|^*}(\xi) \right|^2 \mathrm{d}\xi.$$

Integrating this inequality over $\lambda > 0$ and exchanging the order of integration gives the result. This comes from the layer-cake representation for the left hand side and from the definition of $|\psi|^*$ for the right side. $\qquad\square$

It seems natural to us to conjecture that if the characteristic function χ_Σ of Σ is replaced by the characteristic function of its complement χ_{Σ^c} then the inequality in (5.1) is reversed, or at least that the following holds:

CONJECTURE 5.3. — *Let $d \geqslant 1$ be an integer. Then there exists a constant κ_d such that, for every set $S, \Sigma \subset \mathbb{R}^d$ of finite positive measure and every $\varphi \in L^2(\mathbb{R}^d)$ with support in S,*

$$(5.7) \qquad \int_{\mathbb{R}^d \setminus \Sigma^*} \left| \widehat{|\varphi|^*}(\xi) \right|^2 \mathrm{d}\xi \leqslant \kappa_d e^{\kappa_d (|S| \,|\Sigma|)^{1/d}} \int_{\mathbb{R}^d \setminus \Sigma} \left| \widehat{\varphi}(\xi) \right|^2 \mathrm{d}\xi.$$

Let us note that (5.7) may be rewritten, with $\tilde{\kappa}_d = \kappa_d e^{\kappa_d (|S| \,|\Sigma|)^{1/d}}$,

$$\int_{\mathbb{R}^d} \left| \widehat{|\varphi|^*}(\xi) \right|^2 \mathrm{d}\xi - \int_{\Sigma^*} \left| \widehat{|\varphi|^*}(\xi) \right|^2 \mathrm{d}\xi \leqslant \tilde{\kappa}_d \left(\int_{\mathbb{R}^d} |\widehat{\varphi}(\xi)|^2 \,\mathrm{d}\xi - \int_{\Sigma} |\widehat{\varphi}(\xi)|^2 \,\mathrm{d}\xi \right).$$

As $\left\| \widehat{|\varphi|^*} \right\|_2 = \left\| \widehat{\varphi} \right\|_2$, this is equivalent to

$$\int_{\Sigma} \left| \widehat{\varphi}(\xi) \right|^2 \mathrm{d}\xi \leqslant \left(1 - \frac{1}{\tilde{\kappa}_d} \right) \int_{\mathbb{R}^d} \left| \widehat{\varphi}(\xi) \right|^2 \mathrm{d}\xi + \frac{1}{\tilde{\kappa}_d} \int_{\Sigma^*} \left| \widehat{|\varphi|^*}(\xi) \right|^2 \mathrm{d}\xi.$$

If this conjecture were true, then the following conjecture would follow:

CONJECTURE 5.4. — *Let $d \geqslant 1$ be an integer. Then there exists a constant C_d such that, if S and Σ are two sets of finite measure then, for every $\varphi \in L^2(\mathbb{R}^d)$,*

$$\| \varphi \|_2^2 \leqslant C_d e^{C_d (|S||\Sigma|)^{1/d}} \left(\int_{\mathbb{R}^d \setminus S} \left| \varphi(x) \right|^2 \mathrm{d}x + \int_{\mathbb{R}^d \setminus \Sigma} \left| \widehat{\varphi}(\xi) \right|^2 \mathrm{d}\xi \right).$$

This conjecture has been proved in dimension $d = 1$ by F. Nazarov [23] and for $d \geqslant 2$ and either S or Σ convex by the author in [13]. (The result was stated with a constant of the form $C_d e^{C_d \min(\omega(S)|\Sigma|^{1/d}, \omega(\Sigma)|S|^{1/d})}$ where $\omega(S)$ –resp. $\omega(\Sigma)$– is the mean width of S –resp. Σ– if this set is convex. But, it is a well known fact that $\omega(S) \leqslant C_d |S|^{1/d}$, see e.g. [24]).

Let us now show how Conjecture 5.3 implies Conjecture 5.4. First, as is well known (see *e.g.* [23] or [12]) and easy to prove, it is equivalent to show that

$$\int_{\Sigma} \left| \widehat{\varphi}(\xi) \right|^2 \mathrm{d}\xi \leqslant C_d e^{C_d (|S||\Sigma|)^{1/d}} \int_{\mathbb{R}^d \setminus \Sigma} \left| \widehat{\varphi}(\xi) \right|^2 \mathrm{d}\xi$$

for every $\varphi \in L^2$ with support in S. But, from Theorem 5.1,

$$\int_{\Sigma} \left| \widehat{\varphi}(\xi) \right|^2 \mathrm{d}\xi \leqslant \int_{\Sigma^*} \left| \widehat{|\varphi|^*}(\xi) \right|^2 \mathrm{d}\xi.$$

Now $|\varphi|^*$ is supported in S^*, so that the particular case of Conjecture 5.4 that has already been proved in [13] (and $|S^*| = |S|$, $|\Sigma^*| = |\Sigma|$) implies that

$$\int_{\Sigma} \left| \widehat{\varphi}(\xi) \right|^2 \mathrm{d}\xi \leqslant C_d e^{C_d (|S||\Sigma|)^{1/d}} \int_{\mathbb{R}^d \setminus \Sigma^*} \left| \widehat{|\varphi|^*}(\xi) \right|^2 \mathrm{d}\xi.$$

Finally, once Conjecture 1 is established, this would imply that

$$\int_{\Sigma} \left| \widehat{\varphi}(\xi) \right|^2 \mathrm{d}\xi \leqslant \kappa_d C_d e^{(C_d + \kappa_d)(|S||\Sigma|)^{1/d}} \int_{\mathbb{R}^d \setminus \Sigma} \left| \widehat{\varphi}(\xi) \right|^2 \mathrm{d}\xi.$$

Our original statement of Conjecture 5.3 was without the extra assumption of φ being of support of finite measure and the extra $e^{\kappa_d (|S||\Sigma|)^{1/d}}$ term. This fact is obviously false. Indeed, take $\Sigma = B(0,1) = \Sigma^*$ and assume that $\widehat{\varphi}$ is supported in Σ. Then

$$\int_{\mathbb{R}^d \setminus \Sigma^*} \left| \widehat{|\varphi|^*}(\xi) \right|^2 \mathrm{d}\xi \leqslant \tilde{\kappa}_d \int_{\mathbb{R}^d \setminus \Sigma} \left| \widehat{\varphi}(\xi) \right|^2 \mathrm{d}\xi$$

would imply that $\widehat{|\varphi|^*}$ is also supported in Σ. But this implies that $|\varphi|^*$ is an analytic functions. This is obviously not always the case, as the example $\varphi = \widehat{\chi_{B(0,1)}}$ shows. We thank Aline Bonami for pointing out this fact.

The extra assumption on the support of φ and the fact that the constant $\tilde{\kappa}_d$ with $|S||\Sigma|$ are hoped to be sufficient to take care of this problem.

6. An application to the Free Shrödinger Equation

Let us recall that the solution of the Free Shrödinger Equation

$$(6.1) \qquad \begin{cases} i\partial_t v + \dfrac{1}{4\pi}\Delta_x^2 v = 0 \\ v(x,0) = v_0(x) \end{cases}$$

with initial data $v_0 \in L^2(\mathbb{R}^d)$ has solution

$$
\begin{aligned}
v(x,t) &= \int_{\mathbb{R}^d} e^{-i\pi|\xi|^2 t + 2i\pi\langle x,\xi\rangle}\widehat{v_0}(\xi)\mathrm{d}\xi \\
(6.2) \qquad &= \mathcal{F}^{-1}[e^{-i\pi|\cdot|^2 t}\widehat{v_0}](x) \\
&= \frac{e^{i\pi|x|^2/t}}{(it)^{d/2}}\int_{\mathbb{R}^d} e^{i\pi|y|^2/t}v_0(y)e^{-2i\pi\langle y,x\rangle/t}\mathrm{d}y \\
(6.3) \qquad &= \frac{e^{i\pi|x|^2/t}}{(it)^{d/2}}\mathcal{F}[e^{i\pi|\cdot|^2/t}v_0](x/t).
\end{aligned}
$$

We may thus apply Corollary 4.7 and Theorem 5.1 to obtain a control of $v(x,t)$ over sets of finite measure. This is stated in terms of $\widehat{|\widehat{v_0}|}^*$ for small time and of $\widehat{|v_0|}^*$ for large time. More precisely, using respectively (6.2) and (6.3), we obtain the following:

THEOREM 6.1. — Let v be the solution of (6.1) and let Σ be a set of finite measure and let τ be defined by $|B(0,\tau)| = |\Sigma|$. Let κ_d be the constant in Theorem 1.3 and Υ_d be the constant in Corollary 4.7. Let $\beta \geqslant \frac{d}{2} - 1$.

(1) For every $t > 0$,

$$\int_\Sigma |v(x,t)|^2\,dx \leqslant \kappa_d \int_{B(0,\tau)}\left|\widehat{|\widehat{v_0}|}^*(\xi)\right|^2\,d\xi.$$

Moreover, if $\widehat{v_0}$ has support of finite measure S and if $\Omega > 0$, then

$$\int_\Sigma |v(x,t)|\,dx \leqslant \Upsilon_d|\Sigma|^{1/2}\Omega^{-d/2}(1 + \Omega^d S)^{1/2}\int_{\mathbb{R}^d}(1 - |\xi|^2/\Omega^2)_+^\beta\widehat{|\widehat{v_0}|}^*(\xi)\,d\xi.$$

(2) *For every $t > 0$,*

$$\int_\Sigma |v(x,t)|^2\, dx \leqslant \kappa_d t^{-d} \int_{B(0,\tau)} \left|\widehat{|v_0|}^*(\xi/t)\right|^2 d\xi.$$

(6.4)
$$= \kappa_d \int_{B(0,\tau/t)} \left|\widehat{|v_0|}^*(\xi)\right|^2 d\xi.$$

Moreover, if v_0 has support of finite measure S and if $\Omega > 0$, then

$$\int_\Sigma |v(x,t)|\, dx \leqslant \Upsilon_d |\Sigma|^{1/2} \Omega^{-d/2} (1 + \Omega^d S)^{1/2} \int_{\mathbb{R}^d} (1 - |\xi|^2/\Omega^2)_+^\beta \widehat{|v_0|}^*(\xi)\, d\xi.$$

Here C is a constant that only depends on d and on β.

Sketch of proof. — The first part of each statement results immediately from Theorem 1.3. For the second part of (1), we have used $\psi = \chi_{\Sigma/\Omega}$ in Corollary 4.7 and a trivial change of variables. The second statement of (2) results from the following computation:

$$\int_\Sigma |v(x,t)|\, dx = t^{-d/2} \int_\Sigma |\mathcal{F}[e^{i\pi|\cdot|^2/t} v_0](x/t)|\, dx$$

$$= t^{d/2} \int_{\mathbb{R}^d} |\mathcal{F}[e^{i\pi|\cdot|^2/t} v_0](x)| \chi_{\Sigma/t\Omega}(x/\Omega)\, dx.$$

Now taking $\psi = \chi_{t\Sigma/\Omega}$ in Corollary 4.7 shows that $\int_\Sigma |v(x,t)|\, dx$ is

$$\leqslant \Upsilon_d t^{d/2} \|\chi_{\Sigma/t\Omega}\|_2 (1 + \Omega^d S)^{1/2} \int_{\mathbb{R}^d} \left(1 - \frac{|\xi|^2}{\Omega^2}\right)_+^\beta \widehat{|v_0|^2}(\xi)\, d\xi$$

$$= \Upsilon_d |\Sigma|^{1/2} \Omega^{-d/2} (1 + \Omega^d S)^{1/2} \int_{\mathbb{R}^d} \left(1 - \frac{|\xi|^2}{\Omega^2}\right)_+^\beta \widehat{|v_0|^2}(\xi)\, d\xi.$$

$\square$

Let us remark that, using Hölder, we have for $1 \leqslant q \leqslant +\infty$, $\dfrac{1}{q} + \dfrac{1}{q'} = 1$

$$\int_{B(0,\tau/t)} \left|\widehat{|v_0|}^*(\xi)\right|^2 d\xi = \int_{\mathbb{R}^d} \chi_{B(0,\tau/t)} \left|\widehat{|v_0|}^*(\xi)\right|^2 d\xi$$

$$\leqslant \left\|\chi_{B(0,\tau/t)}\right\|_q \left\|\left|\widehat{|v_0|}^*(\xi)\right|^2\right\|_{q'}$$

$$= |B(0,\tau/t)|^{1/q} \left\|\left|\widehat{|v_0|}^*(\xi)\right|\right\|_{2q'}^2$$

$$\leqslant \frac{|\Sigma|^{1/q}}{t^{d/q}} \left\|\widehat{|v_0|}^*\right\|_{\frac{2q}{q+1}}^2.$$

where, in the last line, we have used Hausdorff-Young's Inequality [11, page 104] with $p := \dfrac{2q}{q+1} \in [1,2]$ and $\dfrac{1}{p} + \dfrac{1}{2q'} = 1$. It follows from (6.4) that

$$\int_\Sigma |v(x,t)|^2 \, \mathrm{d}x \leqslant C \frac{|\Sigma|^{\frac{2}{p}-1}}{t^{d\left(\frac{2}{p}-1\right)}} \left\|v_0\right\|_p^2.$$

This estimate can also be obtained directly from the standard dispersive estimate (see *e.g.* [26])

$$\left(\int_{\mathbb{R}^d} |v(x,t)|^{p'} \, \mathrm{d}x\right)^{1/p'} \leqslant C t^{-\frac{d}{2}\left(\frac{2}{p}-1\right)} \left\|v_0\right\|_p$$

$\left(\dfrac{1}{p} + \dfrac{1}{p'} = 1, \ p \leqslant 2\right)$ and Hölder's inequality. Recall that this estimate is just the interpolation between the trivial $L^1 - L^\infty$-estimates and Parseval applied to (6.3).

The estimate (6.4) is slightly more precise and also shows that the case of radial initial data is somehow the worst case.

7. Concluding remarks

For the clarity of exposition, we have chosen to deal only with the d-dimensional radial decreasing rearrangement of functions. An alternative would have be to deal with the 1-dimensional (resp. k-dimensional) symmetric decreasing rearrangement defined as follows: for a set $E \subset \mathbb{R}^d$ of finite measure, we define $E^\star = [-a,a]$ where $a = |E|/2$ (resp. $E^\star = B(0,r) \subset \mathbb{R}^k$ with $|B(0,r)|_k = |E|_d$). One can then define the symmetric decreasing rearrangement of functions through the layer cake representation:

$$|\varphi|^\star(x) = \int_0^{+\infty} \chi_{\{y\in\mathbb{R}^d \, : \, |\varphi(y)|>\lambda\}^\star}(x) \, \mathrm{d}\lambda.$$

This rearrangement has similar properties to the one we considered so far. One may then adapt directly the proofs of Theorems 4.4 and 5.1 to obtain the following results:

THEOREM 7.1. — *Let $d \geqslant 1$ be an integer.*

(1) *Let $\alpha > -1/2$, $\beta \geqslant 0$. Let $S, \Omega > 0$ and let ζ_α be as in Theorem 4.4. Then there exists a constant $C = C(d,\alpha,\beta)$ such that, for every*

$\varphi \in L^1(\mathbb{R}^d)$ *with support of finite measure* S, *for every* $x, a \in \mathbb{R}^d$,

$$\left| \int_{\mathbb{R}^d} \widehat{\varphi}(\xi)(1 - |\xi - a|^2/\Omega^2)_+^\alpha e^{2i\pi x\xi} \, \mathrm{d}\xi \right|$$

$$\leqslant \quad C\zeta_\alpha(\Omega^d S) \int_{\mathbb{R}} (1 - |\xi|^2/\Omega^2)_+^\beta \widehat{|\varphi|^\star}(\xi) \, \mathrm{d}\xi.$$

(2) *There exists a constant* κ_d *such that, for every* $\Sigma \in \mathbb{R}^d$ *of finite measure and every* $\varphi \in L^2(\mathbb{R}^d)$,

$$\int_\Sigma \left| \widehat{\varphi}(\xi) \right|^2 \, \mathrm{d}\xi \leqslant \kappa_d \int_{-|\Sigma|/2}^{|\Sigma|/2} \left| \widehat{|\varphi|^\star}(\xi) \right|^2 \, \mathrm{d}\xi.$$

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