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CONTACT HOMOLOGY, CAPACITY AND NON-SQUEEZING IN $\mathbb{R}^{2n} \times S^1$ VIA GENERATING FUNCTIONS

by Sheila SANDON

Abstract. — Starting from the work of Bhupal we extend to the contact case the Viterbo capacity and Traynor's construction of symplectic homology. As an application we get a new proof of the Non-Squeezing Theorem of Eliashberg, Kim and Polterovich.

Résumé. — Inspirés par le travail de Bhupal, nous étendons à la géométrie de contact la notion de capacité de Viterbo ainsi que la construction, dûe à Traynor, d'homologie symplectique. Comme application, nous obtenons une démonstration alternative du Théorème de Non-Tassement d'Eliashberg, Kim et Polterovitch.

1. Introduction

Consider the domains $B^{2n}(R) = \{ \pi \sum_{i=1}^{n} x_i^2 + y_i^2 < R \}$ and $C^{2n}(R) = B^2(R) \times \mathbb{R}^{2n-2}$ in the standard symplectic euclidean space $\left(\mathbb{R}^{2n}, \omega_0 = dx \wedge dy\right)$. Gromov's Non-Squeezing Theorem [24] states that if $R_2 < R_1$ then there is no symplectic embedding of $B(R_1)$ into $C(R_2)$. The analogous statement for balls and cylinders in the standard contact euclidean space $\left(\mathbb{R}^{2n+1}, \xi_0 = \ker(dz - ydx)\right)$ is trivially false, because one can use the contact transformation $(x, y, z) \mapsto (\alpha x, \alpha y, \alpha^2 z)$, where α is some positive constant, to squeeze any domain into an arbitrarily small ball[(1)]. However an interesting non-squeezing phenomenon arises if we consider the contact manifold $\mathbb{R}^{2n} \times S^1$ instead of $\mathbb{R}^{2n+1}$ and the following stronger notion of contact squeezing, that was introduced by Eliashberg, Kim and Polterovich [16].

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(1) In fact, as Francisco Presas explained to me, it is even possible to find a contact embedding of the whole $\mathbb{R}^{2n+1}$ into an arbitrarily small ball. A proof of this can be found for example in Chekanov, van Koert and Schlenk [7].

Definition 1.1. — *Given open domains $\mathcal{U}_1$ and $\mathcal{U}_2$ in a contact manifold (V,ξ) we say that $\mathcal{U}_1$ can be squeezed into $\mathcal{U}_2$ if there exists a contact isotopy $\varphi_t : \overline{\mathcal{U}_1} \longrightarrow V$, $t \in [0,1]$, such that φ_0 is the identity and $\varphi_1(\overline{\mathcal{U}_1}) \subset \mathcal{U}_2$. We say that $\mathcal{U}_1$ can be squeezed into $\mathcal{U}_2$ inside a third domain $\mathcal{V}$ if $\varphi_t(\overline{\mathcal{U}_1}) \subset \mathcal{V}$ for all t.*

Note that if $\overline{\mathcal{U}_1}$ is compact then by the isotopy extension theorem (see for example [20]) any contact squeezing of $\mathcal{U}_1$ into $\mathcal{U}_2$ inside $\mathcal{V}$ can be extended to a global contactomorphism of V supported in $\mathcal{V}$.

Given a domain $\mathcal{U}$ in $\mathbb{R}^{2n}$ we will denote by $\widehat{\mathcal{U}}$ the domain $\mathcal{U}\times S^1$ in $\mathbb{R}^{2n}\times S^1$. In [16] it is proved that for any R_1, R_2 there exists a contact embedding of $\widehat{B(R_1)}$ into $\widehat{B(R_2)}$, which if $n>1$ is isotopic through smooth embeddings to the inclusion $\widehat{B(R_1)} \hookrightarrow \mathbb{R}^{2n}\times S^1$. However, this isotopy cannot be made contact if $R_2 < k \leqslant R_1$ for some integer k.

Theorem 1.2 (Non-Squeezing Theorem [16]). — *Assume $R_2 \leqslant k \leqslant R_1$ for some integer k. Then the closure of $\widehat{B(R_1)}$ cannot be mapped into $\widehat{B(R_2)}$ by a compactly supported contactomorphism of $\mathbb{R}^{2n}\times S^1$. In particular, $\widehat{B(R_1)}$ cannot be squeezed into $\widehat{B(R_2)}$.*

Eliashberg, Kim and Polterovich also proved that $\widehat{B(R_1)}$ can be squeezed into $\widehat{B(R_2)}$ if R_1 and R_2 are smaller than 1 and if $n>1$ (in the 3-dimensional case it is never possible to squeeze $\widehat{B(R_1)}$ into a smaller $\widehat{B(R_2)}$, as can be seen using the techniques of Eliashberg [14]). It remains an open question whether $\widehat{B(R_1)}$ can be squeezed into $\widehat{B(R_2)}$ for $n>1$ and $k-1 < R_2 \leqslant R_1 < k$ with $k>1$.

An interesting feature of contact squeezing is that it requires extra room. For example, if $R_2 \leqslant \frac{1}{l} \leqslant R_1$ for some integer l, then any contact squeezing of $\widehat{B(R_1)}$ into $\widehat{B(R_2)}$ must move $\widehat{B(R_1)}$ outside $\widehat{B(\frac{1}{l-1})}$ at a certain time. This is a special case of the following theorem.

Theorem 1.3 ([16]). — *Assume that $R_2 \leqslant \frac{k}{l} \leqslant R_1 < R_3 \leqslant \frac{k}{l-1}$ for some integers k and l. Then the closure of $\widehat{B(R_1)}$ cannot be mapped into $\widehat{B(R_2)}$ by a compactly supported contactomorphism ψ of $\mathbb{R}^{2n}\times S^1$ with $\psi\left(\widehat{B(R_3)}\right) = \widehat{B(R_3)}$. In particular, $\widehat{B(R_1)}$ cannot be squeezed into $\widehat{B(R_2)}$ inside $\widehat{B(\frac{k}{l-1})}$.*

Theorems 1.2 and 1.3 are proved in [16] using contact homology of fiberwise starshaped domains in $\mathbb{R}^{2n}\times S^1$. This is a special instance of the Symplectic Field Theory, and is related to a version of the filtered symplectic

homology of domains in $\mathbb{R}^{2n}$ as used by Biran, Polterovich and Salamon [3], Cieliebak, Ginzburg and Kerman [11] and Ginzburg and Gürel [21]. We will present here a proof of the same results using generating functions instead of holomorphic curves techniques.

Generating functions have been studied extensively by many authors in the 80's and 90's. They provide a powerful tool in symplectic and contact topology, with important applications also to many of the central problems of these subjects (see for instance Chaperon [4, 5], Laudenbach and Sikorav [29], Sikorav [32, 33], Givental [22, 23], Viterbo [41, 39], Traynor [37, 38], Théret [34, 36, 35], Chekanov [6], Eliashberg and Gromov [15], Bhupal [1, 2], Milinković [31], Chekanov and Pushkar [8], Ferrand and Pushkar [18], Jordan and Traynor [28], Colin, Ferrand and Pushkar [12], Chernov and Nemirovski [9, 10], Eiseman, Lima, Sabloff and Traynor [13], Fuchs and Rutherford [19]). In particular, Viterbo [41] applied Morse-theoretical methods to the generating function of a Lagrangian submanifold L of the cotangent bundle of a closed manifold B to define invariants $c(u, L) \in \mathbb{R}$ for any $u \in \mathrm{H}^*(B)$. Using this he could then define an invariant $c(\phi)$ for compactly supported Hamiltonian symplectomorphisms ϕ of $\mathbb{R}^{2n}$, which in turn led to the definition of a symplectic capacity for domains in $\mathbb{R}^{2n}$. Among the applications discussed by Viterbo there is in particular the definition of a partial order and a bi-invariant metric on the group of compactly supported Hamiltonian symplectomorphisms of $\mathbb{R}^{2n}$.

Extending the work of Viterbo, Traynor [37] defined homology groups for Hamiltonian symplectomorphisms and, via a limit process, domains of $\mathbb{R}^{2n}$. More precisely, for any domain $\mathcal{U}$ in $\mathbb{R}^{2n}$ and any interval $(a, b]$ of $\mathbb{R}$ she defined homology groups $G_*{}^{(a,b]}(\mathcal{U})$. She proved that these groups are symplectic invariants and calculated them in the case of open ellipsoids.

Some of the above results have been extended to contact topology. In particular, Bhupal [2] defined invariants $c\,(u, L)$ for a Legendrian submanifold L of the 1-jet bundle of a closed manifold B and $u \in \mathrm{H}^*(B)$. Proceeding as in [41] he then associated a number $c(\phi)$ to each compactly supported contactomorphisms ϕ of $\mathbb{R}^{2n+1}$ isotopic to the identity, and used this construction to define a partial order on the groups of all such contactomorphisms. In contrast with the symplectic case, the number $c(\phi)$ is not invariant by conjugation of ϕ with another contactomorphism ψ. For this reason it is not possible to mimic Viterbo's construction of a symplectic capacity to obtain a contact invariant for domains in $\mathbb{R}^{2n+1}$. However Bhupal could prove that $c(\psi\phi\psi^{-1}) = 0$ if and only if $c(\phi) = 0$, which was all he needed to define

the partial order. Our contribution to this problem is the observation that if we consider contactomorphisms of $\mathbb{R}^{2n} \times S^1$, regarded as contactomorphisms of $\mathbb{R}^{2n+1}$ that are 1-periodic in the z-coordinate, then the methods of Bhupal can be used to show that $c(\psi\phi\psi^{-1}) = k$ if and only if $c(\phi) = k$, where k is any positive integer. In particular this implies that the integer part of $c(\phi)$ is invariant by conjugation, and this fact can be used to define an integral contact invariant for domains in $\mathbb{R}^{2n} \times S^1$. In analogy with the symplectic case we call this invariant a contact capacity. Given a domain $\mathcal{U}$ in $\mathbb{R}^{2n}$, we prove that the contact capacity of $\widehat{\mathcal{U}}$ equals the integer part of the Viterbo capacity of $\mathcal{U}$. This then easily yields a proof of Theorem 1.2 (see 3.6).

Similar observations can be made about homology groups. Using the set-up of Bhupal, it is possible to extend the construction of Traynor to the contact case and get homology groups ${G_*}^{(a,b]}(\mathcal{V})$ for a domain $\mathcal{V}$ of $\mathbb{R}^{2n+1}$. These groups however are not contact invariant, but they become so in the 1-periodic case if we consider only integer values of a and b.

The crucial fact that explains the special role played by the integers in the contact 1-periodic case is the following. In the symplectic case there is a 1-1 correspondence between critical points of the generating function of a Hamiltonian symplectomorphism ϕ and the fixed points of ϕ. Moreover, critical values are given by the symplectic action of the corresponding fixed points. Since the symplectic action is invariant by conjugation it follows that the generating functions of ϕ and of $\psi\phi\psi^{-1}$ have the same critical values. This fundamental fact can be used to prove that the Viterbo capacity and Traynor's homology groups are symplectic invariants (see 2.6 and 2.7). The same argument does not apply to the contact case. Given a contactomorphism ϕ of $\mathbb{R}^{2n+1}$ we will see in 3.2 that critical points of the generating function of ϕ with critical value c correspond to points (x, y, z) of $\mathbb{R}^{2n+1}$ such that $\phi(x, y, z) = (x, y, z + c)$. Thus the generating functions of ϕ and of $\psi\phi\psi^{-1}$ do not have the same critical values in general. However, if one of the two functions has 0 as critical value then so does the other as well, because critical points with critical value 0 correspond to fixed points. Similarly, in the 1-periodic case the same holds if we replace 0 by any integer k. We will explain in 3.6 and 3.7 how this observation implies that our homology groups and integral capacity for domains of $\mathbb{R}^{2n} \times S^1$ are contact invariants.

We will now show how one can use our construction of contact homology to prove Theorems 1.2 and 1.3, referring to 3.7 for all technical details.

Assume we have R_1, R_2, R_3 with $R_2 \leqslant \frac{k}{l} < R_1 < R_3 \leqslant \frac{k}{l-1}$. We have to show that $\widehat{B(R_1)}$ cannot be mapped into $\widehat{B(R_2)}$ by a contactomorphism ψ of $\mathbb{R}^{2n} \times S^1$ such that $\psi\left(\widehat{B(R_3)}\right) = \widehat{B(R_3)}$. Suppose this can be done. Then we can consider the following commutative diagram:

$$\begin{array}{ccccc} {G_*}^{(k,\infty]}(\widehat{B(R_3)}) & \longrightarrow & {G_*}^{(k,\infty]}(\widehat{B(R_1)}) & & \\ \uparrow{\scriptstyle \psi_*} & & & \nwarrow{\scriptstyle \psi_*} & \\ {G_*}^{(k,\infty]}(\widehat{B(R_3)}) & \longrightarrow & {G_*}^{(k,\infty]}(\widehat{B(R_2)}) & \longrightarrow & {G_*}^{(k,\infty]}\left(\psi(\widehat{B(R_1)})\right) \end{array}$$

where the horizontal arrows denote the homomorphisms induced by inclusions (see Theorem 3.24) and the vertical ones are isomorphisms induced by ψ (see Theorem 3.23). Consider $\mathbb{Z}_2$-coefficients, and $* = 2nl$. Then by Theorems 2.20 and 3.25 we know that ${G_*}^{(k,\infty]}(\widehat{B(R_2)}) = 0$, ${G_*}^{(k,\infty]}(\widehat{B(R_1)}) = {G_*}^{(k,\infty]}(\widehat{B(R_3)}) = \mathbb{Z}_2$, and that the horizontal map on the top is an isomorphism. Thus the diagram gives a contradiction, yielding the proof of Theorem 1.3. Theorem 1.2 can be proved similarly, considering $* = 2n$ and a big enough R_3.

This article is organized as follows.

In Section 2 we describe the constructions by Viterbo and Traynor of a symplectic capacity and symplectic homology for domains in $\mathbb{R}^{2n}$. In 2.7 we define homology groups for compactly supported Hamiltonian symplectomorphisms of $\mathbb{R}^{2n}$ and use them to construct, via a limit process, symplectic homology of domains. The limit process is based on the Viterbo partial order on $\mathrm{Ham}^c(\mathbb{R}^{2n})$, which we discuss in 2.5. The Viterbo capacity is described in 2.6. The partial order and the capacity are defined using the invariants for Hamiltonian symplectomorphisms introduced by Viterbo. We discuss these invariants in 2.3 and 2.4. In 2.1 and 2.2 we give the needed preliminaries on generating functions. In this section we always follow Traynor [37] and Viterbo [41] except for the following points: we give a different proof of symplectic invariance of the homology groups (Proposition 2.16); monotonicity of the invariant $c(\phi)$ is proved directly in [Vit92, Proposition 4.6] while for us is an immediate consequence of Proposition 2.10.

In Section 3 we generalize the results of Section 2 to the contact case. In 3.6 and 3.7 respectively we construct a contact capacity and contact homology groups for domains in $\mathbb{R}^{2n} \times S^1$. The limit process to define contact homology of domains is based on the Bhupal partial order on the group of

contactomorphisms of $\mathbb{R}^{2n+1}$, which we discuss in 3.5. All the constructions in this section use the generalization of the Viterbo's invariants to contactomorphisms of $\mathbb{R}^{2n+1}$ and $\mathbb{R}^{2n} \times S^1$. We discuss these invariants in 3.3 and 3.4. In 3.1 and 3.2 we give respectively some preliminaries on generating functions in contact topology, and a more detailed discussion of generating functions for contactomorphisms of $\mathbb{R}^{2n+1}$.

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2. Symplectic Capacity and Homology for Domains in $\mathbb{R}^{2n}$

We refer to McDuff and Salamon [30] for preliminaries on symplectic topology. Here we only discuss some basic concepts that are needed for the rest of the article.

A symplectic manifold is an even dimensional manifold M endowed with a symplectic form, i.e a non-degenerate closed 2-form $\omega \in \Omega^2(M)$. A symplectic manifold (M, ω) is said to be exact if $\omega = -d\lambda$ for some 1-form λ which is then called a Liouville form. In this paper we will only deal with the following two (exact) symplectic manifolds: the standard symplectic euclidean space $\left(\mathbb{R}^{2n},\, \omega_0 = -d\left(ydx\right)\right)$ and the cotangent bundle T^*B of a manifold B, endowed with the canonical symplectic form $\omega_{\text{can}} = -d\left(pdq\right)$ where q is the coordinate on the base and p on the fiber. A diffeomorphism ϕ of a symplectic manifold (M, ω) is called a symplectomorphism if $\phi^*\omega = \omega$. Given a time-dependent function H_t on M, the flow ϕ_t of the time-dependent vector field X_t defined by the condition $i_{X_t}\omega = -dH_t$ consists of symplectomorphisms. The isotopy ϕ_t is called a Hamiltonian isotopy, with Hamiltonian function H_t. A Hamiltonian symplectomorphism of (M, ω) is a symplectomorphism that can be obtained as the time-1 map of a Hamiltonian isotopy. An immersion $i : L \to (M, \omega)$ is called isotropic if $i^*\omega = 0$ and Lagrangian if moreover the dimension of L is maximal, i.e. half of the dimension of M. If (M, ω) is exact with Liouville form λ, then a Lagrangian immersion $i : L \to (M, \omega)$ is called exact if $i^*\lambda = df$ for some function f.

Consider an exact symplectic manifold $(M, \omega = -d\lambda)$. The **action functional** $\mathcal{A}_H$ with respect to a time-dependent Hamiltonian H is defined by

$$\mathcal{A}_H(\gamma) := \int_{t_0}^{t_1} \Big(\lambda\big(\dot{\gamma}(t)\big) + H_t\left(\gamma(t)\right)\Big)\, dt$$

for a path $\gamma : [t_0, t_1] \to M$. A crucial fact is that γ is a critical point of $\mathcal{A}_H$ (with respect to variations with fixed endpoints) if and only if it is an integral curve of the Hamiltonian flow of H. Moreover we have the following lemma.

LEMMA 2.1 ([30], 9.19). — *Let ϕ_t, $t \in [0, 1]$, be a symplectic isotopy of an exact symplectic manifold $\big(M, \omega = -d\lambda\big)$, starting at the identity. Then ϕ_t is a Hamiltonian isotopy if and only if ${\phi_t}^*\lambda - \lambda = dF_t$ for a smooth family of functions $F_t : M \longrightarrow \mathbb{R}$. In this case the F_t are given by*

$$F_t = \int_0^t \big(\lambda(X_s) + H_s\big) \circ \phi_s\, ds$$

where X_t is the vector field generating ϕ_t, and $H_t : M \longrightarrow \mathbb{R}$ the corresponding Hamiltonian function. In other words, the value of F_t at a point q of M is the action functional with respect to H of the path $\phi_s(q)$, $s \in [0, t]$.

The action functional plays a central role in symplectic topology. It is also related in a crucial way to generating functions and thus to the invariants defined by Traynor and Viterbo that we are going to discuss in this section.

2.1. Generating functions for Lagrangian submanifolds of T^*B

Consider a smooth manifold B. Given a function $f : B \to \mathbb{R}$, the graph of its differential is a Lagrangian submanifold L_f of T^*B. Many geometric properties of L_f can be inferred by looking at f, the most immediate instance of this being the fact that critical points of f correspond to intersection points of L_f with the 0-section. The idea of generating functions is to generalize this construction in order to be able to associate a function to a more general class of Lagrangian submanifolds of T^*B. This can be achieved by considering functions defined on the total space of a fiber bundle over B, and by using the following construction which is due to Hörmander.

Definition 2.2 ([27]). — *A* variational family (E, S) *over a manifold* B *is a function* $S : E \longrightarrow \mathbb{R}$ *defined on the total space of a fiber bundle* $p : E \longrightarrow B$. (E, S) *is a* transverse variational family *if* $dS : E \longrightarrow T^*E$ *is transverse to* $N_E := \{ (e, \eta) \in T^*E \mid \eta \equiv 0 \text{ on } \ker \mathrm{d}p\,(e) \}$.

Consider the set Σ_S of *fiber critical points* of S, i.e. points e of E that are critical points of the restriction of S to the fiber through e. Note that $\Sigma_S = (dS)^{-1}(N_E)$, so if the variational family (E, S) is transverse then Σ_S is a submanifold of E of dimension equal to the dimension of B. To any e in Σ_S we can associate an element $v^*(e)$ of $T^*_{p(e)}B$ (the *Lagrange multiplier*) defined by $v^*(e)\,(X) := dS\,(\widehat{X})$ for $X \in T_{p(e)}B$, where $\widehat{X}$ is any vector in T_eE with $p_*(\widehat{X}) = X$.

Proposition 2.3. — *If* (E, S) *is a transverse variational family over* B, *then* $i_S : \Sigma_S \longrightarrow T^*B$, $e \mapsto \big(p(e), v^*(e)\big)$ *is a Lagrangian immersion.*

In this case, $S : E \longrightarrow \mathbb{R}$ is called a **generating function** for the Lagrangian submanifold $L_S := i_S\,(\Sigma_S)$ of T^*B. Note that (non-degenerate) critical points of S correspond under i_S to (transverse) intersection points of L_S with the 0-section. Note also that i_S is an exact Lagrangian immersion, with ${i_S}^*\,\lambda_{\mathrm{can}} = d\,(S_{|\Sigma_S})$. A proof of Proposition 2.3 can be found for instance in [30, 9.34].

A crucial example of this construction is given by the case in which E is the space of paths $\gamma : [0, 1] \to T^*B$ that begin at the 0-section. E can be

seen as a fiber bundle over B with projection $\gamma \mapsto \pi\big(\gamma(1)\big)$, where π is the projection of T^*B into B. Given a time-dependent Hamiltonian H on T^*B we can define a function $S : E \to \mathbb{R}$ by $S(\gamma) := \mathcal{A}_H(\gamma)$. Then Σ_S is the set of orbits of the Hamiltonian flow of H and the Lagrange multiplier of an element γ of Σ_S is the vertical component of $\gamma(1)$. Thus S generates the image of the 0-section under the time-1 map of the Hamiltonian flow of H. Note that S is not a generating function in the sense of the above definition because E has infinite dimensional fibers. However, it is possible to approximate E by a finite dimensional space and prove in this way that any Lagrangian submanifold of T^*B which is Hamiltonian isotopic to the 0-section has a generating function. This was done by Sikorav, using the broken Hamiltonian trajectories idea of Chaperon [4] and Laudenbach and Sikorav [29]. It was also proved that by this construction one can obtain in fact a generating function which is quadratic at infinity in the following sense.

DEFINITION 2.4. — *A generating function $S : E \longrightarrow \mathbb{R}$ for a Lagrangian submanifold of T^*B is* ***quadratic at infinity*** *if $p : E \longrightarrow B$ is a vector bundle and if there exists a non-degenerate quadratic form $Q_\infty : E \longrightarrow \mathbb{R}$ such that $dS - \partial_v Q_\infty : E \longrightarrow E^*$ is bounded, where ∂_v denotes the fiber derivative.*

This condition is important because it makes possible to apply to generating functions all arguments of Morse theory, even though the functions are not defined on a compact manifold.

THEOREM 2.5 ([32], [33]). — *If B is closed, then any Lagrangian submanifold of T^*B which is Hamiltonian isotopic to the 0-section has a generating function quadratic at infinity (g.f.q.i.). More generally, if $L \subset T^*B$ has a g.f.q.i. and ψ_t is a Hamiltonian isotopy of T^*B, then there exists a continuous family of g.f.q.i. $S_t : E \longrightarrow \mathbb{R}$ such that each S_t generates the corresponding $\psi_t(L)$.*

A second fundamental result is the uniqueness theorem of Viterbo and Théret, which says that all generating functions of a Lagrangian submanifold of T^*B which is Hamiltonian isotopic to the 0-section are related by some basic operations that do not affect the Morse theory of the function. As a consequence, all the invariants defined using generating functions do not depend on the choice of the specific generating function used to calculate them.

THEOREM 2.6 ([41], [36]). — *Suppose that B is closed, and let L be a Lagrangian submanifold of T^*B Hamiltonian isotopic to the 0-section.*

If $S : E \longrightarrow \mathbb{R}$ is a g.f.q.i. for L then any other g.f.q.i. S' for L can be obtained from S by the following operations:

- *addition of a constant: $S' = S + c : E \longrightarrow \mathbb{R}$, for some $c \in \mathbb{R}$;*
- *fiber-preserving diffeomorphism: $S' = S \circ \phi$, for some fiber-preserving diffeomorphism $\phi : E' \longrightarrow E$;*
- *stabilization (assuming $p : E \longrightarrow B$ is a vector bundle): $S' = S + Q : E' = E \oplus F \longrightarrow \mathbb{R}$, where $F \longrightarrow B$ is a vector bundle and $Q : F \longrightarrow \mathbb{R}$ is a non-degenerate quadratic form.*

A g.f.q.i. $S : E \longrightarrow \mathbb{R}$ is said to be *special* if $E = B \times \mathbb{R}^N$ and $S = S_0 + Q_\infty$, where S_0 is compactly supported and Q_∞ is the same quadratic form on each fiber.

PROPOSITION 2.7 ([36]). — *If B is closed, then any g.f.q.i. can be modified to a special one by applying the operations in Theorem 2.6.*

In the following we will always consider generating functions which are quadratic at infinity, and we will assume that they are special whenever this is needed.

2.2. Generating functions for Hamiltonian symplectomorphisms of $\mathbb{R}^{2n}$

We will now apply the results of 2.1 to compactly supported Hamiltonian symplectomorphisms of $\mathbb{R}^{2n}$. We do this by associating to such a symplectomorphism ϕ of $\mathbb{R}^{2n}$ a Lagrangian submanifold of T^*S^{2n}, as we will now explain. We first drop the condition of ϕ being compactly supported, and construct a Lagrangian submanifold Γ_ϕ of $T^*\mathbb{R}^{2n}$. Note first that the graph of ϕ can be seen as a Lagrangian embedding $\mathrm{gr}_\phi : \mathbb{R}^{2n} \longrightarrow \overline{\mathbb{R}^{2n}} \times \mathbb{R}^{2n}$, where $\overline{\mathbb{R}^{2n}}$ denotes the symplectic manifold $(\mathbb{R}^{2n}, -\omega_0)$. We identify $\overline{\mathbb{R}^{2n}} \times \mathbb{R}^{2n}$ with $T^*\mathbb{R}^{2n}$ by the symplectomorphism[(2)] $\tau : (x, y, X, Y) \mapsto (x, Y, Y - y, x - X)$ and define $\Gamma_\phi : \mathbb{R}^{2n} \longrightarrow T^*\mathbb{R}^{2n}$ by $\Gamma_\phi = \tau \circ \mathrm{gr}_\phi$. Since τ sends the diagonal of $\overline{\mathbb{R}^{2n}} \times \mathbb{R}^{2n}$ to the 0-section of $T^*\mathbb{R}^{2n}$, fixed points of ϕ correspond to intersection points of Γ_ϕ with the 0-section. Note that Γ_ϕ can also be written as $\Gamma_\phi = \Psi_\phi\,(0\text{-section})$ where Ψ_ϕ is the symplectomorphism of $T^*\mathbb{R}^{2n}$

(2) One can use in fact any other symplectomorphism that sends the diagonal to the 0-section. Traynor and Viterbo use respectively $\tau' : (x, y, X, Y) \mapsto (y, X, x - X, Y - y)$ and $\tau'' : (x, y, X, Y) \mapsto (\frac{x+X}{2}, \frac{y+Y}{2}, Y - y, x - X)$. We use τ because it is consistent with the formula in the contact case given by Bhupal (see 3.2).

defined by the diagram

$$\begin{array}{ccc} \overline{\mathbb{R}^{2n}} \times \mathbb{R}^{2n} & \xrightarrow{\mathrm{id}\times\phi} & \overline{\mathbb{R}^{2n}} \times \mathbb{R}^{2n} \\ \downarrow{\scriptstyle \tau} & & \downarrow{\scriptstyle \tau} \\ T^*\mathbb{R}^{2n} & \xrightarrow[\Psi_\phi]{} & T^*\mathbb{R}^{2n}. \end{array}$$

This shows in particular that Γ_ϕ is Hamiltonian isotopic to the 0-section. Observe that the above diagram behaves well with respect to composition: for all Hamiltonian symplectomorphisms ϕ, ϕ_1 and ϕ_2 we have namely that $\Psi_{\phi_1} \circ \Psi_{\phi_2} = \Psi_{\phi_1\phi_2}$ (in particular $\Gamma_{\phi_1 \circ \phi_2} = \Psi_{\phi_1}(\Gamma_{\phi_2})$) and ${\Psi_\phi}^{-1} = \Psi_{\phi^{-1}}$. Note moreover that Γ_ϕ is in fact an exact Lagrangian embedding, with

$${\Gamma_\phi}^*(\lambda_{\mathrm{can}}) = d(x\phi_2 - \phi_1\phi_2 + F)$$

where ϕ_1 and ϕ_2 denote the first and last n components of ϕ and F is a function satisfying $\phi^*(\lambda_0) - \lambda_0 = dF$ for $\lambda_0 = ydx$ (see Lemma 2.1).

Assume now that ϕ is compactly supported. Then Γ_ϕ coincides with the 0-section outside a compact set, so (by regarding S^{2n} as the 1-point compactification of $\mathbb{R}^{2n}$) it can be seen as Lagrangian submanifold T^*S^{2n}, Hamiltonian isotopic to the 0-section. By Theorems 2.5 and 2.6 it follows that Γ_ϕ has a g.f.q.i. $S : E \longrightarrow \mathbb{R}$, which is unique up to addition of a constant, fiber-preserving diffeomorphism and stabilization. We may and will always assume that S is special. Note that this assumption in particular normalizes S, removing the indeterminacy by a constant.

A crucial property of any generating function of a Hamiltonian symplectomorphism ϕ of $\mathbb{R}^{2n}$ is that its set of critical values coincides with the action spectrum of ϕ.

Definition 2.8. — *Let ϕ be a Hamiltonian symplectomorphism of $\mathbb{R}^{2n}$. The* ***symplectic action*** *of a fixed point q of ϕ is defined by*

$$\mathcal{A}_\phi(q) := \mathcal{A}_H\left(\phi_t(q)\right) = \int_0^1 \left(\lambda(X_t) + H_t\right)\left(\phi_t(q)\right) dt$$

where ϕ_t is a Hamiltonian isotopy joining ϕ to the identity, X_t the vector field generating it and H_t the corresponding Hamiltonian. The action spectrum *of ϕ is the set $\Lambda(\phi)$ of all values of $\mathcal{A}_\phi$ at fixed points of ϕ.*

Let $F : \mathbb{R}^{2n} \to \mathbb{R}$ be the compactly supported function satisfying $\phi^*\lambda_0 - \lambda_0 = dF$. Then by Lemma 2.1 we have $\mathcal{A}_\phi(q) = F(q)$, so in particular we see that the definition of $\mathcal{A}_\phi(q)$ does not depend on the choice of the Hamiltonian isotopy ϕ_t connecting ϕ to the identity.

LEMMA 2.9. — *Let ϕ be a compactly supported Hamiltonian symplectomorphism of $\mathbb{R}^{2n}$, with g.f.q.i. S. Then a point q of $\mathbb{R}^{2n}$ is a fixed point of ϕ if and only if $(q,0) \in \Gamma_\phi$, and thus if and only if ${i_S}^{-1}(q,0)$ is a critical point of S. In this case the corresponding critical value is the symplectic action $\mathcal{A}_\phi(q)$.*

Proof. — The first statement is immediate. Suppose now that we have a fixed point q of ϕ, and take a point p in $\mathbb{R}^{2n}$ outside the support of ϕ. We claim that

$$S\left({i_S}^{-1}(q,0)\right) = -\int_{\gamma \sqcup \phi(\gamma)^{-1}} \lambda_0 = \mathcal{A}_\phi(q)$$

where γ is any path in $\mathbb{R}^{2n}$ joining p to q. The second equality is proved in [30, 9.30]. As for the first, it can be seen as follows. Note that

$$-\int_{\gamma \sqcup \phi(\gamma)^{-1}} \lambda_0 = \int_{\gamma \times \phi(\gamma)} (-\lambda_0) \times \lambda_0$$

where $(-\lambda_0)\times\lambda_0$ is the Liouville form of $\overline{\mathbb{R}^{2n}}\times\mathbb{R}^{2n}$ and $\gamma\times\phi(\gamma)$ a path in the Lagrangian submanifold gr_ϕ of $\overline{\mathbb{R}^{2n}}\times\mathbb{R}^{2n}$. After identifying $\overline{\mathbb{R}^{2n}}\times\mathbb{R}^{2n}$ with $T^*\mathbb{R}^{2n}$ the result will follow from the following more general fact. Suppose that a Lagrangian submanifold L of T^*B is generated by $S : E \to \mathbb{R}$, i.e. L is the image of $i_S : \Sigma_S \to T^*B$. Since ${i_S}^*\lambda_{\mathrm{can}} = d\left(S_{|\Sigma_S}\right)$ we have that $\int_\gamma \lambda_{\mathrm{can}} = S\left({i_S}^{-1}(y)\right) - S\left({i_S}^{-1}(x)\right)$ for any path γ in L joining two points x and y. In our situation this gives

$$\begin{aligned} -\int_{\gamma \sqcup \phi(\gamma)^{-1}} \lambda_0 = \int_{\gamma\times\phi(\gamma)} (-\lambda_0)\times\lambda_0 &= \int_{\tau\left(\gamma\times\phi(\gamma)\right)} \lambda_{\mathrm{can}} \\ &= S\left({i_S}^{-1}(q,0)\right) - S\left({i_S}^{-1}(p,0)\right) = S\left({i_S}^{-1}(q,0)\right). \end{aligned}$$

The last equality holds because $S\left({i_S}^{-1}(p,0)\right) = 0$, since p is outside the support of ϕ. The second follows from $\tau^*\lambda_{\mathrm{can}} = (-\lambda_0)\times\lambda_0 + d(-XY+xY)$ and the fact that the function $-XY + xY$ vanishes at the endpoints (p,p) and (q,q) of the path $\gamma \times \phi(\gamma)$. □

In 2.4 and 2.7 respectively we are going to associate to any compactly supported Hamiltonian symplectomorphism ϕ of $\mathbb{R}^{2n}$ a real number $c(\phi)$ and, for real parameters a and b, homology groups ${G_k}^{(a,b]}(\phi)$. The number $c(\phi)$ is obtained by selecting a critical value of the generating function S of ϕ, while the groups ${G_*}^{(a,b]}(\phi)$ are defined to be the relative homology of sublevel sets of S at a and b. Both $c(\phi)$ and ${G_*}^{(a,b]}(\phi)$ are invariant by conjugation of ϕ with another Hamiltonian symplectomorphism of $\mathbb{R}^{2n}$. As we will see, this is an immediate consequence of Lemma 2.9 and the fact that the action spectrum of a Hamiltonian symplectomorphism is invariant

by conjugation. In 2.7 we will then apply a limit process in order to associate to any domain $\mathcal{U}$ of $\mathbb{R}^{2n}$ symplectic homology groups $G_*{}^{(a,b]}(\mathcal{U})$, by looking at the corresponding groups for Hamiltonian symplectomorphisms supported in $\mathcal{U}$. The limit process will be with respect to the following partial order $\leqslant$ on the group $\mathrm{Ham}^c(\mathbb{R}^{2n})$ of compactly supported Hamiltonian symplectomorphisms of $\mathbb{R}^{2n}$: we say that $\phi_1 \leqslant \phi_2$ if $\phi_2\phi_1^{-1}$ is the time-1 flow of some non-negative Hamiltonian (Hamiltonian functions of compactly supported symplectomorphism are normalized to be 0 outside the support). The fact that $\leqslant$ is indeed a partial order, in particular that if $\phi_1 \leqslant \phi_2$ and $\phi_2 \leqslant \phi_1$ then $\phi_1 = \phi_2$, will be proved in 2.5 by comparing $\leqslant$ with the partial order on $\mathrm{Ham}^c(\mathbb{R}^{2n})$ defined by Viterbo [41]. We will need the following proposition.

PROPOSITION 2.10. — *If $\phi_1 \leqslant \phi_2$, then there are generating functions S_1, $S_2 : E \longrightarrow \mathbb{R}$ for Γ_{ϕ_1}, Γ_{ϕ_2} respectively such that $S_1 \leqslant S_2$.*

This proposition is proved by Traynor [37, 5.3]. It will also follow as a special case of the corresponding result in contact geometry, that we will prove in 3.2.

2.3. Invariants for Lagrangian submanifolds

In the next four sections we will follow Viterbo [41] very closely. We will first define invariants for Lagrangian submanifolds of T^*B and discuss their properties. Then we will apply these invariants to compactly supported Hamiltonian symplectomorphisms of $\mathbb{R}^{2n}$, and use them to define a partial order $\leqslant_V$ on $\mathrm{Ham}^c(\mathbb{R}^{2n})$ and a capacity for domains in $\mathbb{R}^{2n}$.

Let B be a closed manifold and fix a point P on it. Denote by 0_B the 0-section of T^*B and by $\mathcal{L}_P$ the set of all Lagrangian submanifolds of T^*B which are Hamiltonian isotopic to 0_B and such that $P \in L \cap 0_B$. We normalize generating functions by requiring that the critical point corresponding to P has critical value 0. In this way the set of critical values of a generating function for a Lagrangian submanifold L depends only on L, and not on the choice of the generating function. Given L in $\mathcal{L}_P$, we will now explain how to use a cohomology class u of B to select a critical value of the generating function of L, in order to define an invariant $c(u, L)$.

Let L be an element of $\mathcal{L}_P$ with g.f.q.i. $S = S_0 + Q_\infty : E \longrightarrow \mathbb{R}$. We denote by E^a, for $a \in \mathbb{R} \cup \infty$, the sublevel set of S at a, i.e. $E^a = \{\, x \in E \mid S(x) \leqslant a \,\}$, and by $E^{-\infty}$ the set E^{-a} for a big (note that up to homotopy

equivalence $E^{-\infty}$ is the same for all L in $\mathcal{L}_P$). We will study the inclusion $i_a : (E^a, E^{-\infty}) \hookrightarrow (E, E^{-\infty})$, and the induced map on cohomology

$$i_a{}^* : H^*(B) \equiv H^*(E, E^{-\infty}) \longrightarrow H^*(E^a, E^{-\infty}).$$

Here $H^*(B)$ is identified with $H^*(E, E^{-\infty})$ via the Thom isomorphism

$$T : H^*(B) \xrightarrow{\cong} H^*\big(D(E^-), S(E^-)\big)$$

where E^- denotes the subbundle of E where Q_∞ is negative definite. Note that this isomorphism shifts the grading by the index of Q_∞. Note also that by excision $H^*\big(D(E^-), S(E^-)\big)$ is isomorphic to $H^*(E, E^{-\infty})$. For $|a|$ big enough we have $H^*(E^a, E^{-\infty}) \equiv 0$ if $a < 0$, and $i_a{}^* = \mathrm{id}$ if $a > 0$. So we can define

$$c(u, L) := \inf\{\, a \in \mathbb{R} \mid i_a{}^*(u) \neq 0 \,\}$$

for any $u \neq 0$ in $H^*(B)$. It follows from Theorem 2.6 that $c(u, L)$ is well-defined, i.e. it does not depend on the choice of the generating function used to calculate it. Note also that $c(u, L)$ is a critical value of S. The other relevant properties of $c(u, L)$ are contained the following lemma.

LEMMA 2.11. — *Let $\mu \in H^n(B)$ denote the orientation class of B. The map $H^*(B) \times \mathcal{L}_P \longrightarrow \mathbb{R}$, $(u, L) \longmapsto c(u, L)$ satisfies the following properties:*

(i) *If L_1, L_2 have generating functions S_1, $S_2 : E \longrightarrow \mathbb{R}$ with $|S_1 - S_2|_{C^0} \leqslant \varepsilon$, then for any u in $H^*(B)$ it holds that*

$$|c(u, L_1) - c(u, L_2)| \leqslant \varepsilon.$$

(ii) $$c\big(u \cup v, L_1 + L_2\big) \geqslant c(u, L_1) + c(v, L_2)$$

where $L_1 + L_2$ is defined by

$$L_1 + L_2 := \{\, (q, p) \in T^*B \mid p = p_1 + p_2,\ (q, p_1) \in L_1,\ (q, p_2) \in L_2 \,\}.$$

(iii) $$c(\mu, \bar{L}) = -c(1, L),$$

where $\bar{L}$ denotes the image of L under the map $T^*B \to T^*B$, $(q, p) \mapsto (q, -p)$.

(iv) *$c(\mu, L) = c(1, L)$ if and only if L is the 0-section. In this case we have*

$$c(\mu, L) = c(1, L) = 0.$$

(v) *For any Hamiltonian symplectomorphism Ψ of T^*B such that $\Psi(P) = P$, it holds*

$$c\big(u, \Psi(L)\big) = c\big(u, L - \Psi^{-1}(0_B)\big).$$

The first property is immediate. For $a \in \mathbb{R}$ and $j = 1, 2$ denote by $\left(E^a\right)_j$ the sublevel set of S_j at a, and by $(i_a{}^*)_j$ the map on cohomology induced by the inclusion of the pair $\left((E^a)_j\,,\, E^{-\infty}\right)$ into $\left(E\,,\, E^{-\infty}\right)$. If $|S_1 - S_2|_{C^0} \leqslant \varepsilon$, then we have inclusions of sublevel sets $\left(E^{a-\varepsilon}\right)_2 \subset \left(E^a\right)_1 \subset \left(E^{a+\varepsilon}\right)_2$. For any $a > c(u, L_1)$ we have $(i_a{}^*)_1(u) \neq 0$ which implies $(i_{a+\varepsilon}{}^*)_2(u) \neq 0$ and so $c(u, L_2) \leqslant a + \varepsilon$. Similarly, for any $a' < c(u, L_1)$ we have that $c(u, L_2) > a' - \varepsilon$. It follows that $c(u, L_1) - \varepsilon \leqslant c(u, L_2) \leqslant c(u, L_1) + \varepsilon$ as we wanted.

Properties (ii), (iii) and (iv) require more elaborated arguments of algebraic topology, and we refer to [41] for a proof[(3)]. We will present here only the proof of (v), because it is the only point that needs arguments of symplectic geometry. We will see in 3.5 that the analogue statement is not true in the contact case.

We first need to introduce some preliminaries from [41] and [40]. Given Lagrangian submanifolds L_1, L_2 of T^*B and points x, y in $L_1 \cap L_2$, define

$$l\,(x, y; L_1, L_2) := \int_{\gamma_1 \gamma_2^{-1}} \lambda_{\mathrm{can}}$$

where γ_1 and γ_2 are paths in L_1, L_2 respectively joining x and y. Note that $l(x, y; L_1, L_2) = S_1\big({i_{S_1}}^{-1}(y)\big) - S_1\big({i_{S_1}}^{-1}(x)\big) + S_2\big({i_{S_2}}^{-1}(y)\big) - S_2\big({i_{S_2}}^{-1}(x)\big)$, where S_1, S_2 are g.f.q.i. for L_1, L_2. In particular, for any L in $\mathcal{L}_P$ and u in $H^*(B)$ there exist points x, y in $L \cap 0_B$ such that $c(u, L) = l\,(x, y, ; L, 0_B)$: just take $x = P$ and y such that $S\left({i_S}^{-1}(y)\right) = c(u, L)$, where S is a g.f.q.i. for L. Note that if Ψ_t is an Hamiltonian isotopy of T^*B then $l\,(x, y; L_1, L_2) = l\left(\Psi_t(x), \Psi_t(y); \Psi_t(L_1), \Psi_t(L_2)\right)$, as can be easily checked using the fact that ${\Psi_t}^*\lambda_{\mathrm{can}} - \lambda_{\mathrm{can}}$ is exact. For $L \in \mathcal{L}_P$, define a subset $\Lambda(L)$ of $\mathbb{R}$ by $\Lambda(L) := \{\, l(x, y, ; L, 0_B) \,|\, x, y \in L \cap 0_B \,\}$.

Proof of Lemma 2.11(v). — Let Ψ be the time-1 flow of a Hamiltonian isotopy Ψ_t, and consider the map $t \longmapsto c\big(u, \Psi_t^{-1}\Psi(L) - \Psi_t^{-1}(0_B)\big)$. We know by Lemma 2.11(i) and Theorem 2.5 that this map is continuous, and we claim that it takes values in $\Lambda(L)$. Since $\Lambda(L)$ is a totally disconnected set, it will follow that $t \longmapsto c\big(u, \Psi_t^{-1}\Psi(L) - \Psi_t^{-1}(0_B)\big)$ is independent of t and thus in particular $c\big(u, \Psi(L)\big) = c\big(u, L - \Psi^{-1}(0_B)\big)$. To prove the claim, let x_t, y_t be points in the intersection of $\Psi_t^{-1}\Psi(L) - \Psi_t^{-1}(0_B)$ with 0_B such that

$$c(u, \Psi_t^{-1}\Psi(L) - \Psi_t^{-1}(0_B)) = l(x_t, y_t; \Psi_t^{-1}\Psi(L) - \Psi_t^{-1}(0_B), 0_B),$$

(3) See also Milinković [31] for an alternative definition and proof of the main properties of the invariants $c(u, L)$, based on Morse homology.

and let x'_t, y'_t be the corresponding points in $\Psi_t^{-1}\Psi(L) \cap \Psi_t^{-1}(0_B)$. Then we have

$$\begin{aligned} c\left(u, \Psi_t^{-1}\Psi(L) - \Psi_t^{-1}(0_B)\right) &= l(x_t, y_t; \Psi_t^{-1}\Psi(L) - \Psi_t^{-1}(0_B), 0_B) \\ &= l(x'_t, y'_t; \Psi_t^{-1}\Psi(L), \Psi_t^{-1}(0_B)) = l(\Psi_t x'_t, \Psi_t y'_t; \Psi(L), 0_B) \in \Lambda(L) \end{aligned}$$

as we wanted. □

2.4. Invariants for Hamiltonian symplectomorphisms of $\mathbb{R}^{2n}$

We will now apply the construction of 2.3 to the special case of a compactly supported Hamiltonian symplectomorphism ϕ of $\mathbb{R}^{2n}$. We define

$$c(\phi) := c(\mu, \Gamma_\phi)$$

where Γ_ϕ is the Lagrangian submanifold of T^*S^{2n} constructed in 2.2 and μ the orientation class of S^{2n}. Note that Γ_ϕ intersects the 0-section at the point at infinity of S^{2n}. This point plays the role of the point P in 2.3. We know that $c(\phi)$ is a critical value for any g.f.q.i. of Γ_ϕ, and hence that $c(\phi) = \mathcal{A}_\phi(q)$ for some fixed point q of ϕ. Note also that $c(\mathrm{id}) = 0$. Moreover we have the following properties.

PROPOSITION 2.12. — *For all ϕ, ψ in $Ham^c(\mathbb{R}^{2n})$ it holds:*

(i) $c(\phi) \geqslant 0$.
(ii) *If $c(\phi) = c(\phi^{-1}) = 0$ then ϕ is the identity.*
(iii) $c(\phi\psi) \leqslant c(\phi) + c(\psi)$.
(iv) $c(\phi) = c(\psi\phi\psi^{-1})$.
(v) *If $\phi_1 \leqslant \phi_2$ in the sense of 2.2, then $c(\phi_1) \leqslant c(\phi_2)$.*

Proof. —

(i) We will prove that $c(1, \overline{\Gamma_\phi}) \leqslant 0$ for any ϕ, and then use Lemma 2.11(iii) to conclude that

$$c(\phi) = c(\mu, \Gamma_\phi) = -c(1, \overline{\Gamma_\phi}) \geqslant 0.$$

Since $c(1, \Gamma_\phi) = \inf\{\, a \in \mathbb{R} \mid {i_a}^*(1) \neq 0 \,\}$, we need to prove that ${i_0}^*(1) \neq 0$. Let $S : E \to \mathbb{R}$ be a g.f.q.i. for $\overline{\Gamma_\phi}$, and recall that we regard S^{2n} as the 1-point compactification $\mathbb{R}^{2n} \cup \{P\}$. Consider the commutative diagram

$$\begin{array}{ccc} H^*(E^0, E^{-\infty}) & \longrightarrow & H^*({E_P}^0, {E_P}^{-\infty}) \\ \uparrow{\scriptstyle (i_0)^*} & & \uparrow{\scriptstyle \cong} \\ H^*(S^{2n}) & \longrightarrow & H^*(\{P\}) \end{array}$$

where the horizontal maps are induced by the inclusions $\{P\} \hookrightarrow S^{2n}$ and $E_P \hookrightarrow E$. Since ϕ is compactly supported, Γ_ϕ and hence $\overline{\Gamma_\phi}$ coincide with the 0-section on a neighborhood of P, so $S_{|E_P} : E_P \to \mathbb{R}$ is a quadratic form. It follows that the vertical map on the right hand side is an isomorphism. Since the horizontal map on the bottom sends 1 to 1, we see that ${i_0}^*(1) \neq 0$ as we wanted.

(ii) Note first that $c(u, \Gamma_\phi) = c(u, \overline{\Gamma_{\phi^{-1}}})$ for all u (apply Lemma 2.11(v) to $L = 0_B$ and $\Psi = \Psi_\phi$). Using this, the result then follows from Lemma 2.11(iii)-(iv).

(iii) Using (ii), (v) and (iii) of Proposition 2.11 we have

$$\begin{aligned} c(\psi) = c(\mu, \Gamma_\psi) &= c\big(\mu \cup 1, \Psi_{\phi^{-1}}(\Gamma_{\phi\psi})\big) = c\big(\mu \cup 1, \Gamma_{\phi\psi} - \Psi_\phi(0_B)\big) \\ &\geqslant c(\mu, \Gamma_{\phi\psi}) + c\big(1, \overline{\Psi_\phi(0_B)}\big) = c(\mu, \Gamma_{\phi\psi}) + c(1, \overline{\Gamma_\phi}) \\ &= c(\mu, \Gamma_{\phi\psi}) - c(1, \Gamma_\phi) = c(\phi\psi) - c(\phi) \end{aligned}$$

i.e. $c(\phi\psi) \leqslant c(\phi) + c(\psi)$ as we wanted.

(iv) Let ψ be the time-1 map of a Hamiltonian isotopy ψ_t, and consider the map $t \mapsto c(\psi_t \phi {\psi_t}^{-1})$. We know that this map is continuous (by Lemma 2.11(i) and Theorem 2.5) and that it takes values in the totally disconnected set $\Lambda(\phi)$, since $\Lambda(\psi_t \phi {\psi_t}^{-1}) = \Lambda(\phi)$ (see for instance [26, 5.2]). It follows that it is independent of t, so in particular $c(\phi) = c(\psi \phi \psi^{-1})$.

(v) We know by Proposition 2.10 that there are generating functions S_{ϕ_1}, S_{ϕ_2} for Γ_{ϕ_1}, Γ_{ϕ_2} respectively such that $S_{\phi_1} \leqslant S_{\phi_2}$. So for any a we have inclusion of sublevel sets $(E^a)_{S_{\phi_2}} \subset (E^a)_{S_{\phi_1}}$ and this easily implies that $c(u, \Gamma_{\phi_1}) \leqslant c(u, \Gamma_{\phi_2})$ for any u. In particular, $c(\phi_1) \leqslant c(\phi_2)$ as we wanted.

□

2.5. The Viterbo partial order

The Viterbo partial order $\leqslant_V$ on $\mathrm{Ham}^c(\mathbb{R}^{2n})$ is defined as follows. Given ϕ_1, ϕ_2 in $\mathrm{Ham}^c(\mathbb{R}^{2n})$ we set

$$\phi_1 \leqslant_V \phi_2 \quad \text{if} \quad c(\phi_1 {\phi_2}^{-1}) = 0.$$

Using the properties in Proposition 2.12 it is immediate to see that $\leqslant_V$ is indeed a partial order, that it is bi-invariant (i.e. if $\phi_1 \leqslant_V \phi_2$ and $\psi_1 \leqslant_V \psi_2$ then $\phi_1\psi_1 \leqslant_V \phi_2\psi_2$), and that if $\phi_1 \leqslant \phi_2$ in the sense of 2.2 then $\phi_1 \leqslant_V \phi_2$. In particular this implies that $\leqslant$ is also a partial order.

2.6. The Viterbo capacity

Given an open and bounded domain $\mathcal{U}$ of $\mathbb{R}^{2n}$, its **Viterbo capacity** is defined by $c(\mathcal{U}) := \sup\{\, c(\phi) \mid \phi \in \mathrm{Ham}\,(\mathcal{U})\,\}$ where $\mathrm{Ham}\,(\mathcal{U})$ denotes the set of time-1 maps of Hamiltonian functions supported in $\mathcal{U}$. By the following lemma, $c(\mathcal{U})$ is a finite real number.

LEMMA 2.13. — *If $\phi \in Ham\,(\mathcal{U})$ and ψ is such that $\psi(\mathcal{U}) \cap \mathcal{U} = \emptyset$, then $c(\phi) \leqslant \gamma(\psi)$ where $\gamma(\psi) := c(\psi) + c(\psi^{-1})$.*

Proof. — We first show that under the hypotheses of the lemma we have $c(\psi\phi) = c(\psi)$. Let x_t be a fixed point for $\psi\phi_t$ such that $c(\psi\phi_t) = \mathcal{A}_{\psi\phi_t}(x_t)$. Since $\psi(\mathcal{U}) \cap \mathcal{U} = \emptyset$, we see that $x_t \notin \mathcal{U}$. It follows that x_t is also a fixed point for all ϕ_t, hence for ψ. Moreover $\mathcal{A}_{\psi\phi_t}(x_t) = \mathcal{A}_{\psi}(x_t)$. Thus the continuous map $t \mapsto c(\psi\phi_t)$ takes values in $\Lambda(\psi)$ and hence is independent of t. In particular we get that $c(\psi\phi) = c(\psi)$ as we claimed. Using this and Proposition 2.12(iii) it then follows that

$$c(\phi) \leqslant c(\psi\phi) + c(\psi^{-1}) = c(\psi) + c(\psi^{-1}) = \gamma(\psi).$$

□

We can extend the definition to arbitrary domains of $\mathbb{R}^{2n}$ by setting

$$c(\mathcal{V}) := \sup\{\, c(\mathcal{U}) \mid \mathcal{U} \subset \mathcal{V},\ \mathcal{U} \text{ bounded}\,\}$$

if $\mathcal{V}$ is open, and

$$c(A) := \inf\{\, c(\mathcal{V}) \mid \mathcal{V} \text{ open},\ A \subset \mathcal{V}\,\}$$

for an arbitrary domain A.

THEOREM 2.14. — *c is a (relative) capacity in $\mathbb{R}^{2n}$, i.e. it satisfies the following properties:*

(i) *(Symplectic Invariance) For any Hamiltonian symplectomorphism ψ of $\mathbb{R}^{2n}$ we have*
$$c(\psi(\mathcal{U})) = c(\mathcal{U}).$$
(ii) *(Monotonicity) If $\mathcal{U}_1 \subset \mathcal{U}_2$, then $c(\mathcal{U}_1) \leqslant c(\mathcal{U}_2)$.*
(iii) *(Conformality) $c(\alpha\mathcal{U}) = \alpha^2 c(\mathcal{U})$ for any positive constant α.*
(iv) *(Non-triviality) $c\,(B^{2n}(1)) > 0$ and $c\,(C^{2n}(1)) < \infty$.*

Proof. — If $\phi \in \mathrm{Ham}\,(\mathcal{U})$ then $\psi\phi\psi^{-1} \in \mathrm{Ham}\,\big(\psi(\mathcal{U})\big)$, thus symplectic invariance follows from Proposition 2.12(iv). Monotonicity is immediate from the definition, and non-triviality will be discussed in the example below. As for conformality, it can be seen as follows. Consider first a conformal symplectomorphism ψ of $\mathbb{R}^{2n}$, i.e. $\psi^*\omega = \alpha\,\omega$ for some constant α.

Then $\Lambda(\psi\phi\psi^{-1}) = \alpha\,\Lambda(\phi)$ (see Hofer and Zehnder [26, 5.2]). Suppose that ψ is isotopic to the identity through conformal symplectomorphisms, i.e. $\psi = \psi_t|_{t=1}$ with $\psi_t^*\omega = \alpha(t)\,\omega$ for some function $\alpha(t)$ with $\alpha(0) = 1$ and $\alpha(1) = \alpha$. The continuous map $t \mapsto \frac{1}{\alpha(t)}\, c(\psi_t\phi\psi_t^{-1})$ takes values in the totally disconnected set $\Lambda(\phi)$, thus it is independent of t and so in particular $c(\psi\phi\psi^{-1}) = \alpha\, c(\phi)$. Applying this to the conformal symplectomorphism ψ: $(x, y) \mapsto (\alpha x, \alpha y)$ we get $c(\psi\phi\psi^{-1}) = \alpha^2\, c(\phi)$. Since $\psi\phi\psi^{-1} \in \mathrm{Ham}(\alpha\mathcal{U})$ if $\phi \in \mathrm{Ham}(\mathcal{U})$, it follows that $c(\alpha\mathcal{U}) = \alpha^2 c(\mathcal{U})$ as we wanted. □

Example 2.15. — Consider the ellipsoid

$$E(\alpha_1, \cdots, \alpha_n) := \{\, \frac{1}{\alpha_1}|z_1|^2 + \cdots + \frac{1}{\alpha_n}|z_n|^2 < 1 \,\} \subset \mathbb{R}^{2n} \equiv \mathbb{C}^n$$

where $0 < \alpha_1 \leqslant \alpha_2 \leqslant \cdots \leqslant \alpha_n < \infty$. Using Traynor's calculations of symplectic homology of $E(\alpha_1, \cdots, \alpha_n)$ it is easy to see that $c\big(E(\alpha_1, \cdots, \alpha_n)\big) = \pi\alpha_1$ (see also Hermann [25]), in particular $c\big(B(R)\big) = R$. Since any bounded domain contained in $C^{2n}(R)$ is also contained in some ellipsoid $E(\alpha_1, \cdots, \alpha_n)$ with $\alpha_1 = R$, it follows by monotonicity that $c(C^{2n}(R)) = R$.

2.7. Symplectic homology

We will now associate homology groups first to a compactly supported Hamiltonian symplectomorphism of $\mathbb{R}^{2n}$, by considering relative homology of sublevel sets of its generating function, and then, by a limit process, to domains of $\mathbb{R}^{2n}$. In this section we follow Traynor [37] although we give a different proof of symplectic invariance of the homology groups (Proposition 2.16).

Let ϕ be a compactly supported Hamiltonian symplectomorphism of $\mathbb{R}^{2n}$. Given real numbers a, b not belonging to the action spectrum of ϕ and such that $-\infty < a < b \leqslant \infty$, we define the **$k$-th symplectic homology group** of ϕ with respect to the values a, b by

$$G_k^{\,(a,b]}(\phi) := H_{k+\iota}(E^b, E^a)$$

where E^c, for $c \in \mathbb{R} \cup \infty$, denotes the sublevel set $\{\, x \in E \,|\, S(x) \leqslant c \,\}$ of a generating function $S : E \to \mathbb{R}$ for ϕ and ι is the index of the quadratic at infinity part of S. It follows from Theorem 2.6 that the $G_k^{\,(a,b]}(\phi)$ are well-defined, i.e. do not depend on the choice of the generating function (see also [37, 3.6]). Moreover, we will prove now that they are invariant by conjugation with a Hamiltonian symplectomorphism.

PROPOSITION 2.16. — *For any ϕ and ψ in $Ham^c(\mathbb{R}^{2n})$ we have an induced isomorphism*

$$\psi_* : {G_*}^{(a,b]}(\psi\phi\psi^{-1}) \longrightarrow {G_*}^{(a,b]}(\phi).$$

To prove this we will need the following lemma.

LEMMA 2.17. — *Let f_t, $t \in [0,1]$, be a continuous 1-parameter family of functions defined on a compact manifold M. Suppose that $a \in \mathbb{R}$ is a regular value of all f_t. Then there exists an isotopy θ_t of M such that $\theta_t({M^a}_0) = {M^a}_t$, where ${M^a}_t := \{\, x \in M \mid f_t(x) \leqslant a \,\}$.*

Proof. — Since a is a regular value of f_t for all $t \in [0,1]$, there exists an $\varepsilon > 0$ such that there are no critical values of any f_t in the interval $(a-\varepsilon, a+\varepsilon)$. Take a $\delta > 0$ such that if $|t-s| < \delta$ then $|f_t(x) - f_s(x)| < \varepsilon$ for all $x \in M$, and consider a sequence $0 = t_0 < t_1 < \cdots < t_{k-1} < t_k = 1$ with $|t_i - t_{i-1}| < \delta$ for all $i = 1, \cdots, k$. For $t_{i-1} < t < t_i$ define a diffeomorphism ${\theta_t}^i : {f_{t_{i-1}}}^{-1}(a) \to {f_t}^{-1}(a)$ by sending a point x of ${f_{t_{i-1}}}^{-1}(a)$ to the point obtained by following the flow of the (normalized) gradient $\bigtriangledown f_t$ for a time $a - f_t(x)$. Note that by construction $\bigtriangledown f_t$ will never be 0 in this process. Note also that (after taking a smaller subdivision if needed) $\bigtriangledown f_t$ is transverse to ${f_{t_{i-1}}}^{-1}(a)$, so ${\theta_t}^i$ is indeed a diffeomorphism. We can now define a 1-parameter family of diffeomorphisms $\theta_t : {f_0}^{-1}(a) \to {f_t}^{-1}(a)$ by defining inductively $\theta_t = {\theta_t}^i \circ \theta_{t_{i-1}}$ for $t_{i-1} < t < t_i$. A global isotopy as in the statement is now obtained by applying the isotopy extension theorem. □

Proof of Proposition 2.16. — Let ψ_t be a Hamiltonian isotopy starting at the identity and ending at $\psi_1 = \psi$. We have $\Lambda\left(\psi_t\phi\psi_t^{-1}\right) = \Lambda(\phi)$ for all t thus if we consider a continuous family $S_t : \mathbb{R}^{2n} \times \mathbb{R}^N \longrightarrow \mathbb{R}$ of generating functions, each S_t generating the corresponding $\psi_t\phi\psi_t^{-1}$, then by Lemma 2.9 the set $\Lambda\left(\psi_t\phi\psi_t^{-1}\right)$ of critical values of S_t is independent of t. Since a and b are regular values for S_0 it follows that they are regular values for all S_t, and so we can conclude using an analogue of Lemma 2.17 for pairs of sublevel sets. Note that we can do it even though $\mathbb{R}^{2n} \times \mathbb{R}^N$ is not compact, because the functions S_t are (special) quadratic at infinity. □

Consider now a domain $\mathcal{U}$ of $\mathbb{R}^{2n}$. Given $a, b \in \mathbb{R}$ we denote by ${\mathrm{Ham}_{a,b}}^c(\mathcal{U})$ the set of compactly supported Hamiltonian symplectomorphisms of $\mathbb{R}^{2n}$ that are the time-1 map of a Hamiltonian function which is supported in $\mathcal{U}$ and whose action spectrum does not contain a and b. Note that ${\mathrm{Ham}_{a,b}}^c(\mathcal{U})$ is directed with respect to the partial order $\leqslant$, i.e. for any ϕ, ψ in ${\mathrm{Ham}_{a,b}}^c(\mathcal{U})$ there is a φ in ${\mathrm{Ham}_{a,b}}^c(\mathcal{U})$ such that $\phi \leqslant \varphi$ and $\psi \leqslant \varphi$. Recall that if $\phi_1 \leqslant \phi_2$ we have an induced homomorphism

${\lambda_1}^2 : {G_k}^{(a,b]}(\phi_2) \longrightarrow {G_k}^{(a,b]}(\phi_1)$. Note that given ϕ_1, ϕ_2, ϕ_3 in ${\mathrm{Ham}_{a,b}}^c(\mathcal{U})$ with $\phi_1 \leqslant \phi_2 \leqslant \phi_3$, it holds ${\lambda_1}^2 \circ {\lambda_2}^3 = {\lambda_1}^3$ and ${\lambda_i}^i = \mathrm{id}$. This means in particular that $\{{G_k}^{(a,b]}(\phi_i)\}_{\phi_i \in {\mathrm{Ham}_{a,b}}^c(\mathcal{U})}$ is an inversely directed family of groups, so we can define the **k-th symplectic homology group** ${G_k}^{(a,b]}(\mathcal{U})$ of $\mathcal{U}$ with respect to the values a, b to be the inverse limit of this family. Note that ${G_k}^{(a,b]}(\mathcal{U})$ can be calculated by any sequence $\phi_1 \leqslant \phi_2 \leqslant \phi_3 \leqslant \cdots$ such that the associated Hamiltonians get arbitrarily large.

THEOREM 2.18 (Symplectic Invariance). — *For any domain $\mathcal{U}$ in $\mathbb{R}^{2n}$ and any Hamiltonian symplectomorphism ψ we have an induced isomorphism $\psi_* : {G_*}^{(a,b]}\big(\psi(\mathcal{U})\big) \longrightarrow {G_*}^{(a,b]}(\mathcal{U})$.*

Proof. — Let $\phi_1 \leqslant \phi_2 \leqslant \phi_3 \leqslant \cdots$ be an unbounded ordered sequence supported in $\mathcal{U}$. Then $\psi\phi_1\psi^{-1} \leqslant \psi\phi_2\psi^{-1} \leqslant \psi\phi_3\psi^{-1} \leqslant \cdots$ is an unbounded ordered sequence supported in $\psi(\mathcal{U})$. By Proposition 2.16 we have isomorphisms ${\psi_i}^* : {G_*}^{(a,b]}(\psi\phi_i\psi^{-1}) \longrightarrow {G_*}^{(a,b]}(\phi_i)$, commuting with the ${\lambda_i}^j$ of the limit process. Thus we get an induced isomorphism between ${G_*}^{(a,b]}(\psi(\mathcal{U}))$ and ${G_*}^{(a,b]}(\mathcal{U})$. □

THEOREM 2.19 (Monotonicity). — *Every inclusion of domains induces a homomorphism of homology groups (reversing the order) with the following functorial properties:*

(i) *If $\mathcal{U}_1 \subset \mathcal{U}_2 \subset \mathcal{U}_3$ then the following diagram commutes*

$$\begin{array}{ccc} {G_*}^{(a,b]}(\mathcal{U}_3) & \longrightarrow & {G_*}^{(a,b]}(\mathcal{U}_2) \\ & \searrow & \downarrow \\ & & {G_*}^{(a,b]}(\mathcal{U}_1). \end{array}$$

(ii) *If $\mathcal{U}_1 \subset \mathcal{U}_2$, then for any Hamiltonian symplectomorphism ψ the following diagram commutes*

$$\begin{array}{ccc} {G_*}^{(a,b]}(\mathcal{U}_2) & \longrightarrow & {G_*}^{(a,b]}(\mathcal{U}_1) \\ {\scriptstyle \psi_*}\uparrow & & \uparrow{\scriptstyle \psi_*} \\ {G_*}^{(a,b]}\big(\psi(\mathcal{U}_2)\big) & \longrightarrow & {G_*}^{(a,b]}\big(\psi(\mathcal{U}_1)\big). \end{array}$$

Proof. — Suppose $\mathcal{U}_1 \subset \mathcal{U}_2$. Given an unbounded ordered sequence ${\phi_1}^2 \leqslant {\phi_2}^2 \leqslant {\phi_3}^2 \leqslant \cdots$ supported in $\mathcal{U}_2$, there exists an unbounded ordered sequence ${\phi_1}^1 \leqslant {\phi_2}^1 \leqslant {\phi_3}^1 \leqslant \cdots$ supported in $\mathcal{U}_1$ such that ${\phi_i}^1 \leqslant {\phi_i}^2$. The

homomorphisms ${G_*}^{(a,b]}(\phi_i^2) \longrightarrow {G_*}^{(a,b]}(\phi_i^1)$ induce a homomorphism of the inverse limits ${G_*}^{(a,b]}(\mathcal{U}_1) \to {G_*}^{(a,b]}(\mathcal{U}_2)$. The functorial properties are easy to check. □

Traynor [37] calculated the homology groups with $\mathbb{Z}_2$-coefficients of ellipsoids in $\mathbb{R}^{2n}$. We will need the following special case of her calculations.

THEOREM 2.20. — *Consider $B(R) \subset \mathbb{R}^{2n}$ and let a be a positive real number. Then for $* = 2nl$ we have*

$$ {G_*}^{(a,\infty]}\big(B(R)\big) = \begin{cases} \mathbb{Z}_2 & \text{if } \frac{a}{l} < R \leqslant \frac{a}{l-1} \\ 0 & \text{otherwise} \end{cases} $$

where l is any positive integer. In particular for $l = 1$ we have

$$ {G_{2n}}^{(a,\infty]}\big(B(R)\big) = \begin{cases} \mathbb{Z}_2 & \text{if } R > a \\ 0 & \text{otherwise.} \end{cases} $$

For all other values of $*$ the corresponding homology groups are zero. Moreover, given R_1, R_2 with $\frac{a}{l} < R_2 < R_1 \leqslant \frac{a}{l-1}$, the homomorphism ${G_*}^{(a,\infty]}\big(B(R_1)\big) \longrightarrow {G_*}^{(a,\infty]}\big(B(R_2)\big)$ induced by the inclusion $B(R_2) \subset B(R_1)$ is an isomorphism.

3. Contact Capacity and Homology for Domains in $\mathbb{R}^{2n} \times S^1$

We refer to Geiges [20] for an introduction to Contact Topology, and discuss here only some basic preliminaries.

A contact manifold is an odd dimensional manifold V^{2n+1} endowed with a hyperplanes field ξ which is maximally non-integrable, i.e. it is locally the kernel of a 1-form η such that $\eta \wedge (d\eta)^n$ never vanishes. We will always assume that the contact manifold is cooriented, i.e. that η is globally defined. Standard examples of contact manifolds can be obtained by considering the prequantization space of an exact symplectic manifold $\big(M, \omega = -d\lambda\big)$, i.e. the manifold $M \times \mathbb{R}$ endowed with the contact structure $\xi = \ker(dz - \lambda)$ where z is the coordinate on $\mathbb{R}$. Special instances of this construction are the standard contact euclidean space $\big(\mathbb{R}^{2n+1}, \xi_0 = \ker(dz - ydx)\big)$, which is the prequantization of $(\mathbb{R}^{2n}, \omega_0)$, and the 1-jet bundle J^1B of a manifold B, which is the prequantization of $(T^*B, \omega_{\text{can}})$.

A diffeomorphism ϕ of a contact manifold $\big(V, \xi = \ker(\eta)\big)$ is called a contactomorphism if its differential preserves ξ and its coorientation. It is called a strict contactomorphism if $\phi^*\eta = \eta$. A time-dependent vector field X_t

on V is called a contact vector field if its flow consists of contactomorphisms. Given a time-dependent function H_t on V there exists a unique contact vector field X_t such that $\eta(X_t) = H_t$ (see [20, Section 2.3]). The function H_t is then called the contact Hamiltonian of the flow ϕ_t of X_t, with respect to the contact form η. For instance notice that if φ is an exact symplectomorphism of the exact symplectic manifold $\big(M, \omega = -d\lambda\big)$, i.e. a symplectomorphism of M satisfying $\varphi^*\lambda - \lambda = dF$ for some function $F : M \to \mathbb{R}$, then φ can be lifted to a strict contactomorphism $\widetilde{\varphi}$ of the prequantization $M \times \mathbb{R}$ by defining $\widetilde{\varphi}(q, z) = (\varphi(q), z + F(q))$. If φ_t is a Hamiltonian isotopy of M with Hamiltonian function $H_t : M \to \mathbb{R}$, then the contact isotopy $\widetilde{\varphi}_t$ of $M \times \mathbb{R}$ is generated by the contact Hamiltonian $\widetilde{H}_t : M \times \mathbb{R} \to \mathbb{R}$ defined by $\widetilde{H}_t(q, z) = H_t(q)$.

An immersion $i : L \to \big(V, \xi = \ker(\eta)\big)$ is called isotropic if $i^*\eta = 0$ and Legendrian if moreover the dimension of L is maximal, i.e. half of $(\dim(M) - 1)$. For example, if V is the prequantization of an exact symplectic manifold $\big(M, \omega = -d\lambda\big)$ and $i : L \to M$ is an exact Lagrangian immersion with $i^*\lambda = df$, then the lift $i \times f$ is a Legendrian immersion of L into $V = M \times \mathbb{R}$. Note that in particular, up to addition of a constant in the $\mathbb{R}$-coordinate, this gives a 1-1 correspondence between Legendrian submanifolds of V and exact Lagrangian submanifolds of M.

In the contact case, generating functions are defined for Legendrian submanifolds of J^1B. A Lagrangian submanifold of T^*B that is Hamiltonian isotopic to the 0-section is in particular exact, and we will see that it has the same generating function as its lift to J^1B. This basic fact is what is behind the relation between the symplectic invariants defined in the previous section and the contact invariants that we are going to define now.

3.1. Generating functions for Legendrian submanifolds of J^1B

Consider a real function f defined on a smooth manifold B. The 1-jet of f is the Legendrian immersion $j^1f : B \to J^1B$ defined by $x \mapsto \big(x, df(x), f(x)\big)$. Note that j^1f is the lift of the differential of f, seen as an exact Lagrangian immersion $B \to T^*B$. More generally, given a transverse variational family (E, S) over B denote by $j_S : \Sigma_S \to J^1B$ the lift of the exact Lagrangian immersion $i_S : \Sigma_S \longrightarrow T^*B$ defined in 2.1, i.e. $j_S(e) = \big(p(e), v^*(e), S(e)\big)$. Then $S : E \longrightarrow \mathbb{R}$ is called a **generating function** for the Legendrian submanifold $\widetilde{L_S} := j_S(\Sigma_S)$ of J^1B. Note that critical points of S correspond under j_S with intersection points of $\widetilde{L_S}$ with the

0-wall of J^1B (which is defined to be the product of the 0-section of T^*B with $\mathbb{R}$), and that the corresponding critical value is the $\mathbb{R}$-coordinate of the intersection point with the 0-wall. Moreover, non-degenerate critical points correspond to transverse intersections (see [6, Proposition 2.1]). Note also that if two functions differ by an additive constant, then they generate different Legendrian submanifolds of J^1B (in fact different lifts of the same Lagrangian submanifold of T^*B).

The existence and uniqueness theorems for generating functions have been generalized to the contact case by Chaperon, Chekanov and Théret.

THEOREM 3.1 ([5], [6], [34]). — *If B is closed, then any Legendrian submanifold of J^1B contact isotopic to the 0-section has a g.f.q.i., which is unique up to fiber-preserving diffeomorphism and stabilization. If $L \subset J^1B$ has a g.f.q.i. and ψ_t is a contact isotopy of J^1B, then there exists a continuous family of g.f.q.i. $S_t : E \longrightarrow \mathbb{R}$ such that each S_t generates the corresponding $\psi_t(L)$.*

As in the symplectic case, any g.f.q.i. is equivalent to a special one. We will always assume generating functions to be special whenever this is needed.

3.2. Generating functions for contactomorphisms of $\mathbb{R}^{2n+1}$

In order to apply the results of the previous section to contactomorphisms of $\mathbb{R}^{2n+1}$ we need to associate to a contactomorphism of $\mathbb{R}^{2n+1}$ a Legendrian submanifold in some 1-jet bundle. Moreover, we should do this in a way which is compatible with the construction given in the symplectic case. By this we mean the following. Recall that any Hamiltonian symplectomorphism φ of $\mathbb{R}^{2n}$ can be lifted to a contactomorphism $\widetilde{\varphi}$ of $\mathbb{R}^{2n+1}$. To get a simple relation between the contact invariants that we will define in this section and the symplectic ones defined before, we need the generating function of $\widetilde{\varphi}$ to be essentially the same as the generating function of φ. We now explain how this can be done, following Bhupal [2]. Let ϕ be a contactomorphism of $\mathbb{R}^{2n+1}$, with $\phi^*(dz - ydx) = e^g(dz - ydx)$ for some function $g : \mathbb{R}^{2n+1} \longrightarrow \mathbb{R}$. Consider the graph of ϕ, i.e. the embedding

$$\mathrm{gr}_\phi : \mathbb{R}^{2n+1} \longrightarrow \mathbb{R}^{2(2n+1)+1}, \quad q \mapsto (q, \phi(q), g(q)).$$

If we endow $\mathbb{R}^{2(2n+1)+1}$ with the contact structure given by the kernel of $e^\theta\,(dz - ydx) - (dZ - YdX)$, then gr_ϕ becomes a Legendrian embedding. Define now $\Gamma_\phi : \mathbb{R}^{2n+1} \longrightarrow J^1\mathbb{R}^{2n+1}$ to be the composition $\Gamma_\phi = \tau \circ \mathrm{gr}_\phi$,

where $\tau : \mathbb{R}^{2(2n+1)+1} \longrightarrow J^1\mathbb{R}^{2n+1}$ is the contact embedding defined by

$$(x, y, z, X, Y, Z, \theta) \mapsto \big(x, Y, z, Y - e^\theta y, x - X, e^\theta - 1, xY - XY + Z - z\big).$$

Thus

$$\Gamma_\phi(x, y, z) = \big(x, \phi_2, z, \phi_2 - e^g y, x - \phi_1, e^g - 1, x\phi_2 - \phi_1\phi_2 + \phi_3 - z\big). \tag{3.1}$$

To motivate this formula, consider the case of the lift $\widetilde{\varphi}$ of a Hamiltonian symplectomorphism φ of $\mathbb{R}^{2n}$. Recall that $\widetilde{\varphi}$ is defined by $\widetilde{\varphi}(x, y, z) = \big(\varphi_1(x, y), \varphi_2(x, y), z + F(x, y)\big)$ where F is the compactly supported function satisfying $\varphi^*\lambda_0 - \lambda_0 = dF$. In 2.2 we associated to φ the Lagrangian embedding $\Gamma_\varphi : \mathbb{R}^{2n} \longrightarrow T^*\mathbb{R}^{2n}$, $(x, y) \mapsto \big(x, \varphi_2, \varphi_2 - y, x - \varphi_1\big)$. This embedding is exact with ${\Gamma_\varphi}^*\lambda_{\text{can}} = d(x\varphi_2 - \varphi_1\varphi_2 + F)$, thus it can be lifted to the Legendrian embedding $\widetilde{\Gamma_\varphi} : \mathbb{R}^{2n} \longrightarrow J^1\mathbb{R}^{2n}$, $(x, y) \mapsto \big(x, \varphi_2, \varphi_2 - y, x - \varphi_1, x\varphi_2 - \varphi_1\varphi_2 + F\big)$. Identify now $J^1\mathbb{R}^{2n+1}$ with $J^1\mathbb{R}^{2n} \times T^*\mathbb{R}$ via $(x, y, z, X, Y, Z, \theta) \mapsto \big((x, y, X, Y, \theta)\,, (z, Z)\big)$ and consider the Legendrian embedding $\widetilde{\Gamma_\varphi} \times 0$-section, $\mathbb{R}^{2n+1} \longrightarrow J^1\mathbb{R}^{2n+1} : (x, y, z) \mapsto \big(x, \varphi_2, z, \varphi_2 - y, x - \varphi_1, 0, x\varphi_2 - \varphi_1\varphi_2 + F\big)$. Note that, since φ is a strict contactomorphism, $\widetilde{\Gamma_\varphi} \times 0$-section coincides with the Legendrian embedding $\Gamma_{\widetilde{\varphi}} : \mathbb{R}^{2n+1} \longrightarrow J^1\mathbb{R}^{2n+1}$ given by (3.1). Besides shedding some light to the formula (3.1) the above discussion proves the following lemma.

LEMMA 3.2. — *If φ is a compactly supported Hamiltonian symplectomorphism of $\mathbb{R}^{2n}$ with generating function $S : \mathbb{R}^{2n} \times \mathbb{R}^N \longrightarrow \mathbb{R}$, then the function $\widetilde{S} : \mathbb{R}^{2n+1} \times \mathbb{R}^N \longrightarrow \mathbb{R}$ defined by $\widetilde{S}(x, y, z; \xi) = S(x, y; \xi)$ is a generating function for the lift $\widetilde{\varphi}$.*

Similarly to the symplectic case, for a contactomorphisms ϕ of $\mathbb{R}^{2n+1}$ we can write Γ_ϕ also as $\Gamma_\phi = \Psi_\phi(0\text{-section})$, with Ψ_ϕ denoting the local contactomorphism of $J^1\mathbb{R}^{2n+1}$ defined by the diagram

$$\begin{array}{ccc} \mathbb{R}^{2(2n+1)+1} & \xrightarrow{\overline{\phi}} & \mathbb{R}^{2(2n+1)+1} \\ \Big\downarrow{\scriptstyle \tau} & & \Big\downarrow{\scriptstyle \tau} \\ J^1\mathbb{R}^{2n+1} & \xrightarrow[\Psi_\phi]{} & J^1\mathbb{R}^{2n+1} \end{array} \tag{3.2}$$

where $\overline{\phi}$ is the contactomorphism $(p, P, \theta) \mapsto (p, \phi(P), g(P)+\theta)$. This shows in particular that if ϕ is contact isotopic to the identity then Γ_ϕ is contact isotopic to the 0-section. Suppose indeed that ϕ is the time-1 map of a contact isotopy ϕ_t. Then we get a local contact isotopy Ψ_{ϕ_t} of $J^1\mathbb{R}^{2n+1}$ connecting Ψ_ϕ to the identity. By the contact isotopy extension theorem (see [20, Section 2.6]) we can extend this local isotopy to a global one, so

we see that Γ_ϕ is contact isotopic to the 0-section. Notice that, as in the symplectic case, diagram (3.2) behaves well with respect to composition: for all contactomorphisms ϕ, ϕ_1 and ϕ_2 we have namely that $\Psi_{\phi_1} \circ \Psi_{\phi_2} = \Psi_{\phi_1\phi_2}$ (in particular $\Gamma_{\phi_1 \circ \phi_2} = \Psi_{\phi_1}(\Gamma_{\phi_2})$) and $\Psi_\phi^{-1} = \Psi_{\phi^{-1}}$.

If ϕ is compactly supported then the Legendrian embedding $\Gamma_\phi : \mathbb{R}^{2n+1} \longrightarrow J^1\mathbb{R}^{2n+1}$ coincides with the 0-section outside a compact set, so it can be seen as a Legendrian submanifold of J^1S^{2n+1}, which is contact isotopic to the 0-section if ϕ is contact isotopic to the identity. By Theorem 3.1, it follows that Γ_ϕ has a generating function, which is unique up to fiber-preserving diffeomorphism and stabilization. The same is true if ϕ is a contactomorphism of $\mathbb{R}^{2n+1}$ which is 1-periodic in the z-coordinate and compactly supported in the (x, y)-plane, because then Γ_ϕ can be seen as a Legendrian submanifold of $J^1(S^{2n} \times S^1)$. We will denote by $\mathrm{Cont}_0^c(\mathbb{R}^{2n+1})$ the group of compactly supported contactomorphisms of $\mathbb{R}^{2n+1}$ that are isotopic to the identity, and by $\mathrm{Cont}^c_{1\text{-per}}(\mathbb{R}^{2n+1})$ the group of contactomorphisms of $\mathbb{R}^{2n+1}$ that are 1-periodic in the z-coordinate, compactly supported in the (x, y)-plane and isotopic to the identity through contactomorphisms of this form. Note that $\mathrm{Cont}^c_{1\text{-per}}(\mathbb{R}^{2n+1})$ can be identified with the group $\mathrm{Cont}_0^c(\mathbb{R}^{2n} \times S^1)$ of compactly supported contactomorphisms of $\mathbb{R}^{2n} \times S^1$ isotopic to the identity.

Recall that in the symplectic case the set of critical values of a generating function coincides with the action spectrum of the generated Hamiltonian symplectomorphism. Before stating the contact analogue of this crucial result we need to introduce the following terminology. Given a contactomorphism ϕ of $\mathbb{R}^{2n+1}$ with $\phi^*(dz - ydx) = e^g(dz - ydx)$, we say that $q = (x, y, z)$ is a **translated point** for ϕ if $\phi_1(q) = x$, $\phi_2(q) = y$ and $g(q) = 0$. In analogy to the symplectic case we will call $\phi_3(q) - z$ the **contact action** of ϕ at the translated point q.

LEMMA 3.3. — *Let ϕ be a contactomorphism of $\mathbb{R}^{2n+1}$ with generating function S. Then a point $q = (x, y, z)$ of $\mathbb{R}^{2n+1}$ is a translated point of ϕ if and only if $\big(q, 0, \phi_3(q) - z\big) \in \Gamma_\phi$, and so if and only if $i_S^{-1}\big(q, 0, \phi_3(q) - z\big)$ is a critical point of S. In this case the corresponding critical value is the contact action $\phi_3(q) - z$.*

Proof. — If q is a translated point then $\big(q, 0, \phi_3(q) - z\big) = \Gamma_\phi(q) \in \Gamma_\phi$. Conversely, it is easy to see that if $\big(q, 0, \phi_3(q) - z\big) = \Gamma_\phi(q_0)$ for some $q_0 \in \mathbb{R}^{2n+1}$ then $q_0 = q$ and q is a translated point. Recall then from 3.1 that intersections of Γ_ϕ with the 0-wall correspond to critical points of the generating function S, with critical value given by the last coordinate. □

Consider for example the lift $\widetilde{\varphi}$ of a Hamiltonian symplectomorphism φ of $\mathbb{R}^{2n}$. Recall that $\widetilde{\varphi}$ is defined by $\widetilde{\varphi}(x,y,z) = \big(\varphi_1(x,y), \varphi_2(x,y), z+F(x,y)\big)$. A point (x,y,z) of $\mathbb{R}^{2n+1}$ is a translated point for $\widetilde{\varphi}$ if and only if (x,y) is a fixed point of φ. In this case, using Lemma 2.1, we see that the contact action is given by $F(x,y) = \mathcal{A}_\varphi(x,y)$. Note that this gives an alternative proof of the fact that the set of critical values of the generating function of φ coincides with the action spectrum of φ.

Similarly to the symplectic case we can define a relation $\leqslant$ on the groups $\mathrm{Cont}_0^{\,c}(\mathbb{R}^{2n+1})$ and $\mathrm{Cont}^{\,c}_{1\text{-per}}(\mathbb{R}^{2n+1})$ by setting $\phi_1 \leqslant \phi_2$ if $\phi_2\phi_1^{-1}$ is the time-1 flow of some non-negative contact Hamiltonian. We will see in 3.5 that this relation is in fact a partial order. In the rest of this section we will show that the analogue of Proposition 2.10 is still true in the contact case. We will only consider compactly supported contactomorphisms, but all arguments go through for elements of $\mathrm{Cont}^{\,c}_{1\text{-per}}(\mathbb{R}^{2n+1})$ as well.

PROPOSITION 3.4. — *Let ϕ_0, ϕ_1 be either in $Cont_0^{\,c}(\mathbb{R}^{2n+1})$ or in $Cont^{\,c}_{1\text{-per}}(\mathbb{R}^{2n+1})$. If $\phi_0 \leqslant \phi_1$, then there are generating functions S_0, $S_1 : E \longrightarrow \mathbb{R}$ for Γ_{ϕ_0}, Γ_{ϕ_1} respectively such that $S_0 \leqslant S_1$.*

Note that, by considering the lift of Hamiltonian symplectomorphisms of $\mathbb{R}^{2n}$, this result contains Proposition 2.10 as a special case. To prove Proposition 3.4 we will use the concept of *Greek generating functions* for contactomorphisms of $J^1\mathbb{R}^m$, which was introduced by Chaperon [5].

Let φ be a contactomorphism of $J^1\mathbb{R}^m$, and assume it is $\mathcal{C}^1$-close to the identity[(4)]. Then the **Greek generating function** of φ is a function $\Phi : \mathbb{R}^m \times (\mathbb{R}^m)^* \times \mathbb{R} \to \mathbb{R}$ defined as follows. For $(p,z) \in (\mathbb{R}^m)^* \times \mathbb{R}$, consider the function $A_{p,z} : \mathbb{R}^m \to \mathbb{R}$ given by $A_{p,z}(q) = z+pq$. Note that $j^1A_{p,z} : \mathbb{R}^m \to J^1\mathbb{R}^m$, for (p,z) varying in $(\mathbb{R}^m)^* \times \mathbb{R}$, form a foliation of $J^1\mathbb{R}^m$. Since φ is $\mathcal{C}^1$-close to the identity, $\varphi\,(j^1A_{p,z})$ is still a section of $J^1\mathbb{R}^m$ and thus it is the 1-jet of a function $\Phi_{p,z} : \mathbb{R}^m \to \mathbb{R}$. The Greek generating function Φ is then defined by $\Phi(Q,p,z) = \Phi_{p,z}(Q)$. The *Latin generating function* of φ is the function $F : \mathbb{R}^m \times (\mathbb{R}^m)^* \times \mathbb{R} \to \mathbb{R}$ defined by $F(Q,p,z) := \Phi(Q,p,z) - (z+pQ)$. Note that F is identically 0 if (and only if) φ is the identity. Moreover one can show that F is independent of z if and only if φ is the lift of an Hamiltonian symplectomorphism of $T^*\mathbb{R}^m$, and that in

(4) Chaperon showed in fact how to construct a Greek generating function $\Phi : J^1\mathbb{R}^m \times \mathbb{R}^N \to \mathbb{R}$ for any compactly supported contactomorphism of $J^1\mathbb{R}^m$ contact isotopic to the identity, in such a way that the corresponding *Latin generating function* is quadratic at infinity. However we will only need the construction of Greek generating functions for $\mathcal{C}^1$-small contactomorphisms.

this case it coincides with the function constructed by Traynor in [37, 4.4] (but we are not going to need this fact in the following).

For the proof of Proposition 3.4 we will need the following three lemmas.

LEMMA 3.5. — *Consider a Legendrian submanifold L of $J^1\mathbb{R}^m$ with generating function $S : \mathbb{R}^m \times \mathbb{R}^N \to \mathbb{R}$, and a compactly supported contact isotopy φ_t of $J^1\mathbb{R}^m$ which is $\mathcal{C}^1$-close to the identity and has Greek generating function $\Phi_t : \mathbb{R}^m \times (\mathbb{R}^m)^* \times \mathbb{R} \to \mathbb{R}$. Then the function $S_t : \mathbb{R}^m \times \left((\mathbb{R}^m)^* \times \mathbb{R}^m \times \mathbb{R}^N\right) \to \mathbb{R}$ defined by*

$$S_t\left(Q; p, q, \xi\right) := \Phi_t\left(Q, p, S(q; \xi) - pq\right)$$

is a generating function for $\varphi_t(L)$.

This lemma can be obtained as a special case of the composition formula (9) in [5] (see also Section III of [34]).

LEMMA 3.6 ([5], 2.2). — *Let φ_t be a contact isotopy of $J^1\mathbb{R}^m$ with contact Hamiltonian $H_t : J^1\mathbb{R}^m \to \mathbb{R}$. Assume that φ_t is $\mathcal{C}^1$-close to the identity and has Greek generating function $\Phi_t : \mathbb{R}^m \times (\mathbb{R}^m)^* \times \mathbb{R} \to \mathbb{R}$. Then given (q, p, z) in $\mathbb{R}^m \times (\mathbb{R}^m)^* \times \mathbb{R}$ it holds*

$$\left.\frac{d\Phi_t}{dt}\right|_{t=t_0}(Q_{t_0}, p, z) = H_{t_0}\left(Q_{t_0}, P_{t_0}, Y_{t_0}\right)$$

where $(Q_{t_0}, P_{t_0}, Y_{t_0}) = \varphi_t\left(q, p, z + pq\right)$.

The next lemma is a special case for $t = 0$ of Lemma 3.5, and can also be easily verified directly.

LEMMA 3.7. — *Consider a Legendrian submanifold L of $J^1\mathbb{R}^m$. If $S : \mathbb{R}^m \times \mathbb{R}^N \to \mathbb{R}$ is a generating function for L, then so is the function $S_0 : \mathbb{R}^m \times (\mathbb{R}^m)^* \times \mathbb{R} \to \mathbb{R}$ defined by $S_0\left(Q; p, q, \xi\right) := S(q; \xi) + p(Q - q)$.*

Proof of Proposition 3.4. — Let $\phi_1{\phi_0}^{-1}$ be the time-1 map of a contact isotopy ψ_t of $\mathbb{R}^{2n+1}$. We will first prove the result assuming that ψ_t is $\mathcal{C}^1$-close to the identity. Consider the contact isotopy Ψ_{ψ_t} of $J^1\mathbb{R}^{2n+1}$: we know that it is $\mathcal{C}^1$-close to the identity and has non-negative Hamiltonian, because so does ψ_t by assumption. Thus by Lemma 3.6 if $\Psi_t : J^1\mathbb{R}^{2n+1} \to \mathbb{R}$ is a Greek generating function for Ψ_{ψ_t} then $\dfrac{d}{dt}\Psi_t \geqslant 0$. Take now a generating function $S : \mathbb{R}^{2n+1} \times \mathbb{R}^N \to \mathbb{R}$ for $\Gamma_{\phi_0} \subset J^1\mathbb{R}^{2n+1}$. Then, by Lemma 3.5, $\Gamma_{\psi_t\phi_0} = \Psi_{\psi_t}(\Gamma_{\phi_0})$ has generating function $S_t\left(Q; p, q, \xi\right) := \Psi_t\left(Q, p, S(q; \xi) - pq\right)$. Thus $\dfrac{d}{dt}S_t \geqslant 0$, in particular $S_1 \geqslant S_0$. Note that S_1 is a generating function for Γ_{ϕ_1}, and S_0 is a generating function for Γ_{ϕ_0}

related to S as in Lemma 3.7. For the general case the result follows by repeating this process and applying Lemma 3.7 at every step. This can be done because it can be proved (see Lemma 1 in Section 2.4 of [5]) that there exists a $\delta > 0$ such that every $\psi_t \psi_s^{-1}$ with $|s-t| < \delta$ is $\mathcal{C}^1$-small enough to have a Greek generating function. □

3.3. Invariants for Legendrian submanifolds

Let B be a closed manifold, and denote by $\mathcal{L}$ the set of all Legendrian submanifolds of J^1B contact isotopic to the 0-section. As in the symplectic case, for any $L \in \mathcal{L}$ and $u \neq 0$ in $H^*(B)$ we can define a real number $c(u, L)$ by

$$c(u, L) := \inf\{\, a \in \mathbb{R} \mid {i_a}^*(u) \neq 0 \,\}$$

where i_a is the inclusion $(E^a, E^{-\infty}) \longrightarrow (E, E^{-\infty})$ of sublevel sets of any generating function for L.

LEMMA 3.8. — *Let $\mu \in H^n(B)$ denote the orientation class of B. The map $H^*(B) \times \mathcal{L} \longrightarrow \mathbb{R}$, $(u, L) \longmapsto c(u, L)$ satisfies the following properties:*

(i) *If L_1, L_2 have generating functions S_1, $S_2 : E \longrightarrow \mathbb{R}$ with $|S_1 - S_2|_{\mathcal{C}^0} \leqslant \varepsilon$, then for any u in $H^*(B)$ it holds that*

$$|c(u, L_1) - c(u, L_2)| \leqslant \varepsilon.$$

(ii) $$c\big(u \cup v, L_1 + L_2\big) \geqslant c(u, L_1) + c(v, L_2)$$

where $L_1 + L_2$ is defined by

$$L_1 + L_2 := \{\, (q, p, z) \in J^1B \mid p = p_1 + p_2,\ z = z_1 + z_2, \\ (q, p_1, z_1) \in L_1,\ (q, p_2, z_2) \in L_2 \,\}.$$

(iii) $$c(\mu, \overline{L}) = -c(1, L),$$

where $\overline{L}$ denotes the image of L under the map $J^1B \to J^1B$, $(q, p, z) \mapsto (q, -p, -z)$.

(iv) *Assume $L \cap 0_B \neq \emptyset$. Then $c(\mu, L) = c(1, L)$ if and only if L is the 0-section. In this case we have*

$$c(\mu, L) = c(1, L) = 0.$$

Proof. — If S is a generating function for $L \subset J^1B$ then S also generates $\pi(L)$, where π denotes the projection $J^1B = T^*B \times \mathbb{R} \to T^*B$. So $c(u, L) = c(u, \pi(L))$ and thus all the results follow from the symplectic case. □

Property (v) of Lemma 2.11 does not hold in the contact case. However Bhupal [2] showed that the following weaker statement is still true.

LEMMA 3.9. — *For any contactomorphism Ψ of J^1B contact isotopic to the identity, $u \neq 0$ in H^*B and $L \in \mathcal{L}$ it holds*

$$c\big(u, \Psi(L)\big) = 0 \quad \Leftrightarrow \quad c\big(u, L - \Psi^{-1}(0_B)\big) = 0.$$

Proof. — Let Ψ_t be a contact isotopy of J^1B with $\Psi = \Psi_t|_{t=1}$, and for every t consider the Legendrian submanifold $\Lambda_t = {\Psi_t}^{-1}\Psi(L) - \Psi_t^{-1}(0_B)$. We have $\Lambda_0 = \Psi(L)$ and $\Lambda_1 = L - \Psi^{-1}(0_B)$. Let $c_t = c(u, \Lambda_t)$. We will prove that if $c_{t_0} = 0$ for some $t \in [0,1]$ then $c_t = 0$ for all t. Let $S_t : E \to \mathbb{R}$ be a 1-parameter family of generating functions for Λ_t. Consider a path x_t in E such that each x_t is a critical point of S_t with critical value c_t, for t in some subinterval of $[0,1]$ containing t_0. Recall that x_t corresponds to an intersection of Λ_t with the 0-wall of J^1B. Since by hypothesis $c_{t_0} = 0$, x_{t_0} corresponds in fact to an intersection of Λ_{t_0} with the 0-section. We will first assume that this intersection is transverse, so that x_{t_0} is a non-degenerate critical point of S_{t_0}. The idea of the proof now is to construct a path y_t in E such that $y_{t_0} = x_{t_0}$ and each y_t is a non-degenerate critical point of S_t with critical value 0. It will then follow from Morse theory that the two paths x_t and y_t must coincide, so that $c_t = 0$ for all t. The path y_t can be constructed as follows. The key observation is that (non-degenerate) critical points of S_t with critical value 0 are in 1-1 correspondence with (transverse) intersection points of Λ_t with 0_B. Moreover (transverse) intersections of Λ_t with 0_B correspond to (transverse) intersections of ${\Psi_t}^{-1}\Psi(L)$ with ${\Psi_t}^{-1}(0_B)$ (by projecting to 0_B), and the last correspond to (transverse) intersections of $\Psi(L)$ with 0_B (by applying Ψ_t), i.e. of Λ_0 with 0_B. Using this we see that $y'_t := \pi\left(\Psi_t^{-1}\Psi_{t_0}\big(\widetilde{i_{S_0}(x_0)}\big)\right)$ is a transverse intersection of Λ_t with 0_B, where $\widetilde{i_{S_0}(x_0)}$ denotes the point in ${\Psi_{t_0}}^{-1}\Psi(L) \cap {\Psi_{t_0}}^{-1}(0_B)$ that projects to $i_{S_0}(x_0) \in {\Psi_{t_0}}^{-1}\Psi(L) - {\Psi_{t_0}}^{-1}(0_B)$. Thus $y_t := {i_{S_t}}^{-1}(y'_t)$ is the desired 1-parameter family of critical points of S_t. This finishes the proof under the assumption that x_{t_0} is a non-degenerate critical point of S_{t_0}. The general case follows from an approximation argument (see [2]). □

In [2] Bhupal realized that this result is enough to extend Viterbo's partial order to the contact case. We will review his construction in 3.5. However, Lemma 3.9 is too weak to give an interesting generalizations to the contact case of the Viterbo capacity. We will now give a stronger version of Lemma 3.9, which is only available in the 1-periodic case and will enable us to define in 3.6 a contact capacity for domains in $\mathbb{R}^{2n} \times S^1$.

We will denote by $\lceil \cdot \rceil$ the integer part of a real number, i.e. the smallest integer that is greater or equal to the given number.

LEMMA 3.10. — *Let Ψ be a contactomorphism of J^1B which is 1-periodic in the $\mathbb{R}$-coordinate of $J^1\mathbb{R}^{2n+1} = T^*\mathbb{R}^{2n+1} \times \mathbb{R}$ and isotopic to the identity through 1-periodic contactomorphisms. Then for every $u \neq 0$ in $H^*(B)$ and $L \in \mathcal{L}$ it holds*

$$\lceil c\big(u, \Psi(L)\big)\rceil = \lceil c\big(u, L - \Psi^{-1}(0_B)\big)\rceil.$$

Proof. — Let Ψ_t be a contact isotopy of J^1B with $\Psi = \Psi_t|_{t=1}$, and consider $c_t = c(u, \Lambda_t)$ where $\Lambda_t = {\Psi_t}^{-1}\Psi(L) - {\Psi_t}^{-1}(0_B)$. We will show that if k is an integer and $c_{t_0} = k$ for some t_0, then $c_t = k$ for all t. Let $S_t : E \to \mathbb{R}$ be a family of generating functions for Λ_t. Then c_t is a critical value of S_t. As in the proof of Proposition 3.9 the result follows if we prove that if x_{t_0} is a (non-degenerate) critical point of S_{t_0} with critical value k then there is a 1-parameter family of (non-degenerate) critical points y_t of S_t with $y_{t_0} = x_{t_0}$ and all with critical value k. The idea to prove this is that, since the Ψ_t are 1-periodic, the construction of the proof of Lemma 3.9 can be adapted to the case in which the critical value 0 is replaced by an integer k. More precisely, it is easy to check that if $\widetilde{x_{t_0}}$ is a critical point of S_{t_0} with critical value k then $y'_t := \pi\left(\Psi_t^{-1}\Psi_{t_0}\big(\widetilde{i_{S_0}(x_0)}\big)\right) + (0, 0, k)$ is in the intersection of Λ_t with $0_B \times \{k\}$. Thus $y_t := {i_{S_t}}^{-1}(y'_t)$ is the desired 1-parameter family of critical points of S_t. □

3.4. Invariants for contactomorphisms of $\mathbb{R}^{2n+1}$

Consider a contactomorphism ϕ either in ${\mathrm{Cont}_0}^c(\mathbb{R}^{2n+1})$ or in $\mathrm{Cont}^c_{1\text{-per}}(\mathbb{R}^{2n+1})$, and define

$$c(\phi) := c(\mu, \Gamma_\phi)$$

where Γ_ϕ is regarded as a Legendrian submanifold either of J^1S^{2n+1} or $J^1(S^{2n} \times S^1)$ and μ is the orientation class either of S^{2n+1} or $S^{2n} \times S^1$. Note that $c(\phi)$ is a critical value of any generating function for Γ_ϕ, so by Lemma 3.3 we have that $c(\phi) = \phi_3(q) - z$ for some translated point $q = (x, y, z)$ of ϕ. Note also that $c(\mathrm{id}) = 0$. Moreover c satisfies the following properties.

PROPOSITION 3.11. — *For all ϕ, ψ in ${\mathrm{Cont}_0}^c(\mathbb{R}^{2n+1})$ or $\mathrm{Cont}^c_{1\text{-per}}(\mathbb{R}^{2n+1})$ it holds:*

(i) $c(\phi) \geqslant 0$.
(ii) *If $c(\phi) = c(\phi^{-1}) = 0$ then ϕ is the identity.*
(iii) *If $c(\phi) = c(\psi) = 0$ then $c(\phi\psi) = 0$.*
(iv) *If $\phi_1 \leqslant \phi_2$ in the sense of 3.2 then $c(\phi_1) \leqslant c(\phi_2)$.*

Proof. —

(i) As in the symplectic case we have $c(1,\overline{\Gamma_\phi}) \leqslant 0$ for all ϕ. Thus by Lemma 3.8(iii) it holds that $c(\phi) = c(\mu,\Gamma_\phi) = -c(1,\overline{\Gamma_\phi}) \geqslant 0$.

(ii) Note first that, for all u, if $c(u,\Gamma_{\phi^{-1}}) = 0$ then also $c(u,\overline{\Gamma_\phi}) = 0$ (apply Lemma 3.9 to $L = 0_B$ and $\Psi = \Psi_{\phi^{-1}}$). Using this, the result then follows from Lemma 3.8(iii)-(iv).

(iii) We have $c(\mu,\Psi_{\phi^{-1}}(\Gamma_{\phi\psi})) = c(\mu,\Gamma_\psi) = 0$. Thus, by Lemma 3.9 and Lemma 3.8(ii),

$$0 = c(\mu,\Gamma_{\phi\psi} - \Psi_\phi(0_B)) = c(\mu,\Gamma_{\phi\psi} - \Gamma_\phi) \geqslant c(\mu,\Gamma_{\phi\psi}) + c(1,\overline{\Gamma_\phi}).$$

Since by Lemma 3.8(iii) it holds $c(1,\overline{\Gamma_\phi}) = -c(\mu,\Gamma_\phi) = 0$, we have that $c(\phi\psi) = c(\mu,\Gamma_{\phi\psi}) \leqslant 0$, and thus $c(\phi\psi) = 0$.

(iv) As in the symplectic case, using Proposition 3.4.

□

Using Lemma 3.10 we can prove a stronger version of Proposition 3.11(iii), that only holds in the 1-periodic case.

PROPOSITION 3.12. — *For all ϕ, ψ in $Cont^c_{1\text{-}per}(\mathbb{R}^{2n+1})$ it holds*

$$\lceil c(\phi\psi)\rceil \leqslant \lceil c(\phi)\rceil + \lceil c(\psi)\rceil.$$

Proof. — We have $c(\psi) = c(\mu,\Gamma_\psi) = c\big(\mu,\Psi_{\phi^{-1}}(\Gamma_{\phi\psi})\big)$ thus by Lemma 3.10 it holds $\lceil c(\psi)\rceil = \lceil c\big(\mu,\Gamma_{\phi\psi} - \Psi_\phi(0_B)\big)\rceil$. But, by Lemma 3.8(ii)-(iii)

$$\begin{aligned} c\big(\mu,\Gamma_{\phi\psi} - \Psi_\phi(0_B)\big) &= c\big(\mu\cup 1,\Gamma_{\phi\psi} - \Gamma_\phi\big) \geqslant c\big(\mu,\Gamma_{\phi\psi}\big) + c\big(1,\overline{\Gamma_\phi}\big) \\ &= c(\phi\psi) - c\big(\mu,\Gamma_\phi\big) = c(\phi\psi) - c(\phi). \end{aligned}$$

Thus

$$\lceil c(\psi)\rceil \geqslant \lceil c(\phi\psi) - c(\phi)\rceil \geqslant \lceil c(\phi\psi)\rceil - \lceil c(\phi)\rceil$$

as we wanted. □

In contrast with the symplectic case, c is not invariant by conjugation. Recall that in the symplectic case this property follows from the fact that, for every Hamiltonian symplectomorphism φ of $\mathbb{R}^{2n}$, $c(\varphi)$ belongs the action spectrum of φ which is invariant by conjugation. In the contact case the situation is very different since the set of values taken by the contact action $\phi_3(q) - z$ at translated points $q = (x,y,z)$ of a contactomorphism ϕ of $\mathbb{R}^{2n+1}$ is not invariant by conjugation. In fact, not even the property of being a translated point is invariant by conjugation: if q is a translated point for ϕ then in general $\psi_t(q)$ is not a translated point for $\psi_t\phi{\psi_t}^{-1}$. However, we are going to see that this is true if q is a translated point with action 0, and in the 1-periodic case also if the action is any integer. As we

will see this observation is the key to prove that, in the 1-periodic case, the integer part of c is invariant by conjugation.

Recall that a point q of $\mathbb{R}^{2n+1}$ is a translated point for a contactomorphism ϕ if and only if $\Gamma_\phi(q)$ is in the intersection of Γ_ϕ with the 0-wall. We will say that q is a *non-degenerate* translated point if this intersection is transverse and thus if the corresponding critical point of the generating function of ϕ is non-degenerate. Note that this condition can also be expressed by requiring that there is no tangent vector $X \neq 0$ at q such that $(\Gamma_\phi)_*(X)$ is tangent to the 0-wall, or equivalently (see [2]) no tangent vector $X \neq 0$ at q such that $\phi_*(X) = X$ and $X(g) = 0$.

LEMMA 3.13 ([2]). — *Let ϕ and ψ be contactomorphisms of $\mathbb{R}^{2n+1}$. Then $q \in \mathbb{R}^{2n+1}$ is a translated point of ϕ with contact action 0 if and only if $\psi(q)$ is a translated point of $\psi\phi\psi^{-1}$ with contact action 0. Moreover, q is non-degenerate if and only if so is $\psi(q)$.*

Proof. — Note first that if $\phi^*(dz - ydx) = e^g(dz - ydx)$ and $\psi^*(dz - ydx) = e^f(dz - ydx)$ then $(\psi\phi\psi^{-1})^*(dz - ydx) = e^h(dz - ydx)$ with $h = f \circ \phi \circ \psi^{-1} + g \circ \psi^{-1} - f \circ \psi^{-1}$. Suppose that q is a translated point of ϕ with contact action 0, i.e. $\phi(q) = q$ and $g(q) = 0$. Then $\psi\phi\psi^{-1}\big(\psi(q)\big) = \psi(q)$ and $h\big(\psi(q)\big) = f\big(\phi(q)\big) + g(q) - f(q) = 0$ so that $\psi(q)$ is a translated point of $\psi\phi\psi^{-1}$ with contact action 0. To prove the last statement we will show that if q is a degenerate translated point then so is $\psi(q)$. By the discussion above, if q is a degenerate translated point for ϕ then there is a tangent vector $X \neq 0$ at q such that $\phi_*(X) = X$ and $X(g) = 0$. But then

$$\big(\psi\phi\psi^{-1}\big)_* \big(\psi_*(X)\big) = \psi_*(X)$$

and

$$\begin{aligned}\psi_*(X)(h) &= X(f \circ \phi + g - f) = X(f \circ \phi) + X(g) - X(f) \\ &= \phi_*(X)(f) - X(f) = 0\end{aligned}$$

thus $\psi(q)$ is a degenerate translated point for $\psi\phi\psi^{-1}$. □

We now give the 1-periodic version of the previous lemma.

LEMMA 3.14. — *Let ϕ and ψ be 1-periodic contactomorphisms of $\mathbb{R}^{2n+1}$, and k an integer. Then $q \in \mathbb{R}^{2n+1}$ is a translated point of ϕ with contact action k if and only if $\psi(q)$ is a translated point of $\psi\phi\psi^{-1}$ with contact action k. Moreover, q is non-degenerate if and only if so is $\psi(q)$.*

Proof. — The same proof as in Lemma 3.13 goes through in this situation, due to the 1-periodicity of ψ and the fact that k is an integer. Suppose indeed that q is a translated point of ϕ with contact action k,

i.e. $\phi(q) = q + (0,0,k)$ and $g(q) = 0$. Then $\psi\phi\psi^{-1}\big(\psi(q)\big) = \psi\big(\phi(q)\big) = \psi\big(q + (0,0,k)\big) = \psi(q) + (0,0,k)$ and

$$h\big(\psi(q)\big) = f\big(\phi(q)\big) + g(q) - f(q) = f\big(q + (0,0,k)\big) + g(q) - f(q) = 0$$

(note that f is invariant by integer translation in the z-coordinate since ψ is 1-periodic), thus $\psi(q)$ is a translated point of $\psi\phi\psi^{-1}$ with contact action k. The statement about the non-degeneracy can be seen as in the proof of Lemma 3.13. □

The above lemma is the key to prove the following crucial result.

LEMMA 3.15. — *Consider a contactomorphism ϕ and a contact isotopy ψ_t in $Cont^c_{1\text{-per}}(\mathbb{R}^{2n+1})$ and let $S_t : E \to \mathbb{R}$ be a 1-parameter family of generating functions for the conjugation $\psi_t\phi{\psi_t}^{-1}$. If k is an integer and c_t is a path of critical values of S_t with $c_{t_0} = k$ for some $t_0 \in \mathbb{R}$, then $c_t = k$ for all t.*

Proof. — Suppose that c_t is a path of critical values of S_t with $c_{t_0} = k$ for some t_0. Let $x_t = (q_t, \xi_t) \in \mathbb{R}^{2n+1} \times \mathbb{R}^N$ be a 1-parameter family of critical points of S_t, for t in some subinterval of $[0,1]$ containing t_0. Following the model of the proof of Lemma 3.9, the result follows if we construct a path y_t in E such that $y_{t_0} = (q_{t_0}, \xi_{t_0})$ and every y_t is a (non-degenerate) critical point of S_t with critical value k (assuming that x_t is non-degenerate). We know that q_{t_0} is a non-degenerate translated point for ${\psi_{t_0}\phi\psi_{t_0}}^{-1}$ with action $c_{t_0} = k$. By Lemma 3.14 it follows that $\psi_t\big({\psi_{t_0}}^{-1}(q_{t_0})\big)$ is a path of non-degenerate translated points for $\psi_t\phi\psi_t^{-1}$, all with action k. Thus $y_t := i_{{S_t}^{-1}}\Big(\psi_t\big({\psi_{t_0}}^{-1}(q_{t_0})\big), 0, k\Big)$ is the desired path of critical points of S_t. □

Lemma 3.15 immediately implies that in the 1-periodic case the integer part of c is invariant by conjugation, as stated in the following proposition. As we will see, this result will allow us to define in 3.6 an integral invariant for domains in $\mathbb{R}^{2n} \times S^1$.

PROPOSITION 3.16. — *For any ϕ, ψ in $Cont^c_{1\text{-per}}(\mathbb{R}^{2n+1})$ it holds*

$$\lceil c(\phi) \rceil = \lceil c(\psi\phi\psi^{-1}) \rceil.$$

In the case of $\mathrm{Cont}^c(\mathbb{R}^{2n+1})$ only the following weaker statement is true.

PROPOSITION 3.17 ([2]). — *For any ϕ, ψ in ${Cont_0}^c(\mathbb{R}^{2n+1})$ we have that $c(\phi) = 0$ if and only if $c(\psi\phi\psi^{-1}) = 0$.*

Proof. — Let ψ be the time-1 map of the contact isotopy ψ_t and consider $c_t = c(\psi_t\phi\psi_t^{-1})$. As in the proof of Lemma 3.15, Lemma 3.13 implies that if $c_{t_0} = 0$ then $c_t = 0$ for all t. □

We end this section explaining the relation between the invariant c in the symplectic and contact case.

PROPOSITION 3.18. — *Let φ be a compactly supported Hamiltonian symplectomorphism of $\mathbb{R}^{2n}$ and $\widetilde{\varphi}$ its lift to $\mathbb{R}^{2n+1}$ or to $\mathbb{R}^{2n}\times S^1$. Then $c(\widetilde{\varphi}) = c(\varphi)$.*

Proof. — The result follows from Lemma 3.2. The case of $\mathbb{R}^{2n+1}$ is immediate, while the 1-periodic case can be seen as follows. Suppose that $\widetilde{\varphi}$ is the lift of φ to $\mathbb{R}^{2n}\times S^1$. By Lemma 3.2 we know that a generating function for $\widetilde{\varphi}$ is given by $\widetilde{S} : (S^{2n}\times S^1)\times\mathbb{R}^N \to \mathbb{R}$, $\widetilde{S}(q,z;\xi) = S(q;\xi)$ where $S : S^{2n}\times\mathbb{R}^N \to \mathbb{R}$ is a generating function for φ. Denote by $\widetilde{E}^a$ the sublevel set of $\widetilde{S}$ with respect to a, and by $\widetilde{i_a}$ the inclusion $(\widetilde{E}^a,\widetilde{E}^{-\infty}) \hookrightarrow (\widetilde{E},\widetilde{E}^{-\infty})$. Then $\widetilde{E}^a = E^a\times S^1$ and, after identifying $H^*(\widetilde{E},\widetilde{E}^{-\infty})$ with $H^*(S^{2n}\times S^1) = H^*(S^{2n})\otimes H^*(S^1)$ and $H^*(\widetilde{E}^a,\widetilde{E}^{-\infty})$ with $H^*(E^a,E^{-\infty})\otimes H^*(S^1)$, the induced map

$$\widetilde{i_a}^* : H^*(S^{2n})\otimes H^*(S^1) \to H^*(E^a,E^{-\infty})\otimes H^*(S^1)$$

is given by $\widetilde{i_a}^* = {i_a}^*\otimes \mathrm{id}$. In particular we have that $\widetilde{i_a}^*(\mu\otimes\mu_{S^1}) = {i_a}^*(\mu)\otimes\mu_{S^1}$ where μ and μ_{S^1} denote respectively the orientation classes of S^{2n} and S^1, thus $\widetilde{i_a}^*(\mu\otimes\mu_{S^1}) = 0$ if and only if ${i_a}^*(\mu) = 0$. Since $\mu\otimes\mu_{S^1}$ is the orientation class of $H^*(S^{2n}\times S^1)$ we conclude that $c(\widetilde{\varphi}) = c(\varphi)$. □

3.5. The Bhupal partial order on $\mathrm{Cont}_0^c(\mathbb{R}^{2n+1})$ and $\mathrm{Cont}_0^c(\mathbb{R}^{2n}\times S^1)$

Bhupal's partial order $\leqslant_B$ on $\mathrm{Cont}_0{}^c(\mathbb{R}^{2n+1})$ and on $\mathrm{Cont}_0{}^c(\mathbb{R}^{2n}\times S^1)$ is defined by

$$\phi_1 \leqslant_B \phi_2 \quad \text{if} \quad c(\phi_1{\phi_2}^{-1}) = 0.$$

Using the properties in Proposition 3.11 it is immediate to see that $\leqslant_B$ is indeed a partial order, that it is bi-invariant (i.e. if $\phi_1\leqslant_B\phi_2$ and $\psi_1\leqslant_B\psi_2$ then $\phi_1\psi_1\leqslant_B\phi_2\psi_2$), and that if $\phi_1\leqslant\phi_2$ in the sense of 3.2 then $\phi_1\leqslant_B\phi_2$. In particular it follows that $\leqslant$ is also a partial order. Note that in the language of Eliashberg and Polterovich [17] this means that $\mathbb{R}^{2n+1}$ and $\mathbb{R}^{2n}\times S^1$ are *orderable* contact manifolds.

3.6. Contact capacity of domains in $\mathbb{R}^{2n}\times S^1$

We will consider domains in $\mathbb{R}^{2n}\times S^1$ as domains in $\mathbb{R}^{2n+1}$ that are invariant by the action of $\mathbb{Z}$ by translations in the z-coordinate. For an

open and bounded domain $\mathcal{V}$ of $\mathbb{R}^{2n} \times S^1$ we define the **contact capacity** of $\mathcal{V}$ as

$$c(\mathcal{V}) := \sup \{ \lceil c(\phi) \rceil \mid \phi \in \mathrm{Cont}\,(\mathcal{V}) \}$$

where $\mathrm{Cont}\,(\mathcal{V})$ denotes the set of time-1 maps of 1-periodic contact Hamiltonian functions supported in $\mathcal{V}$. By the following lemma, $c(\mathcal{V})$ is a well-defined integer number.

LEMMA 3.19. — *For every contactomorphism ψ in $Cont^c_{1\text{-}per}(\mathbb{R}^{2n+1})$ such that $\psi(\mathcal{V}) \cap \mathcal{V} = \emptyset$ we have $c(\mathcal{V}) \leqslant \gamma(\psi)$, where $\gamma(\psi) := \lceil c(\psi) \rceil + \lceil c(\psi^{-1}) \rceil$.*

Proof. — We will show that $\lceil c(\psi\phi) \rceil = \lceil c(\psi) \rceil$ for all ϕ in $\mathrm{Cont}\,(\mathcal{V})$ and ψ as in the statement of the lemma, and then conclude as in the proof of Lemma 2.13. Let $\phi = \phi_t|_{t=1}$, and consider the map $t \mapsto c(\psi\phi_t)$. Suppose $c_{t_0} = k \in \mathbb{Z}$. Then there is a translated point $q = (x, y, z)$ of $\psi\phi_{t_0}$ such that $(\psi\phi_{t_0})_3 - z = k$. But then we can apply an argument similar to the one in Lemma 2.13 to see that q is also an almost fixed point of $\psi\phi_t$ for all t, with $(\psi\phi_t)_3 - z = k$. We can now conclude, as in Lemma 3.15, that $c(\psi\phi_t) = k$ for all t. It follows that $\lceil c(\psi\phi_t) \rceil$ is independent of t, in particular $\lceil c(\psi\phi) \rceil = \lceil c(\psi) \rceil$. □

As in the symplectic case, we can extend the definition to arbitrary domains of $\mathbb{R}^{2n} \times S^1$.

THEOREM 3.20. — *c satisfies the following properties:*

(i) *(Contact Invariance) For any ψ in $Cont_0^c(\mathbb{R}^{2n} \times S^1)$ we have $c(\psi(\mathcal{V})) = c(\mathcal{V})$.*
(ii) *(Monotonicity) If $\mathcal{V}_1 \subset \mathcal{V}_2$, then $c(\mathcal{V}_1) \leqslant c(\mathcal{V}_2)$.*
(iii) *For any domain $\mathcal{U}$ in $\mathbb{R}^{2n}$ we have $c(\mathcal{U} \times S^1) = \lceil c(\mathcal{U}) \rceil$.*

Proof. — Contact invariance follows from Proposition 3.16, and monotonicity is immediate from the definition. As for the last property, it can be seen as follows. If φ is an Hamiltonian symplectomorphism of $\mathbb{R}^{2n}$ generated by a Hamiltonian $H : \mathbb{R}^{2n} \to \mathbb{R}$ supported in $\mathcal{U}$, then its lift $\widetilde{\varphi}$ is generated by the contact Hamiltonian $\widetilde{H} : \mathbb{R}^{2n} \times S^1 \to \mathbb{R}$, $\widetilde{H}(x, y, z) = H(x, y)$ which is supported in $\mathcal{U} \times S^1$. By Proposition 3.18 we have $c(\widetilde{\varphi}) = c(\varphi)$, so we see that $c(\mathcal{U} \times S^1) \geqslant \lceil c(\mathcal{U}) \rceil$. Equality holds because for every ϕ in $\mathrm{Cont}\,(\mathcal{U} \times S^1)$ there exists a φ in $\mathrm{Ham}\,(\mathcal{U})$ such that $\phi \leqslant \widetilde{\varphi}$. □

Note that the Non-Squeezing Theorem of Eliashberg, Kim and Polterovich follows immediately from Theorem 3.20 and Example 2.15. Indeed, consider $R_2 \leqslant k < R_1$ for $k \in \mathbb{Z}$ and suppose that there is a contactomorphism ψ in $\mathrm{Cont}_0^c(\mathbb{R}^{2n} \times S^1)$ such that $\psi\left(\widehat{B(R_1)}\right) \subset \widehat{B(R_2)}$.

Then by monotonicity we have $c\left(\psi\left(\widehat{B(R_1)}\right)\right) \leqslant c\left(\widehat{B(R_2)}\right)$. But this is impossible since $c\left(\psi\left(\widehat{B(R_1)}\right)\right) = c\left(\widehat{B(R_1)}\right) = \lceil c\left(B(R_1)\right)\rceil > k$ and $c\left(\widehat{B(R_2)}\right) = \lceil c\left(B(R_2)\right)\rceil \leqslant k$. Note that the same argument shows that if $R_2 \leqslant k < R_1$ it is in fact not even possible to squeeze $\widehat{B(R_1)}$ into $\widehat{C(R_2)}$.

3.7. Contact homology of domains in $\mathbb{R}^{2n} \times S^1$

In this last section we generalize to the contact case Traynor's construction of symplectic homology. Similarly to the case of the capacity, we only obtain contact invariant homology groups ${G_*}^{a,b}(\mathcal{V})$ for domains $\mathcal{V}$ in $\mathbb{R}^{2n} \times S^1$ and for *integer* parameters a and b.

Let ϕ be a contactomorphism in $\mathrm{Cont}^c_{1\text{-per}}(\mathbb{R}^{2n+1})$ with generating function $S : E = (S^{2n} \times S^1) \times \mathbb{R}^N \longrightarrow \mathbb{R}$. Given integer numbers a and b that are not critical values of S and such that $-\infty < a < b \leqslant \infty$, we define the **$k$-th contact homology group** of ϕ with respect to the values a and b by

$$G_k{}^{(a,b]}(\phi) := H_{k+\iota}(E^b, E^a)$$

where E^a, E^b denote the sublevel sets of S, and ι is the index of the quadratic at infinity part of S. By the uniqueness part in Theorem 3.1 these groups are well-defined, i.e. do not depend on the choice of S.

The following proposition follows immediately from Lemma 3.2.

PROPOSITION 3.21. — *For any φ in $Ham^c(\mathbb{R}^{2n})$ we have*

$$G_*{}^{(a,b]}(\widetilde{\varphi}) = G_*{}^{(a,b]}(\varphi) \otimes H_*(S^1).$$

The definition of $G_k{}^{(a,b]}(\phi)$ would in fact make sense for all real numbers a and b and also for contactomorphisms of $\mathrm{Cont}^c(\mathbb{R}^{2n+1})$. However, the facts that a and b are integers and ϕ is 1-periodic are crucial to prove the following proposition.

PROPOSITION 3.22. — *For any ϕ and ψ in $Cont^c_{1\text{-per}}(\mathbb{R}^{2n+1})$ we have an induced isomorphism*

$$\psi_* : G_*{}^{(a,b]}(\psi\phi\psi^{-1}) \longrightarrow G_*{}^{(a,b]}(\phi).$$

Proof. — Let ψ be the time-1 map of an isotopy ψ_t of 1-periodic contactomorphisms of $\mathbb{R}^{2n+1}$, and let $S_t : \mathbb{R}^{2n+1} \times \mathbb{R}^N \longrightarrow \mathbb{R}$ be generating functions for $\psi_t\phi\psi_t^{-1}$. In contrast to the symplectic case the critical values of S_t are not fixed. However we will now see that, due to Lemma 3.15, we can still find an isotopy conjugating the preimages ${S_t}^{-1}(a)$ and ${S_t}^{-1}(b)$.

Recall that ${G_*}^{(a,b]}(\phi)$ is only defined in the case that a and b are not critical values of the generating function S_0 of ϕ. Since a and b are integers, it follows from Lemma 3.15 that a and b are not critical values of S_t, for any t. Thus we can apply an analogue of Lemma 2.17 for pairs of sublevel sets to find an isotopy θ_t of $\mathbb{R}^{2n+1}\times\mathbb{R}^N$ such that $\theta_t\left({S_0}^{-1}((\infty,a])\right) = {S_t}^{-1}((\infty,a])$ and $\theta_t\left({S_0}^{-1}((\infty,b])\right) = {S_t}^{-1}((\infty,b])$. In particular for $t=1$ this induces the desired isomorphism $\psi_* : {G_*}^{(a,b]}(\psi\phi\psi^{-1}) \longrightarrow {G_*}^{(a,b]}(\phi)$. □

Consider now a domain $\mathcal{V}$ in $\mathbb{R}^{2n}\times S^1$. Given integer numbers a and b, we denote by ${\mathrm{Cont}_{a,b}}^c(\mathcal{V})$ the set of ϕ in $\mathrm{Cont}^c_{1\text{-per}}(\mathbb{R}^{2n+1})$ with support contained in $\mathcal{V}$ and whose generating function does not have a, b as critical values. Note that ${\mathrm{Cont}_{a,b}}^c(\mathcal{V})$ is directed with respect to the partial order $\leqslant$ defined by the Hamiltonians, i.e. for any ϕ, ψ in ${\mathrm{Cont}_{a,b}}^c(\mathcal{V})$ there is a φ in ${\mathrm{Cont}_{a,b}}^c(\mathcal{V})$ such that $\phi\leqslant\varphi$ and $\psi\leqslant\varphi$. Suppose now that $\phi_1\leqslant\phi_2$. Then by Proposition 3.4 we know that there are generating functions S_1, $S_2 : E\longrightarrow\mathbb{R}$ for Γ_{ϕ_1}, Γ_{ϕ_2} respectively such that $S_1\leqslant S_2$. Thus we have inclusions of sublevel sets $E_2^a\subset E_1^a$ and $E_2^b\subset E_1^b$, and so an induced homomorphism ${\lambda_1}^2 : {G_k}^{(a,b]}(\phi_2)\longrightarrow {G_k}^{(a,b]}(\phi_1)$. Note that given ϕ_1, ϕ_2, ϕ_3 in ${\mathrm{Cont}_{a,b}}^c(\mathcal{V})$ with $\phi_1\leqslant\phi_2\leqslant\phi_3$, it holds ${\lambda_1}^2\circ{\lambda_2}^3 = {\lambda_1}^3$ and ${\lambda_i}^i = \mathrm{id}$. This means in particular that $\{{G_k}^{(a,b]}(\phi_i)\}_{\phi_i\in{\mathrm{Cont}_{a,b}}^c(\mathcal{V})}$ is an inversely directed family of groups, so we can define the **k-th contact homology group** ${G_k}^{(a,b]}(\mathcal{V})$ of $\mathcal{V}$ with respect to the values a an b to be the inverse limit of this family. Note that ${G_k}^{a,b}(\mathcal{V})$ can be calculated by any sequence $\phi_1\leqslant\phi_2\leqslant\phi_3\leqslant\cdots$ such that the associated contact Hamiltonians get arbitrarily large.

The next two theorems are proved as in the symplectic case (using Proposition 3.22 for the first).

THEOREM 3.23 (Contact invariance). — *For any domain $\mathcal{V}$ in $\mathbb{R}^{2n}\times S^1$ and any contactomorphism ψ of $\mathbb{R}^{2n}\times S^1$ isotopic to the identity we have an induced isomorphism $\psi_* : {G_k}^{(a,b]}\left(\psi(\mathcal{V})\right)\longrightarrow {G_k}^{(a,b]}(\mathcal{V})$.*

THEOREM 3.24 (Monotonicity). — *Every inclusion of domains induces a homomorphism of homology groups (reversing the order), with the following functorial properties:*

(i) *If $\mathcal{V}_1\subset\mathcal{V}_2\subset\mathcal{V}_3$ then the following diagram commutes*

$$\begin{array}{ccc} {G_*}^{(a,b]}(\mathcal{V}_3) & \longrightarrow & {G_*}^{(a,b]}(\mathcal{V}_2) \\ & \searrow & \downarrow \\ & & {G_*}^{(a,b]}(\mathcal{V}_1). \end{array}$$

(ii) *If $\mathcal{V}_1 \subset \mathcal{V}_2$, then for any contactomorphism ψ the following diagram commutes*

$$\begin{array}{ccc} {G_*}^{(a,b]}(\mathcal{V}_2) & \longrightarrow & {G_*}^{(a,b]}(\mathcal{V}_1) \\ {\scriptstyle\psi_*}\uparrow & & \uparrow{\scriptstyle\psi_*} \\ {G_*}^{(a,b]}\big(\psi(\mathcal{V}_2)\big) & \longrightarrow & {G_*}^{(a,b]}\big(\psi(\mathcal{V}_1)\big). \end{array}$$

The relation between symplectic and contact homology is given by the following theorem.

THEOREM 3.25. — *For any domain $\mathcal{U}$ of $\mathbb{R}^{2n}$ we have ${G_*}^{(a,b]}(\mathcal{U}\times S^1) = {G_*}^{(a,b]}(\mathcal{U}) \otimes H_*(S^1)$. Moreover, this correspondence is functorial in the following sense. Let $\mathcal{U}_1$, $\mathcal{U}_2$ be domains in $\mathbb{R}^{2n}$ with $\mathcal{U}_1 \subset \mathcal{U}_2$, and for $i = 1, 2$ identify ${G_*}^{(a,b]}(\mathcal{U}_i \times S^1)$ with ${G_*}^{(a,b]}(\mathcal{U}_i) \otimes H_*(S^1)$. Then the homomorphism ${G_*}^{(a,b]}(\mathcal{U}_2 \times S^1) \to {G_*}^{(a,b]}(\mathcal{U}_1 \times S^1)$ induced by the inclusion $\mathcal{U}_1 \times S^1 \hookrightarrow \mathcal{U}_2 \times S^1$ is given by $\mu \otimes id$, where $\mu : {G_*}^{(a,b]}(\mathcal{U}_2) \to {G_*}^{(a,b]}(\mathcal{U}_1)$ is the homomorphism induced by $\mathcal{U}_1 \hookrightarrow \mathcal{U}_2$.*

Proof. — If $\varphi_1 \leqslant \varphi_2 \leqslant \varphi_3 \leqslant \cdots$ is an unbounded ordered sequence supported in $\mathcal{U}$ then $\widetilde{\varphi_1} \leqslant \widetilde{\varphi_2} \leqslant \widetilde{\varphi_3} \leqslant \cdots$ in an unbounded ordered sequence supported in $\mathcal{U} \times S^1$, thus the first statement follows from Proposition 3.21. Suppose now that $\mathcal{U}_1 \subset \mathcal{U}_2$, and consider unbounded ordered sequences ${\varphi_1}^1 \leqslant {\varphi_2}^1 \leqslant {\varphi_3}^1 \leqslant \cdots$ and ${\varphi_1}^2 \leqslant {\varphi_2}^2 \leqslant {\varphi_3}^2 \leqslant \cdots$ supported in $\mathcal{U}_1$ and $\mathcal{U}_2$ respectively and such that ${\varphi_i}^1 \leqslant {\varphi_i}^2$. Then the homomorphism ${G_*}^{(a,b]}(\mathcal{U}_2) \to {G_*}^{(a,b]}(\mathcal{U}_1)$ is induced by the homomorphisms ${G_*}^{(a,b]}({\varphi_i}^2) \to {G_*}^{(a,b]}({\varphi_i}^1)$. If we calculate the contact homology of $\mathcal{U}_1 \times S^1$ and $\mathcal{U}_2 \times S^1$ using the sequences $\widetilde{{\varphi_1}^1} \leqslant \widetilde{{\varphi_2}^1} \leqslant \widetilde{{\varphi_3}^1} \leqslant \cdots$ and $\widetilde{{\varphi_1}^2} \leqslant \widetilde{{\varphi_2}^2} \leqslant \widetilde{{\varphi_3}^2} \leqslant \cdots$ then the homomorphism ${G_*}^{(a,b]}(\mathcal{U}_2 \times S^1) \to {G_*}^{(a,b]}(\mathcal{U}_1 \times S^1)$ is induced by the homomorphisms ${G_*}^{(a,b]}(\widetilde{{\varphi_i}^2}) = {G_*}^{(a,b]}({\varphi_i}^2) \otimes H_*(S^1) \to {G_*}^{(a,b]}(\widetilde{{\varphi_i}^1}) = {G_*}^{(a,b]}({\varphi_i}^1) \otimes H_*(S^1)$ which are obtained by tensoring ${G_*}^{(a,b]}({\varphi_i}^2) \to {G_*}^{(a,b]}({\varphi_i}^1)$ with the identity on $H_*(S^1)$. □

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