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QUASI-REDUCTIVE (BI)PARABOLIC SUBALGEBRAS IN REDUCTIVE LIE ALGEBRAS.

by Karin BAUR & Anne MOREAU (*)

Abstract. — We say that a finite dimensional Lie algebra is quasi-reductive if it has a linear form whose stabilizer for the coadjoint representation, modulo the center, is a reductive Lie algebra with a center consisting of semisimple elements. Parabolic subalgebras of a semisimple Lie algebra are not always quasi-reductive (except in types A or C by work of Panyushev). The classification of quasi-reductive parabolic subalgebras in the classical case has been recently achieved in unpublished work of Duflo, Khalgui and Torasso. In this paper, we investigate the quasi-reductivity of biparabolic subalgebras of reductive Lie algebras. Biparabolic (or seaweed) subalgebras are the intersection of two parabolic subalgebras whose sum is the total Lie algebra. As a main result, we complete the classification of quasi-reductive parabolic subalgebras of reductive Lie algebras by considering the exceptional cases.

Résumé. — Une algèbre de Lie de dimension finie est dite quasi-réductive si elle possède une forme linéaire dont le stablisateur pour la représentation coadjointe, modulo le centre, est une algèbre de Lie réductive avec un centre formé d'éléments semi-simples. Les sous-algèbres paraboliques d'une algèbre de Lie semi-simple ne sont pas toujours quasi-réductives (sauf en types A ou C d'après un résultat de Panyushev). Récemment, Duflo, Khalgui and Torasso ont terminé la classification des sous-algèbres paraboliques quasi-réductives dans le cas classique. Dans cet article nous étudions la quasi-réductivité des sous-algèbres biparaboliques des algèbres de Lie réductives. Les sous-algèbres biparaboliques sont les intersections de deux sous-algèbres paraboliques dont la somme est l'algèbre de Lie ambiante. Notre principal résultat est la complétion de la classification des sous-algèbres paraboliques quasi-réductives des algèbres de Lie réductives.

Keywords: Reductive Lie algebras, quasi-reductive Lie algebras, index, biparabolic Lie algebras, seaweed algebras, regular linear forms.
Math. classification: 17B20, 17B45, 22E60.
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Introduction

Let G be a complex connected linear algebraic Lie group. Denote by $\mathfrak{g}$ its Lie algebra. The group G acts on the dual $\mathfrak{g}^*$ of $\mathfrak{g}$ by the coadjoint action. For $f \in \mathfrak{g}^*$, we denote by $G(f)$ its stabilizer in G; it always contains the center Z of G. One says that a linear form $f \in \mathfrak{g}^*$ has *reductive type* if the quotient $G(f)/Z$ is a reductive subgroup of $\mathrm{GL}(\mathfrak{g})$. The Lie algebra $\mathfrak{g}$ is called *quasi-reductive* if it has linear forms of reductive type. This notion goes back to M. Duflo. He initiated the study of such Lie algebras because of applications in harmonic analysis, see [3]. For more details about linear forms of reductive type and quasi-reductive Lie algebras we refer the reader to Section 1.

Reductive Lie algebras are obviously quasi-reductive Lie algebras since in that case, 0 is a linear form of reductive type. Biparabolic subalgebras form a very interesting class of non-reductive Lie algebras. They naturally extend the classes of parabolic subalgebras and of Levi subalgebras. The latter are clearly quasi-reductive since they are reductive subalgebras.. Biparabolic subalgebras were introduced by V. Dergachev and A. Kirillov in the case $\mathfrak{g} = \mathfrak{sl}_n$, see [2]. A *biparabolic* subalgebra or *seaweed* subalgebra (of a semisimple Lie algebra) is the intersection of two parabolic subalgebras whose sum is the total Lie algebra.

In this article, we are interested in the classification of quasi-reductive (bi)parabolic subalgebras. Note that it is enough to consider the case of (bi)parabolic subalgebras of the simple Lie algebras, cf. Remark 1.4.

In the classical cases, various results are already known: All biparabolic subalgebras of $\mathfrak{sl}_n$ and $\mathfrak{sp}_{2n}$ are quasi-reductive as has been proven by D. Panyushev in [19]. The case of orthogonal Lie algebras is more complicated: On one hand, there are parabolic subalgebras of orthogonal Lie algebras which are not quasi-reductive, as P. Tauvel and R.W.T. Yu have shown (Section 3.2 of [20]). On the other hand, D. Panyushev and A. Dvorsky exhibit many quasi-reductive parabolic subalgebras in [6] and [19] by constructing linear forms with the desired properties. Recently, M. Duflo, M.S. Khalgui and P. Torasso have obtained the classification of quasi-reductive parabolic subalgebras of the orthogonal Lie algebras in unpublished work, [4]. They were able to characterize quasi-reductive parabolic subalgebras in terms of the flags stabilized by the subalgebras.

The main result of this paper is the completion of the classification of quasi-reductive parabolic subalgebras of simple Lie algebras. This is done in Section 5 (Theorem 5.1 and Theorem 5.2). Our goal is ultimately to describe all quasi-reductive biparabolic subalgebras. Thus, in the first sections

we present results concerning biparabolic subalgebras to remain in a general setting as far as possible. For the remainder of the introduction, $\mathfrak{g}$ is a finite dimensional complex semisimple Lie algebra.

The paper is organized as follows:

In Section 1 we introduce the main notations and definitions. We also include in this section a short review of known results about biparabolic subalgebras, including the description of quasi-reductive parabolic subalgebras in the classical Lie algebras (Subsection 1.4). In Section 2, we describe two methods of reduction, namely the transitivity property (Theorem 2.1) and the additivity property (Theorem 2.11). As a first step of our classification, we exhibit in Section 3 a large collection of quasi-reductive biparabolic subalgebras of $\mathfrak{g}$ (Theorem 3.6). Next, in Section 4, we consider *the* non quasi-reductive parabolic subalgebras of $\mathfrak{g}$, for simple $\mathfrak{g}$ of exceptional type (Theorems 4.1, 4.3 and 4.6). This is a crucial part of the paper. Indeed, to study the quasi-reductivity, we can make explicit computations (cf. Section 5) while it is much trickier to prove that a Lie algebra is not quasi-reductive. Using the results of Sections 2, 3 and 4, we are able to cover a large number of parabolic subalgebras. The remaining cases are dealt with in Section 5 (Theorem 5.6, Propositions 5.8 and 5.9). This completes the classification of quasi-reductive parabolic subalgebras of $\mathfrak{g}$ (Theorems 5.1 and 5.2, see also Tables 5.1 and 5.2).

At this place, we also want to point out that in [16], O. Yakimova and the second author study the *maximal reductive stabilizers* of quasi-reductive parabolic subalgebras of $\mathfrak{g}$. This piece of work yields an alternative proof of Proposition 5.9 which is not based on the computer programme `GAP`, see Remark 5.10.

1. Notations, definitions and basic facts

In this section, we recall a number of known results that will be used in the sequel.

1.1. Let $\mathfrak{g}$ be a complex Lie algebra of a connected linear algebraic Lie group G. Denoting by $\mathfrak{g}(f)$ the Lie algebra of $G(f)$, we have $\mathfrak{g}(f) = \{x \in \mathfrak{g} \mid (\mathrm{ad}^* x)(f) = 0\}$ where ad^* is the coadjoint representation of $\mathfrak{g}$. Recall that a linear form $f \in \mathfrak{g}^*$ is of *reductive type* if $G(f)/Z$ is a reductive Lie subgroup of $\mathrm{GL}(\mathfrak{g})$. We can reformulate this definition as follows:

DEFINITION 1.1. — *An element f of $\mathfrak{g}^*$ is said to be of reductive type if $\mathfrak{g}(f)/\mathfrak{z}$ is a reductive Lie algebra whose center consists of semisimple elements of $\mathfrak{g}$ where $\mathfrak{z}$ is the center of $\mathfrak{g}$.*

Recall that a linear form $f \in \mathfrak{g}^*$ is *regular* if the dimension of $\mathfrak{g}(f)$ is as small as possible. By definition, the *index* of $\mathfrak{g}$, denoted by $\operatorname{ind}\mathfrak{g}$, is the dimension of the stabilizer of a regular linear form. The index of various special classes of subalgebras of reductive Lie algebras has been studied by several authors, cf. [18], [25], [14], [15]. For the index of seaweed algebras, we refer to [17], [6], [20], [22], [10] and [11].

Recall that $\mathfrak{g}$ is called *quasi-reductive* if it has linear forms of reductive type. From Duflo's work [3, §§I.26-27] one deduces the following result about regular linear forms of reductive type:

PROPOSITION 1.2. — *Suppose that $\mathfrak{g}$ is quasi-reductive. The set of regular linear forms of reductive type forms a Zariski open dense subset of $\mathfrak{g}^*$.*

1.2. From now on, $\mathfrak{g}$ is a complex finite dimensional semisimple Lie algebra. The dual of $\mathfrak{g}$ is identified with $\mathfrak{g}$ through the Killing form of $\mathfrak{g}$. For $u \in \mathfrak{g}$, we denote by φ_u the corresponding element of $\mathfrak{g}^*$. For $u \in \mathfrak{g}$, the restriction of φ_u, to any subalgebra $\mathfrak{a}$ of $\mathfrak{g}$ will be denoted by $(\varphi_u)|_{\mathfrak{a}}$.

Denote by π the set of simple roots with respect to a fixed triangular decomposition

$$\mathfrak{g} = \mathfrak{n}^+ \oplus \mathfrak{h} \oplus \mathfrak{n}^-$$

of $\mathfrak{g}$, and by Δ_π (respectively Δ_π^+, Δ_π^-) the corresponding root system (respectively positive root system, negative root system). If π' is a subset of π, we denote by $\Delta_{\pi'}$ the root subsystem of Δ_π generated by π' and we set $\Delta_{\pi'}^\pm = \Delta_{\pi'} \cap \Delta_\pi^\pm$. For $\alpha \in \Delta_\pi$, denote by $\mathfrak{g}_\alpha$ the α-root subspace of $\mathfrak{g}$ and let h_α be the unique element of $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ such that $\alpha(h_\alpha) = 2$. For each $\alpha \in \Delta_\pi$, fix $x_\alpha \in \mathfrak{g}_\alpha$ so that the family $\{x_\alpha, h_\beta \ ; \ \alpha \in \Delta_\pi, \beta \in \pi\}$ is a Chevalley basis of $\mathfrak{g}$. In particular, for non-colinear roots α and β, we have $[x_\alpha, x_\beta] = \pm(p+1)x_{\alpha+\beta}$ if $\beta - p\alpha$ is the source of the α-string through β.

We briefly recall a classical construction due to B. Kostant. It associates to a subset of π a system of strongly orthogonal positive roots in Δ_π. This construction is known to be very helpful to obtain regular forms on biparabolic subalgebras of $\mathfrak{g}$. For a recent account about the *cascade* construction of Kostant, we refer to [22, §1.5] or [21, §40.5].

For λ in $\mathfrak{h}^*$ and $\alpha \in \Delta_\pi$, we shall write $\langle \lambda, \alpha^\vee \rangle$ for $\lambda(h_\alpha)$. Recall that two roots α and β in Δ_π are said to be *strongly orthogonal* if neither $\alpha+\beta$ nor $\alpha-\beta$ is in Δ_π. Let π' be a subset of π. The *cascade* $\mathcal{K}_{\pi'}$ of π' is defined by induction on the cardinality of π' as follows:

(1) $\mathcal{K}(\emptyset) = \emptyset$,

(2) If $\pi'_1,\ldots,\pi'_r$ are the connected components of π', then $\mathcal{K}_{\pi'} = \mathcal{K}_{\pi'_1} \cup \cdots \cup \mathcal{K}_{\pi'_r}$,

(3) If π' is connected, then $\mathcal{K}_{\pi'} = \{\pi'\} \cup \mathcal{K}_T$ where $T = \{\alpha \in \pi' \mid \langle \alpha, \varepsilon_{\pi'}^\vee \rangle = 0\}$ and $\varepsilon_{\pi'}$ is the highest positive root of $\Delta^+_{\pi'}$.

For $K \in \mathcal{K}_{\pi'}$, set

$$\Gamma_K = \{\alpha \in \Delta_K \mid \langle \alpha, \varepsilon_K^\vee \rangle > 0\} \quad \text{and} \quad \Gamma_K^0 = \Gamma_K \setminus \{\varepsilon_K\}.$$

Notice that the subspace $\sum_{K \in \Gamma_K} \mathfrak{g}_\alpha$ is a Heisenberg Lie algebra whose center is $\mathfrak{g}_{\varepsilon_K}$.

The cardinality k_π of $\mathcal{K}_\pi$ only depends on $\mathfrak{g}$; it is independent of the choices of $\mathfrak{h}$ and π. The values of k_π for the different types of simple Lie algebras are given in Table 1.1; in this table, for a real number x, we denote by $[x]$ the largest integer $\leqslant x$.

$\mathrm{A}_\ell, \ell \geqslant 1$	$\mathrm{B}_\ell, \ell \geqslant 2$	$\mathrm{C}_\ell, \ell \geqslant 3$	$\mathrm{D}_\ell, \ell \geqslant 4$	G_2	F_4	E_6	E_7	E_8
$\left[\frac{\ell+1}{2}\right]$	ℓ	ℓ	$2\left[\frac{\ell}{2}\right]$	2	4	4	7	8

Table 1.1. k_π *for the simple Lie algebras.*

For π' a subset of π, we denote by $\mathcal{E}_{\pi'}$ the set of the highest roots ε_K where K runs over the elements of the cascade of π'. By construction, the subset $\mathcal{E}_{\pi'}$ is a family of pairwise strongly orthogonal roots in $\Delta_{\pi'}$. For the convenience of the reader, the set $\mathcal{E}_\pi$, for each simple Lie algebra of type π, is described in the Tables 1.2 and 1.3. We denote by $E_{\pi'}$ the subspace of $\mathfrak{h}^*$ which is generated by the elements of $\mathcal{E}_{\pi'}$.

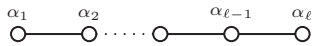
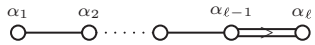
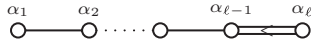
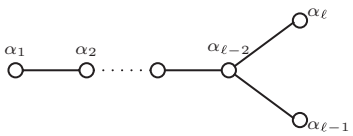
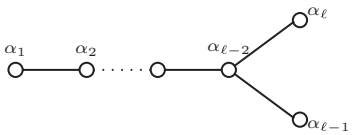
A_ℓ, $\ell \geqslant 1$:		$\{\varepsilon_i = \alpha_i + \cdots + \alpha_{i+(\ell-2i+1)},\ i \leqslant \left[\frac{\ell+1}{2}\right]\}$
B_ℓ, $\ell \geqslant 2$:		$\{\varepsilon_i = \alpha_{i-1} + 2\alpha_i + \cdots + 2\alpha_\ell,\ i \text{ even},\ i \leqslant \ell\} \cup \{\varepsilon_i = \alpha_i,\ i \text{ odd}, i \leqslant \ell\}$
C_ℓ, $\ell \geqslant 3$:		$\{\varepsilon_i = 2\alpha_i + \cdots + 2\alpha_{\ell-1} + \alpha_\ell,\ i \leqslant \ell-1\} \cup \{\varepsilon_\ell = \alpha_\ell\}$
D_ℓ, ℓ even, $\ell \geqslant 4$:		$\{\varepsilon_i = \alpha_{i-1} + 2\alpha_i + \cdots + 2\alpha_{\ell-2} + \alpha_{\ell-1} + \alpha_\ell,\ i \text{ even},\ i < \ell-1\} \cup \{\varepsilon_i = \alpha_i,\ i \text{ odd},\ i < \ell\} \cup \{\varepsilon_\ell = \alpha_\ell\}$
D_ℓ, ℓ odd, $\ell \geqslant 5$:		$\{\varepsilon_i = \alpha_{i-1} + 2\alpha_i + \cdots + 2\alpha_{\ell-2} + \alpha_{\ell-1} + \alpha_\ell,\ i \text{ even },\ i < \ell-1\} \cup \{\varepsilon_i = \alpha_i,\ i \text{ odd },\ i < \ell\} \cup \{\varepsilon_{\ell-1} = \alpha_{\ell-2} + \alpha_{\ell-1} + \alpha_\ell\}$

Table 1.2. $\mathcal{E}_\pi$ for the classical Lie algebras.

1.3. A *biparabolic* subalgebra of $\mathfrak{g}$ is defined to be the intersection of two parabolic subalgebras whose sum is $\mathfrak{g}$. This class of algebras has first been studied in the case of $\mathfrak{sl}_n$ by Dergachev and Kirillov [2] under the name of *seaweed* algebras.

For a subset π' of π, we denote by $\mathfrak{p}^+_{\pi'}$ the standard parabolic subalgebra of $\mathfrak{g}$ which is the subalgebra generated by $\mathfrak{b}^+ = \mathfrak{h} \oplus \mathfrak{n}^+$ and by $\mathfrak{g}_{-\alpha}$, for $\alpha \in \pi'$. We denote by $\mathfrak{p}^-_{\pi'}$ the "opposite parabolic subalgebra" generated by $\mathfrak{b}^- = \mathfrak{n}^- \oplus \mathfrak{h}$ and by $\mathfrak{g}_\alpha$, for $\alpha \in \pi'$. Set $\mathfrak{l}_{\pi'} = \mathfrak{p}^+_{\pi'} \cap \mathfrak{p}^-_{\pi'}$. Then $\mathfrak{l}_{\pi'}$ is a Levi factor of both parabolic subalgebras $\mathfrak{p}^+_{\pi'}$ and $\mathfrak{p}^-_{\pi'}$ and we can write $\mathfrak{l}_{\pi'} = \mathfrak{n}^+_{\pi'} \oplus \mathfrak{h} \oplus \mathfrak{n}^-_{\pi'}$ where $\mathfrak{n}^\pm_{\pi'} = \mathfrak{n}^\pm \cap \mathfrak{l}_{\pi'}$. Let $\mathfrak{m}^+_{\pi'}$ (respectively $\mathfrak{m}^-_{\pi'}$) be the nilradical of $\mathfrak{p}^+_{\pi'}$ (respectively $\mathfrak{p}^-_{\pi'}$). We denote by $\mathfrak{g}_{\pi'}$ the derived Lie algebra of $\mathfrak{l}_{\pi'}$ and by $\mathfrak{z}(\mathfrak{l}_{\pi'})$ the center of $\mathfrak{l}_{\pi'}$. The Cartan subalgebra $\mathfrak{h} \cap \mathfrak{g}_{\pi'}$ of $\mathfrak{g}_{\pi'}$ will be denoted by $\mathfrak{h}_{\pi'}$.

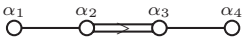
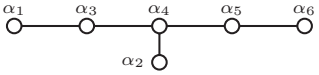
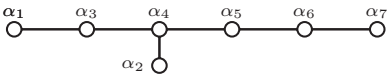
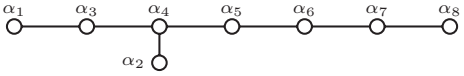
G_2: 	$\{\varepsilon_1 = 23,\ \varepsilon_2 = 01\}$
F_4: 	$\{\varepsilon_1 = 2342,\ \varepsilon_2 = 0122,$ $\varepsilon_3 = 0120,\ \varepsilon_4 = 0100\}$
E_6: 	$\{\varepsilon_1 = \underset{2}{12321},\ \varepsilon_2 = \underset{0}{11111},$ $\varepsilon_3 = \underset{0}{01110},\ \varepsilon_4 = \underset{0}{00100}\}$
E_7: 	$\{\varepsilon_1 = \underset{2}{234321},\ \varepsilon_2 = \underset{1}{012221},$ $\varepsilon_3 = \underset{1}{012100},\ \varepsilon_4 = \underset{0}{000001},$ $\varepsilon_5 = \underset{1}{000000},\ \varepsilon_6 = \underset{0}{010000},$ $\varepsilon_7 = \underset{0}{000100}\}$
E_8: 	$\{\varepsilon_1 = \underset{3}{2465432},\ \varepsilon_2 = \underset{2}{2343210},$ $\varepsilon_3 = \underset{1}{0122210},\ \varepsilon_4 = \underset{1}{0121000},$ $\varepsilon_5 = \underset{0}{0000010},\ \varepsilon_6 = \underset{1}{0000000},$ $\varepsilon_7 = \underset{0}{0100000},\ \varepsilon_8 = \underset{0}{0001000}\}$

Table 1.3. $\mathcal{E}_\pi$ for the exceptional Lie algebras.

DEFINITION 1.3. — *The subalgebra $\mathfrak{q}_{\pi_1,\pi_2}$ of $\mathfrak{g}$ given as follows by the subsets $\pi_1, \pi_2 \subset \pi$*

$$\mathfrak{q}_{\pi_1,\pi_2} := \mathfrak{p}^+_{\pi_1} \cap \mathfrak{p}^-_{\pi_2} = \mathfrak{n}^+_{\pi_2} \oplus \mathfrak{h} \oplus \mathfrak{n}^-_{\pi_1}$$

is called the standard *biparabolic subalgebra (associated to π_1 and π_2). Its nilpotent radical is $\mathfrak{u}_{\pi_1,\pi_2} := (\mathfrak{n}^+_{\pi_2} \cap \mathfrak{m}^+_{\pi_1}) \oplus (\mathfrak{n}^-_{\pi_1} \cap \mathfrak{m}^-_{\pi_2})$ and $\mathfrak{l}_{\pi_1,\pi_2} := \mathfrak{l}_{\pi_1 \cap \pi_2}$ is the standard Levi factor of $\mathfrak{q}_{\pi_1,\pi_2}$.*

Any biparabolic subalgebra is conjugate to a standard one, see [22, §2.3] or [10, §2.5]. So, for our purpose, it will be enough to consider standard biparabolic subalgebras.

Remark 1.4. — The classification of quasi-reductive (bi)parabolic subalgebras of reductive Lie algebras can be deduced from the classification of quasi-reductive (bi)parabolic subalgebras of simple Lie algebras: A stabilizer of a linear form on $\mathfrak{g}$ is the product of its components on each of the simple factors of $\mathfrak{g}$ and of the center of $\mathfrak{g}$. As a consequence, we may assume that $\mathfrak{g}$ is simple without loss of generality.

Let π_1, π_2 be two subsets of π. The dual of $\mathfrak{q}_{\pi_1,\pi_2}$ is identified to $\mathfrak{q}_{\pi_2,\pi_1}$ via the Killing form of $\mathfrak{g}$. For $\underline{a} = (a_K)_{K\in\mathcal{K}_{\pi_2}} \in (\mathbb{C}^*)^{\mathrm{k}_{\pi_2}}$ and $\underline{b} = (b_L)_{L\in\mathcal{K}_{\pi_1}} \in (\mathbb{C}^*)^{\mathrm{k}_{\pi_1}}$, set

$$u(\underline{a},\underline{b}) = \sum_{K\in\mathcal{K}_{\pi_2}} a_K x_{-\varepsilon_K} + \sum_{L\in\mathcal{K}_{\pi_1}} b_L x_{\varepsilon_L}$$

It is an element of $\mathfrak{u}_{\pi_2,\pi_1}$ and the linear form $(\varphi_u)|_{\mathfrak{q}_{\pi_1,\pi_2}}$ is a regular element of $\mathfrak{q}^*_{\pi_1,\pi_2}$ for any $(\underline{a},\underline{b})$ running through a nonempty open subset of $(\mathbb{C}^*)^{\mathrm{k}_{\pi_2}+\mathrm{k}_{\pi_1}}$, cf. [22, Lemma 3.9].

We denote by E_{π_1,π_2} the subspace generated by the elements ε_K, for $K \in \mathcal{K}_{\pi_1} \cup \mathcal{K}_{\pi_2}$. Thus, $\dim E_{\pi_1,\pi_2} = \mathrm{k}_{\pi_1} + \mathrm{k}_{\pi_2} - \dim(E_{\pi_1} \cap E_{\pi_2})$. As it has been proved in [10, §7.16], we have

$$\text{(1.1)}\quad \operatorname{ind}\mathfrak{q}_{\pi_1,\pi_2} = (\operatorname{rk}\mathfrak{g} - \dim E_{\pi_1,\pi_2}) + (\mathrm{k}_{\pi_1} + \mathrm{k}_{\pi_2} - \dim E_{\pi_1,\pi_2})$$

Remark 1.5. — By (1.1), the index of $\mathfrak{q}_{\pi_1,\pi_2}$ is zero if and only if $E_{\pi_1} \cap E_{\pi_2} = \{0\}$ and $\mathrm{k}_{\pi_1} + \mathrm{k}_{\pi_2} = \operatorname{rk}\mathfrak{g}$. For example, in type E_6, there are exactly fourteen standard parabolic subalgebras $\mathfrak{p}^+_{\pi'}$ with index zero. The corresponding subsets $\pi' \subset \pi$ of the simple roots are the following:

$$\{\alpha_1,\alpha_5\};\{\alpha_3,\alpha_6\};\{\alpha_1,\alpha_4,\alpha_5\};\{\alpha_3,\alpha_4,\alpha_6\};\{\alpha_1,\alpha_5,\alpha_6\};$$
$$\{\alpha_1,\alpha_3,\alpha_6\};\{\alpha_1,\alpha_3,\alpha_5\};\{\alpha_3,\alpha_5,\alpha_6\};\{\alpha_1,\alpha_3,\alpha_4\};\{\alpha_4,\alpha_5,\alpha_6\};$$
$$\{\alpha_1,\alpha_3,\alpha_4,\alpha_5\};\{\alpha_3,\alpha_4,\alpha_5,\alpha_6\};\{\alpha_1,\alpha_2,\alpha_3,\alpha_4\};\{\alpha_2,\alpha_4,\alpha_5,\alpha_6\}.$$

This was already observed in the unpublished work [7] of A. Elashvili (with a small error).

In the sequel, we will often make use of the following element of $\mathfrak{u}_{\pi_2,\pi_1}$ on our way to construct reductive forms:

$$u^-_{\pi_1,\pi_2} \quad = \sum_{\varepsilon\in\mathcal{E}_{\pi_2},\ \varepsilon\notin\Delta^+_{\pi_1}} x_{-\varepsilon}$$

If $\pi_2 = \pi$, we simply write $u^-_{\pi_1}$ for $u^-_{\pi_1,\pi}$ and, in the special case of $\pi_1 = \emptyset$ and $\pi_2 = \pi$, we write u^- for $u^-_{\emptyset}$. Let B be the Borel subgroup of G whose Lie algebra is $\mathfrak{b}^+$. We summarize in the following proposition useful results of Kostant concerning the linear form $(\varphi_{u^-})|_{\mathfrak{b}^+}$. They can be found in [21, Proposition 40.6.3].

PROPOSITION 1.6. — (i) *The linear form* $(\varphi_{u^-})|_{\mathfrak{b}^+}$ *is of reductive type for* $\mathfrak{b}^+$. *More precisely, the stabilizer of* φ_{u^-} *in* $\mathfrak{b}^+$ *is the subspace* $\bigcap_{K\in\mathcal{K}_\pi} \ker \varepsilon_K$ *of* $\mathfrak{h}$ *of dimension* $\mathrm{rk}\,\mathfrak{g} - \mathrm{k}_\pi$.

(ii) *Let* $\mathfrak{m}$ *be an ideal of* $\mathfrak{b}^+$ *contained in* $\mathfrak{n}^+$. *The* B*-orbit of* $(\varphi_{u^-})|_{\mathfrak{m}}$ *in* $\mathfrak{m}^*$ *is an open dense subset of* $\mathfrak{m}^*$.

1.4. We end the section by reviewing what is known in the classical case. First recall that the biparabolic subalgebras of simple Lie algebras of type A and C are always quasi-reductive as has been shown by D. Panyushev in [19].

The classification of quasi-reductive parabolic subalgebras of the orthogonal Lie algebras is given in the recent work [4] of Duflo, Khalgui and Torrasso. Since we will use this result repeatedly, we state it below.

Let E be a complex vector space of dimension N endowed with a non-degenerate symmetric bilinear form. Denote by $\mathfrak{so}_N$ the Lie algebra of the corresponding orthogonal group. Let $\mathcal{V} = \{\{0\} = V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_s = V\}$ be a flag of isotropic subspaces in E, with $s \geqslant 1$. Its stabilizer in $\mathfrak{so}_N$ is a parabolic subalgebra of $\mathfrak{so}_N$ and any parabolic subalgebra of $\mathfrak{so}_N$ is obtained in this way. We denote by $\mathfrak{p}_{\mathcal{V}}$ the stabilizer of $\mathcal{V}$ in $\mathfrak{so}_N$.

THEOREM 1.7. — [4] *Let* $\mathcal{V} = \{\{0\} = V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_s = V\}$ *be a flag of isotropic subspaces in* E *with* $s \geqslant 1$. *Denote by* $\mathcal{V}'$ *the flag of isotropic subspaces in* E *which is equal to* $\mathcal{V} \setminus \{V\}$ *if* $\dim V$ *is odd and equal to* $N/2$, *and equal to* $\mathcal{V}$ *otherwise.*

The Lie algebra $\mathfrak{p}_{\mathcal{V}}$ *is quasi-reductive if and only if the sequence* $\mathcal{V}'$ *does not contain two consecutive subspaces of odd dimension.*

Example 1.8. — For $\mathfrak{g} = \mathrm{D}_6$ there are twelve standard parabolic subalgebras $\mathfrak{p} = \mathfrak{p}^+_{\pi'}$ which are not quasi-reductive. The corresponding subsets $\pi' \subset \pi$ of the simple roots are the following:

$$\{\alpha_2\}, \{\alpha_4\}, \{\alpha_1, \alpha_4\}, \{\alpha_2, \alpha_4\}, \{\alpha_2, \alpha_5\}, \{\alpha_2, \alpha_6\},$$
$$\{\alpha_1, \alpha_2, \alpha_4\}, \{\alpha_2, \alpha_3, \alpha_4\}, \{\alpha_2, \alpha_4, \alpha_5\}, \{\alpha_2, \alpha_4, \alpha_6\},$$
$$\{\alpha_2, \alpha_5, \alpha_6\}, \{\alpha_2, \alpha_4, \alpha_5, \alpha_6\}.$$

Among these, the connected π' are $\{\alpha_2\}, \{\alpha_4\}, \{\alpha_2, \alpha_3, \alpha_4\}$.

Thus it remains to determine the quasi-reductive parabolic subalgebras of the exceptional Lie algebras. This is our goal.

2. Methods of reduction

In this section, we develop methods of reduction to deduce the quasi-reductivity of a parabolic subalgebra from the quasi-reductivity of other subalgebras. We assume that $\pi_2 = \pi$. Nevertheless we keep the notations of biparabolic subalgebras where it is convenient.

2.1. The following theorem seems to be standard. As there is no proof to our knowledge, we give a short proof here:

THEOREM 2.1 (Transitivity). — *Let π'', π' be subsets of π with $\pi'' \subset \pi'$. Suppose that $\mathcal{K}_{\pi'} \subset \mathcal{K}_\pi$. Then, $\mathfrak{q}_{\pi'',\pi}$ is quasi-reductive if and only if $\mathfrak{q}_{\pi'',\pi'}$ is.*

Proof. — Note that the assumption $\mathcal{K}_{\pi'} \subset \mathcal{K}_\pi$ implies $\operatorname{ind} \mathfrak{q}_{\pi'',\pi'} = \operatorname{ind} \mathfrak{q}_{\pi'',\pi} + (\mathrm{k}_\pi - \mathrm{k}_{\pi'})$ by formula (1.1). Since $\mathfrak{u}_{\pi',\pi}$ is an ideal of $\mathfrak{b}^+$ contained in $\mathfrak{n}^+$, Proposition 1.6(ii) enables to choose w' in $\mathfrak{l}_{\pi'}$ such that both $(\varphi_{w'+u^-_{\pi'}})|_{\mathfrak{q}_{\pi'',\pi}}$ and $(\varphi_{w'})|_{\mathfrak{q}_{\pi'',\pi'}}$ are regular linear forms of $\mathfrak{q}_{\pi'',\pi}$ and $\mathfrak{q}_{\pi'',\pi'}$ respectively. Then one can show that $\mathfrak{q}_{\pi'',\pi'}(\varphi_{w'}) = \mathfrak{q}_{\pi'',\pi}(\varphi_{w'+u^-_{\pi'}}) \oplus \sum_{K \in \mathcal{K}_\pi \setminus \mathcal{K}_{\pi'}} \mathbb{C} h_{\varepsilon_K}$. By Proposition 1.2, if $\mathfrak{q}_{\pi'',\pi}$ (respectively $\mathfrak{q}_{\pi'',\pi'}$) is quasi-reductive, then we can assume furthermore that $(\varphi_{w'+u^-_{\pi'}})|_{\mathfrak{q}_{\pi'',\pi}}$ (respectively $(\varphi_{w'})|_{\mathfrak{q}_{\pi'',\pi'}}$) has reductive type. Hence the equivalence of the theorem follows. □

Suppose that $\mathfrak{g}$ is simple and let $\widetilde{\pi}$ be the subset of π defined by $\mathcal{K}_\pi = \{\pi\} \cup \mathcal{K}_{\widetilde{\pi}}$. If $\mathfrak{g}$ is of exceptional type, $\pi \setminus \widetilde{\pi}$ only consists of one simple root which we denote by α_π. Note that α_π is the simple root which is connected to the lowest root in the extended Dynkin diagram.

As a consequence of Theorem 2.1, to describe all the quasi-reductive parabolic subalgebras of $\mathfrak{g}$, for $\mathfrak{g}$ of exceptional type, it suffices to consider the case of parabolic subalgebras $\mathfrak{p}^+_{\pi'}$ with $\alpha_\pi \in \pi'$. This will be an important reduction in the sequel.

Remark 2.2. — If $\mathfrak{g}$ has type F_4 (resp. E_6, E_7, E_8), then $\mathfrak{g}_{\widetilde{\pi}}$ has type C_3 (resp. A_5, D_6, E_7). In particular, if $\mathfrak{g}$ has type F_4 or E_6, then $\mathfrak{p}^+_{\pi'}$ is quasi-reductive for any π' which does not contain α_π because in types A and C all (bi)parabolic subalgebras are quasi-reductive.

2.2. As a next step we now focus on a property that we call "additivity" to relate the quasi-reductivity of different parabolic subalgebras (cf. Theorem 2.11). Throughout this paragraph, $\mathfrak{g}$ is assumed to be simple.

DEFINITION 2.3. — *Let π', π'' be subsets of π. We say that π' is* not connected to π'' *if α' is orthogonal to α'', for all (α', α'') in $\pi' \times \pi''$.*

Notation 2.4. — For a positive root α, we denote by $K_\pi^+(\alpha)$ the only element L of $\mathcal{K}_\pi$ such that $\alpha \in \Gamma_L$. Note that unless $\alpha \in \mathcal{E}_\pi$, $K_\pi^+(\alpha)$ is the only element L of $\mathcal{K}_\pi$ for which $\varepsilon_L - \alpha$ is a positive root. For $K \in \mathcal{K}_\pi$, we have $K_\pi^+(\varepsilon_K) = K$.

Remark 2.5. — It can be checked that $K_\pi^+(\alpha) = K_\pi^+(\beta)$ for α, β simple if and only if α and β are in the same orbit of $-w_0$ where w_0 is the longest element of the Weyl group of $\mathfrak{g}$. This suggests that w_0 should play a role in these questions, as may be guessed from a result of Kostant which says that $\mathcal{E}_\pi$ is a basis of the space of fixed points of $-w_0$ and from work of Joseph and collaborators ([10, 11]).

DEFINITION 2.6. — *We shall say that two subsets π', π'' which are not connected to each other* satisfy the condition $(*)$ *if:*

$$(*) \qquad K_\pi^+(\alpha') \neq K_\pi^+(\alpha'') \quad \forall\ (\alpha', \alpha'') \in \pi' \times \pi''\,.$$

Note that if $\mathrm{k}_\pi = \mathrm{rk}\,\mathfrak{g}$ (that is if $-w_0$ acts trivially on π), the condition $(*)$ is always satisfied. Moreover, by using Table 1.3, a case-by-case discussion shows:

LEMMA 2.7. — *Assume that $\mathfrak{g}$ is simple of exceptional type and let π' be a connected subset of π containing α_π. Then, for any subset π'' of π which is not connected to π', the two subsets π', π'' satisfy the condition $(*)$, unless $\mathfrak{g} = \mathrm{E}_6$, $\pi' = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ and $\pi'' = \{\alpha_6\}$ or by symmetry $\pi' = \{\alpha_2, \alpha_4, \alpha_5, \alpha_6\}$ and $\pi'' = \{\alpha_1\}$.*

Remark 2.8. — If $\mathfrak{g} = \mathrm{E}_6$, with $\pi' = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ and $\pi'' = \{\alpha_6\}$, then $K_\pi^+(\alpha_1) = K_\pi^+(\alpha_6) = \{\{\alpha_1, \alpha_3, \alpha_4, \alpha_5, \alpha_6\}\}$, so π' and π'' do not satisfy the condition $(*)$. As a matter of fact, the parabolic subalgebra $\mathfrak{p}^+_{\pi' \cup \pi''}$ will appear as a very special case (see Remark 2.12).

Let π', π'' be two subsets of π which are not connected to each other and assume that π', π'' satisfy condition $(*)$. By Proposition 1.6(ii), we can let w' be in $\mathfrak{l}_{\pi'}$ such that $(\varphi_w)|_{\mathfrak{p}^+_{\pi'}}$ is regular where $w = w' + u^-_{\pi'}$. Denote by $\mathfrak{s}'$ be the image of $\mathfrak{p}^+_{\pi'}(\varphi_w)$ by the projection map from $\mathfrak{p}^+_{\pi'}$ to its derived Lie algebra $\mathfrak{g}_{\pi'} \oplus \mathfrak{m}^+_{\pi'}$ with respect to the decomposition $\mathfrak{p}^+_{\pi'} = \mathfrak{z}(\mathfrak{l}_{\pi'}) \oplus \mathfrak{g}_{\pi'} \oplus \mathfrak{m}^+_{\pi'}$. Let $\mathfrak{k}'$ be the intersection of $\mathfrak{z}(\mathfrak{l}_{\pi'})$ with $\displaystyle\bigcap_{\varepsilon \in \mathcal{E}_\pi,\, \varepsilon \notin \Delta^+_{\pi'}} \ker \varepsilon$.

LEMMA 2.9. — (i) $\mathrm{ind}\,\mathfrak{p}^+_{\pi'} = \dim \mathfrak{s}' + \dim \mathfrak{k}'$.
(ii) $[\mathfrak{s}', \mathfrak{p}^+_{\pi' \cup \pi''}] \subset \mathfrak{p}^+_{\pi'}$ *and* $\varphi_w([\mathfrak{s}', \mathfrak{p}^+_{\pi' \cup \pi''}]) = \{0\}$.

Proof. — (i) We have $\dim \mathfrak{p}^+_{\pi'}(\varphi_w) = \operatorname{ind} \mathfrak{p}^+_{\pi'}$. Since the image of $\mathfrak{p}^+_{\pi'}(\varphi_w)$ by the projection map from $\mathfrak{p}^+_{\pi'}$ to $\mathfrak{g}_{\pi'} \oplus \mathfrak{m}^+_{\pi'}$ is $\mathfrak{s}'$, it suffices to observe that the intersection of $\mathfrak{z}(\mathfrak{l}_{\pi'})$ with $\mathfrak{p}^+_{\pi'}(\varphi_w)$ is $\mathfrak{k}'$. And this follows from the choice of w.

(ii) Let x be an element of $\mathfrak{p}^+_{\pi'}(\varphi_w)$; write $x = x_0 + x' + x^+$ with $x_0 \in \mathfrak{z}(\mathfrak{l}_{\pi'})$, $x' \in \mathfrak{g}_{\pi'}$ and $x^+ \in \mathfrak{m}^+_{\pi'}$. Since $[x^+, w']$ lies in $\mathfrak{m}^+_{\pi'}$, the fact that $x \in \mathfrak{p}^+_{\pi'}(\varphi_w)$ means $[x_0, u^-_{\pi'}] + [x', w'] + [x', u^-_{\pi'}] + [x^+, u^-_{\pi'}] \in \mathfrak{m}^+_{\pi'}$. First, we have to show $[x' + x^+, \mathfrak{p}^+_{\pi'\cup\pi''}] \subset \mathfrak{p}^+_{\pi'}$. As $[x', \mathfrak{p}^+_{\pi'\cup\pi''}] \subset \mathfrak{p}^+_{\pi'}$ since π'', π' are not connected, it suffices to prove that $x^+ \in \mathfrak{m}^+_{\pi'\cup\pi''}$. If not, there are $\gamma \in \Delta^+_{\pi''}$, $K \in \mathcal{K}_\pi$, and $\alpha' \in \Delta^+_{\pi'}$ such that

$$\gamma - \varepsilon_{K^+_\pi(\gamma)} = -(\alpha' + \varepsilon_K) \text{ , i.e. } \varepsilon_{K^+_\pi(\gamma)} = \gamma + (\alpha' + \varepsilon_K)$$

Hence $\gamma, \alpha' \in \Gamma^0_{K^+_\pi(\gamma)}$ that is $K^+_\pi(\alpha') = K^+_\pi(\gamma)$. But this contradicts condition $(*)$. Thus $[x' + x^+, \mathfrak{p}^+_{\pi'\cup\pi''}] \subset \mathfrak{p}^+_{\pi'}$.

It remains to show: $\varphi_w([x' + x^+, \mathfrak{p}^+_{\pi'\cup\pi''}]) = \{0\}$ that is $[x' + x^+, w] \in \mathfrak{m}^+_{\pi'\cup\pi''}$. If $[x' + x^+, w] \notin \mathfrak{m}^+_{\pi'\cup\pi''}$, there must be $\gamma \in \Delta^+_\pi \setminus \Delta_{\pi'}$, $K \in \mathcal{K}_\pi$, and $\alpha'' \in \Delta^+_{\pi''}$ such that $\gamma - \varepsilon_K = \alpha''$. In particular $\alpha'' \in \Gamma^0_{K^+_\pi(\gamma)}$ that is $K^+_\pi(\alpha'') = K^+_\pi(\gamma)$. On the other hand, $[x, w] \in \mathfrak{m}^+_{\pi'}$ implies that there exist $\alpha' \in \Delta^+_{\pi'}$ and $L \in \mathcal{K}_\pi$, such that

$$\gamma - \varepsilon_{K^+_\pi(\gamma)} = -(\alpha' + \varepsilon_L) \text{ , i.e. } \varepsilon_{K^+_\pi(\gamma)} = \gamma + (\alpha' + \varepsilon_L)$$

As before, we deduce that $\alpha' \in \Gamma^0_{K^+_\pi(\gamma)}$, i.e. $K^+_\pi(\alpha') = K^+_\pi(\gamma) = K^+_\pi(\alpha'')$ and this contradicts condition $(*)$. □

COROLLARY 2.10. — *Let π', π'' be two subsets of π which are not connected to each other and satisfy condition $(*)$. If $\mathfrak{p}^+_{\pi'\cup\pi''}$ is quasi-reductive then $\mathfrak{p}^+_{\pi'}$ and $\mathfrak{p}^+_{\pi''}$ are both quasi-reductive.*

Proof. — Suppose that $\mathfrak{p}^+_{\pi'\cup\pi''}$ is quasi-reductive and that any one of the other two parabolic subalgebras is not quasi-reductive and show that this leads to a contradiction. By assumption we can choose $\varphi \in (\mathfrak{p}^+_{\pi'\cup\pi''})^*$ of reductive type for $\mathfrak{p}^+_{\pi'\cup\pi''}$ such that $\varphi' = \varphi|_{\mathfrak{p}^+_{\pi'}}$ and $\varphi'' = \varphi|_{\mathfrak{p}^+_{\pi''}}$ are $\mathfrak{p}^+_{\pi'}$-regular and $\mathfrak{p}^+_{\pi''}$-regular respectively. Suppose for instance that $\mathfrak{p}^+_{\pi'}$ is not quasi-reductive. By Proposition 1.6(ii) we can suppose furthermore that $\varphi' = (\varphi_w)|_{\mathfrak{p}^+_{\pi'}}$ for some $w = w' + u^-_{\pi'}$ with $w' \in \mathfrak{l}_{\pi'}$.

Since we assumed that $\mathfrak{p}^+_{\pi'}$ is not quasi-reductive, $(\varphi_w)|_{\mathfrak{p}^+_{\pi'}}$ contains a nonzero nilpotent element, x, which is so contained in the derived Lie algebra of $\mathfrak{p}^+_{\pi'}$. Then, Lemma 2.9(ii) gives $[x, \mathfrak{p}^+_{\pi'\cup\pi''}] \subset \mathfrak{p}^+_{\pi'}$ and $\{0\} = \varphi_w([x, \mathfrak{p}^+_{\pi'\cup\pi''}]) = \varphi'([x, \mathfrak{p}^+_{\pi'\cup\pi''}]) = \varphi([x, \mathfrak{p}^+_{\pi'\cup\pi''}])$. As a consequence, $\mathfrak{p}^+_{\pi'\cup\pi''}(\varphi)$ contains the nonzero nilpotent element x. This contradicts the

choice of φ. The same line of arguments works if we assume that $\mathfrak{p}^+_{\pi''}$ is not quasi-reductive. □

Under certain conditions, the converse of Corollary 2.10 is also true as we show now. To begin with, let us express the index of $\mathfrak{p}^+_{\pi'\cup\pi''}$ in terms of those of $\mathfrak{p}^+_{\pi'}$ and $\mathfrak{p}^+_{\pi''}$. As $E_{\pi'\cup\pi'',\pi} = E_{\pi',\pi} + E_{\pi',\pi}$, we get: $\dim E_{\pi'\cup\pi'',\pi} = \dim E_{\pi',\pi} + \dim E_{\pi'',\pi} - \dim(E_{\pi',\pi} \cap E_{\pi'',\pi})$. Hence, formula (1.1) implies

$$\operatorname{ind}\mathfrak{p}^+_{\pi'\cup\pi''} = \operatorname{ind}\mathfrak{p}^+_{\pi'} + \operatorname{ind}\mathfrak{p}^+_{\pi''} - (\operatorname{rk}\mathfrak{g} + \mathrm{k}_\pi - 2\dim(E_{\pi',\pi} \cap E_{\pi'',\pi})) \ . \tag{2.11}$$

In case $\operatorname{rk}\mathfrak{g} = \mathrm{k}_\pi$, the intersection $E_{\pi',\pi} \cap E_{\pi'',\pi}$ is equal to E_π and has dimension $\operatorname{rk}\mathfrak{g}$. Hence, the index is additive in that case, as (2.1) shows.

THEOREM 2.11 (Additivity). — *Assume that $\mathfrak{g}$ is simple and of exceptional type and that $\mathrm{k}_\pi = \operatorname{rk}\mathfrak{g}$. Let π', π'' be two subsets of π which are not connected to each other. Then, $\mathfrak{p}^+_{\pi'\cup\pi''}$ is quasi-reductive if and only if both $\mathfrak{p}^+_{\pi'}$ and $\mathfrak{p}^+_{\pi''}$ are quasi-reductive.*

Remark 2.12. — The conclusions of Theorem 2.11 is valid for classical simple Lie algebras, even without the hypothesis $\mathrm{k}_\pi = \operatorname{rk}\mathfrak{g}$. In types A or C this follows from the fact that all biparabolic subalgebras are quasi-reductive. If $\mathfrak{g}$ is an orthogonal Lie algebra, this is a consequence of Theorem 1.7. However, for the exceptional Lie simple algebra E_6, the only one for which $\mathrm{k}_\pi \neq \operatorname{rk}\mathfrak{g}$, the conclusions of Theorem 2.11 may fail. Indeed, let us consider the following subsets of π for $\mathfrak{g}$ of type E_6: $\pi' = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ and $\pi'' = \{\alpha_6\}$.. By Remark 1.5, $\mathfrak{p}^+_{\pi'}$ is quasi-reductive as a Lie algebra of zero index. On the other hand, $\mathfrak{p}^+_{\pi''}$ is quasi-reductive by the transitivity property, cf. Remark 2.2. But, it will be shown in Theorem 4.6 that $\mathfrak{p}^+_{\pi'\cup\pi''}$ is not quasi-reductive.

As a consequence of Lemma 2.7 and Corollary 2.10, even in type E_6 where $\operatorname{rk}\mathfrak{g} \neq \mathrm{k}_\pi$, if $\mathfrak{p}^+_{\pi'\cup\pi''}$ is quasi-reductive, then $\mathfrak{p}^+_{\pi'}$ and $\mathfrak{p}^+_{\pi''}$ are both quasi-reductive.

As a by-product of our classification, we will see that the above situation is the only case which prevents the additivity property to be true for all simple Lie algebras (see Remark 5.3).

Proof. — We argue by induction on the rank of $\mathfrak{g}$. By the transitivity property (Theorem 2.1), Remark 2.12 and the induction, we can assume that $\alpha_\pi \in \pi'$. Then, by Lemma 2.7 and Corollary 2.10, only remains to prove that if both $\mathfrak{p}^+_{\pi'}$ and $\mathfrak{p}^+_{\pi''}$ are quasi-reductive, then so is $\mathfrak{p}^+_{\pi'\cup\pi''}$.

Assume that both $\mathfrak{p}^+_{\pi'}$ and $\mathfrak{p}^+_{\pi''}$ are quasi-reductive. By Proposition 1.2, we can find a linear regular form φ in $(\mathfrak{p}^+_{\pi'\cup\pi''})^*$ such that $\varphi' = \varphi_{|_{\mathfrak{p}^+_{\pi'}}}$ and

$\varphi'' = \varphi_{|_{\mathfrak{p}^+_{\pi''}}}$ are regular and of reductive type for $\mathfrak{p}^+_{\pi'}$ and $\mathfrak{p}^+_{\pi''}$ respectively. By Proposition 1.6(ii), we can assume that $\varphi = (\varphi_{w+u^-})|_{\mathfrak{p}^+_{\pi'\cup\pi''}}$, where $w = h+w'+w''$, with $w' \in \mathfrak{n}^+_{\pi'}$, $w'' \in \mathfrak{n}^+_{\pi''}$ and $h \in \mathfrak{h}$. Hence, $\varphi' = (\varphi_{h+w'+u^-})|_{\mathfrak{p}^+_{\pi'}}$ and $\varphi'' = (\varphi_{h+w''+u^-})|_{\mathfrak{p}^+_{\pi''}}$.

Use the notations of Lemma 2.9. By Lemma 2.9(ii), $\mathfrak{s}'$ is contained in $\mathfrak{p}^+_{\pi'\cup\pi''}(\varphi)$. Show now that $\mathfrak{k}'$ is zero. Let h be an element $\mathfrak{k}'$. Since $h \in \mathfrak{k}'$, we have $\varepsilon(h) = 0$ for any $\varepsilon \in \mathcal{E}_\pi$ which is not in $\Delta^+_{\pi'}$. On the other hand, for any $\varepsilon \in \mathcal{E}_\pi \cap \Delta^+_{\pi'}$, we have $\varepsilon(h) = 0$ since h lies in the center of $\mathfrak{l}_{\pi'}$. Hence, our assumption $\operatorname{rk}\mathfrak{g} = \mathrm{k}_\pi$ implies $h = 0$. As a consequence of Lemma 2.9(i), we deduce that $\operatorname{ind}\mathfrak{p}^+_{\pi'} = \dim\mathfrak{s}'$. Similarly, if $\mathfrak{s}''$ denotes the image of $\mathfrak{p}^+_{\pi''}(\varphi'')$ under the projection from $\mathfrak{p}^+_{\pi''}$ to $\mathfrak{g}_{\pi''} \oplus \mathfrak{m}^+_{\pi''}$, Lemma 2.9(ii) tells us that $\mathfrak{s}''$ is contained in $\mathfrak{p}^+_{\pi'\cup\pi''}(\varphi)$ and that $\operatorname{ind}\mathfrak{p}^+_{\pi''} = \dim\mathfrak{s}''$.

To summarize, our discussion shows that $\mathfrak{s}'+\mathfrak{s}''$ is contained in $\mathfrak{p}^+_{\pi'\cup\pi''}(\varphi)$ and that these two subspaces have the same dimension by equation (2.1). So $\mathfrak{s}'+\mathfrak{s}'' = \mathfrak{p}^+_{\pi'\cup\pi''}(\varphi)$. But by assumption, $\mathfrak{s}'+\mathfrak{s}''$ only consists of semisimple elements. From that we deduce that φ is of reductive type for $\mathfrak{p}^+_{\pi'\cup\pi''}$, whence the theorem. □

3. Some classes of quasi-reductive biparabolic subalgebras

In this section we show that, under certain conditions on the interlacement of the two cascades of π_1 and π_2, we can deduce that $\mathfrak{q}_{\pi_1,\pi_2}$ is quasi-reductive (Theorem 3.6). We assume in this section that $\mathfrak{g}$ is simple.

3.1. We start by introducing the necessary notations. Recall that for a positive root α, $K^+_\pi(\alpha)$ stands for the only element L of $\mathcal{K}_\pi$ such that $\alpha \in \Gamma_L$, cf. Notation 2.4. To any positive root $\alpha \in \Delta^+_\pi$ we now associate the subset $\mathcal{K}^-_\pi(\alpha)$ of the cascade $\mathcal{K}_\pi$ of all L such that the highest root ε_L can be added to α:

$$\mathcal{K}^-_\pi(\alpha) \quad = \quad \{L \in \mathcal{K}_\pi \,|\, \varepsilon_L + \alpha \in \Delta^+_\pi\}\,.$$

Observe that the set $\mathcal{K}^-_\pi(\alpha)$ may be empty or contain more than one element.

Examples 3.1. — (1) If K is in the cascade $\mathcal{K}_\pi$ then $\mathcal{K}^-_\pi(\varepsilon_K)$ is empty.

(2) In type E_7, for $\alpha = \alpha_4 + \alpha_5 + \alpha_6$, the set $\mathcal{K}^-_\pi(\alpha)$ has more than one element: $\varepsilon_4 + \alpha, \varepsilon_5 + \alpha, \varepsilon_6 + \alpha$ are all positive roots.

We need also the following notation:

$$\widetilde{\Delta}^+_\pi = \{\alpha \in \Delta^+_\pi,\, \alpha = \frac{1}{2}(\varepsilon_K - \varepsilon_{K'})\,;\, K, K' \in \mathcal{K}_\pi\}\,.$$

Remark 3.2. — One can check that for $\mathfrak{g}$ a simply-laced simple Lie algebra, no positive root can be written in the way as asked for in the definition of $\widetilde{\Delta}^+_\pi$. Thus $\widetilde{\Delta}^+_\pi$ is empty if $\mathfrak{g}$ is simple of type A, D or E.

We list the sets $\widetilde{\Delta}^+_\pi$ in Table 3.1 for the simple Lie algebras of types B_ℓ, C_ℓ, G_2 and F_4.

B_ℓ, $\ell \geqslant 2$:	$\{\frac{1}{2}(\varepsilon_{2i} - \varepsilon_{2i-1}),\ i = 1,\ \ldots,\ \lceil \frac{\ell}{2} \rceil\}$
C_ℓ, $\ell \geqslant 3$:	$\{\frac{1}{2}(\varepsilon_i - \varepsilon_{i+k+1}),\ 1 \leqslant i \leqslant \ell - 1,\ 0 \leqslant k \leqslant l - i - 1\}$
G_2:	$\{\ 11\ = \frac{1}{2}(\varepsilon_1 - \varepsilon_2)\}$
F_4:	$\{\ 1110\ = \frac{1}{2}(\varepsilon_1 - \varepsilon_2),\ \ 1111\ = \frac{1}{2}(\varepsilon_1 - \varepsilon_3),\ \ 1121\ = \frac{1}{2}(\varepsilon_1 - \varepsilon_4),$
	$0001\ = \frac{1}{2}(\varepsilon_2 - \varepsilon_3),\ \ 0011\ = \frac{1}{2}(\varepsilon_2 - \varepsilon_4),\ \ 0010\ = \frac{1}{2}(\varepsilon_3 - \varepsilon_4)\}$

Table 3.1. $\widetilde{\Delta}^+_\pi$ for the simple Lie algebras.

Part of the following lemma explains that for a root α in $\widetilde{\Delta}^+_\pi$ we can actually describe the two cascades involved in the expression of α:

LEMMA 3.3. — (i) *Whenever $\alpha \in \widetilde{\Delta}^+_\pi$, then $\mathcal{K}^-_\pi(\alpha)$ consists of a unique element $K^-_\pi(\alpha)$.*

(ii) *For any element $\alpha = \frac{1}{2}(\varepsilon_K - \varepsilon_{K'})$ of $\widetilde{\Delta}^+_\pi$ we have $K = K^+_\pi(\alpha)$ and $K' = K^-_\pi(\alpha)$.*

Proof. — One can deduce (i) from Table 3.1.

(ii) By (i), we have $\mathcal{K}^-_\pi(\alpha) = \{K^-_\pi(\alpha)\}$. Furthermore, $< \alpha, \varepsilon^\vee_K >= 1$ so $\varepsilon_K - \alpha$ is a root (cf. [21, Proposition 18.5.3(iii)]). Since $\varepsilon_K - \alpha = \varepsilon_{K'} + \alpha$, these two are both positive roots, forcing $K^+_\pi(\alpha) = K$ and $K^-_\pi(\alpha) = K'$. □

Let π_1 and π_2 be two subsets of π. We define

$$\widetilde{\mathcal{K}}^{(j)}_i = \{M \in \mathcal{K}_{\pi_i} \mid \varepsilon_M \in \widetilde{\Delta}^+_{\pi_j}\}.$$

Thus, for M is in $\widetilde{\mathcal{K}}^{(j)}_i$ we have $\varepsilon_M = \frac{1}{2}(\varepsilon_{K^+_{\pi_j}(\varepsilon_M)} - \varepsilon_{K^-_{\pi_j}(\varepsilon_M)})$ by Lemma 3.3(ii). Note that M is an element of the cascade of π_i while $K^\pm_{\pi_j}(\varepsilon_M)$ belong to the cascade of π_j.

DEFINITION 3.4. — *Let π_1, π_2 be subsets of π. We say that the cascades $\mathcal{K}_{\pi_1}$ and $\mathcal{K}_{\pi_2}$ are* well-interlaced *if* $\dim(E_{\pi_1} \cap E_{\pi_2}) = \#(\mathcal{K}_{\pi_1} \cap \mathcal{K}_{\pi_2}) + \#\widetilde{\mathcal{K}}^{(2)}_1 + \#\widetilde{\mathcal{K}}^{(1)}_2$.

Remark 3.5. — The following subsets π_1, π_2 of π give rise to examples of well-interlaced cascades:

(1) π_1 and π_2 are such that $\mathcal{K}_{\pi_i} \subset \mathcal{K}_{\pi_j}$ or $\mathcal{K}_{\pi_j} \subset \mathcal{K}_{\pi_i}$. In particular, this is the case if π_1 or π_2 is empty.

(2) π_1 and π_2 are such that the collection of all highest roots $\mathcal{E}_{\pi_1} \cup \mathcal{E}_{\pi_2}$ consists of linearly independent elements[(1)].

These two cases have already been studied by Tauvel and Yu in [22].

We are now ready to formulate the main result of this section. It will be proved in Subsection 3.2 below.

THEOREM 3.6. — *Let $\mathfrak{q}_{\pi_1,\pi_2}$ be a biparabolic subalgebra of $\mathfrak{g}$. Assume that the cascades $\mathcal{K}_{\pi_1}$ and $\mathcal{K}_{\pi_2}$ are well-interlaced. Then $\mathfrak{q}_{\pi_1,\pi_2}$ is quasi-reductive.*

More precisely, the linear form $\varphi_{u(\underline{a},\underline{b})}$ is of reductive type for almost all choices of the coefficients $(\underline{a},\underline{b}) \in \mathbb{C}^{\mathrm{k}_{\pi_1}+\mathrm{k}_{\pi_2}}$.

Example 3.7. — Suppose that $\mathfrak{g}$ is simple of type E_6. In the case where $\pi_1 = \{\alpha_2, \alpha_3, \alpha_4\}$ (resp. $\pi_1 = \{\alpha_2, \alpha_3, \alpha_4, \alpha_6\}$, $\pi_1 = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$) and $\pi_2 = \pi$, the union $\mathcal{E}_{\pi_1} \cup \mathcal{E}_{\pi_2}$ consists of linearly independent elements. Hence $\mathfrak{q}_{\pi_1,\pi_2} = \mathfrak{p}^+_{\pi_1}$ is quasi-reductive by Remark 3.5(2) and Theorem 3.6.

We now give an example which is not covered by Remark 3.5:

Example 3.8. — Suppose that $\mathfrak{g}$ is simple of type F_4. The subsets $\pi_1 = \{\alpha_3, \alpha_4\}$ and $\pi_2 = \pi$ are well-interlaced and $\mathfrak{q}_{\pi_1,\pi_2} = \mathfrak{p}^+_{\pi_1}$ is quasi-reductive by Theorem 3.6. Note that Theorem 2.1 provides an alternative way to prove that this parabolic subalgebra is quasi-reductive.

Remark 3.9. — The converse of Theorem 3.6 is not true. For example, we can easily check that the assumption of Theorem 3.6 does not hold for the parabolic subalgebra $\mathfrak{p}^+_{\{\alpha_2,\alpha_4\}}$ of E_6. However, it is quasi-reductive as we will show in Subsection 5.1 (Theorem 5.6).

3.2. This subsection is devoted to the proof of Theorem 3.6. We start with two technical lemmata.

Let $\alpha \in \widetilde{\Delta}^+_\pi$. Recall that by Lemma 3.3(ii), α is written as $\alpha = \frac{1}{2}(\varepsilon_{K^+_\pi(\alpha)} - \varepsilon_{K^-_\pi(\alpha)})$. As an abbreviation we set

$$\overline{\alpha} = \frac{1}{2}(\varepsilon_{K^+_\pi(\alpha)} + \varepsilon_{K^-_\pi(\alpha)})\,.$$

(1) We mean that this collection of roots forms a set of linearly independent roots, neglecting any multiplicities that might occur, cf. Example 3.7 below.

Thus $\alpha+\overline{\alpha}=\varepsilon_{K_\pi^+(\alpha)}$ and $-\alpha+\overline{\alpha}=\varepsilon_{K_\pi^-(\alpha)}$. From the relations between the four roots α, $\overline{\alpha}$, $\varepsilon_{K_\pi^-(\alpha)}$ and $\varepsilon_{K_\pi^+(\alpha)}$ we define the structure constants $\tau_1,\tau_2,\tau_3,\tau_4$ as follows:

$$[x_\alpha, x_{-\varepsilon_{K_\pi^+(\alpha)}}]=\tau_1 x_{-\overline{\alpha}}\ ;\quad [x_{-\alpha}, x_{-\varepsilon_{K_\pi^-(\alpha)}}]=\tau_2 x_{-\overline{\alpha}}\ ;$$
$$[x_\alpha, x_{\overline{\alpha}}]=\tau_3 x_{\varepsilon_{K_\pi^+(\alpha)}}\ ;\quad [x_{-\alpha}, x_{\overline{\alpha}}]=\tau_4 x_{\varepsilon_{K_\pi^-(\alpha)}}\ .$$

LEMMA 3.10. — *Assume that $\mathfrak{g}$ is of type B_ℓ ($\ell\geqslant 2$), C_ℓ ($\ell\geqslant 3$) or F_4. Let α be in $\widetilde{\Delta}_\pi^+$.*

(i) *The only roots of the form $k\alpha+l\overline{\alpha}$ are $\{\pm\alpha,\ \pm\overline{\alpha},\ \pm(\alpha\pm\overline{\alpha})\}$.*

(ii) *We have $\tau_1,\tau_2\in\{-1,1\}$, $\tau_3,\tau_4\in\{-2,2\}$ and $\tau_1\tau_4=\tau_2\tau_3$.*

Proof. — (i) By assumption, the four linear combinations $\pm(\alpha\pm\overline{\alpha})$ are all roots. The claim then follows since root strings have at most length 2 in types B, C and F.

(ii) We explain how to obtain $\tau_1=\pm 1$, the computations of τ_i for $i=2,3,4$ is completely analogous. Consider the α-string through $-\varepsilon_{K_\pi^+(\alpha)}$. It has the form $\{-\varepsilon_{K_\pi^+(\alpha)},-\overline{\alpha},-\varepsilon_{K_\pi^-(\alpha)}\}$, so in particular, $p=0$ in the notation of Subsection 1.2, whence $\tau_1=\pm 1$.

Only remains to proof the equality $\tau_1\tau_4=\tau_2\tau_3$. We compute the bracket $[x_{-\alpha},[x_\alpha,x_{\overline{\alpha}}]]$ in two different ways. We have $[x_{-\alpha},x_{\varepsilon_{K_\pi^+(\alpha)}}]=-\tau_1 x_{\overline{\alpha}}$ (cf. [21, §18.2.2 and Corollary 18.5.5]). Hence

$$[x_{-\alpha},[x_\alpha,x_{\overline{\alpha}}]]=[x_{-\alpha},\tau_3 x_{\varepsilon_{K_\pi^+(\alpha)}}]=-\tau_1\tau_3 x_{\overline{\alpha}}.$$

On the other hand, as $\varepsilon_{K_\pi^+}$ and $\varepsilon_{K_\pi^-}$ have the same length ($\mathfrak{g}$ having type different from G_2), we have: $[h_\alpha,x_{\overline{\alpha}}]=\langle\overline{\alpha},\alpha^\vee\rangle x_{\overline{\alpha}}=0$. So: $[x_{-\alpha},[x_\alpha,x_{\overline{\alpha}}]]=[x_{\overline{\alpha}},[x_\alpha,x_{-\alpha}]]+[x_\alpha,[x_{-\alpha},x_{\overline{\alpha}}]]=-[h_\alpha,x_{\overline{\alpha}}]+\tau_4[x_\alpha,x_{\varepsilon_{K_\pi^-(\alpha)}}]=-\tau_4\tau_2 x_{\overline{\alpha}}$ again by using [21, §18.2.2 and Corollary 18.5.5]. We have so obtained $\tau_1\tau_3=\tau_2\tau_4$. From that the claim follows. □

From now, we let π_1,π_2 be two subsets of π.

LEMMA 3.11. — *Let M be an element of $\widetilde{\mathcal{K}}_i^{(j)}$.*

(i) *$\varepsilon_{K_{\pi_j}^+(\varepsilon_M)}$ and $\varepsilon_{K_{\pi_j}^-(\varepsilon_M)}$ are not roots of π_i.*

(ii) *For $K\in\mathcal{K}_{\pi_j}$, $\varepsilon_M\pm\varepsilon_K$ is a root if and only if $K=K_{\pi_j}^{\mp}(\varepsilon_M)$.*

Proof. — (i) Can be deduced from Tables 1.2, 1.3 and 3.1.

(ii) The fact that $\varepsilon_M-\varepsilon_{K_j^+(\varepsilon_M)}$ and $\varepsilon_M+\varepsilon_{K_j^-(\varepsilon_M)}$ are roots of π_j has been observed in Lemma 3.10(i). Next, by Lemma 3.3(i), we know that $K_{\pi_j}^-(\varepsilon_M)$ is the only element L of $\mathcal{K}_{\pi_j}$ such that $\varepsilon_M+\varepsilon_L$ is a root. Suppose now that there is $L\in\mathcal{K}_{\pi_j}$, $L\neq K_{\pi_j}^+(\varepsilon_M)$, such that $\varepsilon_L-\varepsilon_M$ is a root. By Lemma 3.10(i), we have $L\neq K_{\pi_j}^-(\varepsilon_M)$. So, the fact that $\varepsilon_L-\varepsilon_M$ is a

root forces $\beta = \varepsilon_M - \varepsilon_L$ to be a positive root, by definition of $K^+_{\pi_j}(\varepsilon_M)$. Then the equality $\beta + \varepsilon_L = \varepsilon_M$ implies $\langle \beta, \varepsilon_M^\vee \rangle = 1$. On the other hand, we have $\langle \varepsilon_M, \varepsilon_L^\vee \rangle = \langle \frac{1}{2}(\varepsilon_{K^+_{\pi_j}(\varepsilon_M)} - \varepsilon_{K^-_{\pi_j}(\varepsilon_M)}), \varepsilon_L^\vee \rangle = 0$ since $L \neq K^\pm_{\pi_j}(\varepsilon_M)$. So $\langle \varepsilon_L, \varepsilon_M^\vee \rangle = 0$. As a consequence, $1 = \langle \beta, \varepsilon_M^\vee \rangle = \langle \beta + \varepsilon_L, \varepsilon_M^\vee \rangle = \langle \varepsilon_M, \varepsilon_M^\vee \rangle = 2$. Hence we get a contradiction. □

Recall that for $(\underline{a}, \underline{b}) \in (\mathbb{C}^*)^{\mathrm{k}_{\pi_1} + \mathrm{k}_{\pi_2}}$, we have set

$$u(\underline{a}, \underline{b}) = \sum_{K \in \mathcal{K}_{\pi_2}} a_K x_{-\varepsilon_K} + \sum_{L \in \mathcal{K}_{\pi_1}} b_L x_{\varepsilon_L}$$

Lemma 3.12. — *Let $(\underline{a}, \underline{b})$ be in $(\mathbb{C}^*)^{\mathrm{k}_{\pi_1} + \mathrm{k}_{\pi_2}}$. For $K \in \mathcal{K}_{\pi_i} \cap \mathcal{K}_{\pi_j}$, $M \in \widetilde{\mathcal{K}}_1^{(2)}$ and $N \in \widetilde{\mathcal{K}}_2^{(1)}$, there exist $\rho_K \in \mathbb{C}^*$, $(\lambda_M, \mu_M, \nu_M) \in (\mathbb{C}^*)^3$ and $(\lambda'_M, \mu'_M, \nu'_M) \in (\mathbb{C}^*)^3$ such that the elements y_K, z_M and t_N of $\mathfrak{q}_{\pi_1,\pi_2}$ defined by*

$$y_K = x_{\varepsilon_K} + \rho_K x_{-\varepsilon_K}$$
$$z_M = x_{\varepsilon_M} + \lambda_M x_{-\varepsilon_M} + \mu_M x_{\varepsilon_{K^+_{\pi_2}(\varepsilon_M)}} + \nu_M x_{\varepsilon_{K^-_{\pi_2}(\varepsilon_M)}}$$
$$t_N = x_{-\varepsilon_N} + \lambda'_N x_{\varepsilon_N} + \mu'_N x_{-\varepsilon_{K^+_{\pi_1}(\varepsilon_N)}} + \nu'_N x_{-\varepsilon_{K^-_{\pi_1}(\varepsilon_N)}}$$

are semisimple elements of $\mathfrak{g}$ which stabilize $\varphi_{u(\underline{a},\underline{b})}$ in $\mathfrak{q}_{\pi_1,\pi_2}$.

Proof. — Set $u = u(\underline{a}, \underline{b})$. For $K \in \mathcal{K}_{\pi_i} \cap \mathcal{K}_{\pi_j}$, it is clear that y_K is semisimple. Moreover, for $\rho_K = a_K/b_K$, the element y_K stabilizes $(\varphi_u)|_{\mathfrak{q}_{\pi_1,\pi_2}}$; we even have $[y_K, u] = 0$.

Let now M be in $\widetilde{\mathcal{K}}_1^{(2)}$. If $\mathcal{K}_1^{(2)} \neq \emptyset$ then $\mathfrak{g}$ cannot be of type G_2, since for G_2, $\mathcal{K}_1^{(2)} = \emptyset$ (cf. Table 3.1). So $\mathfrak{g}$ is of type B_ℓ, C_ℓ or F_4 (Remark 5.3). Thus we are in the situation of Lemma 3.10. Let $(\lambda_M, \mu_M, \nu_M)$ be in $(\mathbb{C}^*)^3$. By definition of z_M, we have:

$$\begin{aligned}[z_M, u] = &\sum_{K \in \mathcal{K}_{\pi_2}} a_K([x_{\varepsilon_M}, x_{-\varepsilon_K}] + \lambda_M [x_{-\varepsilon_M}, x_{-\varepsilon_K}]) \\ &+ \mu_M \sum_{L \in \mathcal{K}_{\pi_1}} b_L [x_{\varepsilon_{K^+_{\pi_2}(\varepsilon_M)}}, x_{\varepsilon_L}] + \nu_M \sum_{L \in \mathcal{K}_{\pi_1}} b_L [x_{\varepsilon_{K^-_{\pi_2}(\varepsilon_M)}}, x_{\varepsilon_L}] \\ &- \lambda_M b_M h_{\varepsilon_M} + \mu_M a_{K^+_{\pi_2}(\varepsilon_M)} h_{\varepsilon_{K^+_{\pi_2}(\varepsilon_M)}} + \nu_M a_{K^-_{\pi_2}(\varepsilon_M)} h_{\varepsilon_{K^-_{\pi_2}(\varepsilon_M)}}\end{aligned}$$

Note that $[x_{\varepsilon_M}, x_{-\varepsilon_K}] \neq 0$ if and only if $K = K^+_{\pi_2}(\varepsilon_M)$ by Lemma 3.3. By Lemma 3.11(i), the element $v = \mu_M \sum_{L \in \mathcal{K}_{\pi_1}} b_L [x_{\varepsilon_{K^+_{\pi_2}(\varepsilon_M)}}, x_{\varepsilon_L}] + \nu_M \sum_{L \in \mathcal{K}_{\pi_1}} b_L [x_{\varepsilon_{K^-_{\pi_2}(\varepsilon_M)}}, x_{\varepsilon_L}]$ lies in $\mathfrak{u}_{\pi_1,\pi_2}$. We set $\overline{\varepsilon_M} = \frac{1}{2}(\varepsilon_{K^+_{\pi_j}(\varepsilon_M)} + \varepsilon_{K^-_{\pi_j}(\varepsilon_M)})$, and define the structure constants $\tau_1, \tau_2, \tau_3, \tau_4$ for $\alpha = \varepsilon_M$ and

$\overline{\alpha} = \overline{\varepsilon_M}$. Then, by Lemma 3.3 we have

$$[z_M, u] = (\tau_1 a_{K^+_{\pi_2}(\varepsilon_M)} + \lambda_M \tau_2 a_{K^-_{\pi_2}(\varepsilon_M)}) x_{-\overline{\varepsilon_M}} + v \\ -\lambda_M b_M h_{\varepsilon_M} + \mu_M a_{K^+_{\pi_2}(\varepsilon_M)} h_{\varepsilon_{K^+_{\pi_2}(\varepsilon_M)}} + \nu_M \, a_{K^-_{\pi_2}(\varepsilon_M)} h_{\varepsilon_{K^-_{\pi_2}(\varepsilon_M)}}$$

By Remark 3.2, the elements of $\mathcal{E}_{\pi_2}$ form a basis of $\mathfrak{h}^*_{\pi_2}$ since $\mathcal{K}_1^{(2)} \neq \emptyset$. So, by Lemma 3.3, we can write $h_{\varepsilon_M} = c^+ h_{\varepsilon_{K^+_{\pi_2}(\varepsilon_M)}} - c^- h_{\varepsilon_{K^-_{\pi_2}(\varepsilon_M)}}$ with $c^+, c^- \in \mathbb{C}^*$. Furthermore, $\varepsilon_{K^+_{\pi_2}(\varepsilon_M)}$ and $\varepsilon_{K^-_{\pi_2}(\varepsilon_M)}$ have the same length (they are both long roots, cf. Table 3.1). So, $c^+ = c^-$ (cf. [21, §18.3.3]). Hence

$$[z_M, u] = (\tau_1 a_{K^+_{\pi_2}(\varepsilon_M)} + \lambda_M \tau_2 a_{K^-_2(\varepsilon_M)}) x_{-\overline{\varepsilon_M}} + v \\ + \ (-c\lambda_M b_M + \mu_M a_{K^+_{\pi_2}(\varepsilon_M)}) h_{\varepsilon_{K^+_{\pi_2}(\varepsilon_M)}} + (c\lambda_M \, b_M \\ + \nu_M a_{K^-_{\pi_2}(\varepsilon_M)}) h_{\varepsilon_{K^-_{\pi_2}(\varepsilon_M)}}$$

As a result, if we take for λ_M, $\lambda_M = -\tau_1 a_{K^+_{\pi_2}(\varepsilon_M)}/(\tau_2 a_{K^-_{\pi_2}(\varepsilon_M)})$ and then for μ_M and μ_M, $\mu_M = c\lambda_M b_M / a_{K^+_{\pi_2}(\varepsilon_M)}$ and $\nu_M = -c\lambda_M b_M / a_{K^-_{\pi_2}(\varepsilon_M)}$ we obtain that $[z_M, u] = v \in \mathfrak{u}_{\pi_1,\pi_2}$, i.e. that z_M stabilizes $(\varphi_u)|_{\mathfrak{q}_{\pi_1,\pi_2}}$. In a similar way, one shows that t_N stabilizes $(\varphi_u)|_{\mathfrak{q}_{\pi_1,\pi_2}}$.

It remains to prove that z_M is semisimple (and that t_N is semisimple but this can be done in a similar way). By Lemma 3.10(i), we have

$$\exp(t \operatorname{ad} x_{\overline{\varepsilon_M}})(x_{\varepsilon_M} + \lambda_M x_{-\varepsilon_M}) \\ = x_{\varepsilon_M} + \lambda_M x_{-\varepsilon_M} + t[x_{\overline{\varepsilon_M}}, x_{\varepsilon_M}] + t\lambda_M [x_{\overline{\varepsilon_M}}, x_{-\varepsilon_M}] \\ = x_{\varepsilon_M} + \lambda_M x_{-\varepsilon_M} - t\tau_3 \, x_{\varepsilon_{K^+_{\pi_2}(\varepsilon_M)}} - t\lambda_M \tau_4 x_{\varepsilon_{K^-_{\pi_2}(\varepsilon_M)}}$$

for any $t \in \mathbb{C}^*$. By Lemma 3.10(ii), we have $\tau_1\tau_4 = \tau_2\tau_3$. Therefore it is possible to choose t so that both equalities $-t\tau_3 = c\lambda_M b_M / a_{K^+_{\pi_2}(\varepsilon_M)}$ $(= \ \mu_M)$ and $-t\tau_4 = -cb_M / a_{K^-_{\pi_2}(\varepsilon_M)}$ $(= \ \nu_M)$ hold, because

$$\lambda_M = -\tau_1 a_{K^+_{\pi_2}(\varepsilon_M)} / (\tau_2 a_{K^-_{\pi_2}(\varepsilon_M)}).$$

With such a t, $\exp(t \operatorname{ad} x_{\overline{\varepsilon_M}})(x_{\varepsilon_M} + \lambda_M x_{-\varepsilon_M}) = z_M$. Hence z_M is semisimple since $x_{\varepsilon_M} + \lambda_M x_{-\varepsilon_M}$ is. □

We can now complete the proof of Theorem 3.6:

Let $(\underline{a}, \underline{b}) \in (\mathbb{C}^*)^{\mathrm{k}_{\pi_1} + \mathrm{k}_{\pi_2}}$ such that $(\varphi_u)|_{\mathfrak{q}_{\pi_1,\pi_2}}$ is $\mathfrak{q}_{\pi_1,\pi_2}$-regular where $u = (\underline{a}, \underline{b})$. The orthogonal of E_{π_1,π_2} in $\mathfrak{h}$ is contained in $\mathfrak{q}_{\pi_1,\pi_2}(\varphi_u)$. Then, by Lemma 3.12, it suffices to prove that the elements y_K, z_M, t_N, for $K \in \mathcal{K}_{\pi_1} \cap \mathcal{K}_{\pi_2}$, $M \in \widetilde{\mathcal{K}}_1^{(2)}$, and $N \in \widetilde{\mathcal{K}}_2^{(1)}$, are linearly independent. Indeed, if

so, the stabilizer of $(\varphi_u)|_{\mathfrak{q}_{\pi_1,\pi_2}}$ in $\mathfrak{q}_{\pi_1,\pi_2}$ contains a (commutative) subalgebra which consists of semisimple elements of $\mathfrak{q}_{\pi_1,\pi_2}$ and of dimension $(\mathrm{rk}\,\mathfrak{g} - \dim E_{\pi_1,\pi_2}) + \#(\mathcal{K}_{\pi_1} \cap \mathcal{K}_{\pi_2}) + \#\widetilde{\mathcal{K}}_1^{(2)} + \#\widetilde{\mathcal{K}}_2^{(1)}$. But the hypothesis of Theorem 3.6 tells us that $\#(\mathcal{K}_{\pi_1} \cap \mathcal{K}_{\pi_2}) + \#\widetilde{\mathcal{K}}_1^{(2)} + \#\widetilde{\mathcal{K}}_2^{(1)} = \dim(E_{\pi_1} \cap E_{\pi_2}) = \mathrm{k}_{\pi_1} + \mathrm{k}_{\pi_2} - \dim E_{\pi_1,\pi_2}$ (cf. Definition 3.4). Hence, by formula (1.1), this subalgebra is the stabilizer of $(\varphi_u)|_{\mathfrak{q}_{\pi_1,\pi_2}}$.

Now, by construction, $M, N, K^+_{\pi_2}(\varepsilon_M)$, $K^-_{\pi_2}(\varepsilon_M)$, $K^-_{\pi_1}(\varepsilon_N)$, $K^+_{\pi_1}(\varepsilon_N)$ do not belong to $\mathcal{K}_{\pi_1} \cap \mathcal{K}_{\pi_2}$. Moreover, $M \in \mathcal{K}_{\pi_1} \setminus \mathcal{K}_{\pi_2}$ and $N \in \mathcal{K}_{\pi_2} \setminus \mathcal{K}_{\pi_1}$, whence the expected statement.

4. Non quasi-reductive parabolic subalgebras

So far, our results (Theorem 3.6) only provide examples of quasi-reductive parabolic subalgebras. It is much trickier to prove that a given Lie algebra is *not* quasi-reductive. Indeed, to prove that a given parabolic subalgebra is quasi-reductive, one can make explicit computations, cf. Section 5. In this section we exhibit examples of non quasi-reductive parabolic subalgebras.

4.1. We first discuss the case of the parabolic subalgebras $\mathfrak{p}^+_{\pi'}$ where π' only consists of one simple root. For $\alpha \in \pi$, denote the parabolic subalgebra $\mathfrak{p}^+_{\{\alpha\}}$ simply by $\mathfrak{p}^+_\alpha$. Thanks to Theorem 4.1 we have a criterion for the quasi-reductivity of $\mathfrak{p}^+_\alpha$:

THEOREM 4.1. — *Let α be in π. Then the parabolic subalgebra $\mathfrak{p}^+_\alpha$ is quasi-reductive if and only if one of the following two conditions holds: $\alpha \in \widetilde{\Delta}^+_\pi$ or $\{\alpha\} \cup \mathcal{E}_\pi$ consists of linearly independent elements.*

If one of the above two conditions are satisfied, then the cascades of $\{\alpha\}$ and of π are well-interlaced; so, it is clear that $\mathfrak{p}^+_\alpha$ is quasi-reductive by Theorem 3.6. Thus, Theorem 4.1 provides a converse to Theorem 3.6 for $\pi_1 = \{\alpha\}$ and $\pi_2 = \pi$.

Proof. — We only need to show that if $\mathfrak{p}^+_\alpha$ is quasi-reductive then α satisfies one of the two conditions of the theorem. Suppose that $\mathfrak{p}^+_\alpha$ is quasi-reductive. If α does not satisfy any of the above conditions, then $\alpha \in E_\pi$, and α is not an element of $\widetilde{\Delta}^+_\pi \sqcup \mathcal{E}_\pi$. By Proposition 1.2, we can find w in $\mathfrak{p}^-_\alpha$ such that $(\varphi_w)|_{\mathfrak{p}^+_\alpha}$ is regular and of reductive type for $\mathfrak{p}^+_\alpha$. Moreover, by Proposition 1.6(ii), since $\alpha \notin \mathcal{E}_\pi$, we can suppose that w is of the form: $w = ax_{-\alpha} + h + bx_\alpha + u^-$ with $a, b \in \mathbb{C}$, $h \in \mathfrak{h}$. Let us remind that the stabilizer of $(\varphi_{u^-})|_{\mathfrak{b}^+}$ in $\mathfrak{b}^+$ is the orthogonal of E_π in $\mathfrak{h}$ (Proposition 1.6(i)). Consequently, as $\alpha \in E_\pi$, we have $[\mathfrak{b}^+(\varphi_{u^-}), w] = \{0\}$, whence

$\mathfrak{b}^+(\varphi_{u^-}) \subset \mathfrak{p}^+_\alpha(\varphi_w)$. In addition, by formula (1.1), $\operatorname{ind}\mathfrak{p}^+_\alpha = \operatorname{ind}\mathfrak{b}^+ + 1$. So, $\mathfrak{b}^+(\varphi_{u^-})$ is an hyperplane of $\mathfrak{p}^+_\alpha(\varphi_w)$ (cf. [22, Lemma 4.5]). Now choose x in $\mathfrak{p}^+_\alpha$ such that the decomposition

$$\mathfrak{p}^+_\alpha(\varphi_w) = \mathbb{C}x \oplus \bigcap_{K\in\mathcal{K}_\pi} \ker \varepsilon_K \tag{4.1}$$

holds. By the choice of w, $\mathfrak{p}^+_\alpha(\varphi_w)$ is an abelian Lie algebra consisting of semisimple elements. In particular x must be semisimple. Write the element x as follows: $x = \lambda x_{-\alpha} + h' + \mu x_\alpha + x^+$ with $\lambda, \mu \in \mathbb{C}$, $h' \in \mathfrak{h}$ and $x^+ \in \mathfrak{m}^+_\alpha$. From the fact $[x, w] \in \mathfrak{m}^+_\alpha$, we deduce that $h' \in \bigcap_{K\in\mathcal{K}_\pi} \ker \varepsilon_K$. So we can assume that $h' = 0$ according to (4.1). Hence $\lambda\mu \neq 0$, since x is semisimple.

Since α is not in $\mathcal{E}_\pi$, $\varepsilon_{K^+_\pi(\alpha)} - \alpha$ is a (positive) root. In turn, suppose that $\varepsilon_K - \alpha$ is a root, for $K \in \mathcal{K}_\pi$.. As α is a simple root, $\varepsilon_K - \alpha$ is necessarily a positive root, so $K = K^+_\pi(\alpha)$. Therefore, we have

$$[x, w] = \lambda \sum_{L\in\mathcal{K}^-_\pi(\alpha)} [x_{-\alpha}, x_{-\varepsilon_L}] + \mu[x_\alpha, x_{-\varepsilon_{K^+_\pi(\alpha)}}] + \lambda\alpha(h)x_{-\alpha} + (a\mu - b\lambda)h_\alpha - \mu\alpha(h)x_\alpha + [x^+, w]$$

As $[x, w] \in \mathfrak{m}^+_\alpha$, the bracket $[x_{-\varepsilon_{K^+_\pi(\alpha)}}, x_\alpha]$ must be compensated. This bracket cannot be compensated by the term $[x^+, w]$. Indeed, if this were the case, then there would exist $K \in \mathcal{K}_\pi$ and $\beta \in \Delta^+_\pi \setminus \{\alpha\}$ such that $\varepsilon_{K^+_\pi(\alpha)} - \alpha = \varepsilon_K - \beta$. But this would force $K = K^+_\pi(\alpha)$ and so $\alpha = \beta$, which is impossible. We deduce that there is $L \in \mathcal{K}^-_\pi(\alpha)$ such that $\varepsilon_{K^+_\pi(\alpha)} - \alpha = \varepsilon_L + \alpha$. Thus, $\alpha = \frac{1}{2}(\varepsilon_{K^+_\pi(\alpha)} - \varepsilon_L)$ that is $\alpha \in \widetilde{\Delta}^+_\pi$ which contradicts our assumption on α. □

According to Theorem 4.1, we list the simple roots α corresponding to a *non* quasi-reductive parabolic subalgebra $\mathfrak{p}^+_\alpha$ (for simple $\mathfrak{g}$) in Table 4.1.

B_ℓ, $\ell \geqslant 3$	D_ℓ, $\ell \geqslant 4$	G_2	F_4	E_6	E_7	E_8
α_i, $2 \leqslant i \leqslant \ell - 1$, i even	α_i, $2 \leqslant i \leqslant \ell - 2$, i even	α_1	α_1	α_2	α_1, α_4, α_6	α_1, α_4, α_6, α_8

Table 4.1. The parabolic subalgebras $\mathfrak{p}^+_\alpha$ which are not quasi-reductive.

Remark 4.2. — In the exceptional case, Table 4.1 shows that there is always at least one non quasi-reductive parabolic subalgebra.

4.2. We now exhibit a few more parabolic subalgebras which are not quasi-reductive (Theorem 4.3 and Theorem 4.6), all in type E.

THEOREM 4.3. — (i) *If $\mathfrak{g}$ is of type E_7 and if π' is one of the subsets $\{\alpha_1,\alpha_3,\alpha_4\}$, $\{\alpha_4,\alpha_5,\alpha_6\}$, or $\{\alpha_1,\alpha_3,\alpha_4,\alpha_5,\alpha_6\}$, then $\mathfrak{p}^+_{\pi'}$ is not quasi-reductive.*

(ii) *If $\mathfrak{g}$ is of type E_8 and if π' is one of the subsets $\{\alpha_1,\alpha_3,\alpha_4\}$, $\{\alpha_4,\alpha_5,\alpha_6\}$, $\{\alpha_6,\alpha_7,\alpha_8\}$, $\{\alpha_1,\alpha_3,\alpha_4,\alpha_5,\alpha_6\}$ $\{\alpha_4,\alpha_5,\alpha_6,\alpha_7,\alpha_8\}$, or $\{\alpha_1,\alpha_3,\alpha_4,\alpha_5,\alpha_6,\alpha_7,\alpha_8\}$, then $\mathfrak{p}^+_{\pi'}$ is not quasi-reductive.*

The indices of the parabolic subalgebras considered in Theorem 4.3 are given in Table 5.1. Note that for $\mathfrak{g}$ of type E_7 or E_8, and $\pi'=\{\alpha_4,\alpha_5,\alpha_6\}$, $\mathfrak{p}^+_{\pi'}$ is not quasi-reductive by Theorem 2.1 and Example 1.8.

In the proof of the theorem and in Lemma 4.4 below, we make use of the following notations: If π' is a connected subset of π, $\widetilde{\pi}'$ is defined to be the connected subset of π' satisfying $\mathcal{K}_{\pi'}=\{\pi'\}\cup\mathcal{K}_{\widetilde{\pi}'}$ and $u^+_{\pi'}$ is the element $\sum_{\varepsilon\in\mathcal{E}_\pi\setminus\Delta^+_{\pi'}} x_\varepsilon$.. Note that the element $u^+_{\pi'}+u^-_{\pi'}$ is a semisimple element of $\mathfrak{g}$. Assume that $\mathfrak{g}$ is of type E_8. Set:

$$\begin{aligned}
&\alpha_{11}=\alpha_3+\alpha_4,\ \alpha_{12}=\alpha_4+\alpha_5,\ \alpha_{13}=\alpha_5+\alpha_6,\ \alpha_{14}=\alpha_6+\alpha_7,\\
&\alpha_{19}=\alpha_3+\alpha_4+\alpha_5,\ \alpha_{20}=\alpha_4+\alpha_5+\alpha_6,\ \alpha_{21}=\alpha_5+\alpha_6+\alpha_7,\\
&\alpha_{27}=\alpha_3+\alpha_4+\alpha_5+\alpha_6,\ \alpha_{28}=\alpha_4+\alpha_5+\alpha_6+\alpha_7,\\
&\alpha_{35}=\alpha_3+\alpha_4+\alpha_5+\alpha_6+\alpha_7
\end{aligned}$$

and denote by $I_{\widetilde{\pi}'}$ the set of integers i such that $\alpha_i\in\Delta_{\widetilde{\pi}'}$. Whenever α_i is defined, x_i and y_i stand for x_{α_i} and $x_{-\alpha_i}$ respectively. Consider the following equations:

$$\begin{array}{llcl@{\qquad}llcl}
\text{(E1)} & \mu_4+\nu_{19} & = & 0 & \text{(G1)} & \mu_{11}-\nu_{12} & = & 0\\
\text{(F1)} & \mu_{19}+\nu_4 & = & 0 & \text{(H1)} & \mu_{12}-\nu_{11} & = & 0\\
\text{(E2)} & -\mu_6+\nu_{21} & = & 0 & \text{(G2)} & \mu_{13}+\nu_{14} & = & 0\\
\text{(F2)} & \mu_{21}-\nu_6 & = & 0 & \text{(H2)} & \mu_{14}+\nu_{13} & = & 0\\
\text{(E3)} & \mu_{20}+\nu_{35} & = & 0 & \text{(G3)} & \mu_{27}-\nu_{28} & = & 0\\
\text{(F3)} & \mu_{35}+\nu_{20} & = & 0 & \text{(H3)} & \mu_{28}-\nu_{27} & = & 0
\end{array}$$

in the variables μ_i and ν_i. Set

$$\begin{aligned}
\pi'_1&=\{\alpha_1,\alpha_3,\alpha_4,\alpha_5,\alpha_6\},\\
\pi'_2&=\{\alpha_4,\alpha_5,\alpha_6,\alpha_7,\alpha_8\}\\
\text{and } \pi'_3&=\{\alpha_1,\alpha_3,\alpha_4,\alpha_5,\alpha_6,\alpha_7,\alpha_8\}.
\end{aligned}$$

We now introduce subspaces $\mathfrak{a}_k$ of $\mathfrak{g}^{\sim}_{\pi'_k}$ (for $k = 1, 2, 3$) as follows:
- for $k = 1, 2$, $\mathfrak{a}_k$ is the space of elements $\sum_{\varepsilon \in \mathcal{E}_\pi \cap \widetilde{\pi'_k}} \lambda_\varepsilon h_\varepsilon + \sum_{i \in I^{\sim}_{\pi'_k}} (\mu_i x_i + \nu_i y_i)$ with $(\lambda_\varepsilon)_{\varepsilon \in \mathcal{E}_\pi \cap \widetilde{\pi'_k}}$ in $\mathbb{C}^{|\mathcal{E}_\pi \cap \widetilde{\pi'_k}|}$ and where $((\mu_i)_{i \in I^{\sim}_{\pi'_k}}, (\nu_j)_{j \in I^{\sim}_{\pi'_k}})$ run through the set of the solutions of the homogeneous linear system defined by the equations (Ek), (Fk), (Gk), (Hk).
- $\mathfrak{a}_3$ is the space of elements $\sum_{\varepsilon \in \mathcal{E}_\pi \cap \widetilde{\pi'_3}} \lambda_\varepsilon h_\varepsilon + \sum_{i \in I^{\sim}_{\pi'_3}} (\mu_i x_i + \nu_i y_i)$ with $(\lambda_\varepsilon)_{\varepsilon \in \mathcal{E}_\pi \cap \widetilde{\pi'_3}}$ in $\mathbb{C}^{|\mathcal{E}_\pi \cap \widetilde{\pi'_3}|}$ and where $((\mu_i)_{i \in I^{\sim}_{\pi'_3}}, (\nu_j)_{j \in I^{\sim}_{\pi'_3}})$ runs through the set of the solutions of the homogeneous linear system defined by all twelve equations.

Here is a technical lemma used in the proof of Theorem 4.3:

LEMMA 4.4. — *Assume that $\mathfrak{g}$ is of type E_8. Then $\mathfrak{a}_k$, for $k = 1, 2, 3$, is the centralizer in $\mathfrak{g}^{\sim}_{\pi'_k}$ of the semisimple element $u^+_{\pi'} + u^-_{\pi'}$. It is a reductive Lie algebra and its rank is at most $\operatorname{ind} \mathfrak{p}^+_{\pi'} - 1$.*

Proof. — Let $k \in \{1, 2, 3\}$. The fact that $\mathfrak{a}_k$ centralizes $u^-_{\pi'}$ can be checked without difficulty. As μ and ν play the same role in the equations (Ek), (Fk), (Gk), (Hk), we deduce that $\mathfrak{a}_k$ centralizes $u^+_{\pi'}$ too; hence $\mathfrak{a}_k$ centralizes $u^+_{\pi'} + u^-_{\pi'}$. Then $\mathfrak{a}_k$ is a reductive Lie algebra as an intersection between a reductive Lie algebra and the centralizer in $\mathfrak{g}$ of a semisimple element of $\mathfrak{g}$.

Next we show: $\operatorname{rk} \mathfrak{a}_k \leqslant \operatorname{ind} \mathfrak{p}^+_{\pi'} - 1$. We can readily verify from the equations defining $\mathfrak{a}_k$ that the center of $\mathfrak{a}_k$ is zero. Therefore, the rank of $\mathfrak{a}_k$ is strictly smaller that the one of $\mathfrak{g}^{\sim}_{\pi'_k}$. Indeed, if not, $\mathfrak{a}_k$ is a Levi subalgebra of $\mathfrak{g}^{\sim}_{\pi'_k}$ since $\mathfrak{g}^{\sim}_{\pi'_k}$ has type A. But any proper Levi subalgebra of $\mathfrak{g}^{\sim}_{\pi'_k}$ has a non trivial center. So, for $k = 1, 2$, we get $\operatorname{rk} \mathfrak{a}_k \leqslant 2$ since $\operatorname{rk} \mathfrak{g}^{\sim}_{\pi'_k} = \operatorname{ind} \mathfrak{p}^+_{\pi'_k} = 3$ whence the statement.

For $k = 3$, what foregoes yields $\operatorname{rk} \mathfrak{a}_3 \leqslant 4$ since the rank of $\mathfrak{g}^{\sim}_{\pi'_3}$ is 5. We have to show: $\operatorname{rk} \mathfrak{a}_3 < \operatorname{ind} \mathfrak{p}^+_{\pi'_3} = 4$. The space $\mathfrak{a}_3$ has dimension 21. But there is no reductive Lie subalgebra of rank 4 and of dimension 21 since $21 - 4$ is not even. As a result, we get $\operatorname{rk} \mathfrak{a}_3 < 4$. □

Here is the proof of Theorem 4.3:

Proof of Theorem 4.3. — By the transitivity property (Theorem 2.1), statement (ii) implies (i). So we only consider the case of E_8. Let π' be one of the subsets as described in (ii). Assume that $\mathfrak{p}^+_{\pi'}$ is quasi-reductive. We

will show that this leads to a contradiction. Choose $w \in \mathfrak{p}^-_{\pi'}$ such that the following two conditions are satisfied:

- $(\varphi_w)|_{\mathfrak{p}^+_{\pi'}}$ is $\mathfrak{p}^+_{\pi'}$-regular and of reductive type for $\mathfrak{p}^+_{\pi'}$;
- $(\varphi_w)|_{\mathfrak{m}^+_{\pi'}}$ belongs to the B-orbit of $(\varphi_{u^-_{\pi'}})|_{\mathfrak{m}^+_{\pi'}}$.

This choice of w is possible by Proposition 1.2 and Proposition 1.6(ii). By the second condition, we can assume that $w = w' + u^-_{\pi'}$ with $w' \in \mathfrak{l}_{\pi'}$. Let x be an element of the stabilizer $(\varphi_w)|_{\mathfrak{p}^+_{\pi'}}$ in $\mathfrak{p}^+_{\pi'}$; we write $x = h + x' + x^+$, with $h \in \mathfrak{h}$, $x' \in \mathfrak{n}^-_{\pi'} \oplus \mathfrak{n}^+_{\pi'}$ and $x^+ \in \mathfrak{m}^+_{\pi'}$. The fact $[x, w] \in \mathfrak{m}^+_{\pi'}$ forces $h \in \bigcap_{\varepsilon \in \mathcal{E}_\pi \setminus \Delta^+_{\pi'}} \ker \varepsilon$. From that, we deduce that h belongs to the subspace of $\mathfrak{h}$ generated by the elements h_ε, for $\varepsilon \in \mathcal{E}_\pi \cap \pi' \subset \widetilde{\pi}'$ (use Table 1.3). Now for $\alpha \in \Gamma_{\pi'}$, one obtains that $\varepsilon_{K^+_\pi(\alpha)} \notin \Delta^+_{\pi'}$ and we claim that x' has zero coefficient in $\mathfrak{g}_\alpha$. Otherwise, there must be $\beta \in \Delta^+_{\pi'}$ and $K \in \mathcal{K}_\pi$ such that $\alpha - \varepsilon_{K^+_\pi(\alpha)} = -(\beta + \varepsilon_K)$. One can check that for each of the subsets π' such an equality is not possible (use Table 1.3). To summarize, we obtain the inclusion:

$$\mathfrak{p}^+_{\pi'}(\varphi_w) \quad \subset \quad \mathfrak{g}^\sim_{\pi'} \oplus \mathfrak{H}^-_{\pi'} \oplus \mathfrak{m}^+_{\pi'}, \tag{4.2}$$

where $\mathfrak{H}^-_{\pi'}$ is the Heisenberg Lie algebra generated by the $\mathfrak{g}_{-\alpha}$, $\alpha \in \Gamma_{\pi'}$. Let $\mathfrak{t}$ be the image of $\mathfrak{p}^+_{\pi'}(\varphi_w)$ by the projection map from $\mathfrak{g}^\sim_{\pi'} \oplus \mathfrak{H}^-_{\pi'} \oplus \mathfrak{m}^+_{\pi'}$ to $\mathfrak{g}^\sim_{\pi'}$. As $\mathfrak{p}^+_{\pi'}(\varphi_w)$ is a torus of $\mathfrak{g}$ by hypothesis, (4.2) shows that $\mathfrak{t}$ is a torus of $\mathfrak{g}^\sim_{\pi'}$ of dimension $\operatorname{ind} \mathfrak{p}^+_{\pi'} = \dim \mathfrak{p}^+_{\pi'}(\varphi_w)$.

For the first three subsets, with π' of rank 3, $\mathfrak{p}^+_{\pi'}$ has index 2 but $\mathfrak{g}^\sim_{\pi'}$ has rank 1. So we get a contradiction.

The remaining cases, with π' of rank 5 or 7, require more work. Let us describe the torus $\mathfrak{t}$. To do that, we consider on one hand the roots $\alpha \in \Delta^+_{\widetilde{\pi}'}$ with $\varepsilon_{K^+_\pi(\alpha)} \notin \Delta^+_{\pi'}$ for which there exist $\beta \in \Delta^+_{\pi'}$ and $K \in \mathcal{K}_\pi$ such that $\alpha - \varepsilon_{K^+_\pi(\alpha)} = -(\beta + \varepsilon_K)$. On the other hand, we consider the roots $\alpha \in \Delta^-_{\widetilde{\pi}'}$ for which there is $\varepsilon \in \mathcal{E}_\pi \setminus \Delta_{\pi'}$ such that $\alpha + \varepsilon$ is a root. All the possible roots give rise to equations describing $\mathfrak{t}$. Let $k \in \{1, 2, 3\}$ and use the notations introduced before Lemma 4.4. The equations what we obtained are precisely the equations (Ek), (Fk), (Gk), (Hk) for $k = 1, 2$, and all twelve equations above for $k = 3$. Thereby $\mathfrak{t}$ is contained in the reductive Lie algebra $\mathfrak{a}_k$. But the torus $\mathfrak{t}$ has dimension $\operatorname{ind} \mathfrak{p}^+_{\pi'}$ and this contradicts Lemma 4.4. □

Remark 4.5. — Proceeding with the proof of Lemma 4.4, one readily obtains that $\mathfrak{a}_k$, for $k = 1, 2, 3$, has precisely dimension $\operatorname{ind} \mathfrak{p}^+_{\pi'} - 1$ (note that $\dim \mathfrak{a}_1 = \dim \mathfrak{a}_2 = 10$ and $\dim \mathfrak{a}_3 = 21$). Then, the proof of Theorem 4.3

shows that the dimension of the torus part of generic stabilizers is $\operatorname{ind}\mathfrak{p}^+_{\pi'}-1$. This dimension is given, for each case, in the last column of Table 5.1.

We end the section with an example of non quasi-reductive parabolic subalgebra in E_6. As noticed in Remark 2.12, Theorem 4.6 shows that the additivity property fails in type E_6:

THEOREM 4.6. — *If $\mathfrak{g}$ if of type E_6 and if $\pi'=\{\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_6\}$, then $\mathfrak{p}^+_{\pi'}$ is not quasi-reductive.*

By symmetry, if $\pi'=\{\alpha_1,\alpha_2,\alpha_4,\alpha_5,\alpha_6\}$, then $\mathfrak{p}^+_{\pi'}$ is not quasi-reductive, either.

Proof. — Choose $w\in\mathfrak{p}^-_{\pi'}$ such that the following two conditions are satisfied:

- $(\varphi_w)|_{\mathfrak{p}^+_{\pi'}}$ is $\mathfrak{p}^+_{\pi'}$-regular;
- $(\varphi_w)|_{\mathfrak{n}^+}$ belongs to the B-orbit of $(\varphi_{u^-})|_{\mathfrak{n}^+}$.

This choice is possible by Proposition 1.2 and Proposition 1.6(ii). By the second condition, we can assume that $w=w'+u^-$ where w' is in $\mathfrak{h}\oplus\mathfrak{n}^+_{\pi'}$. For $x\in\mathfrak{p}^+_{\pi'}$, we write $x=h+x'+x^+$ with $h\in\mathfrak{h}$, $x'\in\mathfrak{n}^-_{\pi'}\oplus\mathfrak{n}^+_{\pi'}$ and $x^+\in\mathfrak{m}^+_{\pi'}$. Set:

$$\begin{aligned}&\alpha_7=\alpha_1+\alpha_3,\ \alpha_8=\alpha_2+\alpha_4,\ \alpha_9=\alpha_3+\alpha_4,\\&\alpha_{12}=\alpha_1+\alpha_3+\alpha_4,\ \alpha_{13}=\alpha_2+\alpha_3+\alpha_4,\\&\alpha_{17}=\alpha_1+\alpha_2+\alpha_3+\alpha_4\end{aligned}$$

and let $I_{\pi'}$ be the set of integers i such that $\alpha_i\in\Delta_{\pi'}$. Then, for $i\in I_{\pi'}$, x_i, y_i and h_i stand for x_{α_i}, $x_{-\alpha_i}$ and h_{α_i} respectively. Write $x'=\sum_{i\in I_{\pi'}}\mu_i x_i+\sum_{i=1}^{6}\lambda_i h_i+\sum_{i\in I_{\pi'}}\nu_i y_i$ and $w'=h_0+\sum_{l\in I_{\pi'}}a_l x_l$ with $h_0\in\mathfrak{h}$ and $(\mu_i,\lambda_j,\nu_k,a_l)_{i,j,k,l}\in\mathbb{C}^{3|I_{\pi'}|+6}$.

From $[x,w]\in\mathfrak{m}^+_{\pi'}$, we first deduce that h belongs to $\ker\varepsilon$ for any $\varepsilon\in\mathcal{E}_\pi\setminus\Delta^+_{\pi'}$ whence we get $\lambda_1=-\lambda_6$ and $\lambda_3=-\lambda_5$. Next, we argue as at the end of Theorem 4.3(ii): we use the roots $\alpha\in\Delta^+_{\pi'}$ such that $\varepsilon_{K^+_\pi(\alpha)}\notin\Delta^+_{\pi'}$ and for which there exist $\beta\in\Delta^+_{\pi'}$ and $K\in\mathcal{K}_\pi$ such that $\alpha-\varepsilon_{K^+_\pi(\alpha)}=-(\beta+\varepsilon_K)$. This enables us to show that $\mu_i=0$ for any $i\in I_{\pi'}\setminus\{1,4,6\}$ and that $\nu_6=\mu_1$, $\mu_6=\nu_1$. Now, we consider the terms in x_α for $\alpha\in\Delta_{\pi'}$ and in h_α for $\alpha\in\pi'$ of $[x,w]$. All these terms have to be zero; this gives us equations. Some of them involve the terms in x_α for certain $\alpha\in\Delta^+_\pi\setminus\Delta^+_{\pi'}$ but we can eliminate these variables and obtain equations whose variables are only the $(\lambda_i)_i$'s, $(\mu_j)_j$'s, and $(\nu_k)_k$'s, for $i=1,3,4$, $j=1,4$ and $k\in I_{\pi'}$. Here are these equations:

$$
\begin{aligned}
&(\mathrm{X1}) \quad 2a_1\lambda_1 - a_1\lambda_3 + (\alpha_6-\alpha_6)(h_0)\mu_1 + a_7\nu_3 + a_{12}\nu_9 + a_{17}\nu_{13} &&= 0\\
&(\mathrm{X2}) \quad -a_2\lambda_4 + a_8\nu_4 + 2\nu_8 + a_{13}\nu_9 + a_{17}\nu_{12} &&= 0\\
&(\mathrm{X3}) \quad -a_3\lambda_1 + 2a_3\lambda_3 - a_3\lambda_4 - a_7\nu_1 + a_9\nu_4 + a_{13}\nu_8 &&= 0\\
&(\mathrm{X4}) \quad 2a_4\lambda_4 - \alpha_4(h_0)\mu_4 - a_8\nu_2 - a_9\nu_3 - a_{12}\nu_7 &&= 0\\
&(\mathrm{X5}) \quad -2\lambda_4 + \alpha_4(h_0)\nu_4 - a_2\nu_8 - a_3\nu_9 - a_7\nu_{12} &&= 0\\
&(\mathrm{X6}) \quad -2a_6\lambda_1 + a_6\lambda_3 + (\alpha_1-\alpha_6)(h_0)\nu_1 + a_3\nu_7 + a_9\nu_{12} + a_{13}\nu_{17} &&= 0\\
&(\mathrm{X7}) \quad a_7\lambda_1 + a_7\lambda_3 - a_7\lambda_4 - a_3\mu_1 + a_{12}\nu_4 + a_{17}\nu_8 &&= 0\\
&(\mathrm{X8}) \quad a_8\lambda_4 + a_2\mu_4 - 2\nu_2 - a_{13}\nu_3 - a_{17}\nu_7 &&= 0\\
&(\mathrm{X9}) \quad -a_9\lambda_1 + 2a_9\lambda_3 + a_9\lambda_4 + a_3\mu_4 - a_{12}\nu_1 - a_{13}\nu_2 &&= 0\\
&(\mathrm{X12}) \quad a_{12}\lambda_1 + a_{12}\lambda_3 + a_{12}\lambda_4 - a_9\mu_1 + a_7\mu_4 - a_{17}\nu_2 &&= 0\\
&(\mathrm{X13}) \quad -a_{13}\lambda_1 - a_{17}\nu_1 &&= 0\\
&(\mathrm{X17}) \quad a_{17}\lambda_1 + a_{17}\lambda_3 - a_{13}\mu_1 &&= 0\\
&(\mathrm{H1}) \quad a_6\mu_1 - a_1\nu_1 - a_7\nu_7 - a_{12}\nu_{12} - a_{17}\nu_{17} &&= 0\\
&(\mathrm{H3}) \quad -a_3\nu_3 - a_7\nu_7 - a_9\nu_9 - a_{12}\nu_{12} - a_{13}\nu_{13} - a_{17}\nu_{17} &&= 0\\
&(\mathrm{H4}) \quad -2\mu_4 - a_2\nu_2 + 2a_4\nu_4 + a_8\nu_8 + 2a_9\nu_9 + 2a_{12}\nu_{12} + a_{13}\nu_{13} + a_{17}\nu_{17} &&= 0
\end{aligned}
$$

Using a computer algebra system, we show that for any $((a_i)_{i\in I_{\pi'}}, (\alpha_i(h_0))_{i\in\pi'})$ in an open dense subset of $\mathbb{C}^{|I_{\pi'}|}\times\mathbb{C}^6$, the above homogeneous linear system has rank 14, $a_{13}a_{17}\neq 0$, and any of its solution $((\lambda_i)_{i=1,3,4}, (\mu_j)_{j=1,4}, (\nu_k)_{k\in I_{\pi'}})$ verifies $\lambda_3 = 0$. We can (and do) assume that $((a_i)_{i\in I_{\pi'}}, (\alpha_i(h_0))_{i\in\pi'})$ belongs to this open subset; in particular $a_{13}a_{17}\neq 0$. From the equations (X13) and (X17), we obtain that any solution of this system verifies $\lambda_1^2 + \mu_1\nu_1 = 0$ because $\lambda_3 = 0$. Since $\mu_i = 0$ for any $i \in I_{\pi'}\setminus\{1,4,6\}$ as observed previously, this shows that x' is a nilpotent element of $\mathfrak{l}_{\pi'}$; so x is a nilpotent element of $\mathfrak{g}$. As a consequence, $\mathfrak{p}^+_{\pi'}$ is not quasi-reductive. □

5. Explicit computations and classification

We assume in this part that $\mathfrak{g}$ is simple of exceptional type. Together with Theorem 1.7, the next two theorems (Theorem 5.1 and Theorem 5.2) complete the classification of quasi-reductive parabolic subalgebras of simple Lie algebras. The goal of this section is to prove these theorems.

THEOREM 5.1. — *Assume that $\mathfrak{g}$ is of type G_2, F_4, E_7 or E_8. Let π' be a subset of π.*

(i) *If $\mathfrak{g}$ is of type G_2, then $\mathfrak{p}^+_{\pi'}$ is quasi-reductive if and only if π' is different from $\{\alpha_1\}$.*

(ii) *If $\mathfrak{g}$ is of type F_4, then $\mathfrak{p}^+_{\pi'}$ is quasi-reductive if and only if each connected component of π' is different from $\{\alpha_1\}$.*

(iii) *If $\mathfrak{g}$ is of type* E_7, *then $\mathfrak{p}^+_{\pi'}$ is quasi-reductive if and only if each connected component of π' is different from the subsets $\{\alpha_1\}$, $\{\alpha_4\}$, $\{\alpha_6\}$, $\{\alpha_1,\alpha_3,\alpha_4\}$, $\{\alpha_4,\alpha_5,\alpha_6\}$ and $\{\alpha_1,\alpha_3,\alpha_4,\alpha_5,\alpha_6\}$.*

(iv) *If $\mathfrak{g}$ is of type* E_8, *then $\mathfrak{p}^+_{\pi'}$ is quasi-reductive if and only if each connected component of π' is different from the subsets $\{\alpha_1\}$, $\{\alpha_4\}$, $\{\alpha_6\}$, $\{\alpha_8\}$, $\{\alpha_1,\alpha_3,\alpha_4\}$, $\{\alpha_4,\alpha_5,\alpha_6\}$, $\{\alpha_6,\alpha_7,\alpha_8\}$, $\{\alpha_1,\alpha_3,\alpha_4,\alpha_5,\alpha_6\}$, $\{\alpha_4,\alpha_5,\alpha_6,\alpha_7,\alpha_8\}$ and $\{\alpha_1,\alpha_3,\alpha_4,\alpha_5,\alpha_6,\alpha_7,\alpha_8\}$.*

THEOREM 5.2. — *Assume that $\mathfrak{g}$ is of type* E_6 *and let π' be a subset of π. Then $\mathfrak{p}^+_{\pi'}$ is quasi-reductive except in the following three cases:*

1) *$\{\alpha_2\}$ is a connected component of π';*
2) *$\pi' = \{\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_6\}$;*
3) *$\pi' = \{\alpha_1,\alpha_2,\alpha_4,\alpha_5,\alpha_6\}$.*

Table 5.1 and Table 5.2 below summarize the results of Theorems 5.1 and 5.2 ; indeed, whenever $\mathrm{rk}\,\mathfrak{g} = \mathrm{k}_\pi$, only the cases where π' is connected

Type F_4:

π'	$\mathrm{ind}\,\mathfrak{p}^+_{\pi'}$	dim. of torus part
$\{\alpha_1\}$	1	0

Type E_7:

π'	$\mathrm{ind}\,\mathfrak{p}^+_{\pi'}$	dim. of torus part
$\{\alpha_1\}$	1	0
$\{\alpha_4\}$	1	0
$\{\alpha_6\}$	1	0
$\{\alpha_1,\alpha_3,\alpha_4\}$	2	1
$\{\alpha_4,\alpha_5,\alpha_6\}$	2	1
$\{\alpha_1,\alpha_3,\alpha_4,\alpha_5,\alpha_6\}$	3	2

Type E_8:

π'	$\mathrm{ind}\,\mathfrak{p}^+_{\pi'}$	dim. of torus part
$\{\alpha_1\}$	1	0
$\{\alpha_4\}$	1	0
$\{\alpha_6\}$	1	0
$\{\alpha_8\}$	1	0
$\{\alpha_1,\alpha_3,\alpha_4\}$	2	1
$\{\alpha_4,\alpha_5,\alpha_6\}$	2	1
$\{\alpha_6,\alpha_7,\alpha_8\}$	2	1
$\{\alpha_1,\alpha_3,\alpha_4,\alpha_5,\alpha_6\}$	3	2
$\{\alpha_4,\alpha_5,\alpha_6,\alpha_7,\alpha_8\}$	3	2
$\{\alpha_1,\alpha_3,\alpha_4,\alpha_5,\alpha_6,\alpha_7,\alpha_8\}$	4	3

Table 5.1. The non quasi-reductive parabolic subalgebras $\mathfrak{p}^+_{\pi'}$ with connected π' in F_4, E_7 and E_8 and their indices.

need to be dealt with thanks to Theorem 2.11. In these tables, the last column gives the dimension of the torus part of a generic stabilizer; we refer to Remark 4.5 for explanations in the types E_7 and E_8. For the type E_6, let us roughly explain our computations : in most cases, the subspaces $\bigcap_{\varepsilon\in\mathcal{E}_\pi\cup\Delta_{\pi'}} \ker\varepsilon$ of $\mathfrak{h}$ yield elements of the generic stabilizers of the regular linear forms of the form $(\varphi_{w'+u^-})|_{\mathfrak{p}^+_{\pi'}}$ with $w'\in\mathfrak{l}_{\pi'}$. For the cases $\{\alpha_1,\alpha_2,\alpha_6\}$, $\{\alpha_2,\alpha_3,\alpha_5\}$ and $\{\alpha_1,\alpha_2,\alpha_3,\alpha_5,\alpha_6\}$, one can show that the generic stabilizers of these forms also contain nonzero semisimple elements which do not belong to $\mathfrak{h}$. Since this is not a central point for our work, we omit the details.

$\pi'\subset\pi$, π of type E_6	$\operatorname{ind}\mathfrak{p}^+_{\pi'}$	dim. of torus part
$\{\alpha_2\}$	3	2
$\{\alpha_1,\alpha_2\}$, $\{\alpha_2,\alpha_6\}$	2	1
$\{\alpha_3,\alpha_2\}$, $\{\alpha_2,\alpha_5\}$	2	1
$\{\alpha_1,\alpha_2,\alpha_5\}$, $\{\alpha_2,\alpha_3,\alpha_6\}$	1	0
$\{\alpha_1,\alpha_2,\alpha_6\}$	3	2
$\{\alpha_2,\alpha_3,\alpha_5\}$	3	2
$\{\alpha_1,\alpha_2,\alpha_3\}$, $\{\alpha_2,\alpha_5,\alpha_6\}$	2	1
$\{\alpha_1,\alpha_2,\alpha_3,\alpha_5\}$, $\{\alpha_2,\alpha_3,\alpha_5,\alpha_6\}$	1	0
$\{\alpha_1,\alpha_2,\alpha_3,\alpha_6\}$, $\{\alpha_2,\alpha_1,\alpha_5,\alpha_6\}$	1	0
$\{\alpha_1,\alpha_2,\alpha_3,\alpha_5,\alpha_6\}$	3	2
$\{\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_6\}$, $\{\alpha_1,\alpha_2,\alpha_4,\alpha_5,\alpha_6\}$	1	0

Table 5.2. The non quasi-reductive parabolic subalgebras $\mathfrak{p}^+_{\pi'}$ in E_6 and their indices.

Remark 5.3. — Theorems 5.1 and 5.2 confirm what was announced in Remark 2.12: The only cases where the additivity property fails is for $\mathfrak{g}=\mathrm{E}_6$ and $\pi'=\{\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_6\}$ (where $\{\alpha_1,\alpha_2,\alpha_3,\alpha_4\}$ is not connected to $\{\alpha_6\}$), resp. for $\mathfrak{g}=\mathrm{E}_6$ and $\pi'=\{\alpha_1,\alpha_2,\alpha_4,\alpha_5,\alpha_6\}$.

By Theorem 4.1, Theorem 4.3 and Theorem 4.6, in order to prove Theorems 5.1 and 5.2, it is enough to show that if π' is different from the subsets listed in Theorems 5.1 and 5.2 then $\mathfrak{p}^+_{\pi'}$ is quasi-reductive. This is our goal until the end of the paper. Recall that α_π is the simple root connected to the lowest root in the extended Dynkin diagram. By Theorem 2.1, we can assume that π' contains α_π. Moreover, whenever $\operatorname{rk}\mathfrak{g}=\mathrm{k}_\pi$, we can assume that π' is connected by Theorem 2.11.

The case where π' has rank 1 was dealt with in Theorem 4.1. In the next subsection, we study the case where π' is connected and of rank 2. Then we discus the remaining cases in Subsection 5.2.

5.1. Assume that $\mathfrak{g}$ is of type F_4, E_6, E_7 or E_8 and let π' be a connected subset of π of rank 2 which contains α_π. Write $\pi' = \{\alpha_{i_1}, \alpha_{i_2}\}$ with $\alpha_{i_2} = \alpha_\pi$. Lemma 5.4 shows that the roots of π' have common properties:

LEMMA 5.4. — *The subset π' has type A_2 and there are four integers j_0, j_1, j_2, j_3 in $\{1, \dots, \mathrm{k}_\pi\}$ and a quadruple $(c_0, c_1, c_2, c_3) \in \mathbb{C}^4$ such that the following properties are satisfied:*
$\alpha_{i_1} = \varepsilon_{j_1}$, $\alpha_{i_2} = \frac{1}{2}(\varepsilon_{j_0} - \varepsilon_{j_1} - \varepsilon_{j_2} - \varepsilon_{j_3})$ *and* $h_{\varepsilon_{\pi'}} = \sum_{k=0}^{3} c_k h_{\varepsilon_{j_k}}$.

Proof. — We verify the properties for each type:

Type F_4: $\pi' = \{\alpha_2, \alpha_1\}$, with $\alpha_1 (= \alpha_\pi) = \frac{1}{2}(\varepsilon_1 - \varepsilon_4 - \varepsilon_2 - \varepsilon_3)$ and $\alpha_2 = \varepsilon_4$.
Moreover $h_{\alpha_1+\alpha_2} = \frac{1}{2}(h_{\varepsilon_1} + h_{\varepsilon_4} - h_{\varepsilon_2} - h_{\varepsilon_3})$.
Type E_6: $\pi' = \{\alpha_4, \alpha_2\}$, with $\alpha_2 (= \alpha_\pi) = \frac{1}{2}(\varepsilon_1 - \varepsilon_4 - \varepsilon_2 - \varepsilon_3)$ and $\alpha_4 = \varepsilon_4$.
Moreover $h_{\alpha_2+\alpha_4} = \frac{1}{2}(h_{\varepsilon_1} + h_{\varepsilon_4} - h_{\varepsilon_2} - h_{\varepsilon_3})$.
Type E_7: $\pi' = \{\alpha_1, \alpha_3\}$, with $\alpha_1 (= \alpha_\pi) = \frac{1}{2}(\varepsilon_1 - \varepsilon_6 - \varepsilon_2 - \varepsilon_3)$ and $\alpha_3 = \varepsilon_6$.
Moreover $h_{\alpha_1+\alpha_3} = \frac{1}{2}(h_{\varepsilon_1} + h_{\varepsilon_6} - h_{\varepsilon_2} - h_{\varepsilon_3})$.
Type E_8: $\pi' = \{\alpha_7, \alpha_8\}$, with $\alpha_8 (= \alpha_\pi) = \frac{1}{2}(\varepsilon_1 - \varepsilon_5 - \varepsilon_2 - \varepsilon_3)$ and $\alpha_7 = \varepsilon_5$.
Moreover $h_{\alpha_7+\alpha_8} = \frac{1}{2}(h_{\varepsilon_1} + h_{\varepsilon_5} - h_{\varepsilon_2} - h_{\varepsilon_3})$. □

Recall that there exist $\underline{a} = (a_1, \dots, a_{\mathbf{k}_\mathfrak{g}}) \in (\mathbb{C}^*)^{\mathrm{k}_\pi}$ and $b \in \mathbb{C}^*$, such that the linear form $(\varphi_{u(\underline{a},b)})|_{\mathfrak{p}^+_{\pi'}}$ is $\mathfrak{p}^+_{\pi'}$-regular. Since $\varepsilon_{\pi'} = \frac{1}{2}(\varepsilon_{j_0} + \varepsilon_{j_1} - \varepsilon_{j_2} - \varepsilon_{j_3})$, the element $\varepsilon_{j_0} - \varepsilon_{\pi'}$ is a positive root. Denote by β_2 and β_3 the two positive roots $\beta_2 = (\varepsilon_{j_0} - \varepsilon_{\pi'}) - \varepsilon_{j_2}$ and $\beta_3 = (\varepsilon_{j_0} - \varepsilon_{\pi'}) - \varepsilon_{j_2}$.. For $\underline{\lambda} = (\lambda_2, \lambda_3, \mu_0, \mu_1, \mu_2, \mu_3, \nu) \in (\mathbb{C}^*)^7$, we set $x(\underline{\lambda}) = x_{-\varepsilon_{\pi'}} + \lambda_2 x_{\beta_2} + \lambda_3 x_{\beta_3} + \sum_{k=0}^{3} \mu_k x_{\varepsilon_{j_k}} + \nu x_{-\varepsilon_{j_1}}$.

LEMMA 5.5. — *Let $(\underline{a}, b)$ be in $(\mathbb{C}^*)^{\mathrm{k}_\pi} \times \mathbb{C}^*$ such that $(\varphi_{u(\underline{a},b)})|_{\mathfrak{p}^+_{\pi'}}$ is $\mathfrak{p}^+_{\pi'}$-regular.*

For a suitable choice of $\underline{\lambda} = (\lambda_2, \lambda_3, \mu_0, \mu_1, \mu_2, \mu_3, \nu) \in (\mathbb{C}^*)^7$, the element $x(\underline{\lambda})$ lies in the stabilizer of $(\varphi_{u(\underline{a},b)})|_{\mathfrak{p}^+_{\pi'}}$ in $\mathfrak{p}^+_{\pi'}$. Moreover, for such a $\underline{\lambda}$, we have $\mathfrak{p}^+_{\pi'}(\varphi_{u(\underline{a},b)}) = \bigcap_{K \in \mathcal{K}_\pi} \ker \varepsilon_K \oplus \mathbb{C}x(\underline{\lambda})$ and the element $x(\underline{\lambda})$ is semisimple. In particular $\varphi_{u(\underline{a},b)}$ is of reductive type for $\mathfrak{p}^+_{\pi'}$.

Proof. — By definition, we have $\varepsilon_{j_2} + \varepsilon_{\pi'} = \varepsilon_{j_0} - \beta_2$, $\varepsilon_{j_3} + \varepsilon_{\pi'} = \varepsilon_{j_0} - \beta_3$, $\beta_2 - \varepsilon_{j_3} = \beta_3 - \varepsilon_{j_2} = \alpha_{i_2}$ and $\varepsilon_{\pi'} - \varepsilon_{j_1} = \alpha_{i_2}$. We define the structure constants $\tau_1, \tau_2, \tau_3, \tau_4, \tau_5, \tau_6, \tau_0$ by the following equations:

$$[x_{\beta_2}, x_{-\varepsilon_{j_0}}] = \tau_1 x_{-(\varepsilon_{\pi'}+\varepsilon_{j_2})} \quad ; \quad [x_{-\varepsilon_{\pi'}}, x_{-\varepsilon_{j_2}}] = \tau_2 x_{-(\varepsilon_{\pi'}+\varepsilon_{j_2})}$$
$$[x_{\beta_3}, x_{-\varepsilon_{j_0}}] = \tau_3 x_{-(\varepsilon_{\pi'}+\varepsilon_{j_3})} \quad ; \quad [x_{-\varepsilon_{\pi'}}, x_{-\varepsilon_{j_3}}] = \tau_4 x_{-(\varepsilon_{\pi'}+\varepsilon_{j_3})}$$
$$[x_{\beta_2}, x_{-\varepsilon_{j_3}}] = \tau_5 x_{\alpha_{i_2}} \quad ; \quad [x_{\beta_3}, x_{-\varepsilon_{j_2}}] = \tau_6 x_{\alpha_{i_2}} \quad ;$$
$$[x_{-\varepsilon_{j_1}}, x_{\varepsilon_{\pi'}}] = \tau_0 x_{\alpha_{i_2}}.$$

Set $u = u(\underline{a}, b)$ and $x = x(\underline{\lambda})$. We have:

$$[x,u] = b[x_{-\varepsilon_{\pi'}}, x_{\varepsilon_{\pi'}}] + \sum_{k=0}^{3} \mu_k a_{j_k} [x_{\varepsilon_{j_k}}, x_{-\varepsilon_{j_k}}] + a_{j_2}[x_{-\varepsilon_{\pi'}}, x_{-\varepsilon_{j_2}}]$$
$$+ a_{j_3}[x_{-\varepsilon_{\pi'}}, x_{-\varepsilon_{j_3}}] + \ \lambda_3 a_{j_0}[x_{\beta_3}, x_{-\varepsilon_{j_0}}] + \lambda_2 a_{j_0}[x_{\beta_2}, x_{-\varepsilon_{j_0}}]$$
$$+ \nu b[x_{-\varepsilon_{j_1}}, x_{\varepsilon_{\pi'}}] + \lambda_2 a_{j_3}[x_{\beta_2}, x_{-\varepsilon_{j_3}}] + \lambda_3 a_{j_2}[x_{\beta_3}, x_{-\varepsilon_{j_2}}] + v$$

where v is in $\mathfrak{m}^+_{\pi'}$. In the above notations, this gives:

$$[x,u] = b(-h_{\varepsilon_{\pi'}}) + \sum_{k=0}^{3} \mu_k a_{j_k} h_{\varepsilon_{j_k}} + (a_{j_2}\tau_2 + \lambda_2 a_{j_0}\tau_1) x_{-(\varepsilon_{\pi'}+\varepsilon_{j_2})}$$
$$+ (a_{j_3}\tau_4 + \lambda_3 a_{j_0}\tau_3) x_{-(\varepsilon_{\pi'}+\varepsilon_{j_3})} + (\nu b \tau_0 + \lambda_2 a_{j_3}\tau_5 + \lambda_3 a_{j_2}\tau_6) x_{\alpha_{i_2}} + v$$

Set $\mu_k = (bc_k)/a_{j_k}$, for $k = 0,1,2,3$. By Lemma 5.4 we get $b(-h_{\varepsilon_{\pi'}}) + \sum_{k=0,1,2,3} \mu_k a_{j_k} h_{\varepsilon_{j_k}} = 0$. Next, we set $\lambda_2 = -a_{j_2}\tau_2/(a_{j_0}\tau_1)$ and $\lambda_3 = -a_{j_3}\tau_4/(a_{j_0}\tau_3)$ so that the terms in $x_{-(\varepsilon_{\pi'}+\varepsilon_{j_2})}$ and $x_{-(\varepsilon_{\pi'}+\varepsilon_{j_3})}$ in $[x,u]$ are both equal to zero. At last, we choose ν so that the term in $x_{\alpha_{i_2}}$ in $[x,u]$ is 0. Then the element x stabilizes $(\varphi_u)|_{\mathfrak{p}^+_{\pi'}}$.

Let $\underline{\lambda}$ be as above. We have thus obtained the inclusion $\bigcap_{K\in\mathcal{K}_\pi} \ker \varepsilon_K \oplus \mathbb{C}x \subset \mathfrak{p}^+_{\pi'}(\varphi_u)$. By equation (1.1), $\operatorname{ind} \mathfrak{p}^+_{\pi'} = \operatorname{rk}\mathfrak{g} - \mathrm{k}_\pi + 1$ whence the equality $\bigcap_{K\in\mathcal{K}_\pi} \ker \varepsilon_K \oplus \mathbb{C}x = \mathfrak{p}^+_{\pi'}(\varphi_u)$.

We now show that $x = x(\underline{\lambda})$ is semisimple. To start with, we prove that x is semisimple if and only if $(\tau_2\tau_5)/\tau_1 + (\tau_4\tau_6)/\tau_3 \neq 0$. As β_2 and β_3 are both different from $\alpha_{i_1}, \alpha_{i_2}$ and $\alpha_{i_1} + \alpha_{i_2}$, the component of x on $\mathfrak{l}_{\pi'}$ in the decomposition $\mathfrak{p}^+_{\pi'} = \mathfrak{l}_{\pi'} \oplus \mathfrak{m}^+_{\pi'}$ is $x_{-\varepsilon_{\pi'}} + \mu_1 x_{\varepsilon_{j_1}} + \nu x_{-\varepsilon_{j_1}}$. By what foregoes, $\mu_1 = (bc_1)/a_{j_1} \neq 0$. Therefore, x is semisimple if and only if $\nu \neq 0$. We have $\nu b\tau_0 + \lambda_2 a_{j_3}\tau_5 + \lambda_3 a_{j_2}\tau_6 = 0$, that is, by the choices of λ_2 and λ_3: $\nu b\tau_0 - (a_{j_2}\tau_2 a_{j_3}\tau_5)/(a_{j_0}\tau_1) - (a_{j_3}\tau_4 a_{j_2}\tau_6)/(a_{j_0}\tau_3) = 0$ Hence $\nu = 1/(b\tau_0) \times (a_{j_2}a_{j_3})/a_{j_0} \times ((\tau_2\tau_5)/\tau_1 + (\tau_4\tau_6)/\tau_3)$. As a result, $\nu \neq 0$ if and only $(\tau_2\tau_5)/\tau_1 + (\tau_4\tau_6)/\tau_3 \neq 0$.

It remains to check that the condition $(\tau_2\tau_5)/\tau_1 + (\tau_4\tau_6)/\tau_3 \neq 0$ holds. We check the condition for the all cases considered in the proof of Lemma 5.4. Note that the computations of the integers τ_i can be done using `GAP`.

Type F_4: One checks that $\tau_1 = \tau_3 = 1$ and $\tau_2 = \tau_4 = \tau_5 = \tau_6 = -1$.

Type E_6: One checks that $\tau_1 = \tau_2 = \tau_3 = \tau_4 = 1$ and $\tau_5 = \tau_6 = -1$.

Type E_7: One checks that $\tau_1 = \tau_2 = \tau_3 = \tau_4 = \tau_5 = \tau_6 = -1$.
Type E_8: One checks that $\tau_1 = \tau_2 = \tau_3 = \tau_4 = \tau_5 = \tau_6 = 1$. □

To summarize, this gives us:

THEOREM 5.6. — *For simple $\mathfrak{g}$ of exceptional type, and simple $\pi' \subset \pi$ of rank 2 containing α_π, the parabolic subalgebra $\mathfrak{p}^+_{\pi'}$ is quasi-reductive.*

Using Theorem 5.6, we obtain new cases of quasi-reductive parabolic subalgebras in E_6:

THEOREM 5.7. — *For simple $\mathfrak{g}$ of type E_6 and $\pi'' = \{\alpha_1, \alpha_2, \alpha_4\}$ or $\{\alpha_1, \alpha_2, \alpha_4, \alpha_6\}$, $\mathfrak{p}^+_{\pi''}$ is quasi-reductive.*

Note that Theorem 5.7 cannot be deduced from Theorem 2.11 even though π'' is not connected. Indeed Theorem 2.11 fails in type E_6 as explained in Remark 2.12.

Proof. — We approach the two cases in the same way.

Let π' be the subset $\{\alpha_2, \alpha_4\}$. Then π' is a connected component of π''. Hence, one can choose $u'' = u(\underline{a}, \underline{b})$ such that both $(\varphi_{u''})|_{\mathfrak{p}^+_{\pi''}}$ and $(\varphi_{u'})|_{\mathfrak{p}^+_{\pi'}}$ are regular (for $\mathfrak{p}^+_{\pi''}$ and $\mathfrak{p}^+_{\pi'}$ respectively) where $u' = u(\underline{a}, b_{\pi'})$. Let $\underline{\lambda} = (\lambda_2, \lambda_3, \mu_0, \mu_1, \mu_2, \mu_3, \nu)$ be an element of $\mathbb{C}^7$ such that $x = x(\underline{\lambda})$ stabilizes $(\varphi_{u'})|_{\mathfrak{p}^+_{\pi'}}$ (cf. Lemma 5.5). One can readily check that x belongs to $\mathfrak{p}^+_{\pi''}(\varphi_{u''})$, too. On the other hand, in both cases, the orthogonal of $E_{\pi'',\pi}$ in $\mathfrak{h}$ has dimension 1, is contained in $\mathfrak{p}^+_{\pi''}(\varphi_{u''})$, and does not contain x. Hence, as x is semisimple (by Lemma 5.5), we have found a torus a dimension 2 which is contained in $\mathfrak{p}^+_{\pi''}(\varphi_{u''})$.

We distinguish now the two cases:
Case $\pi'' = \{\alpha_1, \alpha_2, \alpha_4\}$: by (1.1), $\operatorname{ind} \mathfrak{p}^+_{\pi''} = 2$. Then, the above discussion shows that $(\varphi_{u''})|_{\mathfrak{p}^+_{\pi''}}$ is of reductive type.
Case $\pi'' = \{\alpha_1, \alpha_2, \alpha_4, \alpha_6\}$: by (1.1), $\operatorname{ind} \mathfrak{p}^+_{\pi''} = 3$. So, it suffices to provide a nonzero semisimple element in $\mathfrak{p}^+_{\pi''}(\varphi_{u''})$ which does not lie in the preceding torus. We claim that the (semisimple) element $y = (a_{K_3}/b_{\{\alpha_6\}})x_{\alpha_1} + (a_{K_2}/b_{\{\alpha_1\}})x_{-\alpha_1} + (a_{K_3}/b_{\{\alpha_1\}})x_{\alpha_6} + (a_{K_2}/b_{\{\alpha_6\}})x_{-\alpha_6} + x^+$ does the job, where x^+ is an element of $\mathfrak{m}^+_{\pi''}$ and where K_i (for $1 \leqslant i \leqslant 4$) corresponds to the highest root ε_i. □

5.2. Using the results of Sections 2, 3 and 4, we are able to deal with a large number of parabolic subalgebras. Unfortunately, the results obtained so far do not cover all parabolic subalgebras. There remains a small number of cases. We consider these here. This will complete the proof of Theorems 5.1 and 5.2.

We first consider examples which do not need of the computer programme GAP.

It is well known that minimal parabolic subalgebras of a real simple (finite dimensional) Lie algebra are quasi-reductive, see e.g. [13]. Moreover, the complexified subalgebras give rise to quasi-reductive subalgebras of the corresponding complex simple Lie algebra. In type F_4 and type E_6 the so-obtained parabolic subalgebras of $\mathfrak{g}$ correspond to the subsets $\pi' = \{\alpha_1, \alpha_2, \alpha_3\}$ and $\pi' = \{\alpha_2, \alpha_3, \alpha_4, \alpha_5\}$ of π respectively. As a result, we have:

PROPOSITION 5.8. — (i) *If $\mathfrak{g}$ is of type F_4 and if π' is $\{\alpha_1, \alpha_2, \alpha_3\}$ then $\mathfrak{p}^+_{\pi'}$ is quasi-reductive.*

(ii) *If $\mathfrak{g}$ is of type E_6 and if π' is $\{\alpha_2, \alpha_3, \alpha_4, \alpha_5\}$ then $\mathfrak{p}^+_{\pi'}$ is quasi-reductive.*

We consider now the remaining cases. For all these cases, we are able to find $(\underline{a}, \underline{b}) \in \mathbb{C}^{*(\mathrm{k}_\pi + \mathrm{k}_{\pi'})}$ such that $\varphi_{u(\underline{a},\underline{b})}$ is of reductive type for $\mathfrak{p}^+_{\pi'}$. We have used the computer programme GAP to check that the stabilizer of such a form is a torus of $\mathfrak{g}$. The commands we have used are presented in Appendix A.

PROPOSITION 5.9. — (i) *If $\mathfrak{g}$ is of type E_6 and if π' is $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5\}$ or $\{\alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6\}$ then $\mathfrak{p}^+_{\pi'}$ is quasi-reductive.*

(ii) *If $\mathfrak{g}$ is of type E_7 and if π' is one the subsets $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$, $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5\}$, $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6\}$, $\{\alpha_1, \alpha_3, \alpha_4, \alpha_5\}$ or $\{\alpha_1, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7\}$ then $\mathfrak{p}^+_{\pi'}$ is quasi-reductive.*

(iii) *If $\mathfrak{g}$ is of type E_8 and if π' is one the subsets $\{\alpha_5, \alpha_6, \alpha_7, \alpha_8\}$, $\{\alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8\}$, $\{\alpha_2, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8\}$ or $\{\alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8\}$ then $\mathfrak{p}^+_{\pi'}$ is quasi-reductive.*

This proposition completes the proof of Theorems 5.1 and 5.2; the other cases are dealt with either in Remark 1.5, or in Example 3.7, or in Theorems 4.1, 5.6 and 5.7 (or deduced from Theorem 2.1 or Theorem 2.11 as explained before).

Remark 5.10. — As noticed in the introduction, Proposition 5.9 can be proved without the help of GAP; this is done in a joint work of the second author and O. Yakimova, [16] where the authors consider the *maximal reductive stabilizers* of quasi-reductive parabolic subalgebras of simple Lie algebras.

Appendix A.

In this appendix, we explain how to use GAP to verify that for suitable $u = u(\underline{a}, \underline{b})$ and π' as described in Proposition 5.9 the linear form $(\varphi_u)|_{\mathfrak{p}^+_{\pi'}}$ is of reductive type. We do this for the example $\mathfrak{g} = \mathrm{E}_7$ and $\pi' = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5\}$, the other cases work similarly. First, we define the simple Lie algebra L (= $\mathfrak{g}$), a root system R and a Chevalley Basis (h,x,y) of L, and then the parabolic subalgebra P (= $\mathfrak{p}^+_{\pi'}$) generated by gP; its dimension is dP:

```
>L:=SimpleLieAlgebra("E",7,Rationals);;R:=RootSystem(L);;
>x:=PositiveRootVectors(R);;y:=NegativeRootVectors(R);;
>g:=CanonicalGenerators(R);;h:=g[3];;
>gP:=Concatenation(g[1],h,y{[1..5]});;P:=Subalgebra(L,gP);;
>dP:=Dimension(P);
90
```

Next we choose numbers (a1,a2,a3,a4,a5,a6,a7,b1,b2,b3,b4) $\in (\mathbb{C}^*)^{(\mathrm{k}_\pi + \mathrm{k}_{\pi'})}$ and we define the element u=u1+u2 (= u($\underline{a}$,$\underline{b}$)) of $\mathfrak{p}^-_{\pi'}$:

```
>a1:=-3;;a2:=5;;a3:=7;;a4:=11;;a5:=13;;a6:=-17;;a7:=19;;
>b1:=23;;b2:=-29;;b3:=31;;b4:=37;;
>u2:=a1*y[63]+a2*y[49]+a3*y[28]+a4*y[7]+a5*y[2]+a6*y[3]+a7*y[5];;
>u1:=b1*x[37]+b2*[16]+b3*[4]+b4*x[1];;u:=u1+u2;;
```

We are now ready to compute the stabilizer of $(\varphi_u)|_{\mathfrak{p}}$. To start with, we calculate the vector space V generated by the brackets u*bP[i], for $i = 1, \ldots,$ dP, where bP is a basis of P. We obtain the orthogonal K of V with respect to the Killing form thanks to the command KappaPerp. Then, the stabilizer S of $(\varphi_u)|_{\mathfrak{p}}$ is the intersection of K and P:

```
>bP:=List(Basis(P));; l:=[];;
>for i in [1..dP] do l[i]:=u*bP[i];od;;l;
>V:=Subspace(L,l);;K:=KappaPerp(L,V);;S:=Intersection(K,P);;
>dS:=Dimension(S);
4
```

The fact dim S=4 shows that $(\varphi_u)|_{\mathfrak{p}}$ is regular, since ind P = 4. It remains to check that S is a reductive subalgebra of L.. To process, we check that the restriction of the Killing form to S × S is nondegenerate. For that it suffices to compute the intersection between S and its orthogonal in L. The result has to be a vector space of dimension 0:

```
>KS:=Intersection(KappaPerp(L,S),S);
<vector space of dimension 0 over Rationals>
```

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