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PRESCRIBING GAUSS CURVATURE OF SURFACES IN 3-DIMENSIONAL SPACETIMES APPLICATION TO THE MINKOWSKI PROBLEM IN THE MINKOWSKI SPACE

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Abstract. — We study the existence of surfaces with constant or prescribed Gauss curvature in certain Lorentzian spacetimes. We prove in particular that every (non-elementary) 3-dimensional maximal globally hyperbolic spatially compact spacetime with constant non-negative curvature is foliated by compact spacelike surfaces with constant Gauss curvature. In the constant negative curvature case, such a foliation exists outside the convex core. The existence of these foliations, together with a theorem of C. Gerhardt, yield several corollaries. For example, they allow to solve the Minkowski problem in Min_3 for data that are invariant under the action of a co-compact Fuchsian group.

Résumé. — Nous étudions l'existence de surfaces à courbure de Gauss constante ou prescrite dans certains espaces-temps lorentziens. Nous montrons en particulier que tout espace-temps (non-élémentaire) globalement hyperbolique spatialement compact maximal à courbure constante positive ou nulle de dimension 3 est feuilleté en surfaces de Cauchy à courbure de Gauss constante. Dans le cas des espaces-temps à courbure constante strictement négative, le complémentaire du cœur convexe est feuilleté par des surfaces de Cauchy à courbure de Gauss constante. On combinant ces résultats d'existence de feuilletages avec un théorème de C. Gerhardt, on obtient un certain nombre de corollaires. Par exemple, on résout le problème de Minkowski dans Min_3 pour des données qui sont invariantes par l'action d'un groupe fuchsien cocompact.

1. Introduction

Surfaces with constant Gauss curvature embedded in a three-dimensional space have always played a important role in Riemannian geometry and

geometric analysis. It seems however that — unlike surfaces with constant mean curvature — they have not attracted much attention in mathematical relativity. One reason might be that people think that these surfaces do not generalize appropriately in higher dimensions: Hypersurfaces with constant sectional curvature embedded in a n-dimensional space are rigid and rare for $n \geqslant 4$. Observe nevertheless that, from the point of view of geometrical analysis, Gauss curvature should not be considered as a scalar intrinsic curvature (the sectional curvature), but rather as a scalar extrinsic curvature: The K-curvature, *i.e.* the product of the eigenvalues of the second fundamental form. From this viewpoint, the mean curvature and the Gauss curvature both belong to the family of so-called *F-curvatures* (see below), and hypersurfaces with constant "Gauss" curvature are not rigid anymore in dimension $n \geqslant 4$. In spite of the privileged connection between hypersurfaces with constant mean curvature and the Dirichlet problem for the Einstein equation, we think that it could be interesting to consider other kinds of hypersurfaces with constant F-curvature in mathematical relativity. In particular, we think that surfaces with constant Gauss curvature could play an interesting role in (2+1)-gravity.

In the present paper, we study the existence of surfaces with constant Gauss curvature (which we call K-surfaces) in 3-dimensional maximal globally hyperbolic spatially compact vacuum spacetimes. We prove that each such spacetime with non-negative curvature admits a unique foliation by K-surfaces. This foliation will provide us with a canonical time function on the spacetime under consideration.

Let us start by some general introduction, first on the problematic of existence of privileged time functions on spacetimes, and then on the class of spacetimes we are interested in.

1.1. Geometric times on MGHC spacetimes

MGHC spacetimes. —   Recall that a Lorentz manifold M is *globally hyperbolic* if it admits a *Cauchy hypersurface*, *i.e.* a spacelike hypersurface which intersects every inextendible causal curve at exactly one point. A classical result of R. Geroch states that the existence of a single Cauchy hypersurface implies the existence of a foliation by such hypersurfaces. More precisely, recall that a *time function* on a spacetime is a submersion $t \colon M \to \mathbb{R}$ strictly increasing along every future-oriented causal curves. Geroch's result states that a globally hyperbolic spacetime admits a time function whose levels are Cauchy hypersurfaces.

A globally hyperbolic spacetime is said to be *spatially compact* if its Cauchy hypersurfaces are compact. This is equivalent to having a time function which is a proper map, or equivalently, a time function with compact levels. A globally hyperbolic spacetime (M, g) is *maximal* if every isometric embedding of M in another globally hyperbolic spacetime of the same dimension is onto. For short, we shall write MGHC for "maximal globally hyperbolic spatially compact". MGHC spacetimes are the tamest Lorentz manifolds from the geometric analysis viewpoint. These spacetimes appear as cosmological models in mathematical Relativity (MGHC spacetimes satisfying the strong positivity energy condition are sometimes called *cosmological spacetimes*).

Geometric time functions. —   The simplest examples of MGHC spacetimes are metric products, *i.e.* spacetimes of the type $M = (\mathbb{R}, -dt^2) \oplus (\Sigma, h)$, where (Σ, h) is a compact Riemannian manifold.

Let (M, g) be a MGHC spacetime. From a topological viewpoint, M is always homeomorphic to the product $\mathbb{R} \times \Sigma$, where Σ is a Cauchy hypersurface of M. More precisely, a MGHC spacetime M can always be written as a topological product $\mathbb{R} \times \Sigma$, where the first projection $(t, x) \to t$ is a time function, and $(\{t\} \times \Sigma)_{t \in \mathbb{R}}$ is a foliation of M by spacelike hypersurfaces.

From a metrical viewpoint, (M, g) is in general far from being isometric to a direct product. In the (t, x) coordinates given by the topological splitting $M \simeq \mathbb{R} \times \Sigma$, the metric on M has the ADM form: $g = -N(t, x)dt^2 \oplus (\omega_t dt + h_t)$, where h_t is a one parameter family of Riemannian metrics on Σ, ω_t a one parameter family of 1-forms on Σ, $N(t, x)$ a function (called the lapse function), and X_t the dual (with respect to h_t) of $(1/2)\omega_t$ is the shift (non-autonomous) vector field. (One can roughly say at this stage that a Lorentz structure on a MGHC space with topology $\mathbb{R} \times \Sigma$, is a curve in $\mathcal{M}et(\Sigma)$, the space of Riemannian metrics of Σ up to isotopy).

The topological splitting $M \simeq \mathbb{R} \times \Sigma$ are by no means unique. Nonetheless, it is natural and worthwhile to ask if there are privileged splittings (or time function) for a given MGHC spacetime? It is specially exciting to ask what remains from the couple of the orthogonal foliations after perturbation of a direct product metric $-dt^2 \oplus h$?

So the general question we are posing is to produce canonical geometric foliations by Cauchy hypersurfaces, or equivalently geometric time functions, which yield a kind of measurement of the default for (M, g) to be a metric product. Actually, asymptotic behaviour, singularities, shocks, and similar questions are meaningful only in a "natural" coordinates system (which does not create artificial singularities).

Rigid time functions. —   Let us first give some examples of local geo-
metric conditions on time functions (we keep the same notations as above):

- *Static time:* $\frac{\partial}{\partial t}$ is a Killing vector field and is orthogonal to the level
  sets of the time function $(t, x) \mapsto t$. In this case $g = -N(x)dt^2 \oplus h_0$.
- *Static geodesic time:* Same as above, with the additional require-
  ment that $\frac{\partial}{\partial t}$ has geodesic trajectories. This characterize the direct
  product case, $g = -dt^2 \oplus h_0$.
- *Homothetically static-geodesic case:* $g = -dt^2 \oplus w(t)h_0$ for some
  function w.

All these examples are very rigid. Observe that, in these examples, the
time function $t: M \to \mathbb{R}$ is a solution of a *system* of PDEs describing the
extrinsic geometry of its level sets. A less rigid situation is the case where
the time function is assumed to be a solution of a single (scalar) geometric
PDE; This is the case of so-called *F-time functions*.

F-time functions. —   Let $F: \mathbb{R}^n \to \mathbb{R}$ be a map which is invariant by
permutation of the coordinates. The F-curvature of a hypersurface Σ em-
bedded in a $(n+1)$-dimensional spacetime M is defined by evaluating the
function F on the eigenvalues of the second fundamental form of Σ.

In the particular case where F is the (arithmetic) mean of eigenvalues, the
F-curvature is the *mean curvature*. We are going here to consider the case
where F is the opposite of the product of eigenvalues; In the particular case,
the F-curvature is called the Gauss-Killing-Kronecker-Lipschitz-curvature
or K-curvature. Such functions are automatically unique (see §1.3).

A *F-time function* is a time function $t: M \to \mathbb{R}$ such that each level set of
$t^{-1}(a)$ has constant F-curvature, and such that the F-curvature of the level
set $t^{-1}(a)$ is an increasing function of a (in other words, the F-curvature
of the level sets increases along future-directed causal curves).

Constant mean curvature versus constant K-curvature. —   A CMC-
hypersurface is a hypersurface with a constant mean curvature. A K-
hypersurface is a hypersurface with a constant K-curvature.

From a PDE point of view, since one takes a linear sum in the definition
of the mean curvature, the equation defining CMC hypersurfaces is the
simplest one (among all the equation defining hypersurfaces with constant
F-curvature). It is quasi-linear. In all the other cases, the PDE is fully
non-linear of Monge-Ampère type.

Following G. Darboux, one could say that CMC hypersurfaces play a crucial role in physics, whereas K-hypersurfaces in geometry![1] Despite the marriage of geometry and physics via the Theory of General Relativity (some decades after Darboux), K-hypersurfaces remain mostly ignored by physicists nowadays, whereas CMC hypersurfaces are the subject of much attention. In fact one of the central and natural questions in Relativity is whether a given class of spacetimes admits CMC foliations?

One explanation of the success of CMC hypersurfaces in General Relativity is that the Cauchy problem for Einstein equations (in vacuum) can be formulated as a hyperbolic-elliptic well posed PDE system, in a CMC gauge. Roughly speaking, one can incorporate the CMC condition in the ADM representation of the metric, and set a well posed system, that is if initial data satisfy "CMC constraints", then, the system has a solution. This solution is a Ricci flat Lorentz manifold extending the given initial Riemannian manifold, together with a *canonical* CMC foliation on a neighbourhood of it. In other words, one has a kind of a CMC flow.

It is natural to try to extend this consideration to the K-curvature case, that is to write the Einstein equations in a K-gauge. This does not seem easy because of the the non-linear coupling of equations.

One beautiful corollary of CMC gauges is the regularity of global CMC foliations. If one knows that the ambient Lorentz metric is (real) analytic, then the locally defined CMC foliation given by the CMC flow is analytic (by analyticity of solutions of analytic hyperbolic systems with analytic initial data). Therefore, if one knows that the spacetime has a CMC foliation, and moreover, any CMC hypersurface is a leaf of it, then this foliation must be analytic (since it is locally defined by the CMC flow). One achievement of the present article is to show existence of K-foliations as well as uniqueness of K-hypersurfaces, but we can not yet deduce its analyticity, since we were not able to treat the Einstein equations in a K-gauge.

Other geometric time functions. —   A time function $(x, t) \mapsto t$ is *of Gauss type* if the metric has the form $-dt^2 \oplus g_t$. If a level $\{t\} \times \Sigma$ is given, then the others levels are obtained by pushing along orthogonal geodesics. The cosmological time (CT for short) enjoys this property. It will be one important tool in our study of K-times.

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[1] "On peut dire que la courbure totale a plus d'importance en Géométrie ; comme elle ne dépend que de l'élément linéaire, elle intervient dans toutes les questions relatives à la déformation des surfaces. En Physique mathématique, au contraire, c'est la courbure moyenne qui paraît jouer le rôle prépondérant", [29, Livre V, chapitre II].

1.2. MGHC spacetimes of constant curvature

In the present paper, we restrict ourselves to MGHC spacetimes of constant curvature. Such spacetimes have locally trivial geometry, being locally isometric to the Minkowski space Min_3, de Sitter space dS_3 or anti-de Sitter space AdS_3. However, the topology and the global geometry of these spacetimes may be highly nontrivial. Actually, all the questions studied in the present paper concern the global geometry of the spacetimes under consideration.

Focussing our attention on spacetimes with constant curvature is clearly an important restriction. Nevertheless, let us recall that, from a physical viewpoint, every 3-dimensional solution of the vacuum the Einstein equations has constant curvature. Actually, 3-dimensional spacetimes with constant curvature have received much attention in the last fifteen years because of the role they play in quantum gravity (see for instance [25]), and since the discovery of the so-called BTZ black holes models ([8]). Moreover, from a purely mathematical viewpoint, 3-dimensional MGHC spacetimes with constant curvature are the Lorentzian analogs of the Riemannian *hyperbolic ends*, which play a fundamental role in 3-dimensional topology. More formally, there exists a duality between hyperbolic ends and 3-dimensional MGHC spacetimes of positive constant curvature, which allows to translate some of our results on MGHC spacetimes into results on hyperbolic ends (see §12).

The theory of 3-dimensional MGHC spacetimes of constant curvature naturally splits into two cases: The spacetimes whose Cauchy surfaces have genus 0 or 1 are called *elementary*, whereas the spacetimes whose Cauchy surfaces have genus at least 2 are called *non-elementary*. As suggested by the terminology, the global geometry of elementary spacetimes is much easier to understand than those of non-elementary spacetimes. In some sense, elementary spacetimes can be considered as "particular" or "exact" solutions of the Einstein equations, for which all geometrical problems can be solved explicitly "by hand" (see *e.g.* [2, §10.2 and §10.3]). Nevertheless, since tools are quite different, we will restrict ourselves to non-elementary spacetimes.

All the spacetimes we shall consider in the sequel are not only time-orientable, but actually time-oriented. If M is a non-elementary 3-dimensional MGHC spacetime of constant curvature $\Lambda \geqslant 0$, then M is always either past complete (all past directed causal geodesic rays are complete), or future complete. Reversing the time-orientation if necessary, we will always assume we are in the second case. Note that a non-elementary 3-dimensional

MGHC spacetime of constant curvature $\Lambda < 0$ is always neither past nor future complete.

1.3. Some definitions: K-surfaces, K-slicings, K-times

Let (M, g) be a (time-oriented) 3-dimensional spacetime. Given a space-like surface Σ in M, the second fundamental form k_x^Σ of Σ at x is defined by $k_x^\Sigma(X, Y) = -\langle \nabla_X n, Y \rangle$ where n is the future oriented unit normal vector of Σ. The K-curvature (or Gauss-Killing-Kronecker-Lipschitz curvature) of Σ at x is $\kappa^\Sigma(x) = -\lambda_1(x)\lambda_2(x)$, where $\lambda_1(x)$ and $\lambda_2(x)$ are the principal curvatures of Σ at x, *i.e.* the eigenvalues of the second fundamental form k_x^Σ. Note that, in the case where the spacetime M has constant (sectional) curvature Λ, the Gaussian (*i.e.* sectional) curvature of Σ is $R^\Sigma = \Lambda + \kappa^\Sigma$, and its scalar curvature is $2R^\Sigma$.

A K-surface in M is a spacelike surface with constant K-curvature. Observe that, in the case where M has constant curvature, being a K-surface is equivalent to having constant Gaussian curvature.

From now on, we assume (M, g) to be globally hyperbolic and spatially compact. We define a K-slicing of M as a foliation of M by compact K-surfaces. Note that a K-slicing of M is always a trivial foliation, all the leaves of which are Cauchy surfaces of M. Equivalently, the leaves of a K-slicing of M are always the level sets of a certain time function.

Remark 1.1 (Slicing versus Foliation). — Our choice of the term "slicing" instead of the familiar one "foliation" (in the mathematical literature) is to emphasize the fact that we are dealing only with topologically trivial foliations, *e.g.* with all leaves compact. However, one could imagine to study general K-foliations with non-trivial dynamics. Although we will not investigate the question here, we guess that, in our context of MGHC spacetimes of constant curvature, any K-foliation is in fact a slicing. A similar question about CMC-foliations of the 3-dimensional Euclidean space is handled in [50].

A K-slicing need not be unique in general. In order to get some uniqueness, we need to define a more restrictive notion. A C^2 spacelike surface Σ in M is said to be *convex* (resp. *strictly convex*) if it has non-positive (resp. negative) principal curvatures. Similarly, Σ is *concave* or *strictly concave* if the principal curvatures are non-negative or positive. A K-time on M is a time function $\tau \colon M \to \mathbb{R}$ such that, for every $a \in \tau(M)$, the level set $\tau^{-1}(a)$ is a compact strictly convex K-surface with K-curvature equal to

a. Of course, the level sets of any K-time define a K-slicing of M. Nevertheless, the K-slicings defined by K-times are quite specific since all the leaves have negative principal curvatures, and since the K-curvature of the leaves increases with time. Note that the range $\tau(M)$ of a K-time is always included in $(-\infty, 0)$.

The maximum principle implies that K-times are unique. More precisely, if M admits a K-time $\tau\colon M \to \mathbb{R}$, then, for every $a \in \tau(M)$, the only compact K-surface of K-curvature a in M is the level set $\tau^{-1}(a)$. In particular, the only K-slicing on M is the one defined by the level sets of τ (see §4.4).

1.4. Purpose of the paper

In the present paper, we are going here to consider the following questions:

– the existence of K-time (or K-slicings) in 3-dimensional MGHC spacetimes of constant curvature,
– the existence of Cauchy surfaces of prescribed K-curvature in 3-dimensional MGHC spacetimes of constant curvature,
– the Minkowski problem in the 3-dimensional Minkowski space.

The results concerning the last two items are essentially application of the first one.

2. Statements of results

2.1. K-slicings of MGHC spacetimes with constant curvature

The following is our main result:

THEOREM 2.1. — *Let M be a 3-dimensional non-elementary MGHC spacetime with constant curvature Λ. If $\Lambda \geqslant 0$, reversing the time orientation if necessary, we assume that M is future complete.*

- *If $\Lambda \geqslant 0$ (flat case or locally de Sitter case), then M admits a unique K-slicing. The leaves of this slicing are the level sets of a K-time ranging over $(-\infty, -\Lambda)$.*
- *If $\Lambda < 0$ (locally anti-de Sitter case), then M does not admit any global K-slicing, but each of the two connected component of the complement of the convex core[2] of M admits a unique K-slicing. The leaves of the K-slicing of the past of the convex core are the*

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[2] About the notion of convex core, see §5.3.

level sets of a K-time ranging over $(-\infty, 0)$. The leaves of the K-slicing of the future of the convex core are the level sets of a reverse K-time[3] ranging over $(-\infty, 0)$.

Let us make a few comments on this result.

Gaussian curvature. —   Since the Gaussian curvature of a surface is $R = \Lambda + \kappa$, the Gaussian curvature of the leaves of the K-slicings provided by Theorem 2.1 varies in $(-\infty, 0)$ when $\Lambda \geqslant 0$, and in $(-\infty, \Lambda)$ when $\Lambda < 0$.

CMC times. —   It is our interest on CMC-times that led us to extend our attention to more general geometric times. Existence of CMC times on MGHC spacetimes of constant non-positive curvature and any dimension was proved in [1, 2]. For spacetimes locally modelled on the de Sitter space, there are some restrictions but not in dimension 3.

Regularity. —   The slicings provided by Theorem 2.1 are continuous. It follows from their uniqueness (*i.e.* they are canonical). Extra smoothness is not automatic (*e.g.* the cosmological time is $C^{1,1}$, but not C^2). Here, we can hope that our K-slicings are (real) analytic, and even more, they depend analytically on the spacetime (within the space of MGHC spacetimes of curvature Λ and fixed topology). All this depends on consideration of the Einstein equations in a K-gauge.

Non-standard isometric immersions of $\mathbb{H}^2$ in Min_3. —   It was observed by Hano and Nomizu [41], that the hyperbolic plane $\mathbb{H}^2$ admits non standard isometric immersions in the Minkowski space Min_3 (*i.e.* different from the hyperbola, up to a Lorentz motion).

Theorem 2.1, applied in the flat case, yields a K-slicing with (exactly) one isometric copy of $\mathbb{H}^2$ in $\mathbb{R}^{1,2}$, invariant by the holonomy group $\mathbf{\Gamma} \subset SO(1,2) \ltimes \mathbb{R}^3$ (since the curvature varies from $-\infty$ to 0). This isometric immersion is different from the standard one as soon as $\mathbf{\Gamma} \not\subset SO(1,2) \times \{0\}$.

Observe that, in higher dimension, $\mathbb{H}^n$ is rigid in Min_{n+1}. Indeed, the theory of rigidity of submanifolds in the Euclidean space extends straightforwardly to the Minkowski space. From this theory, a hypersurface with a second fundamental form of rank $\geqslant 3$ is rigid. By Gauss equation, any isometric immersion of a hyperbolic space in a Minkowski space is non-degenerate, and hence has the required condition on the rank, whence $n \geqslant 3$.

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[3] A reverse K-time on N is a function $\tau : N \to \mathbb{R}$ which is strictly decreasing along every future oriented causal curve, and such that, for every a, the level set $\tau^{-1}(a)$ is a compact locally strictly concave surface of constant K-curvature equal to a.

Spacetimes as curves in the Teichmüller space. — For the sake of simplicity, let us restrict ourselves here to the case $\Lambda = 0$, *i.e.* that of flat spaces. Denote by Σ the abstract topological surface homeomorphic to a Cauchy surface of M, and by $\mathcal{M}et(\Sigma)$ the space of Riemannian metrics on Σ up to isotopy. Any time function of M gives rise to a family (Σ_t, g_t) of Riemannian metrics on Σ (well-defined up to isotopy) parametrized by the given time. We have in particular, associated to our three favourite, K, CMC and CT times, three curves in $\mathcal{M}et(\Sigma)$:

$$\text{Met}^K \ (\text{resp. Met}^{cmc} \text{ and Met}^{ct})\colon \ t \to g_t \in \mathcal{M}et(\Sigma).$$

Now, let $Teich(\Sigma)$ be the Teichmüller space of Σ, *i.e.* the space of hyperbolic metrics (of curvature-1) on Σ, up to isotopy. Then, we have associated curves:

$$\text{Teich}^K \ (\text{resp. Teich}^{cmc} \text{ and Teich}^{ct})\colon \ t \to [g_t] \in Teich(\Sigma).$$

Thus, Met(t) is the Riemannian metric of the t-level, and Teich(t) is its underlying complex structure.

Let us note the following facts (but we cannot give here a complete overview):

(1) The curves Met^K, Met^{cmc}, Met^{ct}, Teich^K, Teich^{cmc}, Teich^{ct} encode the metric properties and the geometry of the spacetime M.

(2) These curves all coincide exactly when the spacetime (M, g) is static (*i.e.* its universal cover is the solid lightcone).

(3) In the vein of V. Moncrief's work on the "reduction of the Einstein equations to the Teichmüller space" in a CMC gauge in dimension $2 + 1$ [52], one can show in general that all the above curves in $Teich(\Sigma)$ have canonical lift in $T^*(Teich(\Sigma))$ or alternatively $T(Teich(\Sigma))$. There, they define (semi-)flows. In the CT-case, this is nothing but the grafting [18].

(4) It is an essential question to study the behaviour of these curves in both $\mathcal{M}et(\Sigma)$ and $Teich(\Sigma)$ when the parameter tends to one or the other extremity of the existence interval. A delicate point is that there are many compactifications for $Teich(\Sigma)$ and also many notions of convergences for sequences in $\mathcal{M}et(\Sigma)$. We have for instance, equivariant Gromov topology, convergence of spectrum, where in general the limit is a real tree. For existing asymptotic study, see for instance [22] in the CT-case and [4] in the CMC-case, and a forthcoming paper.

(5) One advantage of the K-slicing is that $\text{Teich}^K(t)$ is directly given by $\text{Met}^K(t)$, that is we don't need to uniformize it since it has constant

curvature t (by definition of the parameter). In this case, all notions of convergence in $\mathcal{M}et(\Sigma)$ (up to scaling) and in $Teich(\Sigma)$, coincide.

Affine foliations. — It is interesting to compare our results (existence of CMC-slicings and K-slicings) with similar results in the context of (equi-)affine geometry, in particular with the beautiful theory of affine spheres (see [48]): For every hypersurface Σ in the affine space $\mathbb{R}^{n+1}$, endowed with a parallel volume form, one can define its affine principal curvatures $\lambda_1, \lambda_2, \ldots, \lambda_n$. The hypersurface is an affine hypershere if all these affine principal curvatures are equal and constant along Σ. It is equivalent to require that all the affine normals either intersect at one point, or are mutually parallel. If $\lambda_1 = \lambda_2 = \cdots = \lambda_n < 0$, the affine sphere is hyperbolic. Calabi's conjecture states that:

(1) Every hyperbolic affine sphere which is complete for its "affine" metric is asymptotic to the boundary of a proper open convex cone,

(2) Conversely, for every $\lambda < 0$, every proper open convex contains a unique hyperbolic affine sphere of affine principal curvatures λ.

The first part has been solved by Cheng and Yau under the hypothesis that the affine sphere is complete for an Euclidean metric on $\mathbb{R}^{n+1}$, hypothesis removed later by Li. Most interesting for us is the second part, proved by Sasaki and Gigena. It shows that any proper open cone admits a natural foliation by hyperbolic affine spheres, which can be considered as hypersurfaces of (affinely) constant scalar curvature, or as well of constant mean curvature.

For more details, see [48, Chapter 2], where the authors also discuss the method of C. P. Wang for constructing all hyperbolic affine 2-spheres which admit the action of a discrete subgroup of the equiaffine group with compact quotient.

2.2. Duality hyperbolic-de Sitter, K-slicings of hyperbolic ends

The duality between the de Sitter space dS_3 and the hyperbolic space $\mathbb{H}^3$ (see §12) will allow us to deduce the existence of K-surfaces in hyperbolic ends from the existence of K-surfaces in MGHC spacetimes of constant positive curvature. So we will be able to recover the following result of F. Labourie:

THEOREM 2.2 ([44]). — *A 3-dimensional hyperbolic end possesses a K-slicing with K-curvature ranging in $(0, 1)$.*

Some explanations on hyperbolic ends can be found in §12. Note that, in the Riemannian setting, the K-curvature of a surface is just the product $\lambda_1 \lambda_2$ of the principal curvatures.

Remark 2.3.

(1) It was observed by Mess that the dS$_3$-case of our main Theorem 2.1 can be deduced from Labourie's Theorem, using the duality between dS$_3$ and $\mathbb{H}^3$.

(2) Our proof here of Theorem 2.1 in the dS$_3$-case is completely different from Labourie's one for hyperbolic ends. Actually, we think that our method gives the natural framework for approaching many results of existence of "geometric" surfaces or foliation, even in the hyperbolic setting.

2.3. Surfaces with prescribed K-curvature

The barriers method (see §3.1) allows one to find a surface with constant K-curvature κ in between two surfaces with (non-constant) K-curvatures bounded respectively from below and above by κ. This is an important ingredient in our proof of Theorem 2.1. This method generalizes to the case where κ is not a constant any more, but rather a function (see [35] or section 13). Using this generalization and Theorem 2.1, we will get the following existence result:

THEOREM 2.4. — *Let M be a 3-dimensional non-elementary MGHC spacetime with constant curvature Λ. If $\Lambda \geqslant 0$, reversing the time-orientation if necessary, we assume that M is future complete. Let f be a smooth function on M admitting a range contained in a compact interval $[a,b] \subset]-\infty, \min(0,-\Lambda)[$. Then, there is a Cauchy surface Σ is M with K-curvature $\kappa^\Sigma(x) = f(x)$ for every $x \in \Sigma$.*

In particular:

COROLLARY 2.5. — *Let M be as above. Let Σ_0 be the abstract topological surface homeomorphic to the Cauchy surfaces of M, and let f_0 be a smooth real-valued function on Σ_0, such that the range of f_0 is a contained in a compact interval $[a,b] \subset]-\infty, \min(0,-\Lambda)[$. Then, there exists an embedding of $\phi\colon \Sigma_0 \hookrightarrow M$ such that $\Sigma = \phi(\Sigma_0)$ is a Cauchy surface in M, and the K-curvature of Σ at $\phi(x)$ is equal to $f_0(x)$ for every $x \in \Sigma_0$.*

2.4. The Minkowski problem

Let us recall the classical formulation of the Minkowski problem (see for instance [27, 33, 44, 53]). If S is a closed convex smooth surface in the Euclidean space $\mathbb{R}^3$, then its Gauss map $\nu_S \colon S \to \mathbb{S}^2$ is a diffeomorphism, and one can consider the map $K^S := \kappa^S \circ \nu_S^{-1} \colon \mathbb{S}^2 \to (0, +\infty)$, where κ^S is the K-curvature of S. The Minkowski problem consists in characterizing the functions on the sphere $\mathbb{S}^2$ which have the form K^S for some surface S.

This problem can be transposed in the Lorentzian setting, by replacing the Euclidean space $\mathbb{R}^3$ by the Minkowski space Min_3, requiring the surface S to be spacelike (but not compact!), and replacing the sphere $\mathbb{S}^2$ by the hyperbolic space $\mathbb{H}^2$. Using Theorem 2.1 and a barrier theorem of Gerhardt ([35]), we are able to solve the Minkowski problem in Min_3 in the particular case where the function K^S is invariant under a co-compact Fuchsian group $\Gamma \subset SO(1, 2) = \mathrm{Isom}(\mathbb{H}^2)$:

THEOREM 2.6. — *Let Γ be a co-compact Fuchsian subgroup of $SO(1,2)$, and $f \colon \mathbb{H}^2 \to (-\infty, 0)$ be a Γ-invariant smooth function. Then there exists a strictly convex spacelike surface S in Min_3 such that $f = \kappa^S \circ (\nu^S)^{-1}$, where $\nu^S \colon S \to \mathbb{H}^2$ is the Gauss map of S and $\kappa^S \colon S \to (0, +\infty)$ is the K-curvature (= Gauss curvature) of S. Moreover, if $\boldsymbol{\Gamma}$ is a subgroup of $SO(1,2) \ltimes \mathbb{R}^3 = \mathrm{Isom}(\mathrm{Min}_3)$ which projects bijectively on Γ, then there exists a unique such (convex) surface S which is $\boldsymbol{\Gamma}$-invariant.*

A few comments on the Minkowski problem in Min_3:

(1) There is no uniqueness in general for the Minkowski problem in Min_3. For example, for $f = -1$, any isometric copy of $\mathbb{H}^2$ in Min_3 is a solution of the Minkowski problem. The uniqueness in Theorem 2.6 strongly relies on the convexity and $\boldsymbol{\Gamma}$-invariance hypotheses.

(2) Despite the obvious and natural interest of the Minkowski problem in the Lorentzian setting, there seems to be very little work on this problem. We know essentially two substantial contributions: [47] by A. M. Li and [40] by B. Guan, H.-Y. Jian and R. Schoen. In the last paper, the authors proved a result similar to ours (equally in dimension $2 + 1$), with the invariance condition replaced by prescribing the asymptotic behaviour at infinity. Finally, the referee points to us the unpublished work of Igor Iskhakov who studied similar questions in a polyhedral framework, see [42].

(3) There are generalizations of the Minkowski problem in other directions, still in a Riemannian setting, but for ambient spaces different from $\mathbb{R}^n$. The point here is to define a substitute of the Gauss map. This is indeed possible for universal spaces of constant curvature $\mathbb{S}^{n+1}$ and $\mathbb{H}^{n+1}$. One can quote here recent works by C. Gerhardt which solve the corresponding Minkowski problem (for convex bodies)[36, 33], and a previous work of F. Labourie on an equivariant Minkowski problem for surfaces in the hyperbolic space $\mathbb{H}^3$ [44]. Finally, the Minkowski problem admits generalizations to other curvatures (see for instance [39, 58]).

2.5. Related works

The general mathematical framework unifying our contributions here is that of prescribing (extrinsic) curvatures of hypersurfaces in M. Such a hypersurface N is sometimes called of Weingarten type, *i.e.* it is an F-hypersurface in the sense that: $F(\overrightarrow{\lambda}(x)) = F(\lambda_1(x), \dots, \lambda_n(x)) = f(x)$, where f is a given function on N, and F is a function of the principal curvatures. For instance, one asks the following general questions about them:

– A Dirichlet problem,
– A Dirichlet problem at infinity, also said "entire hypersurfaces problem" (*i.e.* existence of complete hypersurfaces with a prescribed asymptotic behaviour),
– Existence of F-slicings. This requires M to be topologically trivial, or that the slicing is defined on an end of M,
– A natural generalization consists in considering curvature functions $F(\overrightarrow{\lambda}, \nu)$, i.e F also depends on a normal vector ν.
– Finally, regarding any of these questions, one specifies whether M is Riemannian or Lorentzian.

As said previously, there is a wide literature on the CMC case (also called the H-curvature). Let us quote some recent achievements concerning the K-curvature:

– The Dirichlet problem for the K-curvature in the Minkowski space was studied in particular by P. Delanoë [30] and B Guan [38].
– Entire K-hypersurfaces were studied by B. Guan, H.-Y Jian and R. Schoen [40]. (The corresponding CMC case was considered by A. Treibergs [60], and that on the "finite" Dirichlet problem, by Bartnik and Simon [14]).

– In [15, 61] P. Bayard and afterwards J. Urbas solved the Dirichlet problem for the scalar curvature in the Minkowski space of any dimension. Remark that for our dimension $2+1$, the scalar and K-curvatures coincide. Also [16] solved the entire hypersurface problem for the scalar curvature.

– C. Gerhardt and O. Schnürer [34, 57] proved criteria for the solvability of the Dirichlet problem on general Lorentz manifolds, and for various curvature functions,

– Existence of K-slicings of hyperbolic ends in 3-dimension was proved by F. Labourie (see §2.2), and generalized by G. Smith for higher dimensional quasi-Fuchsian ends, where the K-curvature is replaced by a variant, governed by the "special Lagrangian" equation [59],

– Recently, R. Mazzeo and F. Pacard [49] considered manifolds that are merely asymptotically hyperbolic and admitting a conformal compactification. They prove existence of various geometric foliations near their boundary.

3. Ingredients of the proofs

Before going into the details, we present the main ingredients of the proofs of our results. Let (M, g) be a non-elementary 3-dimensional MGHC spacetime with constant curvature Λ. If $\Lambda \geqslant 0$, reversing the time-orientation if necessary, we can (and we do) assume that M is future complete. To prove our main Theorem 2.1, we have to construct a K-slicing on M (or on the complement of the convex core of M if $\Lambda < 0$). Here are the main ingredients of this construction:

3.1. Existence of barriers

A very general principle states that the existence of a surface with prescribed curvature should follow from the existence of so-called *barriers*. In our context, the suitable result was proved by C. Gerhardt:

DEFINITION 3.1. — *For $\kappa \in \mathbb{R}$, a pair of κ-barriers is a pair of disjoint strictly convex Cauchy surfaces Σ^-, Σ^+ in M such that:*

a. Σ^- *is in the past of* Σ^+,

b. *the K-curvature of* Σ^- *is bounded from above by* κ,

c. *the K-curvature of* Σ^+ *is bounded from below by* κ.

THEOREM 3.2 (Gerhardt, [34][4]). — *Given a real number* $\kappa < -\Lambda$, *if* (M, g) *admits a pair of* κ-*barriers* (Σ^-, Σ^+), *then* (M, g) *admits a strictly convex Cauchy surface* Σ *with constant K-curvature* κ. *Moreover, the Cauchy surface* Σ *is in the future of* Σ^- *and in the past of* Σ^+.

Remark 3.3. —   The sign convention of Gerhardt for the K-curvature is the opposite of ours (this is the reason why the main result of [34] asserts that one can find a Cauchy surface with constant Gauss curvature k for any $k > 0$ provided that there is a pair of barriers).

Also note Gerhardt uses the *past* directed unit normal vector to define the principal curvatures; It follows that a surface which is called convex in [34, 35, 33] is called concave here, and vice-versa. This is not a problem since all the result are valid both for convex and concave surfaces (one needs to be careful with the definition of the barriers).

Thanks to this barriers Theorem, to find surfaces of constant K-curvature reduces to find κ-barriers. We will use it to exhibit in §9.1 at least one K-surface, and also to find local K-slicings (see §3.5). It will also be useful to find global K-slicing, but this last step will decompose in two cases, the case $\Lambda \geqslant 0$ and the case $\Lambda < 0$.

In the former case, we will prove directly:

THEOREM 3.4. — *Assume* Λ *is non-negative. There exists* $\varepsilon > 0$ *such that there exists a pair of* κ-*barriers for every* $\kappa \in (-\Lambda - \varepsilon, -\Lambda)$.

Theorem 3.4 will be obtained by using previously known results on the existence of locally strictly convex Cauchy surfaces in flat spacetimes, and some estimates on the behaviour of the K-curvature when one pushes such a Cauchy surface along the orthogonal geodesics. Theorem 3.4 together with Gerhardt's theorem show the existence of a convex Cauchy surface Σ_κ with constant K-curvature κ in M for every $\kappa \in (-\Lambda - \varepsilon, -\Lambda)$. Using classical arguments and some informations on our barriers, we will prove that the family of Cauchy surfaces $(\Sigma_\kappa)_{\kappa \in (-\Lambda - \varepsilon, -\Lambda)}$ is a foliation of a neighbourhood of the future end of M. This will provide us with a "local K-slicing" on a neighbourhood of the future end of M (in the case $\Lambda \geqslant 0$).

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[4] The main result of [34] is stronger than the result stated above (see §13). For instance, it remains true in Lorentzian spacetimes of non-constant curvature, and admits natural generalizations in higher dimensions.

The case $\Lambda < 0$ will be treated differently, using some duality argument, see §3.6.

3.2. Systole and distance to the past singularity

Let τ be the *cosmological time* on M (see §4.1 for more details). A major (and the most original) ingredient of our proof of Theorem 2.1 is the following result, which relates the systole of a Cauchy surface Σ in M and the "distance from Σ to the past singularity of M".

THEOREM 3.5. — *For every $\varepsilon > 0$, there exists a constant $\alpha > 0$ such that, for any Cauchy surface Σ, if $\inf \tau_{|\Sigma}$ is smaller than α then the systole of Σ is smaller than ε.*

This statement should be understood as follows: *If the systole of a Cauchy surface Σ in M is not too small, then no point of Σ is close to the past singularity of M.* We will use this to prove that a sequence $(\Sigma_n)_{n \in \mathbb{N}}$ of Cauchy surfaces with K-curvature bounded away $-\infty$ remains far from the initial singularity (see §3.4).

The proof of Theorem 3.5 strongly relies on some fine knowledge of the geometry of MGHC spacetimes with constant curvature. We think that this Theorem 3.5 is not only a crucial step in the proof of our main result, but is also interesting in its own right. It can be used in various type of situations to prove that a sequence of Cauchy surfaces is relatively compact.

3.3. Decreasing sequences of convex Cauchy surfaces

Another ingredient of our proof of Theorem 2.1 is the fact that the limit of a decreasing sequence of convex Cauchy surfaces is always "spacelike". More precisely:

THEOREM 3.6. — *Let $(\Sigma_n)_{n \in \mathbb{N}}$ be a sequence of convex Cauchy surfaces[5] in M. Assume that this sequence is decreasing (i.e. $I^+(\Sigma_{n+1}) \supset I^+(\Sigma_n)$ for every n), and that the set $\Omega = \bigcup_{n \geqslant 0} I^+(\Sigma_n)$ is not the whole spacetime M. Let $\Sigma_\infty = \partial\Omega$ (note that the Ω is a locally geodesically convex set, and thus Σ_∞ is a convex topological surface). Then all the support planes of Σ_∞ are spacelike.*

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[5] or generalized Cauchy surfaces, see §4.2.

Note that Σ_∞ is not a Cauchy surface in general. An easy (but important) corollary of Theorem 3.6 is the fact that a decreasing sequence of convex Cauchy surfaces is always "uniformly spacelike". To make this precise, let us denote by $T^{-1}M$ the set of all couples (x, P) where x is a point of M and P is a totally geodesic spacelike plane passing through x. The set $T^{-1}M$ is naturally identified with the subset of the tangent bundle TM of M made of all couples (x, v) where v is a future-directed tangent vector of squared norm -1 (the identification is of course through the correspondence between a totally geodesic spacelike plane and its future-directed unit normal vector). This endows $T^{-1}M$ with a natural topology. Clearly, $T^{-1}M$ is not compact (a sequence of spacelike planes may converge to a null plane!).

DEFINITION 3.7. — *A sequence of spacelike surfaces $(\Sigma_n)_{n\in\mathbb{N}}$ is uniformly spacelike if, for every sequence $(x_n)_{n\in\mathbb{N}}$ with $x_n \in \Sigma_n$:*

  – *either the sequence $(x_n)_{n\in\mathbb{N}}$ escapes from any compact subset of M,*
  – *or the sequence $(x_n, P_n)_{n\in\mathbb{N}}$, where P_n is the tangent plane of the surface Σ_n at x_n, stays in a compact subset of $T^{-1}M$.*

COROLLARY 3.8. — *Let $(\Sigma_n)_{n\in\mathbb{N}}$ be a sequence of convex Cauchy surfaces in M. Assume that this sequence is decreasing, and that the set $\Omega = \bigcup_{n\geqslant 0} I^+(\Sigma_n)$ is not the whole spacetime M. Then $(\Sigma_n)_{n\in\mathbb{N}}$ is uniformly spacelike.*

3.4. Sequences of convex Cauchy surfaces with constant K-curvature

Using the two results discussed above, we will get a quite precise description of the possible asymptotic behaviour of a decreasing sequence of convex Cauchy surfaces with constant K-curvature.

THEOREM 3.9. — *Let $(\Sigma_n)_{(n\in\mathbb{N})}$ be a sequence of strictly convex Cauchy surfaces in M, such that, for every $n \in \mathbb{N}$, the surface Σ_n has constant K-curvature κ_n. Assume that this sequence is bounded away from the future end of M (i.e. the cosmological time is bounded from above on $\bigcup_n \Sigma_n$).*

If $\kappa_n \to \kappa$ with $-\infty < \kappa < \min(0, -\Lambda)$, then the sequence $(\Sigma_n)_{n\in\mathbb{N}}$ is precompact in the C^∞ topology. In particular, (Σ_n) is bounded away from the past singularity of M (there exists a Cauchy surface Σ such that $\Sigma_n \subset I^+(\Sigma)$ for all n), and there exists a subsequence of (Σ_n) converging to a smooth surface Σ_∞ with constant K-curvature κ.

On the contrary, if $\kappa_n \to -\infty$, then Σ_n is covering, that is the union over $n \in \mathbb{N}$ of the sets $I^+(\Sigma_n)$ equals the whole spacetime M.

The proof of this theorem involves several ingredients:

– an easy lemma which provides a uniform upper bound for the diameter of the Σ_n's,
– Margulis' Lemma which tells that a uniform upper bound for the diameter of the Σ_n's yields some uniform lower bound for the systole of the Σ_n's (in the case where the sequence $(\kappa_n)_{n\in\mathbb{N}}$ is bounded from below),
– Theorem 3.5 which implies that the Σ_n's remain far from the initial singularity (in the case where he sequence $(\kappa_n)_{n\in\mathbb{N}}$ is bounded from below),
– Theorem 3.6 which roughly says that any limit point Σ_∞ of the sequence $(\Sigma_n)_{n\in\mathbb{N}}$ is uniformly spacelike,
– a result of Schlenker which says that a C^0 limit of convex spacelike surfaces either is a smooth surface or contains a complete geodesic ray,
– the geometric description of non-elementary MGHC spacetimes of constant curvature which entails the fact that such spacetimes never contain any complete geodesic ray.

3.5. Perturbation of Cauchy surfaces with constant K-curvature

A classical calculation yields:

PROPOSITION 3.10. — *Assume that (M, g) admits a strictly convex Cauchy surface Σ with constant K-curvature $\kappa < -\Lambda$. Then, one can find a strictly convex Cauchy surface Σ^- in the past of Σ such that the K-curvature is strictly bounded from above by κ (i.e. such that (Σ^-, Σ) is a pair of κ'-barriers for every $\kappa' < \kappa$ such that κ' is close enough to κ).*

The surface Σ^- can be obtained simply by pushing slightly Σ along the geodesics orthogonal to Σ. Proposition 3.10 and Theorem 3.2 show that, if M admits a convex Cauchy surface Σ with constant K-curvature $\kappa < -\Lambda$, then some neighbourhood of Σ in the past of Σ is foliated by convex Cauchy surfaces with constant K-curvature. Together with Theorem 3.9, this will allow us to prove that any local K-slicing on M can be extended towards the past until it reaches the past singularity of M (*i.e.* until one gets a K-slicing on a neighbourhood of the past end of M). Since, in the case $\Lambda \geqslant 0$ (flat and de Sitter case), we have already explained how to get a local K-slicing on a neighbourhood of the future end of M (see §3.1), this will complete the proof of Theorem 2.1 in the flat case and in the de Sitter case.

3.6. Duality between convex and concave surfaces in the anti-de Sitter space

All the arguments developed above in the case $\Lambda \geqslant 0$ (flat and de Sitter case) apply in the case $\Lambda < 0$ (anti-de Sitter case), except for one: If $\Lambda < 0$, then M is neither future complete nor past complete, and pushing along orthogonal geodesics does not provide, as explained in §3.1, a family of κ-barriers foliating a neighbourhood of the future end of M.

So, in the anti-de Sitter case, one needs an alternative argument to complete the proof of Theorem 2.1. The proof goes as follows (see §11): From [12] we know that the spacetime contains a maximal Cauchy surface Σ. The set of points at Lorentzian distance $\pi/4$ from Σ is an union of two Cauchy surfaces $\Sigma^{\pi/4}$, $\Sigma^{-\pi/4}$, one in the future of Σ, the other in the past of Σ. These two surfaces have both constant K-curvature -1, and are thus κ-barriers. As discussed just above, we cannot prove that the future of $\Sigma^{-\pi/4}$ is foliated by K-surfaces. However, the arguments used in the flat and de Sitter cases ensure that the past $I^-(\Sigma^{-\pi/4})$ admits a K-time ranging over $(-\infty, -1)$. Similarly, the future $I^+(\Sigma^{\pi/4})$ of $\Sigma^{\pi/4}$ admits a reverse K-time. Now the key point is that there is a natural duality between convex and concave spacelike surfaces in AdS_3, preserving the property of having constant K-curvature. Therefore, the dual of the K-slicing of $I^+(\Sigma^{\pi/4})$ is a K-slicing of the future of $\Sigma^{-\pi/4}$ in the past of the convex core, extending the K-slicing of $I^-(\Sigma^{-\pi/4})$.

3.7. Ingredients of the proofs of Theorems 2.2, 2.4 and 2.6

As already explained in the introduction, Theorem 2.2 is an easy corollary of the existence of K-slicing for non-elementary 3-dimensional MGHC spacetimes of de Sitter type, and of the duality between the de Sitter space dS_3 and the hyperbolic space $\mathbb{H}^3$. Theorems 2.2 and 2.4 will follow from a generalized version of the Gerhardt's barriers theorem cited above and from Theorem 2.1: The barriers needed to apply Gerhardt's Theorem will be certain leaves of the K-slicing provided by Theorem 2.1.

3.8. Organization of the paper

Section 4 is devoted to some general preliminaries on time functions, surfaces with constant curvature, etc. In Section, 5, we recall the main geometrical properties of MGHC spacetimes with constant curvature. In Section 6,

we establish the desired relations between the systole of a Cauchy surface and the distance from this Cauchy surface to the initial singularity (Theorem 3.5). In Section 7, we prove that the limit of a decreasing sequence of convex Cauchy surfaces is always uniformly spacelike (Theorem 3.6). In Section 8, we prove Theorem 3.9 concerning decreasing sequence of Cauchy surfaces with constant K-curvature. In Section 9, we explain how to construct barriers; iIn particular, we prove Theorem 3.4 and Proposition 3.10. In Section 10 and 11, we prove our main Theorem 2.1 concerning the existence of K-slicings of 3-dimensional non-elementary MGHC spacetimes of constant curvature. In Section 12, we explain the duality between dS$_3$ and $\mathbb{H}^3$, and how to deduce Theorem 2.2 from Theorem 2.1. In Section 13, we deduce Theorem 2.4 from Theorem 2.1 and Gerhardt's barrier theorem. In Section 14, we consider the Minkowski problem and prove Theorem 2.6.

4. Cosmological time, Cauchy surfaces, maximum principle

For all basic Lorentzian notions such as chronological orientation, past/future, causal past/future, achronal subsets, acausal subsets, edgeless achronal subsets, Cauchy surfaces, global hyperbolicity, and maximal globally hyperbolic space, we refer to [54, 17]. Recall that MGHC is the acronym for "maximal globally hyperbolic spatially compact".

4.1. Cosmological time

In any spacetime (M, g), we can define the *cosmological time* (see [5]).

DEFINITION 4.1. — *The cosmological time of a spacetime (M, g) is the function $\tau \colon M \to [0, +\infty]$ defined by*

$$\tau(x) = \mathrm{Sup}\{\mathrm{Length}(\gamma) \mid \gamma \in \mathcal{R}^-(x)\},$$

where $\mathcal{R}^-(x)$ is the set of all past-oriented causal curves starting at x, and Length(γ) *the Lorentzian length of the causal curve γ.*

In general, this function may have a very bad behaviour: For example, if (M, g) is Minkowski space or the de Sitter space, then $\tau(x) = +\infty$ for every x.

DEFINITION 4.2. — *A spacetime (M, g) is said to have regular cosmological time, if*

(1) M has finite existence time, i.e. $\tau(x) < +\infty$ for every x in M,
(2) for every past-oriented inextendible causal curve $\gamma \colon [0, +\infty) \to M$, $\lim_{t\to\infty} \tau(\gamma(t)) = 0$.

Regular spacetimes admit many interesting properties ([5, Theorem 1.2]): In particular, they are globally hyperbolic, and their cosmological time is a locally Lipschitz time function. A common and important feature of all non-elementary MGHC spacetimes of constant curvature is that up to reversal of the time orientation, they have regular cosmological time ([1, 2, 18]).

Remark 4.3. — For every spacetime (M, g) one can also introduce the reverse cosmological time $\check{\tau} \colon M \to [0, +\infty]$ of a spacetime (M, g), defined by $\check{\tau}(x) = \mathrm{Sup}\{\mathrm{Length}(\gamma) \mid \gamma \in \mathcal{R}^+(x)\}$, where $\mathcal{R}^+(x)$ is the set of all future-oriented causal curves starting at x. In other words, $\check{\tau}$ is the cosmological time of the spacetime obtained by reversing the time-orientation of (M, g).

4.2. Generalized Cauchy surfaces

Let (M, g) be a globally hyperbolic spacetime.

First of all, we recall how Geroch proved that M is diffeomorphic to a product $\Sigma \times \mathbb{R}$ (see [37]). Select a Cauchy-time $t \colon M \to \mathbb{R}$, i.e. a time function such that every level $\{t = a\}$ is a Cauchy surface. Let X be the vector field $-\nabla t$, i.e. minus the gradient of t, and let ϕ^t be the flow of X. Let finally Σ be the level set $\{t = 0\}$. Then, the map $F \colon \Sigma \times \mathbb{R} \to M$ defined by $F(x, t) = \phi^t(x)$ is the required diffeomorphism.

It follows then nearly immediately from the definitions that every achronal subset E of M is the image by F of the graph of a locally Lipschitz function $u \colon \Lambda \to \mathbb{R}$, where Λ is a subset of Σ:

$$E = \big\{(x, t) \in \Lambda \times \mathbb{R} \mid t = u(x)\big\}.$$

If the achronal subset is edgeless then Λ is open and u is proper. Furthermore, if E is edgeless achronal and compact, then $\Lambda = \Sigma$. In particular, Σ is compact too (i.e. M is spatially compact), and every inextendible causal curve intersects E.

Following the spirit of [37], we thus can define:

DEFINITION 4.4. — A generalized Cauchy surface in a MGHC spacetime is a compact achronal edgeless subset.

Now, let us observe that, in the above discussion, X can be replaced by any future-oriented Lipschitz-regular timelike vector field on M. The only difference is that the associated flow ϕ^t might be non-complete, but this does not matter: Applying the process above we obtain an (bilipschitz) identification $F \colon \Omega \to M$ where Ω is an open domain in $\Sigma \times \mathbb{R}$ defined by:

$$\Omega = \left\{ (x,t) \in \Sigma \times \mathbb{R} \mid u_-(x) < t < u_+(x) \right\}$$

where Σ is any Cauchy surface in M and u_- (respectively u_+) is upper semi-continuous (respectively lower semi-continuous). We still have at our disposal a nice description of compact achronal edgeless subsets, *i.e.* generalized Cauchy surfaces: They are graphs of locally Lipschitz functions $u \colon \Sigma \to \mathbb{R}$ satisfying $u_- < u < u_+$. In the sequel, we will use this later remark with X being the opposite of the gradient of a cosmological time function, and the surface Σ being the level set $\tau^{-1}(1)$.

4.3. Convex surfaces

Let (M, g) be a time-oriented spacetime. Recall that a C^2 spacelike surface $\Sigma \subset M$ is said to be *convex* (resp. *strictly convex*) if its second fundamental form is a negative (resp. negative definite) quadratic form. It is not hard to prove that Σ is convex if and only if its future $I^+(\Sigma)$ is locally geodesically convex (every point in M has an arbitrarily small neighbourhood U such that $U \cap I^+(\Sigma)$ is geodesically convex). The advantage of the later characterization is that its extends to non-smooth hypersurfaces:

DEFINITION 4.5. — *An achronal edgeless set $\Sigma \subset M$ (e.g. a generalized Cauchy surface) is said to be* convex *if its future $I^+(\Sigma)$ is locally geodesically convex.*

The three model spacetimes Min_3, dS_3 and AdS_3 admits some locally projective models (usually called *Klein models*). These models allow to reformulate the definition of convexity in the case where (M, g) is a 3-dimensional spacetime with constant curvature. Indeed in this particular case, every point $p_0 \in M$ admits arbitrarily small *convex charts centered at p_0*, *i.e.* causally convex and geodesically convex neighborhood U_0 such that there is a diffeomorphism $\varphi_0 \colon U_0 \to V_0$ where V_0 is an open domain in the vector space $\mathbb{R}^3$ with coordinates (x, y, z) satisfying the following properties:

  (1) $\varphi_0(p_0) = 0$,
  (2) φ_0 maps nonparametrized geodesics in U_0 to affine segments in V_0,

(3) $\varphi^* \frac{\partial}{\partial x_0}$ is a future oriented timelike vector field,

(4) V_0 is the domain $\{x^2 + y^2 < 1, -1 < z < 1\}$,

(5) for every $t \in]-1, +1[$ the hypersurface $\varphi_0^{-1}(\{z = t\})$ is a totally geodesic spacelike surface acausal in U_0,

(6) the image by φ_0 of $I_{U_0}^+(p_0)$ (resp. $I_{U_0}^-(p_0)$) is the open domain $\{z > 0, z^2 > x^2 + y^2\}$ (resp. $\{z < 0, z^2 > x^2 + y^2\}$).

Moreover, for any spacelike Cauchy surface Σ,

(7) for any convex chart centered at $p_0 \in \Sigma$, the image under φ_0 of the intersection $\Sigma \cap U_0$ is the graph of a Lipschitz map $(x, y) \mapsto f(x, y)$,

(8) Σ is convex (resp. concave) if and only if for any sufficiently small convex chart centered at a point p_0 of Σ the function f is concave (resp. convex).

Another important property of spacetimes with constant curvature is the abundance of totally geodesic subspaces. Let M be a 3-dimensional spacetime with constant curvature, and Σ be a spacelike surface in M. A *lower support plane* of Σ is a 2-dimensional (necessarily spacelike) totally geodesic space $P \subset M$ such that:

(1) P intersects Σ at some point p,

(2) Σ is locally contained in the future of P; More precisely, there exists a convex neighbourhood U of p such that $\Sigma \cap U \subset J_U^+(P)$.

One defines similarly the notion of *upper support plane* (by replacing "future" by "past" in the above definition). Using convex charts, it is easy to see that a spacelike surface Σ is convex (resp. concave) if and only if it admits a lower (resp. upper) support plane at each of its points.

4.4. Maximum principle and K-times

Let (M, g) be a (time-oriented) MGHC spacetime. Recall that a K-time on M is a time function $\kappa \colon M \to \mathbb{R}$ such that every level set $\kappa^{-1}(a)$ is a hypersurface of constant K-curvature a. One of the most important features of the K-times is their uniqueness. This property is a consequence of the so-called *maximum principle*. In the following proposition, don't forget that in our convention, principal curvatures of convex spacelike surfaces are negative.

PROPOSITION 4.6 (Maximum Principle). — *Let Σ_1, Σ_2 be two convex spacelike surfaces in M. Assume that Σ_1, Σ_2 meet at some point x, and that Σ_2 is in the future of Σ_1. Then, the principal curvatures of Σ_1 are greater than or equal to those of Σ_2.*

To prove this, one only needs to write the surfaces Σ_1, Σ_2 as the graphs of some functions f_1, f_2 over the exponential of their common tangent plane at x, and to compare the Hessians of f_1 and f_2 at x. See, $e.g.$ [12, Lemma 2.3].

COROLLARY 4.7 (Uniqueness of K-times). — *Assume that M admits a K-time $\kappa \colon M \to \mathbb{R}$. Then, for every $a \in \tau(M)$, the level set $C_a := \kappa^{-1}(a)$ is the unique compact K-surface of K-curvature a in M. In particular, the unique K-slicing on M is the one defined by the level sets of κ.*

Proof. — Let $\Sigma \subset M$ be a compact K-surface with K-curvature a_0. For every $a \in \kappa(M)$, let $C_a := \kappa^{-1}(a)$. Define

$$a^- := \inf_{\Sigma} \kappa, \;\; a^+ = \sup_{\Sigma} \kappa.$$

Since Σ is compact, Σ meets C_{a^-} and C_{a^+}. Moreover, Σ is in the future of C_{a^-}, and in the past of C_{a^+}.

Let x^- be a point where Σ meets C_{a^-}. The maximum principle implies that the principal curvatures Σ at x^- are smaller than or equal to those of C_{a^-}. By definition of a K-time, the surface C_{a^-} is strictly convex, $i.e.$ has negative principal curvatures. It follows that the principal curvatures of Σ at x^- are negative. Moreover, the product of the principal curvatures of Σ (resp. C_{a^-}) at x^- is equal to $-a_0$ (resp. $-a^-$). It follows that $a_0 \leqslant a^- < 0$.

Now, we see that the principal curvature of Σ are negative at every point of Σ. Indeed, they are negative at x^-, and their product is everywhere equal to $-a_0 \neq 0$. Let x^+ be a point where Σ meets C_{a^+}. By definition of a^+ and since κ is a K-time, Σ is in the past of C_{a^+}. Hence, the maximum principle implies that the principal curvatures curvatures of Σ at x^+ are bigger than those of C_{a^+}. Now, recall that the principal curvatures of Σ and C_{a^+} are negative, and the product of principal curvatures of Σ (resp. C_{a^+}) is equal to a_0 (resp. a^+). It follows that $a_0 \geqslant a^+$.

So we have $a^+ \leqslant a \leqslant a^-$ and $a^- \leqslant a^+$. It follows that $a = a^- = a^+$. Since Σ is in the future of C_{a^-} and in the past of C_{a^+}, we obtain the equality $\Sigma = C_{a^-} = C_{a^+}$. □

Remark 4.8. — The strict convexity of the level sets of κ is crucial in the above proof.

5. Geometry of 3-dimensional MGHC spacetimes of constant curvature

Before starting the proof of Theorem 2.1, we need to recall some general facts on the geometry 3-dimensional MGHC spacetimes with constant curvature. In particular, we need to describe the decomposition of every such spacetime into simple "building blocks".

5.1. Geometry of 3-dimensional MGHC flat spacetimes

In [51], Mess has proved that every MGHC flat spacetime can be obtained as a quotient of domain of the Minkowski space Min_3 by a discrete group of isometries. More precisely:

DEFINITION 5.1. — *A future complete regular domain in* Min_3 *is a domain of the form*

$$E = \bigcap_{P \in \Lambda} I^+(P)$$

where Λ *is a set of lightlike affine planes in* Min_3, *and* $I^+(P)$ *denotes the future of the lightlike plane* P. *Such a domain* E *is called* non-elementary *if* Λ *contains three pairwise non-parallel planes. The notion of* (non-elementary) *past complete regular domain is defined analogously.*

Observe that every future (resp. past) complete regular domain $E \subset \mathrm{Min}_3$ is a convex set in Min_3. Moreover, if E is non-elementary, then E is a *proper* convex set, *i.e.* E does not contain any entire affine line of Min_3.

THEOREM 5.2 (Mess, [51]). — *Every non-elementary 3-dimensional MGHC flat spacetime is isometric to the quotient* $\mathbf{\Gamma} \setminus E$ *of a non-elementary future or past complete regular domain* $E \subset \mathrm{Min}_3$ *by a torsion-free discrete subgroup* $\mathbf{\Gamma}$ *of* $\mathrm{Isom}_0(\mathrm{Min}_3) = \mathrm{SO}_0(1,2) \ltimes \mathbb{R}^3$. *Moreover, the linear part of* $\mathbf{\Gamma}$ *is a co-compact Fuchsian subgroup* Γ *of* $\mathrm{SO}_0(1,2)$, *and the projection* $p\colon \mathbf{\Gamma} \to \Gamma$ *is one-to-one.*

Conversely, for every torsion-free discrete subgroup $\mathbf{\Gamma}$ *of* $\mathrm{Isom}_0(\mathrm{Min}_3) = \mathrm{SO}_0(1,2) \ltimes \mathbb{R}^3$, *such that the linear part* Γ *of* $\mathbf{\Gamma}$ *is a co-compact Fuchsian subgroup of* $\mathrm{SO}_0(1,2)$, *and such that the projection* $p\colon \mathbf{\Gamma} \to \Gamma$ *is one-to-one, there exists a non-elementary future (resp. past) complete regular domain* $E \subset \mathrm{Min}_3$ *such that* $\mathbf{\Gamma}$ *preserves* E, *acts properly discontinuously on* E, *and such that* $\mathbf{\Gamma} \setminus E$ *is a non-elementary 3-dimensional MGHC flat spacetime.*

The index 0 in $\mathrm{Isom}_0(\mathrm{Min}_3)$ and $\mathrm{SO}_0(1,2)$ means "identity component of the Lie group".

Remark 5.3. — This theorem entails the fact that every non-elementary 3-dimensional MGHC flat spacetime is either future complete (and past incomplete) or past complete (and future incomplete).

The cosmological time of future complete 3-dimensional MGHC flat spacetimes admits nice properties:

PROPOSITION 5.4 (Bonsante, [23]). — *The cosmological time* $\widetilde{\tau} \colon E \to (0, +\infty)$ *is a* $C^{1,1}$ *function, that is* $\widetilde{\tau}$ *is a* C^1 *function, whose gradient* $\nabla\widetilde{\tau}$ *is a (locally) Lipschitz vector field. The levels of* $\widetilde{\tau}$ *are* $C^{1,1}$ *Cauchy surfaces in* E.

5.2. Geometry of MGHC dS$_3$-spacetimes

5.2.1. The 3-dimensional de Sitter space

Consider the quadratic form $Q_{1,3} = -x_1^2 + x_2^2 + x_3^2 + x_4^2$ on $\mathbb{R}^4$. The *linear model* dS$_3$ of the 3-dimensional de Sitter space is the hyperboloid $\{Q_{1,3} = 1\}$ endowed with the Lorentzian metric induced by $Q_{1,3}$. This spacetime is time-orientable; We choose the time orientation for which the curve $t \mapsto (\sinh t, \cosh t, 0, 0)$ is future-oriented. Moreover, dS$_3$ is globally hyperbolic and that the coordinate x_1 is a time function on dS$_3$. The group of orientation and time orientation preserving isometries of dS$_3$ is the group $\mathrm{O}_0(1,3)$.

It is sometimes convenient to consider the *Klein model* $\mathbb{DS}_3$ of the de Sitter space. By definition, $\mathbb{DS}_3$ is the image of dS$_3$ by the radial projection $\pi \colon \mathbb{R}^4 \smallsetminus \{0\} \to \mathbb{S}^3$, endowed with the push-forward of the Lorentzian metric of dS$_3$. The projection $\pi \colon \mathrm{dS}_3 \to \mathbb{DS}_3$ is a diffeomorphism[6]. The boundary of $\mathbb{DS}_3$ in $\mathbb{S}^3$ is the image under π of the cone $\{Q_{1,3} = 0\}$. This boundary is the disjoint union of two round 2-spheres: $\mathbb{S}_-^2 := \pi(\{Q_{1,3} = 0\} \cap \{x_1 < 0\})$ and $\mathbb{S}_+^2 := \pi(\{Q_{1,3} = 0\} \cap \{x_1 > 0\})$. Note that every future oriented inextendible causal curve $\gamma \subset \mathbb{DS}_3$ "goes from $\mathbb{S}_-^2$ to $\mathbb{S}_+^2$" (more precisely: The α-limit and ω-limit sets of γ are single points lying respectively in $\mathbb{S}_-^2$ and $\mathbb{S}_+^2$).

Now denote by H_-^3 (resp. H_+^3) the connected components of the hyperboloid $\{Q_{1,3} = -1\}$ contained respectively in the half-space $\{x_1 < 0\}$ (resp. $\{x_1 > 0\}$), endowed with the Riemannian metric induced by $Q_{1,3}$.

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[6] Some authors define the Klein model of the de Sitter space as the projection of dS$_3$ in the projective space $\mathbb{RP}^3$. We prefer to consider the projection in $\mathbb{S}^3$ instead of $\mathbb{RP}^3$, in order to keep a time-orientable spacetime.

Note that H^3_- and H^3_+ are two copies of the 3-dimensional hyperbolic space. The Klein models of H^3_- (resp. H^3_+) is the image $\mathbb{H}^3_-$ (resp. $\mathbb{H}^3_+$) under the projection π. Observe that $\mathbb{S}^3 = \mathbb{H}^3_- \sqcup \mathbb{S}^2_- \sqcup \mathbb{D}\mathbb{S}_3 \sqcup \mathbb{S}^2_+ \sqcup \mathbb{H}^3_+$, and that $\mathbb{S}^2_-$ (resp. $\mathbb{S}^2_+$) is the topological boundary of $\mathbb{H}^3_-$ (resp. $\mathbb{H}^3_+$) in $\mathbb{S}^3$.

The group $O_0(1,3)$ is simultaneously the isometry group of the Lorentzian space $\mathbb{D}\mathbb{S}_3$, the isometry group of the hyperbolic spaces $\mathbb{H}^3_-$ and $\mathbb{H}^3_+$, and the conformal group of the spheres $\mathbb{S}^2_-$ and $\mathbb{S}^2_+$.

A key ingredient in the sequel is the fact that the de Sitter space $\mathbb{D}\mathbb{S}_3$ can be thought of as the space of (non-trivial open) round balls in $\mathbb{S}^2_+$. Indeed, for every point $x \in \mathbb{D}\mathbb{S}_3$, let us denote by $\partial^+ I^+(x)$ the set of the future endpoints in $\mathbb{S}^2_+$ of all the future oriented timelike geodesic rays starting at x. Then, for every $x \in \mathbb{D}\mathbb{S}_3$, the set $\partial^+ I^+(x)$ is an open round ball in $\mathbb{S}^2_+$. One can easily check that the map associating to x the round ball $\partial^+ I^+(x)$ establishes a one-to-one correspondence between the points in $\mathbb{D}\mathbb{S}_3$ and the (non-trivial open) round balls in $\mathbb{S}^2_+$. Of course, there is a similar identification between the points of $\mathbb{D}\mathbb{S}_3$ and the round balls in $\mathbb{S}^2_-$.

5.2.2. Scannell's description of MGHC dS$_3$ spacetimes

We now sketch the correspondence between MGHC dS$_3$ spacetimes and compact Möbius surfaces. For more details (and for proofs of the facts stated below), we refer to [55] or [2].

Let us consider a simply connected Möbius surface S (*i.e.* a surface endowed with a $(O_0(1,3), \mathbb{S}^2)$-structure). We will construct a simply connected spacetime $\mathcal{B}_0^+(S)$ associated to S and locally isometric to dS$_3$. For this purpose, we consider a developing map $d \colon S \to \mathbb{S}^2_+$ (such a map d does exists since S is a Möbius surface).

An (open) *round ball* U in S is an open domain in S such that the developing map d to U is one-to-one in restriction to U, and such that $d(U)$ is an open round ball in $\mathbb{S}^2_+$. A round ball $U \subset S$ is said to be *proper* if the image under d of the closure $\overline{U}$ of U in S is the closure of $d(U)$ in $\mathbb{S}^2_+$.

We will denote by $\mathcal{B}_0(S)$ the set of proper round balls. The set $\mathcal{B}_0(S)$ is naturally ordered by the inclusion. For every element U of $\mathcal{B}_0(S)$, we denote by $W(U)$ the subset of $\mathcal{B}_0(S)$ made of the proper round balls U' such that $\overline{U'} \subset U$. Given two elements U, V of $\mathcal{B}_0(S)$ such that $\overline{U} \subset V$, we denote by $W(U, V)$ the set of all proper round balls U' in S such that $\overline{U} \subset U'$ and $\overline{U'} \subset V$. The sets $W(U, V)$ generate a topology on $\mathcal{B}_0(S)$ that we call the *Alexandrov topology*. It can be proved that the set $\mathcal{B}_0(S)$, equipped with the Alexandrov topology, is a manifold.

We already observed that the de Sitter space $\mathbb{DS}_3$, as a set, is canonically identified with the space $\mathcal{B}_0(\mathbb{S}_+^2)$ of all open round balls in the sphere $\mathbb{S}_+^2$. Actually, the identification between $\mathbb{DS}_3$ and $\mathcal{B}_0(\mathbb{S}_+^2)$ is an homeomorphism, once $\mathcal{B}_0(\mathbb{S}_+^2)$ is endowed with the Alexandrov topology.

The map $\mathcal{D}^+\colon \mathcal{B}_0(S) \rightarrow \mathbb{DS}_3$, composition of the developing map $d\colon \mathcal{B}_0(S) \rightarrow \mathcal{B}_0(\mathbb{S}^2)$ and the identification of $\mathcal{B}_0(\mathbb{S}^2)$ with $\mathbb{DS}_3$, is a local homeomorphism. Hence, we can consider the pull-back by $\mathcal{D}^+$ of the de Sitter metric on $\mathcal{B}_0(S)$. We will denote by $\mathcal{B}_0^+(S)$ the manifold $\mathcal{B}_0(S)$ equipped with the pull-back by $\mathcal{D}^+$ of the de Sitter metric. By construction, $\mathcal{B}_0^+(S)$ is a spacetime locally isometric to $\mathbb{DS}_3$.

Now suppose that the simply connected Möbius surface S considered above is the universal cover of a compact Möbius surface Σ. Denote by Γ the fundamental group of Σ. Then the developing map $d\colon S \rightarrow \mathbb{S}_+^2$ is Γ-equivariant: There exists a representation $\rho\colon \Gamma \rightarrow \mathrm{O}_0(1,3)$ such that $d \circ \gamma = \rho(\gamma) \circ d$ for every $\gamma \in \Gamma$. It follows that $\rho(\Gamma)$ acts by isometries on the spacetime $\mathcal{B}_0^+(S)$. This action is free, and properly discontinuous. So, we can consider the spacetime $M^+(\Sigma) := \rho(\Gamma) \smallsetminus \mathcal{B}_0^+(S)$. Up to isometry, this spacetime does not depend on the choice of the developing map d. The important result for us is the following:

THEOREM 5.5 (Scannell). — *Every MGHC* dS_3-*spacetime is past or future complete. Every future complete MGHC* dS_3-*spacetime* M *is isometric to the spacetime* $M^+(\Sigma)$ *for some compact Möbius surface* Σ.

Remark 5.6. — The correspondence between Möbius surfaces and MGH dS_3-spacetimes is an equivalence of category. In particular, any isometric map $\mathcal{B}_0^+(S_1) \rightarrow \mathcal{B}_0^+(S_2)$ induces a complex projective map $S_1 \rightarrow S_2$. Moreover, a construction similar to the one described above allows to associate a hyperbolic end to every Möbius surface S. It follows that there is a duality between MGH dS_3-spacetimes and hyperbolic ends.

5.3. Geometry of MGHC AdS_3-spacetimes

5.3.1. The 3-dimensional anti-de Sitter space

Consider the quadratic form $Q_{2,2} = -x_1^2 - x_2^2 + x_3^2 + x_4^2$ on $\mathbb{R}^4$. The 3-dimensional anti-de Sitter space AdS_3 is the quadric $\{Q_{2,2} = -1\}$, equipped with the Lorentzian metric induced by $Q_{2,2}$. We choose the time-orientation for which the timelike curve $\theta \rightarrow (\cos\theta, \sin\theta, 0, 0)$ is future-oriented. The

group of isometries of AdS_3 preserving the orientation and the time-orientation is the connected component $O_0(2,2)$ of the identity in $O(2,2)$.

Observe that the vector space of 2 by 2 matrices $\mathfrak{gl}(2,\mathbb{R})$ endowed with the quadratic form $-\det$ is isometric to $(\mathbb{R}^4, Q_{2,2})$. Hence AdS_3 is naturally identified with $SL(2,\mathbb{R})$ endowed with its Killing form. The actions of $SL(2,\mathbb{R})$ on itself by left and right translations preserves this Lorentzian metric and commutes each other. Hence we have a natural isomorphism $O_0(2,2) \approx (SL(2,\mathbb{R}) \times SL(2,\mathbb{R}))/I$ where $I = \{(\mathrm{Id}, \mathrm{Id}), (-\mathrm{Id}, -\mathrm{Id})\}$.

We will also consider the *Klein model* $\mathbb{ADS}_3$ of the anti-de Sitter space. By definition, $\mathbb{ADS}_3$ is the image of AdS_3 under the projection $\pi\colon \mathbb{R}^4 \setminus \{0\} \to \mathbb{RP}^3$. The projection $\pi\colon AdS_3 \to \mathbb{ADS}_3$ is a two-fold covering. The spacetime $\mathbb{ADS}_3$ is time-orientable[7], since the antipodal map of $\mathbb{R}^4$ preserves the time-orientation of AdS_3. The isometric action of $(SL(2,\mathbb{R}) \times SL(2,\mathbb{R}))/I$ on AdS_3 induces an action of $PSL(2,\mathbb{R}) \times PSL(2,\mathbb{R})$ on $\mathbb{ADS}_3$.

We denote by $\mathcal{Q}$ the boundary of $\mathbb{ADS}_3$ in $\mathbb{S}^3$. This is a one-sheeted hyperboloid in $\mathbb{RP}^3$, which notoriously admits two transverse rulings by projective lines. This provides an identification between $\mathcal{Q}$ and $\mathbb{RP}^1 \times \mathbb{RP}^1$. The isometric action of $PSL(2,\mathbb{R}) \times PSL(2,\mathbb{R})$ on $\mathbb{ADS}_3$ extends to the boundary $\mathcal{Q}$, and the identification between $\mathcal{Q}$ and $\mathbb{RP}^1 \times \mathbb{RP}^1$ can be chosen so that $PSL(2,\mathbb{R}) \times PSL(2,\mathbb{R})$ acts on $\mathcal{Q}$ coordinate by coordinate:

$$(\gamma_L, \gamma_R) \cdot (\theta_L, \theta_R) = (\gamma_L \theta_L, \gamma_R \theta_R).$$

5.3.2. Mess' description of non-elementary MGHC AdS_3 spacetimes

We briefly recall the correspondence between non-elementary MGHC AdS_3-spacetimes and pairs of Fuchsian representations of a surface group. For more details, see [51] or [12, 18, 10].

Let Γ be the fundamental group of a closed surface of genus $g \geqslant 2$, and consider a pair of Fuchsian representations $\rho_L, \rho_R\colon \Gamma \to PSL(2,\mathbb{R})$. Two such representations are always topologically conjugate: There is a (unique) homeomorphism $f\colon \mathbb{RP}^1 \to \mathbb{RP}^1$ such that:

$$\forall \gamma \in \Gamma, \quad \rho_R(\gamma) \circ f = f \circ \rho_L(\gamma).$$

The graph of f is a closed topological circle Λ_ρ embedded in $\mathcal{Q} \approx \mathbb{RP}^1 \times \mathbb{RP}^1$ which is invariant under the action of Γ defined by $\rho := (\rho_L, \rho_R)$. Moreover, this topological circle $\Lambda_\rho \subset \mathcal{Q} \subset \mathbb{RP}^3$ is contained in an affine chart of $\mathbb{RP}^3$.

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[7] This is the reason why we can project directly in $\mathbb{RP}^3$, contrary to the de Sitter case, see footnote 6.

Let $E(\Lambda_\rho)$ be the *invisible domain* of the topological circle Λ_ρ: By defini-tion, this is the set of all points $p \in \mathbb{ADS}_3$ such that there is no causal curve $\gamma \subset \mathbb{ADS}_3$, contained in an affine chart of $\mathbb{RP}^3$, starting at p and ending at a point $x \in \Lambda_\rho$. The set $E(\Lambda_\rho)$ is contained in an affine chart of $\mathbb{RP}^3$, and the intersection of the closure of $E(\Lambda_\rho)$ with $\mathcal{Q}$ is exactly the topological circle Λ_ρ.

Remark 5.7. — Since $E(\Lambda_\rho)$ is contained in an affine chart of $\mathbb{RP}^3$, the pre-image of $E(\Lambda_\rho)$ under the projection $\pi\colon \mathrm{AdS}_3 \to \mathbb{ADS}_3$ is the disjoint union of two copies of $E(\Lambda_\rho)$.

The group $\rho(\Gamma)$ acts freely and properly discontinuously on the invisible domain $E(\Lambda_\rho)$. The quotient space $M(\rho) = \rho(\Gamma) \smallsetminus E(\Lambda_\rho)$ is a MGHC spacetime with closed Cauchy surfaces of genus g. It turns out that every non-elementary MGHC AdS$_3$-spacetime can be obtained in this way:

THEOREM 5.8 (Mess). — *For $g \geqslant 2$, let Γ_g be the fundamental group of the compact surface of genus g. Then every MGHC AdS$_3$-spacetime with Cauchy surface of genus g is isometric to $M(\rho) = \rho(\Gamma) \smallsetminus E(\Lambda_\rho)$ for some Fuchsian representation $\rho = (\rho_L, \rho_R)\colon \Gamma_g \to \mathrm{PSL}(2, \mathbb{R}) \times \mathrm{PSL}(2, \mathbb{R})$.*

One can also define the *convex hull* $\mathrm{Conv}(\Lambda_\rho)$: It is simply the convex hull of Λ_ρ in any affine chart $U \simeq \mathbb{R}^3$ of $\mathbb{RP}^3$ containing Λ_ρ (it does not depend on the choice of U). It turns out that $\mathrm{Conv}(\Lambda_\rho) \smallsetminus \Lambda_\rho$ is contained in $\mathbb{ADS}_3$. The complement of Λ_ρ in the convex hull $\mathrm{Conv}(\Lambda_\rho)$ is contained in $E(\Lambda_\rho)$. The projection of $\mathrm{Conv}(\Lambda_\rho) \smallsetminus \Lambda_\rho$ in $M(\rho)$ is a closed region $\mathcal{C}(\rho)$, diffeomorphic to $S \times [-1, +1]$, that we call *convex core* of the spacetime $M(\rho)$. The convex core $\mathcal{C}(\rho)$ can also be defined using the cosmological time and the reverse cosmological time of $M(\rho)$:

- The spacetime $M(\rho)$ has a regular cosmological time $\tau\colon M(\rho) \to (0, \pi)$, as well as a regular reverse cosmological time $\check{\tau}\colon M(\rho) \to (0, \pi)$.
- The set $\{\tau \leqslant \pi/2\}$ (respectively $\{\check{\tau} \leqslant \pi/2\}$) is called the *past tight region* (resp. the *future tight region*) and denoted by $M_-(\rho)$ (resp. $M_+(\rho)$).
- The convex core $\mathcal{C}(M)$ corresponds to the set $\{\tau \leqslant \pi/2, \; \check{\tau} \leqslant \pi/2\}$. Hence, it is the intersection between the past tight region $M_-(\rho)$ and the future tight region $M_+(\rho)$.
- The cosmological time τ (resp. the reverse cosmological time $\check{\tau}$) is a $C^{1,1}$ function in restriction to $M_-(\rho)$ (resp. $M_+(\rho)$).

5.4. Canonical decomposition of MGHC spacetimes
of constant curvature

In this section, M denotes a non-elementary MGHC spacetime of constant curvature Λ. Recalling the metric if necessary, we may (and we do) assume that $\Lambda = 0$ (flat case), $+1$ (dS$_3$ case) or -1 (AdS$_3$ case). Moreover, in the nonnegative curvature case $\Lambda \geqslant 0$ one can assume, up to a time reversal, that M is future complete.

In order to treat simultaneously the three possible cases, we define M_0 to be:

–   the whole spacetime M in the flat and in the dS$_3$ case,
–   the past tight region of M in the anti-de Sitter case,

and we consider the functions $\mathfrak{f}_\Lambda \colon \mathbb{R} \to \mathbb{R}$ and $\mathfrak{g}_\Lambda \colon \mathbb{R} \to \mathbb{R}$ defined as follows:

$$\begin{aligned} \mathfrak{f}_\Lambda(t) &= t && \text{and} && \mathfrak{g}_\Lambda(t) = 1 && \text{if } \Lambda = 0 \text{ (flat case),} \\ \mathfrak{f}_\Lambda(t) &= \sinh t && \text{and} && \mathfrak{g}_\Lambda(t) = \cosh t && \text{if } \Lambda = +1 \text{ (dS}_3\text{-case),} \\ \mathfrak{f}_\Lambda(t) &= \sin t && \text{and} && \mathfrak{g}_\Lambda(t) = \cos t && \text{if } \Lambda = -1 \text{ (AdS}_3\text{-case).} \end{aligned}$$

Analyzing the cosmological time τ of M, and following the same train of thoughts as in the flat case ([23]), one gets:

FACT AND DEFINITIONS 5.9. — M_0 admits a canonical decomposition. *Each element of this decomposition is:*

1)   *either a thin block isometric to $\mathbb{R} \times (0, +\infty)$ or $\mathbb{S}^1 \times (0, +\infty)$ endowed with the metric $\mathfrak{f}_\Lambda(t)^2 \mathrm{d}x^2 - \mathrm{d}t^2$,*
2)   *or a solid block isometric to $\Sigma \times (0, +\infty)$ endowed with the Lorentzian metric $\mathfrak{f}_\Lambda(t)^2 \mathrm{ds}_{\mathrm{hyp}}^2 - \mathrm{d}t^2$ where $(\Sigma, \mathrm{ds}_{\mathrm{hyp}}^2)$ is the interior of a hyperbolic surface with boundary.*

Every maximal subset of M_0 foliated by thin blocks of the canonical decomposition is isometric to $\mathbb{S}^1 \times (-\ell/2, \ell/2) \times (0, +\infty)$ endowed with the metric $\mathfrak{f}_\Lambda(t)^2 \mathrm{d}\theta^2 + \mathfrak{g}_\Lambda(t)^2 \mathrm{d}x^2 - \mathrm{d}t^2$, and is called a Misner block of width ℓ. Moreover, the parameter t in the above is the (regular) cosmological time τ.

FACT 5.10. — *For every $a \in (0, \pi/2)$, there exists a measured lamination $\mathcal{L}_a$ in the surface $C_a = \tau^{-1}(a)$, such that:*

1)   *The leaves of $\mathcal{L}_a$ are the intersections of C_a with the thin blocks of the canonical decomposition of M_0.*
2)   *The connected components of $C_a \smallsetminus \mathcal{L}_a$ are the intersections of C_a with the solid blocks of the canonical decomposition of M_0.*
3)   *The intersection of C_a with a Misner block associated with the canonical decomposition of M_0 is a annulus in M_0 foliated by closed leaves of $\mathcal{L}_a$.*

FACT 5.11. — *The gradient of the cosmological time defines a Lipschitz map $N\colon M_0 \to C_\infty^-$ where C_∞^- is a hyperbolic surface. The image under N of the union of the thin blocks of the canonical decomposition is a geodesic lamination $\mathcal{L}_\infty^-$ on C_∞^-. The transverse measures of the laminations $\mathcal{L}_a$ induce a transverse measure on $\mathcal{L}_\infty^-$. Moreover:*

1) *The pre-image (for N) of a non-weighted leaf of $\mathcal{L}_\infty^-$ is a thin block of the canonical decomposition of M_0 which is not included in a Misner block.*

2) *The pre-image of a connected component of $C \smallsetminus \mathcal{L}_\infty^-$ is a solid block of the canonical decomposition of M_0.*

3) *The pre-image of a leaf of $\mathcal{L}_\infty^-$ which carries a weight $\ell > 0$ is a Misner block of width ℓ associated with the canonical decomposition of M_0.*

Remark 5.12. — Note that, since C is compact, a leaf of $\mathcal{L}$ which carries a non-zero weight is necessarily closed, and that there are only finitely many such leaves in $\mathcal{L}$. It follows that there are only finitely many Misner blocks associated with the canonical decomposition of M_0.

Remark 5.13. — In the dS$_3$ and flat case $\Lambda \geqslant 0$, the hyperbolic surface C_∞^- can be interpreted as the future conformal boundary at infinity of M. In the AdS$_3$-case $\Lambda = -1$, the hyperbolic surface C_∞^- admits a (non-smooth) isometric embedding, as a pleated surface, in the MGHC spacetime M in which M_0 is the past tight region. This pleated surface is actually the topological boundary of M_0 in M, *i.e.* the $(\pi/2)$-level set of the cosmological time τ. The pleating lamination is the measured geodesic lamination $\mathcal{L}_\infty^-$ defined above.

Remark 5.14. — Still in the AdS$_3$-case $\Lambda = -1$, the future tight region M_+ of M also admits a canonical decomposition associated with a hyperbolic surface C_∞^+ and a measured geodesic lamination $\mathcal{L}_\infty^+$ which are in general different from the hyperbolic surface C_∞^- and the measured geodesic laminations $\mathcal{L}_\infty^-$.

Remark 5.15. — Supports of measured geodesic laminations in hyperbolic surfaces have zero Lebesgue measure, but it is not true that the complement in $\mathcal{L}_a$ of the Misner blocks has zero Lebesgue measure. Actually, the area of C_a for the induced metric depends continuously on the measured lamination $\mathcal{L}_\infty^-$. In the case where $\mathcal{L}_a$ is simplicial, *i.e.* a simple closed geodesic of length L with a weight l, this area is the sum of the area of C_∞ and the area of the inserted annulus, *i.e.* alL. It follows that in general the area of C_a is $2\pi(2g - 2) + al(\mathcal{L}_\infty)$, where $l(\mathcal{L}_\infty)$ is the *length of the*

measured lamination (*cf* § 7 of [21]). In particular, even if C_a has no Misner block, the lamination $\mathcal{L}_a$ has nonzero Lebesgue measure.

6. Upper bound of the systole

The purpose of this section is to prove Theorem 3.5. We use the notations defined in the preceding section. In particular, (M, g) is a MGHC spacetime of constant curvature Λ, and M_0 is:

– the whole spacetime M in the flat and in the dS$_3$ case,
– the past tight region of M in the anti-de Sitter case.

We denote by $\tau \colon M_0 \to (0, +\infty)$ the cosmological time of M_0. We denote by N the opposite of the gradient of τ, and by $(\phi^t)_{t \in \mathbb{R}}$ the flow associated to N. We denote by $\tau_{\max}$ the supremum of τ over M_0 (recall that $\tau_{\max} = +\infty$ if $\Lambda = 0$ or $+1$, and $\tau_{\max} = \pi/2$ if $\Lambda = -1$). For every $a \in (0, \tau_{\max})$, we denote by C_a the a-level of τ. As explained in §4.2, there is a Lipschitz parametrization of M_0 obtained simply by pushing the surface C_1 along the flow ϕ^t. More precisely, the map

$$\Phi \colon C_1 \times (0, \tau_{\max}) \longrightarrow M_0$$
$$(x, t) \longmapsto \phi^{t-1}(x)$$

is a global bi-Lipschitz homeomorphism. Recall that the cosmological time τ defines a measured lamination $\mathcal{L}_1$ on the surface C_1. The lamination $\mathcal{L}_1$ induces a decomposition of M_0 as a disjoint union of "buildings blocks", each of which being:

– a solid block $\Phi(U \times (0, \tau_{\max}))$ associated to a connected component U of $C_1 \smallsetminus \mathcal{L}_1$,
– or a thin block $\Phi(\gamma \times (0, \tau_{\max}))$ associated to a non-closed leaf γ of $\mathcal{L}_1$,
– or a Misner block $\Phi(A \times (0, \tau_{\max}))$ associated to a maximal annulus $A \subset C_1$ foliated by closed leaves of $\mathcal{L}_1$ (such a Misner block can itself be decomposed as a disjoint union of thin blocks associated to the closed leaves of $\mathcal{L}_1$ foliating A).

6.1. The expanding character of the cosmological flow

Here, we consider a Lipschitz curve $c \colon [a, b] \to M_0$ which is spacelike (*i.e.* the velocity $c'(s)$ is almost everywhere spacelike). We denote by $c_1 \colon [a, b] \to$

M_0 the projection of curve c on the Cauchy surface C_1 along the flow ϕ^t. Observe that

(6.1) $$c_1(s) = \phi^{1-\tau(c(s))}(c(s)).$$

It follows that c_1 is a Lipschitz curve. In particular, the length of the spacelike curve c_1 is well defined.

PROPOSITION 6.1. — *Assume that the curve c is contained in the past of the Cauchy surface C_1 (i.e. the restriction of the cosmological time τ to the curve c is bounded from above by 1). Then:*

(i) *the length of c is smaller than or equal to the length of c_1,*
(ii) *the cosmological time variation $|\tau(c(b)) - \tau(c(a))|$ is smaller than or equal to the length of c_1.*

Proof. — For $s \in [a,b]$, let $t(s) = \tau(c(s))$, so that $c(s) = \phi^{t(s)-1}(c_1(s))$. Since the curve c is spacelike, one has, for almost every $s \in [a,b]$,

$$0 \leqslant |c'(s)|^2.$$

On the other hand, the equality $c(s) = \phi^{t(s)}(c_1(s))$ yields almost everywhere:

$$
\begin{aligned}
|c'(s)|^2 &= \left| t'(s)N(c(s)) + d\left(\phi^{t(s)-1}\right)(c_1(s)) \cdot c_1'(s) \right|^2 \\
&= \left| d\left(\phi^{t(s)-1}\right)(c_1(s)) \cdot c_1'(s) \right|^2 + t'(s)^2 |N(c(s))|^2 \\
&= \left| d\left(\phi^{t(s)-1}\right)(c_1(s)) \cdot c_1'(s) \right|^2 - t'(s)^2.
\end{aligned}
$$

Indeed, $d\left(\phi^{t(s)-1}\right)(c_1(s)) \cdot c_1'(s)$ is tangent to the level $C_{t(s)}$, which is orthogonal to $N = -\nabla\tau$, and $|N|^2 = -1$. In particular, for almost every s, one has

(6.2) $$|c'(s)|^2 \leqslant \left| d\left(\phi^{t(s)-1}\right)(c_1(s)).c_1'(s) \right|^2.$$

Moreover, the map $\phi^{t-1}\colon C_1 \to C_t$ is contracting for every $t < 1$ (see [23, Lemma 7.4]) [8]. It follows that, for almost every s, one has

(6.3) $$\left| d\left(\phi^{t(s)-1}\right)(c_1(s)).c_1'(s) \right|^2 \leqslant |c_1'(s)|^2.$$

Putting together inequalities (6.2) and (6.3), one gets that $|c'(s)| \leqslant |c_1'(s)|$ for almost every s. Integrating over $[a,b]$, one gets that the length of c is smaller than or equal to the length of c_1.

---

[8] It does not follow directly from the fact that it is obviously true on $C_1 \smallsetminus \mathcal{L}_1$ and in every Misner block in C_1, since the union of non-closed leaves in $\mathcal{L}_1$ may have non-zero Lebesgue measure (*cf* Remark 5.15).

From the above inequalities, one gets, for almost every $s \in [a, b]$

$$t'(s)^2 \leqslant \left| d\left(\phi^{t(s)-1}\right)(c_1(s))(c_1'(s)) \right|^2 \leqslant |c_1'(s))|^2.$$

Integrating over the interval $[a, b]$, one gets that $|\tau(c(b)) - \tau(c(a))| = |t(b) - t(a)|$ is smaller than or equal to the length of c_1, as wanted. $\qquad\square$

6.2. Key estimates

As in the previous paragraph, we consider a spacelike Lipschitz curve $c : [a, b] \to M_0$, and we denote by c_1 the projection of c on the level set C_1 along the flow ϕ^t. There is one particular situation where Proposition 6.1 can be dramatically improved: When the curve c is contained in a solid block or in a thin block.

PROPOSITION 6.2. — *Assume that c is contained in a solid block, or in a thin block. Denote by ℓ the length of c_1. Then, one has*

$$\nu_\Lambda^{-1} \cdot \exp(-\ell) \leqslant \frac{\tau(c(b))}{\tau(c(a))} \leqslant \nu_\Lambda \cdot \exp(\ell)$$

where ν_Λ is a constant (depending on Λ but not on M), and

$$\text{length}(c) \leqslant \mu_\Lambda(\tau(c(a)), \ell)$$

where $\mu_\Lambda \colon (0, 1) \times (0, +\infty) \to (0, +\infty)$ is a universal function (depending on Λ but not on M) such that: For every $L > 0$, the map $\tau \mapsto \mu_\Lambda(\tau, L)$ is increasing and $\lim_{\tau \to 0} \mu_\Lambda(\tau, L) = 0$.

Remark 6.3. — The last estimate should be interpreted as follows: *Given the length of the projection of the curve c on the cosmological level set C_1, if the cosmological time takes a small value somewhere on c, then the length of c is small.*

Proof of Proposition 6.2: Static blocks. — Assume that c is contained in a solid block. This block is of the form $M_0(U) := \Phi(U \times (0, \tau_{\max}))$ where U is a connected component of $C_1 \smallsetminus \mathcal{L}_1$. If ds_{hyp}^2 denotes the hyperbolic metric, then $(M_0(U), g)$ is locally isometric to $\mathbb{H}^2 \times (0, \tau_{\max})$ equipped with the metric (see §5.4):

- $g_0 = \tau^2 \text{ds}_{\text{hyp}}^2 - d\tau^2$ in the flat case,
- $g_0 = \sinh^2(\tau) \text{ds}_{\text{hyp}}^2 - d\tau^2$ in the dS$_3$ case,
- $g_0 = \sin^2(\tau) \text{ds}_{\text{hyp}}^2 - \tau^2$ in the AdS$_3$ case.

The flat case. — Let $T = \log(\tau)$. Then $(M_0(U), g)$ is locally isometric to $\mathbb{H}^2 \times (0, +\infty)$ equipped with the Lorentzian metric

$$g_0 = \exp(2T)(\mathrm{ds}^2_{\mathrm{hyp}} - dT^2) = t^2(\mathrm{ds}^2_{\mathrm{hyp}} - dT^2).$$

This metric is conformally equivalent to the metric $k = \mathrm{ds}^2_{\mathrm{hyp}} - dT^2$. In particular, the curve c is spacelike for the metric k. Therefore, the same arguments (for the metric k) as in the proof of Proposition 6.1 show that T-variation $|T(c(a)) - T(c(b))|$ is smaller than or equal to the length of c_1 (note that the length of c_1 for the metric $g_0 = \exp(2T)(\mathrm{ds}^2_{\mathrm{hyp}} - dT^2)$ and for the metric $k = \mathrm{ds}^2_{\mathrm{hyp}} - dT^2$ are equal, since $T = 0$ on c_1). Since $T = \log(\tau)$, this yields the desired inequality:

$$\exp(-\ell) \leqslant \frac{\tau(c(b))}{\tau(c(a))} \leqslant \exp(\ell).$$

For any $s \in [a, b]$, if we replace b by s in the above arguments, one gets $\exp(-\ell) \leqslant \tau(c(s))/\tau(c(a)) \leqslant \exp(\ell)$. In particular,

$$\tau(c(s)) \leqslant \tau(c(a)) \exp(\ell).$$

Now, the same calculations as in the proof of Proposition 6.1 (for the metric k) show that $|c'(s)|^2_k \leqslant |c'_1(s)|^2_k$ where $|\cdot|_k$ denotes the norm of a vector for the metric k. Using $|c'(s)|^2_k = \tau(c(s))^2 . |c'(s)|^2_{g_0}$ and $|c'_1(s)|^2_k = \tau(c_1(s))^2 |c'_1(s)|^2_{g_0} = |c'_1(s)|^2_{g_0}$, we get

$$|c'(s)|_{g_0} \leqslant \tau(c(s)) |c'_1(s)|_{g_0}.$$

Putting together the two last inequalities above, one obtains

$$\mathrm{length}_g(c) = \mathrm{length}_{g_0}(c) = \int_a^b |c'(s)|_{g_0} ds$$

$$\leqslant \tau(c(a)) \exp(\ell) \int_a^b |c'_1(s)|_{g_0} ds$$

$$= \tau(c(a)) \ell \exp(\ell).$$

We have thus proved Proposition 6.2 in the case $\Lambda = 0$, with $\nu_0 = 1$ and $\mu_0(\tau, \ell) = \ell \exp(\ell)\tau$.

The dS$_3$ case. — Let $T = \log(\tanh(\frac{\tau}{2}))$ (note that $\tau \mapsto \log(\tanh(\frac{\tau}{2}))$ is the antiderivative of $\tau \mapsto \frac{1}{\sinh(\tau)}$). Then $(M(U), g)$ is locally isometric to $\mathbb{H}^2 \times]0, T_{\max}[$ endowed with the metric:

$$g_1 = \sinh(\tau)^2(\mathrm{ds}^2_{\mathrm{hyp}} - dT^2).$$

As in the flat case, the T-variation c is therefore at most ℓ. The inequality

$$\exp(-\ell) \leqslant \frac{\tanh(\tau(c(b))/2)}{\tanh(\tau(c(a))/2)} \leqslant \exp(\ell)$$

follows. This yields

$$\nu_1^{-1} \exp(-\ell) \leqslant \frac{\tau(c(b))}{\tau(c(a))} \leqslant \nu_1 \exp(\ell)$$

where $\nu_1 = \frac{e+1}{e-1}$ is the upper bound of $\frac{\tau}{\tanh(\tau/2)}$ on $[0,1]$. Now, by the same arguments as in the flat case, we get for every $s \in [a,b]$

$$\tanh(\tau(c(s)/2)) \leqslant \tanh(\tau(c(a)/2)) \exp(\ell)$$

and

$$|c'(s)|_{g_1} \leqslant \sinh(\tau(c(s)))|c_1'(s)|_{g_1} = \frac{2\tanh(\tau(c(s))/2)}{1 - \tanh^2(\tau(c(s))/2)}|c_1'(s)|_{g_1}.$$

Putting these two last inequalities together, and integrating over $[a,b]$, we obtain as wanted $\mathrm{length}(c) \leqslant \mu_1(\tau(c(a)), \ell)$ for

$$\mu_1(\tau, \ell) = \frac{2l \exp(\ell) \tanh(\tau/2)}{1 - \exp(2\ell) \tanh^2(\tau/2)}.$$

The AdS_3 *case.* — Similar the dS_3 case, after replacing sinh and tanh by tan and sin. The constant ν_{-1} is the supremum of $\frac{\tau}{\tan(\tau/2)}$ over $[0, \pi/2]$, and the universal map μ_{-1} is:

$$\mu_{-1}(\tau, \ell) = \frac{2\ell \exp(\ell) \tan(\tau/2)}{1 + \exp(2\ell) \tan^2(\tau/2)}.$$

This completes the proof of Proposition 6.2 is the case where c is contained in a solid block. $\qquad\square$

Proof of Proposition 6.2: Thin block. — Now, we assume that c is contained in a thin block $M_0(\gamma) := \Phi(\gamma \times (0, \tau_{\max}))$ where γ is leaf of $\mathcal{L}_1$. The block $(M_0(\gamma), g)$ is isometric to $\mathbb{R} \times (0, \tau_{\max})$ (if γ is not closed) or $\mathbb{S}^1 \times (0, \tau_{\max})$ (if γ is closed) equipped with the metric (see §5.4):

- $g_0 = \tau^2 \mathrm{d}\theta^2 - d\tau^2$ in the flat case,
- $g_0 = \sinh^2(\tau)\mathrm{d}\theta^2 - d\tau^2$ in the dS_3 case,
- $g_0 = \sin^2(\tau)\mathrm{d}\theta^2 - d\tau^2$ in the AdS_3 case.

The end of the proof is exactly the same as in the case where c is contained in a solid block, replacing ds^2 by $\mathrm{d}\theta^2$. $\qquad\square$

6.3. Proof of Theorem 3.5

We consider a constant $\varepsilon > 0$ and a spacelike Cauchy surface Σ in M_0, such that the systole of Σ bigger than or equal to ε (*i.e.* that any closed curve in Σ whose length is less than ε is homotopically trivial).

The discussion in §4.2 shows that we can write Σ as a graph over C_1. More precisely, there exists a map $t\colon C_1 \to (0, \tau_{\max})$ such that

$$\Sigma = \left\{ \phi^{t(x)-1}(x) \mid x \in C_1 \right\}.$$

Observe that this definition of the function t yields

$$\tau\left(\phi^{t(x)-1}(x)\right) = t(x)$$

for every $x \in C_1$. We will prove that the function $t\colon C_1 \to (0, \tau_{\max})$ is bounded from below by a constant $\alpha_\varepsilon > 0$. Using the equality above, this will imply that the restriction of the cosmological time τ to the surface S is bounded from below by α_ε. Theorem 3.5 will follow.

Remark 6.4. — Assume that the systole of the cosmological level set C_1 is bigger than or equal to ε. Let $\hat{t}(x) = \min(t(x), 1)$. Then the systole of the surface $\widehat{\Sigma} := \{\phi^{\hat{t}(x)-1}(x) \mid x \in C_1\}$ where $\hat{t}(x) = \min(t(x), 1)$ is also greater than or equal to ε. And of course, if $\hat{t}$ is bounded from below by some constant α_ε, then t is also bounded from below by α_ε. Therefore, in the sequel, we can (and we will) assume than that t is bounded from above by 1. In other words, we will assume that the surface S is in the past of the cosmological level set C_1.

6.3.1. Decomposition of the lamination $\mathcal{L}_1$

We denote by $\mathcal{L}_1^c$ (resp. $\mathcal{L}_1^{nc}$) the union of the closed (resp. non-closed) leaves of the lamination $\mathcal{L}_1$. Since $\mathcal{L}_1$ is a measured lamination, $\mathcal{L}_1^c$ and $\mathcal{L}_1^{nc}$ are closed. Indeed, if $\mathcal{L}_1^{nc}$ were not closed, there would be a bunch of leaves of $\mathcal{L}_1^{nc}$ spiraling on some closed leaf γ of $\mathcal{L}_1^c$. But then, the measure of any small arc transverse to γ would be infinite. Contradiction.

Therefore, $\mathcal{L}_1^c$ and $\mathcal{L}_1^{nc}$ are sublaminations of $\mathcal{L}_1$. So we have a decomposition of $\mathcal{L}_1$ into two disjoint sublaminations $\mathcal{L}_1^c$ and $\mathcal{L}_1^{nc}$, the first one made only of closed leaves, the second one without closed leaves.

Observe that the support of $\mathcal{L}_1^c$ is exactly the intersection of the surface C_1 with the Misner blocks of M_0. In particular, this support of $\mathcal{L}_1^c$ is a finite union of disjoint annuli in C_1, each of these annuli being foliated by leaves of $\mathcal{L}_1^c$.

6.3.2. Estimates of t on $\mathcal{L}_1^c$

LEMMA 6.5. — *There exists a constant $\eta^c > 0$ such that $t(x) \geqslant \eta^c$ for every x in the support of the lamination $\mathcal{L}_1^c$.*

Proof. — Let us first recall that the support of the lamination $\mathcal{L}_1^c$ is exactly the intersection of the surface C_1 with the Misner blocks of M_0. The intersection of C_1 with a Misner block is an annulus foliated by leaves of $\mathcal{L}_1^c$, all of which have the same length. There are only finitely many Misner blocks in M_0. Therefore there are only finitely many possible lengths for the leaves of $\mathcal{L}_1^c$. We denote these lengths by $\ell_1, \ldots, \ell_k$.

Now, let $x \in C_1$ be a point in the support of the lamination $\mathcal{L}_1^c$. Denote by $\gamma_{1,x}$ the leaf of $\mathcal{L}_1^c$ containing x, and by γ_x be the projection of $\gamma_{1,x}$ on the surface Σ along the flow ϕ^t (of course, it follows that $\gamma_{1,x}$ is the projection of γ_x on the surface C_1). From Proposition 6.2, we get

$$\text{length}(\gamma_x) \leqslant \mu_\Lambda(t(x), \ell_i)$$

where ℓ_i is the length of the closed leaf $\gamma_{1,x}$. This closed leaf $\gamma_{1,x}$ is not homotopically trivial in C_1 (since the image of $\gamma_{1,x}$ under the Gauss map $N \colon C_1 \to C_\infty$ is a closed leaf of the geodesic lamination $\mathcal{L}_\infty$, *i.e.* a closed geodesic in the hyperbolic surface C_∞). Hence the closed curve γ_x is not homotopically trivial in Σ. So our assumption on the systole of Σ and the above estimate on the length of γ_x implies

$$\mu_\Lambda(t(x), \ell_i) \geqslant \varepsilon.$$

It remains to choose a constant $\eta^c > 0$ such that $\mu_\Lambda(\tau, \ell_i) < \varepsilon$ for every $\tau \leqslant \eta$ and every $i \in \{1, \ldots, k\}$. The above inequality implies that $t(x) \geqslant \eta^c$ for every x in the support of $\mathcal{L}_1^c$. This completes the proof of Lemma 6.5. $\square$

6.3.3. Estimates of t on $\mathcal{L}_1^{nc}$

We will prove the following:

LEMMA 6.6. — *There exists a constant $\eta^{nc} > 0$ such that $t(x) \geqslant \eta^{nc}$ for every $x \in \mathcal{L}_1^{nc}$.*

Proof. — For every $x \in \mathcal{L}_1$, we denote by $\mathcal{L}_1(x)$ the leaf of $\mathcal{L}_1$ containing x. The idea of the proof is to construct, for every $x \in \mathcal{L}_1^{nc}$, a closed homotopically non-trivial loop containing x, by closing long segment of the leaf of $\mathcal{L}_1(x)$. Recall that a lamination $\mathcal{L}$ is *minimal* if every leaf of it is dense (in its support). We will use the following fact (see for example Proposition (iii) in [46]): *The lamination $\mathcal{L}_1^{nc}$ is a union of finitely many minimal sublaminations $\mathcal{L}_1^1, \ldots, \mathcal{L}_1^k$.*

Let $i \in \{1, \ldots, k\}$. Going to a double cover if necessary, we assume that the lamination $\mathcal{L}_1^i$ is orientable. Since $\mathcal{L}_1^i$ is minimal (and compact), there exists $\ell_i \in \mathbb{R}$ such that every $x \in \mathcal{L}_1^i$ is $(\ell_i, \frac{\varepsilon}{2})$-*recurrent*: Tthere is a

segment $\alpha_{1,x}$ starting at x in the leaf $\mathcal{L}_1(x)$, such that the length of $\alpha_{1,x}$ is at most ℓ_i, and such that the ends of $\alpha_{1,x}$ can be joined by a small arc $\beta_{1,x}$ transverse[9] to $\mathcal{L}_1$ of length at most $\frac{\varepsilon}{2}$.

For any $x \in \mathcal{L}_1^i$, we consider the concatenation of the arcs $\alpha_{1,x}$ and $\beta_{1,x}$. This is a closed loop $\gamma_{1,x}$. This loop $\gamma_{1,x}$ can be perturbed to get a loop transverse to the lamination $\mathcal{L}_1$. So, by Poincaré-Bendixon Theorem, $\gamma_{1,x}$ is homotopically non trivial in C_1. We denote by γ_x the projection of the loop $\gamma_{1,x}$ on Σ along the flow ϕ^t. Of course, γ_x is the concatenation of the projections α_x and β_x of the arcs $\alpha_{1,x}$ and $\beta_{1,x}$.

Since $\alpha_{1,x}$ is contained in a thin geodesic cone, we can apply Proposition 6.2 to this arc. We get that the length of α_x is bounded from above by $\mu_\Lambda(t(x), \ell_x)$, where ℓ_x is the length of α_x. On the other hand, according to the Proposition 6.1 and Remark 6.4, the length of the arc β_x is less than the length of the arc $\beta_{1,x}$. In particular, it is less than $\varepsilon/2$. So we get an upper bound for the length of the loop γ_x:

$$\operatorname{length}(\gamma_x) \leqslant \sup_{\ell \leqslant \ell_i} \mu_\Lambda(t(x), \ell) + \varepsilon/2.$$

Since γ_x is a non-homotopically trivial loop in Σ, and since the systole is bounded from below by ε, we get

$$\varepsilon \leqslant \varepsilon/2 + \sup_{\ell \leqslant \ell_i} \mu_\Lambda(t(x), \ell),$$

which leads to

$$\varepsilon/2 \leqslant \sup_{\ell \leqslant \ell_i} \mu_\Lambda(t(x), \ell).$$

Now, let η^{nc} be a positive number such that, if $\tau \leqslant \eta^{nc}$, then

$$\sup_{\ell \leqslant \ell_i} \mu_\Lambda(\tau, \ell) < \varepsilon/2$$

for every $i \in \{1, \ldots, k\}$. The inequalities above imply that $t(x) \geqslant \eta^{nc}$ for every $x \in \mathcal{L}_1^{nc}$. $\qquad \square$

6.3.4. Estimates on $C_1 \smallsetminus \mathcal{L}_1$

Let δ be the diameter of the Cauchy surface C_1 (for the Riemannian metric induced by g on C_1).

LEMMA 6.7. — *Let $\eta = \min(\eta^c, \eta^{nc})$, where η^c, η^{nc} are the positive numbers defined in Lemma 6.5 and 6.6. Then*

$$t(x) \geqslant \nu_\lambda \exp(\delta)\eta \quad \text{for every} \quad x \in C_1 \smallsetminus \mathcal{L}_1.$$

---

[9] By such, we mean that $\beta_{1,x}$ always intersects $\mathcal{L}_1$ with the same orientation.

Proof. — Any point $x \in C_1 \setminus \mathcal{L}_1$ can be joined to a point $y \in \mathcal{L}_1$ by a path c_1 of length at most δ. We can assume the whole of c_1 but y is contained in $C_1 \setminus \mathcal{L}_1$, that is the curve c_1 minus y is contained in a solid block. The first inequality in Proposition 6.2 yields:

$$t(x) \geqslant \nu_\Lambda \exp(\delta) t(y).$$

Since $y \in \mathcal{L}_1 = \mathcal{L}_1^c \cup \mathcal{L}_1^{nc}$, Lemmas 6.5 and 6.6 yield

$$t(y) \leqslant \eta.$$

Putting together the two inequalities above, we get $t(x) \geqslant \nu_\lambda \exp(\delta) \eta$, as wanted. □

6.3.5. Conclusion

Using Lemmas 6.5, 6.6 and 6.7 together with the decomposition

$$C_1 := \mathcal{L}_1^c \cup \mathcal{L}_1^{nc} \cup (C_1 \setminus \mathcal{L}_1),$$

we get a uniform positive lower bound for t on C_1. Since $\phi^{t(x)-1}(x)$ ranges over Σ when x ranges over C_1, and since $\tau\left(\phi^{t(x)-1}(x)\right) = t(x)$, this yields a uniform positive lower bound for the cosmological time τ on Σ. This completes the proof of Theorem 3.5. □

7. Sequences of convex Cauchy surfaces

The aim of this section is to prove Theorem 3.6 and Corollary 3.8.

Let us recall what are the objects and the hypotheses. We consider a 3-dimensional non-elementary MGHC spacetime (M, g) of constant curvature, and a sequence $(\Sigma_n)_{(n \in \mathbb{N})}$ of (locally) convex generalized Cauchy surfaces in M (see §4.2 and 4.3). We assume that this sequence is *decreasing* (i.e. $I^+(\Sigma_{n+1}) \supset I^+(\Sigma_n)$ for every n), and we assume that the set $\Omega = \bigcup_{n \in \mathbb{N}} I^+(\Sigma_n)$ is not the whole M.

We want to prove that every support plane of Ω is spacelike (Theorem 3.6), and that the Σ_n's are uniformly spacelike (Corollary 3.8; See definition 3.7).

Observe that Ω is a future domain, that is, for every x in Ω, we have $I^+(x) \subset \Omega$. It follows that the boundary $\Sigma_\infty = \partial\Omega$ is a closed (but non-compact in general) edgeless achronal subset (see [54, §14, Corollary 27]), hence a topological surface.

LEMMA 7.1. — *The set Σ_∞ is the set of limits in M of sequences $(p_n)_{n\in\mathbb{N}}$ with $p_n \in \Sigma_n$.*

Proof. — Let p be an element of Σ_∞, and let $c\colon]-\infty,+\infty[\to M$ be an inextendible future timelike curve with $c(0) = p$. Since every Σ_n is a Cauchy surface, for every n there is a unique real number t_n such that $c(t_n)$ belongs to Σ_n. Then $c(]t_n,+\infty[)$ is contained in $I^+(\Sigma_n)$. Since p does not belong to Ω_n we have $t_n \geqslant 0$. Since the sequence $(\Sigma_n)_{n\in\mathbb{N}}$ is decreasing, the sequence $(t_n)_{n\in\mathbb{N}}$ is also decreasing and admits a limit $\bar{t}$. Then $c(\bar{t})$ belongs to Σ_∞: We obtain $\bar{t} = 0$ since Σ_∞ is achronal.

Conversely, if p is a limit of a sequence $(p_n)_{n\in\mathbb{N}}$ with $p_n \in \Sigma_n$, it belongs to the closure of Ω. If p belongs to Ω, then it belongs to some Ω_{n_0}. The same then would be true for all the p_n with n sufficiently big, but this is impossible for $n > n_0$. Hence, p belongs to the boundary $\Sigma_\infty = \partial\Omega$. $\square$

Let $p \in \Sigma_\infty$, and let $\varphi\colon U \to V$ be a convex chart centered at p. By our discussion in §4.3, the surfaces $\varphi(\Sigma_n \cap U)$ are graphs of convex functions. Using Lemma 7.1, it follows that there is a convex map $(x,y) \to z = f_\infty(x,y)$ such that $\varphi(U \cap \Sigma_\infty)$ is the graph of f_∞. In other words, the topological surface Σ_∞ is convex.

Proof of Theorem 3.6. — Since Σ_∞ is achronal, it is quite immediate that a support plane of Σ_∞ cannot contain a timelike curve. Hence, we must show that no support plane to Σ_∞ at p is lightlike.

Assume by contradiction that Σ_∞ admits a lightlike hyperplane H at p in a convex chart U centered at p. On one hand, $\Sigma_\infty \cap U$ is contained in $J_U^+(H)$. On the other hand, $\Sigma_\infty \cap U$ is contained in the causal past of the future $I_U^+(p)$. But $H \cap J_U^+(p) = H^\perp$ is a null segment, and hence this null segment is contained in Σ_∞.

More formally, we have proved that there is a future-oriented null geodesic segment $c\colon [0,1] \to M$ with $c(0) = p$ such that every $c(t)$ belongs to Σ_∞. Let $\hat{c}\colon [0,T[\to M$ be the inextendible geodesic ray extending c in the future. Let T' be the sup of times t such that $\hat{c}(t)$ belongs to Σ_∞. Observe that $T' > 0$. Assume $T' < T$. Since Σ_∞ is closed, $q = \hat{c}(T')$ belongs to Σ_∞.

Since there is a null geodesic segment $[p,q]$ contained in Σ_∞, there is only one support plane to Σ_∞ at q: The null plane containing $\hat{c}(t)$ for $0 < t$ near T'. The argument above then implies that $\hat{c}(t)$ belongs to Σ_∞ for $t > T'$ and close enough to T'. This contradicts the definition of T'. Therefore, $T' = T$, that is, the whole future oriented null ray $\hat{c}$ is contained in Σ_∞.

Now, for every n, the point $p = c(0)$ is in the past of Σ_n, and Σ_n is a Cauchy surface, thus, the inextendible geodesic ray $\hat{c}$ intersects $I^+(\Sigma_n)$. Since $I^+(\Sigma_n) \subset I^+(\Sigma_\infty)$, it follows that the geodesic ray $\hat{c}$ intersects

$I^+(\Sigma_\infty)$. This contradicts the fact that Σ_∞ is achronal and $\hat{c}(t)$ is contained in Σ_∞. □

Proof of Corollary 3.8. — Assume that the sequence $(\Sigma_n)_{n\in\mathbb{N}}$ is not uniformly spacelike. Then one can find a sequence of points $(x_n)_{n\in\mathbb{N}}$, with $x_n \in \Sigma_n$, such that:

- extracting a subsequence if necessary, the sequence $(x_n)_{n\in\mathbb{N}}$ converges towards a point $x \in M$,
- if P_n is the unique support plane of Σ_n at x, then, extracting a subsequence if necessary, the sequence $(x_n, P_n)_{n\in\mathbb{N}}$ converges towards a couple (x, P) where P is a null plane at x.

By Lemma 7.1, the point x is on the topological surface Σ. For every n, since P_n is the support plane of Σ_n at x_n, the surface Σ_n is in the future of the totally geodesic plane P_n. Using again Lemma 7.1, it follows that Σ is in the future of the null plane P. Since $x \in \Sigma \cap P$, this implies that P is a support plane of Σ at x, contradicting Theorem 3.6.

As a consequence, the sequence $(\Sigma_n)_{n\in\mathbb{N}}$ is uniformly spacelike. □

8. Sequences of Cauchy surfaces of constant K-curvature

The present section is devoted to the proof of Theorem 3.9. Recall that we are considering a 3-dimensional non-elementary MGHC spacetime (M, g) of constant curvature Λ, and a decreasing sequence of Cauchy surfaces $(\Sigma_n)_{n\in\mathbb{N}}$ in M, such that, for every n, the surface Σ_n is locally strictly convex and has a constant K-curvature $\kappa_n < -\Lambda$ (note that this condition is automatically satisfied if $\Lambda \leqslant 0$ since locally strictly convex surfaces have negative K-curvature).

We will see later that $(\kappa_n)_{n\in\mathbb{N}}$ is automatically a decreasing sequence of real numbers (see Remark 10.3). Anyway, we do not really need this here: We only need to assume that

$$\kappa_n \to \kappa \in [-\infty, \min(-\Lambda, 0)) \quad \text{when} \quad n \to \infty.$$

If $\kappa > -\infty$, we have to prove that the sequence Σ_n converges to a smooth surface Σ_∞ of constant K-curvature κ. If $\kappa = -\infty$, we have to prove that $M = \bigcup_{n\in\mathbb{N}} I^+(\Sigma_n)$.

8.1. The $\kappa > -\infty$ case

The main idea of the proof is to apply a result of Schlenker which describes the asymptotic behaviour of a sequence of immersions of a disc

in a Lorentzian space of constant curvature, when the sequence of immersions itself does not converge, although the sequence of metrics obtained by pulling-back the Lorentzian metric of constant curvature by the immersions does converge (see Theorem 8.5 below for a precise statement).

We consider the surfaces Σ_n as the images of embeddings $f_n \colon \Sigma \hookrightarrow M$, where Σ is a fixed abstract compact surface. For every n, we denote by g_n the Riemannian metric induced on the Cauchy surface Σ_n by the Lorentzian metric g. We can write

$$g_n = |\kappa_n|^{\frac{1}{2}} . \hat{g}_n$$

where $\hat{g}_n$ is a metric on Σ_n with constant K-curvature -1. We denote by $[\hat{g}_n]$ the class of the pull-back metric $f_n^* \hat{g}_n$ in the moduli space $\mathrm{Mod}(\Sigma)$, i.e. the space of hyperbolic metrics on Σ up to isometries.

LEMMA 8.1. — *The sequence $([\hat{g}_n])_{n \in \mathbb{N}}$ is relatively compact in $\mathrm{Mod}(\Sigma)$.*

Proof. — Since $(\Sigma_n)_{n \in \mathbb{N}}$ is a decreasing sequence of Cauchy surfaces, all of them lie in the past of Σ_0. They lie in the past of a level set of the cosmological time of M, say the level set C_a. By Proposition 6.1, this implies that the diameter of (Σ_n, g_n) is uniformly bounded from above by the diameter of the level set $\mathcal{C}_a$. Since $\kappa \neq 0$, deleting a finite number of elements of the sequence $(\Sigma_n)_{n \in \mathbb{N}}$ if necessary, we can assume that the K-curvatures κ_n are bounded away from 0. It follows that the diameter of $(\Sigma_n, \hat{g}_n)$ is also uniformly bounded from above. By the collar neighbourhood lemma, this implies that the systole of $(\Sigma_n, \hat{g}_n)$ is uniformly bounded away from 0. It follows that the sequence $([\hat{g}_n])_{n \in \mathbb{N}}$ is relatively compact in $\mathrm{Mod}(\Sigma)$ (see for example [24, Theorem 6.6.5]). $\qquad\square$

Lemma 8.1 can be reformulated as follows:

COROLLARY 8.2. — *For every $n \in \mathbb{N}$, one can select the smooth embedding $f_n \colon \Sigma \hookrightarrow M$ such that $f_n(\Sigma) = \Sigma_n$ and such that the sequence $(f_n^* \hat{g}_n)_{n \in \mathbb{N}}$ is relatively compact in the space of the Riemannian metrics on Σ endowed with C^∞ topology.* $\qquad\square$

Since the sequence $(\kappa_n)_{n \in \mathbb{N}}$ converges, the sequence $(f_n^* g_n)_{n \in \mathbb{N}}$ is also relatively compact in the space of the Riemannian metrics on S endowed with C^∞ topology. Extracting a subsequence if necessary, we get a sequence of embeddings (that we still denote by $(f_n)_{n \in \mathbb{N}}$) of Σ in M such that the sequence of metrics $(f_n^* g_n)_{n \in \mathbb{N}}$ converges in the C^∞ topology towards a metric g_∞ on Σ.

Now, let us consider a point x in Σ. For every n, let $y_n := f_n(x)$.

LEMMA 8.3. — *The sequence of points* $(y_n)_{n\in\mathbb{N}} = (f_n(x))_{n\in\mathbb{N}}$ *has a convergent subsequence.*

Proof. — For every $n \in \mathbb{N}$, the point y_n belongs to the surface Σ_n. So, it is enough to prove that all the Σ_n's are contained in a compact region of M.

On the one hand, since the sequence $(\Sigma_n)_{n\in\mathbb{N}}$ is decreasing, all the Σ_n's are contained in the past of the Cauchy surface Σ_0. On the other hand, we have seen that the sequence of the systoles of the $(\Sigma_n, \hat{g}_n)$'s is bounded away from 0 (proof of Lemma 8.1). Since the sequence $(\kappa_n)_{n\in\mathbb{N}}$ is bounded away from 0 (deleting a finite number of surfaces Σ_n if necessary), it follows that the sequence of the systoles of the (Σ_n, g_n)'s is also bounded away from 0 . By Theorem 3.5, this implies that all the Σ_n's are contained in the future of some level set C_a of the cosmological time of M. As a consequence, all the Σ_n's are contained in the compact subset $I^+(C_a) \cap I^-(\Sigma_0)$ of M.   □

LEMMA 8.4. — *The sequence of 1-jets* $(j^1 f_n(x))_{n\in\mathbb{N}}$ *has a convergent subsequence.*

Proof. — By Lemma 8.3, up to extracting a subsequence, we may assume that the sequence of points $(y_n)_{n\in\mathbb{N}}$ converges towards a point $y \in M$. For every $n \in \mathbb{N}$, the positive definite quadratic form $q_n := (f_n^* g_n)_{|T_x\Sigma}$ is the pull back by $df_n(x)$ of the positive definite quadratic form $s_n := g_{n|T_{y_n}\Sigma_n}$. On the one hand, the sequence $(q_n)_{n\in\mathbb{N}}$ converges towards the positive definite quadratic form q on $g_{\infty|T_x\Sigma}$. On the other hand, Corollary 3.8 implies that, up to extracting a subsequence, the sequence of subspaces $T_{y_n}\Sigma_n$ converges towards a spacelike plane P in T_yM, and thus the sequence of quadratic forms $(s_n)_{n\in\mathbb{N}}$ converges towards the positive definite quadratic form $s = g_{|P}$. Since the space of linear maps leaving invariant a positive definite quadratic form is compact, it follows that, up to extracting a subsequence, the sequence of linear maps $df_n(x)$ converges towards a linear map $d\colon T_x\Sigma \to P$ such that q is the pull-back of s by d.   □

Schlenker's result only concerns embeddings of discs in a simply connected Lorentzian space. So we need to lift everything to the universal covering. The universal covering of Σ is the 2-disc $\mathbb{D}$. The universal covering $\widetilde{M}$ of M is a convex open domain in $X = \text{Min}_3$, dS_3 or $\widetilde{\text{AdS}}_3$. Corollary 8.2 and Lemma 8.4 implies that, for every $n \in \mathbb{N}$, we can find a lift $\widetilde{\Sigma}_n$ of Σ_n in X, a lift $\widetilde{f}_n\colon \mathbb{D} \hookrightarrow X$ of f_n such that:

  – the sequence of metrics $(\widetilde{f}_n^* \widetilde{g}_n)_{n\in\mathbb{N}}$ converges towards a Riemannian metric $\widetilde{g}_\infty$ on $\mathbb{D}$ in the C^∞ topology (where $\widetilde{g}_n$ is the Riemannian metric induced on $\widetilde{\Sigma}_n$ by the Lorentzian metric of X),

– for every $\widetilde{x} \in \mathbb{D}$ the sequence of 1-jets $(j^1 \widetilde{f}_n(\widetilde{x}))_{n \in \mathbb{N}}$ has a convergent subsequence.

Now, we argue by contradiction: *We assume that the sequence of surfaces Σ_n does not converge in the C^∞ topology towards a smooth surface S of constant curvature κ.* It follows in particular that the sequence of embeddings $(f_n)_{n \in \mathbb{N}}$ does not converge in the C^∞ topology. A fortiori, the sequence of embeddings $(\widetilde{f}_n)_{n \in \mathbb{N}}$ does not converge in the C^∞ topology. So we are under the assumption of Schlenkers result:

THEOREM 8.5 (see [56], Théorème 5.6). — *Let $(\widetilde{f}_n)_n \in \mathbb{N} \colon \mathbb{D} \to X$ be a sequence of uniformly elliptic immersions[10] of a disc $\mathbb{D}$ in a simply connected Lorentzian spacetime of constant curvature $(X, \widetilde{g})$. On the one hand, assume that the metrics $\widetilde{f}_n^* \widetilde{g}$ converges in the C^∞ topology towards a Riemannian metric $\widetilde{g}_\infty$ on $\mathbb{D}$, and that there exists a point $x \in \mathbb{D}$ such that the sequence of 1-jets $(j^1 \widetilde{f}_n(x))_{n \in \mathbb{N}}$ converges. On the other hand, assume that the sequence $(\widetilde{f}_n)_{n \in \mathbb{N}}$ does not converge in the C^∞ topology in a neighbourhood of x. Then there exists a maximal geodesic γ of $(\mathbb{D}, \widetilde{g}_\infty)$ and a geodesic arc Γ of $(X, \widetilde{g})$ such that the sequence $(\widetilde{f}_{n|\gamma})_{n \in \mathbb{N}}$ converges towards an isometry $f_\infty \colon \gamma \to \Gamma$.* $\square$

By this theorem, there exists a maximal geodesic γ of $(\mathbb{D}, \widetilde{g}_\infty)$ and a geodesic segment Γ in X, such that $f_{n|\gamma}$ converges towards an isometry from $f_\infty \colon \gamma \to \Gamma$. Since $\widetilde{g}_\infty$ is obtained by lifting a Riemannian metric on a compact surface, it is geodesically complete. In particular, the geodesic γ has infinite length. And, as a consequence, the geodesic arc Γ also has infinite length. But since $f_n(\Sigma) \subset \widetilde{M}$ for every n, the geodesic arc Γ must be contained in the closure of $\widetilde{M}$ in X. Moreover, in the AdS$_3$-case, Γ must be contained in the closure of the complement of the lift of the convex core of M (since all the Σ_n's are contained in the complement of the convex core of M). This contradicts the following proposition:

PROPOSITION 8.6. — *Let (M, g) be a non-elementary 3-dimensional MGHC spacetime with constant curvature Λ. If $\Lambda \geqslant 0$ (flat case or locally de Sitter case), then there is no complete spacelike geodesic in M. If $\Lambda < 0$ (locally anti-de Sitter case), then every complete spacelike geodesic of M is contained in the convex core of M.*

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[10] An immersion f of a $(n-1)$-dimensional manifold N in a n-dimensional Lorentzian space X of constant curvature Λ such that $f(N)$ is spacelike and has constant K-curvature $\kappa < -\Lambda$ is a typical example of *uniformly elliptic immersion*.

Proof. — In the flat case, we know that the spacetime is a quotient $\bar{\Gamma} \smallsetminus E$ where E is a *proper* convex domain, *i.e.* contains no complete affine line (see [51]): The proposition follows.

In the locally anti-de Sitter case, we observe that any complete spacelike geodesic c in M lifts to a complete spacelike geodesic in $E(\Lambda_\rho) \subset \mathrm{ADS}_3$. Such a geodesic admits two extremities in $\mathcal{Q} = \partial \mathrm{ADS}_3$. But these two extremities must belong to the closure of $E(\Lambda_\rho)$ in ADS_3. The intersection between this closure and $\mathcal{Q}$ is reduced to Λ_ρ: Hence, the spacelike geodesic under consideration has extremities in Λ_ρ. It follows that c is contained in the convex core.

The remaining case is the locally de Sitter case. Let $c \colon \mathbb{R} \to M$ be the spacelike geodesic, parametrized by arc-length, and let $\widetilde{c} \colon \mathbb{R} \to \widetilde{M}$ be its lifting in the universal covering. The composition $\widetilde{\zeta} = \mathcal{D} \circ \widetilde{c}$ with the developing map is a complete spacelike geodesic in dS_3. In particular, the image of $\widetilde{\zeta}$ is contained in a 2-plane P of $\mathrm{dS}_3 \subset \mathbb{R}^{1,3}$.

Reversing the time-orientation if necessary, we can assume that (M, g) is future complete. Then $(\widetilde{M}, g)$ is isometric to the spacetime $\mathcal{B}_0(S)$ associated to a simply connected Möbius surface S (naturally homeomorphic to the universal cover of the compact Σ) (see §5.2.2). It follows that the future of the image of $\widetilde{c}$ in $\widetilde{M}$ is isometric to the universal covering $\widetilde{\Omega}(\Delta)$ - here the geodesic Δ of $\mathbb{H}^3$ is the intersection between $\mathbb{H}^3$ and the orthogonal $P^\perp$ in $\mathbb{R}^{1,3}$.

Now we observe that $\widetilde{\Omega}(\Delta)$ can also be described as $\mathcal{B}_0(\mathbb{CP}^1 - \{0, +\infty\})$ if $0, \infty$ denotes the extremities of Δ. In other words, we have an embedding $F \colon \mathcal{B}_0(\mathbb{C}^*) \to \mathcal{B}_0(S)$. According to Remark 5.6, we have a map $f \colon \mathbb{C}^* \to S$, inducing a holomorphic map $\bar{f} \colon \mathbb{C}^* \to \Sigma$. But since Σ has genus $\geqslant 2$ such an holomorphic map must be constant: This is a contradiction. $\square$

The contradiction we have obtained shows that our assumption was absurd. Therefore, the sequence of Cauchy surfaces $(\Sigma_n)_{n \in \mathbb{N}}$ towards a surface Σ_∞ in the C^∞ topology. Clearly, this implies that Σ_∞ is a convex Cauchy surface with constant K-curvature $\kappa = \lim \kappa_n$. This completes the proof of Theorem 3.9 in the case where $\kappa_n \to \kappa > -\infty$.

8.2. The $\kappa = -\infty$ case

We are left to consider the case where $\kappa_n \to -\infty$. In this case, we want to prove: $\bigcup_n I^+(\Sigma_n) = M$.

We argue by contradiction: We suppose that the convex domain $\Omega :=$ $\bigcup_n I^+(\Sigma_n)$ is not the whole spacetime M. Then $\Sigma_\infty := \partial\Omega$ is non-empty. As discussed in the beginning of §7, Σ_∞ is a topological surface. Consider an open set with compact closure N in Σ_∞, and choose a local time function on a neighbourhood of N. This local time function allows to decompose some neighbourhood (with compact closure) W of N as a product $W = N \times I$ where I is an interval in $\mathbb{R}$ and $\{x\} \times I$ is timelike for every $x \in N$. Using Lemma 7.1, we see that this allows to write, for every n large enough, the surface $\Sigma_n \cap W$ as a graph over N. By Corollary 3.8, this sequence of graphs is uniformly spacelike. Obviously, this implies that the area of the local surface $\Sigma_n \cap W$ does not tend to 0 when $n \to \infty$. In particular, the area of the Cauchy surface Σ_n does not tend to 0 when $n \to \infty$. This contradicts the Gauss-Bonnet formula: Up to a multiplicative constant (depending on the genus of the Cauchy surfaces of M) the area of Σ_n equals $-\frac{1}{\Lambda+\kappa_n}$.

This completes the proof of Theorem 3.9.

9. Construction of barriers

This section is essentially devoted to the proof of Theorem 3.4 and Proposition 3.10. The proof of Theorem 3.4 can be divided into two steps:

– first, we show that every 3-dimensional future complete non-elementary MGHC spacetime with constant curvature $\Lambda \geqslant 0$ contains a convex Cauchy surface Σ with K-curvature strictly bounded from above by $-\Lambda$,
– then we push Σ along the geodesics orthogonal to Σ in order to get convex Cauchy surfaces whose K-curvature is arbitrarily close to $-\Lambda$.

Proposition 3.10 will be an easy consequence of our estimates on the behaviour of the principal curvatures when one pushes a surface along the orthogonal geodesics.

It is important at this stage to recall our conventions. Let Σ be a spacelike surface in (M, g). Recall that the second fundamental form of Σ is defined by $II(X, Y) = -\langle \nabla_X n, Y \rangle$ where n is the future oriented unit normal of Σ. The principal curvatures λ_1, λ_2 of Σ are the eigenvalues of the second fundamental form, and the K-curvature of Σ equals $-\lambda_1\lambda_2$. For example, with these conventions, if M is the Minkowski space $\mathbb{R}^{1,2}$ endowed with the Lorentzian metric $-dt^2+dx^2+dy^2$, the surface $\{-t^2+x^2+y^2 = -c^2, t > 0\}$ has principal curvatures $\lambda = \mu = -c$.

9.1. Existence of a Cauchy surface with controlled curvature

9.1.1. The flat case

We will prove the following result:

PROPOSITION 9.1. — *Every non-elementary 3-dimensional future complete flat MGHC spacetime contains a locally strictly convex Cauchy surface (in particular, a Cauchy surface with negative K-curvature).*

Equivalently, every non-elementary 3-dimensional flat MGHC spacetime contains a Cauchy surface with negative principal curvatures. In order to prove this Proposition 9.1, we will use the existence of constant mean curvature surfaces in flat MGHC spacetimes and the following proposition which is essentially due to A. Treibergs:

PROPOSITION 9.2. — *Let (M, g) be a non-elementary MGHC flat spacetime. Then, every constant mean curvature Cauchy surface in M has negative principal curvatures.*

Proof. — Any Cauchy surface with constant mean curvature Σ in M lifts as a complete spacelike surface with constant mean curvature $S \simeq \widetilde{\Sigma}$ in the Minkowski space. It is shown in [60] (see also [27]) that such a surface S is either *strictly convex* (*i.e.* has negative principal curvatures), or splits as a direct product of a curve by an affine line. The splitting case is excluded in our context, since we have assumed that our spacetime is non-elementary. The proposition follows. □

Proof of Proposition 9.1. — In [3] and [1], it was shown that every future complete flat spacetime contains a Cauchy surface S with constant mean curvature -1 (see [3, 1]). Together with Proposition 9.2, this proves Proposition 9.1. □

9.1.2. Transferring properties of CMC hypersurfaces from flat to dS$_3$ and AdS$_3$ spacetimes

Now, we would like to get an analog of Proposition 9.1 in dS$_3$ and AdS$_3$-spacetimes. For this purpose, we will use a correspondence between CMC hypersurfaces in flat spacetimes, CMC hypersurface in dS spacetimes and CMC hypersurfaces in AdS spacetimes, very similar to a statement discovered by H. Hopf in the 50's, in the context of CMC surfaces in Euclidean 3-spaces:

PROPOSITION 9.3 (See *e.g.* [7]). — *Let Σ be a compact surface, η be a Riemannian metric on Σ with scalar curvature R_η, and h be a quadratic differential on Σ. Let Λ and H be two real numbers. There exists an isometric embedding of (Σ, η) as a spacelike Cauchy surface with constant mean curvature H having h as the second fundamental form in some MGHC spacetime (M, g) with constant curvature Λ if and only if the two following conditions are satisfied:*

(1) *the quadratic differential $h_0 = h - H \cdot \eta$ (i.e. the trace free part of h) is the real part of a holomorphic quadratic differential,*

(2) *the Gauss equation $R_\eta + \det_\eta(h_0) = \Lambda - H^2$ is satisfied.*

Remark 9.4. — If (Σ, η, h) satisfies conditions (1) and (2), then:

– one can take as a spacetime (M, g) the Cauchy development of the solution of the Einstein equations with a cosmological constant Λ, in a CMC gauge, with initial Cauchy data (Σ, η, h),

– (S, η, h) is called a *CMC (H, Λ)-initial Cauchy data.*

The following immediate corollary of Proposition 9.3 will allow us to transfer properties of CMC hypersurfaces in flat spacetimes into properties of CMC hypersurfaces in dS_3 and AdS_3 spacetimes.

COROLLARY 9.5. — *A CMC (H, Λ)-initial data $(\Sigma, \eta, h = h_0 + H\eta)$ gives rise to a CMC (H', Λ')-initial data of the form $(\Sigma, \eta, h' = h_0 + H'\eta)$ provided that*

$$\Lambda' - H'^2 = \Lambda - H^2.$$

Remark 9.6. — Assume that $(\Sigma, \eta, h = h_0 + H\eta)$ is a CMC (H, Λ)-initial data and $(\Sigma, \eta, h' = h_0 + H'\eta)$ is a CMC (H', Λ')-initial data. Denote by λ, μ (resp. Λ', μ') the principal curvatures of Σ seen as a CMC (H, Λ)-initial Cauchy data (resp. as a CMC (H', Λ')-initial Cauchy data). Then

$$\lambda' = \lambda + (H' - H) \quad \text{and} \quad \mu' = \mu + (H' - H).$$

9.1.3. The dS_3 case

We will prove the following result:

PROPOSITION 9.7. — *Every non-elementary 3-dimensional future complete MGHC spacetime with constant curvature $\Lambda = +1$ contains a convex Cauchy surface with K-curvature strictly bounded from above by $-\Lambda$.*

Proof. — Consider a non-elementary 3-dimensional future complete MGHC spacetime (M, g) with constant curvature $+1$. It was proved in [2] that (M, g) admits a Cauchy hypersurface Σ with constant mean curvature $-\sqrt{2}$. Denote by η and h the first and the second fundamental forms of Σ. Then $(\Sigma, \eta, h = h_0 - \sqrt{2}\eta)$ is a CMC $(-\sqrt{2}, 1)$-initial Cauchy data. So, by Corollary 9.5, $(\Sigma, \eta, h' = h_0 - \eta)$ is a CMC $(-1, 0)$-initial Cauchy data (*i.e.* (Σ, η) admits an isometric embedding in a flat MGHC spacetime with constant mean curvature -1 and second fundamental form h'). Denote by λ and μ (resp. λ' and μ') the principal curvatures of Σ seen as a CMC $(-\sqrt{2}, +1)$ initial Cauchy data (resp. as a CMC $(-1, 0)$ initial Cauchy data). According to Proposition 9.2, the principal curvatures λ' and μ' are negative. Using Remark 9.6, this implies that

$$\lambda < -\sqrt{2} + 1 \quad \text{and} \quad \mu < -\sqrt{2} + 1$$

(in particular, λ and μ are negative, so that Σ is convex in M.) Moreover, since $\lambda + \mu = 2H = -2\sqrt{2}$, the above inequality yields

$$-\sqrt{2} - 1 < \lambda < -\sqrt{2} + 1 \quad \text{and} \quad -\sqrt{2} - 1 < \mu < -\sqrt{2} + 1.$$

These bounds on λ, μ and the equality $\lambda + \mu = -2\sqrt{2}$ imply that

$$-\lambda \cdot \mu < -1,$$

i.e. that the K-curvature of the Cauchy surface Σ in M is strictly bounded from above by $-\Lambda$, as wanted. $\square$

9.1.4. The AdS$_3$ case

The arguments developed above are not sufficient to get a directly a convex Cauchy surface in every AdS$_3$ spacetime. Nevertheless, these argument provide us with a Cauchy surface with controlled principal curvatures:

PROPOSITION 9.8 (See also Lemma 3.11 in [43]). — *Every non-elementary 3-dimensional MGHC spacetime with constant curvature $\Lambda = -1$ contains a maximal Cauchy surface (i.e. a Cauchy surface with constant mean curvature 0). Moreover, the principal curvatures of this maximal surface stay within the interval $(-1, 1)$.*

Proof. — Consider a 3-dimensional non-elementary MGHC spacetime (M, g) with constant curvature -1. It was proved in [12] that (M, g) admits a maximal Cauchy surface Σ. Denote by η and h the first and the second fundamental forms of Σ. Then $(\Sigma, \eta, h = h_0)$ is a CMC $(0, -1)$-initial Cauchy data. So, by Corollary 9.5, $(\Sigma, \eta, h' = h_0 - \eta)$ is a CMC

$(-1, 0)$-initial Cauchy data. Denote by λ and μ (resp. λ' and μ') the principal curvatures of Σ seen as a CMC $(0, -1)$ initial Cauchy data (resp. as a CMC $(-1, 0)$ initial Cauchy data). According to Proposition 9.2, the principal curvatures λ', μ' are negative. Using Remark 9.6, this implies $\lambda < 1$ and $\mu < 1$. Since $H = -(\lambda + \mu) = 0$, it follows that $-1 < \lambda < 1$ and $-1 < \mu < 1$.   □

Remark 9.9. —   In subsection 9.4, we will explain how to show that every non-elementary MGHC spacetime with constant curvature -1 contains a convex (and a concave) Cauchy surface.

9.2. Pushing a Cauchy surface along the orthogonal geodesics

Let (M, g) be an (arbitrary) 3-dimensional spacetime. Let Σ be a spacelike surface in M. For every $x \in \Sigma$, denote by $n(x)$ the future oriented unit normal vector of Σ at x. For t small enough, the map

$$\phi_\Sigma^t \colon \Sigma \longrightarrow M$$
$$x \longmapsto \exp_x(t \cdot n(x))$$

is well-defined, and is an embedding. For such a t, the surface

$$\Sigma^t := \phi_\Sigma^t(\Sigma)$$

is obviously a compact spacelike surface, hence a Cauchy surface. We say that the Cauchy surface $\Sigma^t := \phi_\Sigma^t(\Sigma)$ is obtained by *pushing* Σ *along orthogonal geodesics for a time t*.

In our situation, since the ambient curvature Λ is constant, one can compute explicitly the principal curvatures of the surface Σ^t:

PROPOSITION 9.10. — *Pick a point $x \in \Sigma$ and a real number t. We assume that the map ϕ_Σ^t is well-defined (i.e. that the geodesics that are orthogonal to Σ exists for a time at least t). We denote by λ, μ the principal curvature of Σ at x.*

1. The flat case. *Assume that (M, g) is flat (i.e. locally isometric to Min_3). If $\lambda.t \neq 1$ and $\mu.t \neq 1$ then ϕ_Σ^t is an embedding in the neighborhood of x, and the principal curvature of the surface Σ^t at the point $\phi_t(x)$ are*

$$\lambda_t = \frac{\lambda}{1 - \lambda t} \quad \text{and} \quad \mu_t = \frac{\mu}{1 - \mu t}.$$

2. The dS_3-case. *Assume that (M, g) has positive constant curvature (i.e. is locally isometric to dS_3). If $\lambda \tanh(t) \neq 1$ and $\mu \tanh(t) \neq 1$ then ϕ_Σ^t*

is an embedding in the neighborhood of x, and the principal curvature of the surface Σ^t at the point $\phi_t(x)$ are

$$\lambda_t = \frac{\lambda - \tanh(t)}{1 - \lambda \tanh(t)} \quad \text{and} \quad \mu_t = \frac{\mu - \tanh(t)}{1 - \mu \tanh(t)}.$$

3. The AdS_3-case. *Assume that (M, g) has negative constant curvature (i.e. is locally isometric to AdS_3) If $\lambda \tan(t) \neq 1$ and $\mu \tan(t) \neq 1$ then ϕ^t_Σ is an embedding in the neighborhood of x, and the principal curvature of the surface Σ^t at the point $\phi^t_\Sigma(x)$ are*

$$\lambda_t = \frac{\lambda + \tan(t)}{1 - \lambda \tan(t)} \quad \text{and} \quad \mu_t = \frac{\mu + \tan(t)}{1 - \mu \tan(t)}.$$

Proof. — Straightforward computation. □

As an immediate corollary, we get:

COROLLARY 9.11. — *Pick a point $x \in \Sigma$. Denote by H (resp. κ) the mean curvature (resp. the K-curvature) of the surface Σ at x. For t small enough, denote by κ_t the K-curvature of the surface Σ^t at $\phi^t_\Sigma(x)$. Then,*

$$\frac{\partial \kappa_t}{\partial t}\bigg|_{t=0} = 2(\kappa + \Lambda)H.$$

Remark 9.12. — In particular, the K-curvature of Σ^t at $\phi^t_\Sigma(x)$ increases with t (for t close to 0) provided that the surface Σ is convex and has a K-curvature κ strictly bounded from above by $-\Lambda$.

9.3. Proof of Theorem 3.4

We consider a 3-dimensional non-elementary MGHC spacetime (M, g) with constant K-curvature $\Lambda \geqslant 0$. Reversing the time orientation if necessary, we assume that (M, g) is future complete.

PROPOSITION 9.13. — *If Σ is a convex Cauchy surface in M whose K-curvature is strictly bounded from above by $-\Lambda$, then:*

(1) *for every $t \geqslant 0$, the map $\phi^t_\Sigma : \Sigma \to M$ is well-defined and is an embedding,*

(2) *for every $t \geqslant 0$, the Cauchy surface $\Sigma^t := \phi^t_\Sigma(\Sigma)$ is convex and has a K-curvature strictly bounded from above by $-\Lambda$,*

(3) *the K-curvature of Σ^t tends uniformly towards $-\Lambda$ when $t \to \infty$.*

Proof. — The map ϕ^t_Σ is defined for every $t \geqslant 0$ because the spacetime was assumed to be future complete. The other assertions follow from Proposition 9.10. □

Proposition 9.1 (in the case $\Lambda = 0$) or Proposition 9.7 (in the case $\Lambda = +1$) provide us with a convex Cauchy surface Σ with K-curvature strictly bounded by $-\Lambda$. Choose $\varepsilon > 0$ such that the K-curvature of Σ is bounded from above by $-\Lambda - \varepsilon$. Let $\kappa \in (-\Lambda - \varepsilon, \Lambda)$. By item 1 and 2 of Proposition 9.13, for every $t \geqslant 0$, the surface $\Sigma^t = \phi_\Sigma^t(\Sigma)$ is well-defined, convex, and has K-curvature strictly bounded from above by $-\Lambda$. Moreover, by item 3 of Proposition 9.13, for $t > 0$ large enough, the K-curvature of Σ_t is strictly bounded from below by κ. Therefore, for $t > 0$ large enough, (Σ, Σ_t) is a pair of κ-barriers. This completes the proof of Theorem 3.4.

9.4. Convex and concave Cauchy surfaces in AdS_3 spacetimes

PROPOSITION 9.14. — *Let (M, g) be a non-elementary 3-dimensional MGHC spacetime of AdS_3 type (i.e. with negative constant curvature). Then M contains two Cauchy surfaces with constant K-curvature -1, one strictly convex and the other strictly concave.*

Proof. — Let Γ be the fundamental group of the Cauchy surfaces of M. By Theorem 5.8, there exists a representations $\rho\colon \Gamma \to \mathrm{PSL}(2, \mathbb{R}) \times \mathrm{PSL}(2, \mathbb{R})$, a curve Λ_ρ in $\partial \mathbb{ADS}_3$ associated to ρ, and a open set $E(\Lambda_\rho)$ in $\mathbb{ADS}_3$ such that $M \simeq \rho(\Gamma) \smallsetminus E(\Lambda_\rho)$. We will need the following lemma:

LEMMA 9.15. — *Let Σ be a Cauchy surface in M, and S be a lift of Σ in $E(\Lambda_\rho)$. Let t_0 be a real number such that the map $\phi_S^t\colon S \to \mathrm{AdS}_3$ is an immersion for every t between 0 and t_0. Then $S^{t_0} = \phi_S^{t_0}(S)$ is contained in $E(\Lambda_\rho)$ and projects to a Cauchy surface Σ^{t_0} in M.*

Remark 9.16. — A posteriori, the surface Σ^{t_0} can of course be obtained by pushing Σ along the orthogonal geodesics for a time t_0. Nevertheless, one cannot define Σ^{t_0} directly in M (without passing to the universal cover) since one does not know *a priori* that the geodesics of M that are orthogonal to Σ exist for a long enough time (recall that M is neither past nor future complete).

Let us postpone the proof of Lemma 9.15, and complete the proof of Proposition 9.14. According to Proposition 9.8, the spacetime M admits a maximal Cauchy surface Σ whose principal curvatures stay within the interval $(-1, 1)$. Let S be a lift of Σ in $E(\Lambda_\rho)$. We denote by $\lambda(x), \mu(x)$ the principal curvatures of the surface S at x. Since $|\lambda(x)| < 1$ and $|\mu(x)| < 1$ for every $x \in \Sigma$, Proposition 9.10 implies that the map ϕ_S^t is an immersion

for every $t \in [-\pi/4, 0]$. Therefore, Lemma 9.15 implies that $S^{-\pi/4}$ is embedded, contained in the domain $E(\Lambda_\rho)$, and projects to a Cauchy surface $\Sigma^{-\pi/4}$ in M, which is obtained by pushing the surface Σ along orthogonal geodesics for a time $(-\pi/4)$. Moreover, Proposition 9.10 implies that, for every $x \in S$, the principal curvatures of the surface $S^{-\pi/4}$ at the point $\phi_S^{-\pi/4}(x)$ are

$$\lambda_{-\pi/4}(x) = \frac{\lambda(x) - 1}{1 + \lambda(x)} \quad \text{and} \quad \mu_{-\pi/4}(x) = \frac{\mu(x) - 1}{1 + \mu(x)}.$$

In particular, the Cauchy surface $S^{-\pi/4}$ is strictly convex and has constant K-curvature -1. It follows immediately that the Cauchy surface $\Sigma^{-\pi/4}$ is also strictly convex, and has constant K-curvature -1.

The same arguments show that the Cauchy surface $\Sigma^{\pi/4}$ is well-defined, strictly concave, and has constant K-curvature -1. □

We are left to prove Lemma 9.15. The key point is the following:

SUB-LEMMA 9.17. — *Let $\widetilde{\Sigma}$ be the (abstract) universal cover of the surface Σ and $\phi\colon \widetilde{\Sigma} \to \mathbb{ADS}_3$ be a Γ-equivariant[11] immersion, such that $\phi(\widetilde{\Sigma})$ is spacelike. Then ϕ is a proper embedding, and $\phi(\widetilde{\Sigma})$ is contained in $E(\Lambda_\rho)$.*

Proof of Sub-lemma 9.17. — Since $\phi(\widetilde{\Sigma})$ is spacelike, the Lorentzian metric of AdS_3 induces a Riemannian metric on $\phi(\widetilde{\Sigma})$. Let us denote by g the pull-back (by ϕ) of this Riemannian metric on $\widetilde{\Sigma}$. Since ϕ is Γ-equivariant, this Riemannian metric g is Γ-invariant. And since Γ is co-compact, g is complete. Therefore, ϕ is a locally isometric immersion of the complete Riemannian surface $(\widetilde{\Sigma}, g)$ in $\mathbb{ADS}_3$. By a lemma of Mess (see [51, Lemma 6]), this implies that ϕ is a proper embedding, and that $\phi(\widetilde{\Sigma})$ is achronal[12]. In particular, $\phi(\widetilde{\Sigma})$ is a closed achronal surface in $\mathbb{ADS}_3$. As explained in §5.3.2, this implies that the boundary in $\mathbb{ADS}_3 \cup \partial\mathbb{ADS}_3$ is the curve Λ_ρ (see [10, Theorem 10.13]). This implies that the Cauchy development $D(\phi(\widetilde{\Sigma}))$ is contained in $E(\Lambda_\rho)$ (see [12, Proposition 5.18]); In particular, the surface $\phi(\widetilde{\Sigma})$ is contained in $E(\Lambda_\rho)$. □

Proof of Lemma 9.15. — Let $\widetilde{\Sigma}$ be the (abstract) universal cover of Σ, and choose a Γ-equivariant homeomorphism $\phi_0\colon \widetilde{\Sigma} \to S$. For every t between 0 and t_0, the map $\phi_S^t\colon S \to S^t$ is $\rho(\Gamma)$-equivariant (since the

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[11] By such, we mean that, for every $\widetilde{x} \in \widetilde{\Sigma}$ and every $\gamma \in \Gamma$, one has $\phi(\gamma{\cdot}x) = \rho(\gamma){\cdot}\phi(\widetilde{x})$.
[12] Remember that this means that every timelike curve in $\widehat{AdS_3}$ cannot intersect a lift of $\phi(\widetilde{\Sigma})$ at two different points. We define this property in $\widehat{\mathbb{ADS}_3}$ (rather than in $\mathbb{ADS}_3$) because $\mathbb{ADS}_3$ itself is not a causal space: Any two points of $\mathbb{ADS}_3$ can be joined by a timelike curve in $\mathbb{ADS}_3$.

group $\rho(\Gamma)$ acts by isometries of $\mathbb{ADS}_3$). Therefore, the map $\phi_t := \phi_S^t \circ \phi_0 \colon \widetilde{\Sigma} \to \mathbb{ADS}_3$ is a Γ-equivariant immersion, and $S^t = \phi_t(\widetilde{\Sigma})$ is spacelike. By Lemma 9.17, it follows that S^t is properly embedded and contained in $E(\Lambda_\rho)$. In particular, S^{t_0} is contained in $E(\Lambda_\rho)$. So, we have proved that the map $\phi_{t_0} \colon \widetilde{\Sigma} \to \mathbb{ADS}_3$ is a $\rho(\Gamma)$-equivariant embedding, and that the spacelike surface $S^{t_0} = \phi_{t_0}(\widetilde{\Sigma})$ is contained in $E(\Lambda_\rho)$. It follows that S^{t_0} projects to a compact spacelike surface in $M \simeq \rho(\Gamma) \setminus E(\Lambda_\rho)$. To conclude the proof, we recall that every compact spacelike surface in a MGHC spacetime is a Cauchy surface. $\qquad\square$

9.5. Proof of Proposition 3.10

We consider a 3-dimensional non-elementary MGHC spacetime (M, g) with constant K-curvature Λ. We assume that (M, g) admits a Cauchy surface Σ which is strictly convex (*i.e.* has negative principal curvatures) and has constant K-curvature $\kappa < -\Lambda$. Note that, since Σ is strictly convex, the mean curvature of Σ is negative.

For t small enough, we define the map ϕ_Σ^t and the surface Σ^t as in Subsection 9.2. For $t < 0$, the Cauchy surface Σ^t is of course in the past of Σ. For t small enough, the principal curvatures of Σ^t are close to those of Σ. In particular, for t small enough, the Cauchy surface Σ^t is strictly convex. Moreover, Corollary 9.11 implies that, for $t < 0$ small enough, the K-curvature curvature of Σ^t is strictly bounded from above by κ (see also Remark 9.12). This completes the proof of Proposition 3.10. $\qquad\square$

Remark 9.18. — In the case here Λ is negative (AdS$_3$ spacetimes), there is an analog statement concerning concave surfaces:
Assume that (M, g) admits a strictly concave Cauchy surface Σ with constant K-curvature κ. Then, in the future of Σ (and arbitrarily close to Σ) one can find a strictly concave Cauchy surface whose K-curvature is strictly bounded from below by κ.

10. K-slicings of flat and dS$_3$ spacetimes

We are now ready to prove the existence of K-slicings of non-elementary 3-dimensional MGHC spacetimes of constant curvature curvature. In the present §, we will only consider the cases of flat and locally de Sitter spacetimes (items 1 and 2 of Theorem 2.1). The case of locally anti-de Sitter spacetime (item 3 of Theorem 2.1) will be treated in §11.

All along this section, we consider a non-elementary 3-dimensional MGHC spacetime (M, g) of constant curvature curvature $\Lambda \geqslant 0$. Reversing time-orientation if necessary, we assume that (M, g) is future complete. As explained in section 3, we will first get a local K-slicing (*i.e.* a K-slicing of some open subset of M), and then extend this local K-slicing to a global one.

10.1. Local K-slicings

DEFINITION 10.1. — *A* local K-slicing *in M is a 1-parameter family* $(\Sigma_\kappa)_{\kappa \in I}$ *of Cauchy surfaces in M such that:*

  (1) *the parameter set I is an interval included in $(-\infty, -\Lambda)$,*
  (2) *for every $\kappa \in I$, the Cauchy surface Σ_κ is convex and has constant K-curvature equal to κ.*

The set $U = \bigcup_{\kappa \in I} \Sigma_\kappa$ is called the support *of the local K-slicing.*

Note that, in the above definition, one does not require the Σ_κ's to be pairwise disjoint, or to depend continuously on κ. Actually, these properties are automatically satisfied:

LEMMA 10.2. — *Every local K-slicing $(\Sigma_\kappa)_{\kappa \in I}$ satisfies the following monotonicity property: For every $\kappa_1, \kappa_2 \in I$ such that $\kappa_1 < \kappa_2$, the Cauchy surface Σ_{κ_1} is strictly in the past of the Cauchy surface Σ_{κ_2}.*

In particular, if $(\Sigma_\kappa)_{\kappa \in I}$ is a local K-slicing, then the Σ_κ's are pairwise disjoint.

Proof of Lemma 10.2. — Consider a local K-slicing $(\Sigma_\kappa)_{\kappa \in I}$ in M, pick up two real numbers $\kappa_1, \kappa_2 \in I$ such that $\kappa_1 < \kappa_2$, and assume that Σ_{κ_1} is not strictly in the past of Σ_{κ_2}.

Since M is future complete, the map $\phi^t = \phi^t_{\Sigma_{\kappa_2}}$ does exists for every $t \geqslant 0$. Moreover, since Σ_{κ_2} is convex and $\Lambda = 0$ or 1, Proposition 9.13 implies that $\phi^t_{\Sigma_{\kappa_2}}$ is an embedding for every $t \geqslant 0$. So, for every $t \geqslant 0$, $\Sigma^t_{\kappa_2} := \phi^t_{\Sigma_{\kappa_2}}(\Sigma_{\kappa_2})$ is a smooth Cauchy surface. For $t > 0$ large enough, the Cauchy surface $\Sigma^t_{\kappa_2}$ is strictly in the future of the Cauchy surface Σ_{κ_1}. Therefore, we may consider the infimum $t_{\min}$ of all real numbers $t > 0$ such that $\Sigma^t_{\kappa_2}$ is in the future of Σ_{κ_1}. Clearly, the surfaces $\phi_t(\Sigma_{\kappa_2})$ and Σ_{κ_1} intersect at some point x (otherwise, $t_{\min}$ would not be minimal). From Proposition 9.13 and Remark 9.12, the K-curvature of Σ_{κ_2} is strictly bounded from below by κ_2. Since $\kappa_1 < \kappa_2$, it follows in particular that the K-curvature of the surface $\Sigma^t_{\kappa_2}$ at x is strictly bigger than those of the

surface Σ_{κ_1}. This contradicts the maximum principle (see §4.4) since the surface $\Sigma^t_{\kappa_2}$ is in the future of the surface Σ_{κ_1}. □

Remark 10.3. — Actually, the argument of the above proof shows that: If Σ, Σ' are two convex Cauchy surfaces in M, whose K-curvatures are strictly bounded from above by $-\Lambda$, and such that the supremum of the K-curvature of Σ is smaller than the infimum of the K-curvature of Σ', then Σ is in the past of Σ'.

In particular, for every $\kappa < -\Lambda$, there exists at most one convex Cauchy surface with constant K-curvature κ.

LEMMA 10.4. — *If $(\Sigma_\kappa)_{\kappa \in I}$ is a local K-slicing in M, then the surface Σ_κ depends continuously on κ.*

Proof. — Consider $\kappa_0 \in I$, and a neighbourhood V of the Cauchy surface Σ_{κ_0}. We have to prove that, for κ close enough to κ_0, the Cauchy surface Σ_κ is contained in V.

By definition of a local K-slicing, the Cauchy surface Σ_{κ_0} is convex, and has constant K-curvature κ. For t small enough, the Cauchy surface $\Sigma^t_{\kappa_0} = \phi^t_{\Sigma_{\kappa_0}}(\Sigma_{\kappa_0})$ is well defined, contained in V, convex, has K-curvature strictly bounded from above by $-\Lambda$. Moreover, by Remark 9.12, for $t > 0$ (resp. $t < 0$) small enough, the K-curvature Cauchy surface $\Sigma^t_{\kappa_0}$ is strictly bounded from below by κ . Therefore, there exists $\eta > 0$ and $\varepsilon > 0$, such that the surfaces $\Sigma^{-\eta}_{\kappa_0}$ and $\Sigma^\eta_{\kappa_0}$ are contained in V, and such that $(\Sigma^{-\eta}_{\kappa_0}, \Sigma^\eta_{\kappa_0})$ is a pair of κ-barriers for every $\kappa \in (\kappa_0 - \varepsilon, \kappa_0 + \varepsilon)$. By Theorem 3.2, it follows that, for every $\kappa \in (\kappa_0 - \varepsilon, \kappa_0 + \varepsilon)$, the neighbourhood V contains a convex Cauchy surface with constant K-curvature κ. By the uniqueness explained in Remark 10.3, it follows that the Cauchy surface Σ_κ is contained in V for every $\kappa \in (\kappa_0 - \varepsilon, \kappa_0 + \varepsilon)$. □

Lemma 10.2 and 10.4 show that a local K-slicing is indeed a *slicing* (i.e. a *foliation*) of some open subset U of M. Moreover, Lemma 10.4 implies that the support U of a local K-slicing $(\Sigma_\kappa)_{\kappa \in I}$ has "no hole": For every $\kappa_1, \kappa_2 \in I$ such that $\kappa_1 < \kappa_2$, the intersection $I^+(\Sigma_{\kappa_1}) \cap I^-(\Sigma_{\kappa_2})$ is contained in U.

10.2. Construction of a local K-slicing

Theorem 3.4 provides us with a non-empty open interval $I_0 :=]-\Lambda - \varepsilon, -\Lambda[$ such that, for every $\kappa \in I_0$, there exists a pair of κ-barriers. Using these barriers and Theorem 3.2, we get for every $\kappa \in I_0$ a convex Cauchy surface Σ_κ with constant K-curvature equal to κ. The family $(\Sigma_\kappa)_{\kappa \in I_0}$ is

by definition a local K-slicing. We denote by U_0 the support of this local K-slicing.

LEMMA 10.5. — *The set U_0 is a neighbourhood of the future end of M, i.e. it contains the whole future of a Cauchy surface of M.*

Proof. — Thanks to Lemmas 10.2 and 10.4, it is enough to prove that the Cauchy surface Σ_κ "escapes towards the future end of M" when $\kappa \to -\Lambda$: More precisely, if K is a compact subset of M, then K is in the past of the surface Σ_κ for every κ close enough to $-\Lambda$.

In order to prove this, we consider any convex Cauchy surface $\widehat{\Sigma}$ in M such that the curvature is strictly bounded from above by $-\Lambda$ (for example, one might take $\widehat{\Sigma}$ to be a leaf of our local K-slicing). The map $\phi^t_{\widehat{\Sigma}} : \widehat{\Sigma} \to M$ is well-defined and is an embedding for every $t \geqslant 0$ (item 1 of Proposition 9.13). For $t \geqslant 0$, denote by $\kappa_{\max}(t)$ the supremum of the K-curvature of the surface $\widehat{\Sigma}^t = \phi^t_{\widehat{\Sigma}}(S)$. By item 2 of Proposition 9.13, one has $\kappa_{\max}(t) < -\Lambda$ for every $t \geqslant 0$. By Remark 10.3, the surface Σ_κ is in the future of $\widehat{\Sigma}^t$ for every $\kappa > \kappa_{\max}(t)$.

But of course, by definition of the surface $\widehat{\Sigma}^t$, it "escapes towards the future end of M" when $t \to \infty$. Therefore, the Cauchy surface Σ_κ escapes towards the future end of M when $\kappa \to -\Lambda$.   □

10.3. Getting a global K-slicing

We say that a local K-slicing $(\Sigma'_\kappa)_{\kappa \in I'}$ *extends* a local K-slicing $(\Sigma_\kappa)_{\kappa \in I}$ if $I \subset I'$ and $\Sigma'_\kappa = \Sigma_\kappa$ for $\kappa \in I$. Our goal is to prove that our local K-slicing $(\Sigma_\kappa)_{\kappa \in I_0}$ can be extended to get K-slicing $(\Sigma_\kappa)_{\kappa \in]-\infty, -\Lambda[}$ whose support is the whole manifold M.

For this purpose, we consider a local K-slicing $(\Sigma_\kappa)_{\kappa \in I}$ which extends our initial local K-slicing $(\Sigma_\kappa)_{\kappa \in I_0}$, and is *maximal* among the local K-slicings extending $(\Sigma_\kappa)_{\kappa \in I_0}$. We denote by U the support of $(\Sigma_\kappa)_{\kappa \in I}$. We have to prove that $I = (-\infty, -\Lambda)$ and $U = M$.

Assume that $I \neq (-\infty, -\Lambda)$. Then, there exists $\alpha \in]-\infty, -\Lambda[$ such that $I =]\alpha, -\Lambda[$ or $I = [\alpha, -\Lambda[$. On the one hand, Theorem 3.9 tells us that the first possibility cannot occur. Indeed, if $I =]\alpha, -\Lambda[$, then there exists a convex Cauchy surface Σ_α with constant K-curvature α such that $I^+(\Sigma_\alpha) = U$. Clearly, $(\Sigma_\kappa)_{\kappa \in [\alpha, -\Lambda)}$ is a local K-slicing which extends the local K-slicing $(\Sigma_\kappa)_{\kappa \in I}$, contradicting the maximality of $(\Sigma_\kappa)_{\kappa \in I}$. On the other hand, Proposition 3.10 implies that the second possibility cannot occur either. Indeed, if $I = [\alpha, -\Lambda[$, then Proposition 3.10 provides us

with a surface Σ such that (Σ, Σ_α) is a pair of κ barriers for every κ smaller than α and close enough to α. Using Theorem 3.2, we get a convex Cauchy surface Σ_κ with constant K-curvature κ for every $\kappa \in]\alpha', \alpha[$ for some $\alpha' < \alpha$. So, we get a local K-slicing $(\Sigma_\kappa)_{\kappa \in]\alpha', -\Lambda[}$ which extends $(\Sigma_\kappa)_{\kappa \in I}$, contradicting the maximality of $(\Sigma_\kappa)_{\kappa \in I}$.

So, we have proved that $I =]-\infty, -\Lambda[$. Using once again Theorem 3.9, we see that this implies that $U = M$. Henceforth, $(\Sigma_\kappa)_{\kappa \in I}$ is a global K-slicing of our spacetime M. This completes the proof of Theorem 2.1 in the flat case and in the locally de Sitter case.

11. K-slicings of AdS_3-spacetime

The strategy used in section 10 to get K-slicings on flat and dS_3-spacetimes does not fully apply in the AdS_3-setting, the main problem being the lack of an analog of Proposition 9.13[13]. This will force us to use completely different arguments, based on the duality between strictly convex and strictly concave Cauchy surfaces in MGHC AdS_3-spacetimes.

11.1. Duality in AdS_3-spacetimes

We will now recall the duality between points and totally geodesic hyperplanes in the anti-de Sitter space. Then, we will explain how this duality between points and totally geodesic hyperplanes induces a duality between strictly convex and strictly concave Cauchy surfaces with constant K-curvature. Most (if not all) the material contained in this section is well-known by experts; The duality between point and hyperplanes was already discussed by H. S. M. Coxeter in a 1943 (see [28]).

11.1.1. Duality between points and totally geodesic planes in $\mathbb{ADS}_3$

We consider the quadratic form $Q_{2,2} = -x_1^2 - x_2^2 + x_3^2 + x_4^2$ on $\mathbb{R}^4$. We recall that the three-dimensional anti-de Sitter space AdS_3 is the quadric $\{Q_{2,2} = -1\}$ endowed with the Lorentzian metric induced by $Q_{2,2}$. Duality phenomena are more easily described in the Klein model $\mathbb{ADS}_3$ of the three-dimensional anti-de Sitter space; We recall that $\mathbb{ADS}_3$ is the projection of AdS_3 in $\mathbb{RP}^3$, so that AdS_3 is a two-fold cover of $\mathbb{ADS}_3$.

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[13] There can be no analog of Proposition 9.13 in the AdS_3-setting, simply because MGHC AdS_3-spacetimes are neither future complete, nor past complete.

Given a set $E \subset \mathbb{RP}^3$, let $\widehat{E}$ denote the cone in $\mathbb{R}^4$ whose projection in $\mathbb{RP}^3$ is equal to E. Then, let $\widehat{E}^\perp$ denote the $Q_{2,2}$-orthogonal of $\widehat{E}$ in $\mathbb{R}^4$, and $E^\perp$ denote the projection of $\widehat{E}^\perp$ in $\mathbb{RP}^3$. Finally, let E^* denote the intersection between $E^\perp$ and $\mathbb{AdS}_3$. Equivalently, E^* is the projection in $\mathbb{RP}^3$ of $\widehat{E}^\perp \cap \mathrm{AdS}_3$.

If x is a point in $\mathbb{AdS}_3$, one easily checks that x^* is a spacelike totally geodesic plane in $\mathbb{AdS}_3$. Conversely, if P is a spacelike totally geodesic plane in $\mathbb{AdS}_3$, then P^* is a point in $\mathbb{AdS}_3$. And clearly, for every point x and every spacelike totally geodesic plane P in $\mathbb{AdS}_3$, we have:

$$(11.1) \qquad\qquad (x^*)^* = x \qquad (P^*)^* = P.$$

Remark 11.1. — If P is a spacelike totally geodesic plane in $\mathbb{AdS}_3$, it is easy to verify that every geodesic of $\mathbb{AdS}_3$ which is orthogonal to P passes through the point P^*. Moreover, the length of a geodesic segment, orthogonal to P, and going from P to P^* is exactly $\pi/2$. In other words, P^* is obtained by pushing P along the orthogonal geodesics for a time $\pi/2$ (or $-\pi/2$).

11.1.2. Duality between strictly convex and concave surfaces in $\mathbb{AdS}_3$

Let S be a (smooth) spacelike surface in $\mathbb{AdS}_3$. For every $x \in S$, we denote by $P_{S,x}$ the totally geodesic plane in $\mathbb{AdS}_3$ which is tangent to S at x. Then we define the *dual set* $S^* \subset \mathbb{AdS}_3$ of S as follows:

$$S^* = \big\{ (P_{S,x})^* \mid x \in S \big\}.$$

Remark 11.2. — Let y be a point in $\mathbb{AdS}_3$. Using (11.1), one can easily show that the point y belongs to the set S^* if and only if the totally geodesic plane y^* is tangent to the surface S at some point.

If S is an arbitrary spacelike surface, the set S^* is not a surface in general; For example, if S is a totally geodesic plane, then S^* is a single point. But everything goes well whenever S is strictly convex (or concave) spacelike surface:

PROPOSITION 11.3. — *If S is a strictly convex (resp. strictly concave) spacelike surface, then the set S^* is strictly concave (resp. strictly convex) spacelike surface.*

In order to prove this proposition, we consider the set $\mathcal{P}_{\mathbb{AdS}_3}$ of all spacelike totally geodesic planes in $\mathbb{AdS}_3$; This set has a natural structure of

three-dimensional manifold. For every spacelike surface S, we consider the map

$$P_S \colon S \longrightarrow \mathcal{P}_{\mathbb{ADS}_3}$$
$$x \longmapsto P_{S,x}$$

The following fact is an immediate consequence of the definition of the second fundamental:

FACT 11.4. — *Given a spacelike surface S in $\mathbb{ADS}_3$ and a point $x \in S$, the second fundamental form of S at x is non-degenerate if and only if the derivative at x of the map P_S has maximal rank.*

Proof of Proposition 11.3. — We assume S is a strictly convex spacelike surface in $\mathbb{ADS}_3$.

First step: S^* is an immersed surface. Since S is strictly convex, for every $x \in S$, the second fundamental form of S at x is non-degenerate. Using Fact 11.4, it follows that, for every $x \in S$, the derivative at x of the map P_S has maximal rank. Therefore, the map $P_S \colon x \mapsto P_{S,x}$ is an immersion of S in $\mathcal{P}_{\mathbb{ADS}_3}$. Moreover, the map $P \mapsto P^*$ is obviously a local diffeomorphism between $\mathcal{P}_{\mathbb{ADS}_3}$ and $\mathbb{ADS}_3$. Therefore the map $x \mapsto (P_{S,x})^*$ defines an immersion of S in $\mathbb{ADS}_3$. The range of this map is the set S^*; Therefore S^* is an immersed surface.

Second step: S^* is an injectively immersed spacelike concave surface. Let $y \in S^*$. By definition of S^*, there exists a point $x \in S$ such that $y = (P_{S,x})^*$. We will prove that the totally geodesic plane x^* meets S^* at y, and that S^* is contained in the past of x^* in a neighbourhood of y.

Since the point x belongs to the totally geodesic plane $P_{S,x}$, the point $y = (P_{S,x})^*$ belongs to the totally geodesic plane x^*. In other words, the totally geodesic plane x^* meets the surface S^* at y. Now let us consider a sequence $(y_k)_{k \in \mathbb{N}}$ of points of S^* converging to y when $k \to \infty$. By Remark 11.2, for every $k \geqslant 0$, the totally geodesic plane y_k^* is tangent to the surface S at some point x_k. Since S is strictly convex, this implies that S is in the future of the plane y_k^* in some neighbourhood U_k of the point x_k (and the size of the neighbourhood U_k does not depend of k provided that the x_k's stay in a compact subset of S). Moreover, if k is large, the totally geodesic plane y_k^* is close to the totally geodesic plane $y^* = ((P_{S,x})^*)^* = P_{S,x}$. Since S is *strictly* convex, this implies that x_k is close to x when k is large. It follows that, for k large enough, the point x is in the neighbourhood U_k. In particular, for k large enough, the point x

is in the future of the totally geodesic plane y_k^*. It follows that, for k large enough, the point $y_k = (y_k^*)^*$ is in the past of the totally geodesic plane x^*.

So, we have proved that the totally geodesic plane x^* meets the immersed surface S^* at y, and that there exists a neighbourhood U of the point y in S^* such that S^* is contained in the past of x^* in U. It follows that x^* is the unique totally geodesic plane which is tangent to S^* at y, and that S^* is "locally contained" in the past of this totally geodesic plane. Since y is an arbitrary point in S^*, this shows that S^* is injectively immersed, spacelike and concave (see §4.3).

Third step: S^* is strictly concave. Since we already know that S^* is concave, we only need to prove that the second fundamental form of S^* is non-degenerate at each point of S^*. According to Lemma 11.4, it is equivalent to prove that the map $P_{S^*} : y \mapsto P_{S^*,y}$ is an immersion of S^* in $\mathcal{P}_{\mathbb{ADS}_3}$.

We have proved during the second step that, for every point $y \in S^*$, if x is the unique point of the surface S such that $y = (P_{S,x})^*$, then x^* is the unique totally geodesic plane tangent to S^* at y, that is $P_{S^*,y} = x^*$. Moreover, we have proved in the first and second step above that the map $x \mapsto (P_{S,x})^*$ is an injective immersion of the surface S in $\mathbb{ADS}_3$, and that the range of this immersion is the surface S^*. It follows that the map $(P_{S,x})^* \mapsto x$ is an immersion of S^* in $\mathbb{ADS}_3$. Since the map $x \mapsto x^*$ is obviously a local diffeomorphism from $\mathbb{ADS}_3$ to $\mathcal{P}_{\mathbb{ADS}_3}$, this shows that the map

$$P_{S^*} : \quad S^* \quad \longrightarrow \mathcal{P}_{\mathbb{ADS}_3}$$
$$y = (P_{S,x})^* \longmapsto x^* = P_{S^*,y}$$

is an immersion. Therefore, the surface S^* is strictly concave. $\square$

In the proof above, we have seen that, for every point $y \in S^*$, if x is the unique point of the surface S such that $y = (P_{S,x})^*$, then $P_{S^*,y} = x^*$, and therefore $(P_{S^*,y})^* = (x^*)^* = x$. This shows that:

COROLLARY 11.5. — *If S is a strictly convex (resp. strictly concave) spacelike surface, then $(S^*)^* = S$ and the map* $\begin{vmatrix} S^* & \to & (S^*)^* \\ y & \mapsto & (P_{S,y})^* \end{vmatrix}$ *is the inverse of the map* $\begin{vmatrix} S & \to & S^* \\ x & \mapsto & (P_{S,x})^* \end{vmatrix}$.

PROPOSITION 11.6. — *Let S be a strictly convex or strictly concave spacelike surface. Let x_0 be a point in S, and $y_0 = (P_{S,x_0})^*$ be the corresponding point in S^*. If λ, μ are the principal curvatures of S at x_0, then the principal curvatures of S^* at y_0 are $-\lambda^{-1}, -\mu^{-1}$.*

Proof. — Here, it is easier to work in $\mathbb{R}^4$ instead of $\mathbb{RP}^3$. Let $\widetilde{S}$ be one of the two lifts of S in $\mathrm{AdS}_3 \subset \mathbb{R}^4$. For every $x \in \widetilde{S}$, let $n_{\widetilde{S},x} \in T_x \mathrm{AdS}_3$ be the future-directed unit normal vector of S at x. Since $\mathrm{AdS}_3 \subset \mathbb{R}^4$, we can see $n_{\widetilde{S},x}$ as an element of $T_x \mathbb{R}^4$. The key point is the following observation:

CLAIM 11.7. — *For every $x \in \widetilde{S}$, the canonical isomorphism between $T_x\mathbb{R}^4$ and $\mathbb{R}^4$ maps the vector $n_{\widetilde{S},x} \in T_x\mathbb{R}^4$ to one of the two lifts in $\mathbb{R}^4$ of the point $(P_{S,x})^*$.*

Let $y \in \mathbb{R}^4$ be the image of $n_{\widetilde{S},x}$ under the canonical isomorphism between $T_x\mathbb{R}^4$ and $\mathbb{R}^4$. For every linear subspace V_x of $T_x\mathbb{R}^4$, denote by $\overline{V_x} \subset \mathbb{R}^4$ the image of V_x under this isomorphism. Since $P_{\widetilde{S},x}$ is a totally geodesic plane of AdS_3, there exists a linear hyperplane H of $\mathbb{R}^4$ such that $P_{\widetilde{S},x}$ is a connected component of $H \cap \mathrm{AdS}_3$. Clearly, $H = \mathbb{R}.x \oplus \overline{T_x P_{\widetilde{S},x}}$. Since the vector $n_{\widetilde{S},x}$ belongs to $T_x\mathrm{AdS}_3$, the point y belongs to $\overline{T_x\mathrm{AdS}_3} = x^\perp$. Since the vector $n_{\widetilde{S},x}$ is orthogonal to $T_x\widetilde{S} = T_x P_{\widetilde{S},x}$, the point y belongs to $(\overline{T_x P_{\widetilde{S},x}})^\perp$. Hence the point y belongs to $(\mathbb{R}.x \oplus \overline{T_x P_{\widetilde{S},x}})^\perp = H^\perp = (P_{\widetilde{S},x})^\perp$. Moreover the $Q_{2,2}$-norm of the vector $n_{\widetilde{S},x}$ is -1. Hence the point y belongs to $\mathrm{AdS}_3 = \{z \in \mathbb{R}^4 \mid Q_{2,2}(z) = -1\}$. So we have proved y belongs to $(P_{\widetilde{S},x})^\perp \cap \mathrm{AdS}_3$. The projection of this intersection in $\mathbb{RP}^3$ is by definition the point $(P_{S,x})^*$. This completes the proof of the claim.

The above claim shows that, if we identify $T_x\mathbb{R}^4$ with $\mathbb{R}^4$ for every $x \in \mathbb{R}^4$, then the Weingarten map of S at x_0 is identified with the derivative at x_0 of the map $x \mapsto (P_{S,x})^*$. Of course the same is true if we replace S by S^*: The Weingarten map of S^* at y_0 is identified with the derivative at y_0 of the map $y \mapsto (P_{S^*,y})^*$. Together with Corollary 11.5, this shows that the Weingarten map of S at x_0 is identified with the opposite of the inverse of the Weingarten map of S^* at y_0. The proposition follows. □

COROLLARY 11.8. — *If S is a spacelike surface with constant K-curvature $\kappa \neq 0$, then S^* is a spacelike surface with constant K-curvature κ^{-1}.*

Proof. — Every surface with non-zero constant K-curvature is either strictly convex or strictly concave. Therefore the corollary is an immediate consequence of Proposition 11.6. □

It is interesting to observe that the dual S^* of a spacelike surface S can also be defined by pushing along orthogonal geodesics:

PROPOSITION 11.9. — *For every spacelike surface S in $\mathbb{ADS}_3$, the dual S^* of S is obtained by pushing S along orthogonal geodesic for a time $\pi/2$ (or a time $-\pi/2$).*

Proof. — Let S be a spacelike surface in $\mathbb{ADS}_3$. For every $x \in S$, we denote by $n_{S,x}$ the future-directed unit normal vector of S at x. For every $t \in \mathbb{R}$, let

$$S^t := \{\exp_x (t \cdot n_{S,x}) \mid x \in S\}$$

the set of points at distance t from S along an orthogonal geodesic (S^t is not necessarily a surface when $|t|$ is large). We shall prove that $S^{\pi/2} = S^*$ (the same arguments apply to show that $S^{-\pi/2} = S^*$).

Consider a point y in S^*. There exists a point $x \in S$ such that $y = P^*_{S,x}$. Since the surface S and the totally geodesic plane $P_{S,x}$ are tangent at x, the vector $n_{S,x}$ is orthogonal to $P_{S,x}$. Using Remark 11.1, it follows that the point $y = P^*_{S,x}$ coincides with the point $\exp_x \left(\frac{\pi}{2} \cdot n_{S,x}\right)$. In particular, $y \in S^{\pi/2}$.

Conversely, consider a point $y \in S^{\pi/2}$. There exists a point $x \in S$ such that $y = \exp_x \left(\frac{\pi}{2} \cdot n_{S,x}\right)$. The same argument as above shows that $(P_{S,x})^* = \exp_x(\frac{\pi}{2} \cdot n_{S,x}) = y$. In particular, $y \in S^*$.   □

11.1.3. Duality for convex and concave surfaces in MGHC AdS$_3$-spacetimes

Now we consider a MGHC spacetime M of constant curvature -1. According to Theorem 5.8, there exists a representation $\rho\colon \Gamma \to \mathrm{PSL}(2,\mathbb{R}) \times \mathrm{PSL}(2,\mathbb{R})$ (where Γ is the fundamental group of the Cauchy surfaces of M) and an open set $E(\Lambda_\rho) \subset \mathbb{ADS}_3$ such that M is isometric to the quotient $M(\rho) := \rho(\Gamma) \setminus E(\Lambda_\rho)$.

Recall that $E(\Lambda_\rho)$ is the invisible domain of a topological circle $\Lambda_\rho \subset \partial\mathbb{ADS}_3$. The intersection of the convex hull of Λ_ρ with $\mathbb{ADS}_3$ is contained in $E(\Lambda_\rho)$; We denote it by $\mathrm{Conv}(\Lambda_\rho)$. Of course, $\mathrm{Conv}(\Lambda_\rho)$ is Γ-invariant, and the projection of $\mathrm{Conv}(\Lambda_\rho)$ in $M(\rho) := \rho(\Gamma) \setminus E(\Lambda_\rho)$ is, by definition, the convex core $\mathcal{C}(M)$ of M. The complement of $\mathcal{C}(M)$ in $M \simeq \rho(\Gamma)\setminus E(\Lambda_\rho)$ admits two connected components, one, denoted by B_- and called the *past end region,* contained in the past of $\mathcal{C}(M)$, and the other, denoted by B_+ and called the *future end region,* contained in the future of $\mathcal{C}(M)$. Since $\mathcal{C}(M) = \{x \in M \mid \check{\tau}(x) \leqslant \pi/2,\ \tau(x) \leqslant \pi/2\}$, we have:

$$B_- = \left\{x \in M \mid \check{\tau}(x) > \pi/2\right\},\ \ B_+ = \left\{x \in M \mid \tau(x) > \pi/2\right\}.$$

Let Σ be a strictly convex Cauchy surface in $M \simeq \rho(\Gamma) \smallsetminus E(\Lambda_\rho)$. Consider the lift S of Σ in $E(\Lambda_\rho) \subset \mathbb{ADS}_3$. Then S is a strictly convex spacelike surface in $\mathbb{ADS}_3$. So we can consider the dual S^* of S as defined in the previous paragraph. By Proposition 11.3, S^* is a strictly concave spacelike surface in $\mathbb{ADS}_3$. Since the surface S is $\rho(\Gamma)$-invariant, so is the surface S^* (the duality is clearly a $\mathrm{Isom}(Q_{2,2})$-equivariant operation). Moreover, since S is strictly convex, Proposition 9.10 shows that $\phi_S^t \colon S \to \mathbb{ADS}_3$ is an immersion for every $t \in [0, \pi/2)$, and step 1 of the proof of Proposition 11.3 shows that $\phi_S^{\pi/2} \colon S \to \mathbb{ADS}_3$ is an immersion. Hence, Lemma 9.15 shows that the surface $S^* = S^{\pi/2}$ is contained in $E(\Lambda_\rho)$, and projects to a compact spacelike strictly concave surface $\Sigma^* = \Sigma^{\pi/2}$ in $M \simeq \rho(\Gamma) \smallsetminus E(\Lambda_\rho)$. Note that Σ^* is a Cauchy surface in M since this is the case for every compact spacelike surface. We say that Σ^* is the *dual* of the Cauchy surface Σ. Of course, one defines similarly the dual Σ^* of a strictly concave Cauchy surface in M; In this case, Σ^* is strictly convex Cauchy surface.

Corollary 11.5 implies that $(\Sigma^*)^* = \Sigma$ for every strictly convex (resp. concave) Cauchy surface Σ in M. By Corollary 11.8, if Σ has constant K-curvature κ, then Σ^* has constant K-curvature κ^{-1}.

LEMMA 11.10. — *Let Σ be a strictly convex Cauchy surface in $M \simeq \rho(\Gamma) \smallsetminus E(\Lambda_\rho)$. Then, Σ is contained in B_-. Moreover, the future push-forward Σ^t along the geodesics normal to Σ is a well-defined Cauchy surface in M for every t in $[0, \pi/2]$, and the dual surface Σ^* coincide with $\Sigma^{\pi/2}$.*

Proof. — The first statement follows from the fact that since Σ is strictly convex, its causal future is convex, and the fact that $\mathrm{Conv}(\Lambda_\rho)$ is contained in any convex subset of M. The other statement follows from Lemma 9.15. □

11.2. Proof of Theorem 2.1 in the AdS$_3$-case

Let (M, g) be a non-elementary MGHC spacetime with constant curvature $\Lambda = -1$. Recall that we denote by $B_\pm$ the past end regions of M. In order to prove Theorem 2.1, we have to construct a K-slicing on B_+, and a K-slicing on B_-. The starting point of our construction will be the Cauchy surfaces Σ^- and Σ^+ provided by Proposition 9.14. Recall that Σ^- is a strictly convex Cauchy surface and Σ^+ is a strictly concave Cauchy surface, both with constant K-curvature -1. According to Lemma 11.10, Σ^- is contained in B_-, and similarly, Σ^+ is contained in B_+. It follows from their definition (surfaces at distance $\pm \pi/4$ of a maximal Cauchy surface) that Σ^- and Σ^+ are dual one to the other.

11.2.1. Construction of the K-slicings on $I^-(\Sigma^-)$ and $I^+(\Sigma^+)$

The Cauchy surface Σ^- is convex and has constant K-curvature -1. Using Proposition 11.3, we get a convex Cauchy surface Σ^{--} in the past of Σ^- and a real number $\varepsilon > 0$, such that (Σ^{--}, Σ^-) is a pair of κ-barriers for every $\kappa \in (-1-\varepsilon, -1)$. Together with Theorem 3.2, this shows the existence, for every $\kappa \in (-1-\varepsilon, -1)$, of a convex Cauchy surface Σ_κ^- with constant K-curvature κ. We set $\Sigma_1^- := \Sigma^-$. Then $(\Sigma_\kappa^-)_{\kappa \in (-1-\varepsilon, -1]}$ is a local K-slicing. Now, we consider a local K-slicing $(\Sigma_\kappa)_{\kappa \in I}$ which extends $(\Sigma_\kappa)_{\kappa \in (-1-\varepsilon, -1]}$, and is maximal for this property. The same arguments as in §10.3 show that $(-\infty, -1] \subset I$ and that the support of $(\Sigma_\kappa)_{\kappa \in I}$ contains the past $I^-(\Sigma^-)$ of the Cauchy surface Σ^-. So, we have proved the following:

PROPOSITION 11.11. — *There exists a local K-slicing $(\Sigma_\kappa^-)_{\kappa \in (-\infty, -1]}$ whose support is the past $I^-(\Sigma^-)$ of the Cauchy surface Σ^-.* □

Of course, similar arguments show that:

PROPOSITION 11.12. — *There exists a local K-slicing $(\Sigma_\kappa^+)_{\kappa \in (-\infty, -1]}$ whose support is the future $I^+(\Sigma^+)$ of the Cauchy surface Σ^+.* □

11.2.2. Construction of the K-slicings on $B_- \smallsetminus I^-(\Sigma^-)$ and $B_+ \smallsetminus I^+(\Sigma^+)$

Proposition 11.11 provides us with a local K-slicing $(\Sigma_\kappa^-)_{\kappa \in (-\infty, -1]}$ whose support is the past of the Cauchy surface Σ^-. To complete the proof of Theorem 2.1 in the AdS$_3$-case, we need to extend this local K-slicing, and get a K-slicing $(\Sigma_\kappa^-)_{\kappa \in (-\infty, 0)}$ whose support is the whole B_-. For this purpose, we use the duality between convex and concave Cauchy surfaces: For every $\kappa \in (-1, 0)$, we set

$$\Sigma_\kappa^- := \left(\Sigma_{1/\kappa}^+\right)^*$$

(this definition is not chasing its own tail since the surface $\Sigma_{1/\kappa}^+$ has already been defined for $\kappa \in (0, 1)$; See Proposition 11.12).

CLAIM 11.13. — *The families of Cauchy surfaces $(\Sigma_\kappa^-)_{\kappa \in (-\infty, 0)}$ is a local K-slicing whose support is included in the end region B_-.*

Proof. — For $\kappa \leqslant -1$, we already know that the Cauchy surface Σ_κ^- is a convex and has constant K-curvature κ. Moreover, for $\kappa \in (-1, 0)$, we know that the Cauchy surface $\Sigma_{1/\kappa}^+$ is strictly concave and has constant K-curvature $\frac{1}{\kappa}$. Together with Proposition 11.6 this implies that, for $\kappa \in$

$(-1, 0)$, the Cauchy surface $\Sigma_\kappa^- := (\Sigma_{1/\kappa}^+)^*$ is convex, has constant K-curvature κ, and is contained in B_-. This shows that $(\Sigma_\kappa^-)_{\kappa \in (-\infty, 0)}$ is a local K-slicing. According to Lemma 11.10, the support of this K-slicing must be contained in B_-. $\square$

CLAIM 11.14. — *For every point $x \in B_-$, there exists $\kappa \in (-\infty, 0)$ such that x is in the past of the Cauchy surface Σ_κ^-.*

Proof. — We denote by $\check{\tau} \colon M \to (0, \pi)$ the reverse cosmological time of M. Recall that

(11.2) $$B_- = \{x \in M \mid \check{\tau}(x) > \pi/2\}.$$

The support of the local K-slicing $(\Sigma_{1/\kappa}^+)_{\kappa \in [-1,0)} = (\Sigma_\kappa^+)_{\kappa \in (-\infty, -1]}$ is a neighbourhood of the future end of M (recall that this means that it contains the future of a Cauchy surface; Do not confuse with the end region). Since $\check{\tau}$ is regular, this implies that

(11.3) $$\lim_{\kappa \to 0} \sup_{x \in \Sigma_{1/\kappa}^+} \check{\tau}(x) = 0.$$

Now, recall that, for every $\kappa \in [-1, 0)$, the surface $\Sigma_\kappa^- = (\Sigma_{1/\kappa}^+)^*$ is obtained by pushing the surface $\Sigma_{1/\kappa}^+$ under the time $(-\pi/2)$ along orthogonal geodesics. In particular, every point of Σ_κ^- is the past end of a geodesic segment orthogonal to $\Sigma_{1/\kappa}^+$ of length $-\pi/2$. This implies that, for every $\kappa \in [-1, 0)$, the length of a timelike curve joining the surface Σ_κ^- to the surface $\Sigma_{1/\kappa}^+$ is at most $\pi/2$ (recall that, in a globally hyperbolic spacetime, the supremum of the lengths of the timelike curves joining a surface Σ to a point p is realized by a timelike geodesic orthogonal to Σ). Together with (11.3), this implies that

(11.4) $$\lim_{\kappa \to 0} \sup_{x \in \Sigma_\kappa^-} \check{\tau}(x) \leqslant \frac{\pi}{2}$$

(actually, this inequality turns out to be an equality since, for every κ, the surface Σ_κ^- is convex, thus contained in $B_- = \{x \in M \mid \check{\tau}(x) > \pi/2\}$). Inequality (11.4) and equality (11.2) show that for every point $x \in B_-$, there exists $\kappa \in (-1, 0)$ such that x is in the past of the Cauchy surface Σ_κ^-. $\square$

Claim 11.14 together with Lemma 10.4 show that the support of the local K-slicing $(\Sigma_\kappa^-)_{\kappa \in (-\infty, 0)}$ is the whole past end region B_-. Of course, similar arguments show that the support of the K-slicing $(\Sigma_\kappa^+)_{\kappa \in (-\infty, 0)}$ is the whole future end region B_+. This completes the proof of Theorem 2.1 in the locally anti-de Sitter case.

12. K-slicings of hyperbolic ends

In this section, we explain how Theorem 2.2 can be seen as a corollary of Theorem 2.1 (more precisely, of the existence of K-slicings in non-elementary MGHC dS$_3$-spacetimes). We will not go into too many details for several reasons:

– Theorem 2.2 is not a new result; It was proved by F. Labourie some fifteen years ago (using techniques rather different from ours),
– the main tool which allows to translate a result concerning non-elementary MGHC dS$_3$-spacetimes into a result concerning hyperbolic ends is the so-called dS$_3 \leftrightarrow \mathbb{H}^3$ duality. This duality is completely similar to the AdS$_3 \leftrightarrow$ AdS$_3$ duality described in the previous section,
– this duality was already observed and used by Mess in [51, § 6] to prove that locally de Sitter MGHC spacetimes admit K-slicings using Labourie's result. Here we adopt the reverse point of view: We explain how to deduce Labourie's result from the existence of K-slicings on locally de Sitter MGHC spacetimes.

12.1. dS$_3 \leftrightarrow \mathbb{H}^3$ duality

12.1.1. Duality between points and totally geodesic planes

Consider the quadratic form $Q_{1,3} = -x_1^2 + x_2^2 + x_3^2 + x_4^2$ on $\mathbb{R}^4$. Recall that dS$_3$ is the hyperboloid $\{Q_{1,3} = 1\}$ endowed with the Lorentzian metric induced by $Q_{1,3}$. Also recall the each of the two sheets of the hyperboloid $\{Q_{1,3} = -1\}$ is a copy of the 3-dimensional hyperbolic space; We denote them by H_-^3 and H_+^3. For every set $A \subset \mathbb{R}^4$, we denote by $A^\perp$ the $Q_{2,2}$-orthogonal of A in $\mathbb{R}^4$. We choose an orientation on $\mathbb{R}^4$.

Let x be a point in dS$_3 \subset \mathbb{R}^4$. Then $x^\perp$ is an hyperplane in $\mathbb{R}^4$, and the restriction of $Q_{1,3}$ to $x^\perp$ has signature $(+,+,+)$. It follows that $x^\perp \cap H_3^+$ is a totally geodesic plane in H_3^+. We denote this totally geodesic plane by x^*. The orientation of $\mathbb{R}^4$ and the choice of x (rather than $-x$) defines an orientation on $x^\perp$, and subsequently on x^*. Conversely, let P be an oriented totally geodesic plane in H_+^3. Then $P^\perp$ is a oriented line in $\mathbb{R}^4$, and the restriction of $Q_{1,3}$ to $P^\perp$ is negative definite. It follows that $P^\perp$ intersects dS$_3$ at two (antipodal) points, and the orientation on $P^\perp$ allows to distinguish choose one of these two points that we denote by P^*. Clearly, for every point $x \in$ dS$_3$ and every totally geodesic plane $P \subset H_+^3$, one has

$(x^*)^* = x$ and $(P^*)^* = P$. This defines a duality between the points of dS_3 and the totally geodesic planes in $H_+^3 \simeq \mathbb{H}^3$.

Now, let x be a point in H_+^3. One can check that $x^\perp \cap dS_3$ is a spacelike totally geodesic space in dS_3; We denote this totally geodesic plane by x^*. Conversely, let P be a spacelike totally geodesic plane in dS_3. The intersection $P^\perp \cap H_+^3$ is reduced to a point, that we denote by P^*. Clearly, for every point $x \in H_+^3$ and every spacelike totally geodesic plane $P \subset dS_3$, one has $(x^*)^* = x$ and $(P^*)^* = P$. This defines a duality between the points of $H_+^3 \simeq \mathbb{H}^3$ and the spacelike totally geodesic planes in dS_3.

12.1.2. Duality between convex surfaces in dS_3 and convex surfaces in H_+^3

Let S be a strictly convex spacelike surface in dS_3. Using exactly the same construction as in §11.1.2 (but now using the duality between points and planes described in §12.1.1 instead of those described in §11.1.1), one can to define a strictly convex $S^* \subset H_3^+$ such that the support planes of S (resp. S^*) are the duals of the points of S^* (resp. S). Conversely, given a strictly convex surface in H_+^3, one can define a strictly convex spacelike surface in dS_3 such that the support planes of S (resp. S^* are the duals of the points of S^* (resp. S). Then, for every strictly convex (resp. strictly convex spacelike) surface S in H_+^3 (resp. dS_3), one has $(S^*)* = S$. Moreover, the same arguments as in the proof of Proposition 11.6 allow to prove:

PROPOSITION 12.1. — *Let S be a strictly convex (resp. strictly convex spacelike) surface in H_+^3 (resp. dS_3). If S has constant K-curvature κ, then S^* has constant curvature $-\kappa^{-1}$.*

12.1.3. Duality between MGHC dS_3-spacetimes and hyperbolic ends

DEFINITION 12.2. — *A (geometrically finite) hyperbolic end is a hyperbolic 3-manifold (M, g) such that:*

- *M is homeomorphic to $\Sigma \times (0, +\infty)$ where Σ is a closed surface,*
- *if $(\overline{M}, \bar{g})$ is the metric completion of M, and $\bar{\Sigma} = \overline{M} \smallsetminus M$, then $(\bar{\Sigma}, \bar{g})$ is a hyperbolic surface homeomorphic to Σ,*
- *$(\bar{\Sigma}, \bar{g})$ is concave, i.e. there is no geodesic in M connecting two elements of $\bar{\Sigma}$,*
- *$(\bar{\Sigma}, \bar{g})$ is a pleated surface.*

Let (M, g) be a hyperbolic end. It admits a conformal boundary homeomorphic to Σ. This conformal boundary is naturally endowed with a Möbius structure. As explained in §5.2.2, every compact surface with negative Euler characteristic endowed with a Möbius structure defines a non-elementary MGHC dS_3-spacetime. We shall denote by (M^*, g^*) the non-elementary MGHC dS_3-spacetime associated to $(\bar{\Sigma}, \bar{g})$. We say that (M^*, g^*) is the dual of the hyperbolic end (M, g).

Conversely, for every non-elementary MGHC dS_3-spacetime (M, g), one can define a geometrically finite hyperbolic end (M^*, g^*). We say that (M^*, g^*) is the dual of (M, g).

12.1.4. Duality between convex surfaces in MGHC dS_3-spacetimes and convex surfaces in hyperbolic ends

Consider a non-elementary MGHC dS_3-spacetime (M, g) and the hyperbolic (M^*, g^*) which is dual to (M, g).

Let Σ be a strictly convex Cauchy surface in M. Then Σ lifts to a strictly convex immersed spacelike surface $S \subset dS_3$. The dual surface S^* is a strictly convex immersed surface in H_+^3. It can be checked that S^* projects to a compact (strictly convex) surface Σ^* in M^*.

Conversely, if Σ is a strictly convex surface in M^*, one can lift Σ to a strictly convex surface S in H_+^3, consider the dual S^* of S , and project S^* to a strictly convex Cauchy surface Σ^* in $((M^*)^*, (g^*)^*) = (M, g)$.

For every strictly convex Cauchy (resp. compact) surface Σ in (M, g) (resp. in (M^*, g^*)), one has $(\Sigma^*)^* = \Sigma$. Moreover, if Σ has constant K-curvature a, then Σ^* has constant K-curvature $-1/a$.

12.2. Sketch of proof of Theorem 2.2

Consider a hyperbolic end (M, g). Consider the future complete non-elementary MGHC dS_3-spacetimes (M^*, g^*) dual to the hyperbolic end (M, g). According to Theorem 2.1, (M^*, g^*) admits a K-time $\kappa^* \colon M^* \to (-\infty, -1)$. For every $a \in (-\infty, 0)$, the level set $\Sigma_a^* := (\kappa^*)^{-1}(a)$ is a strictly convex Cauchy surface in M^* with constant K-curvature equal to a. Then $\Sigma_a := (\Sigma_a^*)^*$ is a strictly convex compact surface in M with constant K-curvature $-1/a$. Then $\{\Sigma_a\}_{a \in]-\infty, -1[}$ is a family of compact surfaces with constant K-curvature in M. Using the fact that $\{\Sigma_a^*\}_{a \in]-\infty, -1[}$ is a trivial foliation of M^*, it is quite easy to prove that $\{\Sigma_a\}_{a \in]-\infty, -1[}$ is a trivial foliation of M. Theorem 2.2 follows.

13. Surfaces with prescribed K-curvature

In this section, we prove Theorem 2.4 and Corollary 2.5. This theorem will follow from :

– the existence K-times on MGHC spacetimes with constant curvature (Theorem 2.1) which will provide us with a pair of barriers,
– a generalization of Theorem 3.2 (Theorem 13.2 below), which asserts that surfaces with prescribed K-curvature exist as soon as barriers exist.

DEFINITION 13.1. — *Let (M, g) be a 3-dimensional globally hyperbolic spacetime with compact Cauchy surfaces. Let $f \colon M \to (-\infty, 0)$ be a smooth function. A* pair of f-barriers *is a pair of disjoint strictly convex Cauchy surfaces Σ^-, Σ^+ in M, such that:*

a. *Σ^- is in the past of Σ^+,*
b. *$\kappa^{\Sigma^-}(x) \leqslant f(x)$ for every $x \in \Sigma^-$,*
c. *$\kappa^{\Sigma^+}(x) \geqslant f(x)$ for every $x \in \Sigma^+$.*

THEOREM 13.2 (Gerhardt, [34]). — *Let (M, g) be a 3-dimensional spatially compact globally hyperbolic spacetime, and $f \colon M \to]-\infty, 0[$ be a smooth function. Assume that M admits a pair of f-barriers (Σ^-, Σ^+). Then M admits a strictly convex Cauchy surface Σ such that $\kappa^\Sigma(x) = f(x)$ for all $x \in \Sigma$.*

Remark 13.3. — We recall that the sign conventions of Gerhardt are different from ours (see Remark 3.3). This is the reason why the function f is required to be strictly positive in [34], whereas it is required to be strictly negative in the statement above.

Proof of Theorem 2.4. — Consider a 3-dimensional non-elementary MGHC spacetime (M, g) with constant curvature Λ. Recall that in the $\Lambda \geqslant 0$ case we assume that (M, g) is future complete (see Remark 5.3). Let $f \colon M_0 \to \mathbb{R}$ be a smooth function such that the range of f is contained in a compact interval $[a, b] \subset]-\infty, \min(0, -\Lambda)[$. Define M_0 as follow: $M_0 = M$ is $\Lambda \geqslant 0$ (flat and locally de Sitter case), and M_0 is the past of the convex core of M if $\Lambda < 0$ (anti de Sitter case). By Theorem 2.1, there exists a K-time $\kappa \colon M_0 \to]-\infty, \min(-\Lambda, 0)[$. Consider the level sets of κ:

$$\Sigma^- := \kappa^{-1}(a) \quad \text{and} \quad \Sigma^+ := \kappa^{-1}(b).$$

By definition of a K-time, Σ^-, Σ^+ are two strictly convex Cauchy surfaces with constant K-curvature respectively equal to a and b, and Σ^- is in the past of Σ^+. In particular, (Σ^-, Σ^+) is a pair of f-barriers. Therefore

Theorem 13.2 applies in our situation and provides us with a strictly convex Cauchy surface Σ such that $\kappa^\Sigma(x) = f(x)$ for every x in Σ. $\square$

Proof of Corollary 2.5. — Write M as a topological product $\Sigma_0 \times \mathbb{R}$, consider the function $f \colon M \to \mathbb{R}$ defined by $f(x,t) = f_0(x)$ where $x \in \Sigma_0$ and $t \in \mathbb{R}$, and apply Theorem 2.4 to the function f. $\square$

14. The Minkowski problem

The purpose of this section is to prove Theorem 2.6. As for the proof of Theorem 2.4, we will use Theorem 2.1 to get a pair of barriers, and then, a theorem of Gerhardt which asserts that surfaces with prescribed K-curvature exist as soon as barriers exist. As a preliminary step, we will need to translate Theorem 2.6 into a statement concerning a function defined on the unit tangent bundle of a non-elementary MGHC flat spacetime.

We begin by stating Gerhardt's result:

THEOREM 14.1 (Gerhardt, see [35]; See also Remarks 3.3 and 13.3). — *Let (M, g) be a spatially compact globally hyperbolic spacetime. Let Ω be an open subset of M bounded by two disjoint strictly convex Cauchy surfaces Σ^- and Σ^+, where Σ^- is assumed to be in the past of Σ^+. Denote by $T^{-1}\Omega$ the bundle of future-oriented unit timelike tangent vectors over Ω:*

$$T^{-1}\Omega = \{(x, \nu) \in TM \mid x \in \Omega,\, \nu \text{ is future-oriented and } g(\nu, \nu) = -1\}.$$

Consider a smooth function $\Phi \colon T^{-1}\Omega \to \mathbb{R}$ with the two following properties:

(i) *there is a negative constant c_1 such that, for all $(x, \nu) \in T^{-1}\Omega$,*

$$\Phi(x, \nu) \leqslant c_1 < 0,$$

(ii) *there are some constants $c_2, c_3 \in \mathbb{R}$ such that, for all $(x, \nu) \in T^{-1}\Omega$,*

$$|||d_x\Phi(x, \nu)||| \leqslant c_2 \left(1 + \|\nu\|^2\right) \text{ and } |||d_\nu\Phi(x, \nu)||| \leqslant c_3 \left(1 + \|\nu\|\right)$$

where $d_x\Phi$ and $d_\nu\Phi$ are the derivatives of Φ with respect to x and ν, where $||| \cdot |||$ is the operator norm associated to an arbitrary auxiliary Riemannian metric on M, and $\|\nu\|$ is the norm of ν for this Riemannian metric[14].

Assume that (Σ^-, Σ^+) is a pair of Φ-barriers, that is:

---

[14] More precisely, we choose a Riemannian metric η on M, we observe that for every $(x, \nu) \in TM$ there is a canonical identification between $T_{(x,\nu)}(T_xM) \simeq T_xM$, and we

– for every $x \in \Sigma^-$, one has $\kappa^{\Sigma^-}(x) \leqslant \Phi(x, \nu^{\Sigma^-}(x))$,
– for every $x \in \Sigma^+$, one has $\kappa^{\Sigma^+}(x) \geqslant \Phi(x, \nu^{\Sigma^+}(x))$.

Then there exists a strictly convex Cauchy surface Σ in M such that, for every $x \in \Sigma$, one has

$$\kappa^{\Sigma}(x) = \Phi\left(x, \nu^{\Sigma}(x)\right).$$

In order to show that Theorem 14.1 applies in our situation, we shall need the following elementary lemma:

LEMMA 14.2. — *Let Γ is a co-compact Fuchsian group in $\mathrm{SO}(1,2)$, and $f\colon \mathbb{H}^2 \to \mathbb{R}$ be a Γ-invariant function. See $\mathbb{H}^2$ as the upper sheet of the hyperboloid $x_1^2 + x_2^2 - x_3^2 = -1$ in $\mathbb{R}^3$, and consider an arbitrary Euclidean norm $\|.\|$ on $\mathbb{R}^3$. Then there exists a constant c such that, for every $\mathrm{n} \in \mathbb{H}^2$,*

$$|||df(\nu)||| \leqslant c$$

where $||| \cdot |||$ is the operator norm[15] associated to the Euclidean norm $\| \cdot \|$.

Proof. — Let $\| \cdot \|_{\mathrm{Lor}} := \sqrt{x_1^2 + x_2^2 - x_3^2}$ be the Lorentzian pseudo-norm on $\mathbb{R}^3$ which is preserved by $\mathrm{SO}(1,2)$, and $\langle .,. \rangle_{\mathrm{Lor}}$ be the associated Lorentzian pseudo-scalar product. We will prove the lemma for the Euclidean norm $\| \cdot \|_{\mathrm{Euc}} := \sqrt{x_1^2 + x_2^2 + x_3^2}$; The general case of an arbitrary Euclidean norm will follow, up to a change of the constant c.

We denote by ∇f the Lorentzian gradient of f (*i.e.* the gradient with respect to the hyperbolic norm on $\mathbb{H}^2$ induced by $\| \cdot \|_{\mathrm{Lor}}$). First observe that, for every $\nu \in \mathbb{H}^2$, and every vector $v \in T_\nu \mathbb{H}^2 \subset \mathbb{R}^3$, we have:

$$\|v\|_{\mathrm{Euc}} \geqslant \|v\|_{\mathrm{Lor}}.$$

Now, observe that, for every $\nu \in \mathbb{H}^2$, the restriction of $\langle \cdot, \cdot \rangle_{\mathrm{Lor}}$ to the plane $T_\nu \mathbb{H}^2$ is positive definite. Hence, for every $\nu \in \mathbb{H}^2$ and every $v \in T_\nu \mathbb{H}^2$, the Cauchy Schwarz inequality for the restriction of $\langle \cdot, \cdot \rangle_{\mathrm{Lor}}$ to $T_\nu \mathbb{H}^2$ yields:

$$|\langle (\nabla f)(\nu), v \rangle_{\mathrm{Lor}}| \leqslant \|(\nabla f)(\nu)\|_{\mathrm{Lor}} \, \|v\|_{\mathrm{Lor}}.$$

set:

$$\|v\| := \sqrt{\eta(v,v)}$$

$$|||d_x \Phi(x,\nu)||| := \sup_{v \in T_x M} \frac{|d_x \Phi(x,\nu).v|}{\|v\|}$$

$$|||d_\nu \Phi(x,\nu)||| := \sup_{v \in T_{(x,\nu)}(T_x^{-1}\Omega) \subset T_{(x,\nu)}(T_x M) \simeq T_x M} \frac{|d_\nu \Phi(x,\nu).v|}{\|v\|}$$

[15] More precisely, $|||df(\nu)||| = \displaystyle\sup_{v \in T_\nu \mathbb{H}^2 \subset T_\nu \mathbb{R}^3 \simeq \mathbb{R}^3} \dfrac{|df(\nu).v|}{\|v\|}$.

As a consequence, for every $\nu \in \mathbb{H}^2$ and every $v \in T_\nu \mathbb{H}^2$, we have:

$$
\begin{aligned}
|df(\nu).v| &= |\langle (\nabla f)(\nu), v \rangle_{\text{Lor}}| \\
&\leqslant \|(\nabla f)(\nu)\|_{\text{Lor}} \, \|v\|_{\text{Lor}} \\
&\leqslant \|(\nabla f)(\nu)\|_{\text{Lor}} \, \|v\|_{\text{Euc}}.
\end{aligned}
$$

But since $\nabla f(\nu)$ is Γ-equivariant, where Γ is co-compact, its hyperbolic norm $\|\nabla f(\nu)\|_{\text{Lor}}$ is uniformly bounded from above. The lemma follows. $\square$

Now we are in a position to prove Theorem 2.6. We consider a co-compact Fuchsian subgroup Γ in $\mathrm{SO}(1,2)$, a subgroup $\mathbf{\Gamma}$ of $\mathrm{SO}(1,2) \ltimes \mathbb{R}^3$ which projects bijectively on Γ, and a Γ-invariant function $f \colon \mathbb{H}^2 \to \mathbb{R}$.

Theorem 5.2 states that there exists a future-complete regular domain $E = E(\mathbf{\Gamma})$ in Min_3 such that the action of $\mathbf{\Gamma}$ on E is free and properly discontinuous, and such that $M := \mathbf{\Gamma} \smallsetminus E$ is a non-elementary future-complete MGHC flat spacetime. We consider the bundle

$$
T^{-1} \mathrm{Min}_3 = \big\{ (x, \nu) \in T \, \mathrm{Min}_3 \mid \nu \text{ future-oriented}, \langle \nu, \nu \rangle = -1 \big\},
$$

and we canonically identify $T^{-1} \mathrm{Min}_3$ with $\mathrm{Min}_3 \times \mathbb{H}^2$. Then we consider the function $F \colon T^{-1} \mathrm{Min}_3 \simeq \mathrm{Min}_3 \times \mathbb{H}^2 \to (-\infty, 0)$ defined by

$$
F(x, \nu) := f(\nu).
$$

The group $\mathbf{\Gamma}$ acts on $T^{-1} \mathrm{Min}_3$ by

$$
\mathrm{g} \cdot (x, \nu) = (\mathbf{g} \cdot (x), d\mathbf{g} \cdot x(\nu)) = (\mathbf{g} \cdot (x), \mathrm{g}(\nu))
$$

(where $\mathbf{g} \in \mathbf{\Gamma} \subset \mathrm{SO}(1,2) \ltimes \mathbb{R}^3$ and $g \in \Gamma \subset \mathrm{SO}(1,2)$ is its linear part) and

$$
\mathbf{\Gamma} \smallsetminus (T^{-1} \mathrm{Min}_3) = T^{-1}M = \big\{ (x, \nu) \in TM \mid \nu \text{ future-oriented}, \langle \nu, \nu \rangle = -1 \big\}.
$$

Since the function f is Γ-invariant, the function F is $\mathbf{\Gamma}$-invariant. Hence F induces a function $\Phi \colon T^{-1}M \to (-\infty, 0)$.

Consider a Cauchy surface Σ in $M = \mathbf{\Gamma} \smallsetminus E$. Then Σ lifts to a $\mathbf{\Gamma}$-invariant spacelike surface S in $E \subset \mathrm{Min}_3$. Let ν^S, ν^Σ, κ^S, κ^Σ be respectively the Gauss map of S, the Gauss map of Σ, the K-curvature of S and the K-curvature of Σ. We have the following equivalences

$$
\begin{aligned}
& & f(\nu) &= \kappa^S \circ (\nu^S)^{-1}(\nu) & &\text{for every } \nu \in \mathrm{range}(\nu^S) \subset \mathbb{H}^2 \\
\iff & & F(x, \nu^S(x)) &= \kappa^S(x) & &\text{for every } x \in S \\
\iff & & \Phi(x, \nu^\Sigma(x)) &= \kappa^\Sigma(x) & &\text{for every } x \in \Sigma.
\end{aligned}
$$

Therefore, in order to prove Theorem 2.6, it is enough to find a strictly convex Cauchy surface Σ in M such that

$$
\tag{14.1} \Phi\big(x, \nu^\Sigma(x)\big) = \kappa^\Sigma(x) \quad \text{for every } x \in \Sigma.
$$

We want to use Theorem 14.1 to get such a Cauchy surface Σ. So, we have to prove that the function Φ satisfies the conditions (i) and (ii) in the statement of Theorem 14.1, and we have to find a pair of Φ-barriers.

The function $f\colon \mathbb{H}^2 \to (-\infty, 0)$ is continuous and Γ-invariant. The group Γ is co-compact. Hence, the range of f is a compact interval $[a, b] \subset (0, +\infty)$. The range of the function $\Phi\colon T^{-1}M \to (-\infty, 0)$ is the same compact interval $[a, b]$. In particular, condition (i) of Theorem 14.1 is satisfied.

By Theorem 2.1, the spacetime M admits a K-time $\kappa\colon M \to (-\infty, 0)$. We consider the Cauchy surfaces $\Sigma^- := \kappa^{-1}(a)$ and $\Sigma^+ := \kappa^{-1}(b)$. By definition of a K-time, the Cauchy surface Σ^- is strictly convex and has constant K-curvature a, the surface Σ^+ is strictly convex Cauchy and has constant K-curvature b, and Σ^- is in the past of Σ^+. In particular, (Σ^-, Σ^+) is a pair of Φ-barriers in the sense defined in the statement of Theorem 14.1.

We denote by Ω the open subset of M bounded by the Cauchy surfaces Σ^- and Σ^+. Observe that Ω is relatively compact in M. We denote by O the lift of Ω in M.

Since the function F is independent of x, we have $d_x F(x, \nu) = 0$ for every $(x, \nu) \in T^{-1} \mathrm{Min}_3$. Hence we have $d_x \Phi(x, \nu) = 0$ for every $(x, \nu) \in T^{-1} \mathrm{Min}_3$. In particular, the first inequality in condition (ii) of Theorem 14.1 is satisfied.

Let η be a Riemannian metric on M. One can lift η to a $\boldsymbol{\Gamma}$-invariant Riemannian metric h on $E \subset \mathrm{Min}_3$. For every $(x, \nu) \in T^{-1}M$, if $(\widetilde{x}, \widetilde{\nu}) \in T^{-1} \mathrm{Min}_3 \simeq \mathrm{Min}_3 \times \mathbb{H}^2$ is a lift of (x, ν), then

$$(14.2) \qquad d_\nu \Phi(x, \nu) = d_{\widetilde{\nu}} F(\widetilde{x}, \widetilde{\nu}) = df(\widetilde{\nu}).$$

According to Lemma 14.2, for every $\widetilde{x} \in O$, there exists a constant $c_{\widetilde{x}}$, such that, for every $\widetilde{\nu} \in \mathbb{H}^2$,

$$|||df(\widetilde{\nu})|||_{\widetilde{x}} \leqslant c_{\widetilde{x}}$$

where $||| \cdot |||_{\widetilde{x}}$ is the operator norm associated to the Euclidean norm $h_{\widetilde{x}}$ on $\mathbb{R}^3 \simeq T_{\widetilde{x}} \mathrm{Min}_3$. Now observe that $\sup_{\widetilde{\nu} \in \mathbb{H}^2} |||df(\widetilde{\nu})|||_{\widetilde{x}}$ depends in a bounded way of $\widetilde{x}$ as far as $\widetilde{x}$ stays in O; Indeed the Riemannian metric h and the derivative df are $\boldsymbol{\Gamma}$-equivariant and $\boldsymbol{\Gamma} \setminus O = \Omega$ is relatively compact. Hence, there exists a constant c such that, for every $\widetilde{x} \in O$ and every $\widetilde{\nu} \in \mathbb{H}^2$,

$$|||df(\widetilde{\nu})|||_{\widetilde{x}} \leqslant c.$$

Using equality (14.2), this implies that, for every $(x, \nu) \in T^{-1}\Omega$,

$$|||d_\nu \Phi(x, \nu)||| \leqslant c$$

where $|||d_\nu \Phi(x, \nu)|||$ is the operator norm of $d_\nu \Phi(x, \nu)$ associated to the Riemannian metric η. In particular, the second inequality in condition (ii) of Theorem 14.1 is satisfied.

So all the hypotheses of Theorem 14.1 are satisfied. Hence this theorem provides us with a strictly convex Cauchy surface Σ in M satisfying (14.1). In order to achieve he proof of Theorem 2.6 the only remaining point is to prove the uniqueness of $\mathbf{\Gamma}$-solution for a given uniform lattice $\mathbf{\Gamma}$ of $SO(1, 2) \ltimes \mathbb{R}^3$.

Let S_1, S_2 be two such $\mathbf{\Gamma}$-invariant solutions. For any t, let S_i^t be the surface obtained by pushing S_i along the normal geodesics during the time t. Let t_1 be the minimal time such that $S_1^{t_1}$ lies in the future of S_2. Since S_1 and S_2 projects in the quotient as compact Cauchy surfaces, t_1 is well-defined and finite. Exchanging S_1 with S_2 if necessary, one can assume $t_1 \geqslant 0$. Then S_2 and $S_1^{t_1}$ are tangent at a common point x. This point is at Lorentzian distance t_1 from a point y in S_1 along a timelike geodesic orthogonal to S_1 at y. This geodesic is also orthogonal to S^{t_1}, and thus to S_2, at x. Hence $\nu^{S_1}(y) = \nu^{S_2}(x)$. Since they are both solutions of the same Minkowski problem, $\kappa^{S_1}(y) = \kappa^{S_2}(x)$. But the K-curvature strictly increases by pushing along normal geodesics (Remark 9.12): At one hand, the K-curvature of $S_1^{t_1}$ is bounded from above by κ^{S_2} (since it lies in the future of S_2); On the other hand, if $t_1 > 0$, it should be strictly bigger than $\kappa^{S_1}(y)$. It follows that t_1 is 0: S_1 lies in the future of S_2. In particular, $t_2 \geqslant 0$. Permuting the role of S_2 and S_1, we prove similarly that S_2 lies in the future of S_1. Hence $S_1 = S_2$. Theorem 2.6 follows.

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