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SEMI-CLASSICAL FUNCTIONAL CALCULUS ON MANIFOLDS WITH ENDS AND WEIGHTED L^p ESTIMATES

by Jean-Marc BOUCLET

Abstract. — For a class of non compact Riemannian manifolds with ends, we give semi-classical expansions of bounded functions of the Laplacian. We then study related L^p boundedness properties of these operators and show in particular that, although they are not bounded on L^p in general, they are always bounded on suitable weighted L^p spaces.

Résumé. — Pour une classe de variétés riemanniennes à bouts, nous donnons des développements semi-classiques de fonctions bornées du Laplacien. Nous étudions ensuite des propriétés de continuité L^p de ces opérateurs et montrons en particulier que, bien qu'ils ne soient en général pas bornés sur L^p, ils le sont toujours sur des espaces L^p à poids convenables.

1. Introduction and Results

In this paper we describe semi-classical expansions of functions of the Laplacian on a class of non compact manifolds of bounded geometry. We also derive certain weighted $L^p \to L^p$ boundedness properties of such operators. Further applications to Littlewood-Paley decompositions [5] and Strichartz estimates [4] will be published separately. Needless to say, the range of applications of the present functional calculus goes beyond Strichartz estimates; there are many problems which naturally involve spectral cutoffs at high frequencies in linear and non linear PDEs (Littlewood-Paley decompositions, paraproducts) or in spectral theory (trace formulas).

Consider a non compact Riemannian manifold $(\mathcal{M}, g)$ with ends, ie whose model at infinity is a product $(R, +\infty) \times S$ with metric $g = dr^2 + d\theta^2/w(r)^2$,

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where $R \gg 1$, $(S, d\theta^2)$ is a compact Riemannian manifold and $w(r)$ a bounded positive function. For instance, $w(r) = r^{-1}$ corresponds to conical ends, $w(r) = 1$ to cylindrical ends and $w(r) = e^{-r}$ to hyperbolic ends. We actually consider more general metrics (see Definition 1.2 below for precise statements) but these are the typical examples we have in mind. If Δ_g denotes the Laplacian on $\mathcal{M}$ and φ is a symbol of negative order, we are interested in decompositions of the form

$$\varphi(-h^2\Delta_g) = \mathcal{Q}_N(\varphi, h) + h^{N+1}\mathcal{R}_N(\varphi, h), \qquad h \in (0, 1], \tag{1.1}$$

where $N \geqslant 0$ is fixed and arbitrary, $\mathcal{Q}_N(\varphi, h)$ has an expansion in powers of h in terms h-pseudo-differential operators and $h^{N+1}\mathcal{R}_N(\varphi, h)$ is a 'nice' remainder. We recall that, for such semi-classical expansions, even the case of $\varphi \in C_0^\infty(\mathbb{R})$ is of interest, by opposition to the classical case ($h = 1$) where C_0^∞ functions of Δ_g are often treated as negligible operators.

There is a large literature devoted to the pseudo-differential analysis of functions of closed operators on manifolds so we only give references which are either classical or close to our framework. For $h = 1$, the case of compact manifolds (ie, essentially, the local interior case) was considered by Seeley [19] (see also [20, pp. 917-920]). For boundary value problems, we refer to [20, 12] and for non compact or singular manifolds to [18, 1]. We also quote [8, 22, 15] where general manifolds of bounded geometry are studied in connection with the problem of the $L^p \to L^p$ boundedness of functions the Laplacian (to which we come back below). The semi-classical case is treated for very general operators on $\mathbb{R}^n$ in [14, 17, 11] and in [7] for a compact manifold. Besides, one of our initial motivations is to extend the functional calculus used in [7] to non compact manifolds and thus to provide a convenient tool to prove Strichartz estimates, as for instance in [13, 6].

Although the general picture is quite clear, at least from the L^2 point of view, the problem of getting expansions of the form (1.1) requires some care. By opposition to the compact case (or to $\mathbb{R}^n$ for uniformly elliptic operators), one has to take into account certain off diagonal effects possibly leading to the unboundedness of the operators on $L^p(\mathcal{M}, dg)$, when $p \neq 2$, if dg denotes the Riemannian measure.

By considering properly supported operators, namely with kernels supported close to the diagonal of $\mathcal{M} \times \mathcal{M}$, we may insure that the principal part of the expansion $\mathcal{Q}_N(\varphi, h)$ is bounded on $L^p(\mathcal{M}, dg)$, for all $p \in [1, \infty]$, uniformly with respect to h. However, the boundedness of the remainder $\mathcal{R}_N(\varphi, h)$ on $L^p(\mathcal{M}, dg)$ remains equivalent to the one of the full operator $\varphi(-h^2\Delta_g)$ and it is well known that the latter may fail for non holomorphic

φ, as first noticed by Clerc and Stein [9] for symmetric spaces. The latter question is treated (with $h = 1$) for a large class of manifolds by Taylor in [22] (see also the references therein and the extension [15] to systems of properly supported operators). Taylor proves that, if A denotes the bottom of the spectrum of $-\Delta_g$ and $L = (-\Delta_g - A)^{1/2}$, the boundedness of $\varphi(L)$ on $L^p(\mathcal{M}, dg)$ is guaranteed if φ is even and holomorphic in a strip of width at least $\kappa|1/p - 1/2|$, with κ the exponential rate of the volume growth of balls. This is typically relevant in the hyperbolic case. To illustrate this fact (as well as some of our results), we recall a short proof of the L^p-unboundedness of $(z - \Delta_{\mathbb{H}^n})^{-1}$ in Appendix A, $\Delta_{\mathbb{H}^n}$ being the Laplacian on the hyperbolic space.

In summary, our first goal is to provide a fairly explicit and precise description of expansions of the form (1.1). For $h = 1$, this result is essentially contained in [8, 22] but we feel that it is worth giving complete proofs for the semi-classical case too, first because we shall use it extensively in subsequent papers and second because of the subtleties due to L^p-unboundedness.

Our second point is to prove weighted L^p estimates on $\mathcal{R}_N(\varphi, h)$ or, equivalently, on the resolvent $(z - \Delta_g)^{-1}$. The basic strategy is to use the expansion (1.1) to get L^2 estimates on commutators of the resolvent with natural first order differential operators and show that $(z - \Delta_g)^{-1}$ is a pseudo-differential operator, using the Beals criterion. At this stage, the meaning of pseudo-differential operator is rather vague but we emphasize that the point is not (only) to control the singularity of the kernel close to the diagonal but also the decay far from the diagonal. As a consequence of this analysis, we obtain in particular that, although $(z - \Delta_g)^{-1}$ is not necessarily bounded on $L^p(\mathcal{M}, dg)$, we always have

$$||w(r)^{\frac{n-1}{p}-\frac{n-1}{2}}(z - \Delta_g)^{-1}w(r)^{\frac{n-1}{2}-\frac{n-1}{p}}||_{L^p(\mathcal{M},dg)\to L^p(\mathcal{M},dg)} < \infty,$$

for all $p \in (1, \infty)$ and $z \notin \operatorname{spec}(\Delta_g)$. More generally, if W is a temperate weight (see Definition 1.6 below), we have

$$||W(r)^{-1}w(r)^{\frac{n-1}{p}-\frac{n-1}{2}}(z - \Delta_g)^{-1}w(r)^{\frac{n-1}{2}-\frac{n-1}{p}}W(r)||_{L^p(\mathcal{M},dg)\to L^p(\mathcal{M},dg)} < \infty.$$

This works in particular for the hyperbolic case where $(z - \Delta_g)^{-1}$ is not bounded on $L^p(\mathcal{M}, dg)$ in general. In the conical case, or more generally if w itself is a temperate weight, we recover the natural (unweighted) boundedness on $L^p(\mathcal{M}, dg)$ by choosing $W = w^{\frac{n-1}{p}-\frac{n-1}{2}}$. The latter boundedness can be seen as a consequence of [22] since, if w is temperate, the volume growth of balls is polynomial. The above estimates are therefore complementary to the results of [22]: if z is too close to the spectrum of the

Laplacian, $(z-\Delta_g)^{-1}$ may not be bounded on $L^p = L^p(\mathcal{M}, dg)$ but it is bounded if we accept to replace L^p by weighted L^p spaces. Furthermore, these weighted spaces are natural since they contain L^p itself when w is temperate (ie essentially if w^{-1} is of polynomial growth).

Let us now state our results precisely.

Manifolds, atlas, partition of unity. In the sequel $\mathcal{M}$ will be a smooth manifold of dimension $n \geqslant 2$, without boundary and which is diffeomorphic to a product outside a compact set in the following sense : we assume that there exist a compact subset $\mathcal{K} \Subset \mathcal{M}$, a real number R, a compact manifold S and a function $r \in C^\infty(\mathcal{M}, \mathbb{R})$ such that

(1) r is a coordinate near $\overline{\mathcal{M} \setminus \mathcal{K}}$ such that

$$r(\underline{x}) \to +\infty, \qquad \underline{x} \to \infty,$$

(2) there is a diffeomorphism of the form

$$\Psi : \mathcal{M} \setminus \mathcal{K} \to (R, +\infty) \times S, \tag{1.2}$$

$$\underline{x} \mapsto (r(\underline{x}), \pi_S(\underline{x}))\,. \tag{1.3}$$

Under these assumptions, we can specify an atlas on $\mathcal{M}$ and a partition of unity as follows. If we consider a chart on S,

$$\psi_\iota : U_\iota \subset S \to V_\iota \subset \mathbb{R}^{n-1}, \tag{1.4}$$

with $\psi_\iota(\underline{y}) = (\theta_1(\underline{y}), \dots, \theta_{n-1}(\underline{y}))$, then the open sets

$$\mathcal{U}_\iota = \Psi^{-1}\left((R, +\infty) \times U_\iota\right) \subset \mathcal{M}, \qquad \mathcal{V}_\iota = (R, +\infty) \times V_\iota \subset \mathbb{R}^n, \tag{1.5}$$

and the map

$$\begin{aligned} \Psi_\iota : \mathcal{U}_\iota \to \mathcal{V}_\iota, \qquad \text{with} \quad \Psi_\iota(\underline{x}) &= (r(\underline{x}), \psi_\iota \circ \pi_S(\underline{x})) \\ &= (r(\underline{x}), \theta_1(\pi_S(\underline{x})), \dots, \theta_{n-1}(\pi_S(\underline{x})))\,, \end{aligned}$$

define a coordinate chart on $\mathcal{M} \setminus \mathcal{K}$. With a standard abuse of notation, we will denote for simplicity these coordinates $(r, \theta_1, \dots, \theta_{n-1})$ or even (r, θ).

Definition 1.1. — *We call $\mathcal{U}_\iota$ a coordinate patch at infinity and the triple $(\mathcal{U}_\iota, \mathcal{V}_\iota, \Psi_\iota)$ a chart at infinity.*

Since S is compact, there is a finite set I_∞ such that the family $(\mathcal{U}_\iota, \mathcal{V}_\iota, \Psi_\iota)_{\iota \in I_\infty}$ is an atlas on $\mathcal{M} \setminus \mathcal{K}$. Choosing another finite collection of coordinate charts for a neighborhood of $\mathcal{K}$, which we denote by $(\mathcal{U}_\iota, \mathcal{V}_\iota, \Psi_\iota)_{\iota \in I_{\mathrm{comp}}}$ for some finite set I_{comp}[(1)], we get a finite atlas on $\mathcal{M}$

[(1)] we keep the notation $\mathcal{U}_\iota, \mathcal{V}_\iota, \Psi_\iota$ but, of course, the corresponding new $\mathcal{U}_\iota$ and $\mathcal{V}_\iota$ are not defined by (1.5). In the core of the paper, there should be anyway no confusion for we shall work almost only on $\mathcal{M} \setminus \mathcal{K}$.

by considering $(\mathcal{U}_\iota, \mathcal{V}_\iota, \Psi_\iota)_{\iota\in I}$ with

$$I = I_\infty \cup I_{\rm comp}.$$

In particular, we can find a finite partition of unity

$$\sum_{\iota\in I} f_\iota = 1 \qquad \text{on } \mathcal{M}, \tag{1.6}$$

such that, for all $\iota \in I$, f_ι is supported in $\mathcal{U}_\iota$. We also set

$$\chi_\iota = f_\iota \circ \Psi_\iota^{-1}. \tag{1.7}$$

If $\mathcal{U}_\iota$ is a patch at infinity, we can assume that f_ι is such that

$$\chi_\iota(r,\theta) = \varrho(r)\kappa_\iota(\theta), \tag{1.8}$$

for some smooth functions ϱ and κ_ι such that, for some $R' > R$,

$$\varrho(r) = 1 \ \text{ for } r \gg 1, \ \ \text{supp } \varrho \subset [R', +\infty), \qquad \kappa_\iota \in C_0^\infty(V_\iota). \tag{1.9}$$

DEFINITION 1.2. — *The manifold $(\mathcal{M}, g)$ is called almost asymptotic if g is a Riemannian metric such that, for some function $w : \mathbb{R} \to (0, +\infty)$, the metric reads, in any chart at infinity,*

$$g \ = \ G_{\rm unif}\left(r, \theta, dr, w(r)^{-1}d\theta\right) \tag{1.10}$$

and the following conditions hold:

(1) *if $\theta = (\theta_1, \dots, \theta_{n-1})$ are local coordinates on S (with values in V_ι, see (1.4)),*

$$G_{\rm unif}(r,\theta,v) := \sum_{1\leqslant j,k\leqslant n} G_{jk}(r,\theta)v_jv_k, \qquad v = (v_1,\dots,v_n) \in \mathbb{R}^n,$$

for some symmetric matrix $(G_{jk}(r,\theta))_{1\leqslant j,k\leqslant n}$ with smooth coefficients such that, for all compact subset $K \subset V_\iota$

$$\left|\partial_r^j\partial_\theta^\alpha G_{jk}(r,\theta)\right| \leqslant C_{j\alpha K}, \qquad r > R, \ \theta \in K, \tag{1.11}$$

and which is uniformly positive definite in the sense that, for some $C > 0$ depending on K,

$$C^{-1}|v|^2 \leqslant G_{\rm unif}(r,\theta,v) \leqslant C|v|^2, \qquad r > R, \ \theta \in K, \ v \in \mathbb{R}^n. \tag{1.12}$$

(2) *The function w is smooth and satisfies, for some $C > 0$ and all $k \in \mathbb{N}$,*

$$0 < w(r) \ \leqslant \ C, \tag{1.13}$$

$$C^{-1} \leqslant w(r)/w(r') \ \leqslant \ C, \qquad \text{if } \ |r - r'| \leqslant 1 \tag{1.14}$$

$$\left|d^k w(r)/dr^k\right| \ \leqslant \ C_k w(r), \tag{1.15}$$

for all $r, r' \in \mathbb{R}$.

Note that (1.14) is equivalent to the fact that, for some $C > 0$,

$$C^{-1} e^{-C|r-r'|} \leqslant \frac{w(r)}{w(r')} \leqslant C e^{C|r-r'|}.$$

In particular, this implies that $w(r) \gtrsim e^{-C|r|}$.

Asymptotically conical manifolds, for which $g = dr^2 + r^2 g_S(r, \theta, d\theta)$ (near infinity), or asymptotically hyperbolic manifolds for which $g = dr^2 + e^{2r} g_S(r, \theta, d\theta)$, with $g_S(r, \theta, d\theta)$ a metric on S depending smoothly on r, satisfy our definition. More precisely, for such asymptotic structures one usually requires that $g_S(r, \theta, d\theta)$ is a small perturbation of a metric $g_S^\infty(\theta, d\theta)$ in the sense that $g_S(r, \theta, d\theta) - g_S^\infty(\theta, d\theta) \to 0$ as $r \to \infty$. See for instance [16] for more precise statements. Here we do not require such a condition which is the reason why we use the terminology *almost* asymptotic.

Differential operators on $\mathcal{M}$. We first compute the Laplacian Δ_g in a chart at infinity. Let us define $\partial_1^w, \dots, \partial_n^w$ by

$$\partial_1^w = \partial_r, \qquad \partial_2^w = w(r)\partial_{\theta_1}, \ \dots \ , \ \partial_n^w = w(r)\partial_{\theta_{n-1}}^w.$$

We also set $(G^{jk})_{1 \leqslant j,k \leqslant n} := (G_{jk})^{-1}_{1 \leqslant j,k \leqslant n}$ and $\det G_{\mathrm{unif}} := \det(G_{jk})$ (see (1.10)). We then have

$$\Delta_g = (\det G_{\mathrm{unif}})^{-1/2} \partial_j^w G^{jk} (\det G_{\mathrm{unif}})^{1/2} \partial_k^w + (1-n) \frac{w'(r)}{w(r)} G^{1k} \partial_k^w, \tag{1.16}$$

using the summation convention for $j, k \geqslant 1$. This formula motivates the introduction of the following class of differential operators.

Definition 1.3. — *For* $m \in \mathbb{N}$, $\mathrm{Diff}_w^m(\mathcal{M})$ *is the space of differential operators* P *of order* $\leqslant m$, *acting on functions on* $\mathcal{M}$, *such that, for any chart at infinity* $(\mathcal{U}_\iota, \mathcal{V}_\iota, \Psi_\iota)$,

$$\Psi_{\iota *} P \Psi_\iota^* = \sum_{k + |\alpha| \leqslant m} a_{k\alpha}^\iota(r, \theta) \left(w(r) D_\theta\right)^\alpha D_r^k, \tag{1.17}$$

with

$$\partial_r^j \partial_\theta^\beta a_{k\alpha}^\iota \in L^\infty\left((R, +\infty) \times K_\iota\right),$$

for all j, β *and all* $K_\iota \Subset V_\iota$. *Here we used the standard notation* $\Psi_\iota^* u = u \circ \Psi_\iota$ *and* $\Psi_{\iota *} v = v \circ \Psi_\iota^{-1}$.

By (1.11), (1.15) and (1.16), we see that $-\Delta_g \in \mathrm{Diff}_w^2(\mathcal{M})$ and that its principal symbol takes the following form in $\mathcal{V}_\iota$, for $\iota \in I_\infty$,

$$p_2^\iota(r, \theta, \rho, w(r)\eta) = G^{11}(r, \theta)\rho^2 + 2G^{1k}(r, \theta)\rho w(r)\eta_k + G^{jk}(r, \theta) w(r)^2 \eta_j \eta_k, \tag{1.18}$$

using the summation convention for $j,k \geqslant 2$. Here and below ρ and η denote respectively the dual variables to r and θ. If $\iota \in I_{\rm comp}$, the principal symbol of $-\Delta_g$ in $\mathcal{V}_\iota$ takes the standard form

$$p_2^\iota(x,\xi) = g^{jk}(x)\xi_j\xi_k \tag{1.19}$$

for some smooth $(g^{jk}(x))$ such that $g^{jk}(x)\xi_j\xi_k \gtrsim |\xi|^2$ for $\xi \in \mathbb{R}^n$ locally uniformly with respect to x.

Remark. — Recall that, if $\iota \in I_\infty$, the principal symbol of $-\Delta_g$ is given by (1.18) but not by p_2^ι itself (see the factor $w(r)$ in the left hand side of (1.18)). This notation (which is perhaps confusing) will be convenient to state Theorem 1.5.

Lebesgue spaces. We now describe volume densities. In coordinates (r,θ) at infinity, the Riemannian volume density associated to g, denoted by dg, reads

$$dg = w(r)^{1-n}(\det\, G_{\rm unif}(r,\theta))^{1/2} dr d\theta, \tag{1.20}$$

where, for all $\iota \in I_\infty$ and all $K_\iota \Subset V_\iota$ (see (1.5)), (1.12) shows the existence of $C_{K_\iota} > 0$ such that

$$C_{K_\iota}^{-1} \leqslant \det\, G_{\rm unif}(r,\theta) \leqslant C_{K_\iota}, \qquad \theta \in K_\iota,\ r > R. \tag{1.21}$$

Define another density $\widetilde{dg}$ on $\mathcal{M}$ by

$$\widetilde{dg} = w^{n-1}(r) dg, \tag{1.22}$$

we then have

$$L^p(\mathcal{M}, dg) = w^{\frac{n-1}{p}}(r) L^p(\mathcal{M}, \widetilde{dg}), \qquad p \in [1,\infty). \tag{1.23}$$

The map

$$L^2(\mathcal{M}, \widetilde{dg}) \ni u \mapsto w(r)^{(n-1)/2} u \in L^2(\mathcal{M}, dg), \tag{1.24}$$

is clearly unitary and the operator

$$\widetilde{\Delta}_g := w(r)^{\frac{1-n}{2}} \Delta_g w(r)^{\frac{n-1}{2}}, \tag{1.25}$$

is symmetric on $C_0^\infty(\mathcal{M})$ with respect to $\widetilde{dg}$. By (1.15), we have

$$\widetilde{\Delta}_g \in \mathrm{Diff}^2_w(\mathcal{M}).$$

We also note that Δ_g and $\widetilde{\Delta}_g$ are essentially self-adjoint on $C_0^\infty(\mathcal{M})$ (following the usual method of [17] for instance) respectively with respect to dg and $\widetilde{dg}$. Since (1.24) is unitary, their self-adjoint realizations are unitarily equivalent.

We next record that, for all $\iota \in I_\infty$ and all $K_\iota \Subset V_\iota$ (see (1.5)), we have the equivalence of norms

(1.26)
$$||u||_{L^p(\mathcal{M},\widetilde{dg})} \approx ||u \circ \Psi_\iota^{-1}||_{L^p(\mathbb{R}^n,drd\theta)}, \qquad \mathrm{supp}(u \circ \Psi_\iota^{-1}) \subset (R,+\infty) \times K_\iota,$$

for $p \in [1,\infty]$. This is a simple consequence of (1.21). On compact subsets, the same equivalence holds trivially. For the measure dg, we have, if $\iota \in I_\infty$,

$$||u||_{L^p(\mathcal{M},dg)} \approx \left|\left| w^{(1-n)/p}(r) u \circ \Psi_\iota^{-1} \right|\right|_{L^p(\mathbb{R}^n,drd\theta)}, \quad \mathrm{supp}(u \circ \Psi_\iota^{-1}) \subset (R,+\infty) \times K_\iota. \tag{1.27}$$

Pseudo-differential operators. We now define a class of semi-classical pseudo-differential operators associated to the partition of unity (1.6). We will choose symbols

$$a^\iota \in S^m_\iota(\mathcal{V}_\iota \times \mathbb{R}^n),$$

where $\mathcal{V}_\iota \subset \mathbb{R}^n$ is defined by (1.5) if $\iota \in I_\infty$. By definition, this means, if $\iota \in I_\infty$, that for all $K_\iota \Subset \mathcal{V}_\iota$,

$$|\partial_r^j \partial_\theta^\alpha \partial_\rho^k \partial_\eta^\beta a^\iota(r,\theta,\rho,\eta)| \leqslant C(1+|\rho|+|\eta|)^{m-k-|\beta|}, \quad r > R, \ \ \theta \in K_\iota, \ \rho \in \mathbb{R}, \ \eta \in \mathbb{R}^{n-1},$$

and, if $\iota \in I_{\mathrm{comp}}$, that for all $\mathcal{K}_\iota \Subset \mathcal{V}_\iota$,

$$|\partial_x^\alpha \partial_\xi^\beta a^\iota(x,\xi)| \leqslant C(1+|\xi|)^{m-|\beta|}, \qquad x \in \mathcal{K}_\iota, \ \xi \in \mathbb{R}^n.$$

In both cases, the topology of $S^m_\iota(\mathcal{V}_\iota \times \mathbb{R}^n)$ is given by the best constants C which define semi-norms.

We basically would like to use operators of the form

$$a^\iota(r,\theta,hD_r,hw(r)D_\theta)\chi_\iota, \qquad \text{if } \iota \in I_\infty,$$

(see (2.1) below) and

$$a^\iota(x,hD_x)\chi_\iota, \qquad \text{if } \iota \in I_{\mathrm{comp}},$$

where χ_ι is defined by (1.7) and $h \in (0,1]$ is the semi-classical parameter. Actually, we need to consider properly supported operators so we construct first suitable cutoffs near the diagonal. Choose a function $\zeta \in C_0^\infty(\mathbb{R}^n)$ and $\varepsilon > 0$ such that

$$\zeta(x) = 1 \ \text{ for } \ |x| \leqslant \varepsilon, \qquad \zeta(x) = 0 \ \text{ for } \ |x| > 2\varepsilon. \tag{1.28}$$

For $\iota \in I_\infty$, the function

$$\chi_\iota^\zeta(r,\theta,r',\theta') := \chi_\iota(r',\theta')\zeta\left((r,\theta)-(r',\theta')\right), \tag{1.29}$$

is smooth on $\mathbb{R}^{2n}$ and, if $K_\iota \Subset V_\iota$ is an arbitrarily small neighborhood of $\mathrm{supp}(\kappa_\iota)$ (see (1.9)), we may choose ε small enough such that

$$\mathrm{supp}(\chi_\iota^\zeta) \subset ((R,+\infty)\times K_\iota)^2 . \tag{1.30}$$

Proceeding similarly for $\iota \in I_{\mathrm{comp}}$, we obtain a family of functions $(\chi_\iota^\zeta)_{\iota\in I}$ supported close to the diagonal of $\mathbb{R}^{2n}$, with also $\mathrm{supp}(\chi_\iota^\zeta) \subset \mathcal{V}_\iota \times \mathcal{V}_\iota$, and such that

$$\chi_\iota^\zeta{}_{|_{\mathrm{diagonal}}} = \chi_\iota. \tag{1.31}$$

Definition 1.4. — *For $a^\iota \in S^m_\iota(\mathcal{V}_\iota \times \mathbb{R}^n)$, the pseudo-differential operator*

$$\mathrm{op}^\iota_{w,h}(a^\iota) : C_0^\infty(\mathbb{R}^n) \to C_0^\infty(\mathcal{V}_\iota)$$

is the operator with kernel

(1.32)
$$(2\pi)^{-n} \iint e^{i(r-r')\rho + i(\theta-\theta')\cdot\eta} a^\iota(r,\theta,h\rho,hw(r)\eta) d\rho d\eta \times \chi_\iota^\zeta(r,\theta,r',\theta'), \qquad \text{if } \iota \in I_\infty,$$

(1.33)
$$(2\pi)^{-n} \int e^{i(x-x')\cdot\xi} a^\iota(x,h\xi) d\xi \times \chi_\iota^\zeta(x,x'), \qquad \text{if } \iota \in I_{\mathrm{comp}}.$$

In other words, $\mathrm{op}^\iota_{h,w}(a^\iota)$ is obtained by multiplying the kernel of $a^\iota(r,\theta,hD_r,hw(r)D_\theta)\chi_\iota$ (resp. of $a^\iota(x,hD_x)\chi_\iota$) by $\zeta((r,\theta)-(r',\theta'))$ (resp. by $\zeta(x-x')$).

If $m < -n$ the integrals in (1.32) and (1.33) are absolutely convergent, otherwise they must be understood as oscillatory integrals in the usual way. That $\mathrm{op}^\iota_{w,h}(a_\iota)$ maps $C_0^\infty(\mathbb{R}^n)$ into $C_0^\infty(\mathcal{V}_\iota)$ follows from the construction of χ_ι^ζ. Note also that

$$\mathrm{op}^\iota_{w,h}(1) = \chi_\iota, \tag{1.34}$$

since the oscillatory integral is the Dirac measure along the diagonal and $\chi_\iota^\zeta(r,\theta,r',\theta') = \chi_\iota(r',\theta')$ for $|r-r'|+|\theta-\theta'|$ small enough.

Remark. — Note the factor $w(r)$ in front of η in the amplitude of (1.32). The choice of notation of Definition 1.4 is thus consistent with the expressions of the principal symbol of $-\Delta_g$ given by (1.18) and (1.19).

We are now ready to state our results. We consider

$$\varphi \in S^{-\sigma}(\mathbb{R}), \qquad \sigma > 0,$$

that is $|\varphi^{(k)}(\lambda)| \leqslant C_k\langle\lambda\rangle^{-\sigma-k}$ for all $\lambda \in \mathbb{R}$. The best constants C_k are semi-norms defining the topology of $S^{-\sigma}(\mathbb{R})$.

THEOREM 1.5. — *Let P denote either $-\Delta_g$ or $-\widetilde{\Delta}_g$. For all $N \geqslant 0$, the following holds:*

$$\varphi(h^2 P) = \sum_{\iota \in \iota} \mathcal{Q}_N^\iota(P, \varphi, h) + h^{N+1} \mathcal{R}_N(P, \varphi, h), \qquad h \in (0, 1],$$

where, for all $\iota \in I$,

$$\Psi_{\iota *} \mathcal{Q}_N^\iota(P, \varphi, h) \Psi_\iota^* = \sum_{j=0}^{N} h^j \mathrm{op}_{w,h}^\iota(a_j^\iota)$$

with symbols $a_0^\iota, \dots, a_N^\iota$ of the form

$$a_0^\iota = \varphi \circ p_2^\iota, \qquad a_j^\iota = \sum_{k \leqslant k(j)} d_{jk}^\iota \varphi^{(k)} \circ p_2^\iota, \quad j \geqslant 1, \tag{1.35}$$

using the functions p_2^ι given by (1.18) for $\iota \in I_\infty$ and (1.19) for $\iota \in I_{\mathrm{comp}}$. Here $k(j) < \infty$ and

$$d_{jk}^\iota \in S_\iota^{2k-j}(\mathcal{V}_\iota \times \mathbb{R}^n)$$

is polynomial in the momentum variable ($d_{jk}^\iota \equiv 0$ if $2k - j < 0$) and independent of φ.

In addition, for all $m, m' \in \mathbb{N}$, all $A \in \mathrm{Diff}_w^m(\mathcal{M})$, $B \in \mathrm{Diff}_w^{m'}(\mathcal{M})$, all $p \in [2, \infty]$ and all N such that $N > n - 2\sigma + m + m'$, there exists C such that

$$\left\| h^m A \mathcal{R}_N(-\Delta_g, \varphi, h) h^{m'} B \right\|_{L^2(\mathcal{M}, dg) \to L^p(\mathcal{M}, dg)} \leqslant C h^{-n\left(\frac{1}{2} - \frac{1}{p}\right)}, \tag{1.36}$$

and, for $P = -\widetilde{\Delta}_g$,

$$\left\| w(r)^{\frac{n-1}{2} - \frac{n-1}{p}} h^m A \mathcal{R}_N(-\widetilde{\Delta}_g, \varphi, h) h^{m'} B \right\|_{L^2(\mathcal{M}, \widetilde{dg}) \to L^p(\mathcal{M}, \widetilde{dg})} \leqslant C h^{-n\left(\frac{1}{2} - \frac{1}{p}\right)}, \tag{1.37}$$

for all $h \in (0, 1]$ in both cases.

This theorem roughly means that, near infinity, $\varphi(h^2 P)$ is well approximated by pseudo-differential operators with symbols of the form $a(r, \theta, \rho, w(r)\eta)$. The principal symbol is for instance

$$\varphi(p_2^\iota(r, \theta, \rho, w(r)\eta)).$$

Note that, when $\varphi \in C_0^\infty(\mathbb{R})$, this symbol is compactly supported with respect to ρ but not uniformly with respect to η: if $w(r) \to 0$ as $r \to \infty$, η is not confined in a fixed compact set, since we only have $|\eta| \lesssim w(r)^{-1}$.

The estimates (1.36) and (1.37) follow from the Sobolev embedding $D((-\Delta_g)^k) \subset L^\infty(\mathcal{M})$ for $k > n/4$ (see Proposition 2.11) and, to that extent, Theorem 1.5 is an L^2 theorem.

We now consider the $L^p \to L^p$ properties. Recall first a classical definition.

DEFINITION 1.6. — *A function $W : \mathbb{R} \to (0, +\infty)$ is a temperate weight if, for some positive constants C,M,*

$$W(r') \leqslant CW(r)(1 + |r - r'|)^M, \qquad r, r' \in \mathbb{R}. \tag{1.38}$$

The meaning of this definition is that W can neither grow nor decay too fast. For instance if $d^k w^{-1}/dr^k$ is bounded on $\mathbb{R}$, w is a temperate weight. This is an elementary consequence of Taylor's formula to order k and of the fact that $|d^j w^{-1}/dr^j| \lesssim w^{-1}$, by (1.15).

The operators $\mathrm{op}^{\iota}_{w,h}(a^{\iota}_j)$ of Theorem 1.5 are bounded on $L^p(\mathcal{M}, dg)$, $L^p(\mathcal{M}, \widetilde{dg})$, or more generally on $L^p(\mathcal{M}, W(r)dg)$ and $L^p(\mathcal{M}, W(r)\widetilde{dg})$ for all temperate weight W and all $p \in [1, \infty]$ (see Proposition 2.3). We therefore focus on the remainder terms $\mathcal{R}_N(P, \varphi, h)$.

THEOREM 1.7. — *For all $N \geqslant 0$, all temperate weight W and all $1 < p < \infty$,*

$$\Big|\Big|W(r)^{-1}\mathcal{R}_N(-\widetilde{\Delta}_g, \varphi, h)W(r)\Big|\Big|_{L^p(\mathcal{M},\widetilde{dg})\to L^p(\mathcal{M},\widetilde{dg})} \leqslant C_{N,p,\varphi,W}, \tag{1.39}$$

for all $h \in (0, 1]$. The constant $C_{N,p,\varphi,W}$ depends (linearly) on a finite number of semi-norms of $\varphi \in S^{-\sigma}(\mathbb{R})$.

COROLLARY 1.8. — *For all $1 < p < \infty$ all temperate weight W and all $\varphi \in S^{-\sigma}(\mathbb{R})$ there exists C such that*

$$\Big|\Big|W(r)^{-1}\varphi(-h^2\widetilde{\Delta}_g)W(r)\Big|\Big|_{L^p(\mathcal{M},\widetilde{dg})\to L^p(\mathcal{M},\widetilde{dg})} \leqslant C, \qquad h \in (0, 1].$$

Equivalently, we have

$$\Big|\Big|W(r)^{-1}w(r)^{\frac{n-1}{p}-\frac{n-1}{2}}\varphi(-h^2\Delta_g)w(r)^{\frac{n-1}{2}-\frac{n-1}{p}}W(r)\Big|\Big|_{L^p(\mathcal{M},dg)\to L^p(\mathcal{M},dg)} \leqslant C, \qquad h \in (0, 1].$$

Observe that Theorem 1.7 and Corollary 1.8 hold in particular if $w(r) = e^{-r}$ in which case $\varphi(-h^2\Delta_g)$ is in general not bounded on $L^p(\mathcal{M}, dg)$. Theorem 1.7 is a consequence of a stronger result, namely Proposition 3.8, showing that, in any chart, the resolvent $(z - \widetilde{\Delta}_g)^{-1}$ is a pseudo-differential operators whose full symbol belongs to a suitable class. Since this result is of more technical nature, we prefer not to state it in this part.

If the function w itself is a temperate weight, for instance if $w(r) = r^{-1}$ for r large, Theorem 1.7 also implies the following result.

Corollary 1.9. — *If w is a temperate weight, then for all temperate weight W, all $N \geqslant 0$ and all $1 < p < \infty$,*

(1.40)
$$\left|\left|W^{-1}(r)\mathcal{R}_N(-\Delta_g,\varphi,h)W(r)\right|\right|_{L^p(\mathcal{M},dg)\to L^p(\mathcal{M},dg)} \leqslant C_{N,p,\varphi,W}, \quad h \in (0,1].$$

The constant $C_{N,p,\varphi,W}$ depends (linearly) on a finite number of semi-norms of $\varphi \in S^{-\sigma}(\mathbb{R})$. In particular, for fixed φ and W there exists $C > 0$ such that

$$\left|\left|W(r)^{-1}\varphi(-h^2\Delta_g)W(r)\right|\right|_{L^p(\mathcal{M},dg)\to L^p(\mathcal{M},dg)} \leqslant C, \qquad h \in (0,1]. \tag{1.41}$$

Of course, (1.41) holds with $W = 1$. As explained in the introduction, this last result can be considered as essentially well known (see for instance [22] for $h = 1$). We quote it to emphasize the difference with Corollary 1.8 where w is not assumed to be a temperate weight. It follows directly from Theorem 1.7, using (1.23), (1.25) and the fact that products or real powers of temperate weights are temperate weights.

2. Parametrix of the resolvent and applications

In the main part of this section, namely until (2.19), we work in coordinate patches $\mathcal{U}_\iota$ of the form (1.5) (ie with $\iota \in I_\infty$).

2.1. Elementary pseudo-differential calculus

In this part, we give elementary composition formulas and the related remainder estimates for pseudo-differential operators of the form $\mathrm{op}^\iota_{w,h}(a)$. We will not develop a systematic study of the symbolic calculus but only record the basic results required for the calculation of parametrices of $(z - h^2\Delta_g)^{-1}$ and $(z - h^2\widetilde{\Delta}_g)^{-1}$.

For $\Omega \subset \mathbb{R}^D$, $D \geqslant 1$, $C^\infty_b(\Omega)$ will denote the space of smooth functions which are bounded on Ω as well as their derivatives.

For $b \in S^m_\iota(\mathcal{V}_\iota \times \mathbb{R}^n)$ and $h \in (0,1]$, we set

$$\begin{aligned}[b(r,\theta,hD_r,hw(r)D_\theta)v]\,(r,\theta) \\ = (2\pi)^{-n}\iint e^{i(r\rho+\theta.\eta)}b(r,\theta,h\rho,hw(r)\eta)\widehat{v}(\rho,\eta)d\rho d\eta\end{aligned} \tag{2.1}$$

with $\hat{v}(\rho,\theta) = \iint e^{-ir\rho - i\theta.\eta} v(r,\theta) dr d\theta$ the usual Fourier transform. In the special case of a polynomial symbol in ρ and η, $a(r,y,\rho,\eta) = \sum a_{j\alpha}(r,\theta)\rho^j\eta^\alpha$, we have

$$a(r,\theta,hD_r,hw(r)D_\theta) = \sum a_{j\alpha}(r,\theta)(hw(r)D_\theta)^\alpha (hD_r)^j, \tag{2.2}$$

where one must notice that D_r and $w(r)D_\theta$ don't commute.

We have the following elementary result.

PROPOSITION 2.1. — *Let $a \in S^{m_1}_\iota(\mathcal{V}_\iota \times \mathbb{R}^n)$ be polynomial in (ρ,η) and let $b \in S^{m_2}(\mathcal{V}_\iota \times \mathbb{R}^n)$ with $m_2 \in \mathbb{R}$. We have*

$$a(r,\theta,hD_r,hw(r)D_\theta)b(r,\theta,hD_r,hw(r)D_\theta) = \sum_{l=0}^{m_1} h^l (a\#b)_l(r,\theta,hD_r,hw(r)D_\theta) \tag{2.3}$$

where, if we set

$$D_w = D_r + \frac{w'(r)}{w(r)}\eta\cdot D_\eta,$$

the symbol $(a\#b)_k = (a\#b)_k(r,\theta,\rho,\eta) \in S^{m_1+m_2-k}(\mathcal{V}_\iota \times \mathbb{R}^n)$ is given by

$$(a\#b)_k = \sum_{j+|\beta|=k} \frac{1}{j!\beta!} w(r)^{|\beta|} \left(\partial_\rho^j \partial_\eta^\beta a\right)\left(D_\theta^\beta D_w^j b\right).$$

When $w \equiv 1$, this proposition is of course the usual composition formula for pseudo-differential operators. Note that, since a is polynomial of degree $\leqslant m_1$, we have $(a\#b)_l \equiv 0$ for $l > m_1$ and the composition formula is exact (there is no remainder term).

Remark. — A simple induction shows that the operator D_w^j is a linear combination of

$$\left(\frac{w'(r)}{w(r)}\right)^{(j_1)} \cdots \left(\frac{w'(r)}{w(r)}\right)^{(j_k)} D_r^l \eta^\alpha D_\eta^\alpha \tag{2.4}$$

with $j_1 + \cdots + j_k + k + l = j$, $|\alpha| \leqslant k$ and $k \geqslant 0$. If $k = 0$ then $(w'/w)^{(j_1)} \cdots (w'/w)^{(j_k)} = 1$. The notation $(w'/w)^{(j_i)}$ stands for the j_i-th derivative of w'/w with $j_i \geqslant 0$.

Proof of Proposition 2.1. — Applying the right hand side of (2.2) to (2.1), the result follows from the Leibniz rule and the fact that

$$D_r\left(b(r,\theta,h\rho,hw(r)\eta)\right) = \left(D_w b\right)(r,\theta,h\rho,hw(r)\eta).$$

We omit the standard details of the calculation. That $(a\#b)_k$ belongs to $S^{m_1+m_2-k}(\mathcal{V}_\iota \times \mathbb{R}^n)$ follows from (1.15) using (2.4). □

We next consider the pseudo-differential quantization $\mathrm{op}^{\iota}_{w,h}(\cdot)$ given by (1.32).

PROPOSITION 2.2. — *Let $a \in S^{m_1}_{\iota}(\mathcal{V}_\iota \times \mathbb{R}^n)$ be polynomial in (ρ, η) and let $b \in S^{m_2}(\mathcal{V}_\iota \times \mathbb{R}^n)$ with $m_2 \in \mathbb{R}$. Let W be a positive function on $\mathbb{R}$ such that*

$$W(r) \leqslant CW(r'), \qquad |r - r'| \leqslant 1. \tag{2.5}$$

Then, for all $N > 0$,

$$a(r, \theta, hD_r, hw(r)D_\theta)\mathrm{op}^{\iota}_{w,h}(b) = \sum_{l=0}^{m_1} h^l \mathrm{op}^{\iota}_{w,h}\left((a\#b)_l\right) + h^{N+1}\overline{R}^{\iota}_N(h, a, b),$$

where, for all $k_1, k_2 \in \mathbb{N}$, all $A_1 \in \mathrm{Diff}^{k_1}_w(\mathcal{M})$, $A_2 \in \mathrm{Diff}^{k_2}_w(\mathcal{M})$ and all $p \in [1, \infty]$,

$$\left|\left|W(r)A_1\Psi^*_\iota \overline{R}^{\iota}_N(h, a, b)\Psi_{\iota *}A_2 W(r)^{-1}\right|\right|_{L^p(\mathcal{M},\widetilde{dg})\to L^p(\mathcal{M},\widetilde{dg})} \lesssim 1, \tag{2.6}$$

$$\left|\left|w(r)^{\frac{n-1}{2}}W(r)A_1\Psi^*_\iota \overline{R}^{\iota}_N(h, a, b)\Psi_{\iota *}A_2 W(r)^{-1}\right|\right|_{L^2(\mathcal{M},\widetilde{dg})\to L^\infty(\mathcal{M})} \lesssim 1, \tag{2.7}$$

for $h \in (0, 1]$. More precisely the norms in (2.6) and (2.7) are controlled by a finite number of semi-norms of a and b independent of h.

Note that the condition (2.5) is satisfied if W is a temperate weight but also by any power of w. In particular, $W(r) = e^{\gamma r}$ is a possible choice although it is not a temperate weight. In particular, (2.6) and (2.7) are respectively equivalent to

$$\left|\left|W(r)A_1\Psi^*_\iota \overline{R}^{\iota}_N(h, a, b)\Psi_{\iota *}A_2 W(r)^{-1}\right|\right|_{L^p(\mathcal{M},dg)\to L^p(\mathcal{M},dg)} \lesssim 1, \tag{2.8}$$

$$\left|\left|W(r)A_1\Psi^*_\iota \overline{R}^{\iota}_N(h, a, b)\Psi_{\iota *}A_2 W(r)^{-1}\right|\right|_{L^2(\mathcal{M},dg)\to L^\infty(\mathcal{M})} \lesssim 1, \tag{2.9}$$

They are simply obtained by replacing $W(r)$ respectively by $W(r)w(r)^{\frac{1-n}{p}}$ and $W(r)w(r)^{\frac{1-n}{2}}$ which both satisfy (2.5).

By opposition to Proposition 2.1, we now have a remainder. It is due to the derivatives of the cutoff near the diagonal in the definition of $\mathrm{op}^{\iota}_{w,h}(\cdot)$ but not to the tail of the expansion $\sum_l h^l(a\#b)_l$ for this sum is finite.

Before proving this proposition, we state two lemmas which will be useful further on and whose proofs are very close to the proofs of the estimates (2.6) and (2.7).

LEMMA 2.3. — *Let $c \in S^m_\iota(\mathcal{V}_\iota \times \mathbb{R}^n)$ with $m < 0$ and let W be a positive function satisfying (2.5). Then, for all $p \in [1, \infty]$, we have*

$$\left|\left|W(r)\mathrm{op}^\iota_{w,h}(c)\,W(r)^{-1}\right|\right|_{L^p(\mathbb{R}^n)\to L^p(\mathbb{R}^n)} \lesssim 1, \qquad h \in (0,1].$$

Proof. — Consider first the case $W \equiv 1$. If $\hat{c}$ is the Fourier transform of c with respect to ρ, η, the kernel of $\mathrm{op}^\iota_{w,h}(c)$ reads

$$C_\iota(r,\theta,r',\theta',h) = h^{-n}w(r)^{1-n}\hat{c}\left(r,\theta,\frac{r'-r}{h},\frac{\theta'-\theta}{hw(r)}\right)\frac{W(r)}{W(r')}\chi^\zeta_\iota(r,\theta,r',\theta').$$

For $(r,\theta) \in \mathcal{V}_\iota$, $c(r,\theta,.,.) \in L^{\epsilon+n/|m|}(\mathbb{R}^n_{\rho,\eta})$, with norm uniformly bounded with respect to (r,θ), thus $\hat{c}(r,\theta,.,.)$ belongs to a bounded subset of $L^1_{loc}(\mathbb{R}^n_{\hat\rho,\hat\eta})$ by Young's theorem. Therefore, for all N we can write

$$|\hat{c}(r,\theta,\hat\rho,\hat\eta)| \leqslant C_N(1+f_0(r,\theta,\hat\rho,\hat\eta))(|\hat\rho|+|\hat\eta|+1)^{-N}, \quad (r,\theta)\in\mathcal{V}_\iota,\ \ \hat\rho\in\mathbb{R},\ \hat\eta\in\mathbb{R}^{n-1}, \tag{2.10}$$

with $f_0(r,\theta,.,.)$ bounded in $L^1_{\mathrm{comp}}(\mathbb{R}^n_{\hat\rho,\hat\eta})$ Thus, the family $\hat{c}(r,\theta,.,.)$ is bounded in $L^1(\mathbb{R}^n_{\hat\rho,\hat\eta})$. Elementary changes of variables show that

$$\sup_{(r,\theta)\in\mathbb{R}^n}\int_{\mathbb{R}}\int_{\mathbb{R}^{n-1}}|C_\iota(r,\theta,r',\theta',h)|dr'd\theta' \lesssim 1,$$
$$\sup_{(r',\theta')\in\mathbb{R}^n}\int_{\mathbb{R}}\int_{\mathbb{R}^{n-1}}|C_\iota(r,\theta,r',\theta',h)|drd\theta \lesssim 1,$$

for $h \in (0,1]$. Recall that C_ι is globally defined on $\mathbb{R}^{2n}$ so the above quantities make sense. The result is then a consequence of the standard Schur lemma. For a general W the same proof applies since we only have to multiply the kernel C_ι by the bounded function $W(r)\chi^\zeta_\iota(r,\theta,r',\theta')W(r')^{-1}$ on the support of which $r - r'$ is bounded. □

LEMMA 2.4. — *Let $c \in S^m_\iota(\mathcal{V}_\iota \times \mathbb{R}^n)$ with $m < -n/2$ and let W be a positive function satisfying (2.5). Then*

$$\left|\left|w(r)^{\frac{n-1}{2}}W(r)\mathrm{op}^\iota_{w,h}(c)\,W(r)^{-1}\right|\right|_{L^2(\mathbb{R}^n)\to L^\infty(\mathbb{R}^n)} \lesssim h^{-n/2}, \qquad h\in(0,1].$$

Proof. — With the notation of the proof of Lemma 2.3, the result is a direct consequence of the estimate

$$\sup_{(r,\theta)\in\mathbb{R}^n}\int_{\mathbb{R}}\int_{\mathbb{R}^{n-1}}|w(r)^{\frac{n-1}{2}}W(r)C_\iota(r,\theta,r',\theta',h)W(r')^{-1}|^2dr'd\theta' \lesssim h^{-n}, \quad h\in(0,1]$$

which follows again from elementary changes of variables, using that $\hat{c}(r,\theta,.,.)$ belongs to a bounded subset of $L^2(\mathbb{R}^n)$ as (r,θ) varies and that $W(r)/W(r')$ is bounded on the support of C_ι. □

Remark. — The proofs of both lemmas still hold if the kernel of $\mathrm{op}^\iota_{w,h}(c)$ is multiplied by a bounded function. We shall use it in the following proof.

Proof of Proposition 2.2. — We may clearly assume that (2.2) is reduced to one term. Applying this operator to (1.32) (with $a = b$) on the r,θ variables, we get the kernel of $\sum_k h^k \mathrm{op}^\iota_{w,h}((a\#b)_k)$ (using Proposition 2.1) plus a linear combination of integrals of the form

$$a_{j\alpha}(r,\theta)\iint e^{i(r-r')\rho+i(\theta-\theta').\eta}(h\rho)^{j_1}(h\eta)^{\alpha_1}(\partial_\theta^{\alpha_2}D_w^{j_2}b)(r,\theta,h\rho,hw(r)\eta)\ d\rho d\eta\partial_r^{j_3}\partial_\theta^{\alpha_3}\chi_\iota^\zeta(r,\theta,r',\theta')$$

where $j_1+j_2+j_3=j$, $\alpha_1+\alpha_2+\alpha_3=\alpha$ and $j_3+|\alpha_3|\geqslant 1$. The latter implies that $\partial_r^{j_3}\partial_\theta^{\alpha_3}\chi_\iota^\zeta$ is supported in $|(r,\theta)-(r',\theta')|\geqslant\varepsilon$ which allows to integrate by parts using $|(r,\theta)-(r',\theta')|^{-2}\Delta_{\rho,\eta}$. We thus obtain integrals of the form

(2.11)
$$h^{2N}\iint e^{i(r-r')\rho+i(\theta-\theta').\eta}c_N(r,\theta,h\rho,hw(r)\eta)\ d\rho d\eta\frac{B_N(r,\theta,r',\theta')}{|(r,\theta)-(r',\theta')|^{2N}}$$

with N as large as we want, $c_N\in S^{m+|\alpha|+j-2N}(\mathcal{V}_\iota\times\mathbb{R}^n)$ and $B_N\in C_b^\infty(\mathbb{R}^{2n})$ with support in $\{\varepsilon\leqslant|(r,\theta)-(r',\theta')|\leqslant 2\varepsilon\}$. With no loss of generality, we may assume that

$$\Psi_{\iota *}A_1\Psi_\iota^* = (w(r)D_\theta)^\beta D_r^k,\qquad \Psi_{\iota *}A_2\Psi_\iota^* = (w(r)D_\theta)^{\beta'}D_r^{k'}.$$

Applying $(w(r)D_\theta)^\beta D_r^k$ to (2.11) yields an integral of the same form, using the boundedness of w and its derivatives. To apply (the transpose of) $(w(r')D_{\theta'})^{\beta'}D_{r'}^{k'}$ to the kernel of $\overline{R}^\iota_N(a,b,h)$, we rewrite this operator as $(w(r')/w(r))^{|\beta'|}(w(r)D_{\theta'})^{\beta'}D_{r'}^{k'}$. We still obtain integrals of the same form as (2.11) multiplied by derivatives of $(w(r')/w(r))^{|\beta'|}$. By (1.14), these derivatives are bounded since $|r-r'|\leqslant 2\varepsilon$ on the support of B_N. Then (2.6) and (2.7) follow respectively from the proofs of Lemma 2.3 and 2.4. □

So far, we have considered the composition with differential operators to the left. Since our operators are properly supported, the composition to the right can be also easily considered.

Proposition 2.5. — *Let a and b be as in Proposition 2.2 and let W be a positive function satisfying (2.5). Then, for all $N>m_1+m_2+n$, we have*

$$\mathrm{op}^\iota_{w,h}(b)a(r,\theta,hDr,hw(r)D_\theta)=\sum_{l=0}^{N}h^l\mathrm{op}^\iota_{w,h}(c_l)+h^{N+1}\underline{R}^\iota_N(h,a,b)$$

with $c_l \in S^{m_1+m_2-l}_\iota(\mathcal{V}_\iota \times \mathbb{R}^n)$ *depending continuously on* a *and* b, *and* $\underline{R}^\iota_N(h,a,b)$ *an operator with continuous kernel supported in* $\mathcal{V}_\iota \times \mathcal{V}_\iota$. *Moreover, for all* N, *all* $k_1, k_2 \in \mathbb{N}$ *such that*

$$N > m_1 + m_2 + n + k_1 + k_2,$$

all $A_1 \in \mathrm{Diff}^{k_1}_w(\mathcal{M})$, $A_2 \in \mathrm{Diff}^{k_2}_w(\mathcal{M})$ *and for all* $p \in [1,\infty]$, *we have*

$$\big|\big|W(r)A_1\Psi^*_\iota \underline{R}^\iota_N(h,a,b)\Psi_{\iota *}A_2 W(r)^{-1}\big|\big|_{L^p(\mathcal{M},\widetilde{dg})\to L^p(\mathcal{M},\widetilde{dg})} \lesssim 1,$$
$$\Big|\Big|W(r)w(r)^{\frac{n-1}{2}}A_1\Psi^*_\iota \underline{R}^\iota_N(h,a,b)\Psi_{\iota *}A_2 W(r)\Big|\Big|_{L^2(\mathcal{M},\widetilde{dg})\to L^\infty(\mathcal{M})} \lesssim 1,$$

for $h \in (0,1]$. *More precisely, these norms are controlled by a finite number of semi-norms of* a *and* b *independent of* h.

We will not need the explicit forms of the symbols c_l since we will only use this proposition for the analysis of some remainder terms.

Note also that the estimates on $\underline{R}^\iota_N(h,a,b)$ have analogues with respect to the measure dg, similar to (2.8) and (2.9),

Proof. — We have to apply the transpose of $a(r',\theta',hD_{r'},hw(r')D_{\theta'})$ to the Schwartz kernel of $\mathrm{op}^\iota_{w,h}(b)$. For simplicity we assume first that $a(r',\theta',\rho,\eta) = w(r')\eta_1$. By Taylor's formula, we can write $w(r')$ as

$$w(r)\left(1+\sum_{j=1}^{N}\frac{1}{j!}\frac{w^{(j)}(r)}{w(r)}(r'-r)^j + \frac{(r'-r)^{N+1}}{N!}\int_0^1 (1-t)^N \frac{w^{(N+1)}(r+t(r'-r))}{w(r)}dt\right).$$

Integrating by parts with respect to ρ in the kernel of $\mathrm{op}^\iota_{w,h}(b)$, the principal part of the Taylor expansion yields the expected expansion with

$$c_l(r,\theta,\rho,\eta) = \frac{1}{j!}D^j_\rho b(r,\theta,\rho,\eta)\frac{w^{(j)}(r)}{w(r)}\eta_1.$$

The remainder is given by two types of terms: first by the derivatives $D_{\theta'_1}$ falling on $\chi^\zeta(r,\theta,r',\theta')$, which yields kernels of the form (2.11), and second by the remainder in the Taylor formula thanks to which we can integrate by parts N times with respect to ρ. In this case, we get a kernel of the form (2.11), with N instead of $2N$ and a symbol $c_N \in S^{m_1+m_2-N}_\iota(\mathcal{V}_\iota \times \mathbb{R}^n)$. Since $r - r'$ is bounded on the support of χ^ζ, $w^{(N)}(r+t(r'-r))/w(r)$ is bounded too, uniformly with respect to $t \in [0,1]$, and the study of the remainder is similar to the one of Proposition 2.2. By induction, we obtain the result if $a = (w(r')\eta)^\alpha$. Derivatives with respect to r or multiplication operators are more standard and studied similarly. □

2.2. Parametrix of the resolvent

In this subsection, we construct a parametrix of the semi-classical resolvent of an operator $P \in \mathrm{Diff}^2_w(\mathcal{M})$. Recall that this means that P is a differential operator of order 2 such that, in any chart at infinity,

$$\Psi_{\iota *} P \Psi_\iota^* = \sum_{k=0}^{2} p^\iota_{2-k}(r, \theta, D_r, w(r) D_\theta) \tag{2.12}$$

with $p^\iota_{2-k} \in S^{2-k}_\iota(\mathcal{V}_\iota \times \mathbb{R}^n)$.

We assume that

$$P \text{ is locally elliptic,} \tag{2.13}$$

ie, in any chart, its principal symbol $p^\iota_{\mathrm{pr}}(x, \xi)$ satisfies $|p^\iota_{\mathrm{pr}}(x, \xi)| \gtrsim |\xi|^2$ for $\xi \in \mathbb{R}^n$, locally uniformly with respect to x. If $\iota \in I_\infty$, using the notation (2.12), we furthermore assume that, for all $K_\iota \Subset V_\iota$ (see (1.5)),

$$|p^\iota_2(r, \theta, \rho, \eta)| \gtrsim \rho^2 + |\eta|^2, \qquad r > R,\ \theta \in K_\iota,\ \rho \in \mathbb{R},\ \eta \in \mathbb{R}^{n-1}. \tag{2.14}$$

Note that this is not a lower bound for the principal symbol of $\Psi_{\iota *} P \Psi_\iota^*$, namely $p^\iota_2(r, \theta, \rho, w(r)\eta)$, whose modulus is only bounded from below by $\rho^2 + w(r)^2|\eta|^2$. This is nevertheless the natural (degenerate) global ellipticity condition in this context. We next define $\mathcal{C} \subset \mathbb{C}$ as

$$\mathcal{C} = \text{closure of the range of the principal symbol of } P, \tag{2.15}$$

which is invariantly defined for the principal symbol is a function on $T^*\mathcal{M}$. We assume that $\mathcal{C} \neq \mathbb{C}$. In the final applications, with $P = -\Delta$ or $-\widetilde{\Delta}_g$, we will of course have $\mathcal{C} = [0, +\infty)$.

We now seek an approximate inverse of $h^2 P - z$, for $h \in (0, 1]$ and $z \in \mathbb{C} \setminus \mathcal{C}$.

We work first in a patch at infinity. Using the notation of (2.12), we set for simplicity

$$p_2 = p^\iota_2 - z, \qquad p_1 = p^\iota_1, \qquad p_0 = p^\iota_0.$$

Observe that p_0, p_1 don't depend on z but that p_2 does. We then have

$$h^2 \Psi_{\iota *} P \Psi_\iota^* - z = \sum_{k=0}^{2} h^k p_{2-k}(r, \theta, hD_r, hw(r) D_\theta).$$

For a given $N \geqslant 0$, we look for symbols $q_{-2}, q_{-3}, \ldots, q_{-2-N}$ satisfying

$$\left(\sum_{k=0}^{2} h^k p_{2-k}(r, \theta, hD_r, hw(r) D_\theta) \right) \left(\sum_{j=0}^{N} h^j \mathrm{op}^\iota_{w,h}(q_{-2-j}) \right) = \chi_\iota + \mathcal{O}(h^{N+1}), \tag{2.16}$$

where χ_ι is defined by (1.8) and where $\mathcal{O}(h^{N+1})$ will be given a precise meaning below. Of course, we need to find such a family of symbols for each patch, ie q_{-2-j} depends on ι, but we omit this dependence for notational simplicity. By Proposition 2.2, the left hand side of (2.16) reads

$$\sum_{k+j+l\leqslant N} h^{k+j+l}\mathrm{op}^\iota_{w,h}\left((p_{2-k}\#q_{-2-j})_l\right) + h^{N+1}R^\iota_N(h,z)$$

where

$$R^\iota_N(h,z) = \sum_{k+j+l\geqslant N+1} h^{k+j+l-N-1}\mathrm{op}^\iota_{w,h}\left((p_{2-k}\#q_{-2-j})_l\right) + \sum_{k,j}\overline{R}^\iota_N(h,h^k p_{2-k},h^j q_{-2-j}), \tag{2.17}$$

with $\overline{R}^\iota_N$ defined in Proposition 2.2. In the above sums, we have $0\leqslant k\leqslant 2$, $0\leqslant j\leqslant N$ and $0\leqslant l\leqslant 2$. Thus, by (1.34), requiring (2.16) leads to the following equations for $q_{-2},\dots,q_{-2-N}$

$$\sum_{k+l+j=\nu}(p_{2-k}\#q_{-2-j})_l = \begin{cases}1 & \text{if } \nu=0,\\ 0 & \text{if } \nu\geqslant 1,\end{cases} \qquad 0\leqslant\nu\leqslant N.$$

This system is triangular and, since $(a\#b)_0 = ab$, its unique solution is given recursively by

$$q_{-2} = \frac{1}{p_2}, \qquad q_{-2-j} = -\frac{1}{p_2}\sum_{\substack{k+j_1+l=j\\ j_1<j}}(p_{2-k}\#q_{-2-j_1})_l \qquad \text{for } j\geqslant 1.$$

PROPOSITION 2.6. — *For all $j\geqslant 1$, q_{-2-j} is a finite sum (with a number of terms $k(j)$ depending on j but not on z) of the form*

$$q_{-2-j} = \sum_{k=1}^{k(j)}\frac{d_{jk}}{p_2^{1+k}}$$

where, for each k, $d_{jk}\in S^{2k-j}_\iota(\mathcal{V}_\iota\times\mathbb{R}^n)$ is a polynomial in ρ,η which is independent of z (in particular $d_{jk}\equiv 0$ when $2k-j<0$). More precisely, the coefficients of these polynomials are linear combinations of products of derivatives of w, w'/w and of the coefficients of p_0,p_1 and $\partial^\alpha p_2$ with $\alpha\neq 0$.

Proof. — This follows from an induction using (2.4) and the fact that, for any multi-index $\alpha\neq 0$, $\partial^\alpha(1/p_2^{1+k})$ is a linear combination of

$$\frac{\partial^{\alpha_1}p_2\cdots\partial^{\alpha_{k'}}p_2}{p_2^{1+k+k'}},$$

with $\alpha_1+\cdots+\alpha_{k'}=\alpha$, $1\leqslant k'\leqslant|\alpha|$ and $\alpha_i\neq 0$ for all $i\in\{1,\dots,k'\}$. □

Using the notation (2.17), we set

$$\mathcal{R}_N^\iota(h,z) = \Psi_\iota^* R_N^\iota(h,z)\Psi_{\iota *}.$$

LEMMA 2.7. — *Let $d\mu$ denote either dg or $\widetilde{dg}$. Then, for all positive function W satisfying (2.5), all $p \in [1,\infty]$ and all $N \geqslant 0$, there exists $\nu > 0$ such that, for all $A \in \mathrm{Diff}_w^m(\mathcal{M})$ and $B \in \mathrm{Diff}_w^{m'}(\mathcal{M})$ with $m+m'-N<0$, we have*

$$\left|\left|W(r)h^m A\mathcal{R}_N^\iota(h,z)h^{m'}BW(r)^{-1}\right|\right|_{L^p(\mathcal{M},d\mu)\to L^p(\mathcal{M},d\mu)} \lesssim \left(\frac{1+|z|}{\mathrm{dist}(z,\mathcal{C})}\right)^\nu,$$

for all $h \in (0,1]$ and all $z \notin \mathcal{C}$.

Proof. — We first assume that $A = B = 1$ (and that $m = m' = 0$). By (1.30), the kernel of $R_N^\iota(h,z)$ is supported in $((R,+\infty)\times K_\iota)^2$ for some $K_\iota \Subset V_\iota$. Thus, using the equivalence of norms (1.26), the result, with $d\mu = \widetilde{dg}$, is a direct consequence of the bound

$$\left|\left|W(r)R_N^\iota(h,z)W(r)^{-1}\right|\right|_{L^p(\mathbb{R}^n)\to L^p(\mathbb{R}^n)} \lesssim \left(\frac{1+|z|}{\mathrm{dist}(z,\mathcal{C})}\right)^\nu, \quad h\in(0,1],\ \ z\notin\mathcal{C}, \tag{2.18}$$

which follows from Proposition 2.2 and Lemma 2.3 once noticed that each semi-norm of q_{-2-j} in $S_\iota^{-2-j}(\mathcal{V}_\iota\times\mathbb{R}^n)$ is bounded by some power of $(1+|z|)/\mathrm{dist}(z,\mathcal{C})$. The latter is due to Proposition 2.6 and

$$\left|\frac{1+\rho^2+\eta^2}{p_2^\iota - z}\right| \approx \left|\frac{1+p_2^\iota}{p_2^\iota - z}\right| \lesssim \frac{1+|z|}{\mathrm{dist}(z,\mathcal{C})},$$

in which we used (2.14). When $d\mu = dg$, we use the equivalence (1.27) so that it is now sufficient to get the bound (2.18) with $R_N^\iota(h,z)$ replaced by $w(r)^{\frac{1-n}{p}}R_N^\iota(h,z)w(r)^{\frac{n-1}{p}}$. The latter is clear for this amounts to multiply the kernel of $R_N^\iota(h,z)$ by $(w(r')/w(r))^{(n-1)/p}$ (which is bounded, using the boundedness of $r-r'$ on the support of χ_ι^ζ and (1.14)) so the (proofs of) Proposition 2.2 and Lemma 2.3 still hold.

For general A and B, we use Propositions 2.2 and 2.5 so that we are reduced to the previous case with an operator of the same form as $R_N^\iota(h,z)$ except that the symbols of the first sum in (2.17) now belong to $S_\iota^{-N+m+m'}(\mathcal{V}_\iota\times\mathbb{R}^n)$. We can apply Lemma 2.3 to this term and the result follows. □

Let us now define

$$Q_N^\iota(h,z) = \sum_{j=0}^N h^j \mathrm{op}_{w,h}^\iota(q_{-2-j}), \qquad \mathcal{Q}_N^\iota(h,z) = \Psi_\iota^* Q_N^\iota(h,z)\Psi_{\iota *}.$$

Then, with f_ι given by (1.6), we obtain the relation

$$(h^2P - z)\mathcal{Q}_N^\iota(h,z) = f_\iota + h^{N+1}\mathcal{R}_N^\iota(h,z). \tag{2.19}$$

So far, we have always assumed that $\iota \in I_\infty$, ie worked in patches at infinity, but the same analysis still holds for relatively compact patches, ie for $\iota \in I_{\mathrm{comp}}$. We don't give the details of the construction in the latter case for two reasons: the first is that this is essentially well known for this is like working on a compact manifold and the second is that the proofs are formally the same with the simpler assumptions that $w \equiv 1$ and that χ_ι is compactly supported.

Thus, by setting

$$\mathcal{Q}_N(h,z) = \sum_{\iota\in I} \mathcal{Q}_N^\iota(h,z), \qquad \mathcal{R}_N(h,z) = \sum_{\iota\in I} \mathcal{R}_N^\iota(h,z),$$

then summing the equalities (2.19) over I and using (1.6), Lemma 2.7 gives the following result where we recall that $\mathcal{C}$ is defined by (2.15).

THEOREM 2.8. — *Let $P \in \mathrm{Diff}^2_w(\mathcal{M})$ be a second order differential operator satisfying (2.13) and (2.14). Then, for all $N \geqslant 0$, we have*

$$(h^2P - z)\mathcal{Q}_N(h,z) = 1 + h^{N+1}\mathcal{R}_N(h,z), \qquad h \in (0,1], \quad z \notin \mathcal{C}. \tag{2.20}$$

If $d\mu$ denotes either dg or $\widetilde{dg}$, and $m, m' \in \mathbb{N}$ satisfy $m + m' < N$, then for all $p \in [1,\infty]$ and for all positive function W satisfying (2.5), there exists $\nu \geqslant 0$ such that, for all $A \in \mathrm{Diff}^m_w(\mathcal{M})$ and $B \in \mathrm{Diff}^{m'}_w(\mathcal{M})$, we have

$$\left|\left| W(r)h^m A\mathcal{R}_N(h,z)h^{m'}BW(r)^{-1} \right|\right|_{L^p(\mathcal{M},d\mu)\to L^p(\mathcal{M},d\mu)} \lesssim \left(\frac{1+|z|}{\mathrm{dist}(z,\mathcal{C})}\right)^\nu, \tag{2.21}$$

for all $h \in (0,1]$ and all $z \notin \mathcal{C}$.

This theorem gives a parametrix of the resolvent of h^2P under the natural ellipticity conditions (2.13) and (2.14) (recall that if w is not bounded from below, this corresponds to a degenerate ellipticity).

From now on, we assume that

$$P \text{ is self-adjoint with respect to } d\mu = dg \text{ or } \widetilde{dg}.$$

This condition is actually equivalent to the symmetry of P on $C_0^\infty(\mathcal{M})$. Indeed, (2.20) and (2.21) implies that $h^2P \pm i$ is injective for h small enough, which shows that P is essentially self-adjoint.

The resolvent $(h^2P - z)^{-1}$ is then well defined for all $z \notin \mathbb{R}$ and

$$(h^2P - z)^{-1} = \mathcal{Q}_N(z,h) - h^{N+1}(h^2P - z)^{-1}\mathcal{R}_N(h,z), \qquad z \notin \mathbb{R}, \ \ h \in (0,1]. \tag{2.22}$$

Theorem 2.8 implies, for $z = i$, that in the operator norm on $L^2(\mathcal{M}, d\mu)$, we have

$$||h^{N+1}\mathcal{R}_N(h,i)||_{L^2(\mathcal{M},d\mu)\to L^2(\mathcal{M},d\mu)} \lesssim h^{N+1}, \qquad h \in (0,1].$$

Thus, for some $h_0 > 0$ small enough and some bounded operator B_1 on $L^2(\mathcal{M}, d\mu)$, we get

$$(h_0^2 P - i)^{-1} = \mathcal{Q}_N(i, h_0)B_1.$$

More generally, for $k \geqslant 1$, we can write

$$(h^2P - z)^{-k} = \frac{1}{(k-1)!}\partial_z^{k-1}(h^2P - z)^{-1}$$

so applying $(k-1)!^{-1}\partial_z^{k-1}$ to (2.22) shows that $(h^2P - z)^{-k}$ reads

(2.23)
$$(k-1)!^{-1}\partial_z^{k-1}\mathcal{Q}_N(z,h) + h^{N+1}(h^2P - z)^{-k}\sum_{j=0}^{k-1}\frac{1}{j!}(h^2P - z)^j\partial_z^j\mathcal{R}_N(z,h),$$

using the holomorphy of $\mathcal{Q}_N(z,h)$ and $\mathcal{R}_N(z,h)$ with respect to $z \in \mathbb{C}\setminus\mathbb{R}$ which standardly follows from Proposition 2.6. Therefore, by choosing N large enough so that the sum above is bounded on L^2 (uniformly in h) and choosing then $h = h_0$ small enough, we obtain

$$(h_0^2P - i)^{-k} = (k-1)!^{-1}\partial_z^{k-1}\mathcal{Q}_N(z,h_0)_{|z=i}B_k, \tag{2.24}$$

for some operator B_k bounded on $L^2(\mathcal{M}, d\mu)$.

Lemma 2.9. — *For all $A \in \mathrm{Diff}_w^{2k}(\mathcal{M})$, $A\partial_z^{k-1}\mathcal{Q}_N(z,h_0)$ is bounded on $L^2(\mathcal{M}, dg)$ and $L^2(\mathcal{M}, \widetilde{dg})$.*

Proof. — Consider first the case of $\widetilde{dg}$. By Proposition 2.6, for all $\iota \in I$, $Q_\iota := \partial_z^{k-1}\mathcal{Q}_N^\iota(z,h_0)$ is of the form $\Psi_\iota^*\mathrm{op}_{w,h}^\iota(q_\iota)\Psi_{\iota *}$ for some symbol $q_\iota \in S_\iota^{-2k}(\mathcal{V}_\iota \times \mathbb{R}^n)$. A direct calculation shows that $\Psi_{\iota *}A\Psi_\iota^*\mathrm{op}_{w,h}^\iota(q_\iota)$ has a kernel of the form

$$(2\pi)^{-n}\int e^{i(x-y)\cdot\xi}a^\iota(x,y,\xi)d\xi,$$

with $a^\iota \in C_b^\infty(\mathbb{R}^{3n})$. Hence, the corresponding operator is bounded on $L^2(\mathbb{R}^n)$ by the Calderòn-Vaillancourt theorem and thus its pullback AQ_ι is bounded on $L^2(\mathcal{M}, \widetilde{dg})$. The boundedness of AQ_ι on $L^2(\mathcal{M}, dg)$ is equivalent to the one of $w(r)^{(1-n)/2}AQ_\iota w(r)^{(n-1)/2}$ on $L^2(\mathcal{M}, \widetilde{dg})$. The latter follows from the same reasoning since $w(r)^{(1-n)/2}Aw(r)^{(n-1)/2} \in \mathrm{Diff}_w^{2k}(\mathcal{M})$ and Q_ι is properly supported. □

By setting $||\cdot|| = ||\cdot||_{L^2(\mathcal{M},d\mu)\to L^2(\mathcal{M},d\mu)}$ we obtain

$$\begin{aligned}\left|\left|h^{2k}A(h^2P-z)^{-k}\right|\right| &\leqslant \left|\left|A(h_0^2P-i)^{-k}\right|\right| \left|\left|h^{2k}\right|\right| \left|\left|(h_0^2P-i)^k(h^2P-z)^{-k}\right|\right| \\ &\leqslant Ch^{2k}\sup_{\lambda\in\mathbb{R}}\left|\frac{h_0^2\lambda+i}{h^2\lambda-z}\right|^k,\end{aligned}$$

using (2.24), Lemma 2.9 and the Spectral Theorem in the last line. By estimating the sup in the right hand side, we obtain the following result.

PROPOSITION 2.10. — *Let $P\in \mathrm{Diff}^2_w(\mathcal{M})$ satisfy* (2.13) *and* (2.14)*, and be self-adjoint with respect to $d\mu = dg$ or $\widetilde{dg}$. Then, for all $k\geqslant 1$ and all $A\in \mathrm{Diff}^{2k}_w(\mathcal{M})$, we have*

$$\left|\left|h^{2k}A(h^2P-z)^{-k}\right|\right|_{L^2(\mathcal{M},d\mu)\to L^2(\mathcal{M},d\mu)} \lesssim \frac{\langle z\rangle^k}{|\mathrm{Im}\ z|^k}, \qquad z\notin\mathbb{R},\quad h\in(0,1].$$

In the same spirit, we will prove the following Sobolev embeddings.

PROPOSITION 2.11. — *Let P be as in Proposition 2.10 and let $k > n/4$ be an integer. Then, if P is self-adjoint with respect to $d\mu = dg$, we have*

$$||(h^2P-z)^{-k}||_{L^2(\mathcal{M},dg)\to L^\infty(\mathcal{M})} \lesssim h^{-\frac{n}{2}}\frac{\langle z\rangle^k}{|\mathrm{Im}\ z|^k}, \quad z\notin\mathbb{R},\quad h\in(0,1].$$

If it is self-adjoint with respect to $d\mu = \widetilde{dg}$, we have

$$||w(r)^{\frac{n-1}{2}}(h^2P-z)^{-k}||_{L^2(\mathcal{M},\widetilde{dg})\to L^\infty(\mathcal{M})} \lesssim h^{-\frac{n}{2}}\frac{\langle z\rangle^k}{|\mathrm{Im}\ z|^k}, \quad z\notin\mathbb{R},\ h\in(0,1].$$

Of course, by taking the adjoints, we have the corresponding $L^1\to L^2$ inequalities.

Proof. — We assume that $d\mu = dg$. By Lemma 2.4 with $W(r) = w(r)^{\frac{1-n}{2}}$, we have

$$\begin{aligned}\left|\left|\partial_z^{k-1}\mathcal{Q}_N(i,h_0)v\right|\right|_{L^\infty(\mathcal{M})} &= \left|\left|\left(\partial_z^{k-1}\mathcal{Q}_N(z,h_0)_{|z=i}W(r)^{-1}\right)W(r)v\right|\right|_{L^\infty(\mathcal{M})} \\ &\leqslant C||v||_{L^2(\mathcal{M},dg)},\end{aligned}$$

using the equivalence of norms (1.27). Using (2.24), this implies that $(h_0^2P-i)^{-k}$ is bounded from $L^2(\mathcal{M},dg)$ to $L^\infty(\mathcal{M})$. Therefore, by writting

$$(h^2P-i)^{-k} = (h_0^2P-i)^{-k}\frac{(h_0P-i)^k}{(h^2P-i)^k},$$

we get

(2.25)
$$||(h^2P-i)^{-k}||_{L^2(\mathcal{M},dg)\to L^\infty(\mathcal{M})} \leqslant C \left|\left| \frac{(h_0^2P-i)^k}{(h^2P-i)^k} \right|\right|_{L^2(\mathcal{M},dg)\to L^2(\mathcal{M},dg)} \lesssim h^{-2k},$$

the last upper bound following from the Spectral Theorem. Using (2.23) with $z=i$, we can write

$$(h^2P-i)^{-k} = (k-1)!\partial_z^{k-1}\mathcal{Q}_N(i,h) + h^{N+1}(h^2P-i)^{-k}\mathcal{R}_{N,k}(i,h) \tag{2.26}$$

where, by Lemma 2.7, $||\mathcal{R}_{N,k}(i,h)||_{L^2(\mathcal{M},dg)\to L^2(\mathcal{M},dg)}$ is bounded uniformly in h if N is large enough. On the other hand, we also know by Lemma 2.4 that

$$||\partial_z^{k-1}\mathcal{Q}_N(i,h)|| \lesssim h^{-n/2}.$$

Thus by choosing N large enough so that $N+1-2k \geqslant -n/2$ and by using (2.25) for the resolvent in the right hand side of (2.26), we obtain the improved estimate

$$||(h^2P-i)^{-k}||_{L^2(\mathcal{M},dg)\to L^\infty(\mathcal{M})} \lesssim h^{-n/2}.$$

We then obtain the result from the estimate

$$\begin{aligned} &||(h^2P-z)^{-k}||_{L^2(\mathcal{M},dg)\to L^\infty(\mathcal{M})} \\ &\quad\leqslant ||(h^2P-i)^{-k}||_{L^2(\mathcal{M},dg)\to L^\infty(\mathcal{M})} \left|\left| \frac{(h^2P-i)^k}{(h^2P-z)^k} \right|\right|_{L^2(\mathcal{M},dg)\to L^2(\mathcal{M},dg)} \end{aligned}$$

whose second norm in right hand side is of order $\langle z\rangle^k/|\mathrm{Im}(z)|^k$ independently of h, by the Spectral Theorem. The case of $d\mu = \widetilde{dg}$ is similar. □

2.3. Proof of Theorem 1.5

We shall use the classical Helffer-Sjöstrand formula

$$\varphi(H) = \frac{1}{\pi}\iint_{\mathbb{R}^2} \bar\partial\widetilde\varphi(x+iy)(H-x-iy)^{-1}dxdy \tag{2.27}$$

with $\bar\partial = (\partial_x + i\partial_y)/2$, valid for any self-adjoint operator H. Here $\widetilde\varphi \in C^\infty(\mathbb{C})$ is an almost analytic extension of φ, ie such that $\widetilde\varphi|_{\mathbb{R}} = \varphi$ and $\bar\partial\widetilde\varphi(z)$ vanishes to sufficiently high order on the real axis.

A justification of this formula for $\varphi \in C_0^\infty(\mathbb{R})$ can be found in [11]. It is shown in [10] that, if $\varphi \in S^{-\sigma}(\mathbb{R})$ with $\sigma > 0$, (2.27) holds with $\widetilde{\varphi}_M$ defined by

$$\widetilde{\varphi}_M(x+iy) \quad = \quad \chi_0(y/\langle x\rangle)\sum_{k=0}^{M} f^{(k)}(x)\frac{(iy)^k}{k!}, \tag{2.28}$$

with $M \geqslant 1$ and $\chi_0 \in C_0^\infty(\mathbb{R})$ such that $\chi_0 \equiv 1$ near 0. With this choice, one has

$$|\bar\partial\widetilde{\varphi}_M(x+iy)| \lesssim |y|^M/\langle x\rangle^{\sigma+1+M}, \qquad x, y \in \mathbb{R}. \tag{2.29}$$

This implies in particular that, for all integers $\nu_1 \geqslant 1$, $\nu_2 \geqslant 0$ and $M \geqslant \nu_1 + \nu_2$, we have

$$\iint_{\mathbb{R}^2} |\bar\partial\widetilde{\varphi}_M(x+iy)| \times |y|^{-\nu_1}\left(\frac{1+|x|+|y|}{|y|}\right)^{\nu_2} dxdy < \infty, \tag{2.30}$$

which is easily seen by splitting the integral into two parts, where $|y| \leqslant 1$ or $|y| > 1$, using the fact that $|y|/\langle x\rangle$ is bounded on the support of $\widetilde{\varphi}_M$ in the latter case. If $\sigma > 1$ and $M \geqslant \nu$, we also have

$$\iint_{\mathbb{R}^2} |\bar\partial\widetilde{\varphi}_M(x+iy)|\left(\frac{1+|x|+|y|}{|y|}\right)^{\nu} dxdy < \infty. \tag{2.31}$$

Proof of Theorem 1.5. — Let $\iota \in I$. The form of $\Psi_{\iota *}\mathcal{Q}_N^\iota(P,\varphi,h)\Psi_\iota^*$, namely (1.35), simply follows by plugging the expansion (2.22) into (2.27) and applying Green's formula. For the latter we use Proposition 2.6 (recalling that $p_2 \equiv p_2^\iota - z$). All the integrals make sense by (2.30) if we choose $\widetilde{\varphi}_M$ with $M \geqslant \max_{j\leqslant N}(k(j)+1)$.

Let us now prove (1.36) and (1.37). Since the proofs are very similar we only show (1.37) and thus consider $P = -\widetilde{\Delta}_g$. Fix $N \geqslant 0$. For $N' > N$ and M large enough, both to be chosen later, we set

$$\mathcal{R}_{N'}(P,\varphi,h) = \frac{1}{\pi}\iint_{\mathbb{R}^2} \partial_{\bar z}\widetilde{\varphi}_M(x+iy)(h^2P - x - iy)^{-1}\mathcal{R}_{N'}(x+iy,h)dxdy. \tag{2.32}$$

We next fix two integers $k > n/4$, $\tilde m \geqslant m/2$, and rewrite $h^m A(h^2P - z)^{-1}$ (with $z = x+iy$) as

$$(h^2P - i)^{-k}\left\{(h^2P - i)^k h^m A(h^2P - i)^{-k-\tilde m}\right\}(h^2P - z)^{-1}(h^2P - i)^{k+\tilde m}. \tag{2.33}$$

Using Proposition 2.10, the term $\{\cdots\}$ is bounded on $L^2(\mathcal{M}, \widetilde{dg})$ uniformly with respect to h since, for any $0 \leqslant j \leqslant k$, $P^j A$ belongs to $\mathrm{Diff}_w^{2j+m}(\mathcal{M})$.

On the other hand, by Theorem 2.8, there exists $\nu_2 > 0$ such that

$$\Big|\Big|(h^2P - i)^{k+\tilde{m}}\mathcal{R}_{N'}(z,h)h^{m'}B\Big|\Big|_{L^2(\mathcal{M},\widetilde{dg})\to L^2(\mathcal{M},\widetilde{dg})} \lesssim \langle z\rangle^{\nu_2}/|\mathrm{Im}\ z|^{\nu_2},$$

for $z \notin \mathbb{R}$ and $h \in (0,1]$, provided

$$N' > m' + 2(k + \tilde{m}). \tag{2.34}$$

By Propositions 2.10 and 2.11, we therefore get, for $p \in \{2,\infty\}$,

$$\Big|\Big|w(r)^{\frac{n-1}{2}-\frac{n-1}{p}}h^m A(h^2P - z)^{-1}\mathcal{R}_{N'}(z,h)h^{m'}B\Big|\Big|_{L^2(\mathcal{M},\widetilde{dg})\to L^p(\mathcal{M},\widetilde{dg})} \lesssim \frac{h^{n(1/p-1/2)}}{|\mathrm{Im}\ z|}\left(\frac{\langle z\rangle}{|\mathrm{Im}\ z|}\right)^{n_2},$$

where the extra power of $|\mathrm{Im}(z)|^{-1}$ comes from the term $(h^2P - z)^{-1}$ in (2.33). Using (2.30), this estimate clearly proves that, for $p \in \{2,\infty\}$,

$$\Big|\Big|w(r)^{\frac{n-1}{2}-\frac{n-1}{p}}h^m A\mathcal{R}_{N'}(P,\varphi,h)h^{m'}B\Big|\Big|_{L^2(\mathcal{M},\widetilde{dg})\to L^p(\mathcal{M},\widetilde{dg})} \lesssim h^{-n(1/2-1/p)},$$

if we choose $M \geqslant \nu_2 + 1$ in (2.32). Then, define $\mathcal{Q}_{NN'}(P,\varphi,h)$ by

$$\sum_{\iota\in\iota}\mathcal{Q}^{\iota}_{N'}(P,\varphi,h) = \sum_{\iota\in\iota}\mathcal{Q}^{\iota}_{N}(P,\varphi,h) + h^{N+1}\mathcal{Q}_{NN'}(P,\varphi,h). \tag{2.35}$$

Using the explicit form of $\mathcal{Q}_{NN'}(P,\varphi,h)$, namely the fact that its symbol is a linear combination of terms of the form $a(r,\theta,\rho,w(r)\eta)$ with $a \in S^{-2\sigma-N}$ (this is due to (1.35)), one has

$$\Big|\Big|w(r)^{\frac{n-1}{2}-\frac{n-1}{p}}h^m A\mathcal{Q}_{NN'}(\varphi,h)h^{m'}B\Big|\Big|_{L^2(\mathcal{M},\widetilde{dg})\to L^p(\mathcal{M},\widetilde{dg})} \lesssim h^{-n\left(\frac{1}{2}-\frac{1}{p}\right)}, \quad h \in (0,1],$$

which is a consequence of Propositions 2.2, 2.5 and of Lemmas 2.3 and 2.4. Since

$$\mathcal{R}_N(P,\varphi,h) = h^{N'-N}\mathcal{R}_{N'}(P,\varphi,h) + \mathcal{Q}_{NN'}(P,\varphi,h), \tag{2.36}$$

by choosing N' such that $N' - N - 2k \geqslant -n/2 + n/p$ and (2.34) holds, we get (1.37) for $p = 2$ or ∞. The other cases follow by interpolation. □

3. L^p bounds for the resolvent

Consider a temperate weight W in the sense of Definition 1.6. The main purpose of this section is to prove the following theorem.

THEOREM 3.1. — *For all $1 < p < \infty$, there exists $\nu_p > 0$ such that*

$$||W(r)(z - \widetilde{\Delta}_g)^{-1} W(r)^{-1}||_{\mathcal{L}(L^2(\mathcal{M}, \widetilde{dg}))} \lesssim \left(\frac{\langle z \rangle}{|\mathrm{Im}\ z|} \right)^{\nu_p},$$

for all $z \in \mathbb{C} \setminus \mathbb{R}$.

Recall that $\widetilde{\Delta}_g$ is defined by (1.25) and is self-adjoint with respect to $\widetilde{dg}$ given by (1.22).

Translated in terms of Δ_g, Theorem 3.1 gives

COROLLARY 3.2. — *For all $1 < p < \infty$, there exists $\nu_p > 0$ such that*

$$||W(r)w(r)^{(n-1)(\frac{1}{p}-\frac{1}{2})}(z - \Delta_g)^{-1} w(r)^{(1-n)(\frac{1}{p}-\frac{1}{2})} W(r)^{-1}||_{\mathcal{L}(L^2(\mathcal{M}, dg))} \lesssim \left(\frac{\langle z \rangle}{|\mathrm{Im}\ z|} \right)^{\nu_p},$$

for all $z \in \mathbb{C} \setminus \mathbb{R}$.

Theorem 3.1 is a consequence of Proposition 3.8 showing a stronger result, namely that, in local charts, $(z - \widetilde{\Delta}_g)^{-1}$ is a pseudo-differential operator with symbol in a class that guarantees the L^p boundedness on $L^p(\mathcal{M}, \widetilde{dg})$. Using Proposition 3.8, we also obtain the following result.

THEOREM 3.3. — *If w is itself a temperate weight, then for all temperate weight W and all $1 < p < \infty$, there exists $\nu_p > 0$ such that*

$$||W(r)(z - \Delta_g)^{-1} W(r)^{-1}||_{\mathcal{L}(L^2(\mathcal{M}, dg))} \lesssim \left(\frac{\langle z \rangle}{|\mathrm{Im}\ z|} \right)^{\nu_p},$$

for all $z \in \mathbb{C} \setminus \mathbb{R}$.

This holds in particular if $W \equiv 1$.

3.1. Reduction

In this subsection, we explain how to reduce Theorem 3.1 to Proposition 3.8 below. This reduction rests on classical results on pseudo-differential operators, namely the Calderòn-Zygmund Theorem 3.4 and the Beals Theorem 3.6.

Recall first the definitions of the usual classes of symbols S^0 and S^0_0:

$$a \in S^0(\mathbb{R}^d \times \mathbb{R}^d) \Leftrightarrow |\partial_x^\alpha \partial_\xi^\beta a(x, \xi)| \lesssim \langle \xi \rangle^{-|\beta|}, \tag{3.1}$$

$$a \in S^0_0(\mathbb{R}^d \times \mathbb{R}^d) \Leftrightarrow |\partial_x^\alpha \partial_\xi^\beta a(x, \xi)| \lesssim 1. \tag{3.2}$$

The following theorem is due to Calderòn-Zygmund.

THEOREM 3.4. — *Let $d \geqslant 1$ and $a \in S^0(\mathbb{R}^d \times \mathbb{R}^d)$. Then, for all $1 < p < \infty$,*

$$||a(x,D)v||_{L^p(\mathbb{R}^d)} \leqslant C_p ||v||_{L^p(\mathbb{R}^d)}, \qquad v \in C_0^\infty(\mathbb{R}^d),$$

where the constant C_p depends on a finite number of semi-norms of a in S^0.

For a proof, see for instance [24].

We next introduce the class $S^{-2,0}_{0,1}(\mathbb{R}^{n+1}\times\mathbb{R}^n)$ of functions $b(x_1, x_1', y, \rho, \eta)$ satisfying

$$\left|\partial_{x_1}^j \partial_{x_1'}^{j'} \partial_y^\alpha \partial_\rho^k \partial_\eta^\beta b(x_1, x_1', y, \rho, \eta)\right| \leqslant C_{j\alpha k\beta} \langle\rho\rangle^{-2} \langle\eta\rangle^{-|\beta|}, \tag{3.3}$$

for $x_1, x_1' \in \mathbb{R}$, $y \in \mathbb{R}^{n-1}$, and $(\rho, \eta) \in \mathbb{R} \times \mathbb{R}^{n-1}$. In particular, for fixed x_1, x_1', ρ, these functions belong to $S^0(\mathbb{R}^{n-1}_y \times \mathbb{R}^{n-1}_\eta)$. Consider the pseudo-differential operator B defined on $\mathbb{R}^n$ by the Schwartz kernel

$$K_B(x_1, y, x_1', y') = (2\pi)^{-n} \int e^{i(y-y')\cdot\eta} \hat{b}(x_1, x_1', y, x_1' - x_1, \eta) d\eta \tag{3.4}$$

where $\hat{b}$ is the Fourier transform of b with respect to ρ. This kernel is continuous with respect to x_1, x_1' (with values in $\mathcal{S}'(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1})$). Integrating by parts with $(x_1 - x_1')^{-1}\partial_\rho$ in the integral defining $\hat{b}$, one sees that, for all N and all α, β,

$$|\partial_\theta^\alpha \partial_\eta^\beta \hat{b}(x_1, y, x_1' - x_1, \eta)| \leqslant C_{N\alpha\beta} \langle x_1 - x_1'\rangle^{-N} \langle\eta\rangle^{-|\beta|}. \tag{3.5}$$

Thus, for all $1 < p < \infty$ and $N > 0$, Theorem 3.4 yields the existence of C_{Np} such that

$$||(Bv)(x_1, .)||_{L^p(\mathbb{R}^{n-1})} \leqslant C_{Np} \int \langle x_1 - x_1'\rangle^{-N} ||v(x_1', .)||_{L^p(\mathbb{R}^{n-1})} dx_1', \tag{3.6}$$

for all $v \in C_0^\infty(\mathbb{R}_{x_1'} \times \mathbb{R}^{n-1}_{y'})$. Denoting by p' the conjugate exponent to p, Hölder's inequality yields

$$\begin{aligned} &||(Bv)(x_1, .)||^p_{L^p(\mathbb{R}^{n-1})} \\ &\quad \lesssim \left(\int \langle x_1 - x_1'\rangle^{-N} dx_1'\right)^{\frac{p}{p'}} \left(\int \langle x_1 - x_1'\rangle^{-N} ||v(x_1', .)||^p_{L^p(\mathbb{R}^{n-1})} dx_1'\right) \end{aligned}$$

and thus, if $N > 1$, we conclude that

$$||Bv||^p_{L^p(\mathbb{R}^n)} \lesssim \iint \langle x_1 - x_1'\rangle^{-N} ||v(x_1', .)||^p_{L^p(\mathbb{R}^{n-1})} dx_1' dx_1 \lesssim ||v||^p_{L^p(\mathbb{R}^n)}, \tag{3.7}$$
$$v \in C_0^\infty(\mathbb{R}^n).$$

More generally, if W is a temperate weight, estimates of the form (3.5) still hold if we replace $\hat{b}(x_1, y, x_1' - x_1, \eta)$ by $W(x_1)\hat{b}(x_1, y, x_1' - x_1, \eta)W(x_1')^{-1}$. All this gives the following result.

PROPOSITION 3.5. — *If $b \in S^{-2,0}_{0,1}(\mathbb{R}^{n+1} \times \mathbb{R}^n)$ and B is defined by the kernel (3.4), then for all temperate weight W, $W(x_1)BW(x_1)^{-1}$ is bounded on $L^p(\mathbb{R}^n)$ for all $1 < p < \infty$, and its norm depends on a finite number of constants $C_{j\alpha k\beta}$ in (3.3).*

We shall essentially prove Theorem 3.1 by showing that the pull-backs on $\mathbb{R}^n$ of $(z - \widetilde{\Delta}_g)^{-1}$ by local charts are pseudo-differential operators with symbols in $S^{-2,0}_{0,1}(\mathbb{R}^{n+1} \times \mathbb{R}^n)$. The main tool to characterize these pull-packs as pseudo-differential operators on $\mathbb{R}^n$ is the Beals criterion which we recall in Theorem 3.6 below. Let us fix first some notation. If A and L are operators on suitable spaces, we set

$$ad_L \cdot A = LA - AL.$$

In our case, L will typically belong to

$$\mathcal{L}_{\mathbb{R}^n} = \{x_1, \dots, x_n, \partial_{x_1}, \dots, \partial_{x_n}\}.$$

THEOREM 3.6 (Beals). — *Let $A : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ be a continuous linear map. If A is bounded on $L^2(\mathbb{R}^n)$ and, more generally, for all N and all $L_1, \dots, L_N \in \mathcal{L}_{\mathbb{R}^n}$, if the operator $ad_{L_1} \dots ad_{L_N} \cdot A$ is bounded on $L^2(\mathbb{R}^n)$, then there exists $a \in S^0_0$ such that*

$$A = a^W(x, D),$$

and each semi-norm of a in S^0_0 is controlled by a finite number of $||ad_{L_1} \dots ad_{L_N} \cdot A||_{L^2 \to L^2}$.

Here $a^W(x, D)$ is the Weyl quantization of a namely the operator whose kernel is

$$(2\pi)^{-n} \int e^{i(x-x')\cdot\xi} a\big((x + x')/2, \xi\big) d\xi.$$

Theorem 3.6 is for instance proved in [2, 3, 11].

The characterization of operators with symbols in $S^{-2,0}_{0,1}(\mathbb{R}^{n+1} \times \mathbb{R}^n)$ is easily deduced from this theorem as follows. Recall first the formula

$$(\partial_x^\alpha \partial_\xi^\beta a)^W(x, D) = i^{-|\beta|} ad_{\partial_x}^\alpha ad_x^\beta \cdot a^W(x, D), \tag{3.8}$$

where $ad_x^\alpha = ad_{x_1}^{\alpha_1} \dots ad_{x_n}^{\alpha_n}$ and $ad_{\partial_x}^\beta = ad_{\partial_{x_1}}^{\beta_1} \dots ad_{\partial_{x_n}}^{\beta_n}$ (note that $ad_{L_1} ad_{L_2} = ad_{L_2} ad_{L_1}$ for all $L_1, L_2 \in \mathcal{L}_{\mathbb{R}^n}$). On the other hand, we also have

$$(\xi_j a)^W(x, D) = D_j a^W(x, D) - \frac{1}{2i}(\partial_{x_j} a)^W(x, D). \tag{3.9}$$

PROPOSITION 3.7. — *Let $A : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ be linear and continuous. Assume that, for all $\alpha, \beta \in \mathbb{N}^n$ and all $\gamma \in \mathbb{N}^n$ such that*

$$\gamma_1 \leqslant 2, \qquad \gamma_2 + \cdots + \gamma_n \leqslant \beta_2 + \cdots + \beta_n,$$

the operator

$$A^\gamma_{\alpha\beta} := D^\gamma_x \left(ad^\alpha_{\partial_x} ad^\beta_x \cdot A \right) \tag{3.10}$$

is bounded on $L^2(\mathbb{R}^n)$. Then A is a pseudo-differential operator with symbol $a \in S^{-2,0}_{0,1}(\mathbb{R}^{n+1} \times \mathbb{R}^n)$ (ie has a kernel of the form (3.4)). Each semi-norm of a in $S^{-2,0}_{0,1}(\mathbb{R}^{n+1} \times \mathbb{R}^n)$ depends on a finite number of operator norms $||A^\gamma_{\alpha\beta}||_{L^2 \to L^2}$.

Proof. — Set $B = (1 + D^2_{x_1})A$. By Theorem 3.6, we can write $B = b^W(x, D)$ for some $b \in S^0_0$ since $B = A^0_{00} + A^\gamma_{00}$, with $\gamma = (2, 0, \ldots, 0)$, which is bounded on L^2 as well as $ad^\alpha_{\partial_x} ad^\beta_x B$ since $ad^\alpha_{\partial_x} ad^\beta_x$ commute with the composition with $\partial^2_{x_1}$. Define then $B^\gamma_{\alpha\beta}$ similarly to (3.10) with B instead of A and with $\gamma = (0, \gamma_2, \ldots, \gamma_n)$. By (3.8) and (3.9), $B^\gamma_{\alpha\beta}$ is the sum of

$$i^{-|\beta|} \left(\xi^\gamma \partial^\alpha_x \partial^\beta_\xi b \right)^W (x, D),$$

and of a linear combination of operators of the form

$$\left(\xi^{\gamma'} \partial^{\alpha'}_x \partial^\beta_\xi b \right)^W (x, D), \qquad \gamma' < \gamma, \quad \alpha' \leqslant \alpha + \gamma.$$

On the other hand, by Theorem 3.6 again, $B^\gamma_{\alpha\beta}$ is of the form $(b^\gamma_{\alpha\beta})^W(x, D)$ for some $b^\gamma_{\alpha\beta} \in S^0_0$. Thus

$$b^\gamma_{\alpha\beta}(x, \xi) = i^{-|\beta|} \xi^\gamma \partial^\alpha_x \partial^\beta_\xi b(x, \xi) + \sum_{\substack{\gamma' < \gamma, \\ \alpha' \leqslant \alpha + \gamma}} c_{\gamma'\alpha'} \xi^{\gamma'} \partial^{\alpha'}_x \partial^\beta_\xi b(x, \xi).$$

By induction on β, we deduce that

$$|\partial^\alpha_x \partial^\beta_\xi b(x, \xi)| \lesssim (1 + |\xi_2| + \cdots + |\xi_n|)^{-\beta_2 - \cdots - \beta_n}. \tag{3.11}$$

Using then the standard fact that any $c^W(y, D_y)$, with $c \in S^0(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1})$, can be written $c_1(y, D_y)$ for some $c_1 \in S^0(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1})$ depending continuously on c, we can write $b^W(x, D_x) = b_1(x, D_x)$ for some symbol b_1 satisfying the estimates (3.11) and depending continuously on b. Therefore $A = (1 + D^2_{x_1})^{-1} b_1(x, D_x)$ and its symbol $\langle \xi_1 \rangle^{-2} b_1(x, \xi)$ clearly belongs to $S^{-2,0}_{0,1}(\mathbb{R}^n \times \mathbb{R}^n) \subset S^{-2,0}_{0,1}(\mathbb{R}^{n+1} \times \mathbb{R}^n)$. □

Let us now choose, for each $\iota \in I$, three functions $f^{(1)}_\iota, f^{(2)}_\iota, f^{(3)}_\iota \in C^\infty(\mathcal{M})$ such that, if we set also

$$f^{(0)}_\iota = f_\iota$$

f_ι being the ι-th element of the partition of unity (1.6), we have

$$f_\iota^{(j+1)} \equiv 1 \ \text{ near } \operatorname{supp}(f_\iota^{(j)}), \qquad j = 0, 1, 2, \tag{3.12}$$

and

$$\operatorname{supp}(f_\iota^{(j)}) \subset \mathcal{U}_\iota, \qquad j = 1, 2, 3. \tag{3.13}$$

If $\iota \in I_{\text{comp}}$ we may assume that $f_\iota^{(j)} \in C_0^\infty(\mathcal{U}_\iota)$ and if $\iota \in I_\infty$ we may assume that

$$\Psi_{\iota *} f_\iota^{(j)}(r, \theta) = \varrho^{(j)}(r) \kappa_\iota^{(j)}(\theta),$$

with $\varrho^{(j)}$ and $\kappa_\iota^{(j)}$ supported in small neighborhoods of $\operatorname{supp}(\varrho)$ and $\operatorname{supp}(\kappa_\iota)$ respectively (see (1.8)), $\kappa_\iota^{(j)}$ being compactly supported and $\varrho^{(j)}(r) = 1$ for r large. Therefore, in all cases,

$$f_\iota^{(j)} \in \operatorname{Diff}_w^0(\mathcal{M}).$$

By (1.6) we can write

$$(z - \widetilde{\Delta}_g)^{-1} = \sum_{\iota \in I} f_\iota^{(0)} (z - \widetilde{\Delta}_g)^{-1} f_\iota^{(2)} + \sum_{\iota' \in I} \sum_{\iota \in I} f_\iota^{(0)} (P - z)^{-1} (1 - f_\iota^{(2)}) f_{\iota'}^{(0)}.$$

The first sum corresponds to 'diagonal terms' and the second double one to 'off diagonal terms' since $f_\iota^{(0)}$ and $(1 - f_\iota^{(2)}) f_{\iota'}^{(0)}$ have disjoint supports.

By Proposition 3.5, Theorem 3.1 would be a direct consequence of the following proposition.

PROPOSITION 3.8. — *For all $\iota, \iota' \in I$, the following operators, acting on $L^2(\mathbb{R}^n)$,*

$$R_\iota(z) \equiv \Psi_{\iota *} f_\iota^{(0)} (z - \widetilde{\Delta}_g)^{-1} f_\iota^{(2)} \Psi_\iota^*, \qquad z \notin \mathbb{R},$$

and

$$R_{\iota\iota'}(z) = \Psi_{\iota *} f_\iota^{(0)} (z - \widetilde{\Delta}_g)^{-1} (1 - f_\iota^{(2)}) f_{\iota'}^{(0)} \Psi_{\iota'}^*, \qquad z \notin \mathbb{R},$$

have kernels of the form (3.4) with symbols whose semi-norms in $S_{0,1}^{-2,0}(\mathbb{R}^{n+1} \times \mathbb{R}^n)$ are bounded by $(\langle z \rangle / |\operatorname{Im} z|)^\nu$, for some ν (depending on the semi-norm).

We shall prove Proposition 3.8 using Proposition 3.7. To this end, it is convenient to introduce the set of operators $\mathcal{C}_\iota(z)$ and $\mathcal{C}_{\iota\iota'}(z)$ defined as follows. Consider all operators $R^{\mathcal{M}}(z)$ of the form

$$R^{\mathcal{M}}(z) = \prod_{j=1}^N \left(w(r)^{k_j} A_j (z - \widetilde{\Delta}_g)^{-1} \right) (z - \widetilde{\Delta}_g)^{-1} \tag{3.14}$$

the product standing for the composition, from the left to the right increasingly in j, with $N \geqslant 1$, $k_1, \dots, k_N \geqslant 0$ and

$$A_j \in \mathrm{Diff}^{m_j}_w(\mathcal{M}), \qquad 0 \leqslant m_j \leqslant 2, \quad m_1 = 0.$$

Consider then all $F^{(0)}_\iota, F^{(2)}_\iota \in \mathrm{Diff}^0_w(\mathcal{M})$ such that

$$\mathrm{supp}(F^{(0)}_\iota) \subset \mathrm{supp}(f^{(0)}_\iota), \qquad \mathrm{supp}(F^{(2)}_\iota) \subset \mathrm{supp}(f^{(2)}_\iota). \tag{3.15}$$

We then define the vector space

$$\mathcal{C}_\iota(z) = \mathrm{span}\{\Psi_{\iota *} F^{(0)}_\iota R^{\mathcal{M}}(z) F^{(2)}_\iota \Psi^*_\iota\}$$

obtained by considering all operators of the form (3.14) and all cutoffs $F^{(0)}_\iota, F^{(2)}_\iota$ satisfying (3.15). Clearly

$$R_\iota(z) \in \mathcal{C}_\iota(z). \tag{3.16}$$

Similarly, consider the set of cutoffs $F^{(2)}_{\iota\iota'} \in \mathrm{Diff}^0_w(\mathcal{M})$ such that

$$\mathrm{supp}(F^{(2)}_{\iota\iota'}) \subset \mathrm{supp}\left((1 - f^{(2)}_\iota) f^{(0)}_{\iota'}\right), \tag{3.17}$$

and define

$$\mathcal{C}_{\iota\iota'}(z) = \mathrm{span}\{\Psi_{\iota *} F^{(0)}_\iota R^{\mathcal{M}}(z) F^{(2)}_{\iota\iota'} \Psi^*_\iota\}.$$

We have

$$R_{\iota\iota'}(z) \in \mathcal{C}_{\iota\iota'}(z). \tag{3.18}$$

To compute the commutators with elements of $\mathcal{L}_{\mathbb{R}^n}$, we start with a few remarks. For $k = 1, \dots, n$, we have

$$x_k \Psi_{\iota *} = \Psi_{\iota *} x^\iota_k, \qquad \Psi^*_{\iota'} x_k = x^{\iota'}_k \Psi^*_{\iota'}, \tag{3.19}$$

if we denote by $(x^\iota_1, \dots, x^\iota_n)$ the coordinates in the ι-th chart and by $(x_1, \dots, x_n)$ those of $\mathbb{R}^n$. Similarly

$$\partial_{x_k} \Psi_{\iota *} = \Psi_{\iota *} \partial_{x^\iota_k}, \qquad \Psi^*_{\iota'} \partial_{x_k} = \partial_{x^{\iota'}_k} \Psi^*_{\iota'}. \tag{3.20}$$

Of course, both (3.19) and (3.20) hold only in coordinate patches. If ι and ι' belong to I_∞, (3.19) reads, for $k = 2, \dots, n$,

$$x_k \Psi_{\iota *} = \Psi_{\iota *} \theta^\iota_{k-1}, \qquad \Psi^*_{\iota'} x_k = \theta^{\iota'}_{k-1} \Psi^*_{\iota'},$$

and for $k = 1$,

$$x_1 \Psi_{\iota *} = \Psi_{\iota *} r, \qquad \Psi^*_{\iota'} x_1 = r \Psi^*_{\iota'}, \tag{3.21}$$

where one should note that r is globally defined on $\mathcal{M}$. We don't write the analogous formulas corresponding to (3.20) for $\iota, \iota' \in I_\infty$ but we recall that ∂_r is only defined where r is a coordinate, namely for $r > R$.

By (3.12), (3.20) and (3.15), we have

$$\partial_k \left(\Psi_{\iota *} F_\iota^{(0)} R^{\mathcal{M}}(z) F_\iota^{(2)} \Psi_\iota^* \right) = \Psi_{\iota *} \left[L_{\iota,k},\ F_\iota^{(0)} R^{\mathcal{M}}(z) F_\iota^{(2)} \right] \Psi_\iota^* + \left(\Psi_{\iota *} F_\iota^{(0)} R^{\mathcal{M}}(z) F_\iota^{(2)} \Psi_\iota^* \right) \partial_k,$$

with

$$L_{\iota,k} = f_\iota^{(3)} \Psi_\iota^* \partial_k \Psi_{\iota *}. \tag{3.22}$$

In particular,

$$\left[\partial_k, \Psi_{\iota *} F_\iota^{(0)} R^{\mathcal{M}}(z) F_\iota^{(2)} \Psi_\iota^* \right] = \Psi_{\iota *} \left[L_{\iota,k}, F_\iota^{(0)} R^{\mathcal{M}}(z) F_\iota^{(2)} \right] \Psi_\iota^*. \tag{3.23}$$

For operators in $\mathcal{C}_{\iota\iota'}(z)$, we use (3.17), that $f_\iota^{(1)} \equiv 1$ near $\mathrm{supp}(f_\iota^{(0)})$ and that

$$(1 - f_\iota^{(2)}) \equiv 0 \qquad \text{near} \ \ \mathrm{supp}(f_\iota^{(1)}),$$
$$(1 - f_\iota^{(1)}) \equiv \begin{cases} 1 & \text{near} \ \ \mathrm{supp}(1 - f_\iota^{(2)}) \\ 0 & \text{near} \ \ \mathrm{supp}(f_\iota^{(0)}) \end{cases},$$

which follow from (3.12), to obtain

$$\partial_k \left(\Psi_{\iota *} F_\iota^{(0)} R^{\mathcal{M}}(z) F_{\iota\iota'}^{(2)} \Psi_{\iota'}^* \right) = \Psi_{\iota *} \left[L_{\iota \to \iota',k}, F_\iota^{(0)} R^{\mathcal{M}}(z) F_{\iota\iota'}^{(2)} \right] \Psi_{\iota'}^*,$$
$$\left(\Psi_{\iota *} F_\iota^{(0)} R^{\mathcal{M}}(z) F_{\iota\iota'}^{(2)} \Psi_{\iota'}^* \right) \partial_k = \Psi_{\iota *} \left[F_\iota^{(0)} R^{\mathcal{M}}(z) F_{\iota\iota'}^{(2)}, L_{\iota \leftarrow \iota',k} \right] \Psi_{\iota'}^*,$$

with

$$L_{\iota \to \iota',k} = f_\iota^{(1)} \Psi_\iota^* \partial_k \Psi_{\iota *}, \qquad L_{\iota \leftarrow \iota',k} = (1 - f_\iota^{(1)}) f_{\iota'}^{(1)} \Psi_{\iota'}^* \partial_k \Psi_{\iota' *}. \tag{3.24}$$

The main consequence is that

$$\left[\partial_k, \Psi_{\iota *} F_\iota^{(0)} R^{\mathcal{M}}(z) F_{\iota\iota'}^{(2)} \Psi_{\iota'}^* \right] = \Psi_{\iota *} \left(\left[L_{\iota \to \iota',k}, F_\iota^{(0)} R^{\mathcal{M}}(z) F_{\iota\iota'}^{(2)} \right] - \left[F_\iota^{(0)} R^{\mathcal{M}}(z) F_{\iota\iota'}^{(2)}, L_{\iota \leftarrow \iota',k} \right] \right) \Psi_{\iota'}^*.$$

With the latter formula, (3.23) and the resolvent identity, namely

$$ad_L \cdot (z - \widetilde{\Delta}_g)^{-1} = -(z - \widetilde{\Delta}_g)^{-1} [L, \widetilde{\Delta}_g] (z - \widetilde{\Delta}_g)^{-1}, \tag{3.25}$$

we are equipped to prove the following result.

PROPOSITION 3.9. — *For all $\alpha \in \mathbb{N}^n$ and all $\iota \in I$ (resp. all $\iota, \iota' \in I$), we have*

$$ad_{\partial_x}^\alpha R_\iota(z) \in \mathcal{C}_\iota(z) \qquad \left(resp. \ \ ad_{\partial_x}^\alpha R_{\iota\iota'}(z) \in \mathcal{C}_{\iota\iota'}(z) \right).$$

More precisely, it is a linear combination (with coefficients independent of z) of operators of the form

$$F_\iota^{(0)} R^{\mathcal{M}}(z) F_\iota^{(2)} \qquad (\text{resp. } F_\iota^{(0)} R^{\mathcal{M}}(z) F_{\iota\iota'}^{(2)})$$

with $N \leqslant |\alpha| + 1$ and

$$A_1 \in \mathrm{Diff}^0_w(\mathcal{M}), \quad A_2, \dots, A_N \in \mathrm{Diff}^2_w(\mathcal{M}), \qquad k_1 = k_2 = \cdots = k_N = 0.$$

Proof. — It follows from an elementary induction once observed that, if L is any of the operators in (3.22) or (3.24), we have

$$A \in \mathrm{Diff}^m_w(\mathcal{M}) \qquad \Rightarrow \qquad [L, A] \in \mathrm{Diff}^m_w(\mathcal{M}).$$

Indeed, if L is compactly support this is trivial. Otherwise, if it is supported in chart a infinity, this is a consequence of the identities

$$\begin{aligned}
[\partial_r, \widetilde{\varrho}(r)\widetilde{\kappa}(\theta)\partial_{\theta_k}] &= \left(\frac{\widetilde{\varrho}'(r)}{w(r)}\widetilde{\kappa}(\theta)\right) w(r)\partial_{\theta_k}, \\
\left[w(r)\partial_{\theta_{k'}}, \widetilde{\varrho}(r)\widetilde{\kappa}(\theta)\partial_{\theta_k}\right] &= \left(\widetilde{\varrho}(r)\partial_{\theta_{k'}}\widetilde{\kappa}(\theta)\right) w(r)\partial_{\theta_k}, \\
\left[w(r)\partial_{\theta_{k'}}, \widetilde{\varrho}(r)\widetilde{\kappa}(\theta)\partial_r\right] &= \left(w(r)\widetilde{\varrho}(r)\partial_{\theta_{k'}}\widetilde{\kappa}(\theta)\right)\partial_r - \left(\frac{w'(r)}{w(r)}\widetilde{\varrho}(r)\widetilde{\kappa}(\theta)\right) w(r)\partial_{\theta_{k'}},
\end{aligned}$$

where all the brackets in the right hand sides are bounded as well as their derivatives, if $\widetilde{\varrho}$ and $\widetilde{\kappa}$ are bounded with compactly supported derivatives, also using (1.13) and (1.15). □

To compute $ad_x^\beta ad_{\partial_x}^\alpha R_\iota(z)$ and $ad_x^\beta ad_{\partial_x}^\alpha R_{\iota\iota'}(z)$, we need the following lemma.

Lemma 3.10. — *Let $\widetilde{\rho}$ be a smooth function on $\mathbb{R}$ with compactly supported derivative and supported in $r > R$. Let $\widetilde{\kappa}(\theta)$ be supported in a patch of the manifold at infinity. Then, for any $A \in \mathrm{Diff}^m_w(\mathcal{M})$, we have*

$$[A, \widetilde{\varrho}(r)r] = A', \qquad [A, \widetilde{\varrho}(r)\widetilde{\kappa}(\theta)\theta_k] = w(r)A'',$$

for some $A', A'' \in \mathrm{Diff}^{m-1}_w(\mathcal{M})$. Furthermore, for all $F \in C_0^\infty(\mathcal{M})$ and all $k \in \mathbb{N}$, we can write

$$[A, F] = w(r)^k A_k,$$

with $A_k \in \mathrm{Diff}^{m-1}_w(\mathcal{M})$.

Proof. — The first two identities follow simply from

$$\begin{aligned}
[\partial_r, \widetilde{\varrho}(r)r] &= \left(\widetilde{\varrho}'(r)r + \widetilde{\varrho}(r)\right), \\
[\partial_r, \widetilde{\varrho}(r)\widetilde{\kappa}(\theta)\theta_k] &= w(r)\left(\frac{\widetilde{\varrho}'(r)}{w(r)}\widetilde{\kappa}(\theta)\theta_k\right), \\
\left[w(r)\partial_{\theta_{k'}}, \widetilde{\varrho}(r)\widetilde{\kappa}(\theta)\theta_k\right] &= w(r)\widetilde{\varrho}(r)\left(\theta_k\partial_{\theta_{k'}}\widetilde{\kappa}(\theta) + \delta_{kk'}\widetilde{\kappa}(\theta)\right),
\end{aligned}$$

since all brackets in the right hand sides are smooth and bounded, together with their derivatives. For the third one, we simply observe that $[A, F]$ is a differential operator of order $m-1$ with compact support and can thus be written $w(r)^k(w(r)^{-k}\,[A,F])$ since w doesn't vanish. □

The main sense of this lemma is that commutators of elements of $\mathrm{Diff}^m_w(\mathcal{M})$ with the multiplication operators by coordinates (cut off to be globally defined) are operators in $\mathrm{Diff}^{m-1}_w(\mathcal{M})$. More precisely, we get a factor $w(r)$ when commuting with angular coordinates or compactly supported functions. Note also that it is crucial for the first commutator that we commute A with a function of r only. Otherwise, we would have to consider for instance terms like

$$[w(r)\partial_{\theta_{k'}}, \widetilde{\varrho}(r)\widetilde{\kappa}(\theta)r] = \big(\widetilde{\varrho}(r)\partial_{\theta_{k'}}\widetilde{\kappa}(\theta)\big)\, w(r)r,$$

with $w(r)r$ unbounded in general.

PROPOSITION 3.11. — *For all $\alpha, \beta \in \mathbb{N}^n$ and all $\iota \in I$ (resp. all $\iota, \iota' \in I$), the operator*

$$ad_x^\beta ad_{\partial_x}^\alpha R_\iota(z) \qquad (\textit{resp. } ad_x^\beta ad_{\partial_x}^\alpha R_{\iota\iota'}(z)),$$

is a linear combination (with coefficients independent of z) of operators of the form

$$F_\iota^{(0)} R^{\mathcal{M}}(z) F_\iota^{(2)} \qquad (\textit{resp. } F_\iota^{(0)} R^{\mathcal{M}}(z) F_{\iota\iota'}^{(2)})$$

where $R^{\mathcal{M}}(z)$ is of the form (3.14) with $N \leqslant |\alpha| + |\beta| + 1$,

$$A_j \in \mathrm{Diff}^{m_j}_w(\mathcal{M}), \qquad 0 \leqslant m_j \leqslant 2$$

and

$$k_j = 2 - m_j, \qquad k_2 + \cdots + k_N = \beta_2 + \cdots + \beta_n.$$

Proof. — We repeat essentially the calculations prior to Proposition 3.9 with x_k instead of ∂_k except for x_1 when we work close to infinity. We proceed as follows. If $\iota \in I_{\mathrm{comp}}$, we define

$$X_{\iota,k} = f_\iota^{(3)} \Psi_\iota^* x_k \Psi_{\iota *}, \tag{3.26}$$

$$X_{\iota\to\iota',k} = f_\iota^{(1)} \Psi_\iota^* x_k \Psi_{\iota *}, \tag{3.27}$$

for $1 \leqslant k \leqslant n$. If $\iota' \in I_{\mathrm{comp}}$ and $\iota \in I$, we also set

$$X_{\iota\leftarrow\iota',k} = (1 - f_\iota^{(1)}) f_{\iota'}^{(1)} \Psi_{\iota'}^* x_k \Psi_{\iota' *}, \tag{3.28}$$

for $1 \leqslant k \leqslant n$. In these cases, $X_{\iota,k}$, $X_{\iota\to\iota',k}$ and $X_{\iota\leftarrow\iota',k}$ are smooth functions compactly supported in coordinates patches. If $k \geqslant 2$ and $\iota, \iota' \in I_\infty$,

we still define $X_{\iota,k}$, $X_{\iota\to\iota',k}$ and $X_{\iota\leftarrow\iota',k}$ by the right hand sides of (3.26), (3.27) and (3.28). Setting finally

$$X_{\iota,1} = r, \qquad \iota \in I_\infty \tag{3.29}$$

$$X_{\iota\to\iota',1} = r, \qquad \iota \in I_\infty,\ \iota' \in I, \tag{3.30}$$

$$X_{\iota\leftarrow\iota',1} = r \qquad \iota \in I,\ \iota' \in I_\infty, \tag{3.31}$$

we have defined $X_{\iota,k}$, $X_{\iota\to\iota',k}$ and $X_{\iota\leftarrow\iota',k}$ for all $\iota,\iota' \in I$ and all $1 \leqslant k \leqslant n$. For operators of the form (3.14) and cutoffs satisfying (3.15), (3.19) imply that

$$\left[x_k, \left(\Psi_{\iota *} F_\iota^{(0)} R^{\mathcal{M}}(z) F_\iota^{(2)} \Psi_\iota^*\right)\right] = \Psi_{\iota *} F_\iota^{(0)} \left[X_{\iota,k}, R^{\mathcal{M}}(z)\right] F_\iota^{(2)} \Psi_\iota^*,$$

for all $\iota \in I$ and $1 \leqslant k \leqslant n$. For off diagonal terms, namely with right cutoffs satisfying (3.17), we have

$$\begin{aligned} x_k \left(\Psi_{\iota *} F_\iota^{(0)} R^{\mathcal{M}}(z) F_{\iota\iota'}^{(2)} \Psi_\iota^*\right) &= \Psi_{\iota *} F_\iota^{(0)} X_{\iota\to\iota',k} R^{\mathcal{M}}(z) F_{\iota\iota'}^{(2)} \Psi_\iota^*, \\ &= \Psi_{\iota *} F_\iota^{(0)} \left[X_{\iota\to\iota',k}, R^{\mathcal{M}}(z)\right] F_{\iota\iota'}^{(2)} \Psi_{\iota'}^* \ + \\ &\qquad \Psi_{\iota *} F_\iota^{(0)} R^{\mathcal{M}}(z) X_{\iota\to\iota',k} F_{\iota\iota'}^{(2)} \Psi_{\iota'}^* \end{aligned}$$

where the last term vanishes if $\iota \in I_{\mathrm{comp}}$ or $k \geqslant 2$. In the remaining cases, namely $k = 1$ and $\iota \in I_\infty$, we have $X_{\iota\to\iota',1} = r$ and

$$r F_{\iota\iota'}^{(2)} \Psi_{\iota'}^* = \begin{cases} F_{\iota\iota'} \Psi_{\iota'}^* \ \text{ with } F_{\iota\iota'} \in C_0^\infty(\mathcal{M}) & \text{if } \iota' \in I_{\mathrm{comp}}\ , \\ F_{\iota\iota'}^{(2)} \Psi_{\iota'}^* x_1 & \text{if } \iota' \in I_\infty. \end{cases}$$

Similarly, we have

$$\begin{aligned} \left(\Psi_{\iota *} F_\iota^{(0)} R^{\mathcal{M}}(z) F_{\iota\iota'}^{(2)} \Psi_\iota^*\right) x_k &= \Psi_{\iota *} F_\iota^{(0)} R^{\mathcal{M}}(z) X_{\iota\leftarrow\iota',k} F_{\iota\iota'}^{(2)} \Psi_\iota^*, \\ &= \Psi_{\iota *} F_\iota^{(0)} \left[R^{\mathcal{M}}(z), X_{\iota\leftarrow\iota',k},\right] F_{\iota\iota'}^{(2)} \Psi_{\iota'}^* \ + \\ &\qquad \Psi_{\iota *} F_\iota^{(0)} X_{\iota\leftarrow\iota',k} R^{\mathcal{M}}(z) F_{\iota\iota'}^{(2)} \Psi_{\iota'}^* \end{aligned}$$

where the last term vanishes if $k \geqslant 2$ or $\iota' \in I_{\mathrm{comp}}$ and

$$\Psi_{\iota *} F_\iota^{(0)} X_{\iota\leftarrow\iota',1} = \begin{cases} F_{\iota\iota'} \Psi_{\iota'}^* \ \text{ with } F_{\iota\iota'} \in C_0^\infty(\mathcal{M}) & \text{if } \iota \in I_{\mathrm{comp}}\ , \\ x_1 \Psi_{\iota *} F_\iota^{(0)} & \text{if } \iota \in I_\infty. \end{cases}$$

This shows that, unless $\iota,\iota' \in I_\infty$ and $k = 1$, $\left[x_k, \left(\Psi_{\iota *} F_\iota^{(0)} R^{\mathcal{M}}(z) F_{\iota\iota'}^{(2)} \Psi_\iota^*\right)\right]$ is the sum of

$$\Psi_{\iota *} F_\iota^{(0)} \left(\left[X_{\iota\to\iota',k}, R^{\mathcal{M}}(z)\right] - \left[R^{\mathcal{M}}(z), X_{\iota\leftarrow\iota',k},\right]\right) F_{\iota\iota'}^{(2)} \Psi_{\iota'}^*$$

and of terms of the same form as $\Psi_{\iota *}F_\iota^{(0)}R^{\mathcal{M}}(z)F_{\iota\iota'}^{(2)}\Psi_\iota^*$. If $\iota,\iota'\in I_\infty$ and $k=1$, we simply have

$$\left[x_1,\left(\Psi_{\iota *}F_\iota^{(0)}R^{\mathcal{M}}(z)F_{\iota\iota'}^{(2)}\Psi_\iota^*\right)\right]=\Psi_{\iota *}F_\iota^{(0)}\left[r,R^{\mathcal{M}}(z)\right]F_{\iota\iota'}^{(2)}\Psi_{\iota'}^*.$$

Using lemma 3.10, the resolvent identity (3.25) and a simple induction, we get the result. □

The next proposition is the final step before being in position to use Proposition 3.7.

PROPOSITION 3.12. — *Fix $\iota\in I$ (resp. $\iota,\iota'\in I$). For all $\alpha,\beta\in\mathbb{N}^n$ and all $\gamma\in\mathbb{N}$ satisfying $\gamma_1\leqslant 2$, $\gamma_2+\cdots+\gamma_n\leqslant\beta_2+\cdots+\beta_n$, the operator*

$$D_x^\gamma ad_x^\beta ad_{\partial_x}^\alpha R_\iota(z)\qquad (\text{resp. } D_x^\gamma ad_x^\beta ad_{\partial_x}^\alpha R_{\iota\iota'}(z)),$$

is a linear combination (with coefficients independent of z) of operators of the form

$$F_\iota^{(0)}R^{\mathcal{M}}(z)F_\iota^{(2)}\qquad (\text{resp. } F_\iota^{(0)}R^{\mathcal{M}}(z)F_{\iota\iota'}^{(2)})$$

(see (3.14)) with $N\leqslant|\alpha|+|\beta|+|\gamma|+1$ and

$$A_1,\dots,A_N\in\mathrm{Diff}_w^2(\mathcal{M}),\qquad k_1=\cdots=k_N=0.$$

In particular, they are bounded on $L^2(\mathbb{R}^n)$ with norms controlled by powers of $\langle z\rangle/|\mathrm{Im}(z)|$.

Proof. — We treat the case of $R_\iota(z)$, the one of $R_{\iota\iota'}(z)$ being completely similar. We start with a simple model case. Consider an operator of the form

$$B(z):=\Psi_{\iota *}F_\iota^{(0)}(z-\widetilde{\Delta}_g)^{-1}w(r)A(z-\widetilde{\Delta}_g)^{-1}F_\iota^{(2)}\Psi_\iota^*,$$

with $A\in\mathrm{Diff}_w^1(\mathcal{M})$. Such operators appear in Proposition 3.11 with $N=2$ if $\beta_2+\cdots+\beta_n=1$ and $\alpha=0$. Compute then $\partial_k B(z)$, with $k\geqslant 2$. We get

$$\Psi_{\iota *}\Big(\left[L_{\iota,k},F_\iota^{(0)}(z-\widetilde{\Delta}_g)^{-1}w(r)\right]A(z-\widetilde{\Delta}_g)^{-1}$$
$$+F_\iota^{(0)}(z-\widetilde{\Delta}_g)^{-1}w(r)L_{\iota,k}A(z-\widetilde{\Delta}_g)^{-1}F_\iota^{(2)}\Big)\Psi_\iota^*.$$

The commutator reads

$$\left[L_{\iota,k},F_\iota^{(0)}\right](z-\widetilde{\Delta}_g)^{-1}w(r)+F_\iota^{(0)}(z-\widetilde{\Delta}_g)\left[\widetilde{\Delta}_g,L_{\iota,k}\right](z-\widetilde{\Delta}_g)^{-1}w(r)$$
$$+F_\iota^{(0)}(z-\widetilde{\Delta}_g)^{-1}\left[L_{\iota,k},w(r)\right]$$

and is bounded on $L^2(\mathcal{M},\widetilde{dg})$ since $[L_{\iota,k},\widetilde{\Delta}_g]\in\mathrm{Diff}_w^1(\mathcal{M})$. The simple and crucial remark is that

$$w(r)L_{\iota,k}A\in\mathrm{Diff}_w^2(\mathcal{M}),$$

although $L_{\iota,k}A \notin \mathrm{Diff}^2_w(\mathcal{M})$ in general. Therefore $\partial_k B(z)$ is a linear combination of operators of the form (3.14) with A_1 of order 0. This then implies that $\partial_1^2\partial_k B(z)$ is also of this form with A_1 of order 2. Iteration of this argument give the result since Proposition 3.11 shows there are at least $\gamma_2+\cdots+\gamma_n$ powers $w(r)$ in the expression of $ad_x^\beta ad_{\partial_x}^\alpha R_\iota(z)$ to absorb $\partial_{x_2}^{\gamma_2}\dots\partial_{x_n}^{\gamma_n}$. □

3.2. Proof of Proposition 3.8

Proposition 3.8 follows from Proposition 3.7 since, by Proposition 3.12, the operators $R_\iota(z)$ and $R_{\iota\iota'}(z)$ satisfy the assumptions of Proposition 3.7.

3.3. Proof of Theorem 3.1

This is a direct consequence of Proposition 3.5 and Proposition 3.8 using the equivalence of norms (1.26).

3.4. Proof of Theorem 3.3

The boundedness of $W(r)(z-\Delta_g)^{-1}W(r)^{-1}$ on $L^p(\mathcal{M},dg)$ is equivalent to the one of

$$W(r)w(r)^{\frac{n-1}{2}-\frac{n-1}{p}}(z-\widetilde{\Delta}_g)^{-1}w(r)^{\frac{n-1}{p}-\frac{n-1}{2}}W(r)^{-1}$$

on $L^p(\mathcal{M},\widetilde{dg})$ so the result follows from Proposition 3.5, with the temperate weight $Ww^{\frac{n-1}{2}-\frac{n-1}{p}}$, and Proposition 3.8. □

3.5. Proof of Theorem 1.7

We note first that, by writing $(z-h^2\widetilde{\Delta}_g)^{-1}=h^{-2}(zh^{-2}-\widetilde{\Delta}_g)^{-1}$, Theorem 3.1 implies that

$$||W(r)(z-h^2\widetilde{\Delta}_g)^{-1}W(r)^{-1}||_{L^p(\mathcal{M},\widetilde{dg})\to L^p(\mathcal{M},\widetilde{dg})}\lesssim h^{-2}\frac{\langle z\rangle^{\nu_p}}{|\mathrm{Im}(z)|^{\nu_p}},\quad h\in(0,1],\quad z\in\mathbb{C}\setminus\mathbb{R},\tag{3.32}$$

by using the inequality $\langle h^{-2}z\rangle/|\mathrm{Im}(h^{-2}z)|\lesssim\langle z\rangle/|\mathrm{Im}(z)|$.

Assume next that $\varphi \in S^{-\sigma}(\mathbb{R})$ with $\sigma > 1$ so that we can use (2.31). By Theorem 2.8 and (3.32), there exists $\nu_{p,N}$ such that

$$\left|\left|W(r)(z-h^2\widetilde{\Delta}_g)^{-1}\mathcal{R}_N(z,h)W(r)^{-1}\right|\right|_{L^p(\mathcal{M},\widetilde{dg})\to L^p(\mathcal{M},\widetilde{dg})} \lesssim h^{-2}\left(\frac{\langle z\rangle}{|\mathrm{Im}\ z|}\right)^{\nu_p+\nu_{N,p}},$$

for $h \in (0,1]$ and $z \notin \mathbb{R}$. By choosing $M \geqslant \nu = \nu_p + \nu_{N,p}$, the above estimate and (2.31) give the expected estimate up to a factor h^{-2}. The latter is eliminated in the standard way: by pushing the expansion to the order h^{N+2}, we write $\mathcal{R}_N(-\widetilde{\Delta}_g,\varphi,h)$ as the sum of properly supported pseudo-differential operators bounded on $W(r)^{-1}L^p(\mathcal{M},\widetilde{dg})$ and of $h^2\mathcal{R}_{N+2}(-\widetilde{\Delta}_g,\varphi,h)$. This implies (1.39).

If now $\varphi \in S^{-\sigma}(\mathbb{R})$ with $\sigma > 0$, we cannot use (2.31). We thus write $\varphi(\lambda) = (\lambda+i)\psi(\lambda)$ with $\psi \in S^{-\sigma-1}(\mathbb{R})$ so that

$$\varphi(-h^2\widetilde{\Delta}_g) = (i-h^2\widetilde{\Delta}_g)\psi(-h^2\widetilde{\Delta}_g). \tag{3.33}$$

We then write again $\mathcal{R}_N(-\widetilde{\Delta}_g,\varphi,h)$ as a finite sum of properly supported pseudo-differential operators bounded on $W(r)^{-1}L^p(\mathcal{M},\widetilde{dg})$ and

$$h^{N+2}\iint_{\mathbb{R}^2}\bar{\partial}\widetilde{\psi}_M(z)(z-h^2\widetilde{\Delta}_g)^{-1}(i-h^2\widetilde{\Delta}_g)\mathcal{R}_{N+2}(z,h)dxdy$$

where $z = x+iy$. By Theorem 2.8, we have

$$\left|\left|W(r)(i-h^2\widetilde{\Delta})\mathcal{R}_{N+2}(z,h)W(r)^{-1}\right|\right|_{L^p(\mathcal{M},\widetilde{dg})\to L^p(\mathcal{M},\widetilde{dg})} \lesssim \left(\frac{\langle z\rangle}{|\mathrm{Im}\ z|}\right)^{\nu_{N+2,p}},$$

and we proceed as above. □

Appendix A. Non $L^p \to L^p$ boundedness on the hyperbolic space

Using the hyperboloid model of the hyperbolic space, namely

$$\mathbb{H}^n = \{x = (x_0,\dots,x_n) \in \mathbb{R}^{n+1} \mid x_0^2 - x_1^2 - \cdots - x_n^2 = 1,\ x_0 > 0\},$$

we have polar coordinates by considering

$$x(r,\omega) = (\cosh r, \omega\sinh r), \qquad r > 0,\ \omega \in \mathbb{S}^{n-1}.$$

In this parametrization, the distance between $x = x(r, \omega)$ and $x' = x(r', \omega')$ reads

$$\begin{aligned} d(x, x') &= \operatorname{arccosh}\left(\cosh r \cosh r' - \omega \cdot \omega' \sinh r \sinh r'\right) \\ &= \operatorname{arccosh}\left\{\left(1 - \frac{|\omega - \omega'|^2}{4}\right)\cosh(r - r') + \frac{|\omega - \omega'|^2}{4}\cosh(r + r')\right\} \end{aligned} \tag{A.1}$$

and the volume element is

$$(\sinh r)^{n-1} dr d\omega,$$

where $d\omega$ is the usual Riemannian measure on the sphere. Considering $n = 3$ for simplicity, the resolvent

$$(-\Delta_{\mathbb{H}^3} - 1 + \epsilon^2)^{-1}, \qquad \epsilon > 0, \tag{A.2}$$

is well defined since, in general, $-\Delta_{\mathbb{H}^n} \geqslant (n-1)^2/4$. Its kernel with respect to the volume element is then given by

$$\frac{1}{4\pi} \frac{e^{-\epsilon d(x,x')}}{\sinh d(x, x')}, \tag{A.3}$$

(see for instance [23, p. 105]).

PROPOSITION A.1. — *Fix $p \in (1, \infty)$ with $p \neq 2$. If $0 < \epsilon < \left|1 - \frac{2}{p}\right|$, then $(-\Delta_{\mathbb{H}^3} - 1 + \epsilon^2)^{-1}$ is not bounded on $L^p(\mathbb{H}^3)$.*

We shall proceed by contradiction, using the following simple lemma.

LEMMA A.2. — *Let K_1, K_2 be two locally integrable functions on $(\mathbb{R}_+ \times \mathbb{S}^2)^2$ such that*

$$K_2(r, \omega, r', \omega') \geqslant |K_1(r, \omega, r', \omega')|. \tag{A.4}$$

Denote by A_j be the operator with kernel K_j with respect to $dr d\omega$ and set $L^p = L^p(\mathbb{R}_+ \times \mathbb{S}^2, dr d\omega)$. Then

$$||A_1||_{L^p \to L^p} \leqslant ||A_2||_{L^p \to L^p}.$$

Proof. — By (A.4), we have, for all $u \in C_0^\infty(\mathbb{R}_+ \times \mathbb{S}^2)$,

$$|(A_1 u)(r, \omega)| \leqslant |(A_2|u|)(r, \omega)|$$

so, taking the L^p norm, we obtain

$$||A_1 u||_{L^p} \leqslant \big|\big|A_2|u|\big|\big|_{L^p} \leqslant ||A_2||_{L^p \to L^p} \big|\big||u|\big|\big|_{L^p} = ||A_2||_{L^p \to L^p} ||u||_{L^p}$$

which gives the result. □

Proof of Proposition A.1. — We argue by contradiction and assume that $(-\Delta_{\mathbb{H}^3} - 1 + \epsilon^2)^{-1}$ is bounded on $L^p(\mathbb{H}^3)$. This is equivalent to the boundedness on $L^p(\mathbb{R}_+ \times \mathbb{S}^2, dr d\omega)$ of the operator with kernel

$$K_2(r,\omega,r',\omega') := (\sinh r)^{\frac{2}{p}} \left(\frac{1}{4\pi} \frac{e^{-\epsilon d(x,x')}}{\sinh d(x,x')} (\sinh r')^2 \right) (\sinh r')^{-\frac{2}{p}}$$

with respect to $dr d\omega$. Since $\cosh(r-r') \leqslant \cosh(r+r')$ for $r, r' \in \mathbb{R}^+$, (A.1) gives

$$d(x,x') \leqslant r + r'$$

so, for $r, r' \geqslant 1$, we have

(A.5)

$$K_2(r,\omega,r',\omega') \gtrsim (e^r)^{\frac{2}{p}} \left(\frac{e^{-\epsilon(r+r')}}{e^{r+r'}} (e^{r'})^2 \right) (e^{r'})^{-\frac{2}{p}} = e^{(\frac{2}{p}-1-\epsilon)r} e^{(1-\frac{2}{p}-\epsilon)r'}.$$

Denoting by $K_1(r,\omega,r',\omega') = K_1(r,r')$ the right hand side of (A.5) multiplied by the characteristic function of $[1,+\infty)^2$, Lemma A.2 implies that the corresponding operator A_1 is bounded on $L^p(\mathbb{R}_+ \times \mathbb{S}^2, dr d\omega)$. This is clearly not true if $\frac{2}{p} - 1 > \epsilon$, otherwise $e^{(\frac{2}{p}-1-\epsilon)r}$ should belong to $L^p(\mathbb{R})$. We also obtain a contradiction if $1 - \frac{2}{p} > \epsilon$ by considering the adjoint of A_1. □

We note that the right hand side of (A.5) also reads

$$e^{(\frac{2}{p}-1)(r-r')-\epsilon(r+r')},$$

showing that the above reasoning gives no contradiction for $p = 2$ nor by restricting the kernel close to the diagonal.

We also recall that $(n-1)|\frac{1}{p} - \frac{1}{2}|$ (ie $|\frac{2}{p} - 1|$ if $n = 3$) is exactly the width of the strip around the real axis in which φ has to be holomorphic to ensure the boundedness on $L^p(\mathbb{H}^n)$ of

$$\varphi\left((-\Delta_{\mathbb{H}^n} - (n-1)^2/4)^{1/2} \right),$$

as proved in [22]. The resolvent (A.2) corresponds to $\varphi(\lambda) = (\lambda^2 + \epsilon^2)^{-1}$ which is holomorphic for $|\mathrm{Im}(\lambda)| < \epsilon$.

Remark. — Proposition A.1 is a low frequency counterexample to the extent that it deals with $(-\Delta_{\mathbb{H}^3} + 1 - (\lambda + i\epsilon)^2)^{-1}$ for $\lambda = 0$. However, a similar unboundedness result can be proved for any $\lambda > 0$. In this case, the kernel (A.3) is no longer positive since it is modified by the oscillatory factor $\exp(i\lambda d(x,x'))$ but one can overcome this problem as follows. Using the fact that $r - r' \leqslant d(x,x') \leqslant r + r'$ and by testing the resolvent against positive radial functions φ and ψ_k localized respectively in $|\lambda r'| \sim \epsilon$ and

$|r - 2k\pi\lambda^{-1}| \leqslant \epsilon$ with ϵ small enough (but fixed) and $k \in \mathbb{N}$, one can bound from below $\mathrm{Re}(\psi_k, (-\Delta_{\mathbb{H}^3} + 1 - (\lambda + i\epsilon)^2)^{-1}\varphi)$ using (A.5) and get a contradiction as $k \to \infty$.

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