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Jörg WINKELMANN

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DEGENERACY OF ENTIRE CURVES IN LOG SURFACES WITH $\bar{q} = 2$

by Jörg WINKELMANN (*)

ABSTRACT. — We determine which algebraic surface of logarithmic irregularity 2 admit an algebraically non-degenerate entire curve.

RÉSUMÉ. — Nous déterminons les surfaces algébriques d'irregularité logarithmique 2 qui admettent des courbes entières non-dégénérées.

1. Introduction

We are interested in the question which complex algebraic varieties X admit a non-degenerate entire curve, *i.e.,* for which X does there exists a holomorphic map $f\colon \mathbb{C} \to X$ such that the image $f(\mathbb{C})$ is dense in X with respect to the (algebraic) Zariski topology on X.

Conjecturally, this property is related to algebraic-geometric properties as well as diophantine properties.

The algebraic-geometric properties are related to the concept of *logarithmic differential forms.*

Let X be a (non-compact) smooth algebraic surface. If $X \hookrightarrow \bar{X}$ is a smooth compactification where $D = \bar{X} \setminus X$ is a hypersurface with only simple normal crossings as singularities (such a compactification exists always due to desingularization theorems), then one can define the notion of logarithmic differential forms $\Omega^k(\bar{X}; \log D)$. The dimension of the space of global sections (on $\bar{X}$) of $\Omega^1(\bar{X}; \log D)$ is called "logarithmic irregularity" and denoted by $\bar{q}$. Due to Hodge theory this number $\bar{q}$ equals the dimension

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of the quasi Albanese variety of X. The logarithmic Kodaira dimension $\bar{\kappa}$ can be defined as the D-dimension of the "logarithmic canonical line bundle" $\Omega^{\dim X}(\bar{X}; \log D)$.

Due to Noguchi's log version of the theorem of Bloch-Ochiai ([9]) there is no non-degenerate entire curve if $\bar{q} > \dim(X) = 2$.

By the conjectures of Lang and Vojta there should be no non-degenerate entire curve if $\bar{\kappa} = 2$.

Here we prove this conjecture for the special case of a surface of logarithmic irregularity $\bar{q} = 2$.

We give a precise criterion (Theorem 1.1) which determines completely which surfaces with $\bar{q} = 2$ admit algebraically non-degenerate entire curves. This criterion implies that an algebraic surface with $\bar{q} = 2$ and $\bar{\kappa} = 2$ cannot admit a non-degenerate entire curve. It also implies that every algebraic surface with $\bar{q} = 2$ and $\bar{\kappa} = 0$ does admit a non-degenerate entire curve.

This criterion can be stated as follows.

Theorem 1.1. — *Let X be a smooth algebraic surface of logarithmic irregularity $\bar{q} \geqslant 2$.*

Then there exists a Zariski dense entire curve $f\colon \mathbb{C} \to X$ if and only if all of the following conditions are fulfilled:

1. *The logarithmic irregularity equals 2.*
2. *The quasi Albanese map $\alpha\colon X \to A$ is generically bijective.*
3. *There exists a semi-abelian subvariety $S \subset A$ with*
    a) $0 \leqslant \dim(S) < \dim A = 2$,
    b) *the complement $A \setminus \alpha(X)$ is contained in finitely many S-orbits and*
    c) *for every positive-dimensional S-orbit Ω in A there exist a point $p \in X$ with $\alpha(p) \in \Omega$ but $D\alpha(T_pX) \not\subset T_{\alpha(p)}\Omega$.*

This generalises earlier results by Dethloff, Lu, Noguchi, Yamanoi and the author. Dethloff and Lu proved [4] that there are no algebraically non-degenerate *Brody* curves if $\bar{\kappa} = 2$. For $\bar{\kappa} = 1$ they proved non-existence of a non-degenerate entire curve under a certain additional assumption.

Noguchi, Yamanoi and the author obtained in [10] the result for the special case where the quasi Albanese map is proper (see [10]).

The above stated precise technical criterion for the existence of a non-degenerate entire curve can be shown to be equivalent to a number of other conditions.

Namely:

THEOREM 1.2. — *Let X be a smooth algebraic surface of logarithmic irregularity $\bar{q} \geqslant 2$. Then the following conditions are equivalent:*

(i) *There exists a non-degenerate entire curve (i.e., a holomorphic map $f\colon \mathbb{C} \to X$ whose image $f(\mathbb{C})$ is Zariski dense in X).*
(ii) *There exists a holomorphic map $f\colon \mathbb{C} \to X$ whose image $f(\mathbb{C})$ is dense in X with respect to the ordinary topology.*
(iii) *There exists a holomorphic map $F\colon \mathbb{C}^2 \to X$ with dense image whose Jacobian does not vanish identically.*
(iv) *X is special in the sense of Campana ([3]).*

Note that condition (ii) implies that the Kobayashi pseudodistance vanishes identically. In particular our results show: If S is a special variety of dimension two with $\bar{q} \geqslant 2$, then the Kobayashi pseudodistance d_S vanishes identically.

Campana conjectured that in general an algebraic variey is special if and only if its Kobayashi pseudodistance vanishes identically.

Concerning Brody curves we consider only the case in which the quasi Albanese variety is compact. For this case we obtain a precise criterion determining which algebraic surfaces admit non-degenerate Brody curves (Proposition 5.6 and 5.8).

As a consequence, we see that the condition of admitting a Zariski dense Brody curve is not a closed condition (see Theorem 6.1). This is remarkable, because the condition of admitting a non-constant Brody curve *is* a closed condition.

This paper is organizes as follows: First we prove Theorem 1.1 which gives precise criteria for the existence of non-degenerate entire curves. Then we investigate the relation of our criterion for the existence of a non-degenerate entire curve with the logarithmic Kodaira dimension. Thereafter, we give a statement on Brody curves.

We also discuss and describe certain examples.

These examples include :

(1) A surface defined over $\mathbb{Q}$ with logarithmic Kodaira dimension 1, which admits a non-degenerate entire curve and a Zariski dense set of integral points for every number field except $\mathbb{Q}$ and imaginary quadratic fields (Example 8.8).
(2) A surface with $\bar{\kappa} = 1$ for which there exists a non-degenerate Brody curve (Example 8.4).
(3) A surface with $\bar{\kappa} = 1$ for which there exists a non-degenerate entire curve although every Brody curve is degenerate (Example 8.5).

(4) A surface with $\bar{\kappa} = 1$ without non-degenerate entire curves (Example 8.6).
(5) A surface with $\bar{\kappa} = 0$ for which there exists a non-degenerate entire curve, but no non-degenerate Brody curve (Example 8.2).

2. Proof of the main result

We start proving Theorem 1.1.

Proof.

"only if". Assume that there exists a Zariski dense entire curve $f: \mathbb{C} \to X$. Let $\alpha: X \to A$ be the quasi Albanese map of X. Let $X \hookrightarrow \bar{X}$ be an open embedding of smooth varieties such that α extends to a proper morphism $\bar{\alpha}: \bar{X} \to A$. Let $\bar{X} \xrightarrow{\beta} Y \xrightarrow{\tau} A$ be the Stein factorisation. Now f gives us a Zariski dense entire curve in Y. Due to the main theorem of [10] it follows that $\tau: Y \to A$ is an unramified covering. In particular, Y is a semi-abelian variety. By the universality property of the quasi Albanese variety every morphism from X to a semi-abelian variety fibers through A. Applied to Y, this shows that $Y \simeq A$. Thus α has generically connected fibers. By the universality property of the quasi Albanese variety the image $\alpha(X)$ generates A as an algebraic group. On the other hand, Noguchi's logarithmic Bloch Ochiai theorem ([9]) implies that the Zariski closure of $\alpha(f(\mathbb{C}))$ is a translate of a semi-abelian subvariety of A. Since $f(\mathbb{C})$ is assumed to be Zariski dense in X, these facts together imply that $\alpha(f(\mathbb{C}))$ is Zariski dense in A. Hence $\alpha: X \to A$ must be dominant. For dimension reasons, this implies that $\alpha: X \to A$ is generically finite.

Let $D \subset A$ be the union of the one-codimensional irreducible components of the closure of $W = A \setminus \alpha(X)$. Then D is a divisor in A such that $Z = \alpha(X) \cap D$ is of codimension at least two in A. Due to [11], Theorem 5.1 it follows that

$$N_F(r, D) = N_F(r, Z) \leqslant T_F(r, \omega_Z) \leqslant \epsilon T_F(r)||_\epsilon$$

for $F = \alpha \circ f$.

In combination with the main theorem of [11] we obtain

$$T_F(r, D) \leqslant \epsilon T_F(r)||_\epsilon.$$

It follows that D can not be big. By [11], Proposition 3.9 (ii) this implies that D has a positive-dimensional stabilizer (with respect to the A-action on itself by left multiplication). Therefore D is empty or there is a one-dimensional algebraic subgroup S of A such that D is the union of finitely many S-orbits. As the quasi Albanese of an algebraic variety, A is a semi-abelian variety which in turn implies that S is a semi-abelian variety as well.

In the first case we are done. In the second case let $E = A/S$ and consider the map $g\colon \mathbb{C} \to E$ given by composing $F = \alpha \circ f\colon \mathbb{C} \to A$ with the natural projection map $\tau\colon A \to A/S$.

Observe that A/S is again a semi-abelian variety. (In fact either $\mathbb{C}^*$ or an elliptic curve, because $\dim(E) = 1$.) The entire curve $g\colon \mathbb{C} \to E$ is algebraically non-degenerate. The Second Main Theorem with truncation level one for entire curves with values in E [(1)] thus implies that for every $q \in E$ there is an element $z \in \mathbb{C}$ with $g(z) = q$ and $Dg|_z \neq 0$.

This implies that for every S-orbit Ω in A there is an element $z \in \mathbb{C}$ with $F(z) \in \Omega$ and $F'(z) \notin T\Omega$. Since $F = \alpha \circ f$, it follows that there is a point $p \in X$ with $\alpha(p) \in \Omega$ but $D\alpha(T_pX) \not\subset T\Omega$.

"if". Next we show the other direction. Let $E = A/S$ and let $g\colon \mathbb{C} \to E$ be an entire curve with dense image (which exists, because E is a semi-abelian variety). Let $M = X \times_E \mathbb{C}$ and $B = A \times_E \mathbb{C}$, $V = W \times_E \mathbb{C}$ ($W = A \setminus \alpha(X)$). Then $B \simeq \mathbb{C} \times S$. Let D_1 denote the set of $z \in C$ such that $V \cap \tau^{-1}(z)$ is finite and let D_2 be the set of $z \in C$ such that $\tau^{-1} \setminus (V \cap \tau^{-1}(z))$ is finite and not empty. Then both D_i are discrete. The existence of a Zariski dense entire curve now follows from Proposition 2.3 below. □

We continue with preparing the proof of Proposition 2.3.

Let Δ be the unit disc and let X be a complex space. For two holomorphic map germs $f_i\colon (\Delta, 0) \to X$ we say that they are k-equivalent (written $f_1 \sim_k f_2$) for $k \in \mathbb{N}$ if and only if

- $f_1(0) = f_2(0)$ and
- Using local coordinates on X, the first k terms of the Taylor series development of f_1 agree with those of f_2.

Lemma 2.1. — *Let $\pi\colon \hat{S} \to S$ be the blow-up of a smooth complex surface S in one point $z \in S$. Let $k \in \mathbb{N}$ and let $\phi, \psi\colon (\Delta, 0) \to \hat{S}$ be germs of holomorphic maps with $(\pi \circ \phi)'(0) \neq 0$.*

Assume $\pi \circ \phi \sim_{k+1} \pi \circ \psi$.

Then $\phi \sim_k \psi$.

Proof. — This follows by an elementary calculation in local coordinates. □

(1) We may simply use the Main Theorem of [11] which holds for semi-abelian varieties in arbitrary dimension. However, for one-dimensional E such Second Main Theorem have already been proved by Nevanlinna.

Corollary 2.2. — *Let $\pi\colon S' \to S$ be a bimeromorphic proper holomorphic map of complex surfaces. Then for each point $p \in S$ there is a number $k_0 \in \mathbb{N}$ such that the following property holds:* If $\phi, \psi\colon (\Delta, 0) \to S'$ are two holomorphic map germs with $\pi \circ \phi \sim_{k_0} \pi \circ \psi$ and $(\pi \circ \phi)'(0) \neq 0$, then $\phi(0) = \psi(0)$.

Proof. — Any such π is given as a sequence of blow-ups. Hence the statement follows by induction on the length of this sequence from the above lemma. □

Proposition 2.3. — *Let X be a surface, A a semi-abelian surface, $S \subset A$ a one-dimensional semi-abelian subvariety of A, $E = A/S$. Let $\alpha\colon X \to A$ be a holomorphic map fulfilling condition (3) of the Theorem 1.1. Let $\tau\colon A \to E$ denote the natural projection map.*

Then there exists a entire curve $f\colon \mathbb{C} \to X$ with Zariski dense image.

Proof. — Let $g\colon (\mathbb{C}, +) \to A$ be a one-parameter subgroup for which $\tau \circ g\colon \mathbb{C} \to E$ is surjective.

Claim. — *Let $p \in X$ and $w \in \mathbb{C}$ with $\tau(g(w)) = \tau(\alpha(p))$. Assume $D\alpha(T_pX) \not\subset T_{\alpha(p)}\Omega$ where $\Omega = S+\alpha(p)$ (i.e., Ω denotes the S-orbit through $\alpha(p)$). Then there exists a holomorphic map germ $h\colon (\Delta, 0) \to S$ such that the holomorphic map germ $H\colon (\mathbb{C}, w) \to A$ defined by $H(z) = g(z) + h(z - w)$ lifts to a map germ from $(\mathbb{C}, w)$ to (X, p).*

Since X is smooth and $D\alpha(T_pX) \not\subset T_{\alpha(p)}\Omega$, there exists a holomorphic map germ $\zeta\colon (\Delta, 0) \to (X, p)$ with $(\tau \circ \alpha \circ \zeta)'(0) \neq 0$. Because $\dim(E) = 1$, it follows that there is an automorphism germ $\phi \in \mathrm{Aut}(\Delta, 0)$ such that

$$\tau \circ \alpha \circ \zeta \circ \phi\colon (\Delta, 0 \to (E, \tau(p)$$

coincides with the germ from $(\delta, 0)$ to $(E, \tau(p))$ given by

$$t \mapsto g(z + t).$$

Now let D_0 be the set of $x \in E$ such that $\tau^{-1}(x)$ is not contained in $\alpha(X)$. Then D_0 is finite and $D_1 = (\tau \circ g)^{-1}(D_0)$ is discrete in $\mathbb{C}$. For every $z \in D_1$ we fix a holomorphic map germ $h_x\colon (\Delta, 0) \to S$ such that the map germ $z \mapsto g(z) + h_x(z - x)$ lifts to a map germ with values in X.

Next we fix a sequence of $(s_n)_{n\in\mathbb{N}}$ in $A \setminus \tau^{-1}(D_0)$ such that $\{s_n : n \in \mathbb{N}\}$ is dense in A. We choose $a_n \in \mathbb{C}$ such that

(1) $\tau(g(a_n)) = \tau(s_n)$.
(2) $\lim_{n\to\infty} |a_n| = \infty$.

(This is possible, because each fiber of $\tau \circ g \colon \mathbb{C} \to E$ is the translate of a full lattice in $\mathbb{C}$ and therefore is an unbounded set.)

Next we choose $b_n \in S$ such that $g(a_n) + b_n = s_n$.

Observe that $D = D_1 \cup \{(a_n \colon n \in \mathbb{N}\}$ is discrete. Therefore we can find a holomorphic map from $\mathbb{C}$ to S with some finite jet prescribed at every $c \in D$. In particular, there is a holomorphic map $h \colon \mathbb{C} \to S$ such that $h(a_n) = b_n$ for all $n \in \mathbb{N}$ and such that the germ of h at w is k_w-equivalent to h_w for all $w \in D_1$.

Let us define $G \colon \mathbb{C} \to A$ by $G(z) = g(z) + h(z)$. Since $\alpha \colon X \to A$ is generically bijective, G lifts to a meromorphic map from $\mathbb{C}$ to X. An the other hand, by construction the map g lifts everywhere locally to a map to A. Together these two properties imply that G lifts to an entire curve $f \colon \mathbb{C} \to X$. Moreover, by construction the image $f(\mathbb{C})$ is dense in X. $\square$

3. Arithmetic

Given an algebraic variety V defined over a number field K and a finite set of places S on K (including all the archimedean ones) there is the notion of a *"set of S-integral points"*, see *e.g.* [12]. If V is an affine variety, then a *"set of S-integral points"* is a set H of K-rational points such that there exists a closed embedding j of V into an affine space $\mathbb{A}^N$ with the property that for every K-rational point $p \in V(K)$ every coordinate of $j(p)$ is an *S-integer*. (A number $x \in K$ is an S-integer if $\nu(x) \geqslant 0$ for every place $\nu \notin S$.)

Following the conjectures of Lang and Vojta, for an algebraic variety V defined over a number field K the following conditions should be equivalent:

(1) There is a finite field extension L/K and a finite set S of places on L including all the archimedean ones such that V admits a Zariski-dense set of S-integral points. (This condition is also called "potentially dense".)

(2) There is a holomorphic map $f \colon \mathbb{C} \to V(\mathbb{C})$ ("entire curve") whose image is Zariski dense in V.

In general this is widely unknown. Concerning the surfaces S of logarithmic irregularity $\bar{q} \geqslant 2$ under consideration in this article it is in many cases unknown whether there exists a Zariski dense subset of integral points are not. However, in the cases where we know the answer, the answer is positive, *i.e.,* confirms what is predicted by the above cited conjecture.

Let us list some cases for which it is known whether a surface of logarithmic irregularity $\bar{q} \geqslant 2$ is *potentially dense*. Let S be an algebraic surface with quasi Albanese map $\alpha\colon S \to A$.

- Assume that $\alpha\colon S \to A$ is not dominant. By the universality property of the quasi Albanese the image $\alpha(S)$ must generate A as an algebraic group. Therefore the Zariski closure of $\alpha(S)$ in A can not be a semi-abelian subvariety. Thus results of Faltings ([5]) and Vojta ([13], [14]) imply that integral point sets in S can not be Zariski dense.
- Let A be a product of two one-dimensional semi-abelian varieties ($A \simeq E_1 \times E_2$) and assume that there is a finite subset $Z \subset A$ such that $S \simeq A \setminus Z$. Then S is potentially dense by an argument of McKinnon (see [6]).
- Assume that $A \setminus \alpha(S)$ contains a curve C which is closed in A. Then S can not be potentially dense. This has been proved by Vojta ([13], [14]. (If A is an abelian variety, this has been shown by Faltings ([5]). The case $A \simeq G_m \times G_m$ is due to Laurent.)
- In some special cases the condition can be checked explicitly. For instances, consider Example 8.8.

4. The logarithmic Kodaira dimension

Next we relate our main theorem to conditions on the logarithmic Kodaira dimension.

Theorem 4.1. — *Let S be an algebraic surface of logarithmic irregularity $\bar{q} = 2$.*

If there exists a non-degenerate entire curve, then $\bar{\kappa} \leqslant 1$.

If $\bar{\kappa} = 0$, then there exists a non-degenerate entire curve.

In order to verify this claim, we need to understand how the logarithmic Kodaira dimension of such a surface S is related to its quasi Albanese A and the image of S in A.

Dethloff and Lu proved in [4], generalizing a result of Kawamata ([7]):

Proposition 4.2. — *Let X be a normal algebraic surface, A a semi-abelian variety, $f\colon X \to A$ a finite morphism and $X_0 \subset X$ an open algebraic subvariety.*

If $\bar{\kappa}_{X_0} = 0$, then f is étale and $X \setminus X_0$ is finite.

In combination with our main theorem this has the following consequence:

Corollary 4.3. — *Let S be an algebraic surface with $\bar{q} = 2$ and $\bar{\kappa} = 0$. Then there exists a non-degenerate entire curve.*

Proof. — Let $S \hookrightarrow \bar{S}$ be an open embedding into a smooth variety $\bar{S}$ such that the quasi Albanese map $\alpha\colon S \to A$ extends to a proper morphism $\bar{\alpha}\colon \bar{S} \to A$. Let $\bar{S} \xrightarrow{\beta} Y \xrightarrow{f} A$ be the Stein factorization of $\bar{\alpha}$. The assumption $\bar{\kappa} = 0$ implies that $\alpha\colon S \to A$ is dominant. We observe that

$$0 = \bar{\kappa}(S) \geqslant \bar{\kappa}(\bar{S}) \geqslant \bar{\kappa}(Y) \geqslant \bar{\kappa}(A) = 0.$$

Hence we may apply Proposition 4.2 (with $X_0 = X = Y$) and deduce that $f\colon Y \to A$ is étale. It follows that Y is a semi-abelian variety. By the universality property of the quasi Albanese variety the map $S \to Y$ must fiber through A. This forces $f\colon Y \to A$ to be an isomorphism. It follows that the quasi Albanese map $\alpha\colon S \to A$ is generically bijective. Furthermore the assumption $\bar{\kappa}(S) = 0$ together with Lemma 4.5 below implies that $A \setminus \alpha(S)$ is finite. Thus we may deduce the existence of a non-degenerate entire curve from Theorem 1.1. □

Remark 4.4. — The conditions $\bar{q} = 2$ and $\bar{\kappa} = 0$ do not suffice to imply the existence of a non-degenerate *Brody* curve, *cf.* Example 8.2.

Lemma 4.5. — *Let $P = \{(z, w) \in \mathbb{C}^2\colon |z|,\ |w| < 1\}$ and let S denote the complex surface obtained from P by blowing up the origin and then removing the strict transform of $w = 0$. Let $\pi\colon S \to P$ denote the natural projection. Let $P^* = \{(z, w) \in P\colon w \neq 0\}$.*

Then every logarithmic 2-form on S is a pull-back of a logarithmic 2-form on P^*, while the logarithmic 1-forms on S are given as pull-backs of a one-form on P.

Proof. — We realize S as

$$S = \big\{(z, w, s) \in C^3\colon |z|,\ |w| < 1,\ z = ws\big\}$$

which is isomorphic to

$$S \simeq \big\{(w, s) \in \mathbb{C}^2\colon |w|,\ |ws| < 1\big\}.$$

Now π takes the form

$$S' \ni (w, s) \mapsto (ws, w) \in P.$$

We see that dz pulls back to $d(ws) = sdw + wds$ while dw/w pulls back to dw/w. Thus $dz \wedge dw/w$ pulls back to $ds \wedge dw$. This implies the assertions. □

Proposition 4.6. — *Let $\pi\colon A \to E$ be a surjective connected morphism of semi-abelian varieties of dimension 2 resp. 1. Let X be an algebraic surface with a generically bijective map $\alpha\colon X \to A$ such that*

(1) *$\alpha(X)$ intersects each fiber of π,*
(2) *$\pi(A \setminus \alpha(X))$ is finite.*

Then α is the quasi Albanese map of X.

The logarithmic Kodaira dimension of X equals 1 if $A \setminus \alpha(X)$ is infinite and 0 if $A \setminus \alpha(X)$ is finite.

Proof. — This is an immediate consequence of Lemma 4.5. □

Corollary 4.7. — *If a surface of logarithmic irregularity 2 admits a Zariski dense entire curve, then its logarithmic Kodaira dimension is at most 1.*

Proof. — Follows from Theorem 1.1 in combination with Proposition 4.6. □

Corollary 4.8. — *Let S be a surface of logarithmic irregularity $\geqslant 2$ and logarithmic Kodaira dimension 2.*

Then every entire curve is algebraically degenerate.

Remark 4.9. — Dethloff and Lu proved the statement of Corollary 4.8 for Brody curves ([4]).

5. Brody curves

Given a compact complex manifold X, a "Brody curve" is an entire curve $f\colon \mathbb{C} \to X$ with bounded derivative. Due to compactness of X this condition is independent of the choice of the metric on X. The theorem of Brody ([1]) guarantees that a compact complex manifold admits a Brody curve as soon as it admits a (non-constant) entire curve.

Lemma 5.1. — *Let A be an abelian surface and let $f\colon \mathbb{C} \to A$ be a Zariski dense Brody curve. Let E be a one-dimensional abelian variety (i.e., an elliptic curve) and let $\pi\colon A \to E$ be a surjective morphism.*

Then

$$T_f(r) = O\left(T_{\pi\circ f}(r)\right).$$

Proof. — Let $\mathbb{C} \simeq \mathrm{Lie}(E) \subset \mathrm{Lie}(A) \simeq \mathbb{C}^2$ denote the respective Lie algebras and introduce linear complex coordinates z, w on $\mathrm{Lie}(A)$ such that $\mathrm{Lie}(E) = \{w = 0\}$. Now $\omega = dz \wedge d\bar{z} + dw \wedge d\bar{w}$ is a positive $(1,1)$-form which is invariant under translation. By abuse of notation we denote the induced $(1,1)$-form on A again by ω. Identifying $\mathrm{Lie}(A)$ with the universal covering of A, we can lift f to a map $(f_1, f_2)\colon \mathbb{C} \to \mathbb{C}^2 \simeq \mathrm{Lie}(A)$ and obtain

$$T_f(r,\omega) = \int_1^r \left(\int_{\Delta_t} \left(|f_1'|^2 + |f_2'|^2\right) \frac{i}{2} dz \wedge d\bar{z} \right) \frac{dt}{t}.$$

Since f is Brody, it is of order 2 which implies that f_1' and f_2' are constant. Let $\alpha = f_1'$ and $\beta = f_2'$. Since $f\colon \mathbb{C} \to A$ is Zariski dense, the composed map $\pi \circ f$ is non-constant, which in turn implies $\alpha \neq 0$. It follows that

$$\begin{aligned} T_f(r) &= \left(|\alpha|^2 + |\beta|^2\right) r^2 + O(1), \\ T_{\pi\circ f}(r) &= \left(|\alpha|^2\right) r^2 + O(1) \end{aligned}$$

and therefore

$$\lim_{r\to\infty} \frac{T_f(r)}{T_{\pi\circ f}(r)} = \frac{|\alpha|^2 + |\beta|^2}{|\alpha|^2} < \infty.$$

□

Remark 5.2. — Unfortunately, this does not generalize to semi-abelian varieties.

Lemma 5.3. — *Let $f\colon \mathbb{C} \to A$ be a non-degenerate Brody curve with values in an abelian surface A and let $C \subset A$ be an algebraic curve.*

Then $f(\mathbb{C}) \cap C$ is infinite.

Proof. — There is no loss in generality in assuming that C is irreducible. If C is big as a divisor we may argue in the same way as in the proof of the main theorem. Thus we may assume that C is irreducible and not big. Then C is the translate of a one-dimensional abelian subvariety E of A. We consider the projection $\pi\colon A \to A/E$ onto the quotient by E. Since we assumed f to be non-degenerate, we know that $f(\mathbb{C})$ is the translate of a non-compact one-dimensional complex Lie subgroup H of A. For dimension reasons the restricted homomorphism of complex Lie groups $\pi|_H\colon H \to A/E$ is either constant or surjective. In particular the image $\pi|_H(H)$ is compact. Because $\pi(H) \subset A/E$ is compact while H is not, the kernel of the Lie group homomorphism $\pi|_H\colon H \to A/E$ is non-compact and therefore infinite. It follows that all the fibers of the projection map $\pi|_{f(\mathbb{C})} \to A/E$ are infinite. This implies that $f(\mathbb{C}) \cap C$ is infinite. □

Corollary 5.4. — *Let X be a surface with quasi Albanese $\alpha\colon X \to A$. Let $i\colon X \hookrightarrow \bar{X}$ be an algebraic compactification. Assume that $\bar{q}(X) = \dim(A) = 2$ and that there exists a non-degenerate entire curve $f\colon \mathbb{C} \to X$ for which $i \circ f\colon \mathbb{C} \to \bar{X}$ is a Brody curve. Assume further that A is an abelian surface.*

Then $A \setminus \alpha(X)$ is finite.

Proof. — A meromorphic map from a normal complex space with values in a compact complex torus is necessarily holomorphic. Therefore the quasi Albanese map $\alpha\colon X \to A$ extends to a morphism $\bar{\alpha}\colon \bar{X} \to A$. Since $\bar{X}$ is compact, $i \circ f$ being a Brody curve implies that $\bar{\alpha} \circ i \circ f$ is a Brody curve, too. But $\bar{\alpha} \circ i \circ f = \alpha \circ f$. Hence the statement follows from the lemma above. □

We can now deduce a criterion for the existence of non-degenerate Brody curves. For this we recall the notion of an *"elliptic curve with complex multiplication"*. Let E be an elliptic curve, *i.e.,* a one-dimensional abelian variety. An endomorphism is an algebraic morphism $f\colon E \to E$ with fixes the neutral element e_E. The set of all endomorphisms of E is a ring, called $\mathrm{End}(E)$. The ring $\mathrm{End}(E)$ includes $\mathbb{Z}$, since $x \mapsto nx = x + \ldots + x$ is an endomorphism of E for every $n \in \mathbb{Z}$. The elliptic curve E is said to have "*complex multiplication*" if $\mathrm{End}(E)$ is larger than $\mathbb{Z}$.

As a complex Lie group every complex elliptic curve can be written as $E = \mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$ with $\Im(\tau) > 0$. It is well known (and easy to see) that such an elliptic curve admits complex multiplication if and only if $\tau^2 \in \mathbb{Q} \oplus \tau\mathbb{Q}$, *i.e.,* if and only if $\mathbb{Q}[\tau]$ is a quadratic field extension of $\mathbb{Q}$.

Lemma 5.5. — *Let A be an abelian surface.*

Then there exists a complex one-parameter subgroup with real three-dimensional closure unless A is an abelian variety isogenous to $E \times E$ for an elliptic curve E with complex multipliciation.

Proof. — Assume that A is not isogenous to $E \times E$ where E is an elliptic curve with complex multiplication. Due to [15] this implies that there is a connected complex Lie subgroup H of A whose closure is not complex. For dimension reasons H is complex one-dimensional and its closure $\bar{H}$ is real three-dimensional. □

Proposition 5.6. — *Let X be a surface with a generically bijective morphism $\alpha\colon X \to A$ where A is an abelian surface.*

Assume that A is not isogenous to a product of two copies of an elliptic curve with complex multiplication.

If $A \setminus \alpha(X)$ is finite, then there exists a non-degenerate Brody curve $f\colon \mathbb{C} \to X$.

Proof. — Let Z denote the set of all "trouble points" in A, *i.e.,* let Z denote the union of $A \setminus \alpha(X)$ with the set of all points $a \in A$ for which $\alpha^{-1}(a)$ contains more than one point. By the assumption of α being generically finite it is clear that Z is finite. Note that α restricts to an isomorphism from $X \setminus \alpha^{-1}(Z)$ to $A \setminus Z$.

Due to Lemma 5.5 there is a complex one-parameter subgroup $\gamma\colon \mathbb{C} \to A$ whose topological closure $H = \overline{\gamma(\mathbb{C})}$ is real three-dimensional real Lie subgroup of A.

There is a real quotient map τ from A to A/H where A/H is a one-dimensional real Lie group. Since Z is finite, we may choose an element $q_0 \in A/H$ with an open neighbourhood W_0 such that $\bar{W}_0 \cap \tau(Z) = \{\}$. Next we choose an element $q \in A$ with $\tau(q) = q_0$ and define an entire curve in A by $F(t) = q + \gamma(t)$. This is a Brody curve, because γ defines a one-parameter subgroup. Let $W = \tau^{-1}(W_0)$. Then $W \subset A \setminus Z$ and $F(\mathbb{C}) \subset W$. The map $F\colon \mathbb{C} \to A$ can be lifted to an entire curve $f\colon \mathbb{C} \to X$, because α restricts to an isomorphism over $A \setminus Z$ and $F(\mathbb{C}) \subset A \setminus Z$. This lift is a Brody curve, because $\bar{W}$ is relatively compact in $A \setminus Z$ and $F\colon \mathbb{C} \to A$ is a Brody curve. $\square$

Lemma 5.7. — *Let A be an abelian surface, $X \to A$ the blow up in the neutral element e_A and let $f\colon \mathbb{C} \to A$ be a topologically dense Brody curve. Then f can not be lifted to a Brody curve with values in X.*

Proof. — This follows from [16], §5.3, Proposition 3. $\square$

Proposition 5.8. — *Let X be a surface such that its quasi Albanese variety A is isogenous to a a product $E \times E$ where E is an elliptic curve with complex multiplication.*

Then X admits a non-degenerate Brody curve if and only if the quasi Albanese map $\alpha\colon X \to A$ satisfies the following two conditions:

(1) *α is injective and*
(2) *$A \setminus \alpha(X)$ is finite.*

Proof. — Let us assume that the two conditions (1) and (2) are verified. The map α being injective, we may regard X as an open subset of A. Let $\gamma\colon \mathbb{C} \to A$ be a Zariski dense complex one-parameter subgroup. Since $Z = A \setminus \alpha(X)$ is finite, it is clear that

$$W = \big\{(q \in A\colon \exists t \in \mathbb{C}\colon \gamma(t) + q \in Z\big\}$$

is set of Haar measure zero. Hence there is an element $q \in A \setminus W$. Then $f(t) = \gamma(t) + q$ defines a Brody curve in X.

Conversely, assume that X admits a Brody curve $f\colon \mathbb{C} \to X$ and let $X \hookrightarrow \bar{X}$ be a smooth algebraic compactification such $f\colon \mathbb{C} \to \bar{X}$ has bounded derivative for some hermitian metric on $\bar{X}$. The quasi Albanese map $\alpha\colon X \to A$ extends to a rational map from $\bar{X}$ to A. Because A is an abelian variety, this map is actually a morphism $\bar{\alpha}\colon \bar{X} \to A$. Hence $\alpha \circ f = \bar{\alpha} \circ f\colon \mathbb{C} \to A$ is a Brody curve in A. It follows that there is a one-parameter complex subgroup $\gamma\colon \mathbb{C} \to A$ and an element $q \in A$ such that

$$\alpha(f(t)) = q + \gamma(t).$$

Since A is isogenous to a product of two copies of the same elliptic curve with complex multiplication, it follows from [15] that the (real) topological closure of $\gamma(\mathbb{C})$ is a complex Lie subgroup of A. Because $f\colon \mathbb{C} \to X$ is assumed to have Zariski dense image, it follows that $\gamma(\mathbb{C})$ is actually dense in A. Next we observe that $\alpha\colon X \to A$ is generically bijective due to Theorem 1.1. It follows that the extended map $\bar{\alpha}\colon \bar{X} \to A$ is a sequence of blow ups. Now Lemma 5.7 yields a contradiction unless this sequence is trivial. In other words: The map $\bar{\alpha}$ (and therefore likewise the map α) must be injective. $\square$

6. Brody curves in families

The results of the preceding section imply that the existence of non-degenerate Brody curves does not behave well in families.

Theorem 6.1. — *For $\tau \in H = \{z \in \mathbb{C}\colon \Im(z) > 0\}$ define*

$$E_t = \mathbb{C}/\langle 1, \tau \rangle$$

and let X_τ denote the blow up of the abelian surface $E_\tau \times E_\tau$ in the neutral element.

Then X_τ admits a non-degenerate Brody curve if and only if

$$\deg \mathbb{Q}(\tau)/\mathbb{Q} > 2.$$

This deviates from the behaviour of non-constant Brody curves. Their existence is known to be an open condition:

Theorem 6.2. — *Let $f\colon X \to M$ be a proper holomorphic map. Let W denote the set of all $t \in M$ for which $X_t = f^{-1}(\{t\})$ admits a non-constant Brody curve.*

Then W is closed in M.

Proof. — By the theorem of Brody $M \setminus W$ is the set of all $t \in M$ for which X_t is hyperbolic. If X_t is hyperbolic for a given $t \in M$, then there is an open hyperbolic neighbourhood U of X_t in X (see [8], Corollary 3.6.8 (This is actually another consequence of Brodys theorem)). Since

f is proper, $f(X \setminus U)$ is closed in M. It follows that $M \setminus W$ contains an open neighbourhood of t for each $t \in M \setminus W$, *i.e.*, $M \setminus W$ is open and W is closed. □

7. Relationship to *specialness*

Campana conjectured that an algebraic variety is *special* if and only if it admits a non-degenerate entire curve (see [3]). Our results confirm this conjecture for the special case under consideration here.

Theorem 7.1. — *Let S be an algebraic surface of logarithmic irregularity at least* 2*. Then there exists a non-degenerate entire curve if and only if S is special in the sense of* [3]*.*

Proof. — Let $\alpha\colon S \to A$ denote the quasi Albanese map. First we consider the case where α is not dominant. Let Z denote the (Zariski-)closure of $\alpha(S)$ in A. Let H denote the connected component containing the identity map of the stabilizer group

$$\{g \in A\colon g(Z) = Z\}.$$

We consider the quotient map $\pi\colon A \to A/H$. By construction, the stabilizer group of $\pi(Z)$ in A/H is discrete. On the other hand, α being not dominant implies $\dim(A/H) > 0$. Furthermore due to the universality property of the quasi Albanese the image $\pi(Z)$ generates A/H as algebraic group. It follows that $\pi(Z)$ is positive-dimensional and of log general type. Thus in this case there is a non-trivial morphism from S to a variety of log general type. It follows that S is not special. On the other hand, as mentioned above in Theorem 1.1, there is no non-degenerate entire curve.

The second case to be considered is the case where α is generically finite, but not generically bijective. By our theorem there is no non-degenerate entire curve. Thus we have to show that S is not special. As we have done in the proof of Theorem 1.1, we consider a fiberwise compactification, *i.e.*, we embed S into a larger variety $\bar{S}$ such that $\pi\colon S \to A$ extends to a proper map $\bar{\pi}\colon \bar{S} \to A$. As we have checked in the proof of Theorem 1.1, the universality property of the quasi Albanese variety implies that $\bar{\pi}\colon \bar{S} \to A$ can not be an unramified covering. By a result of Kawamata (Theorem 26/27, [7]), it follows that $\bar{\kappa}(\bar{S}) > 0$ and moreover it follows that there is an unramified covering $\tau\colon \tilde{X} \to \bar{S}$ and a surjective morphism from $\tilde{X}$ to a variety $\tilde{Y}$ of log general type with $\dim(\tilde{Y}) = \bar{\kappa}(\bar{S}) > 0$. Hence $\tilde{Y}$ is not special. As a consequence, neither $\bar{S}$ nor S may be special.

The third case is the case where $\alpha\colon S \to A$ is generically bijective and $A \smallsetminus \alpha(S)$ is finite. Let W denote the set of points in A where α is locally bijective. Then W is open in A and $A \smallsetminus W$ is finite. Now every power of a logarithmic form ω on S restricts to W and then extends to A because $\operatorname{codim}(A \smallsetminus W) \geqslant 2$. Conversely logarithmic forms (and powers thereof) on A pull-back to logarithmic forms on S. Using the characterization of special varieties via *Bogomolov sheaves* and the fact that A is special, we may deduce that W and S are special.

Finally we have to consider the case where α is generically bijective and $A \smallsetminus \alpha(S)$ is not finite. Let D denote the union of one-dimensional components of the closure of $A \smallsetminus \alpha(S)$. If D is ample, Lemma 4.5 above implies that $\bar\kappa(S) = 2$. Thus S is not special if D is ample. If D is not ample, its stabilizer group

$$H = \left\{a \in A\colon a(D) + D\right\}$$

must be positive-dimensional. We then may consider the projection onto the quotient A/H^0 by the connected component H^0 containing the identity map. Now the desired result is contained in a result of Buzzard and Lu ([2], Theorem 1.7). ☐

8. Examples

Example 8.1. — Let E be a complex elliptic curve, $e \in E$ and $S = (E \smallsetminus \{e\}) \times (E \smallsetminus \{e\})$.

Then S is an affine surface of logarithmic Kodaira dimension two. The natural embedding of S into $E \times E$ is the quasi Albanese. S is hyperbolic in the sense of Kobayashi, every holomorphic map from $\mathbb{C}$ to S is constant. If E and thereby S is defined over a number field, then every integral point set is finite.

Example 8.2. — Let $K = \mathbb{Q}[\tau]$ be an imaginary quadratic field, let $E = \mathbb{C}/(\mathbb{Z} \oplus \tau\mathbb{Z})$ and let $A = E \times E$. Let X denote the surface obtained by blowing up A at the neutral element.

Then X is a projective surface of Kodaira dimension zero which admits Zariski dense entire curves but no non-degenerate Brody curves (There are however non-constant Brody curves through almost all point arising from orbits of factors of A not meeting e_A.)

Proof. — By definition E is an elliptic curve with complex multiplication. Hence Proposition 5.8 implies that there is no non-degenerate Brody curve. On the other hand, the existence of a non-degenerate entire curve is rather obvious (though we may also invoke Theorem 1.1). □

Remark 8.3. — In [16] we gave an example of a projective threefold X with a hypersurface E such that $f(\mathbb{C}) \subset E$ for every Brody curve although X does admit non-degenerate entire curves.

Example 8.4. — There is an algebraic surface X with compactification $\bar{X}$ with $\bar{q} = 2$ and $\bar{\kappa} = 1$ such that there exists a non-degenerate Brody curve $f\colon \mathbb{C} \to \bar{X}$ with $f(\mathbb{C}) \subset X$.

Proof. — Let $E = \mathbb{C}/\langle 1, \tau\rangle_{\mathbb{Z}}$ be an elliptic curve and let $A = E \times \mathbb{C}^*$ with compactification $\bar{A} = E \times \mathbb{P}_1$. >From an algebraic point of view the product structure of A is unique, since every algebraic morphism from $\mathbb{C}^*$ to E is constant. Thus, for $(n, m) \in \mathbb{Z}^2 \setminus \{(0, 0)\}$ the holomorphic map $f\colon \mathbb{C} \to A$ defined by

$$f(z) = \left(e^{2\pi i z}, [(n + m\tau)z]\right)$$

has an image which is dense in $\bar{A}$ for the Zariski topology.

Now let us construct a surface S by blowing up the neutral element $(e_E, 1) = e_A$ and removing the strict transform of $E \times \{1\}$. We claim that f induces a non-degenerate Brody curve in S. First note that f is periodic. Hence f is given as $f(z) = g(e^{2\pi i z'})$ for some holomorphic map $g\colon \mathbb{C}^* \to \bar{A}$. Furthermore $f\colon \mathbb{C} \to \bar{A}$ is a Brody curve, because it defines a one-parameter subgroup of A. Let $\hat{g}\colon \mathbb{C}^* \to S$ denote the lift of g. Let $K = \{z \in \mathbb{C}^*\colon |z - 1| < \frac{1}{2}\}$. Then $||D\hat{g}||_{\mathbb{C}^* \setminus K}$ is bounded, because g is Brody, $\bar{S} \to A$ is an isomorphism away from e_A and $g(z)$ stays away from e_A for $z \in \mathbb{C}^* \setminus K$. On the other hand, $||D\hat{g}||_K$ is bounded because K is compact. Therefore we obtain a Brody curve. □

Example 8.5. — There is an algebraic surface X with compactification $\bar{X}$ with $\bar{q} = 2$ and $\bar{\kappa} = 1$ such that there exists a non-degenerate entire curve $f\colon \mathbb{C} \to X$, but there is no non-degenerate entire curve $f\colon \mathbb{C} \to X$ for which $f\colon \mathbb{C} \to \bar{X}$ is a Brody curve.

Proof. — Let E be an elliptic curve. Define $A = E \times E$ and let $\bar{X}$ be A blown up in the neutral element $\{(e, e)\}$ of the abelian variety A. Finally construct X by removing the strict transform of $\{e\} \times E$ from $\bar{X}$.

The natural map from X to A is the quasi Albanese α Furthermore $\bar{q} = 2$ and $\bar{\kappa} = 1$ by Proposition 4.6. Evidently $\alpha(X) = A \setminus ((\{e\} \times E) \setminus \{(e, e)\})$. The existence of a non-degenerate entire curve follows from the main theorem while the existence of a non-degenerate Brody curve is ruled out by Corollary 5.4. □

Example 8.6. — There exists an algebraic surface without non-degenerate entire curve such that $\bar{q} = 2$, $\bar{\kappa} = 1$ and such that condition $(*)$ of Dethloff and Lu (see [4]) does not hold.

Proof. — Let A be a semi-abelian surface with one-dimensional semi-abelian subvariety E (for instance let $A = E \times E'$ where E and E' are isomorphic to the multiplicative group $\mathbb{C}^*$ or isomorphic to an elliptic curve). Define a surface A' by blowing up e, where e denotes the neutral element of A. Let D_1 denote the exceptional divisor and let p be the point on D_1 which corresponds to the direction of E via the natural isomorphism $D_1 \simeq \mathbb{P}T_eA$. Obtain another surface A'' by blowing up A' in p. Let $\tau\colon A'' \to A$ be the projection map. Then $\tau^{-1}(E)$ consist of three irreducible components: the strict transform C_1 of E, the strict transform C_2 of the exceptional locus of the first blow-up and the exceptional locus C_3 of the second blow up. Now we define $S = A'' \setminus (C_1 \cup C_3)$. Let $\alpha = \tau|_S$. By construction $(D\alpha)(T_qS) \subset DE$ for all $q \in C_2 = \alpha^{-1}(E)$. Therefore our theorem implies that every entire curve is algebraically degenerate. On the other hand $\alpha(S) = A \setminus (E \setminus \{e\})$ is not open in A which implies that condition $(*)$ of [4] is not fulfilled. Finally we have logarithmic irregularity 2 and logarithmic Kodaira dimension 1 for S due to Proposition 4.6.

If we choose $A = \mathbb{C}^* \times \mathbb{C}^*$ and $E = \{e\} \times \mathbb{C}^*$, this surface can be described explicitly as follows: $S = A'' \setminus (C_1 \cup C_2)$ with

$$\begin{aligned} A'' &= \{(z, w, [x_0 \colon x_1], [y_0 \colon y_1]) \in \mathbb{C}^* \times \mathbb{C}^* \times \mathbb{P}_1 \times \mathbb{P}_1 \\ &\qquad\qquad : (w-1)x_0 = (z-1)x_1, x_0 y_1 = y_0 x_1 (w-1)\} \\ C_1 &= \{z = 1, w = 1, y_1 = 0\} \\ C_2 &= \{z = 1, x_0 = 0, y_0 = 0\} \end{aligned}$$

□

Example 8.7. — An easier but more artificial example arises as follows: Let E be a one-dimensional semi-abelian variety (*i.e.,* each is isomorphic to $\mathbb{C}^*$ or an elliptic curve) and let E' be an elliptic curve. Fix two distinct points $\alpha, \beta \in E'$. Let A' denote the blow up of $A = E \times E'$ in (e, α). Let S be the surface obtained from A' by removing both the strict transform of $E \times \{\alpha\}$ and the preimage of $E \times \{\beta\}$. Then by the main theorem every entire curve is algebraically degenerate. The condition $(*)$ of Dethloff and Lu (as defined in [4]) does not hold. One can verify that $\bar{q} = 2$ using the fact that the logarithmic one-forms on $E' \setminus \{\beta\}$ agree with the regular one-forms on E', because on an elliptic curve there are no rational functions with exactly one simple pole.

Example 8.8. — Let K be a number field and let

$$X = \left\{(z,w;s) \in G_m^2 \times \mathbb{A}^1 \colon z-1 = s(w-1)\right\}.$$

Then X is a two-dimensional affine variety of logarithmic Kodaira dimension 1. It admits a Zariski dense integral subset unless K is an imaginary quadratic extension of $\mathbb{Q}$ or $K \simeq \mathbb{Q}$.

The quasi Albanese map is given by

$$\alpha_X \colon (z,w;s) \mapsto (z,w) \in G_m^2.$$

There exists a non-degenerate entire curve $f \colon \mathbb{C} \to X(\mathbb{C})$.

Proof. — Most assertions are immediate. For the existence of the non-degenerate entire curve we may invoke Theorem 1.1. However, we can also give an explicit description of such a curve:

$$f \colon z \mapsto \left(e^{2\pi i z^2}, e^{2\pi i z}, \frac{e^{2\pi i z^2}-1}{e^{2\pi i z}-1} \right).$$

First observe that $e^{2\pi i z}-1=0$ iff $z \in \mathbb{Z}$. Since $e^{2\pi i z^2}=1$ for $z \in \mathbb{Z}$, the above formula describes a holomorphic map.

If P is a polynomial, the order of $z \mapsto e^{P(z)}$ equals the degree of P. This easily implies that there can not be any algebraic relation between the functions $e^{2\pi i z^2}$ and $e^{2\pi i z}$. It follows that $\alpha(f(\mathbb{C}))$ is Zariski dense in $\mathbb{C}^* \times \mathbb{C}^*$ which in turn implies that $f \colon \mathbb{C} \to X$ is non-degenerate.

In order to verify the statement about the Zariski dense integral subset, first note that by the theorem of Dirichlet $\operatorname{rank}_{\mathbb{Z}}(\mathcal{O}_K^*) \geqslant 1$ if K is neither isomorphic to $\mathbb{Q}$ nor an imaginary quadratic field extension of $\mathbb{Q}$. Let $\tau \in \mathcal{O}_K^*$ be an element of infinite order, *i.e.*, not a root of unity. Since

$$\tau^{kn}-1 = \left(\sum_{j=0}^{n-1} \tau^{kj}\right)(\tau^k - 1)$$

for all $k,n \in \mathbb{N}$ and

$$X(\mathcal{O}_K) = \left\{(z,w;s) \in \mathcal{O}_K^* \times \mathcal{O}_K^* \times \mathcal{O}_K \colon z-1 = s(w-1)\right\},$$

we see that $X(\mathcal{O}_K)$ includes

$$\left(\tau^{kn}, \tau^k, \sum_{j=0}^{n-1} \tau^{kj}\right)$$

for all $k,n \in \mathbb{N}$. Therefore each curve

$$C_n = \left\{(z,w;s) \in X \colon z = w^n\right\}$$

contains infinitely many integral points. Thus the Zariski closure of $X(\mathcal{O}_K)$ contains all the curves C_n and consequently is at least two-dimensional. Since $\dim(X) = 2$, it follows that $X(\mathcal{O}_K)$ is Zariski dense in X. □

Remark 8.9. — It is well known that algebraic varieties X with $0 < \bar{\kappa} < \dim(X)$ may contain Zariski dense integral sets.

Example 8.10. — Let S be an algebraic surface whose quasi Albanese variety is a simple abelian variety.

Then S admits a non-degenerate entire curve if and only if it admits a non-degenerate Brody curve.

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Jörg WINKELMANN
Rhur-Universität Bochum
Mathematisches Institut
Lehrstuhl Analysis II
NA 4/73 44780 Bochum (Germany)
jwinkel@member.ams.org