ANNALES

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Tome 61, nº 4 (2011), p. 1539-1555.

<http://aif.cedram.org/item?id=AIF_2011__61_4_1539_0>

Ann. Inst. Fourier, Grenoble
61, 4 (2011) 1539-1555

NUMBER OF SINGULAR POINTS OF AN ANNULUS IN $\mathbb{C}^2$

by Maciej BORODZIK & Henryk ZOŁĄDEK (*)

Abstract. — Using BMY inequality and a Milnor number bound we prove that any algebraic annulus $\mathbb{C}^*$ in $\mathbb{C}^2$ with no self-intersections can have at most three cuspidal singularities.

Résumé. — Utilisant l' inégalité BMY et une évaluation pour le nombre de Milnor nous prouvons que chaque anneau $\mathbb{C}^*$ dans $\mathbb{C}^2$ sans auto-intersections ne peut avoir qu' au plus trois singularités cuspidalles

1. Introduction

The problem of classification of curves in $\mathbb{C}^2$ of fixed topological type up to an algebraic automorphism of $\mathbb{C}^2$ is in general very difficult. One of the most important results in this domain is the Abhyankar–Moh–Suzuki theorem ([1, 9]) stating that any algebraic curve in $\mathbb{C}^2$ that is diffeomorphic to a disk is in fact algebraically isomorphic to a line. Another one, due to M. Zaidenberg and V. Lin [11], says that any curve homeomorphic to a disk is algebraically equivalent to a curve of the type $x^p = y^q$ for p, q coprime.

In [3, 5] we developed an efficient method in some other particular cases: namely we studied rational curves with one place at infinity and one double point (topological immersions of $\mathbb{C}$ in $\mathbb{C}^2$ with one finite self-intersection) in [3] and annuli (topological embeddings of $\mathbb{C}^*$ in $\mathbb{C}^2$) in [5]. A list of 44 possible cases was found and it was claimed that the list is complete. The claim boils down to the validity of certain conjecture, strongly related

Keywords: Annulus, cuspidal singular point, codimension.
Math. classification: 14H50, 14R10, 14B05.
(*) The first author is supported by Foundation for Polish Science and both authors are supported by Polish MNiSzW Grant No 2 P03A 015 29.

to the unobstructedness problem of [7]. (We also refer the reader to the paper [6] where a partial classification of annuli is given.)

It turns out that our method, even without assuming the above–mentioned conjecture, can be applied to prove a conjecture by Lin and Zaidenberg [12] specified to annuli. The latter conjecture states that any algebraic curve in $\mathbb{C}^2$ with the first Betti number equal to r can have at most $2r+1$ singular points. In the present paper we prove the following theorem, which confirms the Zaidenberg–Lin conjecture for annuli.

Theorem 1.1. — *Any algebraic curve in $\mathbb{C}^2$ homeomorphic to $\mathbb{C}^*$ has at most three singular points.*

The method of the proof is as follows. We use a notion of codimension of a singular point (see [4]). This is the number of conditions for a parametric curve required so that this curve has a given singularity (up to a topological equivalence). A parameter count argument would give the bound for the sum of codimensions over all singular points of the given curve by the dimension of the space of parametric curves. This dimension depends linearly on the degree of the curve under consideration. In [5, Conjecture 3.7] we conjectured such bounds. While we do not have the proof of these bounds, we noted that a slightly weaker codimension bound can be obtained using Bogomolov–Miyaoka–Yau (BMY) inequality (compare [4]). This bound being insufficient to prove that the list in [5] is complete, at least without an additional work, yet is suitable to verify that an annulus cannot have more than three finite singular points.

We believe that our methods can settle the conjecture for all rational curves in $\mathbb{C}^2$. However the computations in the general case seem to be highly complex. In the case of affine plane curves of arbitrary genus with one place at infinity some estimates for the number of singular points have been recently obtained in [2].

Acknowledgments. — This paper was motivated and partially written during the workshop “Affine algebraic geometry” in Oberwolfach. We are grateful to the organizers for inviting us to this conference. We would like to thank M. Koras and V. Lin for stimulating discussions. The first author thanks P. Russell for interesting discussions and an invitation to visit the McGill University.

2. Invariants of singular points

Here we present some notions and estimates from [3, 5, 4].

2.1. Local invariants of singularities of curves

Let $(A,0), A=\{f(x,y)=0\}\subset\mathbb{C}^2$, be a germ of a reduced plane curve near its singular point.

The first invariants of this singularity are: the *number of branches* (irreducible components), denoted by k, and the *multiplicity*, denoted by $\mathrm{mult}_0 A = m$. The latter is the order of the first nonzero term in the Taylor expansion of the defining function f. In this work we consider only the cases with $k=1$ (cuspidal singularities) and $k=2$ (for an annulus it may occur at infinity).

Next invariant is the *external codimension* of a singularity, denoted *ext*ν and defined as follows.

Let

$$x=\tau^m,\quad y=c_1\tau+c_2\tau^2+\dots \tag{2.1}$$

be the Puiseux expansion of A in the cuspidal case ($k=1$). In the space of germs as above (i.e. with fixed multiplicity m) strata of topological equivalence (or so-called μ =const strata) are defined by vanishing of some number of certain Puiseux quantities c_j and by nonzero some other Puiseux quantities; in [5, Section 2.II] and [4, Section 2.1], the quantities c_j which appear in descriptions of these strata are called the *essential Puiseux quantities*. The number of vanishing essential Puiseux quantities is the $y-$*codimension* denoted by ν (see [4, Section 2.1]). This can be explained in terms of the so-called *topologically arranged Puiseux expansion*

$$y=\left(d_0x^{\frac{n_0}{1}}+\dots\right)+\left(d_1x^{\frac{n_1}{m_1}}+\dots\right)+\dots+\left(d_rx^{\frac{n_r}{m_1\cdots m_r}}+\dots\right),$$

where n_j and m_j are positive integers (with $1=m_0$ and $m_j>1$ for $j\geqslant 1$) such that $\gcd(m_j,n_j)=1$, $m_1\dots m_r=m$, the nonzero coefficients $d_1,\dots,d_r$ constitute a part of the essential Puiseux quantities and the dots in the j^{th} summand mean terms with $x^{\frac{i}{m_0\cdots m_j}}$. Here the first (inessential) summand can be absent and the pairs (m_j,n_j), $j\geqslant 1$, are known as the *Puiseux pairs*. The other essential Puiseux quantities, i.e. other than $d_j=c_{n_jm_{j+1}\dots m_r}$ for $j=1,\dots,r$, correspond to those terms $c_ix^{i/m}$ whose potential presence would change the essential part of the above topologically arranged Puiseux expansion.

For example, in the case $m=2$ the strata of topological equivalence are defined by $c_1=c_3=\dots=c_{2\nu-1}=0\neq c_{2\nu+1}$ in (2.1). In the case $m=4$ the conditions $c_1=c_2=c_3=0\neq c_5$, $c_1=c_2=c_3=c_5=c_7=c_9=0\neq c_6c_{11}$ and $c_1=c_2=c_3=c_5=c_6=c_7=c_9=0\neq c_{10}c_{11}$ define three μ =const strata with $\nu=3$, $\nu=6$ and $\nu=7$ respectively. We see that the name "essential" for a Puiseux quantity sometimes depends on the stratum (like for c_{10} above), but the quantities $c_m, c_{2m}, \dots$ are always inessential.

In the cuspidal case we put

$$ext\nu = \nu + m - 2;$$

the additional contribution to $ext\nu$ arises from the conditions for the $m-1$ first derivatives of x (with respect to a parameter t on the curve) to vanish at the singular point.

(Formally one obtains $m-1$ independent conditions, i.e. in the space of local parametric curves $(\mathbb{C}, 0) \longmapsto (\mathbb{C}^2, 0)$. However, in the space of global parametric curves, like in (2.3) below, the positions of their singular points are not fixed. So the condition $dx/dt = 0$ is just the equation for values of the parameter at the singular points and, as such, it does not enter into the collection of "external conditions" for the singularity.)

In the two branches case, $A = A_1 + A_2$ with the multiplicities m_1 and m_2, besides the $y-$codimensions $\nu(A_1)$ and $\nu(A_2)$, we have also the *tangency codimension* $\nu_{\tan}$ between the two branches. It is the number of inessential Puiseux quantities and nonzero essential Puiseux quantities in the common part of the Puiseux expansions of the two branches (we choose the roots of unity of orders m_1 and m_2 to make this common part as long as possible). $\nu_{\tan}$ is a topological invariant of the singularity, because it controls the intersection index of the branches.

For example, if the Puiseux expansions of the two branches are $y = \alpha x^{3/2} + x^2 + \dots$ and $y = \beta x^{3/2} - x^2 + \dots$ (with $m_1 = m_2 = 2$) then $\nu(A_1) = \nu(A_2) = 1$ and $\nu_{\tan} = 1$ when $\alpha^2 \neq \beta^2$ and $\nu_{\tan} = 2$ otherwise.

Here we put

$$ext\nu = \nu(A_1) + \nu(A_2) + \nu_{\tan} + m_1 + m_2 - 2.$$

There exists another interpretation of the external codimension. Namely, we take the *minimal normal crossing resolution* of the singular point $\pi : (V, D) \to (U, A)$, where U is a neighborhood of the origin in $\mathbb{C}^2$. Letting $E = E_1 + \dots + E_l$ be the *exceptional divisor* with components E_j, we consider the vector space

$$\mathrm{Vect}(E) = \mathbb{Q}E_1 \oplus \dots \oplus \mathbb{Q}E_l$$

equipped with the intersection form. Then the strict transform $\widetilde{A}$ of A, as well as D, the reduced total transform of A, are interpreted as elements of $\mathrm{Vect}(E)$. We have also the *local canonical divisor* K defined by the relations $E_j(K + E_j) = -2$ (see [14]).

The following result was proved by S. Orevkov [8] in the cuspidal case and in [4, Proposition 4.1] in general. Orevkov calls the quantity $K(K+D)$ the *rough M–number.*

Proposition 2.1. — *We have*

$$ext\nu = K(K+D).$$

A classical invariant of singularity is the *number of double points*, denoted by δ (sometimes called the delta invariant). In the cuspidal case it equals $\mu/2$, where μ is the *Milnor number* of the singularity. Generally it is the number of double points of a parametric deformation of the curve A : we take a map from a disjoint union $\coprod\{|z|<\varepsilon\}$ of complex discs to $\mathbb{C}^2$ which is a small generic perturbation of the normalization map. In this sense we can interpret δ as the number of double points which are hidden at the singularity.

For example, for the $\mathbf{A}_\mu$ singularity $y^2 = x^{\mu+1}$ we have $\delta = \mu/2$ if μ is even and $\delta = (\mu+1)/2$ if μ is odd.

The following inequality was proved in [3, Proposition 2.9 and Proposition 2.16].

Proposition 2.2. — *If the number of branches is $k = 1$ or $k = 2$ and $m = \mathrm{mult}_0 A$ then*

$$2\delta \leqslant m(ext\nu - m + k + 1).$$

In the above vector space $\mathrm{Vect}(E)$, related with the resolution of singularity, we can use the local *Zariski–Fujita decomposition* [14]

$$K + D = P + N,$$

where P is the positive and N the negative part of $K+D$ (with respect to the intersection form). Then we define the *excess* of the singular point as

$$\eta := -N^2 \geqslant 0. \tag{2.2}$$

This is also a topological invariant, because it is defined via the intersection form on the space $\mathrm{Vect}(E)$.

The following result follows from a rather subtle analysis of the intersection form via dual graph by Orevkov and Zaidenberg [14] (see also [4, Proposition 4.2]). Below we use the notations $\lfloor x\rfloor = \max\{n : n\in\mathbb{Z}, n\leqslant x\}$, $\lceil x\rceil = \min\{n : n\in\mathbb{Z}, x\leqslant n\}$.

Proposition 2.3. — *If (m,n) is the first characteristic pair of an unibranched singularity then its excess (2.2) satisfies*

$$\eta \geqslant (\lceil m/n\rceil - m/n) + (\lceil n/m\rceil - n/m)\,.$$

2.2. Invariants of the annuli

Consider an annulus C given in parametric form by

$$
\begin{aligned}
x &= \varphi(t) = t^p + a_1 t^{p-1} + \cdots + a_{p+r} t^{-r} \\
y &= \psi(t) = t^q + b_1 t^{q-1} + \cdots + b_{q+s} t^{-s},
\end{aligned} \tag{2.3}
$$

where $a_{p+r} b_{q+s} \neq 0$. The numbers p, q, r and s are integers and we can assume that $p, s > 0$ (since we have a topological embedding of $\mathbb{C}^*$).

Such an annulus may have several finite singular points corresponding to the values $t_1, \ldots, t_N$ of t. They are all cuspidal. The above invariants associated with each point t_i are denoted by m_i, $ext\nu_i$, δ_i and η_i.

We denote by ν_∞ the so-called *subtle codimension* of the branch of C as t goes to infinity, which is the codimension of the topological equivalence stratum in the space of germs of the form

$$x = \tau^{-p}, y = \tau^{-q} + c_1 \tau^{-q+1} + \ldots, \tau \to 0$$

(compare [4, Definition 2.6]). Analogously we define the *subtle codimension* ν_0 of the branch of C as $t \to 0$.

The last invariant of the curve C is the *tangency codimension* $\nu_{\tan}$ at infinity. More precisely, if $ps \neq rq$ then the two branches of C do not intersect and we put $\nu_{\tan} = 0$. If $ps = rq$ then $\nu_{\tan}$ is defined as above for a two branches singularity. We use the notion of $\nu_{\tan}$ only in Section 3.2 (Case B2). Sometimes we will use the notation

$$\nu_{\inf} = \nu_0 + \nu_\infty + \nu_{\tan}. \tag{2.4}$$

For the purpose of proving the Main Theorem, the above quantities are not that important as the inequalities that relate them. The first identity, which is a direct consequence of the standard genus formula (or the Poincaré–Hopf formula), can be found in [5, Proposition 2.9 and Eq. (2.11)].

Proposition 2.4. — *A generic curve of the form* (2.3) *has*

$$2\delta_{\max} := (p + r - 1)(q + s - 1) + |ps - rq| - p' - r' + 1$$

finite simple double points, where

$$p' = \gcd(p, q), \quad r' = \gcd(r, s). \tag{2.5}$$

Since we are interested in the annuli, which by definition do not have self-intersections, the $\delta_{\max}$ double points must be hidden at singular points and/or at infinity:

$$\delta_{\max} = \sum \delta_i + \delta_{\inf}.$$

The numbers δ_i are estimated directly in Proposition 2.2,

$$2\delta_i \leqslant m_i(ext\nu_i - m_i + 2).$$

From that proposition we find also a bound for the number of double points hidden at infinity (see [5, Proposition 2.29]):

$$\begin{aligned} 2\delta_{\inf} &\leqslant p'\nu_\infty + r'\nu_0 \quad \text{if} \quad ps \neq qr. \\ 2\delta_{\inf} &\leqslant (p'+r')(\nu_{\inf}+1) \quad \text{if} \quad ps = qr. \end{aligned}$$

We introduce the following quantity:

$$(2.6) \qquad \begin{array}{ll} \mathcal{E} = \sum_{i=1}^N m_i(ext\nu_i - m_i + 2) + p'\nu_\infty + r'\nu_0 & \text{if } ps \neq qr, \\ \mathcal{E} = \sum_{i=1}^N m_i(ext\nu_i - m_i + 2) + (p'+r')(\nu_{\inf}+1) & \text{if } ps = qr. \end{array}$$

By the above local estimates the inequality

$$(2.7) \qquad \Delta := 2\delta_{\max} - \mathcal{E} \leqslant 0$$

holds for an annulus of the form (2.3). The quantity Δ is called the *reserve* in [5, Section 2.1].

Next we would like to bound the sum of codimensions. The bound depends on values of the exponents p, q, r and s.

Definition 2.5. — *A curve C given in* (2.3) *is of*

- *type* $\binom{+}{+}$ *if* $0 < p < q$ *and* $0 < r < s$, $p + r < q + s$*;*
- *type* $\binom{-+}{+-}$ *if* $0 < q < p$ *and* $0 < r < s$, $p + r \leqslant q + s$*;*
- *type* $\binom{-}{+}$ *if* $r < 0$ *and* $q > 0$*;*
- *type* $\binom{-}{-}$ *if* $r < 0$ *and* $q < 0$, $p + r \leqslant q + s$.

Recall that with the open surface $V_0 = \mathbb{C}^2 \setminus C$ we can associate its *logarithmic Kodaira dimension* $\bar\kappa(\mathbb{C}^2 \setminus C)$. It is defined via the normal crossing completion F of V_0 such that $V = V_0 \cup F$ is smooth projective surface. Then

$$\bar\kappa(V_0) = \limsup \log h^0(V, n(K_V + F))/\log n.$$

If $\bar\kappa(V_0) = 2$ then we say that the surface V_0 is of *general type*. I. Wakabayashi [Wa] calculated the logarithmic Kodaira dimension of $\mathbb{C}^2 \setminus C$ in some important cases. From [10] one can deduce, in particular, the following fact.

Proposition 2.6. — *If an annulus C has more than three finite singular pints then the surface $\mathbb{C}^2 \setminus C$ is of general type.*

The codimension bounds we give below were proved in [4, Theorem 4.3]; they essentially rely upon the Bogomolov–Miyaoka–Yau inequality (which was also used by Zaidenberg and Orevkov [8, 14]). Here we state only the result.

Introduce the quantity

$$S := \sum_{i=1}^{N} (ext\nu_i + \eta_i) + \nu_{\mathrm{inf}}, \tag{2.8}$$

which can be regarded roughly as the sum of local codimensions.

Proposition 2.7. — *Let C be an annulus given by (2.3) homeomorphic to $\mathbb{C}^*$ and such that its complement in $\mathbb{C}^2$ is of general type. Then depending on the type of the annulus we have:*

(a) for type $\binom{+}{+}$

$$S \leqslant p+r+q+s+1-\min(\lfloor q/p \rfloor, \lfloor s/r \rfloor) \leqslant p+r+q+s; \tag{2.9}$$

(b) for type $\binom{-+}{+-}$

$$S \leqslant p+r+q+s+1; \tag{2.10}$$

(c) for type $\binom{-}{+}$

$$S \leqslant p-|r|+q+s+2+\lfloor (|r|-1)/s \rfloor - \lfloor q/p \rfloor; \tag{2.11}$$

(d) for type $\binom{-}{-}$

$$S \leqslant p-|r|-|q|+s+3+\lfloor (|r|-1)/s \rfloor + \lfloor (|q|-1)/p \rfloor. \tag{2.12}$$

For the multiplicities m_i and the excesses η_i (see (2.2)) of singular points we have the following bounds.

Lemma 2.8. —

$$\sum_{i=1}^{N} (m_i - 1) \leqslant \min(p+r, q+s).$$

In particular, $N \leqslant p+r$.

Proof. — Assume that $p+r \leqslant q+s$. Then $\dot{x} = d\varphi/dt = R(t)t^{-r-1}$, where $R(t)$ is a polynomial of degree $p+r$ (see (2.3)). If $n_i - 1$ is the order of $d\varphi/dt$ at the i^{th} singular point, then clearly $\sum(n_i-1) \leqslant p+r$ and $m_i \leqslant n_i$. The second statement is obvious. □

Lemma 2.9. — *(a) $\eta_i > 1/2$, thus if $N \geqslant 4$ then $\sum \eta_i > 2$.*

(b) If the multiplicity of a singular point is $m_i = 2$ then $\eta_i \geqslant 5/6$.

(c) If $N \geqslant 4$ and $\min(p+r, q+s) \leqslant 5$ then $\sum \eta_i > 3$.

Proof. — The first two assertions follow directly from Proposition 2.3. In *(c)* we must have $m_i = 2$ for at least three singular points if their number $N = 4$ and all $m_i = 2$ if $N = 5$. Therefore, by Proposition 2.3, either $\sum \eta_i > 3 \cdot \frac{5}{6} + \frac{1}{2} = 3$ or $\sum \eta_i \geqslant 5 \cdot \frac{5}{6} > 4$. $\square$

Two technical statements below turned out useful. The first one is often used in [5, Lemma 5.3].

LEMMA 2.10. — *If $ps - rq \neq 0$ then the quantity*

$$\det{}' := |ps - rq| - p' - r' + 1$$

is a non-negative integer.

The second lemma gives a partial answer to the problem of finding the best parametrization of an annulus given by (2.3). In fact if, say, $x = t^2 + \cdots + t^{-6}$ and $y = t^4 + \cdots + t^{-9}$ we can ask whether it is reasonable to apply a de Jonquière transform $y \to y - x^2$ to reduce the order of y at $t \to \infty$ at the cost of increasing its order as $t \to 0$. We prove that there exists (maybe not unique) way of choosing an automorphism of $\mathbb{C}^2$ that suits best to our estimates.

DEFINITION 2.11. — *A curve C is called ugly if one of the following holds:*

- *it is of type $\binom{+}{+}$, $q/p \in \mathbb{Z}$ and $r < p$;*
- *it is of type $\binom{-+}{+-}$ and either $p/q \in \mathbb{Z}$ and $s < q$ or $s/r \in \mathbb{Z}$ and $p < r$;*
- *it is of type $\binom{-}{+}$, $p/q \in \mathbb{Z}$ and $s < q$.*

Otherwise the curve C is called handsome.

LEMMA 2.12. — *Any curve as in* (2.3) *can be transformed to a handsome one by applying a Cremona automorphism of $\mathbb{C}^2$ and, possibly, the change $t \to 1/t$.*

A straightforward proof is presented in [5, Proposition 2.45].

2.3. Scheme of the proof of Main Theorem

We can order the singular points of C so that $m_1 \geqslant m_2 \cdots \geqslant m_N$.

Recall that we must rule out the possibility $N \geqslant 4$. But one quickly realizes that considering the case $N = 4$ is sufficient. As in [5] the estimates become easier when N grows. For example, the codimension bound is stronger already for $N = 5$.

We split the proof into following five cases:

A Type $\binom{+}{+}$ with $ps \neq rq$,
B Type $\binom{+}{+}$ with $ps = rq$,
C Type $\binom{-+}{+-}$,
D Type $\binom{-}{+}$,
E Type $\binom{-}{-}$.

Each case can be split in turn into two subcases.

(1) We assume that double points hide at finite singular points. This means that the quantity $\mathcal{E}$ from (2.6), which we try to maximize, is greatest when $\delta_{\text{inf}} = 0$. Then it is easy to see that $\mathcal{E}$ is maximal possible, when the multiplicity m_1 and the external codimension $ext\nu_1$ are maximal, and the other multiplicities and $ext\nu$ numbers, including $\nu_{\text{inf}} = \nu_0 + \nu_\infty + \nu_{\tan}$ (see (2.4)) are minimal. Here we have $m_1 \geqslant \max(p', r')$ (see (2.5)) in cases A, C, D, E and $m_1 \geqslant p' + r'$ in case B.

(2) We assume that ν_{inf} is large, so double points hide at infinity (i.e. $\sum \delta_j$ is small relatively to δ_{inf}). Then $\mathcal{E}$ is maximal if all codimensions of singularities at finite distance are minimal and the codimension at infinity is maximal possible. Here either p' or r' exceeds m_1 in cases A, C, D, E or $p' + r' > m_1$ in case B.

In all cases we shall strive to prove that the reserve $\Delta > 0$, which contradicts inequality (2.7). To simplify arguments we will assume that the curve has precisely $N = 4$ singular points.

3. Proof of Main Theorem

3.1. Cases A1 and B1.

These two cases are very similar. Here $\mathcal{E}$ is maximal if $m_2 = m_3 = m_4 = 2$, $ext\nu_2 = ext\nu_3 = ext\nu_4 = 1$ and $\nu_\infty = \nu_0 = 0$. Hence $m_1 \leqslant p+r-2$, and $ext\nu_1 \leqslant p + r + q + s - 6$ (-3 coming from $ext\nu_2 + ext\nu_3 + ext\nu_4$ and another -3 from $\sum \eta_i > 2$, see (2.8)–(2.9) and Lemma 2.9 (a)). Therefore, by (2.6),

$$\mathcal{E} \leqslant (p + r - 2)(q + s - 2) + 6.$$

Let $ps \neq rq$. We have $2\delta_{\max} \geqslant (p + r - 1)(q + s - 1)$ (see Proposition 2.4 and Lemma 2.10). Thus

$$\Delta = 2\delta_{\max} - \mathcal{E} = (p + r + q + s) - 9.$$

But $p+r \geqslant 4$ by Lemma 2.8. Moreover, $q > p$ and $s > r$ by Definition 2.5. Hence $q+s \geqslant 6$ and $\Delta > 0$, so that there is no such curve C in this case (compare (2.7)).

Let $ps = rq$, so $2\delta_{\max} = (p+r-1)(q+s-1) - (p'+r') + 1$ and $\mathcal{E}$ is bounded as above. Therefore

$$\Delta = (p+r) + (q+s) - (p'+r') - 8$$

where $p+r = p_1(p'+r')$, $q+s = q_1(p'+r')$ and $2 \leqslant p_1 < q_1$. We find that the only possibility for $\Delta \leqslant 0$ is $p+r=4$, $q+s=6$. But then Lemma 2.9 (c) gives $\sum \eta_i > 3$. Repeating the above procedure, we get $\mathcal{E} \leqslant 12$ and $2\delta_{\max} = 14$.

3.2. Case A2

Here we assume that the contribution from finite singular points is small and the contribution from infinity is maximal possible, so that $ext\nu_1 = \cdots = ext\nu_4 = 1$ and ν_∞ is maximal (the case with ν_0 maximal is analogous). By Proposition 2.7 (inequality (2.9)) and Lemma 2.9 (a) we get

$$\nu_\infty \leqslant p+r+q+s-7.$$

Hence, by formula (2.6) for $ps \neq qr$,

$$\mathcal{E} \leqslant p'(p+r+q+s-7) + 8$$

where $p' = \gcd(p,q)$ (see (2.5)). Therefore

$$\Delta = 2\delta_{\max} - \mathcal{E} \geqslant (p+r-1-p')(q+s-1-p') - (p')^2 + 5p' - 8.$$

We have $p' \geqslant 2$, since otherwise ($p'=1$) the singularity at infinity is quasi–homogeneous and $\nu_\infty = 0$ by definition. Therefore it is enough to prove that $(p+r-1-p')(q+s-1-p') \geqslant (p')^2 - 1$. This is obviously true if $p \geqslant 2p'$, since $q \geqslant p'+p$. Otherwise, the handsomeness property ensures that $r \geqslant p'$, so $p+r-1-p' \geqslant p'-1$ and $q+s-1-p' \geqslant 2p'-1$. Hence we ask whether $(p'-1)(2p'-1) \geqslant (p')^2 - 1$. But this is always true for $p' \geqslant 2$.

3.3. Case B2

Let us denote $p+r = e$, $q+s = f$, $\gcd(e,f) = p'+r' = e'$. From inequality (2.9) in Proposition 2.7 we get (as in case A2)

$$\nu_{\inf} = \nu_0 + \nu_\infty + \nu_{\tan} \leqslant e+f-7.$$

Hence $\mathcal{E} \leqslant e'(e+f-6)+8$ (see (2.6)), whereas $2\delta_{\max} = (e-1)(f-1)+1-e'$ (see Proposition 2.4). Therefore $\Delta \geqslant (e-e'-1)(f-e'-1)-7-(e')^2+3e'$. Since $e \geqslant 2e'$ and $f \geqslant 3e'$, we get $\Delta > 0$ if $(e')^2 \geqslant 7$. So we assume that $e' = 2$ and then $\Delta \geqslant (e-3)(f-3)-5$. If $e \geqslant 6$ then $f \geqslant 8$ and we get $\Delta > 0$. Hence $e = 4$ (it must be even). But then Lemma 2.9 (c) implies that $\nu_{\mathrm{inf}} \leqslant e+f-8$, so $\Delta \geqslant (e-e'-1)(f-e'-1)-7-(e')^2+4e'$. We observe that $\Delta > 0$, unless $e = 4$ and $f = 6$ in which case we obtain $\Delta = 0$.

We have to exclude the latter possibility. This can be done by computing the sum of δ–invariants of singularities (numbers of double points) of the curve C explicitly. If $e = 4$ and $f = 6$ then $p = r = 2$, $q = s = 3$. As $t \to \infty$ (respectively $t \to 0$) we have $x \sim t^2$, $y \sim t^3$ (respectively $x \sim t^{-2}$, $y \sim t^{-3}$). In the local coordinates $u = x/y$, $w = 1/y$, $s = t^{-1}$ and $s \to 0$ we have $u = s + \dots$, $w = s^3 + \dots$. Thus the both branches are smooth at infinity. Then $\nu_0 = \nu_\infty = 0$ and $\nu_{\mathrm{inf}} = \nu_{\mathrm{tan}}$. The requirement $\Delta = 0$ implies $\nu_{\mathrm{tan}} = 2$. Therefore, if we consider Puiseux expansions

$$y = c_0 x^{3/2} + c_1 x + c_2 x^{1/2} + \dots \qquad \text{as } t \to \infty,$$
$$y = d_0 x^{3/2} + d_1 x + d_2 x^{1/2} + \dots \qquad \text{as } t \to 0,$$

then we must have $c_0 = d_0$ and $c_1 = d_1$ and the codimension bound prohibits that $c_2 = d_2$ if earlier terms agree. It follows that the intersection index of the two branches at infinity is 5. So the δ–invariant of the singularity at infinity is 5. Adding 4 from the cusps at finite distance we obtain 9. But C is rational of degree 6, so the sum of its δ–invariants is $\frac{1}{2}\,5\cdot 4$. Hence it must have an additional double point at finite distance.

3.4. Case C1

Similarly as in case A1 (using the bound (2.10) and Lemma 2.9 (a)) we get

$$ext\nu_1 \leqslant p+q+r+s-5.$$

We get then $\mathcal{E} \leqslant (p+r-2)(q+s-1)+6$ and hence $\Delta \geqslant (q+s-1)+\det{}'-6$, where $\det{}'$ is defined in Lemma 2.10.

Lemma 3.1. — *We have* $\det{}' \geqslant p'r'+1 \geqslant 2$.

Proof. — As $p \geqslant q+p'$ and $s \geqslant r+r'$, we infer that

$$\det{}' \geqslant p'r' + (q-1)r' + (r-1)p' + 1 \geqslant p'r'+1.$$

□

By Lemma 3.1 we get $\Delta \geqslant q+s-5$. So, if $q+s \geqslant 6$, then we are done. Suppose $q+s \leqslant 5$. Then $p+r \leqslant 5$ and we apply Lemma 2.9 (c), so that $ext\nu_1 \leqslant p+r+q+s-6$ and $\Delta \geqslant p+r+q+s+\det{}'-9 \geqslant 1$.

3.5. Case C2

As in case B2 we assume that ν_∞ is maximal. We get $\nu_\infty \leqslant p+r+q+s-6$ so $\mathcal{E} \leqslant p'(p+r+q+s-6)+8$ and

$$\Delta \geqslant (p+r-1-p')(q+s-1-p') - (p')^2 + 4p' - 8 + \det{}'.$$

As $p' \geqslant 2$, using Lemma 3.1, it suffices to show that

$$(p+r-p'-1)(q+s-p'-1) \geqslant (p')^2 - p'.$$

If $q \geqslant 2p'$ then $p \geqslant 3p'$. Thus $p+r-p'-1 \geqslant 2p'$ and $q+s-p'-1 \geqslant p'+1$ $(s > r > 0)$ and we are done. So assume that $q = p'$. By the handsomeness $s \geqslant p'$ (see Definition 2.11 and Lemma 2.12). This implies that $p+r-p'-1 \geqslant p'$ and $q+s-p'-1 \geqslant p'-1$. Hence $\Delta > 0$.

3.6. Case D1

We use the bound $\sum ext\nu_i + \nu_\infty \leqslant p-|r|+q+s+2+\lfloor (|r|-1)/s \rfloor - \sum \eta_i$ from (2.11). We will treat only the case $p+r \leqslant q+s$; the computations are almost identical in the opposite case.

Assume firstly that $m_1 \geqslant m_2 \geqslant 3$. Then $m_1 \leqslant p+r-3$ and $p+r \geqslant 6$. Moreover, $ext\nu_2 \geqslant 3$ (because the coefficients before $(t-t_2)$ and $(t-t_2)^2$ in both $\varphi(t)$ and $\psi(t)$ in (2.3) must vanish). Therefore $ext\nu_1 \leqslant p-|r|+q+s-6+\lfloor (|r|-1)/s \rfloor$. Hence

$$\mathcal{E} = (p-|r|-3)\left(q+s-1+\left\lfloor \frac{|r|-1}{s} \right\rfloor\right) + 3\cdot(3-1)+2+2.$$

Thus

$$\Delta = 2(q+s) - p' - r' - 11 + ps + |r|q - (p-|r|-3)\cdot \left\lfloor \frac{|r|-1}{s} \right\rfloor.$$

Now, since $p-|r| \leqslant q+s$, we get

$$|r|q - (p-|r|)\cdot \left\lfloor \frac{|r|-1}{s} \right\rfloor \geqslant 1 - |r| + |r|q\left(1-\frac{1}{s}\right) + \frac{q}{s}. \tag{3.1}$$

Substituting this into Δ we obtain

$$\Delta \geqslant 2(q+s) - p' - r' - 10 + ps - |r| + |r|q\left(1-\frac{1}{s}\right) + \frac{q}{s}.$$

But $|r|q(1-\frac{1}{s})+\frac{q}{s}\geqslant 0$. It follows that

$$\Delta \geqslant (q+s)+(p-|r|)-10+p(s-1)+(q+s-p'-r')\geqslant 2,$$

since $q+s\geqslant 6$, $p-|r|\geqslant 6$ and the last two terms in the above formula are non-negative.

We are left with the case $m_2=2$. Then $m_3=m_4=2$, so $\sum \eta_i > 3$ by Lemma 2.9 (b). We obtain $ext\nu_1 \leqslant p-|r|+q+s-5+\lfloor (|r|-1)/s\rfloor$. Hence

$$\mathcal{E}=(p-|r|-2)\left(q+s-1+\left\lfloor\frac{|r|-1}{s}\right\rfloor\right)+6.$$

Therefore

$$\Delta = q+s-p'-r'-6+ps+|r|q-(p-|r|-2)\cdot\left\lfloor\frac{|r|-1}{s}\right\rfloor.$$

If $s=1$ then $\Delta = q-p'-8+|r|(q-p+|r|+3)+2p\geqslant q+2p-p'+2|r|-8$, because $p-|r|\leqslant q+s=q+1$. But $p'\leqslant p/2$ and $p\geqslant 5$, as $p-|r|\geqslant 4$. So $\Delta>0$.

Finally, let us assume that $s\geqslant 2$. By (3.1) we have

$$\Delta\geqslant q+s-p'-r'-5+ps-|r|.$$

Then $q+s-p'-r'\geqslant 0$ and $ps-|r|\geqslant p+p-|r|\geqslant 9$, so $\Delta\geqslant 4$.

3.7. Case D2

Here ν_∞ (or ν_0) is bounded from above by $p-|r|+q+s+\lfloor(|r|-1)/s\rfloor+2-7$. Assume that $p'\geqslant r'$. It follows that

$$\mathcal{E}\leqslant 8+p'(p-|r|+q+s+\lfloor(|r|-1)/s\rfloor-5).$$

So

$$\begin{aligned}\Delta =&(p-|r|-1)(q+s-1)+ps+|r|q-p'-r'-7\\ &-p'\left(p-|r|+q+s+\left\lfloor\frac{|r|-1}{s}\right\rfloor-5\right)\end{aligned}$$

This can be transformed into

$$\begin{aligned}\Delta =&(p-p'-1)(q+s-p'-1)+(p-|r|)s\\ &+\left(|r|-p'-\left\lfloor\frac{|r|-1}{s}\right\rfloor+2\right)p'+(|r|-r')-7.\end{aligned}\tag{3.2}$$

Assume that $q\geqslant 2p'$. Obviously, also $p\geqslant 2p'$ and $p\neq q$; thus either p or q is at least $3p'$. Then $(p-p'-1)(q+s-p'-1)\geqslant 2p'(p'-1)$. Moreover,

$(p-|r|)s \geqslant 4$ and $|r|-p'-\lfloor(|r|-1)/s\rfloor+2 \geqslant 3-p'$. Hence $\Delta \geqslant (p')^2+p'-3>0$, as $p' \geqslant 2$.

Therefore $q=p'$. By the handsomeness $s \geqslant p'$ (see Definition 2.11 and Lemma 2.12). It follows that $(p-p'-1)(q+s-p'-1) \geqslant (p'-1)^2$. Thus

$$\Delta \geqslant (p'-1)^2+(p-|r|)s+(3-p')p'-7.$$

But $(p-|r|)s \geqslant 8$, so $\Delta>0$.

Now let us turn to the case $r'>p'$. Equation (3.2) then becomes

$$\Delta \geqslant (p-r'-1)(q+s-r'-1)+(p-|r|)s + \left(|r|-r'-\left\lfloor\frac{|r|-1}{s}\right\rfloor+2\right)r'+|r|-p'-7.$$

Here $s \geqslant r' \geqslant 2$. If $|r| \geqslant 2r'$ then $|r|-r'-\lfloor(|r|-1)/s\rfloor>0$. Since $p-r' \geqslant p-|r|$ we have $(p-r'-1)(q+s-r'-1) \geqslant 0$. We infer that $\Delta \geqslant (p-|r|)s+2r'+|r|-p'-7>1$ for $|r|>r'>p'$. So let $|r|=r'$. Then $|r| \leqslant s$, so $\lfloor(|r|-1)/s\rfloor=0$. Hence $\Delta \geqslant (p-|r|)s-7>0$.

3.8. Case E1

We have here

$$\sum ext\nu_i+\nu_\infty \leqslant p-|r|+s-|q|+\lfloor(|q|-1)/p\rfloor+\lfloor(|s|-1)/r\rfloor+3-\sum\eta_i,$$

i.e. the bound (2.12) holds. It is easy to observe that at most one of the $\lfloor(|q|-1)/p\rfloor$ and $\lfloor(|s|-1)/r\rfloor$ can be non–zero. Following [5] we introduce the quantities $K=p-|r|$ and $L=s-|q|$ with $K \leqslant L$.

Subcase (i): $\lfloor(|q|-1)/p\rfloor>0$. Putting $ext\nu_i=1$ for $i \geqslant 2$, we get $ext\nu_1<K+L+\lfloor(|q|-1)/p\rfloor+3-3-2$. Since all terms in this inequality are integers, we have $ext\nu_1 \leqslant K+L+\lfloor(|q|-1)/p\rfloor-3$ and $m_1 \leqslant K-2$. So

$$\mathcal{E} \leqslant (K-2)\left(L+\left\lfloor\frac{|q|-1}{p}\right\rfloor+1\right)+6.$$

Since $2\delta_{\max}=(K-1)(L-1)+|r|L+|q|K+KL-p'-r'+1$ we get

$$\Delta \geqslant KL+K(|q|-2)+L(|r|+1)-(K-2)\cdot\left\lfloor\frac{|q|-1}{p}\right\rfloor-p'-r'-2. \tag{3.3}$$

Now $p>K-2$. Therefore $(K-2)\cdot\lfloor(|q|-1)/p\rfloor \leqslant |q|-1$. Thus

$$\Delta \geqslant KL-3+(K-1)(|q|-2)+L(|r|+1)-p'-r'.$$

The above inequality can be rewritten as

$$\Delta \geqslant KL - 4 + (K-2)(|q|-2) + (L-1)(|r|+1) + (|q|-p') + (|r|-r').$$

We have $K, L \geqslant 4$ and $|q| \geqslant 3$. Therefore

$$\Delta \geqslant KL - 4 - (K-2) + 3 \cdot 2 > 0.$$

Subcase (ii): $\lfloor (|q|-1)/p \rfloor = 0$. Then we get an equation similar to (3.3)

$$\Delta \geqslant KL + K(|q|-2) + L(|r|+1) - (K-2) \cdot \left\lfloor \frac{|r|-1}{s} \right\rfloor - p' - r' - 2.$$

We have $K-2 \leqslant L-2 < s$ and hence $(K-2) \cdot \lfloor (|r|-1)/s \rfloor \leqslant |r|-1$. Using this we transform the above inequality into

$$\Delta \geqslant KL - 1 + (K-1)(|q|-2) + (L-2)(|r|+1) + (|q|-p') + (|r|-r').$$

As $|q| \geqslant 1$ we get $\Delta > 0$.

3.9. Case E2

Assume that $p' \geqslant r'$. We will not impose, however, the inequality $K \leqslant L$. Then $\mathcal{E} \leqslant p'(K + L - 4 + \lfloor (|q|-1)/p \rfloor + \lfloor (|r|-1)/s \rfloor) + 8$. On the other hand, $2\delta_{\max} = KL - K - L + 2 + K|q| + pL - p' - r'$. Henceforth

$$\Delta \geqslant KL - K - L - 6 + (p-p')L + (|q|-p')K - p' \left(\left\lfloor \frac{|q|-1}{p} \right\rfloor + \left\lfloor \frac{|r|-1}{s} \right\rfloor - 2 \right).$$

If $|r|-1 < s$ then $p' \cdot \lfloor (|q|-1)/p \rfloor \leqslant |q|-1$. Hence we are left with $\Delta \geqslant KL - K - L - 5 + p' + (|q|-p')(K-1)$, where the latter expression is positive.

Therefore $|r|-1 \geqslant s = |q| + L$. Since $p' \leqslant |q| < s$ we infer that $p' \cdot \lfloor (|r|-1)/s \rfloor \leqslant |r|-1$. Reminding that $(p-p')L > (p-p')$ we obtain

$$\Delta \geqslant KL - L - 5 + (|q|-p')K + p' + (p - K - |r|).$$

As $K + |r| = p$ we get $\Delta \geqslant (K-1)L - 5 + (|q|-p')K + p'$. Since $(K-1)L \geqslant 12$ we get $\Delta > 0$.

Now the proof of Main Theorem is complete. $\square$

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Manuscrit reçu le 28 février 2009,
accepté le 5 mars 2010.

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