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THE LOWER BOUND OF THE RICCI CURVATURE THAT YIELDS AN INFINITE DISCRETE SPECTRUM OF THE LAPLACIAN

by Hironori KUMURA

ABSTRACT. — This paper discusses the question whether the discrete spectrum of the Laplace-Beltrami operator is infinite or finite. The borderline-behavior of the curvatures for this problem will be completely determined.

RÉSUMÉ. — Ce document traite de la question si le spectre discret de l'opérateur de Laplace-Beltrami est infini ou fini. La ligne de démarcation du comportement des courbures de ce problème sera complètement déterminée.

1. Introduction

The Laplace-Beltrami operator Δ on a noncompact complete Riemannian manifold (M, g) is essentially self-adjoint on $C_0^\infty(M)$ and its self-adjoint extension to $L^2(M)$ has been studied by several authors from various points of view. In many cases, the bottom of the essential spectrum of $-\Delta$ will be positive (see Brooks [2]), and the discrete spectrum will appear below this bottom number. The purpose of this paper is to determine the borderline-behavior of the curvatures for the question whether the Laplace-Beltrami operator $-\Delta$ has a finite or infinite number of the discrete spectrum. The Rellich's lemma (see, for example, M. Taylor [12]) suggests that this problem depends on the geometry of manifolds *at infinity*. In the case of Schrödinger operators $-\Delta + V$ on the Euclidean space $\mathbf{R}^n$, the borderline-behavior $-\frac{(n-2)^2}{4r^2}$ of the potential V is determined by the *uncertainty principle lemma* $-\Delta \geqslant \frac{(n-2)^2}{4r^2}$ (see Reed-Simon [11] p. 169 and Kirsh-Simon [9]), which is equivalent to the Hardy's inequality $-\frac{d^2u}{dx^2} \geqslant \frac{1}{4x^2}$

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for $u \in C_0^\infty(0,\infty)$ (see, for example, [1]). Our proof will be concerned with this borderline-behavior of the Hardy's inequality (see Proposition 2.1 in Section 2).

Main theorems of this paper is the following:

Theorem 1.1. — *Let (M,g) be an n-dimensional noncompact complete Riemannian manifold and W a relatively compact open subset of M with C^∞-boundary ∂W. We set $r(*) := \mathrm{dist}(*, \partial W)$ on $M \smallsetminus W$. Let $\exp_{\partial W} : \mathcal{N}^+(\partial W) \to M \smallsetminus W$ be the outward normal exponential map and $\mathcal{C}ut(\partial W)$ the corresponding cut locus of ∂W in $M \smallsetminus W$, where*

$$\mathcal{N}^+(\partial W) := \left\{ v \in TM\big|_{\partial W} \mid v \textit{ is outward normal to } \partial W \right\}.$$

Assume that

$$\min \sigma_{\mathrm{ess}}(-\Delta) = \frac{(n-1)^2\kappa}{4}$$

for some constant $\kappa > 0$ and that there exist positive constants R_0 and β, satisfying $\beta > \frac{1}{(n-1)^2}$, such that

$$\mathrm{Ric}_g\left(\nabla r, \nabla r\right)(y) \geqslant (n-1)\left(-\kappa + \frac{\beta}{r(y)^2}\right) \qquad \textit{for } y \in M \smallsetminus (W \cup \mathcal{C}ut(\partial W)) \quad \textit{with} \quad r(y) \geqslant R_0,$$

where Ric_g and ∇r respectively stand for the Ricci curvature of (M,g) and the gradient of the function r. Then, the set

$$\sigma_{\mathrm{disc}}(-\Delta) \cap \left[0, \frac{(n-1)^2\kappa}{4}\right)$$

is infinite, where $\sigma_{\mathrm{disc}}(-\Delta)$ stands for the discrete spectrum of $-\Delta$.

Note that we do not assume that $M \smallsetminus W$ is connected in Theorem 1.1: hence, ∂W may have several but finite number of components.

Similarly, we get the following:

Theorem 1.2. — *Let (M,g) be an n-dimensional complete Riemannian manifold and p_0 be a point of M. We set $r(*) := \mathrm{dist}(*, p_0)$ and denote by $\mathcal{C}ut(p_0)$ the cut locus of p_0. Assume that*

$$\min \sigma_{\mathrm{ess}}(-\Delta) = \frac{(n-1)^2\kappa}{4}$$

for some constant $\kappa > 0$ and that there exist positive constants R_0 and β, satisfying $\beta > \frac{1}{(n-1)^2}$, such that

$$\mathrm{Ric}_g(\nabla r, \nabla r)(y) \geqslant (n-1)\left(-\kappa + \frac{\beta}{r(y)^2}\right) \qquad \text{for } y \in M \setminus \mathcal{C}ut(p_0) \text{ with } r(y) \geqslant R_0.$$

Then, the set

$$\sigma_{\mathrm{disc}}(-\Delta) \cap \left[0, \frac{(n-1)^2\kappa}{4}\right)$$

is infinite.

Although the topological property of manifolds is reflected in that of the cut locus, the theorem above does *not concern the property of the cut locus at all* but only the *Ricci curvatures of the radial direction on the complement of the cut locus.*

The following proposition shows that the curvature assumption in Theorem 1.1 and 1.2 are sharp:

Proposition 1.3. — *Let $\left(\mathbf{R}^n, dr^2 + h^2(r) g_{S^{n-1}(1)}\right)$ be a rotationally symmetric Riemannian manifold and assume that the radial curvature $K(r) = -\frac{h''(r)}{h(r)}$ satisfies*

$$K(r) \leqslant 0 \quad \text{for all } r \geqslant 0$$

and there exists constants $\kappa > 0$, $R_0 > 0$ and $\beta \neq \frac{1}{(n-1)^2}$ such that

$$K(r) = -\kappa + \frac{\beta}{r^2} \quad \text{for } r \geqslant R_0.$$

Then, $\sigma_{\mathrm{ess}}(-\Delta) = \left[\frac{(n-1)^2\kappa}{4}, \infty\right)$, and furthermore, $\sigma_{\mathrm{disc}}(-\Delta) \cap \left[0, \frac{(n-1)^2\kappa}{4}\right)$ is infinite if and only if $\beta > \frac{1}{(n-1)^2}$.

Indeed, under the assumptions in Proposition 1.3, $\mathrm{Ric}_g(\nabla r, \nabla r) = (n-1)K(r) = (n-1)\left(-\kappa + \frac{\beta}{r^2}\right)$, and hence, the lower bound of the Ricci curvature in Theorem 1.1 and 1.2 are sharp. That is, the borderline-behavior of curvatures for our problem can be said to be $-\kappa + \frac{1}{\{(n-1)r\}^2}$. See also [1], Theorem 3.1, for the finiteness-result on not necessarily rotationally symmetric manifolds.

2. Construction of a model space and eigenfunction

In this section, we shall construct a model space and study the property of an eigenfunction, which will be transplanted on M to prove Theorem 1.1.

Let $R_{\min}\colon [0,\infty) \to (-\infty, 0]$ be a nonpositive-valued continuous function satisfying

$$\mathrm{Ric}_g\,(\nabla r, \nabla r)(x) \geqslant (n-1)R_{\min}\,(r(x)) \;\text{ for } x \in M \smallsetminus \big(W \cup \mathcal{C}ut(\partial W)\big)$$

and

$$R_{\min}(r) = -\kappa + \frac{\beta}{r^2} \text{ for } r \geqslant R_1, \tag{1}$$

where $\kappa > 0$ and $R_1 > R_0$ are constants.

Using this function $R_{\min}(t)$, consider the solution $J(t)$ to the following classical Jacobi equation:

$$J''(t) + R_{\min}(t)J(t) = 0;\; J(0) = 0;\; J'(0) = 1$$

and set

$$S(t) = \frac{J'(t)}{J(t)}.$$

Using this function J, let us consider a model space:

$$M_{\text{model}} := \big(\mathbf{R}^n, dr^2 + J(r)^2 g_{S^{n-1}(1)}\big),$$

where r is the Euclidean distance to the origin and $g_{S^{n-1}(1)}$ stands for the standard metric on the unit sphere $S^{n-1}(1)$.

Since $\lim_{t\to+0} S(t) = \infty$, the Laplacian comparison theorem (see Kasue [8]) implies that

$$\Delta r = \Delta_{(M,g)}\, r \leqslant (n-1)S(r) \quad \text{on} \quad M \smallsetminus \big(W \cup \mathcal{C}ut(\partial W)\big). \tag{2}$$

This inequality (2) is known to hold on $M \smallsetminus W$ in the sense of distribution. Note that $J(t) \geqslant t > 0$ due to the non-positivity of $R_{\min}$, and hence, $S(t) = \frac{J'(t)}{J(t)}$ exists for all $t \in (0, \infty)$.

Since $S(t) = \frac{J'(t)}{J(t)}$ satisfies the Riccati equation

$$S'(t) + S^2(t) + R_{\min}(t) = 0 \tag{3}$$

and $R_{\min}(t)$ satisfies (1), it is not hard to see that the solution $S(t)$ to this equation (3) has the asymptotic behavior

$$S(t) = \sqrt{\kappa} - \frac{\beta}{2\sqrt{\kappa}\,t^2} + O\left(\frac{1}{t^3}\right). \tag{4}$$

The following proposition, which also plays an important role in [1], serves to construct an eigenfunction on our model space M_{model}:

Proposition 2.1. — *For any $R > 0$ and $\delta > 0$, consider the following eigenvalue problem* $(*)$*:*

$$\begin{cases} -\varphi''(x) - (1+\delta)\frac{1}{4x^2}\varphi(x) = \lambda\,\varphi(x) \text{ on } [R, 2kR]; \\ \varphi(R) = \varphi(2kR) = 0. \end{cases} \tag{$*$}$$

Then, the first eigenvalue $-\lambda_1 = -\lambda_1(\delta, R, k)$ *of this problem* $(*)$ *is negative, if* $k > 2\left\{\exp\left(\frac{12}{\delta}\right) \wedge 1\right\}$. *Here, we write* $\exp\left(\frac{12}{\delta}\right) \wedge 1 = \min\left\{\exp\left(\frac{12}{\delta}\right), 1\right\}$.

Proof. — We set

$$\chi(x) := \begin{cases} \frac{1}{R}(x-R) & \text{if } x \in [R, 2R], \\ 1 & \text{if } x \in [2R, kR], \\ -\frac{1}{kR}(x - 2kR) & \text{if } x \in [kR, 2kR], \end{cases}$$

where $k > 2$ is a large positive constant defined later. Set $\varphi(x) := \chi(x)x^{\frac{1}{2}}$. Then, the direct computation shows that

$$|\varphi'(x)|^2 - (1+\delta)\frac{1}{4x^2}|\varphi(x)|^2 = |\chi'(x)|^2 x - \frac{\delta}{4x^2}|\varphi(x)|^2 + \frac{1}{2}\left(\chi(x)^2\right)'.$$

Integrating the both sides over $[R, 2kR]$, we have

$$\begin{aligned} &\int_R^{2kR} \left\{ |\varphi'|^2 - (1+\delta)\frac{1}{4x^2}|\varphi|^2 \right\} dx \\ &\quad = \int_R^{2kR} |\chi'(x)|^2 x \, dx - \frac{\delta}{4}\int_R^{2kR} \frac{\chi^2(x)}{x}\, dx \\ &\quad \leqslant \frac{1}{R^2}\int_R^{2R} x\, dx + \frac{1}{(kR)^2}\int_{kR}^{2kR} x\, dx - \frac{\delta}{4}\int_{2R}^{kR} \frac{\chi^2(x)}{x}\, dx \\ &\quad = 3 - \frac{\delta}{4}\log\left(\frac{k}{2}\right). \end{aligned}$$

Hence,

$$\int_R^{2kR} \left\{ |\varphi'|^2 - (1+\delta)\frac{1}{4x^2}|\varphi|^2 \right\} dx < 0 \quad \text{if} \quad k > 2\left\{\exp\left(\frac{12}{\delta}\right) \wedge 1\right\}.$$

Therefore, mini-max principle implies that the first eigenvalue of the problem $(*)$ is negative, if $k > 2\left\{\exp\left(\frac{12}{\delta}\right) \wedge 1\right\}$. □

For $t > 0$, we denote by $B(t)_{M_{\text{model}}}$ the open ball of M_{model} centered at the origin 0 with radius t, and by $\lambda_D\big(B(t)_{M_{\text{model}}}\big)$ the first Dirichlet eigenvalue of $-\Delta_{M_{\text{model}}}$ on $B(t)_{M_{\text{model}}}$. Then, we have the following:

Proposition 2.2. — *Assume that* $\beta(n-1)^2 > 1$ *and choose small constant* $\delta > 0$ *so that* $\beta(n-1)^2 > 1+\delta$. *For a fixed constant* $k > 2\left\{\exp\left(\frac{12}{\delta}\right) \wedge 1\right\}$, *let* $-\lambda_1 = -\lambda_1(k, R, \delta) < 0$ *be the first Dirichlet eigenvalue of the problem* $(*)$. *Then, there exists a positive constant* $R_0(n, \beta, \kappa, \delta, R_{\min})$ *such that*

$$\lambda_D\big(B(2kR)_{M_{\text{model}}}\big) < \frac{(n-1)^2\kappa}{4} - \lambda_1 \tag{5}$$

holds for any $R \geqslant R_0(n, \beta, \kappa, \delta, R_{\min})$.

Proof. — Let $\varphi_1(x)$ be an eigenfunction of the problem $(*)$ with the first Dirichlet eigenvalue $-\lambda_1(k,R,\delta)<0$. Then, we have

(6)
$$\int_R^{2kR}|\varphi_1'(x)|^2\,dx=(1+\delta)\int_R^{2kR}\frac{1}{4x^2}|\varphi_1(x)|^2\,dx-\lambda_1\int_R^{2kR}|\varphi_1(x)|^2\,dx.$$

We set
$$f(x)=\varphi_1(x)J^{-\frac{n-1}{2}}(x).$$

Then, direct computations show that
$$f'(x)=J^{-\frac{n-1}{2}}(x)\left\{\varphi_1'(x)-\frac{n-1}{2}S(x)\varphi_1(x)\right\}$$

and
$$\begin{aligned}&|f'(x)|^2J^{(n-1)}(x)\\&\qquad=|\varphi_1'(x)|^2+\frac{(n-1)^2}{4}S^2(x)|\varphi_1(x)|^2-\frac{n-1}{2}S(x)\left\{\varphi_1(x)^2\right\}'.\end{aligned}$$

As for the last term $-\frac{n-1}{2}S(x)\left\{\varphi_1(x)^2\right\}'$, we calculate
$$-\frac{n-1}{2}\int_R^{2kR}S(x)\left\{\varphi_1(x)^2\right\}'\,dx=\frac{n-1}{2}\int_R^{2kR}S'(x)|\varphi_1(x)|^2\,dx,$$

and hence,
$$\begin{aligned}&\int_R^{2kR}|f'(x)|^2J^{n-1}(x)\,dx\\&\quad=\int_R^{2kR}\left\{|\varphi_1'(x)|^2+\frac{n-1}{2}\left(\frac{n-1}{2}S^2(x)+S'(x)\right)|\varphi_1(x)|^2\right\}dx\\&\quad=\int_R^{2kR}\left\{|\varphi_1'(x)|^2+\frac{n-1}{2}\left(\frac{n-3}{2}S^2(x)-R_{\min}(x)\right)|\varphi_1(x)|^2\right\}dx\\&\quad=\int_R^{2kR}\left\{\frac{1+\delta}{4x^2}-\lambda_1+\frac{n-1}{2}\left(\frac{n-3}{2}S^2(x)-R_{\min}(x)\right)\right\}|\varphi_1(x)|^2\,dx,\end{aligned}$$

where we have used equations (3) and (6). Here, by (1) and (4),
$$\begin{aligned}&\frac{n-1}{2}\left(\frac{n-3}{2}S^2(x)-R_{\min}(x)\right)\\&\qquad=\frac{n-1}{2}\left\{\frac{n-3}{2}\left(\sqrt{\kappa}-\frac{\beta}{2\sqrt{\kappa}\,x^2}+O\left(\frac{1}{x^3}\right)\right)^2+\kappa-\frac{\beta}{x^2}\right\}\\&\qquad=\frac{(n-1)^2\kappa}{4}-\frac{\beta(n-1)^2}{4x^2}+O\left(\frac{1}{x^3}\right)\end{aligned}$$

and, therefore,

$$\int_R^{2kR} |f'(x)|^2 J^{n-1}(x)\,dx$$
$$= \int_R^{2kR} \left\{ \frac{(n-1)^2\kappa}{4} - \lambda_1 - \frac{1}{4x^2}\big(\beta(n-1)^2 - 1 - \delta\big) + O\left(\frac{1}{x^3}\right) \right\} |\varphi_1(x)|^2\,dx.$$

Since $\beta(n-1)^2 - 1 - \delta > 0$ and $|\varphi_1(x)|^2 = |f(x)|^2 J^{n-1}(x)$, we see that

(7) $$\int_R^{2kR} |f'(x)|^2 J^{n-1}(x)\,dx < \left(\frac{(n-1)^2\kappa}{4} - \lambda_1\right) \int_R^{2kR} |f(x)|^2 J^{n-1}(x)\,dx$$

for $R \geqslant R_0(n, \beta, \kappa, \delta, R_{\min})$.

Now, for $y \in M_{\mathrm{model}}$, we set

$$\phi(y) := \begin{cases} f(r(y)), & \text{if } r(y) \in [R, 2kR], \\ 0, & \text{otherwise.} \end{cases}$$

Then, integrating (7) over $S^{n-1}(1)$ with its standard measure, we have

$$\int_{M_{\mathrm{model}}} |\nabla \phi|^2 dv_{M_{\mathrm{model}}} < \left(\frac{(n-1)^2\kappa}{4} - \lambda_1\right) \int_{M_{\mathrm{model}}} |\phi|^2 dv_{M_{\mathrm{model}}}.$$

Hence, mini-max principle implies our desired inequality (5) for $R \geqslant R_0(n, \beta, \kappa, \delta, R_{\min})$. □

Let ψ_1 denote the first Dirichlet eigenfunction of ball $B(2kR)_{M_{\mathrm{model}}}$ for $R \geqslant R_0(n, \beta, \kappa, \delta, R_{\min})$. Then, since the metric is rotationally symmetric, ψ_1 is radial, that is,

(8) $$\psi_1(y) = h_1(r(y))$$

for some function $h_1 \colon [0, 2kR] \to \mathbf{R}$ and h_1 satisfies the equation

(9) $$-h_1''(x) - (n-1)S(x)h_1'(x) = \lambda_D\big(B(2kR)_{M_{\mathrm{model}}}\big)h_1(x)$$

on the interval $(0, 2kR]$. Since h_1 takes the same sign on $[0, 2kR)$ (by maximum principle, or see Prüfer [10]), we may assume that

(10) $$h_1(x) > 0 \quad \text{on} \quad [0, 2kR).$$

Here, we claim the following crucial fact for our proof:

Lemma 2.3. — *Under the assumption* (10), h_1 *satisfies*

(11) $$h_1'(x) < 0 \quad \text{on} \quad (0, 2kR].$$

Proof. — The proof is by contradiction.

First, let us assume that $h_1'(2kR) = 0$. Then, since h_1 satisfies (9) and $h_1(2kR) = 0$, uniqueness of solution implies that $h_1(x) \equiv 0$ which contradicts our assumption (10). Therefore, we see that $h_1'(2kR) < 0$ by (10) and $h_1(2kR) = 0$.

Next, let us assume that $h_1'(x_0) > 0$ for some $x_0 \in (0, 2kR)$. Then, h_1 must takes a minimal value at a point, say x_1, in $[0, x_0)$. If $x_1 \in (0, x_0)$,

$$-h_1''(x_1) = \lambda_D\big(B(2kR)_{M_{\mathrm{model}}}\big)h_1(x_1) > 0 \tag{12}$$

by our assumption (10). However, this contradicts our assumption that h_1 takes a minimal value at x_1. Therefore, $x_1 = 0$. Since $h_1'(0) = 0$, $f(0) = 0$, $f'(0) = 1$, and $S(x) = \frac{f'(x)}{f(x)}$, we see that

$$\lim_{x \to +0} S(x)h_1'(x) = h_1''(0),$$

and hence, by (9),

$$-nh_1''(0) = \lambda_D\big(B(2kR)_{M_{\mathrm{model}}}\big)h_1(0) > 0. \tag{13}$$

Two equations $h_1'(0) = 0$ and (13) imply that 0 is a maximal point of h_1. However, this contradicts our assertion, proved above, that $x_1 = 0$ is a minimal point of h_1.

Thus, we have proved that

$$h_1'(x) \leqslant 0 \quad \text{on} \quad (0, 2kR).$$

However, if $h_1'(x_2) = 0$ for some $x_2 \in (0, 2kR)$, x_2 must be a maximal point of h_1 by the same reason as is seen in (12). Therefore, $h_1'(x_2 - \varepsilon) > 0$ for small $\varepsilon > 0$. This also leads to a contradiction as is seen above. Thus, we have proved (11). $\square$

3. Proof of Theorem 1.1 and 1.2

Let us start with notations involving the cut locus $\mathcal{C}ut(\partial W)$ of the boundary ∂W in $M \setminus W$. Assume that W be a relatively compact open subset of M with C^∞-boundary ∂W and let $\exp_{\partial W} : \mathcal{N}^+(\partial W) \to M \setminus W$ be the outward exponential map. Let $\overrightarrow{n}$ be the outward unit normal vector field along ∂W and set

$$\begin{aligned} U\mathcal{N}^+(\partial W) &= \big\{v \in \mathcal{N}^+(\partial W) \mid |v| = 1\big\}, \\ \mathcal{B}(\partial W, \delta) &= \big\{v \in \mathcal{N}^+(\partial W) \mid |v| < \delta\big\}, \\ B(\partial W, \delta) &= \big\{y \in M \setminus W \mid \operatorname{dist}(W, y) < \delta\big\}. \end{aligned}$$

Moreover, for each $v \in U\mathcal{N}^+(\partial W)$, define

$$\rho(v) = \sup\bigl\{t > 0 \mid \mathrm{dist}\,(W, \exp_{\partial W}(tv)) = t\bigr\}$$

and

$$\mathcal{D}_{\partial W} = \bigl\{tv \in \mathcal{N}^+(\partial W) \mid 0 \leqslant t < \rho(v),\, v \in U\mathcal{N}^+(\partial W)\bigr\}.$$

Then, $\mathcal{C}ut(\partial W) = \bigl\{\exp_{\partial W}(\rho(v)v) \mid v \in U\mathcal{N}^+(\partial W)\bigr\}$. Let dA denote the induced measure on the boundary ∂W and write the Riemannian measure dv_g on the domain $\exp_{\partial W}(\mathcal{D}_{\partial W})$ as follows:

$$dv_g = \sqrt{g}(r,\xi)\, dr\, dA \quad (\xi \in \partial W), \tag{14}$$

where $r = \mathrm{dist}\,(W, *)$.

We shall use the transplantation method as follows: first, for $(t, v) \in [0, \infty) \times U\mathcal{N}^+(\partial W)$ satisfying $tv \in \overline{\mathcal{B}(\partial W, R)} \cap \overline{\mathcal{D}_{\partial W}}$, define a function H_R on $\overline{B(\partial W, R)}$ by

$$H_R\bigl(\exp_{\partial W}(tv)\bigr) = h_1(t),$$

where h_1 is the function defined by (8). Next, using this function H_R, define a function F_R on M by

$$F_R(y) = \begin{cases} h_1(0), & \text{if} \quad y \in \overline{W} \\ H_R(y), & \text{if} \quad r(y) \in (0, R], \\ 0, & \text{otherwise.} \end{cases}$$

Then $F = F_R \in W_c^{1,2}(W \cup \overline{B(\partial W, R)})$, and we get

$$\begin{aligned} \int_{W \cup \overline{B(\partial W,R)}} |\nabla F|^2 dv_g &= \int_{\overline{B(\partial W,R)}} |\nabla F|^2 dv_g \\ &= \int_{\partial W} dA(\xi) \int_0^{\rho(\overrightarrow{n}(\xi))\wedge R} |h_1'(r)|^2 \sqrt{g}(r,\xi)\, dr \end{aligned} \tag{15}$$

and

$$\begin{aligned} \int_{W \cup \overline{B(\partial W,R)}} |F|^2 dv_g = {} & |h_1(0)|^2 \cdot \mathrm{Vol}(W) \\ & + \int_{\partial W} dA(\xi) \int_0^{\rho(\overrightarrow{n}(\xi))\wedge R} |h_1(r)|^2 \sqrt{g}(r,\xi)\, dr, \end{aligned} \tag{16}$$

where $\rho\,(\overrightarrow{n}(\xi)) \wedge R = \min\bigl\{\rho\,(\overrightarrow{n}(\xi))\,, R\bigr\}$.

Now, for each $\xi \in \partial W$,

$$
\begin{aligned}
(17)\quad & \int_0^{\rho\left(\overrightarrow{n}(\xi)\right)\wedge R} |h_1'|^2(r)\sqrt{g}(r,\xi)\,dr \\
&= \left[h_1(r)h_1'(r)\sqrt{g}(r,\xi)\right]_{r=0}^{r=\rho\left(\overrightarrow{n}(\xi)\right)\wedge R} \\
&\qquad\qquad - \int_0^{\rho\left(\overrightarrow{n}(\xi)\right)\wedge R} h_1(r)\left\{h_1'(r)\sqrt{g}(r,\xi)\right\}'\,dr \\
&= (h_1h_1')\left(\rho\left(\overrightarrow{n}(\xi)\right)\wedge R\right)\cdot\sqrt{g}\left(\rho\left(\overrightarrow{n}(\xi)\right)\wedge R,\xi\right) \\
&\qquad\qquad - \int_0^{\rho\left(\overrightarrow{n}(\xi)\right)\wedge R} h_1\left\{h_1'\sqrt{g}(r,\xi)\right\}'\,dr \\
&\leqslant -\int_0^{\rho\left(\overrightarrow{n}(\xi)\right)\wedge R} h_1(r)\left\{h_1'(r)\sqrt{g}(r,\xi)\right\}'\,dr \\
&= -\int_0^{\rho\left(\overrightarrow{n}(\xi)\right)\wedge R} h_1(r)\left\{h_1''(r)+\frac{\partial_r\sqrt{g}(r,\xi)}{\sqrt{g}(r,\xi)}h_1'(r)\right\}\sqrt{g}(r,\xi)\,dr \\
&\leqslant -\int_0^{\rho\left(\overrightarrow{n}(\xi)\right)\wedge R} h_1(r)\left\{h_1''(r)+(n-1)S(r)h_1'(r)\right\}\sqrt{g}(r,\xi)\,dr \\
&= \lambda_D\left(B(2kR)_{M_{\mathrm{model}}}\right)\int_0^{\rho\left(\overrightarrow{n}(\xi)\right)\wedge R} |h_1|^2(r)\sqrt{g}(r,\xi)\,dr,
\end{aligned}
$$

where we have used the fact $h_1'(0) = 0$ at the first equality; we have used (10) and (11) at the first inequality; we have used (10), (11), $\Delta r = \frac{\partial_r\sqrt{g}}{\sqrt{g}}$, and (2) at the second inequality; we have used (9) at the last equality.

Integrating both side of the inequality (17) over ∂W and combining (15) and (16), we see that

$$
\begin{aligned}
&\int_{\overline{B(\partial W,R)}} |\nabla F|^2 dv_g \\
&\quad = \int_{\partial W} dA(\xi)\int_0^{\rho(\overrightarrow{n}(\xi))\wedge R} |h_1'(r)|^2\sqrt{g}(r,\xi)\,dr \\
&\quad \leqslant \lambda_D\left(B(R)_{\widehat{M}_{\mathrm{model}}}\right)\left\{\int_{W\cup\overline{B(\partial W,R)}} |F|^2 dv_g - |h_1(0)|^2\cdot\mathrm{Vol}(W)\right\}.
\end{aligned}
$$

Hence, we have

$$\begin{aligned}(18)\qquad \frac{\int_M |\nabla F_R|^2\,dv_g}{\int_M |F_R|^2\,dv_g} &\leqslant \lambda_D\left(B(R)_{\widehat{M}_{\rm model}}\right)\left\{1-\frac{|h_1(0)|^2\cdot \mathrm{Vol}(W)}{\int_M |F_R|^2\,dv_g}\right\}\\ &< \lambda_D\left(B(R)_{\widehat{M}_{\rm model}}\right).\end{aligned}$$

This inequality(18) holds for all $R \geqslant R_0(n,\beta,\kappa,\delta)$, and hence, setting $R_i = R_0(n,\beta,\kappa,\delta)+i$ and considering the corresponding functions F_{R_i} as above, we get the sequence $\{F_{R_i}\}$ of functions in $W_c^{1,2}(M)$ satisfying

$$\frac{\int_M |\nabla F_{R_i}|^2 dv_g}{\int_M |F_{R_i}|^2 dv_g} < \frac{(n-1)^2\kappa}{4};$$

$$\operatorname{supp} F_{R_i} = \overline{B(\partial W, R_i)}.$$

Since $\{F_{R_i}\}_{i=1}^{\infty}$ spans the infinite dimensional subspace in $W_c^{1,2}(M)$, we obtain the conclusion of Theorem 1.1 by mini-max principle.

Taking $W = \{y \in M \mid \operatorname{dist}(y,p_0) < \varepsilon\}$ for $0<\varepsilon<\min\{\mathrm{inj}(p_0), R_0\}$ in Theorem 1.1, we get Theorem 1.2, where $\mathrm{inj}(p_0)$ stands for the injectivity radius at p_0.

4. Proof of Proposition 1.3

In order to prove Proposition 1.3, we first quote the following theorem from [1]:

THEOREM 4.1. — *Let (M,g) be a complete noncompact Riemannian n-manifold, where $n \geqslant 2$. Assume that one of ends of M, denoted by E, has a compact connected C^∞ boundary $W := \partial E$ such that the outward normal exponential map $\exp_W : \mathcal{N}^+(W) \to E$ is a diffeomorphism, where*

$$\mathcal{N}^+(W) := \left\{ v \in TM\big|_W \;\middle|\; v \ \text{ is outward normal to } W\right\}.$$

Assume also that the mean curvature H_W of W with respect to the inward unit normal vector is positive. Take a positive constant $R>0$ satisfying

$$H_W \geqslant \frac{1}{R} \quad \text{on} \quad W,$$

and set

$$\rho(x) := \mathrm{dist}_g(x,W), \quad \widehat{r}\,(x) := \rho(x) + R \quad \text{for} \quad x \in E.$$

Then, for all $u \in C_0^\infty(M)$, we have

(19)
$$\begin{aligned}\int_E |\nabla u|^2\, dv_g \geqslant \int_E \Big\{ \frac{1}{4\,\widehat{r}^2} + \frac{1}{4}(\Delta\,\widehat{r})^2 - \frac{1}{2}|\nabla d\,\widehat{r}\,|^2 - \frac{1}{2}\operatorname{Ric}_g(\nabla\,\widehat{r}, \nabla\,\widehat{r}) \Big\} u^2\, dv_g \\ + \frac{1}{2}\int_W \Big(\Delta\,\widehat{r} - \frac{1}{R}\Big) u^2\, d\sigma_g \\ \geqslant \int_E \Big\{ \frac{1}{4\,\widehat{r}^2} + \frac{1}{4}(\Delta\,\widehat{r})^2 - \frac{1}{2}|\nabla d\,\widehat{r}|^2 - \frac{1}{2}\operatorname{Ric}_g(\nabla\,\widehat{r}, \nabla\,\widehat{r}) \Big\} |u|^2 dv_g,\end{aligned}$$

where $d\sigma_g$ denote the $(n-1)$-dimensional Riemannian measure of $(W, g\big|_W)$. In particular, if (M, g) has a pole $p_0 \in M$, then

$$\int_M |\nabla u|^2\, dv_g \geqslant \int_M \Big\{ \frac{1}{4r^2} + \frac{1}{4}(\Delta r)^2 - \frac{1}{2}|\nabla dr|^2 - \frac{1}{2}\operatorname{Ric}_g(\nabla r, \nabla r) \Big\} |u|^2\, dv_g,$$

where $r(x) := \operatorname{dist}_g(x, p_0)$ for $x \in M$. Recall that a point p_0 of a Riemannian manifold (M, g) is called a pole if the exponential map $\exp_{p_0} : T_{p_0}M \to M$ at p_0 is a diffeomorphism.

In view of Theorem 1.1, it suffices to prove the following: if $\beta < 1/(n-1)^2$, $\sigma_{\mathrm{disc}}(-\Delta) \cap \Big[0, \frac{(n-1)^2\kappa}{4}\Big)$ is finite.

Let us set $A(r) = \frac{h'(r)}{h(r)}$. Then, $A(r)$ satisfies the following Ricatti equation

$$A'(r) + A^2(r) + K(r) = 0 \quad \text{ on } (0, \infty).$$

Assume that $K(r)$ satisfies

$$K(r) \leqslant 0 \quad \text{ on } (0, \infty) \tag{20}$$

and

$$K(r) = -\kappa + \frac{\beta}{r^2} \quad \text{ for } r \geqslant R_0, \tag{21}$$

where $\kappa > 0$, $\beta < 1/(n-1)^2$, and $R_0 > 0$ are constants. In view of (16), the comparison theorem implies that

$$A(r) \geqslant \frac{1}{r} > 0 \quad \text{ on } (0, \infty). \tag{22}$$

Using the comparison theorem again together with (17) and(18) makes

$$A(r) = \sqrt{\kappa} - \frac{\beta}{2\sqrt{\kappa}\, r^2} + O\left(\frac{1}{r^3}\right) \quad \text{ as } r \to \infty. \tag{23}$$

Therefore, we have

$$\begin{aligned}\frac{1}{4}(\Delta r)^2-\frac{1}{2}|\nabla dr|^2&=\frac{1}{4}(n-1)^2A^2(r)-\frac{1}{2}(n-1)A^2(r)\\&=\frac{(n-1)(n-3)}{4}A^2(r)\\&=\frac{(n-1)(n-3)}{4}\left(\kappa-\frac{\beta}{r^2}\right)+O\left(\frac{1}{r^3}\right),\end{aligned}$$

and hence,

$$\begin{aligned}&\frac{1}{4r^2}+\frac{1}{4}(\Delta r)^2-\frac{1}{2}|\nabla dr|^2-\frac{1}{2}\operatorname{Ric}_g(\nabla r,\nabla r)\\&\quad=\frac{1}{4r^2}+\frac{(n-1)(n-3)}{4}\left(\kappa-\frac{\beta}{r^2}\right)-\frac{(n-1)}{2}\left(-\kappa+\frac{\beta}{r^2}\right)+O\left(\frac{1}{r^3}\right)\\&\quad=\frac{(n-1)^2\kappa}{4}+\frac{1}{4r^2}\left\{1-(n-1)^2\beta\right\}+O\left(\frac{1}{r^3}\right).\end{aligned}$$

Hence, substituting

$$E=\mathbf{R}^n-B_0(R),\quad \rho(x)=\operatorname{dist}_g(x,\partial B_0(R)),\quad \widehat{r}\,(x)=\rho(x)+R=r(x)$$

into the equation (15) in Theorem 4.1, we see that the following inequality holds for all $u\in C_0^\infty(\mathbf{R}^n)$ and $R>0$, where we set $B_0(R)=\{x\in\mathbf{R}^n\mid \operatorname{dist}(x,0)=R\}$ and 0 represents the origin of $\mathbf{R}^n$:

$$\begin{aligned}&\int_{\mathbf{R}^n-B_0(R)}|\nabla u|^2\,dv_g\\&\quad\geqslant\int_{\mathbf{R}^n-B_0(R)}\left\{\frac{1}{4r^2}+\frac{1}{4}(\Delta r)^2-\frac{1}{2}|\nabla dr|^2-\frac{1}{2}\operatorname{Ric}_g(\nabla r,\nabla r,)\right\}|u|^2\,dv_g\\&\qquad+\frac{1}{2}\int_{\partial B_0(R)}\Big(\Delta r-\frac{1}{R}\Big)|u|^2\,d\sigma_g\\&\quad=\int_{\mathbf{R}^n-B_0(R)}\left\{\frac{(n-1)^2\kappa}{4}+\frac{1}{4r^2}\left\{1-(n-1)^2\beta\right\}+O\left(\frac{1}{r^3}\right)\right\}|u|^2\,dv_g\\&\qquad+\frac{1}{2}\int_{\partial B_0(R)}\left\{(n-1)\sqrt{\kappa}-\frac{1}{R}-O\left(\frac{1}{R^2}\right)\right\}|u|^2\,d\sigma_g,\end{aligned}$$

where we have used $\Delta r=(n-1)A(r)=(n-1)\sqrt{\kappa}-O(r^{-2})$ (see (19)). Therefore, since $1>(n-1)^2\beta$, there exits a constant $R_1>R_0$ such that

$$\int_{\mathbf{R}^n-B_0(R)}|\nabla u|^2\,dv_g\geqslant\int_{\mathbf{R}^n-B_0(R)}\frac{(n-1)^2\kappa}{4}|u|^2\,dv_g \tag{24}$$

$$\text{for all } u\in C_0^\infty(\mathbf{R}^n)\ \text{ and } R\geqslant R_1.$$

Now, let $\Delta_{B_0(R_1)}$ be the Laplacian on $(B_0(R_1), dr^2 + h^2(r)g_{S^{n-1}(1)})$ with vanishing Neumann boundary condition and

$$0 = \mu_0 < \mu_1 \leqslant \cdots \leqslant \mu_i \leqslant \mu_{i+1} \leqslant \cdots \nearrow \infty \tag{25}$$

be its eigenvalues with each eigenvalues repeated according to its multiplicity. Also, let $\Delta_{\mathbf{R}^n - B_0(R_1)}$ be the Laplacian on $\left(\mathbf{R}^n - B_0(R_1),\ dr^2 + h^2(r)g_{S^{n-1}(1)}\right)$ with vanishing Neumann boundary condition. Then, from (20), we see that

$$\sigma\left(-\Delta_{\mathbf{R}^n - B_0(R_1)}\right) \subset \left[\frac{(n-1)^2\kappa}{4}, \infty\right). \tag{26}$$

Hence, the domain monotonicity principle (vanishing Neumann boundary data) due to Courant-Hilbert [5] (see also [3] p. 13), together with (21) and (22), implies that

$$\sharp\left\{\lambda \in \sigma_{\mathrm{disc}}(-\Delta) \mid \lambda < \frac{(n-1)^2\kappa}{4}\right\} \leqslant \sharp\left\{\mu_i \mid \mu_i < \frac{(n-1)^2\kappa}{4}\right\} < \infty.$$

Here, "$\sharp$" represents the counting function of eigenvalues with each eigenvalues repeated according to its multiplicity. Thus, we have proved Proposition 1.3.

5. Applications and remarks

Reflecting our proof, we see that the following holds:

PROPOSITION 5.1. — *Let W be a relatively compact open subset of a Riemannian manifold (M, g) of dimension n. Assume that ∂W is C^∞, and that the outward normal exponential map $\exp_{\partial W}\colon N^+(\partial W) \to M \setminus W$ is a diffeomorphism. Moreover, assume that*

$$\Delta r = (n-1)\left\{\sqrt{\kappa} - \frac{\beta}{2\sqrt{\kappa}\, r^2} + O\left(\frac{1}{r^3}\right)\right\} \ (r \to \infty)$$

on $M \setminus W$, where $r = \mathrm{dist}(W, *)$ on $M \setminus W$; κ and β are positive constants. If $\beta > 1/(n-1)^2$, then $\sigma_{\mathrm{ess}}(-\Delta) = \left[\frac{(n-1)^2\kappa}{4}, \infty\right)$ and $\sigma_{\mathrm{disc}}(-\Delta)$ is infinite.

In Proposition 5.1, ∂W may have a finite number of components. Using Proposition 5.1, we can construct examples with infinite number of the discrete spectrum of the Laplacian.

In Theorem 1.1 and 1.2, we assumed that

$$\min \sigma_{\mathrm{ess}}(-\Delta) = \frac{(n-1)^2\kappa}{4}. \tag{27}$$

The condition (27) is satisfied if the inequality

$$\sup\{\mathfrak{h}(M \setminus K) \mid K \subset M \text{ is compact}\} \geqslant (n-1)\sqrt{\kappa} \tag{28}$$

holds under our curvature assumption, where

$$\mathfrak{h}(M \setminus K) := \inf\left\{\frac{\mathrm{Vol}_{n-1}(\partial\Omega)}{\mathrm{Vol}_n(\Omega)} \mid \Omega \subset M \setminus K\right\}$$

is the Cheeger constant of $M \setminus K$; next, the condition (28) holds if there exists a C^∞-function f defined near infinity satisfying

$$\liminf_{M \ni y \to \infty} \Delta f(y) \geqslant (n-1)\sqrt{\kappa} \quad \text{and} \quad |\nabla f| \leqslant 1.$$

Modifying our arguments, we also get the following:

Theorem 5.2. — *Let (M,g) be an n-dimensional noncompact complete Riemannian manifold and W a relatively compact open subset of M with C^∞-boundary ∂W. We set $r(*) := \mathrm{dist}(*, \partial W)$ on $M \setminus W$. Let $\exp_{\partial W} : \mathcal{N}^+(\partial W) \to M \setminus W$ be the outward normal exponential map and $\mathcal{C}ut(\partial W)$ the corresponding cut locus of ∂W in $M \setminus W$.*

Assume that there exist positive constants κ and R_0 and positive-valued continuous function φ of $t \in [R_0, \infty)$ such that

$$\mathrm{Ric}_g(\nabla r, \nabla r)(y) \geqslant -(n-1)\kappa - \varphi(r(y)) \qquad \text{for} \quad y \in M \setminus (W \cup \mathcal{C}ut(\partial W)) \quad \text{with} \quad r(y) \geqslant R_0$$

and

$$\lim_{t\to\infty} \varphi(t) = 0.$$

Then, $\sigma_{\mathrm{ess}}(-\Delta) \cap \left[0, (n-1)^2\kappa/4\right] \neq \emptyset$, where $\sigma_{\mathrm{ess}}(-\Delta)$ stands for the essential spectrum of $-\Delta$.

Theorem 5.2 immediately implies the following

Corollary 5.3. — *Let (M,g) be an n-dimensional noncompact complete Riemannian manifold and p_0 a fixed point of M. We set $r(*) := \mathrm{dist}(*, p_0)$ and denote by $\mathcal{C}ut(p_0)$ the cut locus of p_0. Assume that there exist positive constants κ and R_0 and positive-valued continuous function φ of $t \in [R_0, \infty)$ such that*

$$\mathrm{Ric}_g(\nabla r, \nabla r)(y) \geqslant -(n-1)\kappa - \varphi(r(y)) \qquad \text{for} \quad y \in M \setminus \mathcal{C}ut(p_0) \quad \text{with} \quad r(y) \geqslant R_0$$

and

$$\lim_{t\to\infty} \varphi(t) = 0.$$

Then, $\sigma_{\mathrm{ess}}(-\Delta) \cap \left[0, (n-1)^2\kappa/4\right] \neq \emptyset$.

Corollary 5.3 is a generalization of one of Donnelly's theorems [6] which asserts that $\sigma_{\mathrm{ess}}(-\Delta) \cap \left[0, (n-1)^2\kappa/4\right] \neq \emptyset$ under assumption that $\mathrm{Ric}_g \geqslant -(n-1)\kappa$ on all of (M, g).

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