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A PROOF OF THE STRATIFIED MORSE INEQUALITIES FOR SINGULAR COMPLEX ALGEBRAIC CURVES USING THE WITTEN DEFORMATION

by Ursula LUDWIG

ABSTRACT. — The Witten deformation is an analytic method proposed by Witten which, given a Morse function $f : M \to \mathbf{R}$ on a smooth compact manifold M, allows to prove the Morse inequalities. The aim of this article is to generalise the Witten deformation to stratified Morse functions (in the sense of stratified Morse theory as developed by Goresky and MacPherson) on a singular complex algebraic curve. In a previous article the author developed the Witten deformation for the model of an algebraic curve with cone-like singularities and a certain class of functions called admissible Morse functions. The perturbation arguments needed to understand the Witten deformation on the curve with its metric induced from the Fubini-Study metric of the ambient projective space and for any stratified Morse function are presented here.

RÉSUMÉ. — Soit M une variété Riemannienne compacte et soit $f : M \to \mathbf{R}$ une fonction de Morse sur M. La méthode de Witten utilise une déformation du complexe de de Rham pour démontrer les inegalités de Morse. Le but de cette note est d'étendre cette méthode au cas des courbes algébriques singulières et aux fonctions de Morse stratifiées au sens de la théorie de Goresky/MacPherson.

Dans une note précédente, l'auteur a donné une généralisation de la méthode de Witten pour le cas modèle d'une courbe à singularités coniques et des fonctions de Morse admissibles. Ici on présente les méthodes et arguments nécessaires pour étendre la théorie au courbes équipées de la métrique induite par la métrique de Fubini-Study de l'espace ambiant et à toutes les fonctions de Morse stratifiées.

1. Introduction

Let $X \subset \mathbb{P}^N(\mathbb{C})$ be a singular complex algebraic curve and let $f : X \to \mathbb{R}$ be a stratified Morse function on X in the sense of the theory developed by Goresky/MacPherson in [12]. The singular set of X will be denoted

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by Σ. $X \setminus \Sigma$ is equipped with the Riemannian metric induced from the Fubini-Study metric on $\mathbb{P}^N(\mathbb{C})$.

An important topological invariant of the curve is the so called L^2-cohomology of X, which is defined as follows: Let $(\Omega_0^*(X \setminus \Sigma), d)$ be the de Rham complex of differential forms acting on smooth forms with compact support in $X \setminus \Sigma$. In the case of a singular curve the differential complex $(\Omega_0^*(X \setminus \Sigma), d)$ admits a unique extension into a Hilbert complex $(\mathcal{C}, d, \langle\ ,\ \rangle)$ in the Hilbert space of square integrable forms equipped with the L^2-metric

$$\langle \alpha, \beta \rangle := \int_{X \setminus \Sigma} \alpha \wedge * \beta$$

(see Section 2 for details, note that throughout this paper we use the language of Hilbert complexes introduced in [5].) Another way to state this is to say that the maximal and the minimal closed extension of d coincide, *i.e.*, $d_{\min} = d_{\max}$. The L^2-cohomology of X is defined as the cohomology of the complex $(\mathcal{C}, d, \langle\ ,\ \rangle)$,

$$\begin{aligned} H^i_{(2)}(X) &:= H^i\big((\mathcal{C}, d, \langle\ ,\ \rangle)\big) \\ &= \ker d_{i,\min} / \operatorname{im} d_{i-1,\min} = \ker d_{i,\max} / \operatorname{im} d_{i-1,\max}. \end{aligned} \tag{1.1}$$

The Witten method (see [21], [15]) generalised to our situation consists in deforming the complex $(\Omega_0^*(X \setminus \Sigma), d)$ into

$$0 \to \Omega_0^0(X \setminus \Sigma) \xrightarrow{d_t} \Omega_0^1(X \setminus \Sigma) \xrightarrow{d_t} \Omega_0^2(X \setminus \Sigma) \to 0, \tag{1.2}$$

where the differential d has been deformed by means of the stratified Morse function f into a differential $d_t = e^{-ft} d e^{ft}$; here $t \in (0, \infty)$ is the deformation parameter. We denote by $\delta_t = e^{tf} \delta e^{-tf}$ the formal adjoint of d_t with respect to the L^2-metric $\langle\ ,\ \rangle$. The first important result is the following

Proposition 1.1.

(a) *The complex $(\Omega_0^*(X \setminus \Sigma), d_t, \langle\ ,\ \rangle)$ satisfies the* L^2*-Stokes theorem, i.e.,*

$$\operatorname{dom}(d_{t\max}) = \operatorname{dom}(d_{t\min}). \tag{1.3}$$

In other words the complex $(\Omega_0^*(X \setminus \Sigma), d_t, \langle\ ,\ \rangle)$ admits a unique extension into a Hilbert complex, which we denote by $(\mathcal{C}_t, d_t, \langle\ ,\ \rangle)$.

(b) *The complex $(\mathcal{C}_t, d_t, \langle\ ,\ \rangle)$ is a Fredholm complex whose cohomology is isomorphic to the* L^2*-cohomology of X, i.e., $H^*\big((\mathcal{C}_t, d_t, \langle\ ,\ \rangle)\big) \simeq H^*_{(2)}(X)$. Moreover*

$$H^i\big((\mathcal{C}_t, d_t, \langle\ ,\ \rangle)\big) \simeq \ker d_{t,i} \cap \ker \delta_{t,i-1} \simeq \ker \Delta_{t,i}, \tag{1.4}$$

where $\Delta_t = (d_t + \delta_t)^2$ *denotes the Laplacian associated to the Hilbert complex* $(\mathcal{C}_t, d_t, \langle\ ,\ \rangle)$*, i.e., the closed selfadjoint (non-negative) extension of* $\Delta_{t|\Omega_0}$ *with domain:*

$$\operatorname{dom}(\Delta_t) = \big\{\Phi \;\big|\; \Phi, d_t\Phi, \delta_t\Phi, d_t\delta_t\Phi, \delta_t d_t\Phi \in \mathrm{L}^2\big(\Lambda^*(T^*(X \setminus \Sigma))\big)\big\}. \tag{1.5}$$

(c) *The operator* Δ_t *is a discrete operator. Moreover, for its restriction on* k*-forms,* $k \neq 1$*, we get*

$$\Delta_{t,k} = \Delta^{\mathcal{F}}_{t,k}$$

where $\Delta^{\mathcal{F}}_t$ *denotes the Friedrichs extension of* $\Delta_{t|\Omega_0^*}$*.*

We call the operator Δ_t defined in Part b) of Proposition 1.1 the Witten Laplacian. Part a) of the proposition can be deduced easily from the model case of a curve with cone-like singularities in [16], since the validity of the L^2-Stokes theorem is a quasi-isometry invariant. However we choose a different method of proof here, which is inspired by methods developed in [6] since it also leads to the proof of Part c). Part c) of the proposition will be useful in the sequel since, as a consequence of it, we get that the form domain of the Witten Laplacian and the form domain of the Laplacian associated to the Hilbert complex $(\mathcal{C}, d, \langle\ ,\ \rangle)$ coincide except in the middle degree $k = 1$. Note that unlike the proof of this fact in [16] the proof proposed here can be generalised to the higher dimensional situation.

The advantage of the deformed complex compared to the initial complex is that the spectrum of the Witten Laplacian has "nice" properties for large parameter t. In particular one can show the *spectral gap theorem* below. The restriction $f_{|X\setminus\Sigma}$ of a stratified Morse function is Morse in the usual (smooth) sense. We denote by $c_i(f_{|X\setminus\Sigma})$ the number of critical points of $f_{|X\setminus\Sigma}$ of index i.

Theorem 1.2.

a) *Let* X *be a singular curve and let* $f : X \to \mathbb{R}$ *be a stratified Morse function in the sense of the theory developed by Goresky/MacPherson in* [12]*. Then there exist constants* $C_1, C_2, C_3 > 0$ *and* $t_0 > 0$ *depending on* X *and* f *such that for* $t > t_0$

$$\operatorname{spec}(\Delta_t) \cap (C_1 e^{-C_2 t}, C_3 t) = \emptyset.$$

b) *Let us denote by* $(\mathbb{F}_t, d_t, \langle\ ,\ \rangle)$ *the subcomplex of* $(\mathcal{C}_t, d_t, \langle\ ,\ \rangle)$ *generated by all eigenforms of the Witten Laplacian* Δ_t *to eigenvalues in* $[0, 1]$*. Then, for* $t > t_0$

$$\begin{aligned} \dim \mathbb{F}^i_t &= c_i(f_{|X\setminus\Sigma}) =: c_i(f), && \text{if } i = 0, 2,\\ \dim \mathbb{F}^1_t &= c_1(f_{|X\setminus\Sigma}) + \textstyle\sum_{p\in\Sigma} n_p =: c_1(f), \end{aligned} \tag{1.6}$$

where

$$n_p := \sum (m_p^j - 1). \tag{1.7}$$

and m_p^j *are the multiplicities of the different analytic branches of* X *at the singular point* p. *(The sum in* (1.7) *is taken over all analytic branches at* p.*)*

In the smooth situation one can show that the complex of eigenforms to small eigenvalues converges to the geometric Thom-Smale complex (see [21], [15]). This result has been generalised in [18] to the model case of a curve with cone-like singularities and admissible Morse functions.

As usual the following Morse inequalities follow from the spectral gap theorem by a simple algebraic argument

Corollary 1.3. — *In the situation of Theorem 1.2*

$$\begin{aligned} &\sum_{i=0}^{k} (-1)^{k-i} c_i(f) \geqslant \sum_{i=0}^{k} (-1)^{k-i} b_i^{(2)}(X), \ \textit{for } k = 0, 1, \\ &\sum_{i=0}^{2} (-1)^{i} c_i(f) = \sum_{i=0}^{2} (-1)^{i} b_i^{(2)}(X), \end{aligned} \tag{1.8}$$

where $b_i^{(2)}(X) := \dim H^i_{(2)}(X)$ *denote the* L^2*-Betti numbers of* X.

The key step in the proof of the spectral gap theorem is the construction of a local model operator $\boldsymbol{\Delta}_t^p$ for the Witten Laplacian for each analytic branch near a singular point p of X. Using the model case of a curve with cone-like singularities in [16] and perturbation techniques for regular singular operators as in [7] we show a local version of the spectral gap theorem, namely that $\mathrm{spec}(\boldsymbol{\Delta}_t^p) = \{0\} \cup [Ct^2, \infty)$ for some appropriate constant $C > 0$. The forms in $\ker(\boldsymbol{\Delta}_t^p)$ are 1-forms and $\dim \ker(\boldsymbol{\Delta}_t^p) = m - 1$, where m is the multiplicity of the analytic branch. In the model case of a curve with cone-like singularities (see [16]) the forms in the kernel of the model Witten Laplacian can be computed explicitly and are related to the modified Bessel functions. Therefore in the model case one can deduce from the asymptotic of the modified Bessel functions that the eigenfunctions decay exponentially. In the more general situation treated here however this is not possible. Instead here Agmon type estimates are used to prove the exponential decay of all forms in $\ker(\boldsymbol{\Delta}_t^p)$. Note that the exponential decay of the eigenfunctions is essential for the next step of the proof.

Once the local situation near the singular points of X understood, to complete the proof of the spectral gap theorem one can now proceed by following the steps of the proof in the smooth case. (Here we follow the proof in [2], Section 9).

Since the L^2-cohomology is dual to the intersection homology of the curve from Corollary 1.3 we get back the Morse inequalities of stratified Morse theory (see [12], Section II.6.12 and [11], p. 169). Note that from the point of view of the analytic proof the contribution of the singular points of X to the Morse inequalities is caused by the fact that $\mathrm{dom}(\Delta_1) \neq \mathrm{dom}(\Delta_{t,1})$ and thus is related to the small eigenvalues of the "transversal Laplacian" (*i.e.,* the Laplacian on the link of the singularity). Recall that the small eigenvalues of the transversal Laplacian play an important role for L^2-methods in the presence of singularities, namely for the lack of essential selfadjointness of $\Delta_{|\Omega_0}$ and in the study of index theorems for regular singular operators (see [9] for the general case and [19], [8] for the case of a singular algebraic curve).

Note that obviously Corollary 1.3 could be more quickly deduced directly from the model case treated in [16] by using the fact, that the L^2-cohomology is an invariant of the quasi-isometry class. However our further goal is to extend the Witten method to more general singular situations. The techniques developed here (in particular the proof of Proposition 1.1 and the Agmon type estimates) are useful for the generalisation of the Witten deformation to higher dimensional spaces with cone-like singularities (see [17]).

These notes are organised as follows: In Section 2 we recall the basic facts on the L^2-cohomology of a singular curve and prove Proposition 1.1 for a class of functions which we call admissible functions here (see Definition 2.1). We show that stratified Morse functions in the sense of the theory developed by Goresky/MacPherson are admissible. In Section 3 we develop the local model for the Witten Laplacian near singular points of X and prove a local version of the spectral gap theorem. The Agmon type estimates, used to prove exponential decay of the eigenforms of the model operator, are proved in Section 3.6. Finally in Section 4 we complete the proof of Theorem 1.2 and Corollary 1.3. Note that in Section 2 we consider admissible functions, in particular Proposition 1.1 holds for this class of functions. The results of the next sections hold for stratified Morse functions only.

2. The Witten deformation for singular curves and admissible functions

Let $X \subset \mathbb{P}^n(\mathbb{C})$ be a singular complex algebraic curve. We denote by Σ the singular set of X. Near the singular points of X the metric on X induced by the Fubini-Study metric on $\mathbb{P}^n(\mathbb{C})$ is quasi-isometric to a cone-like metric (see [19], [7]): Let $p \in \Sigma$ be a singular point of X and denote by X_j, $j = 1, \ldots, s$, the analytic branches of X at p. Then for each branch X_j there exist open neighbourhoods $V_j \subset \mathbb{C}$ of 0 resp. $U(p) \subset \mathbb{P}^n(\mathbb{C})$ of p, as well as affine coordinates $z_1, \ldots, z_n$ on $U(p)$ and a normalisation map defined by

(2.1)
$$\begin{array}{llll} \pi : & V_j \subset \mathbb{C} & \to & U(p) \cap X_j \\ & t & \mapsto & (z_1(t), \ldots, z_n(t)) = (t^{m_j}, t^{q_{j2}} f_{j2}(t), \ldots, t^{q_{jn}} f_{jn}(t)), \end{array}$$

such that $\pi_{|V_j - \{0\}}$ is a biholomorphic map. Hereby $m_j < q_{j2} \leqslant q_{j3} \leqslant \ldots \leqslant q_{jn}$. The multiplicity m_j of X_j at p is an analytic invariant, *i.e.*, it does not depend on the choice of local coordinates $z_1, \ldots, z_n$.

We denote by $\widetilde{g}$ the Riemannian metric on X_j induced by the Fubini-Study metric on $\mathbb{P}^n(\mathbb{C})$. Then we have an isometry

$$\Pi : (V_j \setminus \{0\}, \pi^* \widetilde{g}) \simeq (\mathrm{cone}_\epsilon(S^1_m), g(r, \varphi)), \quad t \mapsto (|t|^m, m \cdot \arg(t)),$$

where S^1_m denotes the 1-sphere of length $2\pi m$, $r \in (0, \epsilon)$ and

$$g(r, \varphi) = \widetilde{\alpha}^2 (r^{1/m}, \varphi) dr \otimes dr + \widetilde{\beta}^2 (r^{1/m}, \varphi) r^2 d\varphi \otimes d\varphi, \tag{2.2}$$

with $\widetilde{\alpha}, \widetilde{\beta} \in C^\infty([0, \epsilon) \times S^1_m)$, $\widetilde{\alpha}(0, \cdot) = \widetilde{\beta}(0, \cdot) = 1$. Thus $(X_j, \widetilde{g})$ is a conformally conic Riemannian manifold (see [6] for a definition) and is therefore in particular quasi-isometric to a cone-like singularity.

For the convenience of the reader we recall the basic facts on the L^2-cohomology of the curve. We rephrase them in the language of Hilbert complexes, which has been introduced in [5]. Let $(\Omega^*_0(X \setminus \Sigma), d)$ be the de Rham complex of differential forms acting on smooth forms with compact support. An *ideal boundary condition* for the elliptic complex $(\Omega^*_0(X \setminus \Sigma), d)$ is a choice of closed extensions D_k of d_k in the Hilbert space of square integrable k-forms, such that $D_k(\mathrm{dom}(D_k)) \subset \mathrm{dom}(D_{k+1})$. We then get a Hilbert complex

$$0 \to \mathrm{dom}(D_0) \xrightarrow{D_0} \mathrm{dom}(D_1) \xrightarrow{D_1} \mathrm{dom}(D_2) \to 0. \tag{2.3}$$

The minimal and maximal extension of d

$$\begin{aligned} d_{\min} &:= \overline{d} = \text{ closure of } d, \\ d_{\max} &:= \delta^* = \text{ adjoint of the formal adjoint } \delta \text{ of } d \end{aligned} \tag{2.4}$$

are examples of *ideal boundary conditions*. A priori there may be several distinct *ideal boundary conditions*.

As show in [10] in the case of manifolds with cone-like singularities we have *uniqueness*, *i.e.*, the minimal and the maximal extension coincide. The domains of $d_{\min}$ and $d_{\max}$, and therefore the validity of the L^2-Stokes theorem, are quasi-isometry invariants. Therefore also in the case of conformally conic manifolds (and thus of our curve) we have a *unique ibc*, *i.e.*,

$$d_{k,\min} = d_{k,\max} \text{ for all } k. \tag{2.5}$$

We denote by $(\mathcal{C}, d, \langle\ ,\ \rangle)$ the unique extension of the differential complex $(\Omega_0^*(X \setminus \Sigma), d)$ to a Hilbert complex. The cohomology of this complex is the so called L^2-cohomology of X,

$$H^i_{(2)}(X) := \ker d_{i,\min}/\operatorname{im} d_{i-1,\min} = \ker d_{i,\max}/\operatorname{im} d_{i-1,\max}. \tag{2.6}$$

Note that (2.5) is equivalent to

$$(d_{\max}\alpha, \beta) = (\alpha, \delta_{\max}\beta) \text{ for all } \alpha \in \operatorname{dom}(d_{\max}), \beta \in \operatorname{dom}(\delta_{\max})$$

and is called the L^2-Stokes theorem. Note moreover that the validity of the L^2-Stokes theorem does not imply the essential selfadjointness of the Laplace-Beltrami operator $\Delta_{|\Omega_0} = d\delta + \delta d$ (defined on smooth compactly supported forms). Instead it is equivalent to the selfadjointness of the particular extension $\Delta = d_{\min}\delta_{\min} + \delta_{\min}d_{\min}$ (see [13], Proposition 2.3).

Since the L^2-cohomology of X is a quasi-isometry invariant one could compute it also by replacing the conformally conic metric with a cone-like metric. Therefore it is clear that all L^2-cohomology groups $H^i_{(2)}(X)$ are finite dimensional and the complex $(\mathcal{C}, d, \langle\ ,\ \rangle)$ is Fredholm. Note that the finite dimensionality of $H^i_{(2)}(X)$ also implies that $\operatorname{im} d_i$ is closed and therefore

$$\ker d_{i,\min}/\operatorname{im} d_{i-1,\min} \simeq \ker d_{i,\min}/\overline{\operatorname{im} d_{i-1,\min}}. \tag{2.7}$$

In other words reduced and unreduced L^2-cohomology coincide here.

The *uniqueness* of *ibc* in the case of conformally conic manifolds has also been shown by Brüning and Lesch in [6] by a different argument, which will be useful here.

In this section we perform the Witten deformation of the L^2-complex for a singular curve by means of certain functions $f : X \to \mathbb{R}$, called admissible functions:

Definition 2.1. — *A continuous function $f : X \to \mathbb{R}$ is called admissible if its restriction to $X \setminus \Sigma$ is smooth and moreover locally near a* *singularity $p \in \Sigma$ on each analytic branch of X the function f has the form (in the local coordinates in* (2.2)*)*

$$f(r,\varphi) = f(p) + f_1(r,\varphi) + f_2(r,\varphi), \text{ where } f_1 = rh, f_2 = O(r^{1+\delta}) \tag{2.8}$$

and $h : S^1_m \to \mathbb{R}$ is a smooth function.

Let us first prove the following

Proposition 2.2. — *Any stratified Morse function on a complex singular curve has the following form near a singular point of X (in the local coordinates in* (2.2) *on each branch):*

$$f = r(a\cos\varphi + b\sin\varphi) + O(r^{1+\delta}), \tag{2.9}$$

where $(a,b) \in \mathbb{R}^2 \setminus \{0\}$. In particular any stratified Morse function is admissible.

Proof. — Locally near the singularity any stratified Morse function on X in the sense of the theory developed by Goresky/MacPherson [12] can be written as

$$f = \left(\mathrm{Re}(g) + O(|z|^2)\right)_{|X}, \tag{2.10}$$

where $g : U(p) \to \mathbb{C}$ is a holomorphic function (see [12], Lemma 2.1.4). The affine line $l_j := \{z_2 = \ldots = z_n = 0\}$ is the tangent line to the irreducible branch X_j, therefore the non-degeneracy condition for a stratified Morse function implies that the function g has the form

$$g = \sum a_i z_i + O(|z|^2), \tag{2.11}$$

where $a_1 \neq 0$ and $z_1, \ldots, z_n$ are local coordinates as in (2.1). We get similar conditions for each analytical branch of X at p. An explicit computation using the unitary parameter $t \in \mathbb{C}^*$ and (2.10) and (2.11) shows that

$$f \circ \pi \circ \Pi^{-1} = f_1 + f_2, \tag{2.12}$$

where $f_1 = rh$, $h = a\cos(\varphi) + b\sin(\varphi)$ for some $(a,b) \in \mathbb{R}^2 \setminus \{0\}$ and $f_2 = O(r^{1+\delta})$, $\delta > 0$. □

Let $f : X \to \mathbb{R}$ be an admissible function on the curve X. Let us denote by $(\Omega^*_0(X \setminus \Sigma), d_t, \langle\ ,\ \rangle)$ the differential complex of smooth forms with compact support on $X \setminus \Sigma$, where $d_t = e^{-ft} d e^{ft}$ and $\langle\ ,\ \rangle$ is the L^2-metric, $t \in (0,\infty)$.

Denote by δ_t the formal adjoint of the operator d_t with respect to the metric $\langle\ ,\ \rangle$, and by $\Delta_{t|\Omega_0} = (d_t + \delta_t)^2$ the corresponding Laplacian (acting on smooth compactly supported forms).

LEMMA 2.3. — *The following identities hold for* $\omega \in \Omega_0^*(X \setminus \Sigma)$

$$
\begin{aligned}
d_t\omega &= d\omega + t df \wedge \omega \ , \\
\delta_t\omega &= e^{tf}\delta e^{-tf}\omega = \delta\omega + t\nabla f \lrcorner\, \omega, \\
\Delta_t\omega &= \Delta\omega + t(\mathcal{L}_{\nabla f} + \mathcal{L}^*_{\nabla f})\omega + t^2 \mid \nabla f \mid^2 \omega,
\end{aligned}
\tag{2.13}
$$

where we denote by $\mathcal{L}_{\nabla f} = d(\nabla f \lrcorner\ ) + \nabla f \lrcorner\, d$ *the Lie derivative in the direction of the gradient vector field* ∇f *and by* $\mathcal{L}^*_{\nabla f}$ *its adjoint.*

Proof. — See *e.g.*, Proposition 5.4 in [3]. □

Remark 2.4. — Note that the operator $M_f := \mathcal{L}_{\nabla f} + \mathcal{L}^*_{\nabla f}$ is a zeroth order operator.

In this situation we have two associated Hilbert complexes of interest: the maximal extension $(\mathcal{C}_{t,\max}, d_{t,\max}, \langle\ ,\ \rangle)$, defined by

$$d_{t,\max} = \text{adjoint of the formal adjoint of } d_t \text{ w. r. t. } \langle\ ,\ \rangle, \tag{2.14}$$

and the minimal extension $(\mathcal{C}_{t,\min}, d_{t,\min}, \langle\ ,\ \rangle)$, defined by

$$d_{t,\min} = \text{ closure of } d_t \text{ with respect to } \langle\ ,\ \rangle. \tag{2.15}$$

Let us denote by

$$D_t^{\mathrm{ev}} := d_t + \delta_t : \Omega_0^{\mathrm{ev}}(X) \to \Omega_0^{\mathrm{odd}}(X) \tag{2.16}$$

and by

$$D_t^{\mathrm{odd}} := d_t + \delta_t : \Omega_0^{\mathrm{odd}}(X) \to \Omega_0^{\mathrm{ev}}(X). \tag{2.17}$$

The operator $D_t := D_t^{\mathrm{ev}} + D_t^{\mathrm{odd}}$ is a closable operator with

(2.18)
$$\mathrm{dom}(D_{t,\min}) = \left\{ \begin{array}{l} \omega \in \mathrm{L}^2 \mid \ \text{there exists a sequence} \\ \Phi_n \in \Omega_0^*(X \setminus \Sigma) \text{ s.t. } \Phi_n \to \omega \text{ and } d_t\Phi_n, \delta_t\Phi_n \\ \ \text{are Cauchy sequences in } \mathrm{L}^2(\Lambda^* T^* X) \end{array} \right\}.$$

Thus in particular

$$\mathrm{dom}(D_{t,\min}) \subset \mathrm{dom}(d_{t,\min}) \cap \mathrm{dom}(\delta_{t,\min}). \tag{2.19}$$

LEMMA 2.5. — *With the notations above we get*

(a) *For* $k \neq 1$

$$\mathrm{dom}(d_{t,k,\max}) \cap \mathrm{dom}(\delta_{t,k-1,\max}) \subset \mathrm{dom}(D_{t,\min}). \tag{2.20}$$

(b) *For* $k = 0, 1, 2$

$$d_{t,k,\max} = d_{t,k,\min}, \\ \delta_{t,k,\max} = \delta_{t,k,\min}. \tag{2.21}$$

Moreover if we denote by $\Delta_t^{\mathcal{F}}$ *the Friedrichs extension of* $\Delta_{t|\Omega_0^*}$ *we have*

$$\Delta_{t,k}^{\mathcal{F}} = d_{t,k-1,\min}\delta_{t,k-1,\min} + \delta_{t,k,\min}d_{t,k,\min} \text{ for } k \neq 1. \tag{2.22}$$

Proof. — (a) It is easy to see using the local form of an admissible function f near the singularities and the formulas (3.18) in Section 3 that $df\wedge : \mathrm{L}^2 \to \mathrm{L}^2$ and $\nabla f \lrcorner : \mathrm{L}^2 \to \mathrm{L}^2$ are bounded operators. Denoting by $D = d + \delta$ the Gauss-Bonnet operator for the complex $(\mathcal{C}, d, \langle\ ,\ \rangle)$ we therefore get for all k

$$\begin{aligned} \mathrm{dom}(d_{t,k,\max}) &= \mathrm{dom}(d_{k,\max}), \\ \mathrm{dom}(\delta_{t,k,\max}) &= \mathrm{dom}(\delta_{k,\max}), \\ \mathrm{dom}(D_{t,\min}) &= \mathrm{dom}(D_{\min}). \end{aligned} \tag{2.23}$$

By Theorem 2.1 in [6] we have

$$\mathrm{dom}(d_{k,\max}) \cap \mathrm{dom}(\delta_{k-1,\max}) \subset \mathrm{dom}(D_{\min}),\ \ k \neq 1. \tag{2.24}$$

The claim now follows from (2.23) and (2.24).

(b) From Part a) we get that the hypotheses of Lemma 3.3 in [6] are satisfied for the complex $(\Omega_0^*(X \setminus \Sigma), d_t, \langle\ ,\ \rangle)$. By applying the cited result we get the claim on the domains of $d_{t,k}$ and $\delta_{t,k-1}$ for $k \neq 1$ as well as the claim on $\mathrm{dom}\,\Delta_t^{\mathcal{F}}$. The rest of the claim then also follows since $\delta_{t,\max\,/\,\min}$ is the adjoint of $d_{t,\min\,/\,\max}$. □

Proof of Proposition 1.1. — Part a) has already been shown in Lemma 2.5. We give here a second proof of it, since the below arguments will be used in Section 3 for the local model also. We denote by $\langle\ ,\ \rangle_t$ the twisted L^2-metric:

$$\langle \alpha, \beta \rangle_t = \int_{X\setminus\Sigma} \alpha \wedge *\beta e^{-2tf}. \tag{2.25}$$

Since f is bounded on X the two metrics $\langle\ ,\ \rangle$ and $\langle\ ,\ \rangle_t$ are equivalent. We introduce the following auxiliary differential complex

$$(\widetilde{\mathcal{C}}_t, \widetilde{d}_t, \langle\ ,\ \rangle_t) := (\Omega_0^*(X \setminus \Sigma), d, \langle\ ,\ \rangle_t), \tag{2.26}$$

where $\langle\ ,\ \rangle_t$ is the twisted metric defined above and $\widetilde{d}_t := d$ is the usual differential. The L^2-Stokes theorem holds for this complex: From the discussion at the beginning of the section we know that the L^2-Stokes theorem holds for the complex $(\Omega_0^*(X \setminus \Sigma), d, \langle\ ,\ \rangle)$. As mentioned before the two metrics $\langle\ ,\ \rangle$ and $\langle\ ,\ \rangle_t$ are equivalent and the domains of $d_{\min}$ and $d_{\max}$ are invariant for equivalent metrics. Therefore the complex (2.26) admits a unique extension to a Hilbert complex, which, by abuse of notation, we denote again by

$$(\widetilde{\mathcal{C}}_t, \widetilde{d}_t, \langle\ ,\ \rangle_t) = (\widetilde{\mathcal{C}}_{t\max}, \widetilde{d}_{t\max}, \langle\ ,\ \rangle_t) = (\widetilde{\mathcal{C}}_{t\min}, \widetilde{d}_{t\min}, \langle\ ,\ \rangle_t).$$

Since $d_t(e^{-ft}\omega) = e^{-ft}(d\omega)$ the map

$$e^{-ft} : (\Omega_0^*(X \setminus \Sigma), d, \langle\ ,\ \rangle_t) \to (\Omega_0^*(X \setminus \Sigma), d_t, \langle\ ,\ \rangle), \omega \mapsto e^{-ft}\omega \tag{2.27}$$

is an isomorphism of complexes. It is not difficult to see that the map (2.27) extends to isomorphisms of Hilbert complexes

$$e^{-tf} : (\widetilde{\mathcal{C}}_{t\max\,/\,\min}, \widetilde{d}_{t\max\,/\,\min}, \langle\ ,\ \rangle_t) \simeq (\mathcal{C}_{t\max\,/\,\min}, d_{t\max\,/\,\min}, \langle\ ,\ \rangle).$$

The claim now follows from the validity of the L^2-Stokes theorem for the complex $(\widetilde{\mathcal{C}}_t, \widetilde{d}_t, \langle\ ,\ \rangle_t)$.

(b) Since the Fredholm property of Hilbert complexes is invariant under isomorphism and since $(\mathcal{C}, d, \langle\ ,\ \rangle)$ is Fredholm we deduce that the complex $(\widetilde{\mathcal{C}}_t, \widetilde{d}_t, \langle\ ,\ \rangle_t)$ and therefore by the isomorphism constructed in a) also $(\mathcal{C}_t, d_t, \langle\ ,\ \rangle)$ is Fredholm. The rest of the claim follows again from a) and the general statements for Hilbert complexes in [5] (Theorem 2.4 and Corollary 2.5).

(c) It is well known that Δ (the Laplacian associated to the complex $(\mathcal{C}, d, \langle\ ,\ \rangle)$) is discrete. The claim on the discreteness of Δ_t now follows since the discreteness of the Laplacian associated to a Hilbert complex is invariant under complex isomorphism (see Lemma 2.17 in [5]). The second claim follows from Lemma 2.5. □

Let us denote by $\Delta^{\mathcal{F}}$ (resp. by $\Delta_t^{\mathcal{F}}$) the Friedrichs extension of $\Delta_{|\Omega_0^*(X\setminus\Sigma)}$ (resp. of $\Delta_{t|\Omega_0^*(X\setminus\Sigma)}$).

Corollary 2.6.

(a) *The form domains of* $\Delta^{\mathcal{F}}$ *and* $\Delta_t^{\mathcal{F}}$ *coincide.*
(b) *For* $k \neq 1$ *the form domain of the Witten Laplacian* $\Delta_{t,k}$ *and the form domain of the Laplacian* Δ_k *coincide.*

Proof. — (a) The form domain of $\Delta^{\mathcal{F}}$ is the closure of $\Omega_0^*(X \setminus \Sigma)$ under the norm

$$\|\omega\|_1^2 := \|d\omega\|^2 + \|\delta\omega\|^2 + \|\omega\|^2. \tag{2.28}$$

The form domain of $\Delta_t^{\mathcal{F}}$ is defined similarly. Note moreover that for $\omega \in \Omega_0^*(X \setminus \Sigma)$

$$(2.29)\quad \begin{aligned} \langle \Delta\omega, \omega\rangle &= \|d\omega\|^2 + \|\delta\omega\|^2 \\ &\leqslant 2\big(\|d_t\omega\|^2 + \|\delta_t\omega\|^2 + t^2\langle \mid \nabla f \mid^2 \omega, \omega\rangle\big) \\ &= 2\big(\langle \Delta_t\omega, \omega\rangle + t^2\langle \mid \nabla f \mid^2 \omega, \omega\rangle\big). \end{aligned}$$

And similarly

$$(2.30)\quad \langle \Delta_t\omega, \omega\rangle \leqslant 2\big(\langle \Delta\omega, \omega\rangle + t^2\langle \mid \nabla f \mid^2 \omega, \omega\rangle\big).$$

The claim in (a) now follows easily since $\mid \nabla f \mid^2$ is bounded on X. The claim in (b) is a consequence of Part a), Proposition 1.1 (c) and the fact that $\operatorname{dom}(\Delta_k) = \operatorname{dom}(\Delta_k^{\mathcal{F}})$, $k \neq 1$ (see [6]). □

Remark 2.7. — Note that similarly to Definition 2.1 one can define the class of admissible functions on conformally conic manifolds. The proof of Proposition 1.1 can then be extended to conformally conic manifolds of even dimension and admissible functions on them (see [17]).

3. The local model for the Witten Laplacian

From now on we will always consider the case of a stratified Morse function.

3.1. Definition of the model operator. Main results

Let $p \in \Sigma$ be a singular point of the complex curve. In this section we will develop a local model for the Witten Laplacian for each analytic branch of the curve at p. Recall from Section 2 that the local metric model of a branch of multiplicity m is given by $\operatorname{cone}_\epsilon(S_m^1)$ equipped with the metric g in (2.2).

Let us fix $\epsilon > 0$. Let $\nu_\epsilon : \operatorname{cone}(S_m^1) \to [0,1]$ be a smooth cutoff function, with $\nu_\epsilon = 1$ for $r \in (0, \epsilon/2]$ and $\nu_\epsilon = 0$ for $r \in [\epsilon, \infty)$. We denote by $g_{\text{cone}} := dr^2 + r^2 d\varphi^2$ the conic metric on $\operatorname{cone}(S_m^1)$. We denote by $(\operatorname{cone}(S_m^1), g_{\text{conf}})$ the infinite cone over S_m^1 equipped with the metric

$$(3.1)\quad g_{\text{conf}} = \nu_\epsilon g + (1-\nu_\epsilon) g_{\text{cone}}.$$

Note that by (3.1) and (2.2) $g_{\text{conf}} = \alpha^2(dr^2 + r^2 g_{S_m^1}(r))$ with a conformal factor α, $\alpha \equiv 1$ for $r > \epsilon$. Let $\langle\ ,\ \rangle := \langle\ ,\ \rangle_{\text{conf}}$ be the metric on forms, induced by g_{conf} and let $(\Omega_0^*(\mathrm{cone}(S_m^1)), d, \langle\ ,\ \rangle)$ be the de Rham complex of smooth compactly supported forms on the infinite cone $(\mathrm{cone}(S_m^1), g_{\mathrm{conf}})$. Let $f : \mathrm{cone}(S_m^1) \to \mathbb{R}$ be a function on the infinite cone such that:

$$f = \nu_\epsilon(f_1 + f_2) + (1 - \nu_\epsilon) f_1, \tag{3.2}$$

where $f_1 = rh := r(a\cos(\varphi) + b\sin(\varphi))$, $(a, b) \in \mathbb{R}^2 \setminus \{0\}$ and $f_2 = O(r^{1+\delta})$.

We denote by $(\Omega_0^*(\mathrm{cone}(S_m^1)), d_t, \langle\ ,\ \rangle)$ the complex obtained by deforming the de Rham complex by means of the function f, *i.e.*, $d_t := e^{-tf} d e^{tf}$.

The goal of the sections 3.2 – 3.6 is to show the following two results:

Proposition 3.1. — *Let $t > 0$.*

(a) *There is a unique Hilbert complex $(\mathcal{C}_t^{\mathrm{loc}}, d_t, \langle\ ,\ \rangle)$ extending the complex $(\Omega_0^*(\mathrm{cone}(S_m^1)), d_t, \langle\ ,\ \rangle)$.*

(b) *Let us denote by $\mathbf{\Delta}_t$ the Laplacian associated to the complex $(\mathcal{C}_t^{\mathrm{loc}}, d_t, \langle\ ,\ \rangle)$. The complex $(\mathcal{C}_t^{\mathrm{loc}}, d_t, \langle\ ,\ \rangle)$ is Fredholm and the natural maps*

$$\ker(\mathbf{\Delta}_{t,i}) \to H^i((\mathcal{C}_t^{\mathrm{loc}}, d_t, \langle\ ,\ \rangle)),\ i = 0, 1, 2, \tag{3.3}$$

are isomorphisms. Moreover

$$\dim\ker(\mathbf{\Delta}_t) = \dim\ker(\mathbf{\Delta}_{t,1}) = m - 1. \tag{3.4}$$

Remark 3.2. — We call $\mathbf{\Delta}_t$ the model Witten Laplacian at p. Note that by definition (of the Laplacian associated to a Hilbert complex) the domain of $\mathbf{\Delta}_t$ is $\mathrm{dom}(\mathbf{\Delta}_t) = \{\psi, d_t\psi, \delta_t\psi, d_t\delta_t\psi, \delta_t d_t\psi \in \mathrm{L}^2(\mathrm{cone}(S_m^1))\}$.

In the following we denote by $\mathbf{D}_t^{\mathrm{ev}}, \mathbf{D}_t^{\mathrm{odd}} = (\mathbf{D}_t^{\mathrm{ev}})^*, \mathbf{D}_t := \mathbf{D}_t^{\mathrm{ev}} + \mathbf{D}_t^{\mathrm{odd}}$ the Gauss-Bonnet operators associated to the complex $(\mathcal{C}_t^{\mathrm{loc}}, d_t, \langle\ ,\ \rangle)$.

Theorem 3.3. — *There exists $C > 0$, $t_0 > 0$ such that for $t > t_0$ we have*

(a) $\mathrm{spec}(\mathbf{\Delta}_{t,i}) \subset [Ct^2, \infty)$ *in case $i = 0, 2$.*

(b) $\mathrm{spec}(\mathbf{\Delta}_{t,1}) \subset \{0\} \cup [Ct^2, \infty)$. *Moreover all forms in $\ker(\mathbf{\Delta}_{t,1})$ have exponential decay outside a small neighbourhood of the singularity.*

3.2. Proof of Proposition 3.1

To prove Proposition 3.1 we show the analogous statement for the model case of a cone-like metric g_{cone} and a function $f_1 = r(a\cos\varphi + b\sin\varphi)$ (this case has been studied in [16], but Proposition 3.4 is only implicit there). The general case can then be deduced using the quasi-isometry invariance of the L^2-cohomology.

Let $(\mathrm{cone}(S^1_m), g_{\mathrm{cone}})$ be the infinite cone equipped with the conic metric $g_{\mathrm{cone}} = dr^2 + r^2 d\varphi^2$. We denote by $\langle\ ,\ \rangle^c$ the L^2-metric on forms induced by g_{cone}. Let $(\Omega^*_0(\mathrm{cone}(S^1_m)), d^{f_1}_t, \langle\ ,\ \rangle^c)$ be the deformation of the de Rham complex of smooth forms with compact support, where $d^{f_1}_t = e^{-tf_1} d e^{tf_1}$. We denote by $\delta^{f_1}_t$ its adjoint with respect to the metric $\langle\ ,\ \rangle^{\mathrm{c}}$.

Proposition 3.4. — *There is a unique Hilbert complex $(\mathcal{D}^{f_1}_t, d^{f_1}_t, \langle\ ,\ \rangle^{\mathrm{c}})$ extending the complex $(\Omega^*_0(\mathrm{cone}(S^1_m)), d^{f_1}_t, \langle\ ,\ \rangle^{\mathrm{c}})$. The Laplacian $\boldsymbol{\Delta}^{f_1}_t$ associated to this complex has spectrum*

$$\mathrm{spec}(\boldsymbol{\Delta}^{f_1}_t) = \{0\} \cup [(a^2+b^2)t^2, \infty) \tag{3.5}$$

and

$$\dim\ker(\boldsymbol{\Delta}^{f_1}_t) = \dim\ker(\boldsymbol{\Delta}^{f_1}_{t,1}) = m-1. \tag{3.6}$$

Moreover the complex $(\mathcal{D}^{f_1}_t, d^{f_1}_t, \langle\ ,\ \rangle^{\mathrm{c}})$ is Fredholm and the natural maps

$$\ker(\boldsymbol{\Delta}^{f_1}_{t,i}) \to H^i((\mathcal{D}^{f_1}_t, d^{f_1}_t, \langle\ ,\ \rangle^{\mathrm{c}})), i = 0,1,2 \tag{3.7}$$

are isomorphisms.

Proof. — To prove that there is a unique Hilbert complex extending the complex $(\Omega^0(\mathrm{cone}(S^1_m)), d^{f_1}_t, \langle\ ,\ \rangle^{\mathrm{c}})$ it is enough to show that the minimal extension $d^{f_1}_{t,\min}$ and the maximal extension $d^{f_1}_{t,\max}$ coincide. But this is equivalent to the selfadjointness of the operator $d^{f_1}_{t,\min}\delta^{f_1}_{t,\min} + \delta^{f_1}_{t,\min}d^{f_1}_{t,\min}$. The selfadjointness of the boundary condition at $r \to 0$ follows from the result in Section 2 (applied to the case of a cone-like singularity), moreover the cone is complete at infinity.

The claims on the spectral properties of $\boldsymbol{\Delta}^{f_1}_t$ are shown in [16]. Moreover, since $0 \notin \mathrm{spec}_{ess}(\boldsymbol{\Delta}^{f_1}_t)$ by Theorem 2.4 in [5] we deduce that $(\mathcal{D}^{f_1}_t, d^{f_1}_t, \langle\ ,\ \rangle^{\mathrm{c}})$ is a Fredholm complex and therefore in particular has finite dimensional cohomology groups. Applying Corollary 2.5 in [5] to the complex $(\mathcal{D}^{f_1}_t, d^{f_1}_t, \langle\ ,\ \rangle^{\mathrm{c}})$ one gets that the natural maps (3.7) are isomorphisms. □

Similarly we have a deformed differential complex $(\Omega^*_0(\mathrm{cone}(S^1_m)), d^f_t, \langle\ ,\ \rangle^{\mathrm{c}})$, where $d^f_t = e^{-tf} d e^{tf}$.

Proposition 3.5. — *There is an unique Hilbert complex $(\mathcal{D}^f_t, d^f_t, \langle\ ,\ \rangle^{\mathrm{c}})$ extending the complex $(\Omega^*_0(\mathrm{cone}(S^1_m)), d^f_t, \langle\ ,\ \rangle^{\mathrm{c}})$. The complex $(\mathcal{D}^f_t, d^f_t, \langle\ ,\ \rangle^{\mathrm{c}})$ is Fredholm and the natural maps*

$$\ker(\boldsymbol{\Delta}^f_{t,i}) \to H^i((\mathcal{D}^f_t, d^f_t, \langle\ ,\ \rangle^{\mathrm{c}})), i = 0,1,2 \tag{3.8}$$

are isomorphisms. Moreover:

$$\dim\ker(\boldsymbol{\Delta}^f_t) = \dim\ker(\boldsymbol{\Delta}^f_{t,1}) = m-1. \tag{3.9}$$

Proof. — We have 4 differential complexes of interest: two of them, $C_1 := (\Omega_0^*, d_t^{f_1}, \langle\ ,\ \rangle^{\mathrm{c}})$ resp. $C_2 := (\Omega_0^*, d_t^{f}, \langle\ ,\ \rangle^{\mathrm{c}})$ are obtained by deforming the de Rham complex by means of the function f_1 resp. f. We get two auxiliary complexes $C_3 := (\Omega_0^*, d, \langle\ ,\ \rangle_t^{f_1,\mathrm{c}})$ resp. $C_4 := (\Omega_0^*, d, \langle\ ,\ \rangle_t^{f,\mathrm{c}})$ by twisting the metric $\langle\ ,\ \rangle^c$ by means of f_1 resp. f, more precisely

$$\begin{aligned} \langle\alpha,\beta\rangle_t^{f_1,\mathrm{c}} &= \int_{\mathrm{cone}(S_m^1)} \alpha\wedge *_c\beta e^{-2tf_1}, \\ \langle\alpha,\beta\rangle_t^{f,\mathrm{c}} &= \int_{\mathrm{cone}(S_m^1)} \alpha\wedge *_c\beta e^{-2tf}. \end{aligned} \tag{3.10}$$

Note that the two metrics $\langle\ ,\ \rangle_t^{f_1,\mathrm{c}}$ and $\langle\ ,\ \rangle_t^{f,\mathrm{c}}$ are equivalent.

We will show that all these complexes have unique ibc's and are all isomorphic. The proposition then follows directly from Proposition 3.4 since the complex C_2 "inherits" all properties of C_1.

Let us shortly indicate the relations between the 4 complexes. C_1 and C_3 are isomorphic by an argument as in the alternative proof of Proposition 1.1 (a). This shows that C_3 also satisfies the L^2-Stokes theorem and inherits all properties of the complex C_1 studied in Proposition 3.4. Since the two metrics $\langle\ ,\ \rangle_t^{f_1,\mathrm{c}}$ and $\langle\ ,\ \rangle_t^{f,\mathrm{c}}$ are equivalent and the domains of $d_{\min}$ and $d_{\max}$ are invariant for equivalent metrics we deduce that also C_4 satisfies the L^2-Stokes theorem and has all properties of C_3. Finally C_2 and C_4 are isomorphic again by the argument of the alternative proof of Proposition 1.1 (a) and therefore C_2 inherits all properties of C_4 and hence of C_1. □

Proof of Proposition 3.1. — (a) The metrics g_{cone} and g_{conf} on the infinite cone cone(S_m^1) are quasi-isometric. Since the validity of the L^2-Stokes theorem is an quasi-isometry invariant and, as proved in Proposition 3.5, the complex $(\Omega_0^*, d_t^f, \langle\ ,\ \rangle^{\mathrm{c}})$ has unique ibc so does $(\Omega_0^*, d_t, \langle\ ,\ \rangle)$. Note moreover that

$$H^*((\mathcal{D}_t^f, d_t^f, \langle\ ,\ \rangle^{\mathrm{c}})) \simeq H^*((\mathcal{C}_t^{\mathrm{loc}}, d_t, \langle\ ,\ \rangle)). \tag{3.11}$$

(b) The Fredholm property of the complex $(\mathcal{C}_t^{\mathrm{loc}}, d_t, \langle\ ,\ \rangle)$ follows immediately from the Fredholm property of $(\mathcal{D}_t^f, d_t^f, \langle\ ,\ \rangle^{\mathrm{c}})$. The isomorphism (3.3) follows from standard arguments for Fredholm complexes (see Corollary 2.5 in [5]). In view of (3.11) and Proposition 3.5 we get

$$\begin{aligned} \dim\ker(\boldsymbol{\Delta}_t) = \dim\ker(\boldsymbol{\Delta}_{t,1}) &= \dim H^1((C_t^{\mathrm{loc}}, d_t, \langle\ ,\ \rangle)) \\ &= \dim H^1((\mathcal{D}_t^f, d_t^f, \langle\ ,\ \rangle^{\mathrm{c}})) = m-1. \end{aligned}$$

□

Remark 3.6. — Note that the function f is unbounded on the infinite cone and that therefore, unlike in Section 2, the Hilbert complex associated to the de Rham complex $(\Omega_0^*(\mathrm{cone}(S^1_m), d, \langle\ ,\ \rangle)$ and the Hilbert complex $(\mathcal{C}_t^{\mathrm{loc}}, d_t, \langle\ ,\ \rangle)$, $t > 0$, associated to the deformed complex are not isomorphic.

3.3. A useful unitary transformation

As in [9], Section 5 (see also [6]) for $k = 0, 1, 2$ one can construct linear maps

$$\Psi_k \quad : \quad C_0^\infty(\mathbb{R}^+, \Omega^{k-1}(S^1_m) \oplus \Omega^k(S^1_m)) \quad \to \quad \Omega_0^k(\mathrm{cone}(S^1_m)). \tag{3.12}$$

More precisely with α as in (3.1) and $\beta = \tilde{\beta}$ near the cone point and $\beta \equiv 1$ for $r > \epsilon$ we define

$$\begin{array}{rclcl} \Psi_0 & : & C_0^\infty(\mathbb{R}^+, \Omega^0(S^1_m)) & \longrightarrow & \Omega_0^0(\mathrm{cone}(S^1_m)), \\ & & f & \mapsto & \sqrt{r\alpha\beta}^{-1} f, \\ \\ \Psi_1 & : & C_0^\infty(\mathbb{R}^+, \Omega^0(S^1_m) \oplus \Omega^1(S^1_m)) & \longrightarrow & \Omega_0^1(\mathrm{cone}(S^1_m)), \\ & & (f_0, f_1) & \mapsto & \sqrt{\alpha/\beta r} f_0 dr + \sqrt{\beta r/\alpha} f_1, \\ \\ \Psi_2 & : & C_0^\infty(\mathbb{R}^+, \Omega^1(S^1_m)) & \longrightarrow & \Omega_0^2(\mathrm{cone}(S^1_m)), \\ & & f & \mapsto & \sqrt{r\alpha\beta} f dr. \end{array}$$

The Ψ_k extend to unitary maps

$$\Psi_k : \mathrm{L}^2\big(\mathbb{R}^+, \mathrm{L}^2(\Lambda^{k-1}T^*S^1_m \oplus \Lambda^k T^* S^1_m, g_{S^1_m}(0))\big) \to \mathrm{L}^2\big(\Lambda^k T^*(\mathrm{cone}(S^1_m))\big),$$

which induce unitary maps

$$\Psi_{\mathrm{ev}\,/\,\mathrm{odd}} : C_0^\infty(\mathbb{R}^+, \Omega^0(S^1_m) \oplus \Omega^1(S^1_m)) \to \Omega_0^{\mathrm{ev}\,/\,\mathrm{odd}}(\mathrm{cone}(S^1_m)) \tag{3.13}$$

such that

$$\Psi_{\mathrm{odd}}^{-1} \mathbf{D}^{\mathrm{ev}} \Psi_{\mathrm{ev}} = \alpha^{-1}\left(\partial_r + r^{-1}\big(S_0 + S_1^{\mathrm{ev}}(r)\big)\right) \tag{3.14}$$

and

$$\Psi_{\mathrm{ev}}^{-1} \mathbf{D}^{\mathrm{odd}} \Psi_{\mathrm{odd}} = \alpha^{-1}\left(-\partial_r + r^{-1}\big(S_0 + S_1^{\mathrm{odd}}(r)\big)\right), \tag{3.15}$$

where

$$S_0 = \begin{pmatrix} c_0 & \widetilde{\delta} \\ \widetilde{d} & c_1 \end{pmatrix}, \; c_0 = c_1 = -\frac{1}{2} \tag{3.16}$$

and $S_1^{\mathrm{ev}\,/\,\mathrm{odd}}(r)$ is a family of first order differential operators on $\Omega^*(S^1_m)$, smooth in $\mathbb{R}^+$ satisfying

$$\|S_1\|_{H^1\to \mathrm{L}^2} = O(r^\delta) \text{ and } S_1 = 0 \text{ for } r \geqslant \epsilon. \tag{3.17}$$

Note that S_0 is an elliptic operator. The $\widetilde{\ }$ refers to operators on the link, moreover $\widetilde{\delta}$ is computed with respect to the fixed reference metric $g(0) = d\varphi^2$ on S^1_m. The operators in (3.14), (3.15) are regular singular operators in the sense of [4], [9].

Let $h_l : S^1_m \to \mathbb{R}$ be a function on the link, $p_l := r^l h_l$. Then an explicit computation shows that

$$\Psi_{\mathrm{odd}}^{-1}(dp_l\wedge\ldots+\nabla p_l\lrcorner\ldots)\Psi_{\mathrm{ev}} = \alpha^{-1}\left(r^{l-1}\left(\begin{pmatrix} lh_l & \widetilde{\nabla} h_l\lrcorner \\ \widetilde{d}h_l & -lh_l\end{pmatrix} + O(r^\delta)\right)\right), \tag{3.18}$$

where again $\widetilde{\nabla} h_l$ denotes the gradient with respect to the fixed metric $g(0)$ on S^1_m. A similar formula holds for $\Psi_{\mathrm{ev}}^{-1}(dp_l\wedge\ldots+\nabla p_l\lrcorner\ldots)\Psi_{\mathrm{odd}}$.

Using (3.14), (3.15) and (3.18) we get for the deformed operators (with f as in (3.2))

$$\Psi_{\mathrm{odd}}^{-1}\mathbf{D}_t^{\mathrm{ev}}\Psi_{\mathrm{ev}} = \alpha^{-1}\left(\partial_r + r^{-1}(S_0 + S_1^{\mathrm{ev}}(r)) + t\big(T_0+T_1\big)\right) \tag{3.19}$$

and

$$\Psi_{\mathrm{ev}}^{-1}\mathbf{D}_t^{\mathrm{odd}}\Psi_{\mathrm{odd}} = \alpha^{-1}\left(-\partial_r + r^{-1}(S_0 + S_1^{\mathrm{odd}}(r)) + t\big(T_0+T_1\big)\right) \tag{3.20}$$

where

$$T_0 := \begin{pmatrix} h & \widetilde{\nabla} h\lrcorner \\ \widetilde{d}h & -h\end{pmatrix},\ h := a\cos\varphi + b\sin\varphi, (a,b)\in\mathbb{R}^2\setminus\{0\} \tag{3.21}$$

and $\|T_1\| = O(r^\delta)$ as $r\to 0$.

3.4. The model Witten Laplacian with conformal factor $\alpha = 1$

In this section we focus on the operator (3.19) with conformal parameter $\alpha = 1$. Set

$$P_t^{\mathrm{ev}\,/\,\mathrm{odd}} := \pm\partial_r + r^{-1}\big(S_0 + S_1(r)\big) + tT_0 + tT_1. \tag{3.22}$$

Our aim in this subsection is to prove the proposition below, which will be the main tool in the proof of Part a) of the local spectral gap theorem, Theorem 3.3.

Proposition 3.7. — *There exists $c > 0$, $t_0 > 0$ such that for $u \in C_0^\infty(\mathbb{R}^+, \Omega^*(S^1_m))$ and $t > t_0$ we have*

$$\langle P_t^{\mathrm{odd}} P_t^{\mathrm{ev}} u, u\rangle \geqslant ct^2 \|u\|^2. \tag{3.23}$$

To proof Proposition 3.7 we will use a perturbation argument similar to that in [7], Section 3.

For simplicity we will denote by $H := \mathrm{L}^2(\Lambda^0 T^* S^1_m \oplus \Lambda^1 T^* S^1_m, g_{S^1_m}(0))$ and by $\mathcal{H} := \mathrm{L}^2\big(\mathbb{R}^+, H\big)$. Let us denote by

$$\tau_0 := -\partial_r^2 + r^{-2}(S_0^2 + S_0). \tag{3.24}$$

The differential operator τ_0 is well-defined and symmetric in $\mathcal{H}$ with domain $C_0^\infty\left(\mathbb{R}^+, \Omega^*(S^1_m)\right)$. $C_0^\infty(\mathbb{R}^+, \Omega^*(S^1_m))$ is dense in $\mathcal{H}$ and τ_0 maps $C_0^\infty(\mathbb{R}^+, \Omega^*(S^1_m))$ into itself. Note that

$$A_0 := S_0^2 + S_0 = -\partial_\varphi^2 - \frac{1}{4} \geqslant -\frac{1}{4} \tag{3.25}$$

is a selfadjoint operator, $(A_0 + \mathrm{I})$ has compact resolvent. An explicit computation using (3.16) - (3.22) gives

$$P_t^{\mathrm{odd}} P_t^{\mathrm{ev}} = \tau_0 + r^{-2} R_{A_0} + tN_f + t^2(T_0^2 + R_{T_0^2}) \tag{3.26}$$

where

$$T_0^2 = (a^2 + b^2)\cdot I,\ (a,b) \in \mathbb{R}^2 \setminus \{0\} \text{ as in (3.2)}. \tag{3.27}$$

N_f, $R_{T_0^2}$ are families of zero order operators and R_{A_0} is a family of second order operators supported in $r \in (0, \epsilon)$. Moreover

$$\|N_f\|_H = O(r^{-1+\delta}), \|R_{A_0}\| = O(r^\delta) \text{ and } \|R_{T_0^2}\|_H = O(r^\delta) \text{ as } r \to 0. \tag{3.28}$$

Note that a priori the term N_f in (3.26) would be of order r^{-1} but the leading term

$$S_0 T_0 + T_0 S_0 = \begin{pmatrix} -(h + h'') & (1-1)\nabla h \lrcorner \\ (1-1)dh & h + h'' \end{pmatrix}, \text{ where } h := a\cos\varphi + b \sin\varphi$$

vanishes.

We rewrite the operator $P_t^{\mathrm{odd}} P_t^{\mathrm{ev}}$ as a sum of two terms

$$P_t^{\mathrm{odd}} P_t^{\mathrm{ev}} = L + K_t, \tag{3.29}$$

where

$$L := \frac{1}{2}\tau_0 + r^{-2} R_{A_0},\ K_t := \frac{1}{2}\tau_0 + tN_f + t^2(T_0^2 + R_{T_0^2}). \tag{3.30}$$

Let us first consider L: We treat the perturbation $r^{-2}R_{A_0}$ as in [7], Section 3. As in [7] let us introduce the following operators

$$U_0 := I, \quad U_1(\gamma) := \Omega^\gamma r^{-1}(A_0+I)^{1/2}, \quad U_2(\gamma) := \Omega^\gamma \partial_r, \tag{3.31}$$

where Ω denotes multiplication by $\frac{r}{r+1}$ and $\gamma > 0$. The operators U_i map $C_0^\infty(\mathbb{R}^+, \Omega^*(S_m^1))$ into itself. Recall from [7] that the operators U_i introduced in (3.31) are controlled by τ_0:

Lemma 3.8. — *For all $i = 0, 1, 2$ there exists $c_i > 0$ such that*

$$\|U_i u\|^2 \leqslant c_i \|u\|^2_{\tau_0} \text{ for all } u \in C_0^\infty(\mathbb{R}^+, \Omega^*(S_m^1)), \tag{3.32}$$

where $\|u\|^2_{\tau_0} := \langle \tau_0 u, u\rangle + \|u\|^2$.

Proof. — See proof of Theorem 3.2 in [7]. □

Lemma 3.9.

(a) *The perturbation $r^{-2}R^\epsilon_{A_0}$ has the form*

$$r^{-2}R_{A_0} = \sum_{i,j=0}^{2} U_i^* C_{ij}^\epsilon U_j, \tag{3.33}$$

where the operator functions $C_{ij}^\epsilon \in C(\mathbb{R}^+, \mathcal{L}(H))$ commute with multiplication by functions on $\mathbb{R}_+$ and have support in the interval $[0,\epsilon]$. Moreover

$$\delta(L) := \sum_{i,j=0}^{2} \|C_{ij}^\epsilon\|_{\mathcal{L}(\mathcal{H})} \tag{3.34}$$

can be made arbitrarily small by choosing ϵ small enough.

(b) *For $u \in C_0^\infty\left(\mathbb{R}^+, \Omega^*(S_m^1)\right)$*

$$\langle Lu, u\rangle \geqslant \frac{1}{4}\langle \tau_0 u, u\rangle - \frac{1}{4}\|u\|^2 \geqslant -\frac{1}{4}\|u\|^2. \tag{3.35}$$

Proof. — For the proof of a) see [7]. b) By Part a) we can make the perturbation $\delta(L)$ as small as we like by choosing ϵ small enough. Thus we have, for ϵ small enough and the c_k's as in Lemma 3.8,

$$\delta(L) \max_{k\in\{0,1,2\}} c_k = \sum_{i,j} \|C_{ij}^\epsilon\| \max_{k\in\{0,1,2\}} c_k \leqslant \frac{1}{4}. \tag{3.36}$$

Therefore using Lemma 3.8, (3.33) and (3.36) for $u \in C_0^\infty(\mathbb{R}^+, \Omega^*(S_m^1))$

$$\langle Lu, u\rangle = \frac{1}{2}\langle \tau_0 u, u\rangle + \langle r^{-2}R_{A_0}u, u\rangle \geqslant \frac{1}{2}\langle \tau_0 u, u\rangle - \frac{1}{4}\|u\|^2_{\tau_0} \geqslant -\frac{1}{4}\|u\|^2. \tag{3.37}$$

The last inequality follows since $\tau_0 \geqslant 0$ on $C_0^\infty(\mathbb{R}^+, \Omega^*(S^1_m))$. $\square$

We now treat the term K_t: Note first that $K_t \geqslant \widehat{K}_t$ on $C_0^\infty\left(\mathbb{R}^+, \Omega^*(S^1_m)\right)$, where

$$\widehat{K}_t := \frac{1}{2}\tau_0 - Ctr^{-1+\delta} j(r) + \frac{a^2+b^2}{2} t^2, \tag{3.38}$$

for some $C > 0$ and an appropriate cut-off function $j : \mathbb{R}_+ \to \mathbb{R}$, $j \equiv 0$ for $r > 2\epsilon$.

For $t > 0$ let us denote by U_t the unitary rescaling operator acting by

$$U_t\omega(r,\varphi) = \sqrt{t}\omega(tr,\varphi). \tag{3.39}$$

Rescaling $\widehat{K}_t$ we get

$$\begin{aligned} U_t^{-1}\widehat{K}_t U_t &= t^2\left(\frac{1}{2}\tau_0 - Cr^{-1}\left(t^{-1}r\right)^\delta j\left(t^{-1}r\right) + \frac{a^2+b^2}{2}\right) \\ &=: t^2\left(\frac{1}{2}\tau_0 - R_K^{\epsilon,t} + \frac{a^2+b^2}{2}\right). \end{aligned} \tag{3.40}$$

Lemma 3.10.

(a) *The operator $R_K^{\epsilon,t}$ can be written as*

$$R_K^{\epsilon,t} = U_1 C^{\epsilon,t} U_0, \tag{3.41}$$

with an operator function $C^{\epsilon,t} \in C(\mathbb{R}^+, \mathcal{L}(H))$ such that $\delta_2 := \|C^{\epsilon,t}\|_{\mathcal{L}(\mathcal{H})}$ can be made arbitrarily small by choosing ϵ small enough.

(b) *There exists $c > 0, t_0 > 0$ such that for $t > t_0$ and $u \in C_0^\infty(\mathbb{R}^+, \Omega^*(S^1_m))$ we get*

$$\langle K_t u, u\rangle \geqslant ct^2\|u\|^2. \tag{3.42}$$

Proof. — (a) Set $\gamma := \delta/2$. For $u, v \in C_0^\infty(\mathbb{R}^+, \Omega^*(S^1_m))$ we get

$$\begin{aligned} \left\langle R_K^{\epsilon,t}u, v\right\rangle &= C\left\langle r^{-1}\Omega^\gamma(A_0+I)^{1/2}u, \Omega^{-\gamma}(A_0+I)^{-1/2}\left(t^{-1}r\right)^\delta j(t^{-1}r)v\right\rangle \\ &= \langle U_1 u, C^{\epsilon,t}v\rangle, \end{aligned}$$

with $C^{\epsilon,t} := \Omega^{-\gamma}(A_0+I)^{-1/2}\left(t^{-1}r\right)^\delta j\left(t^{-1}r\right)$. Since $(A_0+I)^{-1/2}$ is bounded and using the support condition for j it is easy to show that $\|C^{\epsilon,t}\| \leqslant 2\epsilon^\delta$ for t chosen big enough.

(b) Proceeding as in Lemma 3.9 we see that Part a) implies that for t large enough

$$t^{-2}U_t^{-1}\widehat{K}_t U_t \geqslant \frac{a^2+b^2}{4} \tag{3.43}$$

and hence by (3.38)

$$K_t \geqslant \widehat{K}_t \geqslant \frac{a^2+b^2}{4}t^2. \tag{3.44}$$

□

The Proposition 3.7 follows combining (3.29), Lemma 3.9 (b) and Lemma 3.10 (b).

3.5. Proof of Theorem 3.3

(a) Let us assume first that $\alpha = 1$. From Proposition 3.7 we deduce that there exists $C > 0$ such that $\langle \mathbf{\Delta}_t^{\mathcal{F}}\omega, \omega\rangle \geqslant Ct^2\|\omega\|^2$ where $\mathbf{\Delta}_t^{\mathcal{F}}$ denotes the Friedrichs extension of the model Witten Laplacian on the infinite cone. Therefore

$$\operatorname{spec}(\mathbf{\Delta}_t^{\mathcal{F}}) \subset [Ct^2, \infty). \tag{3.45}$$

As in Proposition 1.1 (c) one can show that $\mathbf{\Delta}_{t,i} = \mathbf{\Delta}_{t,i}^{\mathcal{F}}$ for $i = 0, 2$. This proves (a) in case $\alpha = 1$. The general case follows from the case $\alpha = 1$ and the inequality

$$\begin{aligned}\langle \mathbf{\Delta}_t\omega, \omega\rangle &= \langle \mathbf{D}_t^{\mathrm{ev}}\omega, \mathbf{D}_t^{\mathrm{ev}}\omega\rangle = \langle \alpha^{-1}P_t^{\mathrm{ev}}\Psi^{-1}\omega, \alpha^{-1}P_t^{\mathrm{ev}}\Psi^{-1}\omega\rangle \\ &\geqslant c\langle P_t^{\mathrm{ev}}\Psi^{-1}\omega, P_t^{\mathrm{ev}}\Psi^{-1}\omega\rangle = c\langle \mathbf{\Delta}_t^{\alpha=1}\omega, \omega\rangle, \text{ for some } c > 0.\end{aligned}$$

(b) We know already from Proposition 3.1 that $0 \in \operatorname{spec}(\mathbf{\Delta}_{t,1})$ is an eigenvalue of multiplicity $m-1$. To show that $\operatorname{spec}(\mathbf{\Delta}_{t,1}) \subset \{0\} \cup [Ct^2, \infty)$ for some $C > 0$ it is enough to show that $(\mathbf{\Delta}_{t,1}\psi, \psi) \geqslant Ct^2\|\psi\|^2$ for all $\psi \in \ker(\mathbf{\Delta}_{t,1})^{\perp}$. Using the Hodge decomposition for the Fredholm complex $(\mathcal{C}_t^{\mathrm{loc}}, d_t, \langle\ ,\ \rangle)$ we can write

$$\psi = d_t\beta + \delta_t\eta \text{ for some } \beta \in \operatorname{dom}(d_t) \cap \Lambda^0, \eta \in \operatorname{dom}(\delta_t) \cap \Lambda^2. \tag{3.46}$$

By Part (a) we know that

$$\langle \mathbf{\Delta}_t\beta, \beta\rangle \geqslant Ct^2\|\beta\|^2. \tag{3.47}$$

Moreover

$$\langle \mathbf{\Delta}_t d_t\beta, d_t\beta\rangle = \langle d_t\delta_t d_t\beta, d_t\beta\rangle = \|\delta_t d_t\beta\|^2 = \|\mathbf{\Delta}_t\beta\|^2. \tag{3.48}$$

Using the Cauchy-Schwarz inequality and (3.47) in (3.48) we get

$$\langle \mathbf{\Delta}_t d_t\beta, d_t\beta\rangle = \|\mathbf{\Delta}_t\beta\|^2 \geqslant \frac{\langle \mathbf{\Delta}_t\beta, \beta\rangle^2}{\|\beta\|^2} \geqslant t^2\langle \mathbf{\Delta}_t\beta, \beta\rangle = Ct^2\|d_t\beta\|^2. \tag{3.49}$$

Similarly one gets

$$\langle \mathbf{\Delta}_t\delta_t\eta, \delta_t\eta\rangle \geqslant Ct^2\|\delta_t\eta\|^2. \tag{3.50}$$

Combining (3.49) and (3.50) we get

$$\begin{aligned}\langle \mathbf{\Delta}_t\psi,\psi\rangle &= \langle \mathbf{\Delta}_t d_t\beta, d_t\beta\rangle + \langle \mathbf{\Delta}_t\delta_t\eta,\delta_t\eta\rangle\\ &\geqslant Ct^2\|d_t\beta\|^2 + Ct^2\|\delta_t\eta\|^2 = Ct^2\|\psi\|^2.\end{aligned} \tag{3.51}$$

The claim about the decay behaviour of the eigenfunctions follows by using Agmon type estimates, which are shown in the next section. □

3.6. Agmon type estimates on the decay of the eigenfunctions

In this section we use Agmon type estimates [1] to prove exponential decay for the forms in $\ker \mathbf{\Delta}_t$. Note that in contrast to the smooth case they hold only outside a small neighbourhood of the singularity, but this is sufficient for what we need. There are two problems we have to take care of, namely the fact that $\operatorname{dom}\Delta_1 \neq \operatorname{dom}\Delta_{t,1}$ and the fact that the potential M_f is not bounded near the singularity. Let us first recall the following formula

Lemma 3.11. — *Let $\phi : \operatorname{cone}(S^1_m) \to \mathbb{R}$ be a smooth function on $\operatorname{cone}(S^1_m)$. Then we have the following identity on forms in $\Omega^*_0(\operatorname{cone}(S^1_m))$:*

$$e^{t\phi}\left(\frac{1}{t^2}\mathbf{\Delta}_t\right)e^{-t\phi} = \frac{1}{t^2}\mathbf{\Delta} + |\nabla f|^2 - |\nabla\phi|^2 + \frac{1}{t}\left(\mathcal{L}_{\nabla f} + \mathcal{L}^*_{\nabla f} + \mathcal{L}_{\nabla\phi} - \mathcal{L}^*_{\nabla\phi}\right). \tag{3.52}$$

Proof. — Like in the smooth case, compare [15], p. 256. □

For the rest of this section let $\varphi \in \ker(\mathbf{\Delta}_t)$, $\|\varphi\| = 1$. Let us choose $\delta > 0$ fixed but arbitrarily small, $\delta << \epsilon$ (ϵ as in (3.1)).

Lemma 3.12. — *Let $\phi : \operatorname{cone}(S^1_m) \to \mathbb{R}$ be a smooth function on $\operatorname{cone}(S^1_m)$. Let $\chi, \rho : \operatorname{cone}(S^1_m) \to [0,1]$ be cutoff functions with the following properties*

$$\operatorname{supp}\chi \subset \operatorname{cone}(S^1_m)\setminus\operatorname{cone}_{\delta/2}(S^1_m),\ \ \chi_{|\operatorname{cone}(S^1_m)\setminus\operatorname{cone}_\delta(S^1_m)} \equiv 1 \tag{3.53}$$

and

$$\operatorname{supp}\rho \subset \operatorname{cone}(S^1_m)\setminus\operatorname{cone}_{\delta}(S^1_m),\ \ \rho_{|\operatorname{cone}(S^1_m)\setminus\operatorname{cone}_{2\delta}(S^1_m)} \equiv 1. \tag{3.54}$$

Denote by $u := \chi\varphi$. Then the following formula holds:

$$\begin{aligned}\left\langle e^{2t\phi}\frac{1}{t^2}\mathbf{\Delta}_t u, \rho^2 u\right\rangle &= \frac{1}{t^2}\langle \mathbf{\Delta} e^{t\phi}u, \rho^2 e^{t\phi}u\rangle\\ &\quad + \left\langle\left(|\nabla f|^2 - |\nabla\phi|^2 + \frac{1}{t}\mathcal{G}_{f,\Phi}\right)e^{t\phi}u, \rho^2 e^{t\phi}u\right\rangle\end{aligned} \tag{3.55}$$

where $M_f := \mathcal{L}_{\nabla f} + \mathcal{L}^*_{\nabla f}$ and $\mathcal{G}_f := M_f + \mathcal{L}_{\nabla\phi} - \mathcal{L}^*_{\nabla\phi}$.

Proof. — The formula (3.55) is a direct consequence of (3.52). Note that u has support outside the cone point. $\square$

LEMMA 3.13. — *The notations are as in the previous lemma. Let* $a_1 = \max_{x\in \mathrm{cone}_{2\delta}(S^1_m)\setminus \mathrm{cone}_\delta(S^1_m)} \phi(x)$. *Then*

(a) $\frac{1}{t^2}\Big\langle D(\rho e^{t\phi}u), D(\rho e^{t\phi}u)\Big\rangle + \Big\langle\Big(\mid \nabla f \mid^2 - \mid \nabla\phi \mid^2 + \frac{1}{t}\mathcal{G}_{f,\Phi}\Big) e^{t\phi}u, \rho^2 e^{t\phi}u\Big\rangle \leqslant C(\rho)\cdot e^{2ta_1}\|\varphi\|^2$,

(b) $\frac{1}{t^2}\left\langle D(\rho e^{t\phi}u), D(\rho e^{t\phi}u)\right\rangle + \left\langle\left(\mid \nabla f \mid^2 - \mid \nabla\phi \mid^2 + \frac{1}{t}M_f\right) e^{t\phi}u, \rho^2 e^{t\phi}u\right\rangle \leqslant C(\rho,\phi)\cdot e^{2ta_1}\|\varphi\|^2$,

where $C(\rho)$ *is a constant depending only on* $\rho_{|\mathrm{cone}_{2\delta}(S^1_m)\setminus \mathrm{cone}_\delta(S^1_m)}$ *and* $C(\rho,\phi)$ *is a constant depending only on* $\rho_{|\mathrm{cone}_{2\delta}(S^1_m)\setminus \mathrm{cone}_\delta(S^1_m)}$ *and* $\phi_{|\mathrm{cone}_{2\delta}(S^1_m)\setminus \mathrm{cone}_\delta(S^1_m)}$.

Proof. — (a) Let us denote by $v := e^{t\phi}u$. We deduce the formula in a) from (3.55). Since $\boldsymbol{\Delta}_t\varphi = 0$ and using that $\operatorname{supp}\rho \subset \{x \mid \chi(x) = 1\}$ we get for the left hand side of (3.55)

$$\left\langle e^{2t\phi}\frac{1}{t^2}\boldsymbol{\Delta}_t u, \rho^2 u\right\rangle = 0, \tag{3.56}$$

and therefore

$$0 = \frac{1}{t^2}\left\langle \boldsymbol{\Delta} v, \rho^2 v\right\rangle + \left\langle\Big(\mid \nabla f \mid^2 - \mid \nabla\phi \mid^2 + \frac{1}{t}\mathcal{G}_{f,\Phi}\Big)v, \rho^2 v\right\rangle. \tag{3.57}$$

We reformulate the term $\left\langle \boldsymbol{\Delta} v, \rho^2 v\right\rangle$ as follows (see [20], Lemma 2.34):

$$\left\langle \boldsymbol{\Delta} v, \rho^2 v\right\rangle = \|D(\rho v)\|^2 - \|[D,\rho]v\|^2, \tag{3.58}$$

where $[D,\rho] = D\rho - \rho D$ is a zeroth order operator with support in $\mathrm{cone}_{2\delta}(S^1_m)\setminus \mathrm{cone}_\delta(S^1_m)$.

By plugging in (3.58) into (3.57) and using again the properties of the cut-off functions ρ and χ we get:

$$\begin{aligned}\frac{1}{t^2}\langle D(\rho v), D(\rho v)\rangle + \left\langle\Big(\mid \nabla f \mid^2 - \mid \nabla\phi \mid^2 + \frac{1}{t}\mathcal{G}_{f,\Phi}\Big)v, \rho^2 v\right\rangle \\ = \frac{1}{t^2}\|[D,\rho]v\|^2 = \frac{1}{t^2}\|[D,\rho]e^{t\phi}\varphi\|^2 \leqslant \frac{1}{t^2}C(\rho)\cdot e^{2a_1 t}\|\varphi\|^2.\end{aligned} \tag{3.59}$$

(b) Note first that

$$\left\langle v, \mathcal{L}_{\nabla\phi}(\rho^2 v)\right\rangle = \left\langle v, \rho^2\mathcal{L}_{\nabla\phi}v\right\rangle + \left\langle v, d\rho^2\wedge(\nabla\phi \lrcorner v)\right\rangle + \left\langle v, \nabla\phi \lrcorner (d\rho^2\wedge v)\right\rangle. \tag{3.60}$$

Now, since $d\rho^2$ has support in $\mathrm{cone}_{2\delta}(S^1_m)\setminus \mathrm{cone}_\delta(S^1_m)$, we get

$$|\langle v, d\rho^2\wedge(\nabla\phi \lrcorner v\rangle| \leqslant C(\rho,\phi)e^{2ta_1}\|\varphi\|^2 \tag{3.61}$$

and

$$|\langle v, \nabla\phi \lrcorner (d\rho^2 \wedge v)\rangle| \leqslant C(\rho,\phi)e^{2ta_1}\|\varphi\|^2 \tag{3.62}$$

for some appropriate constant $C(\rho,\phi)$ depending on $\rho_{|\mathrm{cone}_{2\delta}(S^1_m)\setminus\mathrm{cone}_\delta(S^1_m)}$ and $\phi_{|\mathrm{cone}_{2\delta}(S^1_m)\setminus\mathrm{cone}_\delta(S^1_m)}$.

Thus using (3.60), (3.61) and (3.62)

$$\begin{aligned}\left\langle \mathcal{G}_{f,\phi}v, \rho^2 v\right\rangle &= \langle M_f v, \rho^2 v\rangle + \langle \mathcal{L}_{\nabla\phi}v, \rho^2 v\rangle - \langle v, \mathcal{L}_{\nabla\phi}(\rho^2 v)\rangle \\ &= \langle M_f v, \rho^2 v\rangle - \langle v, d\rho^2 \wedge (\nabla\phi \lrcorner v)\rangle - \langle v, \nabla\phi \lrcorner (d\rho^2 \wedge v)\rangle \\ &\geqslant \langle M_f v, \rho^2 v\rangle - 2C(\rho,\phi)e^{2ta_1}\|\varphi\|^2.\end{aligned} \tag{3.63}$$

The claim in b) follows from Part a) and (3.63). □

The Lithner-Agmon metric on the infinite cone is the (degenerate) metric $|\nabla f|^2 g_{\mathrm{conf}}$. We denote by $d := d(0,\ ) : \mathrm{cone}(S^1_m) \to \mathbb{R}$ the induced Agmon distance (from the cone point). Note that $|\nabla d| = |\nabla f|$ almost everywhere. Recall that near the cone point

$$|\nabla f|^2 = (a^2+b^2) + \text{ higher order terms in } r \tag{3.64}$$

and therefore it is not difficult to see, that there exist constants $0 < c_1 < c_2$ such that

$$\max_{x\in\mathrm{cone}_{2\delta}(S^1_m)\setminus\mathrm{cone}_\delta(S^1_m)} d(x) \leqslant c_1\delta < c_2\delta \leqslant \inf_{x\in\mathrm{cone}(S^1_m)\setminus\mathrm{cone}_{3\delta}(S^1_m)} d(x). \tag{3.65}$$

Proposition 3.14. — *Let $\Omega := \mathrm{cone}(S^1_m) \setminus \mathrm{cone}_{2\delta}(S^1_m)$. For $\tilde\epsilon > 0$ let us denote by $d_{\tilde\epsilon} := (1-\tilde\epsilon)d$. Then there exists $C > 0$, $t_0(\tilde\epsilon) > 0$ such that for $t > t_0(\tilde\epsilon)$ we get*

$$\frac{1}{t^2}\|D(e^{d_{\tilde\epsilon}t}\varphi)\|^2_{\mathrm{L}^2(\Omega)} + \frac{\tilde\epsilon^2}{2}\|e^{d_{\tilde\epsilon}t}\varphi\|^2_{\mathrm{L}^2(\Omega)} \leqslant Ce^{2c_1(1-\tilde\epsilon)t\delta}. \tag{3.66}$$

Proof. — We apply Lemma 3.13 (b) with $\Phi := d_{\tilde\epsilon} = (1-\tilde\epsilon)d$. We denote by

$$V_t := |\nabla f|^2 - |\nabla\Phi|^2 + \frac{1}{t}M_f. \tag{3.67}$$

Since $|\nabla f|^2 = |\nabla d|^2$ we get

$$V_t = |\nabla f|^2 (1-(1-\tilde\epsilon)^2) + t^{-1}M_f = |\nabla f|^2 (2\tilde\epsilon - \tilde\epsilon^2) + t^{-1}M_f. \tag{3.68}$$

Since M_f is a bounded operator outside a neighbourhood of the cone point (indeed $M_f = 0$ for $r > \epsilon$) we get:

$$V_t \geqslant c\frac{\tilde\epsilon^2}{2} \tag{3.69}$$

for t large enough outside a small neighbourhood of the cone point. Applying Lemma 3.13 (b) we now get:

$$\frac{1}{t^2}\|D(\rho e^{td_{\tilde\epsilon}}u)\|^2 + \frac{\tilde\epsilon^2}{2}\left\langle e^{td_{\tilde\epsilon}}u, \rho^2 e^{td_{\tilde\epsilon}}u\right\rangle \leqslant C\cdot e^{2c_1 t\delta(1-\tilde\epsilon)}\|\varphi\|^2, \tag{3.70}$$

and therefore

$$\frac{1}{t^2}\|D(e^{td_{\tilde\epsilon}}\varphi)\|^2_{\mathrm{L}^2(\Omega)} + \frac{\tilde\epsilon^2}{2}\|e^{td_{\tilde\epsilon}}\varphi\|^2_{\mathrm{L}^2(\Omega)} \leqslant C\cdot e^{2c_1(1-\tilde\epsilon)t\delta}\|\varphi\|^2. \tag{3.71}$$

□

The next corollary shows that the L^2-norm of φ is concentrated near the cone point.

Corollary 3.15. — *Let $\Omega\subset \mathrm{cone}(L)\setminus \mathrm{cone}_{3\delta}(L)$. Then there exists $c>0$ such that*

$$\|\varphi\|^2_{\mathrm{L}^2(\Omega)} = O(e^{-ct\delta}). \tag{3.72}$$

Proof. — From Proposition 3.14 we get

$$\|e^{td_{\tilde\epsilon}}\varphi\|^2_{\mathrm{L}^2(\Omega)} \leqslant \frac{C}{\tilde\epsilon^2}e^{2c_1 t\delta(1-\tilde\epsilon)}. \tag{3.73}$$

Since $d\geqslant c_2\delta$ on Ω we get

$$\|\varphi\|^2_{\mathrm{L}^2(\Omega)} \leqslant \frac{C}{\tilde\epsilon^2}e^{-2(c_2-c_1)t} \tag{3.74}$$

and the claim follows. □

As in [14] (p. 24) using a priori estimates for the elliptic operator Δ and Proposition 3.14 we get pointwise estimates for $\varphi\in\ker(\boldsymbol{\Delta}_t)$.

Corollary 3.16. — *There exists $C>0$ such that for $x\in \mathrm{cone}_{\epsilon/2}(L)\setminus \mathrm{cone}_{2\delta}(L)$ we have*

$$\mid\varphi(x)\mid\leqslant Ce^{2ct\delta}e^{-td(x)(1-\tilde\epsilon)}. \tag{3.75}$$

Similar estimates can be shown for the derivatives of φ.

4. Proof of the spectral gap theorem and the Morse inequalities

Proof of Theorem 1.2 (a). — The proof of the spectral gap theorem consists in two steps. The first step, namely the developing of a model operator for Δ_t in the neighbourhood of a singular point $p\in\Sigma$ of X has already been done in Section 3. In the second step of the proof it is now enough to follow the strategy of proof in the smooth case. We follow here the proof in [2], Section 9 and just give a rough outline. (Compare also [16] where the proof is detailed for the model case of a complex curve with cone-like singularities). Recall from the smooth theory that the model Witten Laplacian $\boldsymbol{\Delta}_t^p$ in the neighbourhood of a critical point $p \in X \setminus \Sigma$ has discrete spectrum $\operatorname{spec}(\boldsymbol{\Delta}_t^p) = 2t\mathbb{N}$ and $\dim\ker(\boldsymbol{\Delta}_t^p) = 1$. We denote by $\omega^p(t)$ the generator of $\ker(\boldsymbol{\Delta}_t^p)$. For a singular point $p \in \Sigma$ we denote by $\{\omega_j^p(t) \mid j = 1, \ldots, n_p\}$ the union of the bases of the kernels of all model operators at p. (Recall that we have a model for each branch separately). Let $\mu_\epsilon : \mathbb{R}^+ \to \mathbb{R}$ be a cut-off function with $\mu_\epsilon = 1$ in $[0, \epsilon/4]$, $\operatorname{supp}(\mu_\epsilon) \subset [0, \epsilon/2]$. The forms $\Phi_j^p(t) := \mu_\epsilon(\mid x \mid)\omega_j^p(t)$ can be identified with L^2-forms on X. We denote by

$$\begin{aligned} E(t) :=& \operatorname{span}\big\{\{\Phi_1^p(t) := \mu_\epsilon \omega^p(t) \mid p \in \operatorname{Crit}(f) \setminus \Sigma\} \\ &\cup \{\Phi_j^p(t) \mid p \in \Sigma, j \in I_p := \{1, \ldots, n_p\}\}\big\}. \end{aligned}$$

We get an orthogonal splitting $\mathrm{L}^2(\Lambda^*(T^*(X \setminus \Sigma))) = E(t) \oplus E(t)^\perp$. The closed operator $A_t := d_t + \delta_t$ with $\operatorname{dom}(A_t) = \operatorname{dom}(d_t) \cap \operatorname{dom}(\delta_t) \subset \mathrm{L}^2(\Lambda^*(T^*(X \setminus \Sigma)))$ can be written in matrix form

$$A_t = \begin{pmatrix} A_{t,1} & A_{t,2} \\ A_{t,3} & A_{t,4} \end{pmatrix} \text{ according to the splitting } E(t) \oplus E(t)^\perp.$$

Note that $\operatorname{dom}(A_t)$ equipped with the norm $\|u\|_1 := \sqrt{\|(d+\delta)u\|^2 + \|u\|^2}$ is complete. One can show the following estimates on A_t as $t \to \infty$: □

Proposition 4.1. — *There exist constants $c, C > 0$ and $t_0 > 0$ such that for all $t > t_0$ we have*

(a) *For all $u \in E(t)$ we have $\|A_t u\| = O(e^{-ct})\|u\|$. In particular $\|A_{t,1}u\| = O(e^{-ct})\|u\|$, $\|A_{t,3}u\| = O(e^{-ct})\|u\|$.*
(b) *For all $u \in E(t)^\perp \cap \operatorname{dom}(A_t)$ we get: $\|A_{t,2}u\| \leqslant O(e^{-ct})\|u\|$, $\|A_{t,4}u\| \geqslant C(\|u\|_1 + \sqrt{t}\|u\|)$.*

Remark 4.2. — Note that the second estimate in (b) implies that $\langle \Delta_t u, u\rangle \geqslant t\|u\|^2$ for all $u \in \operatorname{dom}(\Delta_t) \cap E(t)^\perp$.

The proof of Proposition 4.1 is similar to the corresponding statements in the smooth case (see [2], Section 9) and we omit the details here. Note that to prove the estimates for forms s with support in a neighbourhood of a singular point of X Theorem 3.3 on the spectrum of the model Witten Laplacian as well as the decay of the eigenforms in the local model (Corollary 3.15 and Corollary 3.16) are crucial. As in [2], Section 9 (c) and (e), Proposition 4.1 allows to give estimates for the resolvent of $A_t - \lambda : \operatorname{dom}(A_t) \to \mathrm{L}^2\big(\Lambda^*(T^*(X \setminus \Sigma))\big)$, where $\lambda \in \mathbb{C}$, $\mid \lambda \mid \in \left[e^{-ct/2}, \frac{C\sqrt{t}}{2}\right]$, with constants c, C as in Proposition 4.1. We deduce the invertibility of the operator $A_t - \lambda$, and since $\Delta_t - \lambda^2 = (A_t - \lambda)(A_t + \lambda)$ we thus get Theorem 1.2(a).

Proof of Theorem 1.2(b) and Corollary 1.3. — For $i = 0, 1, 2$ we define the $\mathbb{R}$-vector space C_i by

$$C_i := \bigoplus_{p \in \mathrm{Crit}_i(f) \setminus \Sigma} \mathbb{R} \cdot e_1^p \oplus \bigoplus_{p \in \mathrm{Crit}_i(f) \cap \Sigma, j \in I_p} \mathbb{R} \cdot e_j^p. \tag{4.1}$$

We define a linear map

$$J_i(t) : C_i \longrightarrow \mathcal{C}_{t,i}, \ \ J_i(t)(e_j^p) = \Phi_j^p(t). \tag{4.2}$$

We denote by $(\mathbb{F}_t, d_t, \langle\ ,\ \rangle)$ the subcomplex of $(\mathcal{C}_t, d_t, \langle\ ,\ \rangle)$ generated by the eigenforms of Δ_t to eigenvalues lying in $[0, 1]$. We denote moreover by $P(t, [0, 1])$ the orthogonal projection operator from $\mathcal{C}_t$ on $\mathbb{F}_t$ with respect to $\langle\ ,\ \rangle$.

Note first that for all forms in $E(t)$ we have $\langle \Delta_t \varphi, \varphi \rangle \leqslant O(e^{-ct}) \|\varphi\|^2$. Therefore by the Rayleigh-Ritz principle it is clear that $\dim \mathbb{F}_t^i \geqslant \dim E^i(t) = \dim C_i = c_i(f)$. We show now that the linear map $P_i(t, [0, 1]) \circ J_i(t) : C_i \longrightarrow \mathbb{F}_{t,i}$ is a surjective map from C_i onto $\mathbb{F}_{t,i}$, *i.e.*, that $\mathrm{Im}_i(t) := P_i(t, [0, 1]) \circ J_i(t)(C_i) = \mathbb{F}_t$. Let $0 \neq u \in \mathbb{F}_t \cap \mathrm{Im}(t)^{\perp}$. Using the self-adjointness of the projection $P(t, [0, 1])$ we get:

$$0 = \langle u, P(t, [0, 1]) J(t) e_j^p \rangle = \langle P(t, [0, 1]) u, J(t) e_j^p \rangle = \langle u, J(t) e_j^p \rangle. \tag{4.3}$$

Equation (4.3) implies that $u \in E(t)^{\perp}$ and therefore by Proposition 4.1/Remark 4.1

$$\langle \Delta_t u, u \rangle \geqslant t \|u\|^2, \tag{4.4}$$

which is a contradiction to $u \in \mathbb{F}_t$.

Therefore the complex $(\mathbb{F}_t, d_t, \langle\ ,\ \rangle)$ is a finite dimensional subcomplex of $(\mathcal{C}_t, d_t, \langle\ ,\ \rangle)$ with $\dim \mathbb{F}_{t,i} = c_i(f)$. By Proposition 1.1 moreover $H^*_{(2)}(X) \simeq \ker(\Delta_t) \simeq H^*((\mathbb{F}_t, d_t, \langle\ ,\ \rangle))$. The Morse inequalities in Corollary 1.3 now follow by a standard algebraic argument. □

BIBLIOGRAPHY

[1] S. Agmon, *Lectures on exponential decay of solutions of second-order elliptic equations: bounds on eigenfunctions of N-body Schrödinger operators*, Mathematical Notes, vol. 29, Princeton University Press, Princeton, NJ, 1982, 118 pages.

[2] J.-M. Bismut & G. Lebeau, "Complex immersions and Quillen metrics", *Publ. Math. Inst. Hautes Étud. Sci.* **74** (1991), p. 1-197.

[3] J.-M. Bismut & W. Zhang, "An extension of a theorem by Cheeger and Müller", *Astérisque* (1992), no. 205, p. 235, With an appendix by François Laudenbach.

[4] J. Brüning, "L^2-index theorems on certain complete manifolds", *J. Differential Geom.* **32** (1990), no. 2, p. 491-532.

[5] J. Brüning & M. Lesch, "Hilbert complexes", *J. Funct. Anal.* **108** (1992), no. 1, p. 88-132.

[6] ———, "Kähler-Hodge theory for conformal complex cones", *Geom. Funct. Anal.* **3** (1993), no. 5, p. 439-473.

[7] ———, "On the spectral geometry of algebraic curves", *J. Reine Angew. Math.* **474** (1996), p. 25-66.

[8] J. Brüning, N. Peyerimhoff & H. Schröder, "The $\overline{\partial}$-operator on algebraic curves", *Comm. Math. Phys.* **129** (1990), no. 3, p. 525-534.

[9] J. Brüning & R. Seeley, "An index theorem for first order regular singular operators", *Amer. J. Math.* **110** (1988), no. 4, p. 659-714.

[10] J. Cheeger, "On the Hodge theory of Riemannian pseudomanifolds", in *Geometry of the Laplace operator (Proc. Sympos. Pure Math., Univ. Hawaii, Honolulu, Hawaii, 1979)*, Proc. Sympos. Pure Math., XXXVI, Amer. Math. Soc., Providence, R.I., 1980, p. 91-146.

[11] M. Goresky & R. MacPherson, "Morse theory and intersection homology theory", in *Analysis and topology on singular spaces, II, III (Luminy, 1981)*, Astérisque, vol. 101, Soc. Math. France, Paris, 1983, p. 135-192.

[12] ———, *Stratified Morse theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], vol. 14, Springer-Verlag, Berlin, 1988.

[13] D. Grieser & M. Lesch, "On the L^2-Stokes theorem and Hodge theory for singular algebraic varieties", *Math. Nachr.* **246-247** (2002), p. 68-82.

[14] B. Helffer, *Semi-classical analysis for the Schrödinger operator and applications*, Lecture Notes in Mathematics, vol. 1336, Springer-Verlag, Berlin, 1988, vi+107 pages.

[15] B. Helffer & J. Sjöstrand, "Puits multiples en mécanique semi-classique. IV. Étude du complexe de Witten", *Comm. Partial Differential Equations* **10** (1985), no. 3, p. 245-340.

[16] U. Ludwig, "The Witten complex for singular spaces of dimension 2 with cone-like singularities", accepted at Mathematische Nachrichten, see also C. R., Math., Acad. Sci. Paris, 347 (11-12):651-654.

[17] ———, "The Witten deformation for conformal cones", in preparation.

[18] ———, "The geometric complex for algebraic curves with cone-like singularities and admissible Morse function", to appear in Ann. Inst. Fourier, 2010.

[19] M. Nagase, "Gauss-Bonnet operator on singular algebraic curves", *J. Fac. Sci. Univ. Tokyo Sect. IA Math.* **39** (1992), no. 1, p. 77-86.

[20] W. Pardon & M. Stern, "Pure Hodge structure on the L_2-cohomology of varieties with isolated singularities", *J. Reine Angew. Math.* **533** (2001), p. 55-80.

[21] E. Witten, "Supersymmetry and Morse theory", *J. Differential Geom.* **17** (1982), no. 4, p. 661-692 (1983).

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