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On Analytic Vectors for Unitary Representations of Infinite Dimensional Lie Groups

ON ANALYTIC VECTORS
FOR UNITARY REPRESENTATIONS
OF INFINITE DIMENSIONAL LIE GROUPS

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ABSTRACT. — Let G be a connected and simply connected Banach–Lie group. On the complex enveloping algebra of its Lie algebra $\mathfrak{g}$ we define the concept of an analytic functional and show that every positive analytic functional λ is integrable in the sense that it is of the form $\lambda(D) = \langle d\pi(D)v, v \rangle$ for an analytic vector v of a unitary representation of G. On the way to this result we derive criteria for the integrability of ∗-representations of infinite dimensional Lie algebras of unbounded operators to unitary group representations.

For the matrix coefficient $\pi^{v,v}(g) = \langle \pi(g)v, v \rangle$ of a vector v in a unitary representation of an analytic Fréchet–Lie group G we show that v is an analytic vector if and only if $\pi^{v,v}$ is analytic in an identity neighborhood. Combining this insight with the results on positive analytic functionals, we derive that every local positive definite analytic function on a simply connected Fréchet–BCH–Lie group G extends to a global analytic function.

RÉSUMÉ. — Soit G un groupe de Lie–Banach connexe et simplement connexe. Sur l'algèbre enveloppante complexe de son algèbre de Lie $\mathfrak{g}$ nous définissons la notion de fonctionnelle analytique et montrons que chaque fonctionnelle analytique positive λ est integrable au sens où elle est de la forme $\lambda(D) = \langle d\pi(D)v, v \rangle$ pour un vecteur analytique v d'une représentation unitaire de G. Dans la preuve de ce résultat nous obtenons des critères pour l'integrabilité des ∗-representations des algèbres de Lie en représentations de groupe unitaires.

Pour le coefficient matriciel $\pi^{v,v}(g) = \langle \pi(g)v, v \rangle$ d'un vecteur v d'une représentation unitaire d'un groupe de Lie–Fréchet analytique G nous montrons que v est un vecteur analytique si et seulement si $\pi^{v,v}$ est analytique dans un voisinage de l'identité. En combinant ce résultat à ceux sur les fonctionnelles analytiques positives nous obtenons que chaque fonction analytique de type positive locale sur un group de Lie–Fréchet–BCH simplement connexe s'étend en une fonction analytique globale.

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1. Introduction

Let G be an analytic Lie group modeled on a locally convex space and $(\pi, \mathcal{H})$ be a unitary representation of G.[1] We call a vector $v \in \mathcal{H}$ smooth, resp., analytic if its orbit map $\pi^v \colon G \to \mathcal{H}, g \mapsto \pi(g)v$ is smooth, resp., analytic. We write $\mathcal{H}^\infty$, resp., $\mathcal{H}^\omega$ for the subspace of smooth, resp., analytic vectors. Analytic vectors for unitary representations of finite dimensional Lie groups play an important role in the translation process between the representation π of a connected Lie group G and the derived representation $\mathsf{d}\pi \colon \mathfrak{g} \to \mathrm{End}(\mathcal{H}^\infty)$ of its Lie algebra $\mathfrak{g}$. In particular, for a subspace $E \subseteq \mathcal{H}$ consisting of smooth vectors and invariant under $\mathfrak{g}$, we can only conclude its invariance under G if the set of analytic vectors in E is dense ([49, Cor. 4.4.5.5]). The key point in the argument is the Taylor expansion of an analytic function $f \colon G \to E$ with values in a Banach space E near the identity:

$$f(\exp X) = \sum_{n=0}^\infty \frac{1}{n!}(X^n f)(\mathbf{1}),$$

where the element $X \in \mathfrak{g}$ acts as a left invariant differential operator on $C^\infty(G, E)$. The summands in this expansion define a linear map $T(f) \colon U(\mathfrak{g}) \to E, D \mapsto (Df)(\mathbf{1})$. Our motivation for this paper was to understand which functionals on $U(\mathfrak{g})$ correspond in this sense to a matrix coefficient $\pi^{v,v}(g) = \langle \pi(g)v, v \rangle$ of an analytic vector v for a unitary representation $(\pi, \mathcal{H})$ of G.

As we shall see below, all this concerns only representations $(\pi, \mathcal{H})$ for which $\mathcal{H}^\omega$ is dense. For finite dimensional Lie groups, this is always the case for continuous unitary representations on Hilbert spaces. Building on previous work of Harish–Chandra on semisimple groups, Cartier and Dixmier prove in [10] the density of $\mathcal{H}^\omega$ for unitary representations and continuous Banach representations which are bounded on a certain discrete central subgroup. The general case was obtained by Nelson ([36, Thm. 4]) and Gårding ([15]) by convolution with heat kernels (cf. [49, Sect. 4.4.5]). For generalizations of these density results to representations on locally convex spaces, we refer to [31].

For infinite dimensional Lie groups a continuous unitary representation need not possess any differentiable vector. Refining a construction from [3], we have shown in [35] that the continuous unitary representation of the abelian Banach–Lie group $G = (L^p([0,1], \mathbb{R}), +)$, $p \geqslant 1$, on the Hilbert space $\mathcal{H} = L^2([0,1], \mathbb{C})$ by $\pi(g)f = e^{ig}f$ satisfies $\mathcal{H}^\infty = \{0\}$. For $p = 1$, we

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[1] Cf. [34] for a survey on locally convex Lie theory and Section 2 for basic definitions.

thus obtain a continuous representation with no non-zero C^1-vector, and for $p = 2k$, the space of C^k-vectors is non-zero, but there is no non-zero C^{k+1}-vector.

As a consequence of these examples, one cannot expect analytic vectors to form a dense subspace for infinite dimensional Lie groups, and not even for Banach–Lie groups. Up to now, only a few classes of groups are known for which $\mathcal{H}^\omega$ is dense for every continuous unitary representation. If G is the additive group of a nuclear locally convex space, this follows from the Bochner–Minlos Theorem and the regularity of the corresponding measures (cf. [20, Thm. 2.11] and [44, Sect. 2.5] for the special case of the countably dimensional vector group $G = \mathbb{R}^{(\mathbb{N})}$). One also has a corresponding result for Heisenberg groups: Let V be a locally convex space, endowed with a continuous positive definite scalar product. Then we define the corresponding Heisenberg group by

$$\mathrm{Heis}(V) = \mathbb{R} \times V \times V,$$
$$(z, v, w)(z', v', w') := (z + z' + \tfrac{1}{2}(\langle v, w' \rangle - \langle v', w \rangle), v + v', w + w').$$

In [20] Hegerfeldt shows that, if V is separable, barrelled and nuclear, for any continuous unitary representation $(\pi, \mathcal{H})$ of $\mathrm{Heis}(V)$, the space $\mathcal{H}^\omega$ is dense. He even proves the existence of a dense subspace $\mathcal{D} \subseteq \mathcal{H}^\omega$ with the property that for every $v \in \mathcal{D}$ the orbit map $\pi^v(g) = \pi(g)v$ extends to a holomorphic map $\mathrm{Heis}(V)_\mathbb{C} \to \mathcal{H}$ (see also [41] for the special case $V = \mathbb{R}^{(\mathbb{N})}$ and certain Banach completions). Further results for direct limits of finite dimensional matrix groups can be found in [44, Sect. 6.4].

In this paper we are concerned with two aspects of analytic vectors for representations of a 1-connected Lie group G with Lie algebra $\mathfrak{g}$:

(IR) The integrability problem for Lie algebra representations: Given a $*$-representation $(\rho, \mathcal{D})$ of $\mathfrak{g}$ on a pre-Hilbert space $\mathcal{D}$, when is there a continuous unitary representation $(\pi, \mathcal{H})$ of G on the completion $\mathcal{H}$ of $\mathcal{D}$ with $\mathcal{D} \subseteq \mathcal{H}^\infty$ and $\rho(x) = \mathrm{d}\pi(x)|_\mathcal{D}$ for $x \in \mathfrak{g}$? The criteria we discuss here are based on a rich supply of analytic vectors.

(IF) The integrability problem for functionals on $U(\mathfrak{g})$: Which linear functionals $\lambda \colon U_\mathbb{C}(\mathfrak{g}) := U(\mathfrak{g})_\mathbb{C} \to \mathbb{C}$ are *integrable* in the sense that there exists a continuous unitary representation $(\pi, \mathcal{H})$ of G and a smooth vector $v \in \mathcal{H}^\infty$ with $\lambda(D) = \langle \mathrm{d}\pi(D)v, v \rangle$ for $D \in U(\mathfrak{g})$?

As far as we know, the present knowledge on (IR) is limited to finite dimensional Lie algebras and groups. It is based on Nelson's Theorem, asserting that a symmetric operator A on a Hilbert space $\mathcal{H}$ is essentially selfadjoint if it has a dense subspace of vectors v which are analytic in

the sense that $\sum_n \frac{\|A^n v\| t^n}{n!} < \infty$ holds for some $t > 0$. In view of Stone's Theorem, for such operators A, the selfadjoint closure $\overline{A}$ generates a unitary one-parameter group $e^{it\overline{A}}$. The first generalization of this result to finite dimensional Lie algebra $\mathfrak{g}$ can be found in [14], where it is shown that any representation $\rho \colon \mathfrak{g} \to \mathrm{End}(\mathcal{D})$ by skew-symmetric operators on a dense subspace $\mathcal{D}$ of a Hilbert space $\mathcal{H}$ can be integrated to a unitary group representation if $\mathcal{D}$ consists of analytic vectors for the operators $\rho(x_1), \ldots, \rho(x_n)$, where $x_1, \ldots, x_n$ is a basis of $\mathfrak{g}$. In [48] Simon generalizes this result to the situation where $x_1, \ldots, x_n$ are only required to generate $\mathfrak{g}$ as a Lie algebra. These integrability criteria for representations of finite dimensional Lie algebras by unbounded operators generalize to representations on Banach spaces (cf. [18], see also [22, Cor. A.4]). We recommend Section 5 in [28] for a detailed discussion of analytic vectors and integrability conditions for finite dimensional Lie algebras. One of our main results is a solution of (IR) for Banach–Lie algebras asserting the integrability of ρ if $\mathcal{D}$ consists of analytic vectors.

Problem (IF) is an infinite dimensional variant of the classical (non-commutative) moment problem. Since smoothness of a vector $v \in \mathcal{H}$ is equivalent to smoothness of the corresponding positive definite function $\pi^{v,v}(g) = \langle \pi(g)v, v \rangle$ ([35, Thm. 7.2]), the GNS construction implies that (IF) is equivalent to the existence of a smooth positive definite function φ with $\lambda(D) = (D\varphi)(\mathbf{1})$ for $D \in U_{\mathbb{C}}(\mathfrak{g})$.

If $\mathfrak{g}$ is finite dimensional and $x_1, \ldots, x_k$ is a basis of $\mathfrak{g}$, then every linear functional $\lambda \in U_{\mathbb{C}}(\mathfrak{g})^*$ defines a function

$$s_\lambda \colon \mathbb{N}_0^k \to \mathbb{C}, \quad \alpha \mapsto \lambda(x^\alpha), \quad x^\alpha := x_1^{\alpha_1} \cdots x_k^{\alpha_k}.$$

In view of the Poincaré–Birkhoff–Witt Theorem, the correspondence between linear functionals on $U_{\mathbb{C}}(\mathfrak{g})$ and functions on $\mathbb{N}_0^k$ is one-to-one. The *non-commutative moment problem* is to decide for $s \colon \mathbb{N}_0^k \to \mathbb{C}$ when the corresponding functional $\lambda_s \in U_{\mathbb{C}}(\mathfrak{g})^*$ is integrable in the sense of (IF) (cf. [44, Sect. 12.2]). For $G = \mathbb{R}^k$ this specializes to the classical moment problem which, for $(s_\alpha)_{\alpha \in \mathbb{N}_0^k}$, asks for a bounded positive measure μ on $\mathbb{R}^k$ with

$$s_\alpha = \int_{\mathbb{R}^k} x^\alpha \mathsf{d}\mu(x) \quad \text{for} \quad \alpha \in \mathbb{N}_0^k.$$

Here we use Bochner's Theorem which says that the Fourier transform $\mu \mapsto \widehat{\mu}$ defines a bijection between bounded positive Borel measures on $\mathbb{R}^k$ and continuous positive definite functions on $\mathbb{R}^k$ (see [2] for more on the classical moment problem). For results concerning moment problems for

nuclear spaces, we refer to [46, Thm. 12.5.2] and the original papers [7], [21].

An obvious necessary condition for the solvability of (IF) is that λ must be a positive functional on the $*$-algebra $U_{\mathbb{C}}(\mathfrak{g})$ in the sense that $\lambda(D^*D) \geqslant 0$ for $D \in U_{\mathbb{C}}(\mathfrak{g})$, but in general this condition is not sufficient for integrability. For $G = \mathbb{R}^k$, a necessary and sufficient condition for integrability is non-negativity of λ on positive polynomials in $U(\mathfrak{g}) \cong \mathbb{R}[x_1, \ldots, x_k]$, which is stronger than positivity because, for $k \geqslant 2$, not every non-negative polynomial is a sum of squares ([46, Ex. 2.6.11], [4, Sects. 6.1/2]). In the light of these results, it is quite remarkable that, for $G = \mathbb{R}^k$, positivity of λ plus estimates on the s_α which ensure convergence of the series $\sum_\alpha \frac{1}{\alpha!} x^\alpha s_\alpha$ in a zero-neighborhood of $\mathbb{R}^k$ imply unique solvability of (IF) ([44, Lemma 12.2.1], [47, Thm. II.12.7]). A non-commutative variant of this result for Heisenberg groups can be found in [44, Thm. 12.2.3].

For the solvability of (IF) for a connected Lie group G, Schmüdgen shows in [45, Thm. 5.1] that $\lambda \in U_{\mathbb{C}}(\mathbf{L}(G))^*$ is integrable if and only if it is strongly positive in the sense that $\lambda(D) \geqslant 0$ whenever $\mathrm{d}\pi(D) \geqslant 0$ holds for all unitary representations $(\pi, \mathcal{H})$ of G. He also shows in [45, Thm. 4.1] that for $G \not\cong \mathbb{R}$ there exist positive functionals which are not strongly positive. With respect to (IR) for finite dimensional groups, one has a necessary and sufficient criterion in terms of so-called complete positivity ([45], [46, Thm. 11.4.2], [40, Thm. 4.5]). Here we are only concerned with the unique solvability of (IF) for positive functionals which are "analytic" in the sense that they satisfy certain estimates.

The plan of this paper is as follows. After recalling the basic concepts concerning infinite dimensional Lie groups and analytic functions in Section 2, we introduce in Section 3 the concept of an analytic linear map $\beta \colon U_{\mathbb{C}}(\mathfrak{g}) \to E$ to a Banach space E. These are linear maps for which the series $\sum_{n=0}^\infty \frac{\beta(x^n)}{n!}$ converges on a 0-neighborhood of $\mathfrak{g}$. The main result of Section 3 is Theorem 3.6 which asserts that, for any continuous seminorm p on $\mathfrak{g}$ which is submultiplicative in the sense that $p([x,y]) \leqslant p(x)p(y)$ for $x, y \in \mathfrak{g}$, the space of those analytic linear maps for which $\sum_n \frac{\beta(x^n)}{n!}$ converges uniformly on the r-ball with respect to p is invariant under the left and right regular representation of $U(\mathfrak{g})$ on $\mathrm{Hom}(U(\mathfrak{g}), E)$.

In Section 4 we turn to representations $\rho \colon \mathfrak{g} \to \mathrm{End}(V)$ of a Lie algebra $\mathfrak{g}$ on a normed space V. We call such a representation *strongly continuous* if the orbit maps $\rho^v \colon \mathfrak{g} \to V, x \mapsto \rho(x)v$ are continuous and say that $v \in V$ is *analytic* if $\sum_n \frac{\|\rho(x)^n v\|}{n!}$ converges on some 0-neighborhood of $\mathfrak{g}$.

If $\mathfrak{g}$ is a Baire space, v is analytic if and only if it is an analytic vector for all operators $\rho(x)$ separately (Proposition 4.10). In Section 5 we eventually turn to unitary representations of an analytic Lie group G with Lie algebra $\mathfrak{g}$ by showing that $v \in \mathcal{H}$ is analytic if and only if the matrix coefficient $\pi^{v,v}(g) = \langle \pi(g)v, v \rangle$ is analytic in some $\mathbf{1}$-neighborhood. This reduces the analyticity condition for v to a local requirement on a scalar-valued function. In Section 6 we proceed to the infinitesimal picture with the observation that for a $*$-representation of $\mathfrak{g}$ on a pre-Hilbert space $\mathcal{D}$, an element $v \in \mathcal{D}$ is an analytic vector if and only if the linear functional $\rho^{v,v}(D) := \langle \rho(D)v, v \rangle$ on $U(\mathfrak{g})$ is analytic. The main results of Section 6 are integrability results for $*$-representations on pre-Hilbert spaces. Theorem 6.8 asserts that for a 1-connected Banach–Lie group G a $*$-representation $(\rho, \mathcal{D})$ of $\mathfrak{g}$ is integrable if $\mathcal{D}$ consists of analytic vectors. Refining this result for the larger class of BCH–Lie groups, we derive in particular, that every positive analytic functional λ on the $*$-algebra $U_{\mathbb{C}}(\mathfrak{g})$ is integrable in the sense of (IF). We conclude Section 6 with a variant of the integrability theorem on Lie algebra representations that does not require an a priori given Lie group G. The interest in this refinement lies in the fact that not every Banach–Lie algebra is the Lie algebra of some Banach–Lie group ([12]). The class of BCH–Lie groups is substantially larger than the class of Banach–Lie groups. It contains in particular all (projective limits of) nilpotent Lie groups and groups of smooth maps on compact manifolds with values in Banach–Lie groups (see [17] for more details). In particular, all Heisenberg groups are BCH.

From the integrability of positive analytic functionals on $U_{\mathbb{C}}(\mathfrak{g})$, we further derive in Section 7 that any germ of an analytic positive definite function φ on a 1-connected BCH–Lie group extends uniquely to a global analytic positive definite function (Theorem 7.3).

An interesting issue that deserves to be pursued in the future is that the concept of an analytic vector makes sense for any representation (ρ, V) of a locally convex Lie algebra $\mathfrak{g}$ on a normed space V, even when there is no analytic Lie group G with Lie algebra $\mathfrak{g}$. This may open a door to applications for large classes of non-analytic Lie groups such as diffeomorphism groups of compact smooth manifolds.

In his recent paper [29], S. Merigon generalizes our Integrability Theorem 5.2 for simply connected Banach–Lie groups in the sense that he only requires the existence of a direct sum decomposition $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_n$ into closed subspaces, such that all vectors are analytic for every $x \in \mathfrak{g}_j$, $j = 1, \ldots, n$. We expect the results in this paper and [29] to be the key tools

to generalize the theory of analytic extension of unitary group representations ([27], [23, 24], [37, 38], [33]) to the context of Banach–Lie groups.

2. Locally convex Lie groups

In this section we briefly recall the basic concepts related to infinite dimensional Lie groups and calculus on locally convex spaces.

DEFINITION 2.1. — (a) Let E and F be locally convex spaces, $U \subseteq E$ open and $f \colon U \to F$ a map. Then the derivative of f at x in the direction h is defined as

$$\mathrm{d}f(x)(h) := (\partial_h f)(x) := \frac{d}{dt}\Big|_{t=0} f(x + th) = \lim_{t \to 0} \frac{1}{t}(f(x + th) - f(x))$$

whenever it exists. The function f is called differentiable at x if $\mathrm{d}f(x)(h)$ exists for all $h \in E$. It is called continuously differentiable, if it is differentiable at all points of U and

$$\mathrm{d}f \colon U \times E \to F, \quad (x, h) \mapsto \mathrm{d}f(x)(h)$$

is a continuous map. Note that this implies that the maps $\mathrm{d}f(x)$ are linear (cf. [17, Lemma 2.2.14]). The map f is called a C^k-map, $k \in \mathbb{N} \cup \{\infty\}$, if it is continuous, the iterated directional derivatives

$$\mathrm{d}^j f(x)(h_1, \dots, h_j) := (\partial_{h_j} \cdots \partial_{h_1} f)(x)$$

exist for all integers $j \leqslant k$, $x \in U$ and $h_1, \dots, h_j \in E$, and all maps $\mathrm{d}^j f \colon U \times E^j \to F$ are continuous. As usual, C^∞-maps are called smooth.

(b) If E and F are complex locally convex spaces, then f is called complex analytic if it is continuous and for each $x \in U$ there exists a 0-neighborhood V with $x + V \subseteq U$ and continuous homogeneous polynomials $\beta_k \colon E \to F$ of degree k such that for each $h \in V$ we have

$$f(x + h) = \sum_{k=0}^{\infty} \beta_k(h),$$

as a pointwise limit ([5]). The map f is called holomorphic if it is C^1 and for each $x \in U$ the map $\mathrm{d}f(x) \colon E \to F$ is complex linear (cf. [30, p. 1027]). If F is sequentially complete, then f is holomorphic if and only if it is complex analytic ([5, Ths. 3.1, 6.4]).

(c) If E and F are real locally convex spaces, then we call a map $f \colon U \to F$, $U \subseteq E$ open, real analytic or a C^ω-map, if for each point $x \in U$ there exists an open neighborhood $V \subseteq E_{\mathbb{C}}$ and a holomorphic map $f_{\mathbb{C}} \colon V \to F_{\mathbb{C}}$ with $f_{\mathbb{C}}|_{U \cap V} = f|_{U \cap V}$ (cf. [30]). The advantage of this definition, which

differs from the one in [5], *is that it also works nicely for non-complete spaces. Any analytic map is smooth, and the corresponding chain rule holds without any condition on the underlying spaces, which is the key to the definition of analytic manifolds (see* [16] *for details).*

Once the concept of a smooth function between open subsets of locally convex spaces is established (cf. [34], [30], [17]), it is clear how to define a locally convex smooth manifold. A *(locally convex) Lie group G* is a group equipped with a smooth manifold structure modeled on a locally convex space for which the group multiplication and the inversion are smooth maps. We write $\mathbf{1} \in G$ for the identity element and $\lambda_g(x) = gx$, resp., $\rho_g(x) = xg$ for the left, resp., right multiplication on G. Then each $x \in T_{\mathbf{1}}(G)$ corresponds to a unique left invariant vector field x_l with $x_l(g) := T_{\mathbf{1}}(\lambda_g)x, g \in G$. The space of left invariant vector fields is closed under the Lie bracket of vector fields, hence inherits a Lie algebra structure. In this sense we obtain on $\mathfrak{g} := T_{\mathbf{1}}(G)$ a continuous Lie bracket which is uniquely determined by $[x,y]_l = [x_l, y_l]$ for $x, y \in \mathfrak{g}$. We shall also use the functorial notation $\mathbf{L}(G) := (\mathfrak{g}, [\cdot, \cdot])$ for the Lie algebra of G and, accordingly, $\mathbf{L}(\varphi) = T_{\mathbf{1}}(\varphi) \colon \mathbf{L}(G_1) \to \mathbf{L}(G_2)$ for the Lie algebra morphism associated to a morphism $\varphi \colon G_1 \to G_2$ of Lie groups. Then $\mathbf{L}$ defines a functor from the category of locally convex Lie groups to the category of *locally convex Lie algebras*, i.e., locally convex spaces, endowed with continuous Lie brackets. The adjoint action of G on $\mathbf{L}(G)$ is defined by $\mathrm{Ad}(g) := \mathbf{L}(c_g)$, where $c_g(x) = gxg^{-1}$ is the conjugation map. The adjoint action is smooth and each $\mathrm{Ad}(g)$ is a topological isomorphism of $\mathfrak{g}$. If $\mathfrak{g}$ is a Fréchet, resp., a Banach space, then G is called a *Fréchet-*, resp., a *Banach–Lie group*.

A smooth map $\exp_G \colon \mathfrak{g} \to G$ is called an *exponential function* if each curve $\gamma_x(t) := \exp_G(tx)$ is a one-parameter group with $\gamma_x'(0) = x$. Not every infinite dimensional Lie group has an exponential function ([34, Ex. II.5.5]), but exponential functions are unique whenever they exist. A Lie group G is said to be *locally exponential* if it has an exponential function for which there is an open 0-neighborhood U in $\mathbf{L}(G)$ mapped diffeomorphically by $\exp_G$ onto an open subset of G. If, in addition, G is analytic and the exponential function is an analytic local diffeomorphism in 0, then G is called a *BCH–Lie group*. Then the Lie algebra $\mathbf{L}(G)$ is a *BCH–Lie algebra* (for Baker–Campbell–Hausdorff), i.e., there exists an open 0-neighborhood $U \subseteq \mathfrak{g}$ such that for $x, y \in U$ the BCH series

$$x * y = x + y + \frac{1}{2}[x,y] + \cdots$$

converges and defines an analytic function $U \times U \to \mathfrak{g}, (x, y) \mapsto x * y$. The class of BCH–Lie groups contains in particular all Banach–Lie groups ([34, Prop. IV.1.2]).

If $\pi \colon G \to \mathrm{GL}(V)$ is a representation of G on a locally convex space V, the exponential function permits us to associate to each element x of the Lie algebra a one-parameter group $\pi_x(t) := \pi(\exp_G tx)$. We therefore **assume** in the following that G has an exponential function.

3. Analytic functionals

In the following $\mathfrak{g}$ denotes a real locally convex Lie algebra and $U(\mathfrak{g})$ its enveloping algebra. The continuity of the Lie bracket on $\mathfrak{g}$ means that, for every continuous seminorm p on $\mathfrak{g}$, there exists a continuous seminorm $\tilde{p}$ on $\mathfrak{g}$ satisfying

$$(3.1) \qquad p([x, y]) \leqslant \tilde{p}(x)\tilde{p}(y) \quad \text{for} \quad x, y \in \mathfrak{g}.$$

If this relation holds with $\tilde{p} = p$, then we call p is *submultiplicative*. If $\mathfrak{g}$ is a Banach–Lie algebra, then we assume that the norm on $\mathfrak{g}$ is submultiplicative

$$(3.2) \qquad \|[x, y]\| \leqslant \|x\| \cdot \|y\| \quad \text{for} \quad x, y \in \mathfrak{g},$$

which can always be achieved by replacing it by a suitable multiple.

In this section we introduce for each continuous submultiplicative seminorm p on $\mathfrak{g}$ the concept of a p-analytic linear map β on the enveloping algebra $U(\mathfrak{g})$ with values in a Banach space E and define its radius of convergence $R_{\beta,p}$. The main result of this section is Theorem 3.6, asserting that the subspace of elements β with $R_{\beta,p} \geqslant r$ is invariant under the left and right regular action of $U(\mathfrak{g})$. This applies in particular to the norm $p(x) = \|x\|$ of a Banach–Lie algebra.

DEFINITION 3.1. — *Let V be a locally convex space and W be a normed space. We write $\mathrm{Mult}^n(V, W)$ for the space of continuous W-valued n-linear maps on V^n and $\mathrm{Sym}^n(V, W)$ for the subspace of symmetric n-linear maps. We also write $\mathcal{P}(V)$ for the set of continuous seminorms on V.*

(a) For an n-linear map $\beta \colon V^n \to W$ and $p \in \mathcal{P}(V)$, we define

$$\|\beta\|_p := \sup\{\|\beta(v_1, \ldots, v_n)\| \colon v_1, \ldots, v_n \in V, p(v_i) \leqslant 1\} \in [0, \infty].$$

Then β is continuous with respect to the topology on V defined by p if and only if $\|\beta\|_p < \infty$, and we have

$$\|\beta(v_1, \ldots, v_n)\| \leqslant \|\beta\|_p p(v_1) \cdots p(v_n) \quad \text{for} \quad v_i \in V.$$

If V and W are Banach spaces and $p(v) = \|v\|$, then we thus obtain on $\mathrm{Mult}^n(V, W)$ the structure of a Banach space. In this case we simply write $\|\beta\|$ for the norm of β with respect to p.

(b) For $\beta \in \mathrm{Mult}^n(V, W)$, we write β^s for the symmetrization of β:

$$\beta^s(x_1, \ldots, x_n) := \frac{1}{n!} \sum_{\sigma \in S_n} \beta(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$$

and observe that $\|\beta^s\|_p \leqslant \|\beta\|_p$ for every $p \in \mathcal{P}(V)$.

(c) A map $f \colon V \to W$ is called homogeneous of degree n if there exists a $\tilde{f} \in \mathrm{Sym}^n(V, W)$ with

$$f(x) = \tilde{f}(x, x, \ldots, x) \quad \text{for} \quad x \in V.$$

Any such f is smooth and $\tilde{f}$ can be recovered from f by

$$\tilde{f}(v_1, \ldots, v_n) = \frac{1}{n!}(\partial_{v_1} \cdots \partial_{v_n} f)(0).$$

For $p \in \mathcal{P}(V)$, we define

$$\|f\|_p := \sup\{\|f(x)\| \colon p(x) \leqslant 1\}.$$

Then we clearly have $\|f\|_p \leqslant \|\tilde{f}\|_p$, and, in view of [6, Prop. 1.1], we also have

(3.3) $$\|\tilde{f}\|_p \leqslant \frac{(2n)^n}{n!}\|f\|_p.$$

DEFINITION 3.2. — *Let E be a Banach space.*

(a) We call a linear map $\beta \colon U(\mathfrak{g}) \to E$ continuous if, for each $n \in \mathbb{N}_0$, the n-linear map

$$\beta_n \colon \mathfrak{g}^n \to E, \quad (x_1, \ldots, x_n) \mapsto \beta(x_1 \cdots x_n)$$

is continuous and write $\mathrm{Hom}(U(\mathfrak{g}), E)_c$ for the set of all continuous linear maps. If $\mathfrak{g}$ is Banach–Lie algebra, we have on this space a family of seminorms

$$p_n \colon \mathrm{Hom}(U(\mathfrak{g}), E)_c \to \mathbb{R}, \quad \beta \mapsto \|\beta_n\|.$$

(b) We call $\beta \in \mathrm{Hom}(U(\mathfrak{g}), E)_c$ analytic if the series $\sum_{n=0}^{\infty} \frac{\beta(x^n)}{n!}$ converges on a 0-neighborhood of $\mathfrak{g}$. The set of analytic elements is denoted $\mathrm{Hom}(U(\mathfrak{g}), E)_\omega$, and for $E = \mathbb{C}$ the elements of $\mathrm{Hom}(U(\mathfrak{g}), \mathbb{C})_\omega$ are called analytic functionals.

LEMMA 3.3. — ([32, Lemma 2.3]) *If $\mathfrak{g}$ is a Banach–Lie algebra, then the sequence $(p_n)_{n \in \mathbb{N}_0}$ of seminorms turns the space $\mathrm{Hom}(U(\mathfrak{g}), E)_c$ into a Fréchet space for which all evaluation maps $\mathrm{ev}_D \colon \mathrm{Hom}(U(\mathfrak{g}), E)_c \to$*

$E, \beta \mapsto \beta(D)$ are continuous. The left regular representation of $U(\mathfrak{g})$ on this space defined by

$$\mathfrak{g} \times \mathrm{Hom}(U(\mathfrak{g}), E)_c \to \mathrm{Hom}(U(\mathfrak{g}), E)_c, \quad (x, \beta) \mapsto -\beta \circ \lambda_x, \quad \lambda_x(D) = xD,$$

and the right regular representation

$$\mathfrak{g} \times \mathrm{Hom}(U(\mathfrak{g}), E)_c \to \mathrm{Hom}(U(\mathfrak{g}), E)_c, \quad (x, \beta) \mapsto \beta \circ \rho_x, \quad \rho_x(D) = Dx,$$

are continuous bilinear maps.

PROPOSITION 3.4. — If $\mathfrak{g}$ is a Baire space, f.i., if it is Fréchet, then an element $\beta \in \mathrm{Hom}(U(\mathfrak{g}), E)_c$ is analytic if and only if there exists a $p \in \mathcal{P}(\mathfrak{g})$ with

$$(3.4) \qquad \sum_{n=0}^{\infty} \frac{\|\beta_n^s\|_p}{n!} < \infty.$$

If $\mathfrak{g}$ is Banach, this is equivalent to the existence of a $t > 0$ with

$$(3.5) \qquad \sum_{n=0}^{\infty} \frac{\|\beta_n^s\| t^n}{n!} < \infty.$$

Proof. — From (3.4) we obtain for $p(x) < 1$ the estimate

$$\sum_{n=0}^{\infty} \frac{\|\beta(x^n)\|}{n!} = \sum_{n=0}^{\infty} \frac{\|\beta_n^s(x, \ldots, x)\|}{n!} \leqslant \sum_{n=0}^{\infty} \frac{\|\beta_n^s\|_p \, p(x)^n}{n!} \leqslant \sum_{n=0}^{\infty} \frac{\|\beta_n^s\|_p}{n!} < \infty.$$

Therefore (3.4) implies the uniform convergence of the series $\sum_n \frac{\beta(x^n)}{n!}$ for $p(x) < 1$, hence in particular the analyticity of β. If $\mathfrak{g}$ is Banach, we obtain from (3.5) the uniform convergence for $\|x\| < t$.

Suppose, conversely, that β is analytic. Then the functions $f_n(x) := \frac{\beta(x^n)}{n!}$ on $\mathfrak{g}$ are continuous and homogeneous of degree n. In view of [5, Prop. 5.2], the convergence of the sequence $\sum_n f_n(x)$ on some 0-neighborhood of $\mathfrak{g}$ implies the existence of some $q \in \mathcal{P}(\mathfrak{g})$ for which the series converges uniformly for $q(x) \leqslant 1$. In particular, there exists a $C > 0$ with $\|\beta(x^n)\| \leqslant Cn!$ for $q(x) \leqslant 1$. We thus obtain with (3.3) the relation

$$\|\beta_n^s\|_q \leqslant \frac{(2n)^n}{n!} \|n! f_n\|_q \leqslant n^n 2^n \|f_n\|_q \leqslant n^n 2^n C.$$

Stirling's Formula

$$(3.6) \qquad n! \approx \sqrt{2\pi n} \left(\frac{n}{e}\right)^n,$$

leads to $\frac{n^n}{n!} \approx \frac{e^n}{\sqrt{2\pi n}}$, so that we find with some $C' > 0$

$$\sum_{n=0}^{\infty} \frac{\|\beta_n^s\|_q t^n}{n!} \leqslant C' \sum_{n=0}^{\infty} \frac{e^n}{\sqrt{n}} (2t)^n = C' \sum_{n=0}^{\infty} \frac{(2et)^n}{\sqrt{n}}.$$

This series converges for $t < \frac{1}{2e}$, so that the assertion follows for any such t with $p := t^{-1}q$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

DEFINITION 3.5. — *If $\beta \in \mathrm{Hom}(U(\mathfrak{g}), E)_c$ is analytic and $\sum_{n=0}^{\infty} \frac{\|\beta_n^s\|_p t^n}{n!} < \infty$ for some $t > 0$, then*

$$\sum_{n=0}^{\infty} \frac{\|\beta(x^n)\|}{n!} \leqslant \sum_{n=0}^{\infty} \frac{\|\beta_n^s\|_p p(x)^n}{n!} < \infty$$

holds for $p(x) < t$ and

$$R_{\beta,p} := \sup\left\{t > 0 \colon \sum_{n=0}^{\infty} \frac{\|\beta_n^s\|_p t^n}{n!} < \infty\right\} \in]0, \infty]$$

is called the p-radius of convergence of β. If $R_{\beta,p} < \infty$, we say that β is p-analytic, or, analytic with respect to p. According to Hadamard's Formula, we have

$$R_{\beta,p}^{-1} = \limsup \sqrt[n]{\frac{\|\beta_n^s\|_p}{n!}}.$$

Note that, for each $r > 0$, the set $\mathrm{Hom}(U(\mathfrak{g}), E)_{\omega,p,r}$ of analytic maps β with $R_{\beta,p} \geqslant r$ is a linear subspace. We write $\mathrm{Hom}(U(\mathfrak{g}), E)_{\omega,p}$ for the union of all these spaces, which also is a linear subspace.

We now come to the main result of this section.

THEOREM 3.6. — *If $p \in \mathcal{P}(\mathfrak{g})$ is submultiplicative and $r > 0$, then $\mathrm{Hom}(U(\mathfrak{g}), E)_{\omega,p,r}$ is invariant under the left and right regular action of $\mathfrak{g}$. In particular, the space $\mathrm{Hom}(U(\mathfrak{g}), E)_{\omega,p}$ of analytic elements is invariant.*

Proof. — For $\beta \in \mathrm{Hom}(U(\mathfrak{g}), E)_{\omega,p,r}$, $y \in \mathfrak{g}$, $n \in \mathbb{N}_0$, and $1 \leqslant k \leqslant n+1$, we define

$$(i_y^k \beta)_n \in \mathrm{Mult}^n(\mathfrak{g}, E), \quad (i_y^k \beta)_n(x_1, \ldots, x_n) := \beta(x_1 \cdots x_{k-1} y x_k \cdots x_n).$$

For $n \in \mathbb{N}_0$ and $y \in \mathfrak{g}$ we put

$$c_n := \inf\{C > 0 \colon (\forall y \in \mathfrak{g})(\forall k \leqslant n+1)\ \|(i_y^k \beta)_n^s\|_p \leqslant C p(y)\}.$$

Now let $y \in \mathfrak{g}$. Our first goal is an estimate for the numbers c_n. For $n = 0$ we have $k = 1$ and $i_y^1 \beta = \beta(y)$, so that

$$c_0 = \|\beta_1\|_p = \|\beta_1^s\|_p.$$

Our strategy is to compare the two symmetric n-linear maps $(i_y^k \beta)_n^s$ and $i_y(\beta_{n+1}^s) = \beta_{n+1}^s(y, \cdots)$:

$$(i_y^k \beta)_n^s(x_1, \ldots, x_n) - \beta_{n+1}^s(y, x_1, \ldots, x_n)$$
$$= \frac{1}{n!} \Big(\sum_{\sigma \in S_n} \beta(x_{\sigma(1)} \cdots x_{\sigma(k-1)} y x_{\sigma(k)} \cdots x_{\sigma(n)})$$
$$- \frac{1}{n+1} \sum_{j=1}^{n+1} \sum_{\sigma \in S_n} \beta(x_{\sigma(1)} \cdots x_{\sigma(j-1)} y x_{\sigma(j)} \cdots x_{\sigma(n)}) \Big)$$
$$= \frac{1}{(n+1)!} \sum_{j=1}^{n+1} \sum_{\sigma \in S_n}$$
$$\Big(\beta(x_{\sigma(1)} \cdots x_{\sigma(k-1)} y x_{\sigma(k)} \cdots x_{\sigma(n)}) - \beta(x_{\sigma(1)} \cdots x_{\sigma(j-1)} y x_{\sigma(j)} \cdots x_{\sigma(n)}) \Big).$$

With the relation $[y, x_1 \cdots x_a] = \sum_{j=1}^a x_1 \cdots x_{j-1} [y, x_j] x_{j+1} \cdots x_a$ we obtain for $k < j$

$$\beta(x_{\sigma(1)} \cdots x_{\sigma(k-1)} y x_{\sigma(k)} \cdots x_{\sigma(n)}) - \beta(x_{\sigma(1)} \cdots x_{\sigma(j-1)} y x_{\sigma(j)} \cdots x_{\sigma(n)})$$
$$= \beta(x_{\sigma(1)} \cdots x_{\sigma(k-1)} [y, x_{\sigma(k)} \cdots x_{\sigma(j-1)}] x_{\sigma(j)} \cdots x_{\sigma(n)})$$
$$= \sum_{\ell=k}^{j-1} \beta(x_{\sigma(1)} \cdots x_{\sigma(\ell-1)} [y, x_{\sigma(\ell)}] x_{\sigma(\ell+1)} \cdots x_{\sigma(n)}).$$

Next we observe that

$$\sum_{\sigma \in S_n} \beta(x_{\sigma(1)} \cdots x_{\sigma(\ell-1)} [y, x_{\sigma(\ell)}] x_{\sigma(\ell+1)} \cdots x_{\sigma(n)})$$
$$= (n-1)! \sum_{m=1}^n (i_{[y,x_m]}^\ell \beta)_{n-1}^s(x_1, \ldots, \widehat{x_m}, \ldots, x_n),$$

where $\widehat{x_m}$ denotes omission of x_m. The norm of this sum can be estimated by

$$(n-1)! \sum_{m=1}^n c_{n-1} p([y, x_m]) p(x_1) \cdots p(\widehat{x_m}) \cdots p(x_n)$$
$$\leqslant (n-1)! \sum_{m=1}^n c_{n-1} p(y) p(x_1) \cdots p(x_n) = n! c_{n-1} p(y) p(x_1) \cdots p(x_n).$$

For $k > j$ we likewise obtain

$$\beta(x_{\sigma(1)} \cdots x_{\sigma(k-1)} y x_{\sigma(k)} \cdots x_{\sigma(n)}) - \beta(x_{\sigma(1)} \cdots x_{\sigma(j-1)} y x_{\sigma(j)} \cdots x_{\sigma(n)})$$
$$= \beta(x_{\sigma(1)} \cdots x_{\sigma(j-1)} [x_{\sigma(j)} \cdots x_{\sigma(k-1)}, y] x_{\sigma(k)} \cdots x_{\sigma(n)})$$
$$= \sum_{\ell=j}^{k-1} \beta(x_{\sigma(1)} \cdots x_{\sigma(\ell-1)} [x_{\sigma(\ell)}, y] x_{\sigma(\ell+1)} \cdots x_{\sigma(n)})$$

which can be estimated in the same way. This leads to the estimate

$$\|(i_y^k \beta)_n^s(x_1, \ldots, x_n) - \beta_{n+1}^s(y, x_1, \ldots, x_n)\|$$
$$\leqslant \frac{1}{(n+1)!} \Big(\sum_{j>k} \sum_{\ell=k}^{j-1} n! c_{n-1} p(y) p(x_1) \cdots p(x_n)$$
$$+ \sum_{j<k} \sum_{\ell=j}^{k-1} n! c_{n-1} p(y) p(x_1) \cdots p(x_n) \Big)$$
$$\leqslant \frac{1}{n+1} \sum_{j \neq k} |k-j| c_{n-1} p(y) p(x_1) \cdots p(x_n) \leqslant \frac{n^2}{n+1} c_{n-1} p(y) p(x_1) \cdots p(x_n)$$
$$\leqslant n c_{n-1} p(y) p(x_1) \cdots p(x_n).$$

We conclude that

$$\|(i_y^k \beta)_n^s(x_1, \ldots, x_n)\|$$
$$\leqslant \|\beta_{n+1}^s\|_p p(y) p(x_1) \cdots p(x_n) + n c_{n-1} p(y) p(x_1) \cdots p(x_n),$$

i.e.,

$$(3.7) \qquad\qquad c_n \leqslant \|\beta_{n+1}^s\|_p + n c_{n-1}.$$

Iterating this estimate leads to

$$c_n \leqslant \|\beta_{n+1}^s\|_p + n\|\beta_n^s\|_p + n(n-1)\|\beta_{n-1}^s\|_p$$
$$+ \cdots + n(n-1) \cdots 2 \cdot \|\beta_2^s\|_p + n!\|\beta_1^s\|_p.$$

For $0 < t < \min(1, R_{\beta, p})$, there exists a $C > 0$ with

$$\|\beta_n^s\|_p \leqslant C n! t^{-n} \quad \text{for} \quad n \in \mathbb{N}.$$

We thus obtain

$$c_n \leqslant C\big((n+1)! t^{-n-1} + n! n t^{-n} + \cdots + n \cdots 2 \cdot 2! t^{-2} + n! t^{-1}\big)$$
$$\leqslant C(n+1) \cdot (n+1)! t^{-n-1} \leqslant C(n+2)! t^{-n-1}.$$

This implies that

$$\sum_n \frac{c_n s^n}{n!} \leqslant C \sum_n (n+2)(n+1)t^{-1}(s/t)^n,$$

which converges for $s < t$.

Finally we observe that $(\beta \circ \rho_y)_n^s = (i_y^{n+1}\beta)_n^s$, so that

$$\|(\beta \circ \rho_y)_n^s\|_p = \|(i_y^{n+1}\beta)_n^s\|_p \leqslant c_n p(y)$$

shows that $\beta \circ \rho_y$ is analytic with $R_{\beta \circ \rho_y, p} \geqslant R_{\beta, p}$. Similarly $(\beta \circ \lambda_y)_n^s = (i_y^1 \beta)_n^s$ implies that $\beta \circ \lambda_y$ is analytic with $R_{\beta \circ \lambda_y, p} \geqslant R_{\beta, p}$. □

Remark 3.7. — It is an interesting question whether (for Banach–Lie algebras) the condition $\sum_{n=0}^{\infty} \frac{\|\beta_n^s\| t^n}{n!} < \infty$ characterizing analytic elements in $\mathrm{Hom}(U(\mathfrak{g}), E)_c$ also implies the stronger condition $\sum_{n=0}^{\infty} \frac{\|\beta_n\| t^n}{n!} < \infty$ on the non-symmetrized maps β_n. A natural way to verify this would be to compare the norms of β_n and β_n^s. For $x_1, \dots, x_n \in \mathfrak{g}$ we have

$$(\beta_n - \beta_n^s)(x_1, \dots, x_n) = \frac{1}{n!} \sum_{\sigma \in S_n} \beta(x_1 \cdots x_n) - \beta(x_{\sigma(1)} \cdots x_{\sigma(n)}).$$

Writing a permutation in terms of transpositions s_j exchanging j and $j+1$ and using that the corresponding word length satisfies $\ell(\sigma) \leqslant \frac{n(n-1)}{2}$, we obtain as in the proof of Theorem 3.6 an estimate of the form

$$\|\beta_n - \beta_n^s\| \leqslant \frac{n(n-1)}{2}\|\beta_{n-1}\|,$$

which leads to the recursive estimate

$$\|\beta_n\| \leqslant \|\beta_n^s\| + \frac{n(n-1)}{2}\|\beta_{n-1}\|.$$

Iteration now yields

$$\|\beta_n\|$$
$$\leqslant \|\beta_n^s\| + \frac{n(n-1)}{2}\|\beta_{n-1}^s\| + \frac{n(n-1)^2(n-2)}{2^2}\|\beta_{n-2}^s\|$$
$$+ \cdots + \frac{n(n-1)^2 \cdots 2^2 \cdot 1}{2^{n-1}}\|\beta_1\|$$
$$\leqslant \sum_{k=0}^{n-1} \frac{(n!)^2}{((n-k)!)^2 2^k}\|\beta_{n-k}^s\|.$$

From the estimate $\|\beta_n^s\| \leqslant Cn!t^{-n}$, we thus obtain

$$\sum_n \frac{\|\beta_n\|t^n}{n!} \leqslant C \sum_n \frac{t^n}{n!} \sum_{k=0}^{n-1} \frac{(n!)^2}{((n-k)!)^2 2^k} t^{k-n}(n-k)!$$

$$\leqslant C \sum_n \sum_{k=0}^{n-1} \frac{n!}{(n-k)!}(t/2)^k = C \sum_k (t/2)^k \sum_{n>k} \frac{n!}{(n-k)!} = \infty.$$

Apparently this estimate does not lead to the desired convergence, and we do not know if the series $\sum_n \frac{\|\beta_n\|t^n}{n!}$ converges for some $t > 0$.

For finite dimensional Lie algebras, analytic vectors are usually defined by a condition equivalent to the convergence of the series $\sum_n \frac{\|\beta_n\|t^n}{n!}$ for some $t > 0$ (cf. [46, Sect. 10.4]). At first sight this is stronger than the present one (cf. [46, Lemma 10.49]), but according to [36, Lemma 7.1, Thm. 2] it is equivalent. We do not know if this is also true for Banach–Lie algebras.

Remark 3.8. — (a) If the topology on $\mathfrak{g}$ is defined by a set of submultiplicative seminorms, then Theorem 3.6 implies the invariance of the space $\operatorname{Hom}(U(\mathfrak{g}), E)_\omega$ of analytic elements under the left and right regular action.

(b) We do not know if this is true for general locally convex Lie algebras. Here the critical point is that, if c_n^p denotes the constant c_n, defined with respect to the seminorm p, and $p([x,y]) \leqslant q(x)q(y)$ for some $q \in \mathcal{P}(\mathfrak{g})$ with $q \geqslant p$, then the argument in the proof of Theorem 3.6 only yields an estimate of the form

$$c_n^q \leqslant \|\beta_{n+1}^s\|_p + nc_{n-1}^p.$$

We do not see how to use this estimate to conclude the invariance of $\operatorname{Hom}(U(\mathfrak{g}), E)_\omega$.

4. Analytic vectors and continuity of Lie algebra representations

In this section $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and all vector spaces are $\mathbb{K}$-vector spaces. Before we turn to the connection between analytic functionals and unitary representations of Lie groups, we have to clarify some continuity requirements for representations of topological Lie algebras on topological vector spaces.

Continuity of Lie algebra representations

DEFINITION 4.1. — *Let (ρ, V) be a representation of the topological Lie algebra $\mathfrak{g}$ on the topological vector space V.*

(a) We say that ρ is strongly continuous if for every $v \in V$ the map

$$\beta_v \colon \mathfrak{g} \to V, \quad x \mapsto \rho(x)v$$

is continuous. For a finite dimensional Lie algebra $\mathfrak{g}$, this condition is always satisfied.

(b) If $E \subseteq V'$ is a subspace separating the points of the completion $\widehat{V}$ of V, then ρ is called weakly continuous with respect to E if, for $\alpha \in E$ and $v \in V$, the linear functional $x \mapsto \alpha(\rho(x)v)$ is continuous. [2]

(c) If V is a complex pre-Hilbert space, then we call ρ

   (i): weakly continuous if, for any two vectors $v, w \in V$, the map $\mathfrak{g} \to \mathbb{C}, x \mapsto \langle \rho(x)v, w \rangle$ is continuous, i.e., if ρ is weakly continuous with respect to $E = \{\langle \cdot, v \rangle \colon v \in V\}$.

   (ii): a $*$-representation if

$$\langle \rho(x)v, w \rangle = -\langle v, \rho(x)w \rangle \quad \text{for} \quad x \in \mathfrak{g}, v, w \in V.$$

LEMMA 4.2. — *If $\mathfrak{g}$ is a Fréchet space and V is a metrizable locally convex space, then every weakly continuous representation is strongly continuous.*

Proof. — We claim that the graph of β_v, considered as a linear map into the Fréchet completion $\widehat{V}$ of V, is closed. Since every continuous linear functional on V extends uniquely to a continuous linear functional on the completion $\widehat{V}$, we identify the two spaces $\widehat{V}'$ and V'.

Suppose that $x_n \to x$ in $\mathfrak{g}$ and $\beta_v(x_n) \to w \in \widehat{V}$ and let $E \subseteq \widehat{V}'$ be a subspace separating the points of $\widehat{V}$ such that all linear functionals $\mathfrak{g} \to \mathbb{K}, x \mapsto \alpha(\rho(x)v), \alpha \in E, v \in V$, are continuous. Then $\alpha(\rho(x_n)v) \to \alpha(\rho(x)v)$ and $\alpha(\rho(x_n)v) \to \alpha(w)$ implies $w = \rho(x)v$. Therefore the graph of β_v is closed, so that the Closed Graph Theorem ([9, Ch. 1, §3, Cor. 5]) implies that β_v is continuous. $\qquad\square$

The following proposition is classical for $k = 2$, but here we need a more general version.

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[2] If V is a normed space which is not complete and $\widehat{v} \in \widehat{V} \setminus V$, then its annihilator $E := \widehat{v}^\perp \subseteq V' \cong \widehat{V}'$ is a hyperplane with $E^\perp \cap V = \mathbb{K}\widehat{v} \cap V = \{0\}$. Therefore E separates the points of V but not of $\widehat{V}$.

PROPOSITION 4.3. — *If $F_1, \ldots, F_k$ are Fréchet spaces, V a topological vector space, and $\beta\colon F_1 \times \cdots \times F_k \to V$ a k-linear map which is separately continuous, then β is continuous.*

Proof. — We argue by induction over k. The case $k = 1$ is trivial, so that we may assume that $k > 1$. Let $F := F_1 \times \cdots \times F_{k-1}$ and consider the map

$$\Omega\colon F \times F_k \to V, \quad (x, y) = ((x_1, \ldots, x_{k-1}), y) \mapsto \beta(x_1, \ldots, x_{k-1}, y).$$

For $y \in F_k$, our induction hypothesis implies that the $(k-1)$-linear map $\Omega(\cdot, y)\colon F \to V$ is continuous, so that the continuity of Ω follows from [35, Prop. 5.1]. $\qquad\square$

LEMMA 4.4. — *If the representation (ρ, V) is strongly continuous, then, for each $n \in \mathbb{N}$ and every $v \in V$, the n-linear map*

$$\beta_v^n\colon \mathfrak{g}^n \to V, \quad (x_1, \ldots, x_n) \mapsto \rho(x_1) \cdots \rho(x_n)v$$

is separately continuous. If, in addition, $\mathfrak{g}$ is Fréchet, then each β_v^n is continuous.

Proof. — We argue by induction. By definition, the assertion holds for $n = 1$. We assume that each map β_w^{n-1} is continuous in each argument. For $j < n$, the continuity of β_v^n in x_j then follows from the continuity of $\beta_{\rho(x_n)v}^{n-1}$ in each argument. Next we observe that

$$\beta_v^n(x_1, \ldots, x_n) = \beta_v^n(x_1, \ldots, x_{n-2}, x_n, x_{n-1}) + \rho(x_1) \cdots \rho(x_{n-2})\rho([x_{n-1}, x_n])v$$

$$= \beta_v^n(x_1, \ldots, x_{n-2}, x_n, x_{n-1}) + \beta_v^{n-1}(x_1, \ldots, x_{n-2}, [x_{n-1}, x_n]).$$

From the continuity of the Lie bracket on $\mathfrak{g}$ it now follows that both terms on the right are continuous in x_n. This proves the continuity of β_v^n in x_n.

Now $\beta_v^n\colon \mathfrak{g} \to V$ is a separately continuous multilinear map, hence continuous if $\mathfrak{g}$ is a Fréchet space (Proposition 4.3). $\qquad\square$

LEMMA 4.5. — *Suppose that V is a pre-Hilbert space. Then a $*$-representation (ρ, V) of the Fréchet–Lie algebra $\mathfrak{g}$ on V is strongly continuous if and only if for each $v \in V$ the quadratic map $q_v\colon \mathfrak{g} \to \mathbb{R}, x \mapsto \langle \rho(x)^2 v, v \rangle$ is continuous.*

Proof. — If each map q_v is continuous, then $\|\rho(x)v\|^2 = -q_v(x)$ implies that ρ is strongly continuous.

If, conversely, ρ is strongly continuous, then Lemma 4.4 implies that the bilinear maps $\beta_v^2\colon \mathfrak{g}^2 \to V, (x, y) \mapsto \rho(x)\rho(y)v$ are continuous, and this implies the continuity of the maps q_v. $\qquad\square$

PROPOSITION 4.6. — *If (π, V) is a continuous representation of the Lie group G on V, then the derived representation $\mathrm{d}\pi$ of $\mathfrak{g}$ on the space V^∞ of smooth vectors is strongly continuous with respect to the subspace topology induced from V.*

Proof. — If v is a smooth vector and $\pi^v \colon G \to V, g \mapsto \pi(g)v$, its smooth orbit map, then $x \mapsto \mathrm{d}\pi(x)v = T_{(\mathbf{1},0)}(\pi^v)(x, 0)$ is a continuous linear map. $\square$

In [35, Thm. 8.2] we show that the preceding observation can even be sharpened if G and V are Banach. If $\mathcal{D}_{\mathfrak{g}} \subseteq V$ denotes the common domain of the generators $\overline{\mathrm{d}\pi(x)}$ of the one-parameter groups $t \mapsto \pi(\exp tx)$, $x \in \mathfrak{g}$, then, for each $v \in \mathcal{D}_{\mathfrak{g}}$, the map $\beta_v \colon \mathfrak{g} \to V$ is continuous and linear. In this sense the continuity requirement is satisfied for any representation of a Banach–Lie algebra that comes from a continuous Lie group action on a Banach space.

Analytic vectors

We are now ready to turn to the central concept of this article.

DEFINITION 4.7. — *Let A be an operator with domain $\mathcal{D}(A)$ on the Banach space E. An element in the space $\mathcal{D}^\infty(A) := \bigcap_{n=1}^\infty \mathcal{D}(A^n)$ is called a smooth vector for A, and a smooth vector is said to be analytic if there exists a $t > 0$ with*

$$\sum_{n=0}^\infty \frac{\|A^n v\| t^n}{n!} < \infty.$$

We write $\mathcal{D}^\omega(A)$ for the space of analytic vectors for A.

An important application of the concept of an analytic vector is that it provides the following criterion for essential selfadjointness:

THEOREM 4.8. — *(Nelson's Theorem; [36]; [42, Thm. X.39]) A symmetric operator A on a Hilbert space $\mathcal{H}$ for which the space $\mathcal{D}^\omega(A)$ of analytic vectors is dense is essentially selfadjoint.*

DEFINITION 4.9. — *Let (ρ, V) be a strongly continuous representation of $\mathfrak{g}$ on the normed space V. Then an element $v \in V$ is called an analytic vector if $\sum_{n=0}^\infty \frac{\|\rho(x)^n v\|}{n!}$ converges for every x in a 0-neighborhood of $\mathfrak{g}$.*

PROPOSITION 4.10. — *(Automatic simultaneous analyticity) Let (ρ, V) be a strongly continuous representation of the Lie algebra $\mathfrak{g}$ on the normed*

space V. If $\mathfrak{g}$ is a Baire space (which is in particular the case if $\mathfrak{g}$ is Fréchet), then every $v \in V$ which is an analytic vector for all the operators $\rho(x)$, $x \in \mathfrak{g}$, is an analytic vector.

Proof. — In view of Lemma 4.4, the functions $f_n(x) := \frac{\rho(x)^n v}{n!}$ are continuous and homogeneous of degree n. The assumption that v is analytic for every operator $\rho(x)$ implies that the set S of all elements $x \in \mathfrak{g}$ for which the series $\sum_n f_n(x)$ converges in the completion $\widehat{V}$ of V is absorbing, i.e., $\bigcup_{n \in \mathbb{N}} n \cdot S = \mathfrak{g}$. Now [5, Prop. 5.2(1)] implies the uniform convergence of this series on a 0-neighborhood of $\mathfrak{g}$, i.e., v is an analytic vector. $\square$

5. Analytic vectors of unitary representations

The main result of this section asserts that a vector v in a Hilbert space $\mathcal{H}$ is analytic with respect to a continuous unitary representation $(\pi, \mathcal{H})$ of an analytic Lie group G if and only if the matrix coefficient $\pi^{v,v}(g) = \langle \pi(g)v, v \rangle$ is analytic in a **1**-neighborhood. This will be a direct consequence of the following theorem characterizing analyticity of $\mathcal{H}$-valued maps in terms of analyticity of the corresponding kernel. The case of this theorem, where M is one-dimensional, can already be found in [26, p. 81].

THEOREM 5.1. — *Let M be an analytic Fréchet manifold and $\mathcal{H}$ be a Hilbert space. Then a function $\gamma \colon M \to \mathcal{H}$ is analytic if and only if the kernel $K(x,y) := \langle \gamma(x), \gamma(y) \rangle$ is analytic on an open neighborhood of the diagonal in $M \times M$.*

Proof. — **Step 1:** We assume w.l.o.g. that $\mathcal{H}$ is a real Hilbert space. Since compositions of analytic maps are analytic ([16, Prop. 2.8]), the analyticity of γ implies the analyticity of the kernel K. Suppose, conversely, that K is analytic. As our assertion is local, we may assume that M is an open subset of a locally convex space V containing 0, that K is analytic on $M \times M$, and that we have to show the analyticity of γ in a 0-neighborhood. To this end, we may further assume that U is a convex symmetric 0-neighborhood for which K extends to a holomorphic function on $W := (U + iU) \times (U + iU) \subseteq V_{\mathbb{C}} \times V_{\mathbb{C}}$. Shrinking W if necessary, we may further assume that

$$C := \sup\{|K(z,w)| \colon (z,w) \in W\} < \infty.$$

[35, Thm 7.1] implies that γ is smooth.

Step 2: Next we consider the special case $V = \mathbb{R}$. Since K is analytic, it is in particular smooth, so that γ is a smooth curve in $\mathcal{H}$, and

$$\|\gamma^{(n)}(t)\|^2 = \frac{d^n}{dt^n} \frac{d^n}{ds^n} \langle \gamma(t), \gamma(s) \rangle|_{t=s} = (\partial_1^n \partial_2^n K)(t,t).$$

For any $t \in U$ and $r > 0$ for which

$$W_r(t) := \{z \in \mathbb{C}^2 \colon |z_1 - t|, |z_2 - t| \leqslant r\} \subseteq W$$

we have the Cauchy estimates

$$|(\partial_1^n \partial_2^m K)(t,t)| \leqslant C n! m! r^{-n-m}.$$

We conclude that

(5.1) $$\|\gamma^{(n)}(t)\| \leqslant \sqrt{C} n! r^{-n}.$$

Step 3: For a general V, we consider for $x \in U$ and $h \in V$ the curves

$$\gamma_{x,h}(t) := \gamma(x + th)$$

which are analytic in a neighborhood of $t = 0$ by Step 2. To prove the convergence of the Taylor series of γ in a 0-neighborhood, we observe that

$$\gamma_{x,h}^{(n)}(0) = (\mathsf{d}^n \gamma)(x)(h, \ldots, h),$$

so that (5.1) implies that

$$\|(\mathsf{d}^n \gamma)(x)(h, \ldots, h)\| = \|\gamma_{x,h}^{(n)}(0)\| \leqslant n! 2^{-n} \sqrt{C}$$

whenever $(x + z_1 h, x + z_2 h) \in W$ holds for $|z_i| \leqslant 2$. We conclude that the Taylor series

$$\sum_{n=0}^{\infty} \frac{1}{n!} (\mathsf{d}^n \gamma)(x)(h, \ldots, h)$$

of γ in x converges uniformly for all pairs (x, h) in an open neighborhood of $(0,0)$. Since the remainder term in the Taylor expansion satisfies

$$\left\| \gamma(x+h) - \sum_{k=0}^{n} \frac{1}{n!} (\mathsf{d}^n \gamma)(x)(h, \ldots, h) \right\|$$

$$\leqslant \frac{1}{(n+1)!} \int_0^1 \|(\mathsf{d}^{n+1} \gamma)(x + th)(h, \ldots, h)\| \, dt \leqslant 2^{-n-1} \sqrt{C} \to 0,$$

the Taylor series actually converges to $\gamma(x+h)$ for x, h close to 0. In view of [5, Thm. 7.2], this implies that γ is analytic in the sense defined above. $\square$

The following theorem is an analytic version of [35, Thm. 7.2], which is concerned with smooth vectors (cf. [19, Prop. 4.1] for the case $G = \mathbb{R}$).

THEOREM 5.2. — If $(\pi, \mathcal{H})$ is a unitary representation of an analytic Fréchet–Lie group G, then $v \in \mathcal{H}$ is an analytic vector if and only if the corresponding matrix coefficient $\pi^{v,v}(g) := \langle \pi(g)v, v \rangle$ is analytic on a **1**-neighborhood in G.

Proof. — (cf. [35, Thm. 7.2] for smooth vectors) Clearly, $\pi^{v,v}$ is analytic for $v \in \mathcal{H}^\omega$. Suppose, conversely, that $\pi^{v,v}$ is analytic in a **1**-neighborhood $U \subseteq G$ and that U' is a **1**-neighborhood with $h^{-1}g \in U$ for $g, h \in U'$. In view of Theorem 5.1, the analyticity of $\pi^v(g) = \pi(g)v$ on xU', $x \in G$, is equivalent to the analyticity of the function

$$(g, h) \mapsto \langle \pi(xg)v, \pi(xh)v \rangle = \pi^{v,v}(h^{-1}g)$$

on $U' \times U'$, which follows from the analyticity of $\pi^{v,v}$ on U. $\qquad\square$

In view of the GNS construction, we obtain an analog of [35, Cor. 7.4] for analytic functions:

COROLLARY 5.3. — *If φ is a positive definite function on a Lie group G which is analytic in a **1**-neighborhood, then φ is analytic.*

PROBLEM 5.4. — *Suppose that (π, V) is a smooth representation of the Banach–Lie group G on the Banach space V. Is the space V^ω of analytic vectors dense? This is true if $\dim G < \infty$ for any continuous Banach representation ([36, Thm. 4], [15]), but the proof is based on the integrated representation to $L^1(G)$ and mollifying with the heat kernel. All these techniques are not available for general infinite dimensional groups. However, some Lie groups carry so-called heat kernel measures which behave in many respects like Haar measure with a heat kernel density (cf. [11]).*

6. Positive analytic functionals

In this section we turn to the $*$-aspects of analytic functionals on $U(\mathfrak{g})$. Throughout $U_\mathbb{C}(\mathfrak{g}) := U(\mathfrak{g}_\mathbb{C}) \cong U(\mathfrak{g})_\mathbb{C}$ denotes the enveloping algebra of the complexified Lie algebra $\mathfrak{g}_\mathbb{C}$. It coincides with the complexification of the real algebra $U(\mathfrak{g})$ and we have $\mathrm{Hom}_\mathbb{R}(U(\mathfrak{g}), E) \cong \mathrm{Hom}_\mathbb{C}(U_\mathbb{C}(\mathfrak{g}), E)$ for every complex vector space E.

DEFINITION 6.1. — *We write $*$ for the unique antilinear antiautomorphism of $U_\mathbb{C}(\mathfrak{g})$ with $x^* = -x$ for $x \in \mathfrak{g}$. Then $(U_\mathbb{C}(\mathfrak{g}), *)$ is an involutive algebra and a linear functional $\lambda \in U_\mathbb{C}(\mathfrak{g})^*$ is called positive if $\lambda(D^*D) \geqslant 0$ for $D \in U_\mathbb{C}(\mathfrak{g})$.*

Typical examples of positive functional arise from $*$-representations $(\rho, \mathcal{D})$ on pre-Hilbert spaces (see Definition 4.1(b)(ii)) as $\lambda(D) = \langle \rho(D)v, v \rangle$, and a suitable variant of the GNS construction implies that every positive functional on $U_\mathbb{C}(\mathfrak{g})$ is of this from ([39, Thm. 6.3], [46, Thm. 8.6.2]).

DEFINITION 6.2. — *A positive linear functional* $\lambda \in U_{\mathbb{C}}(\mathfrak{g})^*$ *is said to be integrable if there exists a smooth positive definite function* $\varphi \colon G \to \mathbb{C}$ *with*

$$\lambda(D) = (D\varphi)(\mathbf{1}) \quad \text{for} \quad D \in U_{\mathbb{C}}(\mathfrak{g}).$$

If $(\pi, \mathcal{H})$ *is a continuous unitary representation and* $v \in \mathcal{H}$ *with* $\varphi(g) = \langle \pi(g)v, v \rangle$ *(the existence follows from the GNS construction), then* [35, Thm. 7.2] *implies that* v *is a smooth vector. Therefore the integrability of* λ *is equivalent to the existence of a unitary representation* $(\pi, \mathcal{H})$ *and a smooth vector* $v \in \mathcal{H}^\infty$ *with*

$$\lambda(D) = \langle \mathrm{d}\pi(D)v, v \rangle \quad \text{for} \quad D \in U_{\mathbb{C}}(\mathfrak{g}).$$

The following proposition builds a bridge between analytic vectors and analytic functionals.

PROPOSITION 6.3. — *Let* $(\rho, \mathcal{D})$ *be a strongly continuous* ∗*-representation of* $\mathfrak{g}$ *on the pre-Hilbert space* $\mathcal{D}$ *and* $\rho \colon U_{\mathbb{C}}(\mathfrak{g}) \to \mathrm{End}(\mathcal{D})$ *its canonical extension. For* $v \in \mathcal{D}$, *the positive functional* $\beta(D) := \langle \rho(D)v, v \rangle$ *is analytic if and only if* v *is an analytic vector. This implies that* v *is an analytic vector for every operator* $\rho(x)$, $x \in \mathfrak{g}$, *and that* $\sum_{n=0}^{\infty} \frac{\|\rho(x)^n v\|}{n!} < \infty$ *for* $p(x) < R_{\beta, p}/2$.

Proof. — First we observe that, for $x \in \mathfrak{g}$, we have

$$(6.1) \qquad \|\rho(x)^n v\|^2 = \langle \rho(x)^n v, \rho(x)^n v \rangle = (-1)^n \beta(x^{2n}).$$

If β is analytic, then Proposition 3.4 provides an estimate of the form $\|\beta_n^s\|_p \leqslant Cn! t^{-n}$ for every $t < R_{\beta, p}$. This leads to

$$\frac{\|\rho(x)^n v\|}{n!} \leqslant \sqrt{C} \frac{\sqrt{(2n)!}}{n!} p(x)^n t^{-n}.$$

Therefore $\frac{\sqrt{(2n+2)(2n+1)}}{n+1} \to 2$ implies that $\sum_{n=0}^{\infty} \frac{\|\rho(x)^n v\|}{n!}$ converges for $p(x) < t/2$, hence for $p(x) < R_{\beta, p}/2$.

Suppose, conversely, that $\sum_{n=0}^{\infty} \frac{\|\rho(x)^n v\|}{n!}$ converges for every x in a 0-neighborhood U of $\mathfrak{g}$. Then the series

$$\left\langle \sum_{n=0}^{\infty} \frac{\rho(x)^n v}{n!}, v \right\rangle = \sum_{n=0}^{\infty} \frac{\langle \rho(x)^n v, v \rangle}{n!} = \sum_{n=0}^{\infty} \frac{\beta(x^n)}{n!}$$

converges for every $x \in U$.

To see that v is an analytic vector for $\rho(x)$, pick $t > 0$ such that $tx \in U$ and observe that the series

$$\sum_{n=0}^{\infty} \frac{\|\rho(x)^n v\| t^n}{n!} = \sum_{n=0}^{\infty} \frac{\|\rho(tx)^n v\|}{n!}$$

converges. $\square$

DEFINITION 6.4. — Let $(\rho, \mathcal{D})$ be a $*$-representation of $U_{\mathbb{C}}(\mathfrak{g})$ on the pre-Hilbert space $\mathcal{D}$. We call a subset $E \subseteq \mathcal{D}$ equianalytic if there exists a 0-neighborhood $U \subseteq \mathfrak{g}$ such that

$$\sum_{n=0}^{\infty} \frac{\|\rho(x)^n v\|}{n!} < \infty \quad \text{for} \quad v \in E, x \in U.$$

This implies in particular that each $v \in E$ is an analytic vector for every $\rho(x)$, $x \in \mathfrak{g}$.

DEFINITION 6.5. — (cf. [33, Thm. III.1.3], [39, Thm. 6.3], [46, Thm. 8.6.2]) Let $\lambda \in U_{\mathbb{C}}(\mathfrak{g})^*$ be a positive analytic functional. On the dual space $U_{\mathbb{C}}(\mathfrak{g})^*$ we consider the right regular representation of $U_{\mathbb{C}}(\mathfrak{g})$ by $D\beta := \beta \circ \rho_D$. Then there exists a unique Hilbert space $\mathcal{H}_\lambda \subseteq U_{\mathbb{C}}(\mathfrak{g})^*$ containing λ for which the subspace

$$\mathcal{D}_\lambda := U_{\mathbb{C}}(\mathfrak{g})\lambda$$

is dense and on which the scalar product satisfies

(6.2) $\beta(D) = \langle \beta, D^* \lambda \rangle \quad \text{for} \quad \beta \in \mathcal{H}_\lambda, D \in U_{\mathbb{C}}(\mathfrak{g}).$

In particular, we have

$$\langle D_1 \lambda, D_2 \lambda \rangle = \lambda(D_2^* D_1).$$

Now $\rho_\lambda(D)\beta := D\beta := \beta \circ \rho_D$ defines a $*$-representation of $U_{\mathbb{C}}(\mathfrak{g})$ on $\mathcal{D}_\lambda$ by unbounded operators, and for each $x \in \mathfrak{g}$, the operator $\rho_\lambda(x)$ on $\mathcal{D}_\lambda$ is skew-hermitian.

PROPOSITION 6.6. — If λ is a positive analytic functional on $U_{\mathbb{C}}(\mathfrak{g})$, then the following assertions hold:

(a): $\lambda \in \mathcal{D}_\lambda$ is an analytic vector for every $\rho_\lambda(x)$, $x \in \mathfrak{g}$.

(b): If λ is p-analytic for a submultiplicative continuous seminorm p on $\mathfrak{g}$, then $\mathcal{D}_\lambda = \rho_\lambda(U_{\mathbb{C}}(\mathfrak{g}))\lambda$ is equianalytic. More precisely, the series $\sum_{n=0}^{\infty} \frac{\rho_\lambda(x)^n v}{n!}$ converges for $p(x) < R_{\lambda,p}/2$ and $v \in \mathcal{D}_\lambda$.

Proof. — (a) In view of Proposition 6.3, this follows from $\lambda(D) = \langle \rho_\lambda(D)\lambda, \lambda \rangle$ for the representation $(\rho_\lambda, \mathcal{D}_\lambda)$. That this representation is

strongly continuous follows from the continuity of the quadratic maps $x \mapsto \langle \rho_\lambda(x)^2 \rho_\lambda(D)\lambda, \rho_\lambda(D)\lambda \rangle = \lambda(D^* x^2 D)$ (Lemma 4.5).

(b) Let $v := \rho_\lambda(D)\lambda$ for some $D \in U_\mathbb{C}(\mathfrak{g})$ and define

$$\beta(D') := \langle \rho_\lambda(D')v, v \rangle = \lambda(D^* D' D) \quad \text{for} \quad D' \in U_\mathbb{C}(\mathfrak{g}).$$

Then Theorem 3.6 implies that β is an analytic functional whose radius of convergence satisfies $R_{\beta,p} \geqslant r := R_{\lambda,p}$. Therefore Proposition 6.3 implies that v is an analytic vector for every operator $\rho_\lambda(x)$, $x \in \mathfrak{g}$, and the corresponding exponential series converges for $p(x) < R_{\lambda,p}/2 \leqslant R_{\beta,p}/2$. $\square$

LEMMA 6.7. — Let $(\rho, \mathcal{D})$ be a $*$-representation of the BCH–Lie algebra $\mathfrak{g}$ on the dense equianalytic subspace $\mathcal{D}$ of $\mathcal{H}$. Then the closures $\overline{\rho(x)}$, $x \in \mathfrak{g}$, generate unitary one-parameter groups and there exists a 0-neighborhood $U \subseteq \mathfrak{g}$ on which the BCH product $*$ is defined, and the map $\tilde{\rho} \colon \mathfrak{g} \to U(\mathcal{H}), x \mapsto e^{\overline{\rho(x)}}$ satisfies

(6.3) $$\tilde{\rho}(x * y) = \tilde{\rho}(x)\tilde{\rho}(y) \quad \text{for} \quad x, y \in U.$$

Proof. — Nelson's Theorem 4.8 implies that the operators $\rho(x)$ are essentially skew-adjoint, so that their closures generate unitary one-parameter groups and $\tilde{\rho}$ is defined.

Let $U_1 \subseteq \mathfrak{g}$ be an open balanced 0-neighborhood with the property that the function

$$U_1 \to \mathcal{H}, \quad x \mapsto \tilde{\rho}(x) = \sum_{n=0}^{\infty} \frac{\rho(x)^n}{n!}$$

is analytic for every $v \in \mathcal{D}$ (Definition 6.4). Let $U_2 \subseteq U_1$ be a smaller convex 0-neighborhood for which the BCH-multiplication defines an analytic map

$$U_2 \times U_2 \to U_1, \quad (x, y) \mapsto x + y + \frac{1}{2}[x, y] + \cdots.$$

Then we have for $v \in \mathcal{D}$ two analytic functions

$$F_{1/2} \colon U_2 \times U_2 \to \mathbb{C}, \quad F_1(x, y) = \langle \tilde{\rho}(y)v, \tilde{\rho}(-x)v \rangle, \quad F_2(x, y) = \langle \tilde{\rho}(x*y)v, v \rangle.$$

To show that they coincide, it suffices to verify that their Taylor polynomials in $(0, 0)$ agree. Let $\beta(D) := \langle \rho(D)v, v \rangle$. Then, for F_1, the homogeneous term of degree (m, n) is given by

$$T_{m,n}(F_1)(x, y) = \frac{1}{m!n!}\langle \rho_\lambda(y)^n v, \rho_\lambda(-x)^m v \rangle = \frac{1}{m!n!}\langle \rho_\lambda(x^m y^n)v, v \rangle$$
$$= \beta\left(\frac{x^m y^n}{m!n!}\right).$$

For F_2 the homogeneous term of degree (m, n) has the form

$$\sum_{k \leqslant n+m} \frac{1}{k!} T_{m,n}(\langle \rho_\lambda(x * y)^k v, v \rangle) = \beta \Big(\sum_{k \leqslant n+m} \frac{1}{k!} T_{m,n}((x * y)^k) \Big).$$

From the identity $e^x e^y = e^{x*y}$ in the completed free associative algebra $\mathcal{A}$ with two generators x and y we derive that

$$\frac{x^m y^n}{m! n!} = \sum_{k \leqslant n+m} \frac{1}{k!} T_{m,n}((x * y)^k)$$

holds in $\mathcal{A}$, and this implies the corresponding relation in $U(\mathfrak{g})$ ([8, Ch. 2, §6]). We conclude that

$$\langle \tilde{\rho}(x) \tilde{\rho}(y) v, v \rangle = \langle \tilde{\rho}(x * y) v, v \rangle \quad \text{for} \quad x, y \in U_2, v \in \mathcal{D}.$$

Now polarization leads to

$$\langle \tilde{\rho}(x) \tilde{\rho}(y) v, w \rangle = \langle \tilde{\rho}(x * y) v, w \rangle \quad \text{for} \quad v, w \in \mathcal{D}.$$

Since $\mathcal{D}$ is dense in $\mathcal{H}$, we obtain (6.3) for $U := U_2$. $\qquad \square$

The following theorem is our first main result on integrability of Lie algebra representations based on a sufficient supply of analytic vectors.

THEOREM 6.8. — (Integrability Theorem) *Let G be a 1-connected BCH–Lie group with Lie algebra $\mathfrak{g}$ and $(\rho, \mathcal{D})$ be a $*$-representation of $\mathfrak{g}$ on the dense subspace $\mathcal{D}$ of $\mathcal{H}$. Assume that either*

(a): *G is Banach and $\mathcal{D}$ consists of analytic vectors, or*
(b): *$\mathcal{D}$ is equianalytic.*

Then there exists a unique unitary representation $(\pi, \mathcal{H})$ of G on the corresponding Hilbert space $\mathcal{H}$ with $\mathrm{d}\pi(x)|_\mathcal{D} = \rho(x)$ for $x \in \mathfrak{g}$.

Proof. — With Nelson's Theorem 4.8 we see that the operators $\rho(x)$, $x \in \mathfrak{g}$, are essentially skew-adjoint, so that their closures generate unitary one-parameter groups. This leads to a map $\tilde{\rho} : \mathfrak{g} \to \mathrm{U}(\mathcal{H}), x \mapsto e^{\overline{\rho(x)}}$. We claim that

$$(6.4) \qquad\qquad \tilde{\rho}(x * y) = \tilde{\rho}(x) \tilde{\rho}(y)$$

holds for x, y in some open 0-neighborhood $U \subseteq \mathfrak{g}$. It suffices to show that

$$(6.5) \qquad\qquad \tilde{\rho}(x * y) v = \tilde{\rho}(x) \tilde{\rho}(y) v \quad \text{for every} \quad v \in \mathcal{D}.$$

(a) Suppose that G is Banach. Let $\mathcal{H}_v^0 := \rho(U_\mathbb{C}(\mathfrak{g})) v$ and $\mathcal{H}_v$ be the closure of this subspace in $\mathcal{H}$. Then $\lambda(D) := \langle \rho(D) v, v \rangle$ is a positive analytic functional on $U_\mathbb{C}(\mathfrak{g})$ and the map

$$\Gamma : \mathcal{H}_v^0 \to U_\mathbb{C}(\mathfrak{g})^*, \quad \Gamma(w)(D) := \langle \rho(D) w, v \rangle$$

is equivariant with respect to the right regular representation on $U_{\mathbb{C}}(\mathfrak{g})^*$ and satisfies $\Gamma(\mathcal{H}_v^0) = \rho_\lambda(U_{\mathbb{C}}(\mathfrak{g}))\lambda = \mathcal{D}_\lambda$. From $\langle D_1 v, D_2 v \rangle = \lambda(D_2^* D_1)$ we further derive that Γ is isometric. Therefore Proposition 6.6(b) implies that $\mathcal{H}_v^0$ is equianalytic. It therefore suffices to consider case (b) to prove (6.5).

(b) If $\mathcal{D}$ is equianalytic, then Lemma 6.7 applies and provides an open 0-neighborhood $U \subseteq \mathfrak{g}$ with (6.4). Now we derive from [8, Ch. 3, §6, Lemma 1.1] (for Banach–Lie groups) and from [17] (for BCH–Lie groups) that there exists a unique homomorphism $\pi \colon G \to \mathrm{U}(\mathcal{H})$ such that $\pi(\exp_G x) = \tilde\rho(x)$ holds for all elements x in some 0-neighborhood of $\mathfrak{g}$. This relation implies in particular that, for each $x \in \mathfrak{g}$, the unitary one-parameter group $\pi(\exp tx)$ is generated by the closure of the operator $\rho(x)$, so that π is uniquely determined by the representation ρ of $\mathfrak{g}$ on $\mathcal{D}$. $\square$

LEMMA 6.9. — *If λ is a positive analytic functional and $U \subseteq \mathfrak{g}$ a 0-neighborhood, then $\tilde\rho_\lambda(U)\lambda$ is total in $\mathcal{H}_\lambda$.*

Proof. — Let $\mathcal{K}$ denote the closure of the subspace generated by $\tilde\rho(U)\lambda$. From
$$\tilde\rho_\lambda(x)\lambda = \sum_{n=0}^{\infty} \frac{1}{n!}\rho_\lambda(x)^n \lambda$$
for x sufficiently close to 0 we derive that $\rho_\lambda(x)^n \lambda = \frac{d}{dt^n}\Big|_{t=0}\tilde\rho_\lambda(tx)\lambda \in \mathcal{K}$. Now the assertion follows from the fact that $U_{\mathbb{C}}(\mathfrak{g})$ is spanned by the elements of the form x^n, $x \in \mathfrak{g}$, $n \in \mathbb{N}_0$. $\square$

If $(\pi, \mathcal{H})$ is a unitary representation of the analytic Lie group G and $v \in \mathcal{H}^\omega$ an analytic vector, then the corresponding functional $\pi^{v,v}(D) := \langle \mathrm{d}\pi(D)v, v \rangle$ on $U_{\mathbb{C}}(\mathfrak{g})$ is positive and analytic. The following theorem provides a converse which is known for finite dimensional groups (cf. [19, Prop. 2.3]). Its strength lies in the fact that the information about the whole representation $(\pi_\lambda, \mathcal{H}_\lambda)$ is encoded completely on the infinitesimal level in the analytic functional λ.

THEOREM 6.10. — (Integrability Theorem for analytic functionals) *Let G be a 1-connected BCH–Lie group with Lie algebra $\mathfrak{g}$ and $\lambda \in U_{\mathbb{C}}(\mathfrak{g})^*$ be a positive analytic functional. Suppose that $U(\mathfrak{g})\lambda \subseteq \mathrm{Hom}(U(\mathfrak{g}), \mathbb{C})_{\omega,p,r}$ for some $p \in \mathcal{P}(\mathfrak{g})$ and $r > 0$, which is automatically satisfied if $\lambda \in \mathrm{Hom}(U(\mathfrak{g}), \mathbb{C})_{\omega,p}$ for some submultiplicative p. Then there exists a unique unitary representation $(\pi_\lambda, \mathcal{H}_\lambda)$ of G for which the dense subspace $\mathcal{D}_\lambda$ consists of analytic vectors and*
$$\mathrm{d}\pi_\lambda(x)|_{\mathcal{D}_\lambda} = \rho_\lambda(x) \quad \text{for} \quad x \in \mathfrak{g}.$$

Proof. — If $\lambda \in \text{Hom}(U(\mathfrak{g}), \mathbb{C})_{\omega, p}$ for a submultiplicative seminorm p, then Theorem 3.6 implies that $\mathcal{D}_\lambda = U_{\mathbb{C}}(\mathfrak{g})\lambda \subseteq \text{Hom}(U(\mathfrak{g}), \mathbb{C})_{\omega, p, r}$ for some $r > 0$. We assume that this condition is satisfied. Then

$$U_1 := \{x \in \mathfrak{g} : p(x) < r/2\}$$

is an open convex 0-neighborhood on which the series $\sum_{n=0}^{\infty} \frac{1}{n!} \rho_\lambda(x)^n v$ converges (Proposition 6.3). In particular, $\mathcal{D}_\lambda$ is equianalytic and Theorem 6.8(b) applies. $\qquad\square$

Specializing to Banach–Lie groups, the preceding theorem simplifies to:

COROLLARY 6.11. — *Let G be a 1-connected Banach–Lie group with Lie algebra $\mathfrak{g}$ and $\lambda \in U_{\mathbb{C}}(\mathfrak{g})^*$ be a positive analytic functional. Then there exists a unique unitary representation $(\pi_\lambda, \mathcal{H}_\lambda)$ of G for which the dense subspace $\mathcal{D}_\lambda$ consists of analytic vectors and*

$$\mathsf{d}\pi_\lambda(x)|_{\mathcal{D}_\lambda} = \rho_\lambda(x) \quad \text{for} \quad x \in \mathfrak{g}.$$

Integrability without a given group

Since not every Banach–Lie algebra is the Lie algebra of a Lie group ([12]), it is natural to ask to which extent the assumption of the existence of G in Theorem 6.8 is needed to integrate the Lie algebra representation. The following theorem provides a corresponding variant, where no a priori given group is needed (cf. [1] for a non-linear variant of this theorem).

THEOREM 6.12. — (Second Integrability Theorem) *Let $\mathfrak{g}$ be a BCH–Lie algebra and $(\rho, \mathcal{D})$ be a faithful strongly continuous $*$-representation of $\mathfrak{g}$ on the pre-Hilbert space $\mathcal{D}$ by skew-adjoint operators. Assume that $\mathcal{D}$ consists of analytic vectors, so that the closure of each operator $\rho(x)$ generates a unitary one-parameter group on the completion $\mathcal{H}$ of $\mathcal{D}$. Assume that the subgroup*

$$\Gamma := \{z \in \mathfrak{z}(\mathfrak{g}) : e^{\overline{\rho(z)}} = \mathbf{1}\}$$

of the center $\mathfrak{z}(\mathfrak{g})$ of $\mathfrak{g}$ is discrete. Then there exists a connected Lie group G and a continuous unitary representation $(\pi, \mathcal{H})$ of G on the corresponding Hilbert space $\mathcal{H}$ with $\mathsf{d}\pi(x)|_{\mathcal{D}} = \rho(x)$ for $x \in \mathfrak{g}$.

Proof. — With the same argument as in the proof of Theorem 6.8, we obtain a map $\tilde{\rho} : \mathfrak{g} \to \text{U}(\mathcal{H})$ satisfying $\tilde{\rho}(x * y) = \tilde{\rho}(x)\tilde{\rho}(y)$ for x, y in some open balanced 0-neighborhood $U \subseteq \mathfrak{g}$. Shrinking U, we may even assume that

$$\tilde{\rho}(x * y * z) = \tilde{\rho}(x)\tilde{\rho}(y)\tilde{\rho}(z) \quad \text{for} \quad x, y, z \in U.$$

For $z = -x$ this leads to

$$\tilde{\rho}(e^{\operatorname{ad} x} y) = \tilde{\rho}(x * y * (-x)) = \tilde{\rho}(x)\tilde{\rho}(y)\tilde{\rho}(-x) = \tilde{\rho}(x)\tilde{\rho}(y)\tilde{\rho}(x)^{-1},$$

and by passing to the infinitesimal generators, we obtain

$$\rho(e^{\operatorname{ad} x} y) = \tilde{\rho}(x)\rho(y)\tilde{\rho}(x)^{-1}.$$

We consider the subgroup $G := \langle \tilde{\rho}(\mathfrak{g}) \rangle \subseteq \mathrm{U}(\mathcal{H})$ generated by $\tilde{\rho}(\mathfrak{g})$. If $x \in U$ satisfies $\tilde{\rho}(x) = \mathbf{1}$, then the preceding argument, together with the injectivity of ρ implies that $e^{\operatorname{ad} x} = \operatorname{id}_{\mathfrak{g}}$. Since $\mathfrak{z}(\mathfrak{g})$ is open in $\{x \in \mathfrak{g} : e^{\operatorname{ad} x} = \operatorname{id}_{\mathfrak{g}}\}$ (cf. the Adjoint Integrability Theorem in [17]), we may shrink U in such a way that, for $x \in U$, the relation $e^{\operatorname{ad} x} = \operatorname{id}_{\mathfrak{g}}$ implies $x \in \mathfrak{z}(\mathfrak{g})$. In view of the discreteness of Γ, we may further shrink U so that

$$\tilde{\rho}^{-1}(\mathbf{1}) \cap U = \{0\}.$$

If $V \subseteq U$ is an open balanced 0-neighborhood with $V * V \subseteq U$, then $\tilde{\rho}|_V$ is injective. In fact, if $\tilde{\rho}(x) = \tilde{\rho}(y)$ for $x, y \in V$, then

$$\tilde{\rho}((-x) * y) = \tilde{\rho}(-x)\tilde{\rho}(y) = \tilde{\rho}(x)^{-1}\tilde{\rho}(y) = \mathbf{1}$$

leads to $(-x) * y \in \tilde{\rho}^{-1}(\mathbf{1}) \cap U = \{0\}$, We conclude that

$$x = x * 0 = x * ((-x) * y) = (x * (-x)) * y = 0 * y = y$$

and hence that $\tilde{\rho}|_V$ is injective.

This means that $\tilde{\rho}(V) \subseteq G$ carries a manifold structure for which $\tilde{\rho}|_V$ is a diffeomorphism, and that multiplication and inversion are smooth in an identity neighborhood in V. Since G is generated by the image of every 0-neighborhood in V, it carries a unique Lie group structure for which $\tilde{\rho}$ defines a chart in a neighborhood of 0 ([17, Thm. 3.3.2, Rem. 3.3.3]).

The curves $\gamma_x \colon \mathbb{R} \to G$ defined by $\gamma_x(t) := \tilde{\rho}(tx)$ define smooth 1-parameter groups with $\gamma_x'(0) = x$, where we identify $\mathfrak{g}$ via $T_0(\tilde{\rho})$ with $T_{\mathbf{1}}(G)$. Therefore $\tilde{\rho}$ is an exponential function of G. In particular, G is locally exponential.

For every vector $v \in \mathcal{D}$, the map $x \mapsto \tilde{\rho}(x)v$ is analytic in a 0-neighborhood, and since $\tilde{\rho}$ is a local diffeomorphism, the vector v has a smooth orbit map. For the unitary representation $(\pi, \mathcal{H})$ of G given by the inclusion map $\pi \colon G \to \mathrm{U}(\mathcal{H})$, this means that it is smooth, hence in particular continuous. From the construction it follows that, for every $x \in \mathfrak{g}$, the generator $\overline{\mathrm{d}\pi}(x)$ of the unitary one-parameter group $t \mapsto \pi(\exp_G(tx))$ restricts to $\rho(x)$ on $\mathcal{D}$. $\qquad\square$

Remark 6.13. — For a strongly continuous representation (ρ, V) of a Banach–Lie algebra $\mathfrak{g}$, the kernel is a closed ideal, so that $\mathfrak{g}/\ker \rho$ also is

a Banach–Lie algebra. In the situation of Theorem 6.12, we may thus pass to an injective Lie algebra representation for which the theorem can be applied.

Here is an instructive example, showing that the group Γ in Theorem 6.12 is not always discrete.

Example 6.14. — Let $(X, \mathfrak{S}, \mu)$ be a probability space. For a measurable function $f \colon X \to \mathbb{C}$ we define

$$\|f\| := \sup \left\{ \frac{\|f\|_n}{\sqrt[n]{n!}} : n \in \mathbb{N} \right\} \quad \text{for} \quad \|f\|_n := \left(\int_X |f|^n \, d\mu \right)^{1/n}.$$

Writing $\mathcal{M}(X, \mu)$ for the space of measurable functions $f \colon X \to \mathbb{C}$ modulo those vanishing on the complement of a zero set, we obtain a Banach space

$$V := \{ f \in \mathcal{M}(X, \mu) \colon \|f\| < \infty \},$$

which can be viewed as a closed subspace of a weighted ℓ^∞-product of the spaces $L^n(X, \mu)$, $n \in \mathbb{N}$.

For $\|f\| < \frac{1}{2}$ we have

$$\sum_n \frac{\|f^n\|_2}{n!} = \sum_n \frac{\|f\|_{2n}^n}{n!} \leqslant \sum_n \frac{\sqrt{(2n)!}}{n!} \|f\|^n < \infty$$

because $\frac{\sqrt{(2n+2)(2n+1)}}{n+1} \to 2$. Next we observe that for every measurable subset $E \subseteq X$ we have $\chi_E \in V$ because

$$\|\chi_E\| = \sup_n \frac{\|\chi_E\|_n}{\sqrt[n]{n!}} = \sup_n \sqrt[n]{\frac{\mu(E)}{n!}} < \infty$$

follows from $\mu(E)/n! \to 0$. From

$$(n!)^{n+1} = (n!)^n n! \leqslant (n!)^n (n+1)^n = ((n+1)!)^n$$

we derive that $\sqrt[n]{n!} \leqslant \sqrt[n+1]{(n+1)!}$, i.e., the sequence $\sqrt[n]{n!}$ is increasing. The formula for the radius of convergence of the exponential series now yields $\sqrt[n]{n!} \to \infty$.

We claim that $\chi_{E_k} \to 0$ in V if $\mu(E_k) \to 0$. Let $\varepsilon > 0$ and $N \in \mathbb{N}$ with $\sqrt[N]{N!} > \varepsilon^{-1}$. Suppose that $\mu(E_k)^{1/n} \leqslant \varepsilon$ for $k \geqslant N_0$ and every $n \leqslant N$. The relation

$$\sqrt[n]{\frac{\mu(E_k)}{n!}} \leqslant \frac{1}{\sqrt[n]{n!}} \quad \text{for} \quad k \geqslant N_0, n > N$$

and

$$\sqrt[n]{\frac{\mu(E_k)}{n!}} \leqslant \frac{\varepsilon}{\sqrt[n]{n!}} < \varepsilon \quad \text{for} \quad k \geqslant N_0, n \leqslant N$$

then leads to

$$\|\chi_{E_k}\| \leqslant \sup_n \sqrt[n]{\frac{\mu(E_k)}{n!}} \leqslant \varepsilon \quad \text{for} \quad k \geqslant N_0.$$

We conclude that $\chi(E_k) \to 0$ in V. Therefore the closed subgroup

$$\Gamma := \{f \in V : f(X) \subseteq 2\pi\mathbb{Z}\}$$

is not discrete.

The Banach space V acts by unbounded multiplication operators on $L^2(X, \mu)$:

$$\rho(f)\xi := if\xi.$$

The definition of the norm in V implies that each bounded function $\xi \in L^2(X, \mu)$ is an analytic vector for V:

$$\sum_n \frac{\|(if)^n\xi\|_2}{n!} \leqslant \|\xi\|_\infty \sum_n \frac{\|f^n\|_2}{n!} < \infty \quad \text{for} \quad \|f\| < \tfrac{1}{2}.$$

Therefore we have a $*$-representation $(L^\infty(X, \mu), \rho)$ of V by unbounded operators on the dense subspace $L^\infty(X, \mu)$ of the Hilbert space $L^2(X, \mu)$, for which the group

$$\{f \in \mathfrak{g} : e^{\rho(f)} = \mathrm{id}\} = \Gamma$$

is not discrete.

Remark 6.15. — The preceding example shows in particular that there exists an abelian Banach–Lie group G and a continuous unitary representation $(\pi, \mathcal{H})$ for which $\mathcal{H}^\omega$ is dense but the closed subgroup $\ker \pi$ is not discrete and satisfies $\mathbf{L}(\ker \pi) = \{0\}$.

7. Extension of local positive definite functions

DEFINITION 7.1. — Let G be a group and $U \subseteq G$ be a subset. A function $\varphi : UU^{-1} \to \mathbb{C}$ is said to be positive definite if the kernel $K : U \times U \to \mathbb{C}, (x, y) \mapsto \varphi(xy^{-1})$ is positive definite.

The main result of this section is Theorem 7.3 which asserts that any positive definite analytic function defined in a **1**-neighborhood of a simply connected BCH–Lie group G extends to an analytic function on G which is positive definite. This is a quite remarkable result which is far from being true in the C^∞-context. For a discussion of related extension problems for smooth positive definite functions on finite dimensional Lie groups we refer to [25] and the references given there. As we have seen in the introduction,

for $G = \mathbb{R}^n$, such extension problems are part of the classical theory of moments. The corresponding version of Theorem 7.3 can be found in [13, Prop. 3.10].

Before we turn to the extension theorem, we make the following observation on analytic functions on product spaces.

LEMMA 7.2. — *Let X, Y be locally convex spaces and Z a Banach space. Let $U \subseteq X \times Y$ be an open neighborhood of (x_0, y_0) and $f : U \to Z$ be an analytic map. Then there exists an open neighborhood U_{x_0} of x_0 and an open neighborhood U_{y_0} of y_0 such that for every $x \in U_{x_0}$ and $h \in U_{y_0} - y_0$ we have*

$$f(x, y_0 + h) = \sum_{n=0}^{\infty} \frac{1}{n!} (\partial^n_{(0,h)} f)(x, y_0).$$

Proof. — In view of [5, Thm. 5.1, Prop. 5.4], we may assume that f has a complex analytic extension $f : U_{\mathbb{C}} \to Z_{\mathbb{C}}$, where $U_{\mathbb{C}}$ is an open neighborhood of (x_0, y_0) in the complex locally convex space $X_{\mathbb{C}} \times Y_{\mathbb{C}}$. Let V_{x_0}, resp., V_{y_0} be an open balanced neighborhood of x_0 in $U_{\mathbb{C}}$, resp., y_0 in $Y_{\mathbb{C}}$ such that $V_{x_0} \times V_{y_0} \subseteq U_{\mathbb{C}}$. Then [5, Prop. 5.5] implies that

$$f(x, y_0 + h) = \sum_{n=0}^{\infty} \frac{1}{n!} (\partial^n_{(0,h)} f)(x, y_0) \quad \text{for} \quad x \in V_{x_0}, h \in V_{y_0} - y_0.$$

Hence the assertion follows with $U_{x_0} := V_{x_0} \cap X$ and $U_{y_0} := V_{y_0} \cap Y$. $\quad\square$

THEOREM 7.3. — (Extension of local positive definite analytic functions) *Let G be a simply connected Fréchet–BCH–Lie group, $V \subseteq G$ an open connected $\mathbf{1}$-neighborhood and $\varphi : VV^{-1} \to \mathbb{C}$ an analytic positive definite function. Then there exists a unique positive definite analytic function $\tilde{\varphi} : G \to \mathbb{C}$ extending φ.*

Proof. — The uniqueness of $\tilde{\varphi}$ follows from the connectedness of G and the uniqueness of analytic continuation.

Step 1: To obtain its existence, we consider the reproducing kernel Hilbert space $\mathcal{H}_K \subseteq \mathbb{C}^V$ defined by the kernel K via $f(g) = \langle f, K_g \rangle$ for $g \in V$ and $K_g(h) = K(h, g)$. Then the analyticity of the function

$$V \times V \to \mathbb{C}, \quad (g, h) \mapsto \langle K_h, K_g \rangle = K(g, h) = \varphi(gh^{-1})$$

implies that the map $\eta : V \to \mathcal{H}_K, g \mapsto K_g$ is analytic (Theorem 5.1). Here we use that G is assumed to be Fréchet. Hence all functions in $\mathcal{H}_K$ are analytic, and that we obtain for each $x \in \mathfrak{g}$ an operator on $\mathcal{H}_K^0 := \operatorname{span} \eta(V)$ by

$$(\rho(x)f)(g) := \frac{d}{dt}\bigg|_{t=0} f(g \exp_G(tx))$$

because $\frac{d}{dt}|_{t=0} K_y(g \exp_G(tx))$ exists for every $g \in V$ and $x \in \mathfrak{g}$.

Step 2: That we thus obtain a representation $\rho \colon \mathfrak{g} \to \operatorname{End}(\mathcal{H}_K^0)$ follows from the embedding $\mathcal{H}_K^0 \hookrightarrow C^\omega(V, \mathbb{C})$ and the fact that $\mathfrak{g}$ acts by a Lie algebra of (left invariant) vector fields on this space. Next we observe that

$$\langle \rho(x)K_g, K_h \rangle = \frac{d}{dt}\Big|_{t=0} K_g(h \exp_G(tx)) = \frac{d}{dt}\Big|_{t=0} K(h \exp_G(tx), g)$$

$$= \frac{d}{dt}\Big|_{t=0} \varphi(h \exp_G(tx)g^{-1}) = \frac{d}{dt}\Big|_{t=0} K(h, g \exp_G(-tx))$$

$$= \frac{d}{dt}\Big|_{t=0} \overline{K_h(g \exp_G(-tx))}$$

$$= -\overline{(\rho(x)K_h)(g)} = -\overline{\langle \rho(x)K_h, K_g \rangle} = -\langle K_g, \rho(x)K_h \rangle.$$

Therefore ρ extends to a $*$-representation of $U_{\mathbb{C}}(\mathfrak{g})$ on $\mathcal{H}_K^0$.

Step 3: We claim that each $\eta(g) = K_g$, $g \in V$, is an analytic vector with

(7.1) $$\eta(g \exp_G(-x)) = \sum_{n=0}^{\infty} \frac{1}{n!} \rho(x)^n \eta(g)$$

for x sufficiently close to 0. The function $x \mapsto \eta(g \exp_G x)$ is analytic on a 0-neighborhood of $\mathfrak{g}$ and

$$\frac{d^n}{dt^n}\Big|_{t=0} \big(\eta(g \exp_G(tx)))\big)(y)$$

$$= \frac{d^n}{dt^n}\Big|_{t=0} K_{g \exp_G tx}(y) = \frac{d^n}{dt^n}\Big|_{t=0} \varphi(y \exp_G(-tx)g^{-1})$$

$$= \frac{d^n}{dt^n}\Big|_{t=0} K_g(y \exp_G(-tx)) = (\rho(-x)^n \eta(g))(y).$$

This leads to

$$\frac{d^n}{dt^n}\Big|_{t=0} \eta(g \exp_G(-tx)) = \rho(x)^n \eta(g),$$

and hence that (7.1) holds on a 0-neighborhood of $\mathfrak{g}$.

Step 4: Let $W_G \subseteq V$ be an open $\mathbf{1}$-neighborhood and $W_{\mathfrak{g}} \subseteq \mathfrak{g}$ an open balanced 0-neighborhood with $W_G \exp_G(W_{\mathfrak{g}}) \subseteq V$. Next we show that $\eta(W_G)$ is equianalytic and spans a dense subspace of $\mathcal{H}_K$.

Since the map $W_G \times W_{\mathfrak{g}} \to \mathcal{H}_K, (g, x) \mapsto \eta(g \exp_G(x))$ is analytic, Lemma 7.2 shows that, after shrinking W_G and $W_{\mathfrak{g}}$, we may assume that

$$\eta(g \exp_G(-x)) = \sum_{n=0}^{\infty} \frac{1}{n!} \rho(x)^n \eta(g) \quad \text{for} \quad g \in W_G, x \in W_{\mathfrak{g}}.$$

This means that $\eta(W_G)$ is equianalytic. To see that $\mathcal{K} := \overline{\operatorname{span}(\eta(W_G))}$ is dense, we use the analyticity of η to see that $\eta(V) \subseteq \mathcal{K}$, which shows that $\mathcal{H}_K = \mathcal{K}$.

Applying Theorem 6.8 with $\mathcal{D} = \mathrm{span}(\eta(W_G))$, we now obtain a continuous unitary representation $(\pi, \mathcal{H}_K)$ with $\pi(\exp_G(x)) = \overline{e^{\rho(x)}}$ for every $x \in \mathfrak{g}$. Then

$$\tilde{\varphi}(g) := \langle \pi(g) K_{\mathbf{1}}, K_{\mathbf{1}} \rangle = \langle \pi(g)\varphi, \varphi \rangle$$

is an analytic positive definite function on G, and for $x \in W_{\mathfrak{g}}$ we have

$$\tilde{\varphi}(\exp_G x) = \sum_{n=0}^{\infty} \frac{\langle \rho(x)^n \varphi, \varphi \rangle}{n!} = \langle \eta(\exp_G(-x)), \varphi \rangle$$
$$= K_{\exp_G(-x)}(\mathbf{1}) = \varphi(\exp_G x).$$

Therefore $\tilde{\varphi}$ coincides with φ in a $\mathbf{1}$-neighborhood, so that the analyticity of φ and the connectedness of V lead to $\tilde{\varphi}|_V = \varphi$. $\square$

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