ANNALES

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Tome 61, n° 5 (2011), p. 1943-1975.

<http://aif.cedram.org/item?id=AIF_2011__61_5_1943_0>

Ann. Inst. Fourier, Grenoble
61, 5 (2011) 1943-1975

RAMIFICATION AND MODULI SPACES OF FINITE FLAT MODELS

by Naoki IMAI

ABSTRACT. — We determine the type of the zeta functions and the range of the dimensions of the moduli spaces of finite flat models of two-dimensional local Galois representations over finite fields. This gives a generalization of Raynaud's theorem on the uniqueness of finite flat models in low ramifications.

RÉSUMÉ. — Nous déterminons le type des fonctions zéta et la gamme des dimensions des espaces des modules des modèles plats finis des représentations galoisiennes locales à deux dimensions sur corps finis. Cela donne une généralisation du théorème de Raynaud sur l'unicité de modèles plats finis dans les petites ramifications.

Introduction

Let K be a finite extension of the field $\mathbb{Q}_p$ of p-adic numbers. We assume $p > 2$. Let e be the ramification index of K over $\mathbb{Q}_p$, and k be the residue field of K. We consider a two-dimensional continuous representation $V_{\mathbb{F}}$ of the absolute Galois group G_K over a finite field $\mathbb{F}$ of characteristic p. By a finite flat model of $V_{\mathbb{F}}$, we mean a finite flat group scheme $\mathcal{G}$ over $\mathcal{O}_K$, equipped with an action of $\mathbb{F}$, and an isomorphism $V_{\mathbb{F}} \xrightarrow{\sim} \mathcal{G}(\overline{K})$ that respects the action of G_K and $\mathbb{F}$. We assume that $V_{\mathbb{F}}$ has at least one finite flat model. If $e < p-1$, the finite flat model of $V_{\mathbb{F}}$ is unique by Raynaud's result [3, Theorem 3.3.3]. In general, there are finitely many finite flat models of $V_{\mathbb{F}}$, and these appear as the $\mathbb{F}$-rational points of the moduli space of finite flat models of $V_{\mathbb{F}}$, which we denote by $\mathscr{GR}_{V_{\mathbb{F}},0}$. It is natural to ask about the dimension of $\mathscr{GR}_{V_{\mathbb{F}},0}$. In this paper, we determine the type of the zeta functions and the range of the dimensions of the moduli spaces. The main theorem is the following.

Keywords: Group scheme, moduli space, p-adic field.
Math. classification: 11F80, 14D20.

THEOREM. — *Let $d_{V_{\mathbb{F}}} = \dim \mathscr{GR}_{V_{\mathbb{F}},0}$, and $Z(\mathscr{GR}_{V_{\mathbb{F}},0};T)$ be the zeta function of $\mathscr{GR}_{V_{\mathbb{F}},0}$. We put $n=[k:\mathbb{F}_p]$. Then followings are true.*

(1) *After extending the field $\mathbb{F}$ sufficiently, we have*

$$Z(\mathscr{GR}_{V_{\mathbb{F}},0};T)=\prod_{i=0}^{d_{V_{\mathbb{F}}}}(1-|\mathbb{F}|^i T)^{-m_i}$$

for some $m_i \in \mathbb{Z}$ such that $m_{d_{V_{\mathbb{F}}}} > 0$.

(2) *If $n=1$, we have*

$$0 \leqslant d_{V_{\mathbb{F}}} \leqslant \left[\frac{e+2}{p+1}\right].$$

If $n \geqslant 2$, we have

$$0 \leqslant d_{V_{\mathbb{F}}} \leqslant \left[\frac{n+1}{2}\right]\left[\frac{e}{p+1}\right]+\left[\frac{n-2}{2}\right]\left[\frac{e+1}{p+1}\right]+\left[\frac{e+2}{p+1}\right].$$

Here, $[x]$ is the greatest integer less than or equal to x for $x \in \mathbb{R}$.

Furthermore, each equality in the above inequalities can happen for any finite extension K of $\mathbb{Q}_p$.

Raynaud's result says that if $e < p-1$ then $\mathscr{GR}_{V_{\mathbb{F}},0}$ is one point, that is, zero-dimensional and connected. If $e < p-1$, the above theorem also implies that $\mathscr{GR}_{V_{\mathbb{F}},0}$ is zero-dimensional. So it gives a dimensional generalization of Raynaud's result for two-dimensional Galois representations. The connectedness of $\mathscr{GR}_{V_{\mathbb{F}},0}$ is completely false in general. For example, we can check that if $K=\mathbb{Q}_p(\zeta_p)$ and $V_{\mathbb{F}}$ is trivial representations then $\mathscr{GR}_{V_{\mathbb{F}},0}$ consists of $\mathbb{P}^1_{\mathbb{F}}$ and two points (cf. [2, Proposition 2.5.15(2)]). Here $\mathbb{P}^1_{\mathbb{F}}$ denotes the 1-dimensional projective space over $\mathbb{F}$.

In the section 1, we recall the moduli space of finite flat models, and give some Lemmas. We also give an example for any K where the moduli space of finite flat models is one point.

A proof of the main theorem separates into two cases, that is, the case where $V_{\mathbb{F}}$ is not absolutely irreducible and the case where $V_{\mathbb{F}}$ is absolutely irreducible. In section 2, we treat the case where $V_{\mathbb{F}}$ is not absolutely irreducible. In this case, we decompose $\mathscr{GR}_{V_{\mathbb{F}},0}$ into affine spaces in the level of rational points. Then we express the dimensions of these affine spaces explicitly and bound it by combinatorial arguments. In section 3, we treat the case where $V_{\mathbb{F}}$ is absolutely irreducible. A proof is similar to the case where $V_{\mathbb{F}}$ is not absolutely irreducible, but, in this case, we have to decompose $\mathscr{GR}_{V_{\mathbb{F}},0}$ into $\mathbb{A}^d_{\mathbb{F}}$ and $\mathbb{A}^{d-1}_{\mathbb{F}} \times \mathbb{G}_m$ and $\mathbb{A}^{d-2}_{\mathbb{F}} \times \mathbb{G}^2_m$ in the level of rational points. Here $\mathbb{A}^d_{\mathbb{F}}$ denotes the d-dimensional affine space over $\mathbb{F}$, and $\mathbb{G}_m$ is $\mathbb{A}^1_{\mathbb{F}} - \{0\}$.

In the section 4, we state the main theorem and prove it by collecting the results of former sections.

Acknowledgment

The author is grateful to his advisor Takeshi Saito for his careful reading of an earlier version of this paper and for his helpful comments. He would like to thank the referees for their careful reading of this paper and a number of suggestions for improvements.

Notation

Throughout this paper, we use the following notation. Let $p > 2$ be a prime number, and k be the finite field of cardinality $q = p^n$. The Witt ring of k is denoted by $W(k)$. Let K_0 be the quotient field of $W(k)$, and K be a totally ramified extension of K_0 of degree e. The ring of integers of K is denoted by $\mathcal{O}_K$, and the absolute Galois group of K is denoted by G_K. Let $\mathbb{F}$ be a finite field of characteristic p. For a ring A, the formal power series ring of u over A is denoted by $A[[u]]$, and we put $A((u)) = A[[u]](1/u)$. For a field F, the algebraic closure of F is denoted by $\overline{F}$ and the separable closure of F is denoted by F^{sep}. Let v_u be the valuation of $\mathbb{F}((u))$ normalized by $v_u(u) = 1$, and we put $v_u(0) = \infty$. For $x \in \mathbb{R}$, the greatest integer less than or equal to x is denoted by $[x]$. For a positive integer d, the d-dimensional affine space over $\mathbb{F}$ is denoted by $\mathbb{A}^d_{\mathbb{F}}$. Let $\mathbb{G}_m$ be $\mathbb{A}^1_{\mathbb{F}} - \{0\}$.

1. Preliminaries

First of all, we recall the moduli spaces of finite flat models constructed by Kisin in [2].

Let $V_{\mathbb{F}}$ be a continuous two-dimensional representation of G_K over $\mathbb{F}$. We assume that $V_{\mathbb{F}}$ comes from the generic fiber of a finite flat group scheme over $\mathcal{O}_K$. The moduli space of finite flat models of $V_{\mathbb{F}}$, which is denoted by $\mathscr{GR}_{V_{\mathbb{F}},0}$, is a projective scheme over $\mathbb{F}$. An important property of $\mathscr{GR}_{V_{\mathbb{F}},0}$ is the following Proposition.

PROPOSITION 1.1. — *For any finite extension $\mathbb{F}'$ of $\mathbb{F}$, there is a natural bijection between the set of isomorphism classes of finite flat models of $V_{\mathbb{F}'} = V_{\mathbb{F}} \otimes_{\mathbb{F}} \mathbb{F}'$ and $\mathscr{GR}_{V_{\mathbb{F}},0}(\mathbb{F}')$.*

Proof. — This is [2, Corollary 2.1.13]. □

Let $\mathfrak{S} = W(k)[[u]]$, and $\mathcal{O}_{\mathcal{E}}$ be the p-adic completion of $\mathfrak{S}[1/u]$. There is an p-adically continuous action of ϕ on $\mathcal{O}_{\mathcal{E}}$ determined by Frobenius on $W(k)$ and $u \mapsto u^p$. We fix a uniformizer π of $\mathcal{O}_K$, and choose elements $\pi_m \in \overline{K}$ such that $\pi_0 = \pi$ and $\pi_{m+1}^p = \pi_m$ for $m \geqslant 0$, and put $K_\infty = \bigcup_{m \geqslant 0} K(\pi_m)$.

Let $\Phi\mathrm{M}_{\mathcal{O}_{\mathcal{E}},\mathbb{F}}$ be the category of finite $(\mathcal{O}_{\mathcal{E}} \otimes_{\mathbb{Z}_p} \mathbb{F})$-modules M equipped with ϕ-semi-linear map $M \to M$ such that the induced $(\mathcal{O}_{\mathcal{E}} \otimes_{\mathbb{Z}_p} \mathbb{F})$-linear map $\phi^*(M) \to M$ is an isomorphism. Let $\mathrm{Rep}_{\mathbb{F}}(G_{K_\infty})$ be the category of continuous representations of G_{K_∞} over $\mathbb{F}$. Then the functor

$$T : \Phi\mathrm{M}_{\mathcal{O}_{\mathcal{E}},\mathbb{F}} \to \mathrm{Rep}_{\mathbb{F}}(G_{K_\infty});\ M \mapsto \big(k((u))^{\mathrm{sep}} \otimes_{k((u))} M\big)^{\phi=1}$$

gives an equivalence of abelian categories as in [2, (1.1.12)]. Here ϕ acts on $k((u))^{\mathrm{sep}}$ by the p-th power map. We take the ϕ-module $M_{\mathbb{F}} \in \Phi\mathrm{M}_{\mathcal{O}_{\mathcal{E}},\mathbb{F}}$ such that $T(M_{\mathbb{F}})$ is isomorphic to $V_{\mathbb{F}}(-1)|_{G_{K_\infty}}$. Here (-1) denotes the inverse of the Tate twist.

The moduli space $\mathscr{GR}_{V_{\mathbb{F}},0}$ is described via the Kisin modules as in the following.

Proposition 1.2. — *For any $\mathbb{F}$-algebra A, the elements of $\mathscr{GR}_{V_{\mathbb{F}},0}(A)$ naturally correspond to finite projective $(k[[u]] \otimes_{\mathbb{F}_p} A)$-submodules $\mathfrak{M}_A \subset M_{\mathbb{F}} \otimes_{\mathbb{F}} A$ that satisfy the followings:*

(1) *$\mathfrak{M}_A$ generates $M_{\mathbb{F}} \otimes_{\mathbb{F}} A$ over $k((u)) \otimes_{\mathbb{F}_p} A$.*
(2) *$u^e \mathfrak{M}_A \subset (1 \otimes \phi)\big(\phi^*(\mathfrak{M}_A)\big) \subset \mathfrak{M}_A$.*

Proof. — This follows from the construction of $\mathscr{GR}_{V_{\mathbb{F}},0}$ in [2, Corollary 2.1.13]. □

By Proposition 1.2, we often identify a point of $\mathscr{GR}_{V_{\mathbb{F}},0}(\mathbb{F}')$ with the corresponding finite free $k[[u]] \otimes_{\mathbb{F}_p} \mathbb{F}'$-module.

From now on, we assume $\mathbb{F}_{q^2} \subset \mathbb{F}$ and fix an embedding $k \hookrightarrow \mathbb{F}$. This assumption does not matter, because we may extend $\mathbb{F}$ to prove the main theorem. We consider the isomorphism

$$\mathcal{O}_{\mathcal{E}} \otimes_{\mathbb{Z}_p} \mathbb{F} \cong k((u)) \otimes_{\mathbb{F}_p} \mathbb{F} \xrightarrow{\sim} \prod_{\sigma \in \mathrm{Gal}(k/\mathbb{F}_p)} \mathbb{F}((u))\ ;$$
$$\Big(\sum a_i u^i\Big) \otimes b \mapsto \Big(\sum \sigma(a_i) b u^i\Big)_\sigma$$

and let $\epsilon_\sigma \in k((u)) \otimes_{\mathbb{F}_p} \mathbb{F}$ be the primitive idempotent corresponding to σ. Take $\sigma_1, \cdots, \sigma_n \in \mathrm{Gal}(k/\mathbb{F}_p)$ such that $\sigma_{i+1} = \sigma_i \circ \phi^{-1}$. Here we regard ϕ as the p-th power Frobenius, and use the convention that $\sigma_{n+i} = \sigma_i$. In the following, we often use such conventions. Then we have $\phi(\epsilon_{\sigma_i}) = \epsilon_{\sigma_{i+1}}$

and $\phi : M_{\mathbb{F}} \to M_{\mathbb{F}}$ determines $\phi : \epsilon_{\sigma_i} M_{\mathbb{F}} \to \epsilon_{\sigma_{i+1}} M_{\mathbb{F}}$. For $(A_i)_{1\leqslant i\leqslant n} \in GL_2\big(\mathbb{F}((u))\big)^n$, we write

$$M_{\mathbb{F}} \sim (A_1, A_2, \dots, A_n) = (A_i)_i$$

if there is a basis $\{e_1^i, e_2^i\}$ of $\epsilon_{\sigma_i} M_{\mathbb{F}}$ over $\mathbb{F}((u))$ such that $\phi\begin{pmatrix} e_1^i \\ e_2^i \end{pmatrix} = A_i \begin{pmatrix} e_1^{i+1} \\ e_2^{i+1} \end{pmatrix}$. We use the same notation for any sublattice $\mathfrak{M}_{\mathbb{F}} \subset M_{\mathbb{F}}$ similarly. Here and in the following, we consider only sublattices that are $(\mathfrak{S} \otimes_{\mathbb{Z}_p} \mathbb{F})$-modules.

Let A be an $\mathbb{F}$-algebra, and $\mathfrak{M}_A$ be a finite free $(k[[u]] \otimes_{\mathbb{F}_p} A)$-submodules of $M_{\mathbb{F}} \otimes_{\mathbb{F}} A$ that generate $M_{\mathbb{F}} \otimes_{\mathbb{F}} A$ over $k((u)) \otimes_{\mathbb{F}_p} A$. We choose a basis $\{e_1^i, e_2^i\}_i$ of $\mathfrak{M}_A$ over $k[[u]] \otimes_{\mathbb{F}_p} A$. For $B = (B_i)_{1\leqslant i\leqslant n} \in GL_2\big(\mathbb{F}((u)) \otimes_{\mathbb{F}_p} A\big)^n$, the $(\mathfrak{S} \otimes_{\mathbb{Z}_p} A)$-module generated by the entries of $\left\langle B_i \begin{pmatrix} e_1^i \\ e_2^i \end{pmatrix} \right\rangle$ for $1 \leqslant i \leqslant n$ with the basis given by these entries is denoted by $B \cdot \mathfrak{M}_A$. Note that $B \cdot \mathfrak{M}_A$ depends on the choice of the basis of $\mathfrak{M}_A$. We can see that if $\mathfrak{M}_{\mathbb{F}} \sim (A_i)_i$ for $(A_i)_{1\leqslant i\leqslant n} \in GL_2\big(\mathbb{F}((u))\big)^n$ with respect to a given basis, then we have

$$B \cdot \mathfrak{M}_{\mathbb{F}} \sim \big(\phi(B_i) A_i (B_{i+1})^{-1}\big)_i$$

with respect to the induced basis.

LEMMA 1.1. — *Suppose $\mathbb{F}'$ is a finite extension of $\mathbb{F}$, and $x \in \mathscr{GR}_{V_{\mathbb{F}},0}(\mathbb{F}')$ corresponds to $\mathfrak{M}_{\mathbb{F}'}$. Put $\mathfrak{M}_{j,\mathbb{F}'} = \left(\begin{pmatrix} u^{s_{j,i}} & v_{j,i} \\ 0 & u^{t_{j,i}} \end{pmatrix}\right)_i \cdot \mathfrak{M}_{\mathbb{F}'}$ for $1 \leqslant j \leqslant 2$, $s_{j,i}, t_{j,i} \in \mathbb{Z}$ and $v_{j,i} \in \mathbb{F}'((u))$. Assume $\mathfrak{M}_{1,\mathbb{F}'}$ and $\mathfrak{M}_{2,\mathbb{F}'}$ correspond to $x_1, x_2 \in \mathscr{GR}_{V_{\mathbb{F}},0}(\mathbb{F}')$ respectively. Then $x_1 = x_2$ if and only if*

$$s_{1,i} = s_{2,i},\ t_{1,i} = t_{2,i} \text{ and } v_{1,i} - v_{2,i} \in u^{t_{1,i}} \mathbb{F}'[[u]] \text{ for all } i.$$

Proof. — The equality $x_1 = x_2$ is equivalent to the existence of $B = (B_i)_{1\leqslant i\leqslant n} \in GL_2(\mathbb{F}'[[u]])^n$ such that

$$B_i \begin{pmatrix} u^{s_{1,i}} & v_{1,i} \\ 0 & u^{t_{1,i}} \end{pmatrix} = \begin{pmatrix} u^{s_{2,i}} & v_{2,i} \\ 0 & u^{t_{2,i}} \end{pmatrix}$$

for all i. It is further equivalent to the condition that

$$\begin{pmatrix} u^{s_{2,i}-s_{1,i}} & v_{2,i}u^{-t_{1,i}} - u^{s_{2,i}-s_{1,i}-t_{1,i}} v_{1,i} \\ 0 & u^{t_{2,i}-t_{1,i}} \end{pmatrix} \in GL_2(\mathbb{F}'[[u]])$$

for all i. The last condition is equivalent to the desired condition. □

LEMMA 1.2. — *Suppose $V_{\mathbb{F}}$ is absolutely irreducible. If $\mathbb{F}'$ is the quadratic extension of $\mathbb{F}$, then*

$$M_{\mathbb{F}} \otimes_{\mathbb{F}} \mathbb{F}' \sim \left(\begin{pmatrix} 0 & \alpha_1 \\ \alpha_1 u^m & 0 \end{pmatrix}, \begin{pmatrix} \alpha_2 & 0 \\ 0 & \alpha_2 \end{pmatrix}, \dots, \begin{pmatrix} \alpha_n & 0 \\ 0 & \alpha_n \end{pmatrix}\right)$$

for some $\alpha_i \in (\mathbb{F}')^\times$ and a positive integer m such that $(q+1) \nmid m$. Conversely, for each positive integer m such that $(q+1) \nmid m$, there exists an absolutely irreducible representation $V_{\mathbb{F}}$ as above.

Proof. — The first statement is [1, Lemma 1.2], and the second statement follows from the proof of [1, Lemma 1.2]. We have used the assumption $\mathbb{F}_{q^2} \subset \mathbb{F}$ in this Lemma. □

PROPOSITION 1.3. — *If $M_{\mathbb{F}} \sim \left(\begin{pmatrix} u^e & u \\ 0 & 1 \end{pmatrix} \right)_i$, then $\mathscr{GR}_{V_{\mathbb{F}},0}(\mathbb{F}')$ is one point for any finite extension $\mathbb{F}'$ of $\mathbb{F}$.*

Proof. — Let $\mathfrak{M}_{0,\mathbb{F}}$ be the lattice of $M_{\mathbb{F}}$ generated by the basis giving

$$M_{\mathbb{F}} \sim \left(\begin{pmatrix} u^e & u \\ 0 & 1 \end{pmatrix} \right)_i,$$

and let $\mathfrak{M}_{0,\mathbb{F}'} = \mathfrak{M}_{0,\mathbb{F}} \otimes_{\mathbb{F}} \mathbb{F}'$ for finite extensions $\mathbb{F}'$ of $\mathbb{F}$. Then $\mathfrak{M}_{0,\mathbb{F}'}$ gives a point of $\mathscr{GR}_{V_{\mathbb{F}},0}(\mathbb{F}')$. By the Iwasawa decomposition, any point $\mathfrak{M}_{\mathbb{F}'}$ of $\mathscr{GR}_{V_{\mathbb{F}},0}(\mathbb{F}')$ is written as $\left(\begin{pmatrix} u^{-s_i} & v_i \\ 0 & u^{t_i} \end{pmatrix} \right)_i \cdot \mathfrak{M}_{0,\mathbb{F}'}$ for $s_i, t_i \in \mathbb{Z}$ and $v_i \in \mathbb{F}((u))$. Then we have

$$\begin{aligned} \mathfrak{M}_{\mathbb{F}'} &\sim \left(\begin{pmatrix} u^{-ps_i} & \phi(v_i) \\ 0 & u^{pt_i} \end{pmatrix} \begin{pmatrix} u^e & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u^{s_{i+1}} & -v_{i+1}u^{s_{i+1}-t_{i+1}} \\ 0 & u^{-t_{i+1}} \end{pmatrix} \right)_i \\ &= \left(\begin{pmatrix} u^{e-ps_i+s_{i+1}} & u^{1-ps_i-t_{i+1}} + \phi(v_i)u^{-t_{i+1}} - v_{i+1}u^{e-ps_i+s_{i+1}-t_{i+1}} \\ 0 & u^{pt_i-t_{i+1}} \end{pmatrix} \right)_i \end{aligned}$$

with respect to the basis induced from the given basis of $\mathfrak{M}_{0,\mathbb{F}'}$. We put $r_i = -v_u(v_i)$.

By $u^e\mathfrak{M}_{\mathbb{F}'} \subset (1 \otimes \phi)\big(\phi^*(\mathfrak{M}_{\mathbb{F}'})\big) \subset \mathfrak{M}_{\mathbb{F}'}$, we have $e - ps_i + s_{i+1} \leqslant e$ and $pt_i - t_{i+1} \geqslant 0$ for all i, so we get $s_i, t_i \geqslant 0$ for all i.

We are going to show that $1 - ps_i - t_{i+1} \geqslant 0$ for all i. We assume that $1 - ps_{i_0} - t_{i_0+1} < 0$ for some i_0. Then $v_u(v_{i_0+1}u^{e-ps_{i_0}+s_{i_0+1}-t_{i_0+1}}) \leqslant 1-ps_{i_0}-t_{i_0+1}$, because $\phi(v_{i_0})u^{-t_{i_0+1}}$ has no term of degree $1-ps_{i_0}-t_{i_0+1}$. So we get $r_{i_0+1}-s_{i_0+1} \geqslant e-1 \geqslant 0$. Take an index i_1 such that $r_{i_1}-s_{i_1}$ is the maximum. We note that $r_{i_1} - s_{i_1} \geqslant 0$. Then we have $v_u\big(\phi(v_{i_1})u^{-t_{i_1+1}}\big) = v_u(v_{i_1+1}u^{e-ps_{i_1}+s_{i_1+1}-t_{i_1+1}})$, because $v_u\big(\phi(v_{i_1})u^{-t_{i_1+1}}\big) \leqslant -ps_{i_1}-t_{i_1+1}$. So we get $r_{i_1+1}-s_{i_1+1} = p(r_{i_1}-s_{i_1})+er_{i_1}-s_{i_1}$. This is a contradiction. Thus we have proved that $1 - ps_i - t_{i+1} \geqslant 0$ for all i, and this is equivalent to that $s_i = 0$ and $0 \leqslant t_i \leqslant 1$ for all i.

We assume $t_i = 1$ for some i. Then we have $t_i = 1$ for all i, because $pt_{i-1} - t_i \geqslant 0$ for all i. We show that $r_i \leqslant -1$ for all i. We take an index i_2 such that r_{i_2} is the maximum, and assume that $r_{i_2} \geqslant 0$. Then we have $r_{i_2+1} = pr_{i_2} + e > r_{i_2}$, because $v_u\big(1 + \phi(v_{i_2})u^{-1} - v_{i_2+1}u^{e-1}\big) \geqslant 0$. This is a contradiction. So we have $r_i \leqslant -1$ for all i. Then we may assume $v_i = 0$ for all i by Lemma 1.1. Now we have $\mathfrak{M}_{\mathbb{F}'} \sim \left(\begin{pmatrix} u^e & 1 \\ 0 & u^{p-1} \end{pmatrix}\right)_i$, but this contradicts $u^e\mathfrak{M}_{\mathbb{F}'} \subset (1 \otimes \phi)\big(\phi^*(\mathfrak{M}_{\mathbb{F}'})\big)$.

Thus we have proved $s_i = t_i = 0$ for all i. Then we have $r_i \leqslant 0$, because $v_u(u + \phi(v_i) - v_{i+1}u^e) \geqslant 0$. So we may assume $v_i = 0$ for all i by Lemma 1.1, and we have $\mathfrak{M}_{\mathbb{F}'} = \mathfrak{M}_{0,\mathbb{F}'}$. This shows that $\mathscr{GR}_{V_{\mathbb{F}},0}(\mathbb{F}')$ is one point. □

2. The case where $V_{\mathbb{F}}$ is not absolutely irreducible

In this section, we give the maximum of the dimensions of the moduli spaces in the case where $V_{\mathbb{F}}$ is not absolutely irreducible. We put $d_{V_{\mathbb{F}}} = \dim \mathscr{GR}_{V_{\mathbb{F}},0}$. In the proof of the following Proposition, three Lemmas appear.

PROPOSITION 2.1. — *We assume $V_{\mathbb{F}}$ is not absolutely irreducible, and write $e = (p+1)e_0 + e_1$ for $e_0 \in \mathbb{Z}$ and $0 \leqslant e_1 \leqslant p$. Then the followings are true.*

(1) *There are $m_i \in \mathbb{Z}$ for $0 \leqslant i \leqslant d_{V_{\mathbb{F}}}$ such that $m_i \geqslant 0$, $m_{d_{V_{\mathbb{F}}}} > 0$ and*

$$|\mathscr{GR}_{V_{\mathbb{F}},0}(\mathbb{F}')| = \sum_{i=0}^{d_{V_{\mathbb{F}}}} m_i |\mathbb{F}'|^i$$

for all sufficiently large extensions $\mathbb{F}'$ of $\mathbb{F}$.

(2) (a) *In the case $0 \leqslant e_1 \leqslant p-2$, we have $d_{V_{\mathbb{F}}} \leqslant ne_0$. In this case, if*

$$M_{\mathbb{F}} \sim \left(\begin{pmatrix} u^{e_0} & 0 \\ 0 & u^{pe_0} \end{pmatrix}\right)_i,$$

then $d_{V_{\mathbb{F}}} = ne_0$.

(b) *In the case $e_1 = p-1$, we have $d_{V_{\mathbb{F}}} \leqslant ne_0 + 1$. In this case, if*

$$M_{\mathbb{F}} \sim \left(\begin{pmatrix} u^{e_0} & 0 \\ 0 & u^{pe_0+p-1} \end{pmatrix}\right)_i,$$

then $d_{V_{\mathbb{F}}} = ne_0 + 1$.

(c) *In the case $e_1 = p$, we have $d_{V_{\mathbb{F}}} \leqslant ne_0 + \max\{[n/2], 1\}$. In this case, if $n = 1$ and*

$$M_{\mathbb{F}} \sim \begin{pmatrix} u^{e_0} & 0 \\ 0 & u^{pe_0+p-1} \end{pmatrix},$$

then $d_{V_{\mathbb{F}}} = e_0 + 1$, and if $n \geqslant 2$ and

$$M_{\mathbb{F}} \sim \left(\begin{pmatrix} u^{e_{0,i}} & 0 \\ 0 & u^{p(2e_0+1-e_{0,i})} \end{pmatrix} \right)_i,$$

then $d_{V_{\mathbb{F}}} = ne_0 + [n/2]$. Here, $e_{0,i} = e_0$ if i is odd, and $e_{0,i} = e_0 + 1$ if i is even.

Proof. — Extending the field $\mathbb{F}$, we may assume that $V_{\mathbb{F}}$ is reducible. Let $\mathfrak{M}_{0,\mathbb{F}}$ be a lattice of $M_{\mathbb{F}}$ corresponding to a point of $\mathscr{GR}_{V_{\mathbb{F}},0}(\mathbb{F})$. Then we take and fix a basis of $\mathfrak{M}_{0,\mathbb{F}}$ over $k[[u]] \otimes_{\mathbb{F}_p} \mathbb{F}$ such that

$$\mathfrak{M}_{0,\mathbb{F}} \sim \left(\begin{pmatrix} \alpha_i u^{a_{0,i}} & w_{0,i} \\ 0 & \beta_i u^{b_{0,i}} \end{pmatrix} \right)_i$$

for $\alpha_i, \beta_i \in \mathbb{F}^\times$, $0 \leqslant a_{0,i}, b_{0,i} \leqslant e$ and $w_{0,i} \in \mathbb{F}[[u]]$. For any finite extension $\mathbb{F}'$ of $\mathbb{F}$, we put $\mathfrak{M}_{0,\mathbb{F}'} = \mathfrak{M}_{0,\mathbb{F}} \otimes_{\mathbb{F}} \mathbb{F}'$ and $M_{\mathbb{F}'} = M_{\mathbb{F}} \otimes_{\mathbb{F}} \mathbb{F}'$. By the Iwasawa decomposition, any sublattice of $M_{\mathbb{F}'}$ can be written as $\left(\begin{pmatrix} u^{s_i} & v_i' \\ 0 & u^{t_i} \end{pmatrix} \right)_i \cdot \mathfrak{M}_{0,\mathbb{F}'}$ for $s_i, t_i \in \mathbb{Z}$ and $v_i' \in \mathbb{F}'((u))$.

We put

$$I = \left\{ (\underline{a}, \underline{b}) \in \mathbb{Z}^n \times \mathbb{Z}^n \;\middle|\; \underline{a} = (a_i)_{1 \leqslant i \leqslant n},\ \underline{b} = (b_i)_{1 \leqslant i \leqslant n},\ 0 \leqslant a_i, b_i \leqslant e \right\},$$

and

$$\mathscr{GR}_{V_{\mathbb{F}},0,\underline{a},\underline{b}}(\mathbb{F}') = \left\{ \left(\begin{pmatrix} u^{s_i} & v_i' \\ 0 & u^{t_i} \end{pmatrix} \right)_i \cdot \mathfrak{M}_{0,\mathbb{F}'} \in \mathscr{GR}_{V_{\mathbb{F}},0}(\mathbb{F}') \;\middle|\; s_i, t_i \in \mathbb{Z}, v_i' \in \mathbb{F}'((u)), \right.$$
$$\left. a_i = a_{0,i} + ps_i - s_{i+1},\ b_i = b_{0,i} + pt_i - t_{i+1} \right\}$$

for $(\underline{a}, \underline{b}) = \big((a_i)_{1 \leqslant i \leqslant n}, (b_i)_{1 \leqslant i \leqslant n}\big) \in I$. Then we have

$$\mathscr{GR}_{V_{\mathbb{F}},0}(\mathbb{F}') = \bigcup_{(\underline{a},\underline{b}) \in I} \mathscr{GR}_{V_{\mathbb{F}},0,\underline{a},\underline{b}}(\mathbb{F}'),$$

and this is a disjoint union by Lemma 1.1.

Take $\mathfrak{M}_{\mathbb{F}'} = \left(\begin{pmatrix} u^{s_i} & v_i' \\ 0 & u^{t_i} \end{pmatrix}\right)_i \cdot \mathfrak{M}_{0,\mathbb{F}'} \in \mathscr{GR}_{V_{\mathbb{F}},0,\underline{a},\underline{b}}(\mathbb{F}')$ with the basis induced from the basis of $\mathfrak{M}_{0,\mathbb{F}'}$, then $\mathfrak{M}_{\mathbb{F}'} \sim \left(\begin{pmatrix} \alpha_i u^{a_i} & w_i \\ 0 & \beta_i u^{b_i} \end{pmatrix}\right)_i$ for some $(w_i)_{1\leqslant i\leqslant n} \in \mathbb{F}'[[u]]^n$. We note that $a_i + b_i - v_u(w_i) \leqslant e$ for all i by the condition $u^e\mathfrak{M}_{\mathbb{F}'} \subset (1\otimes\phi)\big(\phi^*(\mathfrak{M}_{\mathbb{F}'})\big)$.

Now, any $\mathfrak{M}'_{\mathbb{F}'} \in \mathscr{GR}_{V_{\mathbb{F}},0,\underline{a},\underline{b}}(\mathbb{F}')$ can be written as $\left(\begin{pmatrix} 1 & v_i \\ 0 & 1 \end{pmatrix}\right)_i \cdot \mathfrak{M}_{\mathbb{F}'}$ for some $(v_i)_{1\leqslant i\leqslant n} \in \mathbb{F}'((u))^n$. With the basis induced from $\mathfrak{M}_{\mathbb{F}'}$, we have

$$\mathfrak{M}'_{\mathbb{F}'} \sim \left(\begin{pmatrix} 1 & \phi(v_i) \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \alpha_i u^{a_i} & w_i \\ 0 & \beta_i u^{b_i} \end{pmatrix}\begin{pmatrix} 1 & -v_{i+1} \\ 0 & 1 \end{pmatrix}\right)_i = \left(\begin{pmatrix} \alpha_i u^{a_i} & w_i - \alpha_i u^{a_i} v_{i+1} + \beta_i u^{b_i}\phi(v_i) \\ 0 & \beta_i u^{b_i} \end{pmatrix}\right)_i.$$

We are going to examine the condition for $(v_i)_{1\leqslant i\leqslant n} \in \mathbb{F}'((u))^n$ to give a point of $\mathscr{GR}_{V_{\mathbb{F}},0,\underline{a},\underline{b}}(\mathbb{F}')$ as $\left(\begin{pmatrix} 1 & v_i \\ 0 & 1 \end{pmatrix}\right)_i \cdot \mathfrak{M}_{\mathbb{F}'}$. Extending the field $\mathbb{F}$, we may assume that $\mathscr{GR}_{V_{\mathbb{F}},0,\underline{a},\underline{b}}(\mathbb{F}) = \emptyset$ if and only if $\mathscr{GR}_{V_{\mathbb{F}},0,\underline{a},\underline{b}}(\mathbb{F}') = \emptyset$ for each $(\underline{a},\underline{b}) \in I$ and any finite extension $\mathbb{F}'$ of $\mathbb{F}$.

For $(v_i)_{1\leqslant i\leqslant n} \in \mathbb{F}'((u))^n$, we have

$$\mathfrak{M}'_{\mathbb{F}'} = \left(\begin{pmatrix} 1 & v_i \\ 0 & 1 \end{pmatrix}\right)_i \cdot \mathfrak{M}_{\mathbb{F}'} \in \mathscr{GR}_{V_{\mathbb{F}},0,\underline{a},\underline{b}}(\mathbb{F}')$$

if and only if

$$v_u\big(w_i - \alpha_i u^{a_i} v_{i+1} + \beta_i u^{b_i}\phi(v_i)\big) \geqslant 0$$

and

$$v_u(\alpha_i u^{a_i}) + v_u(\beta_i u^{b_i}) - v_u\big(w_i - \alpha_i u^{a_i} v_{i+1} + \beta_i u^{b_i}\phi(v_i)\big) \leqslant e \text{ for all } i,$$

by the condition $u^e\mathfrak{M}'_{\mathbb{F}'} \subset (1\otimes\phi)\big(\phi^*(\mathfrak{M}'_{\mathbb{F}'})\big) \subset \mathfrak{M}'_{\mathbb{F}'}$. This is further equivalent to

$$v_u\big(\alpha_i u^{a_i} v_{i+1} - \beta_i u^{b_i}\phi(v_i)\big) \geqslant \max\{0, a_i + b_i - e\},$$

because $v_u(w_i) \geqslant \max\{0, a_i + b_i - e\}$. We put $r_i = -v_u(v_i)$, and note that

$$v_u(\alpha_{i-1} u^{a_{i-1}} v_i) \geqslant \max\{0, a_{i-1} + b_{i-1} - e\} \Leftrightarrow r_i \leqslant \min\{a_{i-1}, e - b_{i-1}\},$$

$$v_u\big(\beta_i u^{b_i}\phi(v_i)\big) \geqslant \max\{0, a_i + b_i - e\} \Leftrightarrow r_i \leqslant \min\left\{\frac{e - a_i}{p}, \frac{b_i}{p}\right\}.$$

We define an $\mathbb{F}'$-vector space $\widetilde{N}_{\underline{a},\underline{b},\mathbb{F}'}$ by

$$\widetilde{N}_{\underline{a},\underline{b},\mathbb{F}'} = \big\{(v_1,\dots,v_n) \in \mathbb{F}'((u))^n \;\big|\; v_u\big(\alpha_i u^{a_i} v_{i+1} - \beta_i u^{b_i}\phi(v_i)\big) \geqslant \max\{0, a_i + b_i - e\} \text{ for all } i\big\}.$$

We note that $\widetilde{N}_{\underline{a},\underline{b},\mathbb{F}'} \supset \mathbb{F}'[[u]]^n$, and put $N_{\underline{a},\underline{b},\mathbb{F}'} = \widetilde{N}_{\underline{a},\underline{b},\mathbb{F}'}/\mathbb{F}'[[u]]^n$. Then we have a bijection $N_{\underline{a},\underline{b},\mathbb{F}'} \to \mathscr{GR}_{V_{\mathbb{F}},0,\underline{a},\underline{b}}(\mathbb{F}')$ by Lemma 1.1. We put $d_{\underline{a},\underline{b}} = \dim_{\mathbb{F}'} N_{\underline{a},\underline{b},\mathbb{F}'}$, and note that $\dim_{\mathbb{F}'} N_{\underline{a},\underline{b},\mathbb{F}'}$ is independent of finite extensions $\mathbb{F}'$ of $\mathbb{F}$.

We take a basis $(\mathbf{v}_j)_{1\leqslant j\leqslant d_{\underline{a},\underline{b}}}$ of $N_{\underline{a},\underline{b},\mathbb{F}}$ over $\mathbb{F}$, where $\mathbf{v}_j = (v_{j,1},\dots,v_{j,n}) \in \mathbb{F}((u))^n$. Then, by Proposition 1.2, an $(\mathbb{F}[[u]] \otimes_{\mathbb{F}} \mathbb{F}[X_1,\dots,X_{d_{\underline{a},\underline{b}}}])$-module

$$\mathfrak{M}'_{\mathbb{F}[X_1,\dots,X_{d_{\underline{a},\underline{b}}}]} = \left(\begin{pmatrix} 1 & \sum_j v_{j,i}X_j \\ 0 & 1 \end{pmatrix}\right)_i \cdot (\mathfrak{M}_{\mathbb{F}} \otimes_{\mathbb{F}} \mathbb{F}[X_1,\dots,X_{d_{\underline{a},\underline{b}}}])$$

gives a morphism $f_{\underline{a},\underline{b}} : \mathbb{A}_{\mathbb{F}}^{d_{\underline{a},\underline{b}}} \to \mathscr{GR}_{V_{\mathbb{F}},0}$ such that $f_{\underline{a},\underline{b}}(\mathbb{F}')$ is injective and the image of $f_{\underline{a},\underline{b}}(\mathbb{F}')$ is $\mathscr{GR}_{V_{\mathbb{F}},0,\underline{a},\underline{b}}(\mathbb{F}')$. Then we have (1) and

$$d_{V_{\mathbb{F}}} = \max_{(\underline{a},\underline{b})\in I,\ \mathscr{GR}_{V_{\mathbb{F}},0,\underline{a},\underline{b}}(\mathbb{F})\neq\emptyset} \{d_{\underline{a},\underline{b}}\}.$$

Before going into a proof of (2), we will examine $d_{\underline{a},\underline{b}}$ to evaluate $d_{V_{\mathbb{F}}}$. We put

$$S_{\underline{a},\underline{b},i} = \left\{ (0,\dots,0,v_i,0,\dots,0) \in \mathbb{F}((u))^n \;\middle|\; v_i = u^{-r_i},\ 1 \leqslant r_i \leqslant \min\left\{a_{i-1}, e-b_{i-1}, \frac{e-a_i}{p}, \frac{b_i}{p}\right\}\right\}$$

for $1 \leqslant i \leqslant n$,

$$S_{\underline{a},\underline{b},i,j} = \left\{ (0,\dots,0,v_i,v_{i+1},\dots,v_{i+j},0,\dots,0) \in \mathbb{F}((u))^n \;\middle|\; v_i = u^{-r_i},\ 1 \leqslant r_i \leqslant \min\{a_{i-1}, e-b_{i-1}\},\ \alpha_{i+l}u^{a_{i+l}}v_{i+l+1} = \beta_{i+l}u^{b_{i+l}}\phi(v_{i+l}) \text{ and } -v_u(v_{i+l+1}) > \min\{a_{i+l}, e-b_{i+l}\} \text{ for } 0 \leqslant l \leqslant j-1,\ -v_u(v_{i+j}) \leqslant \min\left\{\frac{e-a_{i+j}}{p}, \frac{b_{i+j}}{p}\right\}\right\}$$

for $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant n-1$, and

$$S_{\underline{a},\underline{b}} = \Big\{(v_1,\dots,v_n) \in \mathbb{F}((u))^n \;\Big|\; \alpha_i u^{a_i} v_{i+1} = \beta_i u^{b_i}\phi(v_i),\ v_1 = u^{v_u(v_1)} \text{ and } -v_u(v_{i+1}) > \min\{a_i, e-b_i\} \text{ for all } i\Big\}.$$

In the above definitions, v_i is on the i-th component. Clearly, all elements of $\bigcup_i S_{\underline{a},\underline{b},i} \cup \bigcup_{i,j} S_{\underline{a},\underline{b},i,j} \cup S_{\underline{a},\underline{b}}$ are in $\widetilde{N}_{\underline{a},\underline{b},\mathbb{F}}$.

LEMMA 2.1. — *The image of* $\bigcup_i S_{\underline{a},\underline{b},i} \cup \bigcup_{i,j} S_{\underline{a},\underline{b},i,j} \cup S_{\underline{a},\underline{b}}$ *in* $N_{\underline{a},\underline{b},\mathbb{F}}$ *forms an* $\mathbb{F}$*-basis of* $N_{\underline{a},\underline{b},\mathbb{F}}$.

Proof. — It is clear that the image of $\bigcup_i S_{\underline{a},\underline{b},i} \cup \bigcup_{i,j} S_{\underline{a},\underline{b},i,j} \cup S_{\underline{a},\underline{b}}$ in $N_{\underline{a},\underline{b},\mathbb{F}}$ are linearly independent over $\mathbb{F}$. So it suffices to show that $\bigcup_i S_{\underline{a},\underline{b},i} \cup \bigcup_{i,j} S_{\underline{a},\underline{b},i,j} \cup S_{\underline{a},\underline{b}}$ and $\mathbb{F}[[u]]^n$ generates $\widetilde{N}_{\underline{a},\underline{b},\mathbb{F}}$. We take $(v_1,\dots,v_n) \in \widetilde{N}_{\underline{a},\underline{b},\mathbb{F}}$. We want to write $(v_1,\dots,v_n)$ as a linear combination of elements of $\bigcup_i S_{\underline{a},\underline{b},i} \cup \bigcup_{i,j} S_{\underline{a},\underline{b},i,j} \cup S_{\underline{a},\underline{b}}$ and $\mathbb{F}[[u]]^n$.

First, we consider the case where there exsits an index i_0 such that $-v_u(v_{i_0}) > \min\{a_{i_0-1}, e-b_{i_0-1}, (e-a_{i_0})/p, b_{i_0}/p\}$. Then there are following two cases:

(i) There are $1 \leqslant i_1 \leqslant n$ and $1 \leqslant j_1 \leqslant n-1$ such that $i_0 \in [i_1, i_1+j_1]$, $1 \leqslant -v_u(v_{i_1}) \leqslant \min\{a_{i_1-1}, e-b_{i_1-1}\}$, $a_{i_1+l} + v_u(v_{i_1+l+1}) = b_{i_1+l} + pv_u(v_{i_1+l})$ and $-v_u(v_{i_1+l+1}) > \min\{a_{i+l}, e-b_{i+l}\}$ for $0 \leqslant l \leqslant j_1-1$ and $-v_u(v_{i_1+j_1}) \leqslant \min\{(e-a_{i_1+j_1})/p, (b_{i_1+j_1})/p\}$.

(ii) $a_i + v_u(v_{i+1}) = b_i + pv_u(v_i)$ and $-v_u(v_{i+1}) > \min\{a_i, e-b_i\}$ for all i.

In the case (i), we can subtract a linear multiple of an element of $S_{\underline{a},\underline{b},i_1,j_1}$ from $(v_1,\dots,v_n)$ so that the u-valuations of the i-th component increase for all $i \in [i_1, i_1+j_1]$. In the case (ii), we can subtract a linear multiple of an element of $S_{\underline{a},\underline{b}}$ from $(v_1,\dots,v_n)$ so that the u-valuations of the i-th component increase for all i.

Repeating such subtractions, we may assume that

$$-v_u(v_i) \leqslant \min\{a_{i-1}, e-b_{i-1}, (e-a_i)/p, b_i/p\}$$

f or all i. Then we can write $(v_1,\dots,v_n)$ as a linear combination of elements of $\bigcup_i S_{\underline{a},\underline{b},i}$ and $\mathbb{F}[[u]]^n$. □

By Lemma 2.1, we have $d_{\underline{a},\underline{b}} = \sum_i |S_{\underline{a},\underline{b},i}| + \sum_{i,j} |S_{\underline{a},\underline{b},i,j}| + |S_{\underline{a},\underline{b}}|$. We note that $0 \leqslant |S_{\underline{a},\underline{b}}| \leqslant 1$ by the definition, and put $d'_{\underline{a},\underline{b}} = \sum_i |S_{\underline{a},\underline{b},i}| + \sum_{i,j} |S_{\underline{a},\underline{b},i,j}|$.

We put

$$T_{\underline{a},\underline{b},i} = \left\{ m \in \mathbb{Z} \;\middle|\; \min\{a_{i-1}, e-b_{i-1}\} < pm + a_{i-1} - b_{i-1} \leqslant \min\left\{ \frac{e-a_i}{p}, \frac{b_i}{p} \right\} \right\},$$

and consider the map

$$\bigcup_{i+j=h} S_{\underline{a},\underline{b},i,j} \to T_{\underline{a},\underline{b},h};\ (v_{i'})_{1\leqslant i' \leqslant n} \mapsto -v_u(v_{h-1}).$$

We can easily check that this map is injective. So we have $\sum_{i+j=h} |S_{\underline{a},\underline{b},i,j}| \leqslant |T_{\underline{a},\underline{b},h}|$ and $d'_{\underline{a},\underline{b}} \leqslant \sum_{1\leqslant i\leqslant n} \big(|S_{\underline{a},\underline{b},i}| + |T_{\underline{a},\underline{b},i}|\big)$.

We take $(\underline{a}', \underline{b}') \in I$ such that $\sum_{1\leqslant i\leqslant n} \big(|S_{\underline{a}',\underline{b}',i}| + |T_{\underline{a}',\underline{b}',i}|\big)$ is the maximum.

LEMMA 2.2. — *$|T_{\underline{a}',\underline{b}',i}| \leqslant 1$ for all i.*

Proof. — We assume there is an index i_0 such that $|T_{\underline{a}',\underline{b}',i_0}| \geqslant 2$. We note that

$$(*) \qquad \min\{a'_{i_0-1}, e-b'_{i_0-1}\} + p + 1 \leqslant \min\left\{ \frac{e-a'_{i_0}}{p}, \frac{b'_{i_0}}{p} \right\}$$

by $|T_{\underline{a}',\underline{b}',i_0}| \geqslant 2$. We are going to show that we can replace a'_{i_0-1}, b'_{i_0-1} so that $\sum_{1\leqslant i\leqslant n} \big(|S_{\underline{a}',\underline{b}',i}| + |T_{\underline{a}',\underline{b}',i}|\big)$ increases. This contradicts the maximality of $\sum_{1\leqslant i\leqslant n} \big(|S_{\underline{a}',\underline{b}',i}| + |T_{\underline{a}',\underline{b}',i}|\big)$. We divide the problem into three cases.

Firstly, if $a'_{i_0-1} + 2 \leqslant e - b'_{i_0-1}$, we replace a'_{i_0-1} by $a'_{i_0-1} + p$, and note that $a'_{i_0-1} + p \leqslant e$ by $(*)$. Then there is no change except for $S_{\underline{a}',\underline{b}',i_0-1}$, $S_{\underline{a}',\underline{b}',i_0}$, $T_{\underline{a}',\underline{b}',i_0-1}$ and $T_{\underline{a}',\underline{b}',i_0}$. We can see that $|S_{\underline{a}',\underline{b}',i_0}|$ increases by at least 2. The condition that there exists $m \in \mathbb{Z}$ such that

$$\min\{a'_{i_0-1}, e-b'_{i_0-1}\} < pm + a'_{i_0-1} - b'_{i_0-1} \leqslant \min\{a'_{i_0-1} + p, e - b'_{i_0-1}\},$$

is equivalent to the condition that there exists $m \in \mathbb{Z}$ such that

$$\min\left\{ \frac{e-a'_{i_0-1}}{p}, \frac{b'_{i_0-1}}{p} \right\} < m \leqslant \min\left\{ \frac{e-a'_{i_0-1}}{p}, \frac{b'_{i_0-1}}{p} + 1 \right\},$$

and further equivalent to the condition that there does not exists $m \in \mathbb{Z}$ such that

$$\min\left\{ \frac{e-a'_{i_0-1}}{p} - 1, \frac{b'_{i_0-1}}{p} \right\} < m \leqslant \min\left\{ \frac{e-a'_{i_0-1}}{p}, \frac{b'_{i_0-1}}{p} \right\}.$$

If the above condition is satisfied, then $|S_{\underline{a}',\underline{b}',i_0-1}|$, $|T_{\underline{a}',\underline{b}',i_0-1}|$ do not change and $|T_{\underline{a}',\underline{b}',i_0}|$ decreases by 1. Otherwise, $|S_{\underline{a}',\underline{b}',i_0-1}| + |T_{\underline{a}',\underline{b}',i_0-1}|$ decreases by at most 1 and $|T_{\underline{a}',\underline{b}',i_0}|$ does not change. In both cases, we have that $\sum_{1\leqslant i\leqslant n} \big(|S_{\underline{a}',\underline{b}',i}| + |T_{\underline{a}',\underline{b}',i}|\big)$ increases by at least 1.

Secondly, if $a'_{i_0-1} \geqslant e - b'_{i_0-1} + 2$, we replace b'_{i_0-1} by $b'_{i_0-1} - p$. Then, by the same arguments, we have that $\sum_{1\leqslant i\leqslant n}\big(|S_{\underline{a}',\underline{b}',i}| + |T_{\underline{a}',\underline{b}',i}|\big)$ increases by at least 1.

In the remaining case, that is the case where $a'_{i_0-1} - 1 \leqslant e - b'_{i_0-1} \leqslant a'_{i_0-1} + 1$, we replace a'_{i_0-1}, b'_{i_0-1} by $a'_{i_0-1} + p$, $b'_{i_0-1} - p$ respectively, and note that $a'_{i_0-1} + p \leqslant e$ and $b'_{i_0-1} - p \geqslant 0$ by $(*)$. Then there is no change except for $S_{\underline{a}',\underline{b}',i_0-1}$, $S_{\underline{a}',\underline{b}',i_0}$, $T_{\underline{a}',\underline{b}',i_0-1}$ and $T_{\underline{a}',\underline{b}',i_0}$. We can see that $|S_{\underline{a}',\underline{b}',i_0-1}| + |T_{\underline{a}',\underline{b}',i_0-1}|$ decreases by at most 1, $|S_{\underline{a}',\underline{b}',i_0}|$ increases by p and $|T_{\underline{a}',\underline{b}',i_0}|$ decreases by 1. Hence $\sum_{1\leqslant i\leqslant n}\big(|S_{\underline{a}',\underline{b}',i}| + |T_{\underline{a}',\underline{b}',i}|\big)$ increases by at least $p - 2 > 0$.

Thus we have proved that $|T_{\underline{a}',\underline{b}',i}| \leqslant 1$ for all i. □

LEMMA 2.3. — *For all i, we have the followings:*

(A_i) *If $|S_{\underline{a}',\underline{b}',i}| + |T_{\underline{a}',\underline{b}',i}| = e_0 + l$ for $l \geqslant 1$, then*

$$|S_{\underline{a}',\underline{b}',i+1}| + |T_{\underline{a}',\underline{b}',i+1}| \leqslant e_0 + e_1 - pl + 1.$$

(B_i) *If $|S_{\underline{a}',\underline{b}',i}| + |T_{\underline{a}',\underline{b}',i}| = e_0 + 1$ and $|S_{\underline{a}',\underline{b}',i+1}| + |T_{\underline{a}',\underline{b}',i+1}| = e_0 + e_1 - p + 1$, then*

$$|S_{\underline{a}',\underline{b}',i+2}| + |T_{\underline{a}',\underline{b}',i+2}| \leqslant e_0 - (p-1)e_1 + 1.$$

Proof. — By the definition of $T_{\underline{a},\underline{b},i}$, we have

$$|T_{\underline{a},\underline{b},i}| \leqslant \max\left\{\min\left\{\frac{e-a_i}{p}, \frac{b_i}{p}\right\} - \min\{a_{i-1}, e - b_{i-1}\}, 0\right\}.$$

Combining this with the definition of $S_{\underline{a},\underline{b},i}$, we get

$$(\star) \qquad |S_{\underline{a},\underline{b},i}| + |T_{\underline{a},\underline{b},i}| \leqslant \min\left\{\left[\frac{e-a_i}{p}\right], \left[\frac{b_i}{p}\right]\right\},$$

and equality happens if and only if in the following two cases:

- $\min\left\{\left[\frac{e-a_i}{p}\right], \left[\frac{b_i}{p}\right]\right\} - \min\{a_{i-1}, e - b_{i-1}\} \leqslant 0.$
- $\min\left\{\left[\frac{e-a_i}{p}\right], \left[\frac{b_i}{p}\right]\right\} - \min\{a_{i-1}, e - b_{i-1}\} = 1$ and $p \mid \big(\min\{e - a_{i-1}, b_{i-1}\} + 1\big)$.

We assume $|S_{\underline{a}',\underline{b}',i_1}| + |T_{\underline{a}',\underline{b}',i_1}| = e_0 + l$ for some i_1 and $l \geqslant 1$. Then we have $p(e_0 + l) \leqslant \min\{e - a'_{i_1}, b'_{i_1}\}$ by $(\star)$. By this inequality, we have

$$\begin{aligned} |S_{\underline{a}',\underline{b}',i_1+1}| &\leqslant \min\{a'_{i_1}, e - b'_{i_1}\} \leqslant \max\{a'_{i_1}, e - b'_{i_1}\} \\ &= e - \min\{e - a'_{i_1}, b'_{i_1}\} \leqslant e - p(e_0 + l) = e_0 + e_1 - pl. \end{aligned}$$

Combining this with $|T_{\underline{a}',\underline{b}',i_1+1}| \leqslant 1$, we get

$$|S_{\underline{a}',\underline{b}',i_1+1}| + |T_{\underline{a}',\underline{b}',i_1+1}| \leqslant e_0 + e_1 - pl + 1.$$

This shows (A_i) for all i.

Further, we examine the case where equality holds in the above inequality, assuming $l = 1$. In this case, we have that $\min\{a'_{i_1}, e - b'_{i_1}\} = e_0 + e_1 - p$, $\min\{e - a'_{i_1}, b'_{i_1}\} = p(e_0 + 1)$ and $|T_{\underline{a}',\underline{b}',i_1+1}| = 1$. Let m be the unique element of $T_{\underline{a}',\underline{b}',i_1+1}$. Then, by the definition of $T_{\underline{a}',\underline{b}',i_1+1}$, we have

$$\min\left\{\frac{e - a'_{i_1+1}}{p}, \frac{b'_{i_1+1}}{p}\right\} - \min\{a'_{i_1}, e - b'_{i_1}\} \geqslant pm - \min\{e - a'_{i_1}, b'_{i_1}\} \geqslant p,$$

because $\min\{e - a'_{i_1}, b'_{i_1}\} = p(e_0 + 1)$ and $pm - \min\{e - a'_{i_1}, b'_{i_1}\} > 0$. Combining this with $\min\{a'_{i_1}, e - b'_{i_1}\} = e_0 + e_1 - p$, we get $p(e_0 + e_1) \leqslant \min\{e - a'_{i_1+1}, b'_{i_1+1}\}$. By the previous argument, we have

$$|S_{\underline{a}',\underline{b}',i_1+2}| + |T_{\underline{a}',\underline{b}',i_1+2}| \leqslant e_0 - (p-1)e_1 + 1.$$

Thus we have proved (B_i) for all i. □

We are going to show (2). Firstly, we treat (a). We note that $e_0 + e_1 - pl + 1 \leqslant e_0 - p(l-1) - 2$ in the case where $0 \leqslant e_1 \leqslant p - 3$, and that $e_0 + e_1 - pl + 1 \leqslant e_0 - p(l-1) - 1$ and $e_0 - (p-1)e_1 + 1 \leqslant e_0 - 1$ in the case where $e_1 = p - 2$. Then (A_i) and (B_i) for all i implies that $\sum_{1 \leqslant i \leqslant n} \big(|S_{\underline{a}',\underline{b}',i}| + |T_{\underline{a}',\underline{b}',i}|\big) \leqslant ne_0$. It further implies that

$$d'_{\underline{a},\underline{b}} \leqslant \sum_{1 \leqslant i \leqslant n} \big(|S_{\underline{a},\underline{b},i}| + |T_{\underline{a},\underline{b},i}|\big) \leqslant ne_0$$

for all $(\underline{a}, \underline{b}) \in I$, and that $d'_{\underline{a},\underline{b}} = ne_0$ only if $|S_{\underline{a},\underline{b},i}| + |T_{\underline{a},\underline{b},i}| = e_0$ for all i. To prove $d_{\underline{a},\underline{b}} \leqslant ne_0$, it suffice to show that $d'_{\underline{a},\underline{b}} = ne_0$ implies $S_{\underline{a},\underline{b}} = \emptyset$, because $|S_{\underline{a},\underline{b}}| \leqslant 1$ for all $(\underline{a}, \underline{b}) \in I$.

We assume that $d'_{\underline{a},\underline{b}} = ne_0$ and $S_{\underline{a},\underline{b}} \neq \emptyset$. By the maximality of

$$\sum_{1 \leqslant i \leqslant n} \big(|S_{\underline{a},\underline{b},i}| + |T_{\underline{a},\underline{b},i}|\big),$$

we have $|T_{\underline{a},\underline{b},i}| \leqslant 1$ for all i. Let $(v_{0,i})_{1 \leqslant i \leqslant n}$ be the unique element of $S_{\underline{a},\underline{b}}$, and we put $r_{0,i} = -v_u(v_{0,i})$. Then we have

$$a_i - r_{0,i+1} = b_i - pr_{0,i} < \max\{0, a_i + b_i - e\}$$

for all i, by the definition of $S_{\underline{a},\underline{b}}$. By $(\star)$ and $e_0 - 1 \leqslant |S_{\underline{a},\underline{b},i}|$ for all i, we have

$$e_0 - 1 \leqslant a_i \leqslant e_0 + e_1,\ pe_0 \leqslant b_i \leqslant pe_0 + e_1 + 1$$

for all i. Take an index i_2 such that r_{0,i_2} is the maximum. Then we have

$$\begin{aligned}(p-1)r_{0,i_2} \leqslant pr_{0,i_2} - r_{0,i_2+1} = b_{i_2} - a_{i_2} &\leqslant (pe_0 + e_1 + 1) - (e_0 - 1)\\ &= (p-1)e_0 + e_1 + 2 \leqslant (p-1)e_0 + p.\end{aligned}$$

So we get $r_{0,i} \leqslant e_0 + 1$ for all i.

If $a_i + b_i - e \leqslant 0$, we have $r_{0,i} \geqslant e_0 + 1$ by $b_i - pr_{0,i} < 0$ and $pe_0 \leqslant b_i$. If $a_i + b_i - e > 0$, we have $r_{0,i} \geqslant e_0 + 1$ by $b_i - pr_{0,i} < a_i + b_i - e$ and $a_i \leqslant e_0 + e_1$. So we have $r_{0,i} = e_0 + 1$ for all i.

By $a_i - r_{0,i+1} = b_i - pr_{0,i}$, we have $(p-1)(e_0+1) = b_i - a_i$ for all i. By the range of a_i and b_i, we have the following two possibilities for each i:

$$(a_i, b_i) = (e_0 - 1, pe_0 + p - 2) \text{ or } (e_0, pe_0 + p - 1).$$

In both cases, we have $|S_{\underline{a},\underline{b},i+1}| = e_0 - 1$.

Now we must have equality in $(\star)$. So we must have

$$p \mid (\min\{e - a_{i-1}, b_{i-1}\} + 1),$$

noting that $|T_{\underline{a},\underline{b},i}| = 1$. This contradicts the possibilities of a_{i-1}, b_{i-1}. Thus we have proved $d_{V_{\mathbb{F}}} \leqslant ne_0$.

For $\underline{a} = (e_0)_{1\leqslant i\leqslant n}$ and $\underline{b} = (pe_0)_{1\leqslant i\leqslant n}$, we have $d_{\underline{a},\underline{b}} \geqslant \sum_{1\leqslant i\leqslant n} |S_{\underline{a},\underline{b},i}| = ne_0$. This shows that $d_{V_{\mathbb{F}}} = ne_0$, if

$$M_{\mathbb{F}} \sim \left(\begin{pmatrix} u^{e_0} & 0 \\ 0 & u^{pe_0}\end{pmatrix}\right)_i.$$

Secondly, we treat (b). In this case, we note that $e_0 + e_1 - pl + 1 = e_0 - p(l-1)$ and $e_0 - (p-1)e_1 + 1 \leqslant e_0 - 3$. Then (A_i) and (B_i) for all i implies $d'_{\underline{a},\underline{b}} \leqslant ne_0$, and further implies $d_{\underline{a},\underline{b}} \leqslant ne_0 + 1$, because $|S_{\underline{a},\underline{b}}| \leqslant 1$. Thus we have proved $d_{V_{\mathbb{F}}} \leqslant ne_0 + 1$.

For $\underline{a} = (e_0)_{1\leqslant i\leqslant n}$ and $\underline{b} = (pe_0 + p - 1)_{1\leqslant i\leqslant n}$, we have

$$d_{\underline{a},\underline{b}} \geqslant \sum_{1\leqslant i\leqslant n} |S_{\underline{a},\underline{b},i}| + |S_{\underline{a},\underline{b}}| = ne_0 + 1,$$

because $(u^{-(e_0+1)})_{1\leqslant i\leqslant n} \in S_{\underline{a},\underline{b}}$. This shows that $d_{V_{\mathbb{F}}} = ne_0 + 1$, if

$$M_{\mathbb{F}} \sim \left(\begin{pmatrix} u^{e_0} & 0 \\ 0 & u^{pe_0+p-1}\end{pmatrix}\right)_i.$$

At last, we treat (c). In this case, we note that $e_0 + e_1 - pl + 1 = e_0 - p(l-1) + 1$ and $e_0 - (p-1)e_1 + 1 \leqslant e_0 - 5$. Then (A_i) and (B_i) for all i implies $d'_{\underline{a},\underline{b}} \leqslant ne_0 + [n/2]$, and that $d'_{\underline{a},\underline{b}} = ne_0 + [n/2]$ only if $e_0 \leqslant |S_{\underline{a},\underline{b},i}| + |T_{\underline{a},\underline{b},i}| \leqslant e_0 + 1$ for all i.

If $n = 1$, then $d'_{\underline{a},\underline{b}} \leqslant e_0$ implies $d_{\underline{a},\underline{b}} \leqslant e_0 + 1$, and the given example for $d_{V_{\mathbb{F}}} = e_0 + 1$ is the same as in (b). So we may assume $n \geqslant 2$ in the following.

To prove $d_{\underline{a},\underline{b}} \leqslant ne_0 + [n/2]$, it suffices to show that $d'_{\underline{a},\underline{b}} = ne_0 + [n/2]$ implies $S_{\underline{a},\underline{b}} = \emptyset$, because $|S_{\underline{a},\underline{b}}| \leqslant 1$ for all $(\underline{a},\underline{b}) \in I$.

We assume that $d'_{\underline{a},\underline{b}} = ne_0 + [n/2]$ and $S_{\underline{a},\underline{b}} \neq \emptyset$. By the maximality of $\sum_{1\leqslant i\leqslant n}\big(|S_{\underline{a},\underline{b},i}| + |T_{\underline{a},\underline{b},i}|\big)$, we have $|T_{\underline{a},\underline{b},i}| \leqslant 1$ for all i. Let $(v_{1,i})_{1\leqslant i\leqslant n}$ be the unique element of $S_{\underline{a},\underline{b}}$, and we put $r_{1,i} = -v_u(v_{1,i})$. Then we have

$$a_i - r_{1,i+1} = b_i - pr_{1,i} < \max\{0, a_i + b_i - e\}$$

for all i by the definition of $S_{\underline{a},\underline{b}}$. By $(\star)$ and $e_0 - 1 \leqslant |S_{\underline{a},\underline{b},i}|$, we have

$$e_0 - 1 \leqslant a_i \leqslant e_0 + p,\ pe_0 \leqslant b_i \leqslant pe_0 + p + 1$$

for all i. Take an index i_3 such that r_{1,i_3} is the maximum. Then we have

$$\begin{aligned}(p-1)r_{1,i_3} &\leqslant pr_{1,i_3} - r_{1,i_3+1} = b_{i_3} - a_{i_3}\\ &\leqslant (pe_0 + p + 1) - (e_0 - 1) = (p-1)e_0 + p + 2.\end{aligned}$$

So we get $r_{1,i} \leqslant e_0 + 2$ for all i.

If $a_i + b_i - e \leqslant 0$, we have $r_{1,i} \geqslant e_0 + 1$ by $b_i - pr_{1,i} < 0$ and $pe_0 \leqslant b_i$. If $a_i + b_i - e > 0$, we have $r_{1,i} \geqslant e_0 + 1$ by $b_i - pr_{1,i} < a_i + b_i - e$ and $a_i \leqslant e_0 + p$. So we have $e_0 + 1 \leqslant r_{1,i} \leqslant e_0 + 2$ for all i.

By $n \geqslant 2$, there is an index i_4 such that $|S_{\underline{a},\underline{b},i_4}| + |T_{\underline{a},\underline{b},i_4}| = e_0 + 1$. Then we have $e_0 + 1 \leqslant \min\big\{(e - a_{i_4})/p, b_{i_4}/p\big\}$ by $(\star)$. We are going to prove that if $e_0 + 1 \leqslant \min\big\{(e - a_i)/p, b_i/p\big\}$, then $|S_{\underline{a},\underline{b},i+1}| + |T_{\underline{a},\underline{b},i+1}| = e_0$ and $e_0 + 1 \leqslant \min\big\{(e - a_{i+1})/p, b_{i+1}/p\big\}$. If we have proved this claim, we have a contradiction by considering i_4.

We assume that $e_0 + 1 \leqslant \min\big\{(e - a_i)/p, b_i/p\big\}$. Then we have $e_0 - 1 \leqslant a_i \leqslant e_0$, $pe_0 + p \leqslant b_i \leqslant pe_0 + p + 1$ and $e_0 - 1 \leqslant |S_{\underline{a},\underline{b},i+1}| \leqslant e_0$. If $|S_{\underline{a},\underline{b},i+1}| = e_0$, we have $a_i = e_0$ and $b_i = pe_0 + p$. However, this contradicts $pr_i - r_{i+1} = b_i - a_i$, because $pr_i - r_{i+1} \neq (p-1)e_0 + p$ by $e_0 + 1 \leqslant r_i, r_{i+1} \leqslant e_0 + 2$. So we have $|S_{\underline{a},\underline{b},i+1}| = e_0 - 1$ and $|T_{\underline{a},\underline{b},i+1}| = 1$. Let m be the unique element of $T_{\underline{a},\underline{b},i+1}$. By the definition of $T_{\underline{a},\underline{b},i+1}$, we have

$$\min\left\{\frac{e - a_{i+1}}{p}, \frac{b_{i+1}}{p}\right\} - \min\{a_i, e - b_i\} \geqslant pm - \min\{e - a_i, b_i\} \geqslant p - 1 \geqslant 2,$$

because $pe_0 + p \leqslant \min\{e - a_i, b_i\} \leqslant pe_0 + p + 1$ and $pm - \min\{e - a_i, b_i\} > 0$. This shows $e_0 + 1 \leqslant \min\big\{(e - a_{i+1})/p, b_{i+1}/p\big\}$. Thus we have proved that $d_{V_{\mathbb{F}}} \leqslant ne_0 + [n/2]$.

For $\underline{a} = (e_{0,i})_{1 \leqslant i \leqslant n}$ and $\underline{b} = \big(p(2e_0 + 1 - e_{0,i})\big)_{1 \leqslant i \leqslant n}$, we have

$$d_{\underline{a},\underline{b}} \geqslant \sum_{1 \leqslant i \leqslant n} |S_{\underline{a},\underline{b},i}| = ne_0 + [n/2],$$

where $e_{0,i}$ is defined in the statement of Proposition 2.1(2)(c). This shows that $d_{V_{\mathbb{F}}} = ne_0 + [n/2]$, if

$$M_{\mathbb{F}} \sim \left(\begin{pmatrix} u^{e_{0,i}} & 0 \\ 0 & u^{p(2e_0+1-e_{0,i})} \end{pmatrix} \right)_i.$$

□

3. The case where $V_{\mathbb{F}}$ is absolutely irreducible

In this section, we give the maximum of the dimensions of the moduli spaces in the case where $V_{\mathbb{F}}$ is absolutely irreducible. In the proof of the following Proposition, three Lemmas appear.

PROPOSITION 3.1. — *We assume $V_{\mathbb{F}}$ is absolutely irreducible, and write $e = (p+1)e_0 + e_1$ for $e_0 \in \mathbb{Z}$ and $0 \leqslant e_1 \leqslant p$. Then the followings are true.*

(1) *There are $m_i \in \mathbb{Z}$ for $0 \leqslant i \leqslant d_{V_{\mathbb{F}}}$ such that $m_{d_{V_{\mathbb{F}}}} > 0$ and*

$$|\mathscr{GR}_{V_{\mathbb{F}},0}(\mathbb{F}')| = \sum_{i=0}^{d_{V_{\mathbb{F}}}} m_i |\mathbb{F}'|^i$$

for all sufficiently large extensions $\mathbb{F}'$ of $\mathbb{F}$.

(2) (a) *In the case $e_1 = 0$, we have $d_{V_{\mathbb{F}}} \leqslant ne_0 - 1$. In this case, if*

$$M_{\mathbb{F}} \sim \left(\begin{pmatrix} 0 & 1 \\ u^{(p+1)e_0-1} & 0 \end{pmatrix}, \begin{pmatrix} u^{e_0} & 0 \\ 0 & u^{pe_0} \end{pmatrix}, \dots, \begin{pmatrix} u^{e_0} & 0 \\ 0 & u^{pe_0} \end{pmatrix} \right),$$

then $d_{V_{\mathbb{F}}} = ne_0 - 1$.

(b) *In the case $1 \leqslant e_1 \leqslant p-1$, we have $d_{V_{\mathbb{F}}} \leqslant ne_0$. In this case, if*

$$M_{\mathbb{F}} \sim \left(\begin{pmatrix} 0 & 1 \\ u^{(p+1)e_0+1} & 0 \end{pmatrix}, \begin{pmatrix} u^{e_0} & 0 \\ 0 & u^{pe_0} \end{pmatrix}, \dots, \begin{pmatrix} u^{e_0} & 0 \\ 0 & u^{pe_0} \end{pmatrix} \right),$$

we have $d_{V_{\mathbb{F}}} = ne_0$.

(c) *In the case $e_1 = p$, we have $d_{V_{\mathbb{F}}} \leqslant ne_0 + [n/2]$. In this case, if*

$$M_{\mathbb{F}} \sim \left(\begin{pmatrix} 0 & 1 \\ u^{(p+1)e_0+1} & 0 \end{pmatrix}, \begin{pmatrix} u^{2e_0+1-e_{0,i}} & 0 \\ 0 & u^{pe_{0,i}} \end{pmatrix}_{2 \leqslant i \leqslant n} \right),$$

then $d_{V_{\mathbb{F}}} = ne_0 + [n/2]$. Here, $e_{0,i} = e_0$ if i is odd, and $e_{0,i} = e_0 + 1$ if i is even.

Proof. — Extending the field $\mathbb{F}$, we may assume that

$$M_{\mathbb{F}} \sim \left(\begin{pmatrix} 0 & \alpha_1 \\ \alpha_1 u^m & 0 \end{pmatrix}, \begin{pmatrix} \alpha_2 & 0 \\ 0 & \alpha_2 \end{pmatrix}, \dots, \begin{pmatrix} \alpha_n & 0 \\ 0 & \alpha_n \end{pmatrix} \right)$$

for some $\alpha_i \in \mathbb{F}^\times$ and a positive integer m such that $(q+1) \nmid m$, by Lemma 1.2. Let $\mathfrak{M}_{0,\mathbb{F}}$ be the lattice of $M_{\mathbb{F}}$ generated by the basis giving the above matrix expression.

For any finite extension $\mathbb{F}'$ of $\mathbb{F}$, we put $\mathfrak{M}_{0,\mathbb{F}'} = \mathfrak{M}_{0,\mathbb{F}} \otimes_{\mathbb{F}} \mathbb{F}'$ and $M_{\mathbb{F}'} = M_{\mathbb{F}} \otimes_{\mathbb{F}} \mathbb{F}'$. By the Iwasawa decomposition, any sublattice of $M_{\mathbb{F}'}$ can be written as $\left(\begin{pmatrix} u^{s_i} & v_i' \\ 0 & u^{t_i} \end{pmatrix} \right)_i \cdot \mathfrak{M}_{0,\mathbb{F}'}$ for $s_i, t_i \in \mathbb{Z}$ and $v_i' \in \mathbb{F}'((u))$.

We put

$$\mathscr{GR}_{V_{\mathbb{F}},0,\underline{a},\underline{b}}(\mathbb{F}') = \left\{ \left(\begin{pmatrix} u^{s_i} & v_i' \\ 0 & u^{t_i} \end{pmatrix} \right)_i \cdot \mathfrak{M}_{0,\mathbb{F}'} \in \mathscr{GR}_{V_{\mathbb{F}},0}(\mathbb{F}') \;\middle|\; s_i, t_i \in \mathbb{Z},\ v_i' \in \mathbb{F}'((u)), \right.$$
$$ps_1 - t_2 = a_1,\ m + pt_1 - s_2 = b_1,$$
$$\left. ps_j - s_{j+1} = a_j,\ pt_j - t_{j+1} = b_j \text{ for } 2 \leqslant j \leqslant n \right\}$$

for $(\underline{a}, \underline{b}) = \big((a_i)_{1 \leqslant i \leqslant n}, (b_i)_{1 \leqslant i \leqslant n}\big) \in \mathbb{Z}^n \times \mathbb{Z}^n$. Then we have

$$\mathscr{GR}_{V_{\mathbb{F}},0}(\mathbb{F}') = \bigcup_{(\underline{a},\underline{b}) \in \mathbb{Z}^n \times \mathbb{Z}^n} \mathscr{GR}_{V_{\mathbb{F}},0,\underline{a},\underline{b}}(\mathbb{F}')$$

and this is a disjoint union by Lemma 1.1. Later, in Lemma 3.1, we will show that there are only finitely many $(\underline{a}, \underline{b})$ such that $\mathscr{GR}_{V_{\mathbb{F}},0,\underline{a},\underline{b}}(\mathbb{F}') \neq \emptyset$.

We take

$$\left(\begin{pmatrix} u^{s_i} & v_i' \\ 0 & u^{t_i} \end{pmatrix} \right)_i \cdot \mathfrak{M}_{0,\mathbb{F}'} \in \mathscr{GR}_{V_{\mathbb{F}},0,\underline{a},\underline{b}}(\mathbb{F}'),$$

and put

$$\mathfrak{M}_{\underline{a},\underline{b},\mathbb{F}'} = \left(\begin{pmatrix} u^{s_i} & 0 \\ 0 & u^{t_i} \end{pmatrix} \right)_i \cdot \mathfrak{M}_{0,\mathbb{F}'}.$$

Then we have

$$\mathfrak{M}_{\underline{a},\underline{b},\mathbb{F}'} \sim \left(\alpha_1 \begin{pmatrix} 0 & u^{a_1} \\ u^{b_1} & 0 \end{pmatrix}, \alpha_2 \begin{pmatrix} u^{a_2} & 0 \\ 0 & u^{b_2} \end{pmatrix}, \dots, \alpha_n \begin{pmatrix} u^{a_n} & 0 \\ 0 & u^{b_n} \end{pmatrix} \right)$$

with respect to the basis induced from $\mathfrak{M}_{0,\mathbb{F}'}$.

Now, any $\mathfrak{M}_{\mathbb{F}'} \in \mathscr{GR}_{V_{\mathbb{F}},0,\underline{a},\underline{b}}(\mathbb{F}')$ can be written as $\left(\begin{pmatrix}1 & v_i\\ 0 & 1\end{pmatrix}\right)_i \cdot \mathfrak{M}_{\underline{a},\underline{b},\mathbb{F}'}$ for some $(v_i)_{1\leqslant i\leqslant n} \in \mathbb{F}'((u))^n$, and we put $r_i = -v_u(v_i)$. We may assume $r_i \geqslant 0$, replacing v_i so that $v_i \notin u\mathbb{F}'[[u]]$ without changing the $(k[[u]]\otimes_{\mathbb{F}_p}\mathbb{F}')$-module $\left(\begin{pmatrix}1 & v_i\\ 0 & 1\end{pmatrix}\right)_i \cdot \mathfrak{M}_{\underline{a},\underline{b},\mathbb{F}'}$ by Lemma 1.1. Then we have

$$\mathfrak{M}_{\mathbb{F}'} \sim \left(\alpha_1\begin{pmatrix}\phi(v_1)u^{b_1} & u^{a_1}-\phi(v_1)v_2u^{b_1}\\ u^{b_1} & -v_2u^{b_1}\end{pmatrix}, \alpha_i\begin{pmatrix}u^{a_i} & \phi(v_i)u^{b_i}-v_{i+1}u^{a_i}\\ 0 & u^{b_i}\end{pmatrix}_{2\leqslant i\leqslant n}\right)$$

with respect to the induced basis, and

$$\begin{pmatrix}\phi(v_1)u^{b_1} & u^{a_1}-\phi(v_1)v_2u^{b_1}\\ u^{b_1} & -v_2u^{b_1}\end{pmatrix} = \begin{pmatrix}\phi(v_1)u^{b_1} & u^{a_1}\\ u^{b_1} & 0\end{pmatrix}\begin{pmatrix}1 & -v_2\\ 0 & 1\end{pmatrix}$$
$$= \begin{pmatrix}v_2^{-1}u^{a_1} & u^{a_1}-\phi(v_1)v_2u^{b_1}\\ 0 & -v_2u^{b_1}\end{pmatrix}\begin{pmatrix}1 & 0\\ -v_2^{-1} & 1\end{pmatrix}.$$

Then the condition $u^e\mathfrak{M}_{\mathbb{F}'} \subset (1\otimes\phi)\big(\phi^*(\mathfrak{M}_{\mathbb{F}'})\big) \subset \mathfrak{M}_{\mathbb{F}'}$ is equivalent to

$$(C_1)\qquad \begin{array}{l} 0 \leqslant a_1 + r_2 \leqslant e,\ 0 \leqslant b_1 - r_2 \leqslant e,\\ \qquad v_u\big(u^{a_1} - \phi(v_1)v_2u^{b_1}\big) \geqslant \max\{0, a_1+b_1-e\},\end{array}$$

$$(C_2)\qquad \begin{array}{l} 0 \leqslant a_i \leqslant e,\ 0 \leqslant b_i \leqslant e,\\ \qquad v_u\big(\phi(v_i)u^{b_i} - v_{i+1}u^{a_i}\big) \geqslant \max\{0, a_i+b_i-e\} \text{ for } 2 \leqslant i \leqslant n.\end{array}$$

We show the following fact:
($\diamondsuit$) If $\mathscr{GR}_{V_{\mathbb{F}},0,\underline{a},\underline{b}}(\mathbb{F}') \neq \emptyset$, there does not exist $(r'_i)_{1\leqslant i\leqslant n} \in \mathbb{Z}^n$ such that $a_1 = b_1 - pr'_1 - r'_2$ and $a_i - r'_{i+1} = b_i - pr'_i$ for $2 \leqslant i \leqslant n$.

We assume that there exists $(r'_i)_{1\leqslant i\leqslant n} \in \mathbb{Z}^n$ satisfying this condition. Changing the basis of $\mathfrak{M}_{\underline{a},\underline{b},\mathbb{F}'}$ by $\left(\begin{pmatrix}1 & u^{-r'_i}\\ 0 & 1\end{pmatrix}\right)_i$, we get

$$M_{\mathbb{F}'} \sim \left(\alpha_1\begin{pmatrix}u^{b_1-pr'_1} & 0\\ u^{b_1} & -u^{b_1-r'_2}\end{pmatrix}, \alpha_i\begin{pmatrix}u^{a_i} & 0\\ 0 & u^{b_i}\end{pmatrix}_{2\leqslant i\leqslant n}\right).$$

This contradicts that $V_{\mathbb{F}}$ is absolutely irreducible.

LEMMA 3.1. — *If $\mathscr{GR}_{V_{\mathbb{F}},0,\underline{a},\underline{b}}(\mathbb{F}') \neq \emptyset$, then*

$$-\frac{e}{p-1} \leqslant a_1 \leqslant e,\ 0 \leqslant b_1 \leqslant \frac{pe}{p-1} \text{ and } 0 \leqslant a_i, b_i \leqslant e \text{ for } 2 \leqslant i \leqslant n.$$

Proof. — We take $\mathfrak{M}_{\mathbb{F}'} \in \mathscr{GR}_{V_{\mathbb{F}},0,\underline{a},\underline{b}}(\mathbb{F}')$ and write it as $\left(\begin{pmatrix}1 & v_i\\ 0 & 1\end{pmatrix}\right)_i \cdot \mathfrak{M}_{\underline{a},\underline{b},\mathbb{F}'}$ for some $(v_i)_{1\leqslant i\leqslant n} \in \mathbb{F}'((u))^n$. We put $r_i = -v_u(v_i)$. We may assume $r_i \geqslant 0$ by Lemma 1.1.

If $r_2 > e/(p-1)$, we have that $a_i - r_{i+1} = b_i - pr_i < 0$ for $2 \leqslant i \leqslant n$ and $r_i > e/(p-1)$ for all i by the condition (C_2), and that $a_1 = b_1 - pr_1 - r_2 < 0$ by the condition (C_1). This contradicts $(\diamondsuit)$, and we have $r_2 \leqslant e/(p-1)$.

Then (C_1) and (C_2) shows the claim. □

To examine $|\mathscr{GR}_{V_{\mathbb{F}},0,\underline{a},\underline{b}}(\mathbb{F}')|$, we consider the case where $0 \leqslant a_1 \leqslant e$ and $0 \leqslant b_1 \leqslant e$, and the case where $\max\{-a_1, b_1 - e\} > 0$.

First, we treat the case where $0 \leqslant a_1 \leqslant e$ and $0 \leqslant b_1 \leqslant e$. In this case, the condition $u^e\mathfrak{M}_{\mathbb{F}'} \subset (1\otimes\phi)\big(\phi^*(\mathfrak{M}_{\mathbb{F}'})\big) \subset \mathfrak{M}_{\mathbb{F}'}$ is equivalent to the condition that $\max\{pr_1 + r_2, pr_1, r_2\} \leqslant \min\{e - a_1, b_1\}$ and (C_2). We put

$$I_{\underline{a},\underline{b}} = \big\{(R_1, R_2) \in \mathbb{Z}\times\mathbb{Z} \;\big|\; pR_1 + R_2 \leqslant \min\{e - a_1, b_1\},\ R_1, R_2 \geqslant 0\big\}$$

and

$$\mathscr{GR}_{V_{\mathbb{F}},0,\underline{a},\underline{b},R_1,R_2}(\mathbb{F}') = \left\{\left(\begin{pmatrix}1 & v_i\\ 0 & 1\end{pmatrix}\right)_i \cdot \mathfrak{M}_{\underline{a},\underline{b},\mathbb{F}'} \in \mathscr{GR}_{V_{\mathbb{F}},0,\underline{a},\underline{b}}(\mathbb{F}') \;\middle|\; v_i \in \mathbb{F}'((u)), r_1 = R_1,\ r_2 = R_2\right\}$$

for $(R_1, R_2) \in I_{\underline{a},\underline{b}}$. Then we have a disjoint union

$$\mathscr{GR}_{V_{\mathbb{F}},0,\underline{a},\underline{b}}(\mathbb{F}') = \bigcup_{(R_1,R_2)\in I_{\underline{a},\underline{b}}} \mathscr{GR}_{V_{\mathbb{F}},0,\underline{a},\underline{b},R_1,R_2}(\mathbb{F}')$$

by Lemma 1.1.

We fix $(R_1, R_2) \in I_{\underline{a},\underline{b}}$. Then the condition that $r_1 = R_1$ and $r_2 = R_2$ implies $\max\{pr_1 + r_2, pr_1, r_2\} \leqslant \min\{e - a_1, b_1\}$. So $\left(\begin{pmatrix}1 & v_i\\ 0 & 1\end{pmatrix}\right)_i \cdot \mathfrak{M}_{\underline{a},\underline{b},\mathbb{F}'}$ gives a point of $\mathscr{GR}_{V_{\mathbb{F}},0,\underline{a},\underline{b},R_1,R_2}(\mathbb{F}')$ if and only if

$$\max\{r_1, 0\} = R_1,\ \max\{r_2, 0\} = R_2 \text{ and } (C_2).$$

We assume $\mathscr{GR}_{V_{\mathbb{F}},0,\underline{a},\underline{b},R_1,R_2}(\mathbb{F}') \neq \emptyset$. Considering $-v_u(v_i)$ for $(v_i)_{1\leqslant i\leqslant n}$ that gives a point of $\mathscr{GR}_{V_{\mathbb{F}},0,\underline{a},\underline{b},R_1,R_2}(\mathbb{F}')$, we have the following two cases:

(i) There are $2 \leqslant n_2 < n_1 \leqslant n+1$ and $R_i \in \mathbb{Z}$ for $3 \leqslant i \leqslant n_2$ and $n_1 \leqslant i \leqslant n$ such that

$$a_i - R_{i+1} = b_i - pR_i < \max\{0, a_i + b_i - e\}$$

for $2 \leqslant i \leqslant n_2 - 1$ and $n_1 \leqslant i \leqslant n$, and

$$R_{n_1} \leqslant \min\{a_{n_1-1}, e - b_{n_1-1}\},\ R_{n_2} \leqslant \min\left\{\frac{e - a_{n_2}}{p}, \frac{b_{n_2}}{p}\right\}.$$

(ii) There are $R_i \in \mathbb{Z}$ for $3 \leqslant i \leqslant n$ such that

$$a_i - R_{i+1} = b_i - pR_i < \max\{0, a_i + b_i - e\}$$

for $2 \leqslant i \leqslant n$.

We note that (ii) includes the case $n = 1$.

We define an $\mathbb{F}'$-vector space $\widetilde{N}_{\underline{a},\underline{b},R_1,R_2,\mathbb{F}'}$ by

$$\widetilde{N}_{\underline{a},\underline{b},R_1,R_2,\mathbb{F}'} = \left\{(v_i)_{1\leqslant i\leqslant n} \in \mathbb{F}'((u))^n \;\middle|\; r_1 \leqslant R_1,\ r_2 \leqslant R_2 \text{ and } (C_2)\right\}.$$

We note that $\widetilde{N}_{\underline{a},\underline{b},R_1,R_2,\mathbb{F}'} \supset \mathbb{F}'[[u]]^n$. We put

$$N_{\underline{a},\underline{b},R_1,R_2,\mathbb{F}'} = \widetilde{N}_{\underline{a},\underline{b},R_1,R_2,\mathbb{F}'}/\mathbb{F}'[[u]]^n$$

and $d_{\underline{a},\underline{b},R_1,R_2} = \dim_{\mathbb{F}'} N_{\underline{a},\underline{b},R_1,R_2,\mathbb{F}'}$. We note that $\dim_{\mathbb{F}'} N_{\underline{a},\underline{b},R_1,R_2,\mathbb{F}'}$ is independent of finite extensions $\mathbb{F}'$ of $\mathbb{F}$. We put

$$\widetilde{N}^{\circ}_{\underline{a},\underline{b},R_1,R_2,\mathbb{F}'} = \left\{(v_i)_{1\leqslant i\leqslant n} \in \widetilde{N}_{\underline{a},\underline{b},R_1,R_2,\mathbb{F}'} \;\middle|\; r_1 = R_1,\ r_2 = R_2\right\}.$$

Let $N^{\circ}_{\underline{a},\underline{b},R_1,R_2,\mathbb{F}'}$ be the image of $\widetilde{N}^{\circ}_{\underline{a},\underline{b},R_1,R_2,\mathbb{F}'}$ in $N_{\underline{a},\underline{b},R_1,R_2,\mathbb{F}'}$. Then we have a bijection

$$N^{\circ}_{\underline{a},\underline{b},R_1,R_2,\mathbb{F}'} \to \mathscr{GR}_{V_{\mathbb{F}},0,\underline{a},\underline{b},R_1,R_2}(\mathbb{F}')$$

by Lemma 1.1. By choosing a basis of $N_{\underline{a},\underline{b},R_1,R_2,\mathbb{F}}$ over $\mathbb{F}$, we have a morphism

$$f_{\underline{a},\underline{b},R_1,R_2} : \mathbb{A}_{\mathbb{F}}^{d_{\underline{a},\underline{b},R_1,R_2}} \to \mathscr{GR}_{V_{\mathbb{F}},0}$$

in the case $R_1 = R_2 = 0$,

$$f_{\underline{a},\underline{b},R_1,R_2} : \mathbb{A}_{\mathbb{F}}^{(d_{\underline{a},\underline{b},R_1,R_2}-2)} \times \mathbb{G}_{m,\mathbb{F}}^2 \to \mathscr{GR}_{V_{\mathbb{F}},0}$$

in the case where $R_1 > 0$, $R_2 > 0$ and (i) holds true, and

$$f_{\underline{a},\underline{b},R_1,R_2} : \mathbb{A}_{\mathbb{F}}^{(d_{\underline{a},\underline{b},R_1,R_2}-1)} \times \mathbb{G}_{m,\mathbb{F}} \to \mathscr{GR}_{V_{\mathbb{F}},0}$$

in the other case, such that $f_{\underline{a},\underline{b},R_1,R_2}(\mathbb{F}')$ is injective and the image of $f_{\underline{a},\underline{b},R_1,R_2}(\mathbb{F}')$ is $\mathscr{GR}_{V_{\mathbb{F}},0,\underline{a},\underline{b},R_1,R_2}(\mathbb{F}')$.

LEMMA 3.2. — *If $0 \leqslant a_1 \leqslant e$ and $0 \leqslant b_1 \leqslant e$, the following hold:*

(a) *In the case $e_1 = 0$, we have $d_{\underline{a},\underline{b},R_1,R_2} \leqslant ne_0 - 1$. In this case, if $a_1 = 0$, $b_1 = (p+1)e_0 - 1$, $a_i = e_0$ and $b_i = pe_0$ for $2 \leqslant i \leqslant n$, then there exists $(R_1, R_2) \in I_{\underline{a},\underline{b}}$ such that $d_{\underline{a},\underline{b},R_1,R_2} = ne_0 - 1$.*

(b) *In the case $1 \leqslant e_1 \leqslant p-1$, we have $d_{\underline{a},\underline{b},R_1,R_2} \leqslant ne_0$. In this case, if $a_1 = 0$, $b_1 = (p+1)e_0 + 1$, $a_i = e_0$ and $b_i = pe_0$ for $2 \leqslant i \leqslant n$, then there exists $(R_1, R_2) \in I_{\underline{a},\underline{b}}$ such that $d_{\underline{a},\underline{b},R_1,R_2} = ne_0$.*

(c) *In the case $e_1 = p$, we have $d_{\underline{a},\underline{b},R_1,R_2} \leqslant ne_0 + [n/2]$. In this case, if $a_1 = 0$, $b_1 = (p+1)e_0 + 1$, $a_i = 2e_0 + 1 - e_{0,i}$ and $b_i = pe_{0,i}$ for $2 \leqslant i \leqslant n$, then there exists $(R_1, R_2) \in I_{\underline{a},\underline{b}}$ such that $d_{\underline{a},\underline{b},R_1,R_2} = ne_0 + [n/2]$. Here, $e_{0,i} = e_0$ if i is odd, and $e_{0,i} = e_0 + 1$ if i is even.*

Proof. — First, we treat the case $n = 1$. In this case, we have

$$R_1 = R_2 \leqslant \left[\frac{\min\{e - a_1, b_1\}}{p+1}\right] \leqslant e_0.$$

So we get $d_{\underline{a},\underline{b},R_1,R_2} \leqslant e_0$ for $(\underline{a},\underline{b}) \in \mathbb{Z}^n \times \mathbb{Z}^n$ and $(R_1, R_2) \in I_{\underline{a},\underline{b}}$ such that $\mathscr{GR}_{V_{\mathbb{F}},0,\underline{a},\underline{b},R_1,R_2}(\mathbb{F}') \neq \emptyset$ and $0 \leqslant a_1, b_1 \leqslant e$. We have to eliminate the possibility of equality in the case $e_1 = 0$. In this case, if we have $d_{\underline{a},\underline{b},R_1,R_2} = e_0$, then $a_1 = 0$ and $b_1 = (p+1)e_0$. This contradicts $(\diamondsuit)$.

We can check that if $e_1 = 0$, $a_1 = 0$, $b_1 = e - 1$ and $R_1 = R_2 = e_0 - 1$, then $d_{\underline{a},\underline{b},R_1,R_2} = e_0 - 1$, and that if $e_1 \neq 0$, $a_1 = 0$, $b_1 = (p+1)e_0 + 1$ and $R_1 = R_2 = e_0$, then $d_{\underline{a},\underline{b},R_1,R_2} = e_0$.

So we may assume $n \geqslant 2$. We put

$$S_{\underline{a},\underline{b},R_1,R_2,1} = \left\{(u^{-r_1}, 0, \dots, 0) \in \mathbb{F}((u))^n \;\middle|\; 1 \leqslant r_1 \leqslant \min\{R_1, a_n, e - b_n\}\right\},$$

$$S_{\underline{a},\underline{b},R_1,R_2,2} = \left\{(0, u^{-r_2}, 0, \dots, 0) \in \mathbb{F}((u))^n \;\middle|\; 1 \leqslant r_2 \leqslant \min\left\{R_2, \frac{e - a_2}{p}, \frac{b_2}{p}\right\}\right\},$$

$$S_{\underline{a},\underline{b},R_1,R_2,i} = \left\{(0, \dots, 0, v_i, 0, \dots, 0) \in \mathbb{F}((u))^n \;\middle|\; v_i = u^{-r_i}, \; \min\left\{a_{i-1}, e - b_{i-1}, \frac{e - a_i}{p}, \frac{b_i}{p}\right\}\right\}$$

for $3 \leqslant i \leqslant n$, and

$$S_{\underline{a},\underline{b},R_1,R_2,i,j} = \left\{(0, \dots, 0, v_i, v_{i+1}, \dots, v_{j+1}, 0, \dots, 0) \in \mathbb{F}((u))^n \;\middle|\; v_i = u^{-r_i}, \; r_i \leqslant \min\{a_{i-1}, e - b_{i-1}\} \text{ if } i \neq 2, \; r_2 \leqslant R_2 \text{ if } i = 2, \; u^{a_l} v_{l+1} = u^{b_l}\phi(v_l) \text{ and } -v_u(v_{l+1}) > \min\{a_l, e - b_l\} \text{ for } i \leqslant l \leqslant j, \; -v_u(v_{j+1}) \leqslant \min\left\{\frac{e - a_{j+1}}{p}, \frac{b_{j+1}}{p}\right\} \text{ if } j \neq n, \; -v_u(v_1) \leqslant R_1 \text{ if } j = n\right\}$$

for $2 \leqslant i \leqslant j \leqslant n$. In the above definitions, v_i is on the i-th component. Then, as in the proof of Lemma 2.1, we can check that $\bigcup_i S_{\underline{a},\underline{b},R_1,R_2,i} \cup \bigcup_{i,j} S_{\underline{a},\underline{b},R_1,R_2,i,j}$ is an $\mathbb{F}$-basis of $N_{\underline{a},\underline{b},R_1,R_2,\mathbb{F}}$. So we have $d_{\underline{a},\underline{b},R_1,R_2} = \sum_i |S_{\underline{a},\underline{b},R_1,R_2,i}| + \sum_{i,j} |S_{\underline{a},\underline{b},R_1,R_2,i,j}|$.

We put

$$T_{\underline{a},\underline{b},R_1,R_2,1} = \big\{m \in \mathbb{Z} \,\big|\, \min\{a_n, e - b_n\} < pm + a_n - b_n \leqslant R_1\big\},$$

$$T_{\underline{a},\underline{b},R_1,R_2,2} = \emptyset \quad \text{and}$$

$$T_{\underline{a},\underline{b},R_1,R_2,i} =$$

$$\left\{m \in \mathbb{Z} \,\middle|\, \min\{a_{i-1}, e - b_{i-1}\} < pm + a_{i-1} - b_{i-1} \leqslant \min\left\{\frac{e - a_i}{p}, \frac{b_i}{p}\right\}\right\}$$

for $3 \leqslant i \leqslant n$. We consider the map

$$\bigcup_{2 \leqslant i \leqslant h-1} S_{\underline{a},\underline{b},R_1,R_2,i,h-1} \to T_{\underline{a},\underline{b},R_1,R_2,h};\ (v_{i'})_{1 \leqslant i' \leqslant n} \mapsto -v_u(v_{h-1})$$

for $3 \leqslant h \leqslant n+1$. We can easily check that this map is injective. So we have

$$\sum_{2 \leqslant i \leqslant h-1} |S_{\underline{a},\underline{b},R_1,R_2,i,h-1}| \leqslant |T_{\underline{a},\underline{b},R_1,R_2,h}|$$

and

$$d_{\underline{a},\underline{b},R_1,R_2} \leqslant \sum_{1 \leqslant i \leqslant n} \big(|S_{\underline{a},\underline{b},R_1,R_2,i}| + |T_{\underline{a},\underline{b},R_1,R_2,i}|\big).$$

We take $(\underline{a}', \underline{b}') \in \mathbb{Z}^n \times \mathbb{Z}^n$ and $(R_1', R_2') \in I_{\underline{a}',\underline{b}'}$ such that $0 \leqslant a_1', b_1' \leqslant e$ and $\sum_{1 \leqslant i \leqslant n} \big(|S_{\underline{a}',\underline{b}',R_1',R_2',i}| + |T_{\underline{a}',\underline{b}',R_1',R_2',i}|\big)$ is the maximum. We can prove that $|T_{\underline{a}',\underline{b}',R_1',R_2',i}| \leqslant 1$ for all i as in the proof of Lemma 2.2.

We can also show that

(A_i) if $|S_{\underline{a}',\underline{b}',R_1',R_2',i}| + |T_{\underline{a}',\underline{b}',R_1',R_2',i}| = e_0 + l$ for $l \geqslant 1$, then

$$|S_{\underline{a}',\underline{b}',R_1',R_2',i+1}| + |T_{\underline{a}',\underline{b}',R_1',R_2',i+1}| \leqslant e_0 + e_1 - pl + 1$$

for $i \neq 1$, and that

(B_i) if $|S_{\underline{a}',\underline{b}',R_1',R_2',i}| + |T_{\underline{a}',\underline{b}',R_1',R_2',i}| = e_0 + 1$ and $|S_{\underline{a}',\underline{b}',R_1',R_2',i+1}| + |T_{\underline{a}',\underline{b}',R_1',R_2',i+1}| = e_0 + e_1 - p + 1$, then

$$|S_{\underline{a}',\underline{b}',R_1',R_2',i+2}| + |T_{\underline{a}',\underline{b}',R_1',R_2',i+2}| \leqslant e_0 - (p-1)e_1 + 1$$

for $2 \leqslant i \leqslant n-1$ as in the proof of Lemma 2.3. By the same argument, we can show that

(A_1) if $|S_{\underline{a}',\underline{b}',R_1',R_2',1}| + |T_{\underline{a}',\underline{b}',R_1',R_2',1}| = e_0 + l$ for $l \geqslant 1$, then

$$|S_{\underline{a}',\underline{b}',R_1',R_2',2}| + |T_{\underline{a}',\underline{b}',R_1',R_2',2}| \leqslant e_0 + e_1 - pl,$$

and that

(B_n) if $|S_{\underline{a}',\underline{b}',R_1',R_2',n}| + |T_{\underline{a}',\underline{b}',R_1',R_2',n}| = e_0 + 1$ and $|S_{\underline{a}',\underline{b}',R_1',R_2',1}| + |T_{\underline{a}',\underline{b}',R_1',R_2',1}| = e_0 + e_1 - p + 1$, then

$$|S_{\underline{a}',\underline{b}',R_1',R_2',2}| + |T_{\underline{a}',\underline{b}',R_1',R_2',2}| \leqslant e_0 - (p-1)e_1,$$

using the followings:

$$|S_{\underline{a}',\underline{b}',R_1',R_2',1}| + |T_{\underline{a}',\underline{b}',R_1',R_2',1}| \leqslant R_1,\ pR_1 + R_2 \leqslant e, |S_{\underline{a}',\underline{b}',R_1',R_2',2}| \leqslant R_2$$

and $T_{\underline{a}',\underline{b}',R_1',R_2',2} = \emptyset$.

Firstly, we treat the case where $0 \leqslant e_1 \leqslant p-1$, that is, (a) or (b). We note that $e_0 + e_1 - pl + 1 \leqslant e_0 - p(l-1) - 1$ in the case $0 \leqslant e_1 \leqslant p-2$, and that $e_0 + e_1 - pl + 1 = e_0 - p(l-1)$ and $e_0 - (p-1)e_1 + 1 \leqslant e_0 - 3$ in the case $e_1 = p-1$. Then (A_i) for all i and (B_i) for $i \neq 1$ implies

$$\begin{aligned} d_{\underline{a},\underline{b},R_1,R_2} &\leqslant \sum_{1 \leqslant i \leqslant n} \left(|S_{\underline{a},\underline{b},R_1,R_2,i}| + |T_{\underline{a},\underline{b},R_1,R_2,i}|\right) \\ &\leqslant \sum_{1 \leqslant i \leqslant n} \left(|S_{\underline{a}',\underline{b}',R_1',R_2',i}| + |T_{\underline{a}',\underline{b}',R_1',R_2',i}|\right) \leqslant ne_0 \end{aligned}$$

for $(\underline{a},\underline{b}) \in \mathbb{Z}^n \times \mathbb{Z}^n$ and $(R_1,R_2) \in I_{\underline{a},\underline{b}}$ such that $\mathscr{GR}_{V_{\mathbb{F}},0,\underline{a},\underline{b},R_1,R_2}(\mathbb{F}') \neq \emptyset$ and $0 \leqslant a_1, b_1 \leqslant e$. So we get the desired bound, if $1 \leqslant e_1 \leqslant p-1$. In the case $e_1 = 0$, we have to eliminate the possibility of equality. In this case, if we have equality, we get that $\sum_{1 \leqslant i \leqslant n}\left(|S_{\underline{a},\underline{b},R_1,R_2,i}| + |T_{\underline{a},\underline{b},R_1,R_2,i}|\right)$ is the maximum and $\left(|S_{\underline{a},\underline{b},R_1,R_2,i}| + |T_{\underline{a},\underline{b},R_1,R_2,i}|\right) = e_0$ for all i by (A_i) for all i. Then we have

$$R_1 = R_2 = e_0,\ e_0 - 1 \leqslant a_i \leqslant e_0,\ pe_0 \leqslant b_i \leqslant pe_0 + 1 \text{ for } 2 \leqslant i \leqslant n$$

by the followings:

$$\begin{aligned} pR_1 + R_2 = e, |S_{\underline{a},\underline{b},R_1,R_2,1}| + |T_{\underline{a},\underline{b},R_1,R_2,1}| \leqslant R_1, |S_{\underline{a},\underline{b},R_1,R_2,2}| \leqslant R_2, \\ |S_{\underline{a},\underline{b},R_1,R_2,i}| + |T_{\underline{a},\underline{b},R_1,R_2,i}| \leqslant \min\{(e-a_i)/p, b_i/p\} \text{ for } 2 \leqslant i \leqslant n \end{aligned}$$

and

$$|S_{\underline{a},\underline{b},R_1,R_2,i}| \geqslant e_0 - 1 \text{ for } i \neq 2.$$

Now we have $a_1 = 0$ and $b_1 = (p+1)e_0$ by $R_1 = R_2 = e_0$. We show that $|T_{\underline{a},\underline{b},R_1,R_2,i}| = 0$ for $3 \leqslant i \leqslant n$. We assume that $|T_{\underline{a},\underline{b},R_1,R_2,i_0}| = 1$ for some $i_0 \neq 1,2$, and let m be the unique element of $T_{\underline{a},\underline{b},R_1,R_2,i_0}$. Then, by the definition of $T_{\underline{a},\underline{b},R_1,R_2,i_0}$, we have

$$\begin{aligned} \min\left\{\frac{e-a_{i_0}}{p}, \frac{b_{i_0}}{p}\right\} - \min\{a_{i_0-1}, e - b_{i_0-1}\} &\geqslant pm - \min\{e - a_{i_0-1}, b_{i_0-1}\} \\ &\geqslant p-1 \geqslant 2, \end{aligned}$$

because

$$pe_0 \leqslant \min\{e - a_{i_0-1}, b_{i_0-1}\} \leqslant pe_0 + 1$$

and

$$pm - \min\{e - a_{i_0-1}, b_{i_0-1}\} > 0.$$

This contradicts the possibilities of a_{i_0-1}, a_{i_0}, b_{i_0-1} and b_{i_0}. The same argument shows that $|T_{\underline{a},\underline{b},R_1,R_2,1}| = 0$. Now we have $|S_{\underline{a},\underline{b},R_1,R_2,i}| = e_0$ for all i, and that

$$a_1 = 0,\ b_1 = (p+1)e_0,\ a_i = e_0,\ b_i = pe_0 \text{ for } 2 \leqslant i \leqslant n.$$

Then we have

$$a_1 = b_1 - pr'_1 - r'_2 \text{ and } a_i - r'_{i+1} = b_i - pr'_i \text{ for } 2 \leqslant i \leqslant n$$

for $(r'_i)_{1\leqslant i\leqslant n} = (e_0)_{1\leqslant i\leqslant n}$. This contradicts $(\diamondsuit)$. So we have $d_{\underline{a},\underline{b},R_1,R_2} \leqslant ne_0 - 1$, if $e_1 = 0$.

We can check that if $e_1 = 0$, $a_1 = 0$, $b_1 = (p+1)e_0 - 1$, $R_1 = e_0$, $R_2 = e_0 - 1$, $a_i = e_0$ and $b_i = pe_0$ for $2 \leqslant i \leqslant n$, then $d_{\underline{a},\underline{b},R_1,R_2} \geqslant \sum_{1\leqslant i\leqslant n} |S_{\underline{a},\underline{b},R_1,R_2,i}| = ne_0 - 1$. We can check also that if $1 \leqslant e_1 \leqslant p-1$, $a_1 = 0$, $b_1 = (p+1)e_0 + 1$, $R_1 = e_0$, $R_2 = e_0 + 1$, $a_i = e_0$ and $b_i = pe_0$ for $2 \leqslant i \leqslant n$, then $d_{\underline{a},\underline{b},R_1,R_2} \geqslant \sum_{1\leqslant i\leqslant n} |S_{\underline{a},\underline{b},R_1,R_2,i}| = ne_0$.

Secondly, we treat (c). In this case, we note that $e_0 + e_1 - pl + 1 = e_0 - p(l-1) + 1$ and $e_0 - (p-1)e_1 + 1 \leqslant e_0 - 5$. Then (A_i) for all i and (B_i) for $i \neq 1$ implies

$$\begin{aligned} d_{\underline{a},\underline{b},R_1,R_2} &\leqslant \sum_{1\leqslant i\leqslant n} \big(|S_{\underline{a},\underline{b},R_1,R_2,i}| + |T_{\underline{a},\underline{b},R_1,R_2,i}|\big) \\ &\leqslant \sum_{1\leqslant i\leqslant n} \big(|S_{\underline{a}',\underline{b}',R'_1,R'_2,i}| + |T_{\underline{a}',\underline{b}',R'_1,R'_2,i}|\big) \leqslant ne_0 + \left[\frac{n}{2}\right] \end{aligned}$$

for $(\underline{a},\underline{b}) \in \mathbb{Z}^n \times \mathbb{Z}^n$ and $(R_1,R_2) \in I_{\underline{a},\underline{b}}$ such that $0 \leqslant a_1, b_1 \leqslant e$. So we get the desired bound.

We can check that if $e_1 = p$, $a_1 = 0$, $b_1 = (p+1)e_0 + 1$, $R_1 = e_0$, $R_2 = e_0 + 1$, $a_i = 2e_0 + 1 - e_{0,i}$ and $b_i = pe_{0,i}$ for $2 \leqslant i \leqslant n$, then $d_{\underline{a},\underline{b},R_1,R_2} \geqslant \sum_{1\leqslant i\leqslant n} |S_{\underline{a},\underline{b},R_1,R_2,i}| = ne_0 + [n/2]$. □

Next, we consider the remaining case, that is, the case where

$$\max\{-a_1, b_1 - e\} > 0.$$

In this case, $v_u\big(u^{a_1} - \phi(v_1)v_2u^{b_1}\big) \geqslant \max\{0, a_1 + b_1 - e\}$ implies $pr_1 + r_2 = b_1 - a_1$, because $a_1 < \max\{0, a_1 + b_1 - e\}$. So the condition $u^e\mathfrak{M}_{\mathbb{F}'} \subset (1\otimes\phi)\big(\phi^*(\mathfrak{M}_{\mathbb{F}'})\big) \subset \mathfrak{M}_{\mathbb{F}'}$ implies

$$pr_1 + r_2 = b_1 - a_1,\ \max\{-a_1, b_1 - e\} \leqslant r_2 \leqslant \min\{e - a_1, b_1\}.$$

We note that if $n = 1$, then $pr_1 + r_2 = b_1 - a_1$ contradicts $(\diamondsuit)$ because $r_1 = r_2$. So we may assume $n \geqslant 2$. We put

$$I_{\underline{a},\underline{b}} = \Big\{(R_1, R_2) \in \mathbb{Z} \times \mathbb{Z} \;\Big|\; pR_1 + R_2 = b_1 - a_1, \\ \max\{-a_1, b_1 - e\} \leqslant R_2 \leqslant \min\{e - a_1, b_1\}\Big\}$$

and $m_{\underline{a},\underline{b}} = \big[\big(\max\{-a_1, b_1 - e\} - 1\big)/p\big]$. We note that $R_1 \geqslant m_{\underline{a},\underline{b}} + 1 > 0$ and $R_2 \geqslant \max\{-a_1, b_1 - e\} > 0$. We put

$$\mathscr{GR}_{V_{\mathbb{F}},0,\underline{a},\underline{b},R_1,R_2}(\mathbb{F}') = \\ \left\{\left(\begin{pmatrix}1 & v_i \\ 0 & 1\end{pmatrix}\right)_i \cdot \mathfrak{M}_{\underline{a},\underline{b},\mathbb{F}'} \in \mathscr{GR}_{V_{\mathbb{F}},0,\underline{a},\underline{b}}(\mathbb{F}') \;\middle|\; v_i \in \mathbb{F}'((u)), \\ v_u(v_1) = -R_1,\ v_u(v_2) = -R_2\right\}$$

for $(R_1, R_2) \in I_{\underline{a},\underline{b}}$. Then we have a disjoint union

$$\mathscr{GR}_{V_{\mathbb{F}},0,\underline{a},\underline{b}}(\mathbb{F}') = \bigcup_{(R_1,R_2)\in I_{\underline{a},\underline{b}}} \mathscr{GR}_{V_{\mathbb{F}},0,\underline{a},\underline{b},R_1,R_2}(\mathbb{F}')$$

by Lemma 1.1. Extending the field $\mathbb{F}$, we may assume that $\mathscr{GR}_{V_{\mathbb{F}},0,\underline{a},\underline{b},R_1,R_2}(\mathbb{F}') \neq \emptyset$ if and only if $\mathscr{GR}_{V_{\mathbb{F}},0,\underline{a},\underline{b},R_1,R_2}(\mathbb{F}) \neq \emptyset$ for each $(R_1, R_2) \in I_{\underline{a},\underline{b}}$, $(\underline{a}, \underline{b}) \in \mathbb{Z}^n \times \mathbb{Z}^n$ and any finite extension $\mathbb{F}'$ of $\mathbb{F}$.

We fix $(R_1, R_2) \in I_{\underline{a},\underline{b}}$, and assume $\mathscr{GR}_{V_{\mathbb{F}},0,\underline{a},\underline{b},R_1,R_2}(\mathbb{F}) \neq \emptyset$. If $v_u(v_1) = -R_1$ and $v_u(v_2) = -R_2$, the condition

$$v_u\big(u^{a_1} - \phi(v_1)v_2u^{b_1}\big) \geqslant \max\{0, a_1 + b_1 - e\}$$

is equivalent to the following: There uniquely exist $\gamma_{1,0}, \gamma_{2,0} \in (\mathbb{F}')^\times$ and $\gamma_{1,i}, \gamma_{2,i} \in \mathbb{F}'$ for $1 \leqslant i \leqslant m_{\underline{a},\underline{b}}$ such that

$$-v_u\left(v_1 - \sum_{0\leqslant i\leqslant m_{\underline{a},\underline{b}}} \gamma_{1,i}u^{-R_1+i}\right) \leqslant R_1 - m_{\underline{a},\underline{b}} - 1,$$

$$-v_u\left(v_2 - \sum_{0\leqslant i\leqslant m_{\underline{a},\underline{b}}} \gamma_{2,i}u^{-R_2+pi}\right) \leqslant R_2 - \max\{-a_1, b_1 - e\},$$

$$\gamma_{1,0}\gamma_{2,0} = 1,\ \sum_{0\leqslant i\leqslant l} \gamma_{1,i}\gamma_{2,l-i} = 0 \text{ for } 1 \leqslant l \leqslant m_{\underline{a},\underline{b}}.$$

We note that $(\gamma_{1,i})_{0\leqslant i\leqslant m_{\underline{a},\underline{b}}}$ determines $(\gamma_{1,i}, \gamma_{2,i})_{0\leqslant i\leqslant m_{\underline{a},\underline{b}}}$.

We prove that for $0 \leqslant i \leqslant m_{\underline{a},\underline{b}}$ there uniquely exist $2 \leqslant n_{2,i} < n_{1,i} \leqslant n+1$, $r_{1,i,j} \in \mathbb{Q}$ for $n_{1,i} \leqslant j \leqslant n+1$ and $r_{2,i,j} \in \mathbb{Z}$ for $2 \leqslant j \leqslant n_{2,i}$ such that $r_{1,0,j} \in \mathbb{Z}$ for $n_{1,0} \leqslant j \leqslant n+1$ and

$$\begin{aligned}
a_j - r_{1,i,j+1} &= b_j - pr_{1,i,j} < \max\{0, a_j + b_j - e\} \text{ for } n_{1,i} \leqslant j \leqslant n,\\
r_{1,i,n+1} &= R_1 - i,\ r_{1,i,n_{1,i}} \leqslant \min\{a_{n_{1,i}-1}, e - b_{n_{1,i}-1}\},\\
a_j - r_{2,i,j+1} &= b_j - pr_{2,i,j} < \max\{0, a_j + b_j - e\} \text{ for } 2 \leqslant j \leqslant n_{2,i} - 1,\\
r_{2,i,2} &= R_2 - pi,\ r_{2,i,n_{2,i}} \leqslant \min\left\{\frac{e - a_{n_{2,i}}}{p}, \frac{b_{n_{2,i}}}{p}\right\}.
\end{aligned}$$

Define $r_{1,i,j} \in \mathbb{Q}$ for $2 \leqslant j \leqslant n+1$ and $r_{2,i,j} \in \mathbb{Z}$ for $2 \leqslant j \leqslant n+1$ such that

$$\begin{aligned}
r_{1,i,n+1} &= R_1 - i,\ a_j - r_{1,i,j+1} = b_j - pr_{1,i,j} \text{ for } 2 \leqslant j \leqslant n,\\
r_{2,i,2} &= R_2 - pi,\ a_j - r_{2,i,j+1} = b_j - pr_{2,i,j} \text{ for } 2 \leqslant j \leqslant n.
\end{aligned}$$

We put

$$\begin{aligned}
n_{1,i} &= \max\Big\{\big\{3 \leqslant j \leqslant n+1 \;\big|\; r_{1,i,j} \leqslant \min\{a_{j-1}, e - b_{j-1}\}\big\} \cup \{2\}\Big\},\\
n_{2,i} &= \min\left\{\left\{2 \leqslant j \leqslant n \;\middle|\; r_{2,i,j} \leqslant \min\left\{\frac{e - a_j}{p}, \frac{b_j}{p}\right\}\right\} \cup \{n+1\}\right\}.
\end{aligned}$$

We consider $(v_i)_{1 \leqslant i \leqslant n}$ that gives a point of $\mathscr{GR}_{V_{\mathbb{F}},0,\underline{a},\underline{b},R_1,R_2}(\mathbb{F})$. Then we have $r_{1,0,j} = -v_u(v_j) \in \mathbb{Z}$ for $n_{1,0} \leqslant j \leqslant n+1$ and $r_{2,0,j} = -v_u(v_j) \in \mathbb{Z}$ for $2 \leqslant j \leqslant n_{2,0}$. It remains to show that $n_{2,i} < n_{1,i}$. We have $n_{2,i} \leqslant n_{2,0}$ and $n_{1,0} \leqslant n_{1,i}$, because $r_{1,i,j} \leqslant r_{1,0,j}$ and $r_{2,i,j} \leqslant r_{2,0,j}$ for $2 \leqslant j \leqslant n+1$. So it suffices to show $n_{2,0,j} < n_{1,0,j}$. If $n_{2,0,j} \geqslant n_{1,0,j}$, we have

$$a_1 = b_1 - pv_u(v_1) - v_u(v_2) \text{ and } a_j - v_u(v_{j+1}) = b_j - v_u(v_j) \text{ for } 2 \leqslant j \leqslant n,$$

and this contradicts $(\diamondsuit)$.

We put

$$M_{\underline{a},\underline{b},R_1,R_2} = \big\{0 \leqslant i \leqslant m_{\underline{a},\underline{b}} \;\big|\; r_{1,i,j} \in \mathbb{Z} \text{ for } n_{1,i} \leqslant j \leqslant n+1\big\}.$$

For $(v_i)_{1 \leqslant i \leqslant n}$ that gives a point of $\mathscr{GR}_{V_{\mathbb{F}},0,\underline{a},\underline{b},R_1,R_2}(\mathbb{F}')$, we take $\gamma_{1,i}$, $\gamma_{2,i}$ and $n_{1,i}$, $n_{2,i}$, $r_{1,i,j}$, $r_{2,i,j}$ as above. We note that $\gamma_{1,i} = 0$ if $i \notin M_{\underline{a},\underline{b},R_1,R_2}$. We put

$$\begin{aligned}
M_{1,\underline{a},\underline{b},R_1,R_2,j} &= \big\{0 \leqslant i \leqslant m_{\underline{a},\underline{b}} \;\big|\; n_{1,i} \leqslant j \leqslant n+1\big\},\\
M_{2,\underline{a},\underline{b},R_1,R_2,j} &= \big\{0 \leqslant i \leqslant m_{\underline{a},\underline{b}} \;\big|\; 2 \leqslant j \leqslant n_{2,i}\big\}
\end{aligned}$$

for $2 \leqslant j \leqslant n+1$, and define $(v_i^*)_{1 \leqslant i \leqslant n} \in \mathbb{F}'((u))^n$ by

$$v_j^* = v_j - \sum_{i \in M_{1,\underline{a},\underline{b},R_1,R_2,j}} \gamma_{1,i} u^{-r_{1,i,j}} - \sum_{i \in M_{2,\underline{a},\underline{b},R_1,R_2,j}} \gamma_{2,i} u^{-r_{2,i,j}}$$

for $2 \leqslant j \leqslant n+1$. This is well-defined by the above remark. We put

$$\widetilde{N}^*_{\underline{a},\underline{b},R_1,R_2,\mathbb{F}'} = \big\{(v_i^*)_{1\leqslant i\leqslant n} \in \mathbb{F}'((u))^n \;\big|\; (v_i)_{1\leqslant i\leqslant n} \in \mathbb{F}'((u))^n \text{ gives a point of } \mathscr{GR}_{V_{\mathbb{F}},0,\underline{a},\underline{b},R_1,R_2}(\mathbb{F}')\big\}.$$

Then we have

$$\widetilde{N}^*_{\underline{a},\underline{b},R_1,R_2,\mathbb{F}'} = \big\{(v_i)_{1\leqslant i\leqslant n} \in \mathbb{F}'((u))^n \;\big|\; -v_u(v_1) \leqslant R_1 - m_{\underline{a},\underline{b}} - 1, \; -v_u(v_2) \leqslant R_2 - \max\{-a_1, b_1 - e\}, (C_2)\big\}$$

by the construction of $(v_i^*)_{1\leqslant i\leqslant n}$ and the conditions (C_1) and (C_2). This implies that $\widetilde{N}^*_{\underline{a},\underline{b},R_1,R_2,\mathbb{F}'} \subset \mathbb{F}'((u))^n$ is an $\mathbb{F}'$-vector subspace, and

$$\widetilde{N}^*_{\underline{a},\underline{b},R_1,R_2,\mathbb{F}'} \supset \mathbb{F}'[[u]]^n.$$

We put

$$N^*_{\underline{a},\underline{b},R_1,R_2,\mathbb{F}'} = \widetilde{N}^*_{\underline{a},\underline{b},R_1,R_2,\mathbb{F}'}/\mathbb{F}'[[u]]^n$$

and $d^*_{\underline{a},\underline{b},R_1,R_2} = \dim_{\mathbb{F}'} N^*_{\underline{a},\underline{b},R_1,R_2,\mathbb{F}'}$. We note that $\dim_{\mathbb{F}'} N^*_{\underline{a},\underline{b},R_1,R_2,\mathbb{F}'}$ is independent of finite extensions $\mathbb{F}'$ of $\mathbb{F}$. By Lemma 1.1, giving an element of $N^*_{\underline{a},\underline{b},R_1,R_2,\mathbb{F}'}$ and $(\gamma_{1,i})_{0\leqslant i\leqslant m_{\underline{a},\underline{b}}}$ such that $\gamma_{1,0} \neq 0$ and $\gamma_{1,i} = 0$ if $i \notin M_{\underline{a},\underline{b},R_1,R_2}$ is equivalent to giving a point of $\mathscr{GR}_{V_{\mathbb{F}},0,\underline{a},\underline{b},R_1,R_2}(\mathbb{F}')$. By choosing a basis of $N^*_{\underline{a},\underline{b},R_1,R_2,\mathbb{F}}$ over $\mathbb{F}$, we have a morphism

$$f_{\underline{a},\underline{b},R_1,R_2} : \mathbb{A}_{\mathbb{F}}^{\left(d^*_{\underline{a},\underline{b},R_1,R_2} + |M_{\underline{a},\underline{b},R_1,R_2}| - 1\right)} \times \mathbb{G}_{m,\mathbb{F}} \to \mathscr{GR}_{V_{\mathbb{F}},0}$$

such that $f_{\underline{a},\underline{b},R_1,R_2}(\mathbb{F}')$ is injective and the image of $f_{\underline{a},\underline{b},R_1,R_2}(\mathbb{F}')$ is equal to $\mathscr{GR}_{V_{\mathbb{F}},0,\underline{a},\underline{b},R_1,R_2}(\mathbb{F}')$. We put $d_{\underline{a},\underline{b},R_1,R_2} = d^*_{\underline{a},\underline{b},R_1,R_2} + |M_{\underline{a},\underline{b},R_1,R_2}|$. Then we have (1) and

$$d_{V_{\mathbb{F}}} = \max_{\mathscr{GR}_{V_{\mathbb{F}},0,\underline{a},\underline{b},R_1,R_2}(\mathbb{F})\neq\emptyset} \big\{d_{\underline{a},\underline{b},R_1,R_2}\big\}.$$

In this maximum, we consider all $(\underline{a},\underline{b}) \in \mathbb{Z}^n \times \mathbb{Z}^n$. We have already examined $d_{\underline{a},\underline{b},R_1,R_2}$ for $(\underline{a},\underline{b})$ such that $a_1 \geqslant 0$ and $b_1 \leqslant e$. So it suffices to bound $d_{\underline{a},\underline{b},R_1,R_2}$ for $(\underline{a},\underline{b})$ such that $\max\{-a_1, b_1 - e\} > 0$.

LEMMA 3.3. — *If $\max\{-a_1, b_1 - e\} > 0$, the followings hold:*

(a) *In the case $e_1 = 0$, we have $d_{\underline{a},\underline{b},R_1,R_2} \leqslant ne_0 - 1$.*
(b) *In the case $1 \leqslant e_1 \leqslant p-1$, we have $d_{\underline{a},\underline{b},R_1,R_2} \leqslant ne_0$.*
(c) *In the case $e_1 = p$, we have $d_{\underline{a},\underline{b},R_1,R_2} \leqslant ne_0 + [n/2]$.*

Proof. — We put

$$S_{\underline{a},\underline{b},R_1,R_2,1} = \big\{(v_1,0,\dots,0) \in \mathbb{F}((u))^n \;\big|\; v_1 = u^{-r_1}, \\ 1 \leqslant r_1 \leqslant \min\{R_1 - m_{\underline{a},\underline{b}} - 1, a_n, e - b_n\}\big\},$$

$$S_{\underline{a},\underline{b},R_1,R_2,2} = \left\{(0,v_2,0,\dots,0) \in \mathbb{F}((u))^n \;\middle|\; v_2 = u^{-r_2}, \right. \\ \left. 1 \leqslant r_2 \leqslant \min\left\{R_2 - \max\{-a_1, b_1 - e\}, \frac{e-a_2}{p}, \frac{b_2}{p}\right\}\right\},$$

$$S_{\underline{a},\underline{b},R_1,R_2,i} = \left\{(0,\dots,0,v_i,0,\dots,0) \in \mathbb{F}((u))^n \;\middle|\; v_i = u^{-r_i}, \right. \\ \left. 1 \leqslant r_i \leqslant \min\left\{a_{i-1}, e - b_{i-1}, \frac{e-a_i}{p}, \frac{b_i}{p}\right\}\right\}$$

for $3 \leqslant i \leqslant n$, and

$$S_{\underline{a},\underline{b},R_1,R_2,i,j} = \\ \left\{(0,\dots,0,v_i,v_{i+1},\dots,v_{j+1},0,\dots,0) \in \mathbb{F}((u))^n \;\middle|\; v_i = u^{-r_i}, \right. \\ r_i \leqslant \min\{a_{i-1}, e - b_{i-1}\} \text{ if } i \neq 2,\ r_2 \leqslant R_2 - \max\{-a_1, b_1 - e\} \text{ if } i = 2, \\ u^{a_l} v_{l+1} = u^{b_l}\phi(v_l) \text{ and } -v_u(v_{l+1}) > \min\{a_l, e - b_l\} \text{ for } i \leqslant l \leqslant j, \\ -v_u(v_{j+1}) \leqslant \min\left\{\frac{e-a_{j+1}}{p}, \frac{b_{j+1}}{p}\right\} \text{ if } j \neq n, \\ \left. -v_u(v_1) \leqslant R_1 - m_{\underline{a},\underline{b}} - 1 \text{ if } j = n\right\}$$

for $2 \leqslant i \leqslant j \leqslant n$. In the above definitions, v_i is on the i-th component. Then, as in the proof of Lemma 2.1, we can check that $\bigcup_i S_{\underline{a},\underline{b},R_1,R_2,i} \cup \bigcup_{i,j} S_{\underline{a},\underline{b},R_1,R_2,i,j}$ is an $\mathbb{F}$-basis of $N^*_{\underline{a},\underline{b},R_1,R_2,\mathbb{F}}$. So we have $d^*_{\underline{a},\underline{b},R_1,R_2} = \sum_i |S_{\underline{a},\underline{b},R_1,R_2,i}| + \sum_{i,j} |S_{\underline{a},\underline{b},R_1,R_2,i,j}|$.

We put

$$T_{\underline{a},\underline{b},R_1,R_2,1} = \big\{m \in \mathbb{Z} \;\big|\; \min\{a_n, e - b_n\} < pm + a_n - b_n \leqslant R_1 - m_{\underline{a},\underline{b}} - 1\big\},$$

$$T_{\underline{a},\underline{b},R_1,R_2,2} = \\ \left\{m \in \mathbb{Z} \;\middle|\; R_2 - \max\{-a_1, b_1 - e\} < R_2 - pm \leqslant \min\left\{R_2, \frac{e-a_2}{p}, \frac{b_2}{p}\right\}\right\}$$

and

$$T_{\underline{a},\underline{b},R_1,R_2,i} = \left\{ m \in \mathbb{Z} \;\middle|\; \min\{a_{i-1}, e-b_{i-1}\} < pm + a_{i-1} - b_{i-1} \leqslant \min\left\{ \frac{e-a_i}{p}, \frac{b_i}{p} \right\} \right\}$$

for $3 \leqslant i \leqslant n$. We note that these definitions for $S_{\underline{a},\underline{b},R_1,R_2,i}$, $S_{\underline{a},\underline{b},R_1,R_2,i,j}$ and $T_{\underline{a},\underline{b},R_1,R_2,i}$ in the case $\max\{-a_1, b_1-e\} > 0$ are compatible with the definitions in the case $\max\{-a_1, b_1-e\} \leqslant 0$, if $\max\{-a_1, b_1-e\} = 0$. So in the following, we can consider also the case $\max\{-a_1, b_1-e\} = 0$. We need to consider this case in the following arguments.

We consider the map

$$\bigcup_{2\leqslant j\leqslant h-1} S_{\underline{a},\underline{b},R_1,R_2,j,h-1} \cup \left\{0 \leqslant i \leqslant m_{\underline{a},\underline{b}} \;\middle|\; n_{2,i} = h\right\} \to T_{\underline{a},\underline{b},R_1,R_2,h};$$
$$(v_i)_{1\leqslant i\leqslant n} \mapsto -v_u(v_{h-1}),\ i \mapsto r_{2,i,h-1}$$

for $3 \leqslant h \leqslant n+1$. We can easily check that this map is injective and that

$$\left\{0 \leqslant i \leqslant m_{\underline{a},\underline{b}} \;\middle|\; n_{2,i} = 2\right\} = T_{\underline{a},\underline{b},R_1,R_2,2}.$$

So we have $\left(\sum_{2\leqslant ß\leqslant j\leqslant n} |S_{\underline{a},\underline{b},R_1,R_2,i,j}|\right) + m_{\underline{a},\underline{b}} + 1 \leqslant \sum_{1\leqslant i\leqslant n} |T_{\underline{a},\underline{b},R_1,R_2,i}|$ and

$$d_{\underline{a},\underline{b},R_1,R_2} \leqslant d^*_{\underline{a},\underline{b},R_1,R_2} + m_{\underline{a},\underline{b}} + 1 \leqslant \sum_{1\leqslant i\leqslant n} \left(|S_{\underline{a},\underline{b},R_1,R_2,i}| + |T_{\underline{a},\underline{b},R_1,R_2,i}|\right).$$

We take $(\underline{a}'', \underline{b}'') \in \mathbb{Z}^n \times \mathbb{Z}^n$ and $(R_1'', R_2'') \in I_{\underline{a}'',\underline{b}''}$ such that $\max\{-a_1'', e-b_1''\} \geqslant 0$ and $\sum_{1\leqslant i\leqslant n}\left(|S_{\underline{a}'',\underline{b}'',R_1'',R_2'',i}| + |T_{\underline{a}'',\underline{b}'',R_1'',R_2'',i}|\right)$ is the maximum. We can prove that $|T_{\underline{a}'',\underline{b}'',R_1'',R_2'',i}| \leqslant 1$ for all $i \neq 2$ as in the proof of Lemma 2.2.

We show that we may take $(\underline{a}'', \underline{b}'') \in \mathbb{Z}^n \times \mathbb{Z}^n$ and $(R_1'', R_2'') \in I_{\underline{a}'',\underline{b}''}$ such that $0 \leqslant -a_1'' = b_1'' - e \leqslant p-1$. If $-a_1'' > b_1'' - e$, then we replace b_1'' by $b_1''+1$ and R_2'' by $R_2''+1$. We again have $(R_1'', R_2'') \in I_{\underline{a}'',\underline{b}''}$ after the replacement. This replacement increases $\sum_{1\leqslant i\leqslant n}\left(|S_{\underline{a}'',\underline{b}'',R_1'',R_2'',i}| + |T_{\underline{a}'',\underline{b}'',R_1'',R_2'',i}|\right)$ by 0 or 1, but by the maximality there is no case where it increases by 1. Similarly, if $-a_1'' < b_1'' - e$, we may replace a_1'' by $a_1''-1$ and R_2'' by $R_2''+1$. So we may assume $-a_1'' = b_1'' - e$.

If $-a_1'' \geqslant p$ and $\min\{b_2''/p, (e-a_2'')/p\} \geqslant R_2''$, we replace R_1'' by $R_1''-1$ and R_2'' by $R_2''+p$. By

$$R_2'' + p \leqslant \frac{e}{p} + p < e + p \leqslant e - a_1'' = b_1'',$$

we again have $(R_1'', R_2'') \in I_{\underline{a}'',\underline{b}''}$ after the replacement. This replacement increases $\sum_{1\leqslant i\leqslant n}\big(|S_{\underline{a}'',\underline{b}'',R_1'',R_2'',i}| + |T_{\underline{a}'',\underline{b}'',R_1'',R_2'',i}|\big)$ by at least $p-2$. This is a contradiction. So if $-a_1'' \geqslant p$, we have $\min\{b_2''/p, (e-a_2'')/p\} < R_2''$. If $-a_1'' \geqslant p$, we replace a_1'' by $a_1''+p$, b_1'' by $b_1''-p$, R_1'' by $R_1''-1$ and R_2'' by $R_2''-p$. We again have $(R_1'', R_2'') \in I_{\underline{a}'',\underline{b}''}$ after the replacement. This replacement does not change $\sum_{1\leqslant i\leqslant n}\big(|S_{\underline{a}'',\underline{b}'',R_1'',R_2'',i}| + |T_{\underline{a}'',\underline{b}'',R_1'',R_2'',i}|\big)$. Iterating these replacements, we may assume $0 \leqslant -a_1'' = b_1'' - e \leqslant p-1$. We already treated the case where $-a_1'' = b_1'' - e = 0$. So we may assume $1 \leqslant -a_1'' = b_1'' - e \leqslant p-1$. We note that $|T_{\underline{a}'',\underline{b}'',R_1'',R_2'',2}| \leqslant 1$ in this case.

Now we can show that

(A_i') if $|S_{\underline{a}'',\underline{b}'',R_1'',R_2'',i}| + |T_{\underline{a}'',\underline{b}'',R_1'',R_2'',i}| = e_0 + l$ for $l \geqslant 1$, then

$$|S_{\underline{a}'',\underline{b}'',R_1'',R_2'',i+1}| + |T_{\underline{a}'',\underline{b}'',R_1'',R_2'',i+1}| \leqslant e_0 + e_1 - pl + 1$$

for $i \neq 1$, and that

(B_i') if $|S_{\underline{a}'',\underline{b}'',R_1'',R_2'',i}| + |T_{\underline{a}'',\underline{b}'',R_1'',R_2'',i}| = e_0 + 1$ and $|S_{\underline{a}'',\underline{b}'',R_1'',R_2'',i+1}| + |T_{\underline{a}'',\underline{b}'',R_1'',R_2'',i+1}| = e_0 + e_1 - p + 1$, then

$$|S_{\underline{a}'',\underline{b}'',R_1'',R_2'',i+2}| + |T_{\underline{a}'',\underline{b}'',R_1'',R_2'',i+2}| \leqslant e_0 - (p-1)e_1 + 1$$

for $2 \leqslant i \leqslant n-1$ as in the proof of Lemma 2.3. By the same argument, we can show that

(A_1') if $|S_{\underline{a}'',\underline{b}'',R_1'',R_2'',1}| + |T_{\underline{a}'',\underline{b}'',R_1'',R_2'',1}| = e_0 + l$ for $l \geqslant 0$, then

$$|S_{\underline{a}'',\underline{b}'',R_1'',R_2'',2}| + |T_{\underline{a}'',\underline{b}'',R_1'',R_2'',2}| \leqslant e_0 + e_1 - pl,$$

and that

(B_n') if $|S_{\underline{a}'',\underline{b}'',R_1'',R_2'',n}| + |T_{\underline{a}'',\underline{b}'',R_1'',R_2'',n}| = e_0 + 1$ and $|S_{\underline{a}'',\underline{b}'',R_1'',R_2'',1}| + |T_{\underline{a}'',\underline{b}'',R_1'',R_2'',1}| = e_0 + e_1 - p + 1$, then

$$|S_{\underline{a}'',\underline{b}'',R_1'',R_2'',2}| + |T_{\underline{a}'',\underline{b}'',R_1'',R_2'',2}| \leqslant e_0 - (p-1)e_1,$$

using the followings:

$$|S_{\underline{a}'',\underline{b}'',R_1'',R_2'',1}| + |T_{\underline{a}'',\underline{b}'',R_1'',R_2'',1}| \leqslant R_1 - 1,\ pR_1 + R_2 = e - 2a_1'',$$
$$|S_{\underline{a}'',\underline{b}'',R_1'',R_2'',2}| \leqslant R_2 + a_1'',\ 1 \leqslant -a_1'' \leqslant p-1$$
$$\text{and } |T_{\underline{a}'',\underline{b}'',R_1'',R_2'',2}| \leqslant 1.$$

Then (A_i') for all i and (B_i') for $i \neq 1$ implies that

$$\sum_{1\leqslant i\leqslant n}\big(|S_{\underline{a}'',\underline{b}'',R_1'',R_2'',i}| + |T_{\underline{a}'',\underline{b}'',R_1'',R_2'',i}|\big) \leqslant ne_0$$

in the case $0 \leqslant e_1 \leqslant p-2$, and that

$$\sum_{1 \leqslant i \leqslant n} \left(|S_{\underline{a}'', \underline{b}'', R_1'', R_2'', i}| + |T_{\underline{a}'', \underline{b}'', R_1'', R_2'', i}| \right) \leqslant ne_0 + \left[\frac{n}{2} \right]$$

in the case $e_1 = p-1$. It remains to eliminate the possibility of equality in the case $e_1 = 0$.

We assume that $e_1 = 0$ and $\sum_{1 \leqslant i \leqslant n} \left(|S_{\underline{a}'', \underline{b}'', R_1'', R_2'', i}| + |T_{\underline{a}'', \underline{b}'', R_1'', R_2'', i}| \right) = ne_0$. Then (A_i') for all i implies that $|S_{\underline{a}'', \underline{b}'', R_1'', R_2'', i}| + |T_{\underline{a}'', \underline{b}'', R_1'', R_2'', i}| = e_0$ for all i. Now we have

$$e_0 = |S_{\underline{a}'', \underline{b}'', R_1'', R_2'', 1}| + |T_{\underline{a}'', \underline{b}'', R_1'', R_2'', 1}| \leqslant R_1 - 1$$

and

$$e_0 - 1 \leqslant |S_{\underline{a}'', \underline{b}'', R_1'', R_2'', 2}| \leqslant R_2 + a_1''.$$

This implies $e + p - 1 - a_1'' \leqslant pR_1 + R_2$. Because $pR_1 + R_2 = e - 2a_1''$, this inequality happens only in the case $-a_1'' = p-1$, and in this case the above inequalities become equality. So we have $e_0 - 1 = |S_{\underline{a}'', \underline{b}'', R_1'', R_2'', 2}|$ and $R_2 = e_0 + p - 2$. By $|T_{\underline{a}'', \underline{b}'', R_1'', R_2'', 2}| = 1$, we have $R_2 \leqslant \min\{(e - a_2'')/p, b_2''/p\}$. So we get $a_2'' \leqslant e_0 - p(p-2) \leqslant e_0 - 3$, but this contradicts $|S_{\underline{a}'', \underline{b}'', R_1'', R_2'', 3}| \geqslant e_0 - 1$. Thus we have eliminated the possibility of equality in the case $e_1 = 0$. □

The claim (2) follows from Lemma 3.2 and Lemma 3.3. □

REMARK 3.4. — *By Lemma 1.2, we can check that there is $V_{\mathbb{F}}$ satisfying the conditions for $M_{\mathbb{F}}$ in Proposition 3.1.*

4. Main theorem

To fix the notation, we recall the definition of the zeta function of a scheme of finite type over a finite field.

DEFINITION 4.1. — *Let X be a scheme of finite type over $\mathbb{F}$. We put $q_{\mathbb{F}} = |\mathbb{F}|$. The zeta function $Z(X;T)$ of X is defined by*

$$Z(X;T) = \exp\left(\sum_{m=1}^{\infty} \frac{\left| X(\mathbb{F}_{q_{\mathbb{F}}^m}) \right|}{m} T^m \right).$$

Here,

$$\exp\big(f(T)\big) = \sum_{m=0}^{\infty} \frac{1}{m!} f(T)^m \in \mathbb{Q}[[T]]$$

for $f(T) \in T\mathbb{Q}[[T]]$.

THEOREM 4.2. — *Let $Z(\mathscr{GR}_{V_{\mathbb{F}},0};T)$ be the zeta function of $\mathscr{GR}_{V_{\mathbb{F}},0}$. Then the followings are true.*

(1) *After extending the field $\mathbb{F}$ sufficiently, we have*

$$Z(\mathscr{GR}_{V_{\mathbb{F}},0};T)=\prod_{i=0}^{d_{V_{\mathbb{F}}}}(1-|\mathbb{F}|^{i}T)^{-m_i}$$

for some $m_i\in\mathbb{Z}$ such that $m_{d_{V_{\mathbb{F}}}}>0$.

(2) *If $n=1$, we have*

$$0\leqslant d_{V_{\mathbb{F}}}\leqslant\left[\frac{e+2}{p+1}\right].$$

If $n\geqslant 2$, we have

$$0\leqslant d_{V_{\mathbb{F}}}\leqslant\left[\frac{n+1}{2}\right]\left[\frac{e}{p+1}\right]+\left[\frac{n-2}{2}\right]\left[\frac{e+1}{p+1}\right]+\left[\frac{e+2}{p+1}\right].$$

Furthermore, each equality in the above inequalities can happen for any finite extension K of $\mathbb{Q}_p$.

Proof. — This follows from Proposition 1.3, Proposition 2.1, Proposition 3.1 and Remark 3.4. □

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Manuscrit reçu le 11 mai 2009,
révisé le 6 avril 2010,
accepté le 31 août 2010.

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