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DEDEKIND SUMS INVOLVING JACOBI MODULAR FORMS AND SPECIAL VALUES OF BARNES ZETA FUNCTIONS

by Abdelmejid BAYAD & Yilmaz SIMSEK (*)

Abstract. — In this paper we study three new shifted sums of Apostol-Dedekind-Rademacher type. The first sums are written in terms of Jacobi modular forms, and the second sums in terms of cotangent and the third sums are expressed in terms of special values of the Barnes multiple zeta functions. These sums generalize the classical Dedekind-Rademacher sums. The main aim of this paper is to state and prove the Dedekind reciprocity laws satisfied by these new sums. As an application of our Dedekind reciprocity law we show how to derive all the well-known results on Dedekind reciprocity law studied by Hall-Wilson-Zagier, Beck-Berndt-Dieter, Katayama and Nagasaka-Ota-Sekine.

Résumé. — Dans ce papier nous étudions trois types nouveaux de sommes shiftées de Dedekind-Apostol-Rademacher. Les premières sommes sont écrites à l'aide des formes modulaires de Jacobi et les deuxièmes sont écrites en termes de valeurs de fonctions cotangentes et les troisièmes sont exprimées à l'aide de valeurs spéciales de fonctions zêta multiples de Barnes. Le résultat principal de cet article est de montrer une loi de réciprocité de Dedekind satisfaites par ces nouvelles sommes. Nos résultats recouvrent ceux de Hall-Wilson-Zagier sur les sommes classiques de Dedekind-Rademacher, ceux de Beck-Berndt-Dieter sur les sommes cotangentes et d'autres résultats obtenus par Ota et Nagasaka sur les sommes de Dedekind, attachées aux dérivées premières de fonctions zêta de Barnes.

1. Statement of main results

Let $\mathcal{H} = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$ the upper half plane. Throught this paper we use the notations:

$$\tau \in \mathcal{H},\ e(z) = e^{2\pi i z}, q_\tau = e(\tau)\ ,\ z \in \mathbb{C}$$

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and

$$\varphi = \varphi_1 \tau + \varphi_2,\ (\varphi_1, \varphi_2) \in \mathbb{R}^2,\ \forall\, \varphi \in \mathbb{C}$$

because $\{\tau, 1\}$ is a $\mathbb{R}$-basis of $\mathbb{C}$.

1.1. Jacobi forms and elliptic Dedekind sums.

The Jacobi's theta function associated to τ is given by

(1.1)

$$\theta_\tau(z) = i q_\tau^{1/8} \left(e(z/2) - e(-z/2)\right) \prod_{n=1}^{\infty} \left(1 - q_\tau^n\right)\left(1 - q_\tau^n e(z)\right)\left(1 - q_\tau^n e(-z)\right).$$

Let $L = \mathbb{Z}\tau + \mathbb{Z}$ be the complex lattice generated by the oriented $\mathbb{Z}$-oriented basis. We associate to L a Jacobi form of two variables as follows:

$$D_L(z;\varphi) = \frac{1}{\omega_2} e\left(\frac{z}{\omega_2}\varphi_1\right) \frac{\theta_\tau'(0)\theta_\tau\left(\frac{z+\varphi}{\omega_2}\right)}{\theta_\tau\left(\frac{z}{\omega_2}\right)\theta_\tau\left(\frac{\varphi}{\omega_2}\right)}. \tag{1.2}$$

Let n, N be fixed integers $\geqslant 2$, let $a_1, ..., a_n$ be positive integers $\geqslant 1$ and pairwise coprime, $\beta_1, ..., \beta_N,\ \alpha_1, \ldots,\ \alpha_N \in \mathbb{C}$, $x_1, \ldots,\ x_n,\ x_1', \ldots,\ x_n' \in \mathbb{R}$ and $\varphi_1, \cdots,\ \varphi_n$ be complex variables. Now, we set

$$z_k = -x_k' + x_k\tau, Z = (z_1, \ldots, z_n), \Phi = (\varphi_1, \ldots, \varphi_n),$$

$$A = (a_1, \ldots, a_n), \alpha = (\alpha_1, \ldots, \alpha_N), \beta = (\beta_1, \ldots, \beta_N)$$

For any $(k, m) \in \mathbb{N}^2$, $1 \leqslant k \leqslant n,\ 1 \leqslant m \leqslant N$, we introduce the higher-multiple elliptic Dedekind sums by

$$\mathfrak{S}_{k,m}(A, \alpha, \beta, Z, \Phi) := \tag{1.3}$$
$$\frac{1}{a_k} \sum_{\tilde{t_k} \in L/a_k L} \prod_{\substack{(j,l) \neq (k,m) \\ 1 \leqslant j \leqslant n, 1 \leqslant l \leqslant N}} D_L\left(\alpha_l \frac{\varphi_j}{a_j}; a_j \frac{z_k + \tilde{t}_k}{a_k} - z_j + a_j(\beta_m - \beta_l)\right)$$

In this paper we prove the following general Dedekind reciprocity law

Theorem 1.1 (main theorem). — *Let $\varphi_1, ..., \varphi_n$ be complex variables with sum zero. If*

$$a_j z_k - a_k z_j + a_j a_k (\beta_m - \beta_l) \notin L\ ,$$

for all $(l, j) \neq (m, k)$ with $1 \leqslant j, k \leqslant n, 1 \leqslant l, m \leqslant N$.
Then

$$\sum_{m=1}^{N} \sum_{k=1}^{n} \mathfrak{S}_{k,m}(A, \alpha, \beta, Z, \Phi) = 0.$$

For instance, if $N = 1, n = 3$, this implies a result of Sczech [15]. In general, the case $N = 1$ can be viewed as an elliptic version of the result of Hall-Wilson-Zagier [9] if $n = 3$. For arbitrary n and $N = 1$, this includes an elliptic version of a result of Beck [5] and a theorem of Dieter [8]. These facts will be specified in what follows.

1.2. Applications

We denote by $a_1, \dots, a_n$, $\alpha_1, \dots, \alpha_N$, $m_1, \dots, m_n$ positive integers, $x_1, \dots, x_n$, $\varphi_1, \dots, \varphi_n$ and $\beta_1, \dots, \beta_N$ real numbers.

1.2.1. Cotangents sums:

We study the expressions

$$(1.4)\quad \mathfrak{C}_{k,m}(A, X, \phi, \beta) := \frac{1}{a_k} \sum_{t_k \bmod a_k} \prod_{\substack{(j,l) \neq (k,m) \\ 1 \leqslant j \leqslant n \\ 1 \leqslant l \leqslant N}} \left(\cot \pi \left(a_j \frac{x_k + t_k}{a_k} - x_j - a_j(\beta_l - \beta_m) \right) - \cot \pi \left(\alpha_j \varphi_j \right) \right).$$

Hence, from our theorem 1.1 we obtain the higher cotangent reciprocity law

Theorem 1.2 (higher-Cotangent sums). — *For all j, k, m such that: $1 \leqslant j \neq k \leqslant n, 1 \leqslant m \leqslant N$, assume that*

$$a_j x_k - a_k x_j \notin \mathbb{Z} \text{ and } \alpha_m \varphi_k \notin \mathbb{Z}.$$

Then

$$(1.5)\qquad \sum_{\substack{1 \leqslant k \leqslant n \\ 1 \leqslant m \leqslant n}} \mathfrak{C}_{k,m}(A, X, \phi, \beta) = Im \left(\prod_{k=1}^{n} \prod_{m=1}^{N} \left(i - cot\left(\pi \alpha_m \varphi_k \right) \right) \right).$$

If we take $N = 1$, $\beta_1 = 0$ and $\alpha_1 = 1$, then we obtain a generalization of a result of Beck [5] as follows. More precisely,

Corollary 1.3 (generalisation of the Beck's cotangent sums). — *For all j, k such that: $1 \leqslant j \neq k \leqslant n$, and*

$$a_j x_k - a_k x_j \notin \mathbb{Z} \text{ and } \varphi_k \notin \mathbb{Z}.$$

We have

$$(1.6)\qquad \sum_{k=1}^{n} \mathfrak{C}_k(A, X, \phi) = Im\Big(\prod_{k=1}^{n} \Big(i - cot\Big(\pi \varphi_k \Big) \Big) \Big).$$

where

$$\mathfrak{C}_k\left(A, X, \phi\right) := \frac{1}{a_k} \sum_{t_k \bmod a_k} \prod_{1 \leqslant j \neq k \leqslant n} \left(\cot \pi \left(a_j \frac{x_k + t_k}{a_k} - x_j \right) - \cot\left(\pi \varphi_j\right) \right).$$

The Beck's original result corresponds to $\varphi_k = \frac{1}{2}$ for all $1 \leqslant k \leqslant n$.

1.2.2. Higher Dedekind-Rademacher sums

We define higher Dedekind-Rademacher sums by

(1.7)
$$S_k(A, X, M, \alpha) := \sum_{t_k \bmod a_k} \prod_{1 \leqslant j \neq k \leqslant n} B_{m_j} \left(s(\alpha) \left\{ a_j \frac{t_k + x_k}{a_k} - x_j \right\}; \alpha \right),$$

where $s(\alpha) = \sum_{m=1}^{N} \alpha_m, \quad M = (m_1, ..., m_k, ..., m_n)$. For the definiton of the Bernoulli polynomials $B_n(x)$, see section 2.1.

Alternately, the higher Dedekind-Rademacher sums can be defined by their generating function

$$\mathfrak{S}_k(A, \alpha, X, \Phi) := \sum_{m_1, \ldots, \check{m}_k, \ldots, m_n \geqslant 0} \frac{S_k(A, X, M, \alpha)}{m_1! \ldots \check{m}_k! \ldots m_n!} \prod_{1 \leqslant j \neq k \leqslant n} \left(\frac{\varphi_j}{a_j} \right)^{m_j - N}.$$

Theorem 1.4 (higher Dedekind-Rademacher Sums). — *For all j, k, m such that: $1 \leqslant j \neq k \leqslant n, 1 \leqslant m \leqslant N$ we assume that*

$$a_j x_k - a_k x_j \notin \mathbb{Z} a_j + \mathbb{Z} a_k \text{ and } \alpha_m \frac{\varphi_k}{a_k} \notin \mathbb{Z}.$$

$$\sum_{k=1}^{n} \mathfrak{S}_k(A, \alpha, X, 2\pi i \Phi).C_{k,N} = 0. \tag{1.8}$$

where

$$C_{k,N} := C_{k,N}(A, \alpha, 2\pi i \Phi) = \begin{cases} \sum_{m=1}^{N} \prod_{1 \leqslant l \neq m \leqslant N} \left(\cot \left(\alpha_l \frac{\varphi_k}{a_k} \right) - i \right) & \text{If } N \geqslant 2 \\ 1 & \text{If } N = 1 \end{cases}$$

Remarks. —
(1) Taking $N = 1$ and $n = 3$ in Theorem 1.4, yields the result of Hall-Lewis-Zagier [9]. The general case can be viewed as a generalization of [9].
(2) The special case $n = 1$, N arbitrary, was first studied by Nagasaka-Ota-Sekine [12] and Katayama [10, 11].

(3) Note that our main Theorems 1.1, 1.2 and 1.4 are valid for N **and** n **both arbitrary.**

2. Special values of the zeta functions of Riemann, Hurwitz and Barnes

It is well-known that the special values of the zeta functions of Riemann, Hurwitz, Barnes and Shintani at non-positive integer values of the complex variable s can be expressed rationally in terms of the Bernoulli numbers and Bernoulli polynomials. For the convenience of the reader, we briefly recall these classical results. We will give a review of Jacobi modular forms which we will use throught this paper. In the following we recall the properties of Riemann, Hurwitz, Barnes zeta functions and the Jacobi modular forms.

Our references for this section are [1-4] and [10-14]

2.1. Riemann-Bernoulli functions

It's well-known that the Riemann zeta function is defined by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, , \text{ for } \Re(s) > 1$$

By Mellin transform, has the contour integral representation

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{t^{s-1}}{e^t - 1} dt, \text{ for } \Re(s) > 1.$$

By modifying the contour Riemann showed that

$$2\sin(\pi s)\Gamma(s)\zeta(s) = i \oint_C \frac{(-x)^{s-1}}{e^x - 1}\, dx$$

for all s, where the contour C starts and ends at $+\infty$ and circles the origin once. Then the function $s \to \zeta(s)$ has analytic continuation to an entire function on the whole complex s-plane. This entire function we denote it by $\zeta(s)$. Moreover, at integral values of s, we have

$$\zeta(1-m) = -\frac{B_m}{m};\ \forall\ m \geqslant 2$$

$$\zeta(2m) = \frac{(-1)^{m+1}(2\pi)^{2m}}{2(2m)!} B_{2m};\ \forall\ m \geqslant 0.$$

The m^{th} Bernoulli polynomial is defined by $B_m(x) = \sum_{n=0}^{m} \binom{m}{n} B_n X^{m-n}$.
More precisely, $B_m(x)$ are defined by the generating function

$$\frac{te^{tx}}{e^t - 1} = \sum_{n=0}^{\infty} \frac{B_n(x)}{n!} t^n. \tag{2.1}$$

Thus, $B_0(x) = 1$, $B_1(x) = x - \frac{1}{2}$, $B_2(x) = x^2 - x + \frac{1}{6}$,... and $B_n := B_n(0)$ is the n-th Bernoulli number.

Let $\{x\}$ be a fractional part of the real number x. Then the Bernoulli functions $\bar{B}_n(x)$ are defined by

$$\bar{B}_n(x) = \begin{cases} 0, & \text{If } n = 1, x \in \mathbb{Z} \\ B_n(\{x\}) & \textit{Otherwise}, \end{cases}$$

where

$$\delta_{x,y} = \begin{cases} 1 & \text{If } x = y \\ 0 & \textit{Otherwise} \end{cases} \quad \forall x, y \in \mathbb{R}$$

is the Kronecker delta function.

In terms of generating function, we have

$$\sum_{n=0}^{\infty} \frac{\bar{B}_n(x)}{n!} t^n = \frac{te^{tx}}{e^t - 1} + \frac{t}{2} \delta_{0,\{x\}}. \tag{2.2}$$

Equivalently,

$$\bar{B}_n(x) = B_n(\{x\}) + \frac{1}{2} \delta_{1,n} \delta_{0,\{x\}}, \forall n \in \mathbb{N}.$$

In addition it has Fourier expansion formula

$$\bar{B}_n(x) = -\frac{n!}{(2\pi i)^n} \sum_{k \in \mathbb{Z}}^{(e)} \frac{e(kx)}{k^n} \ , \ n \in \mathbb{N}^*. \tag{2.3}$$

2.2. Hurwitz-Bernoulli functions

Hurwitz zeta function is given by

$$\zeta(s; x) = \sum_{k=0}^{\infty} (x + k)^{-s},$$

it converges for $\Re s > 1$ so that $\zeta(s; x)$ is an analytic function of s in this region. It has an integral representation in terms of the Mellin transform as

$$\zeta(s; x) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{t^{s-1} e^{-xt}}{1 - e^{-t}} dt, \text{ for } \Re(s) > 1 \text{ and } \Re(x) > 0,$$

and can be used to show that $\zeta(s; x)$ admits an analytic extension to the whole complex plane except for a simple pole at $s = 1$.

In most of the examples discussed here we consider only the range $0 < x \leqslant 1$. Special cases of $\zeta(s;x)$ include the Riemann zeta function.

Alternately,

$$\zeta(-n,x) = -\frac{B_{n+1}(x)}{n+1}, \forall n \in \mathbb{N}.$$

In particular, the relation holds for $n = 0$ and one has

$$\zeta(0,x) = \frac{1}{2} - x.$$

2.3. Barnes-Bernoulli

In about 1900, Barnes developed a comprehensive theory for a new class of special functions, the so-called multiple zeta and gamma functions, defined as a generalisation of Hurwitz zeta function and Euler gamma function. Barnes multiple zeta function $\zeta(s,x;\alpha)$ depends on parameters $\alpha_1,..,\alpha_N$ in $\alpha = (\alpha_1,..,\alpha_N)$ that will be taken strictly positive throught this paper. It may be defined by the series

$$\zeta(s,x;\alpha) = \sum_{\overrightarrow{m}\in\mathbb{N}^N} \frac{1}{(x+\alpha.\overrightarrow{m})^s}, \quad \text{it converges for } \Re(s) > N \text{ and } \ \Re(x) > 0,$$

where

$$\alpha.\overrightarrow{m} = \sum_{i=1}^{N} \alpha_i m_i.$$

We remark that when $N = 1$,

$$\zeta(s,x;(\alpha_1)) = \sum_{m_1\in\mathbb{N}} \frac{1}{(x+\alpha_1 m_1)^s} = \frac{1}{\alpha_1^s}\zeta\left(s, \frac{x}{\alpha_1}\right),$$

where $\zeta\left(s, \frac{x}{\alpha_1}\right)$ is the Hurwitz zeta function. We get from [14] the analytic continuation and special values are given by the contour integral representation can be written as follows:

$$\zeta(s,x;\alpha) = \frac{\Gamma(1-s)e^{-s\pi i}}{2\pi i} \int_{I(\lambda,\infty)} \frac{t^{s-1}e^{-xt}}{\prod_{i=1}^{N}\left(1-e^{-\alpha_i t}\right)} dt,$$

for $\mathfrak{R}(s) > N$ and $\mathfrak{R}(x) > 0$, where $I(\lambda,\infty)$ is the path consisting of the real line from $+\infty$ to λ, the circle around 0 of radius λ counter-clock wise from λ to λ and the real line from λ to ∞.

By using the power series

$$\prod_{i=1}^{N}\left(1-e^{-\alpha_i t}\right)^{-1} = \sum_{\overrightarrow{m}\in\mathbb{N}^N} exp\left(-t\alpha.\overrightarrow{m}\right), \text{ see [14]}.$$

The Barnes multiple zeta functions can be rewritten as follows

$$\zeta(s,x;\alpha)=\frac{1}{\Gamma(s)}\int_0^\infty \frac{t^{s-1}e^{-xt}}{\prod_{i=1}^{N}\left(1-e^{-\alpha_i t}\right)}dt,\ \text{for } \Re(s)>1 \text{ and } \Re(x)>0.$$

The so-called multiple Bernoulli polynomials $B_n(x,;\alpha)$ are defined by

$$\frac{t^N e^{xt}}{\prod_{i=1}^{N}\left(e^{\alpha_i t}-1\right)}=\sum_{n=0}^{\infty}\frac{B_n(x;\alpha)}{n!}t^n. \tag{2.4}$$

The values at $s=1-m$, $m\in\mathbb{N}^*$, are given by

$$\zeta(1-m,x;\alpha)=\frac{(-1)^{m-1}(m-1)!}{(N+m-1)!}B_{N+m-1}(x;\alpha);\forall m\in\mathbb{N}^* \tag{2.5}$$

We have

$$\frac{B_m(x;\alpha)}{m!}=\sum_{0\leqslant m_1,\dots,m_N\leqslant m}\frac{B_{m_1}...B_{m_N}}{m_1!...m_N!}\alpha_1^{m_1}...\alpha_N^{m_N}.\frac{x^{(m-(m_1+...+m_N))}}{(m-(m_1+...+m_N))!}. \tag{2.6}$$

Note that, for $N=1$, we have $B_m\left(x;(\alpha_1)\right)=\alpha_1^m B_m\left(\frac{x}{\alpha_1}\right)$.
The multiple Bernoulli functions type (or Barnes-Bernoulli functions) are defined by

$$\bar{B}_m(x;\alpha)=B_m(\{x\};\alpha)+\frac{1}{2}\delta_{1,m}\delta_{1,N}\delta_{0,\{x\}},\forall m\in\mathbb{N}. \tag{2.7}$$

Then

$$\sum_{n=0}^{\infty}\frac{\bar{B}_n(x;\alpha)}{n!}t^n=\frac{t^N e^{xt}}{\prod_{i=1}^{N}\left(e^{\alpha_i t}-1\right)}+\frac{t}{2}\delta_{1,N}\delta_{0,\{x\}}. \tag{2.8}$$

Using the above generating function one can prove the following properties

Proposition 2.1. —

1. *(**Homogeneity**): $B_n(\lambda x,\lambda\alpha)=\lambda^{n-N}B_n(x,\alpha),\forall\lambda\in\mathbb{C}\backslash\{0\}$.*
2. *(**Difference equation**):*

$$B_n(x+\alpha_N,\alpha)-B_n(x,\alpha)=nB_{n-1}\Big(x,(\alpha_1,...,\alpha_{N-1})\Big),\quad \forall n\geqslant N \text{ and } \quad N\geqslant 2.$$

(3) *(***Distribution formulae 1***):*

$$\sum_{i=0}^{M-1} B_n\left(x+i\frac{\alpha_k}{M};\alpha\right) = M^{N-n}B_n\Big(Mx;(M\alpha_1,...,M\alpha_{k-1},\alpha_k,M\alpha_{k+1},...,M\alpha_N)\Big).$$

(4) *(***Distribution formulae 2***):*

$$\sum_{i=0}^{M-1} B_n\left(\left\{x+i\frac{\alpha_k}{M}\right\};\alpha\right) = M^{N-n}B_n\Big(\{Mx\};(M\alpha_1,...,M\alpha_{k-1},\alpha_k,M\alpha_{k+1},...,M\alpha_N)\Big).$$

(5) *Let $\alpha_N \neq 0$, we have the Fourier expansion*

$$\alpha_N.\bar{B}_n(x,\alpha) = -\frac{n!}{\left(\frac{2\pi i}{\alpha_N}\right)^n}\sum_{l\in\mathbb{Z}}^{(e)}\sum_{k=0}^{n}\frac{B_k(0,(\alpha_1,...,\alpha_{N-1}))}{k!}\left(\frac{2\pi i}{\alpha_N}\right)^k\frac{e^{2\pi ilx}}{l^{n-k}},\ \forall\ N\geqslant 2.$$

In particular, if we assume $\alpha_N = 1$, we obtain the quite formula

$$\bar{B}_n(x,\alpha) = -\frac{n!}{(2\pi i)^n}\sum_{l\in\mathbb{Z}}^{(e)}\sum_{k=0}^{n}\frac{B_k(0,(\alpha_1,...,\alpha_{N-1}))}{k!}(2\pi i)^k\frac{e^{2\pi ilx}}{l^{n-k}},\ \forall\ N\geqslant 2.$$

(6) *(***Distribution formulae 3***): Let $\alpha_N \neq 0$, we have*

$$\sum_{f=0}^{M-1}\bar{B}_n\left(x+\frac{f}{M},\alpha\right) = \alpha_N^{1-n}M^{1-n}\sum_{k=0}^{m}\binom{m}{k}B_k\Big(0,(\alpha_1,...,\alpha_{N-1})\Big)M^k\bar{B}_{n-k}\big(Mx\big).$$

Proof. — To prove i), we use directly the equality (2.4) and the fact

$$\frac{t^Ne^{(\lambda x)t}}{\prod_{i=1}^{N}\left(e^{(\lambda\alpha_i)t}-1\right)} = \lambda^{-N}\frac{(\lambda t)^Ne^{x(\lambda t)}}{\prod_{i=1}^{N}\left(e^{\alpha_i(\lambda t)}-1\right)}. \tag{2.9}$$

For ii), we remark that

$$\frac{t^Ne^{(x+\alpha_N)t}}{\prod_{i=1}^{N}(e^{\alpha_it}-1)} - \frac{t^Ne^{xt}}{\prod_{i=1}^{N}\left(e^{\alpha_it)}-1\right)} = t\times\frac{t^{N-1}e^{xt}}{\prod_{i=1}^{N-1}(e^{\alpha_it}-1)}, \tag{2.10}$$

and we use the equality (2.4) to obtain the desired equality ii).
Again, from the equality (2.4) we deduce the distribution formulae iii). For the second distribution formula iv) we use the equality (2.8). In order to prove the properties v) and vi) we need the following addition formulae for the Barnes-Bernoulli functions

$$\frac{\bar{B}_n\Big(x+y,\alpha\Big)}{n!}=\sum_{k=0}^{n}\frac{\bar{B}_{n-k}(x)}{(n-k)!}.\frac{B_k\Big(y,(\alpha_1,...,\alpha_{N-1})\Big)}{k!},\ \forall N\geqslant 2.$$

which comes from the following obvious identity, for $\alpha_N\neq 0$

$$\begin{aligned}\frac{t^Ne^{(x+y)t}}{\prod_{i=1}^{N}\left(e^{\alpha_i t}-1\right)}&=\frac{t^{N-1}e^{xt}}{\prod_{i=1}^{N-1}\left(e^{\alpha_i t}-1\right)}.\frac{te^{yt}}{e^{\alpha_N t}-1}\\&=\frac{1}{\alpha_N}.\frac{t^{N-1}e^{xt}}{\prod_{i=1}^{N-1}\left(e^{\alpha_i t}-1\right)}.\frac{(\alpha_N t)e^{\frac{y}{\alpha_N}(\alpha_N t)}}{e^{\alpha_N t}-1}.\end{aligned}$$

Now, to obtain v), we take $y=0,\alpha_N=1$ and we use the Fourier expansion (2.3), for Bernoulli functions. To obtain vi) we replace (x,y) in the above addition formula by $\Big(x+\frac{f}{M},0\Big)$ and use the Fourier expansion (2.3) to get the desired result. □

3. Jacobi forms: Analytic point of view

Let $\mathcal{H}=\{z\in\mathbb{C}:\mathrm{Im}(z)>0\}$ be upper half plane. For $\tau\in\mathcal{H}$, we recall the Jacobi's theta function.

(3.1)

$$\theta_\tau(z)=iq_\tau^{1/8}\left(e(z/2)-e(-z/2)\right)\prod_{n=1}^{\infty}\left(1-q_\tau^n\right)\left(1-q_\tau^ne(z)\right)\left(1-q_\tau^ne(-z)\right).$$

Now, for an arbitrary complex lattice L, we fix $\{\omega_1,\omega_2\}$ an oriented $\mathbb{Z}$-basis of L i.e $Im\left(\frac{\omega_1}{\omega_2}\right)\ >\ 0$. We associate to L a Jacobi form of two variables

$$D_L(z;\varphi)=\frac{1}{\omega_2}e\left(\frac{z}{\omega_2}\varphi_1\right)\frac{\theta_\tau'(0)\theta_\tau\left(\frac{z+\varphi}{\omega_2}\right)}{\theta_\tau\left(\frac{z}{\omega_2}\right)\theta_\tau\left(\frac{\varphi}{\omega_2}\right)},\tag{3.2}$$

where $\tau=\frac{\omega_1}{\omega_2}$. We define the following $\mathbb{R}$-alternating bilinear form

$$E_L(z,\varphi)=\frac{\bar{z}\varphi-z\bar{\varphi}}{\omega_1\bar{\omega_2}-\bar{\omega_1}\omega_2}=\frac{\bar{z}\varphi-z\bar{\varphi}}{2i|\omega_2|^2Im\left(\frac{\omega_1}{\omega_2}\right)},$$

which is the symplectic form on $\mathbb{C}$ associated to the oriented complex lattice L. Then we can rewrite D_L as an infinite product

$$D_L(z,\varphi) = \frac{2\pi i}{w_2} q_{\frac{z}{w_2}}^{\frac{\operatorname{Im}\left(\frac{\varphi}{w_2}\right)}{\operatorname{Im}\tau}} \frac{\left(q_{\frac{z+\varphi}{w_2}}^{\frac{1}{2}} - q_{\frac{z+\varphi}{w_2}}^{-\frac{1}{2}}\right)}{\left(q_{\frac{z}{w_2}}^{\frac{1}{2}} - q_{\frac{z}{w_2}}^{-\frac{1}{2}}\right)\left(q_{\frac{\varphi}{w_2}}^{\frac{1}{2}} - q_{\frac{\varphi}{w_2}}^{-\frac{1}{2}}\right)} \tag{3.3}$$

$$\prod_{n\geqslant 1} \frac{(1-q_\tau^n)^2\left(1-q_\tau^n q_{\frac{z+\varphi}{w_2}}\right)\left(1-q_\tau^n q_{\frac{z+\varphi}{w_2}}^{-1}\right)}{\left(1-q_\tau^n q_{\frac{z}{w_2}}\right)\left(1-q_\tau^n q_{\frac{z}{w_2}}^{-1}\right)\left(1-q_\tau^n q_{\frac{\varphi}{w_2}}\right)\left(1-q_\tau^n q_{\frac{\varphi}{w_2}}^{-1}\right)}.$$

This Jacobi form has the following properties:

(1) D_L is meromorphic in the first variable z,
(2) D_L satisfies

$$\begin{cases} D_L(z;\varphi+\rho) = D_L(z;\varphi) \\ D_L(z+\rho;\varphi) = e(E_L(\rho,\varphi))D_L(z;\varphi) \end{cases} \qquad \forall \rho \in L,$$

(3) D_L has the functional equation

$$D_L(z;\varphi)e(-E_L(z,\varphi)) = D_L(\varphi;z).$$

The proofs of these properties are elementary [4, 3, 1].
From the product representation (3.3) we obtain that, for each $z,\varphi \in \mathbb{C}\backslash\mathbb{Z}\tau+\mathbb{Z}$,

$$\begin{aligned} \frac{1}{2\pi i}\lim_{\operatorname{Im}(\tau)\to\infty} D_\tau(z,\varphi) &= e\Big(E_L([z],\{\varphi\})\Big)\delta_{0,\{\varphi_1\}}\delta_{0,\{z_1\}} \\ &+ \frac{e\Big(E_L([z],\{\varphi\})\Big)}{e(\{\varphi\})-1}\delta_{0,\{\varphi_1\}} + \frac{e\Big(E_L([z],\{\varphi\})\Big)e\left(\{z\}\{\varphi_1\}\right)}{e(\{z\})-1}\delta_{0,\{z_1\}}. \end{aligned} \tag{3.4}$$

Here we denote by $\{z\}=\{z_1\}\tau+\{z_2\}$ and $[z]=[z_1]\tau+[z_2]$, where $\{z_1\}$, $\{z_2\}$ are the fractional parts of real numbers z_1,z_2 (resp. $[z_1],[z_2]$ integer parts of real numbers z_1,z_2).
For more details about our Jacobi form D_L see [4, 3].

In particular, for $z\in\mathbb{R}\backslash\mathbb{Z}, \varphi=\varphi_1\tau+\varphi_2$ with $\varphi_1\notin\mathbb{Z}$, we obtain

$$\frac{1}{2\pi i}\lim_{\operatorname{Im}(\tau)\to\infty} D_\tau(z,\varphi) = \frac{e(z\{\varphi_1\})}{e(z)-1}. \tag{3.5}$$

4. Proofs of main results

4.1. Proof of theorem 1.1:

For the parameters

$$z_k = -x'_k + x_k\tau, \tilde{t}_k = -t'_k + t_k\tau, Z = (z_1, \dots, z_n), \phi = (\varphi_1, \dots, \varphi_n),$$

$$A = (a_1, \dots, a_n), \alpha = (\alpha_1, \dots, \alpha_N), \beta = (\beta_1, \dots, \beta_N),$$

we consider the function

$$F(z) = \prod_{m=1}^{N} \prod_{j=1}^{n} D_L\left(a_j z - z_j - a_j\beta_m; \alpha_m \frac{\varphi_j}{a_j}\right).$$

It has the following interesting properties:

(1) F is a meromorphic, with poles at

$$z = \frac{z_i + \tilde{t}_i}{a_i} + \beta_m, \tilde{t}_i \in L \pmod{a_i L}, 1 \leqslant i \leqslant n, 1 \leqslant m \leqslant N.$$

By the hypothesis we have

$$a_j z_k - a_k z_j + a_j a_k(\beta_m - \beta_l) \notin L$$

which implies that all the poles of F are simple.

(2) by using the periodicity of the Jacobi form $D_L(z;\varphi)$, It's easy to see that F is periodic with period lattice L i.e

$$F(z+\rho) = F(z), \forall \rho \in L.$$

In fact, $\forall\ \rho \in L$, we have

$$F(z+\rho) = e\left(E_L(\rho, \sum_{i=1}^{n} \varphi_i)\right) F(z) = F(z).$$

because,

$$\sum_{i=1}^{n} \varphi_i = 0.$$

Then, by the Liouville's residue theorem for an elliptic function with periods containing lattice L, we obtain

$$\sum_{\frac{z_i+\tilde{t}_i}{a_i}+\beta_m} \operatorname{Res}\left(F(z)dz, z = \frac{z_i + \tilde{t}_i}{a_i} + \beta_m\right) = 0.$$

where the summation is over all the poles of F

$$\frac{z_i + \tilde{t}_i}{a_i} + \beta_m,\ \tilde{t}_i \in L \pmod{a_i L},\ 1 \leqslant i \leqslant n,\ 1 \leqslant m \leqslant N.$$

The poles are simple with residue

$$\operatorname{Res}\left(F(z)dz, z=\frac{z_k+\tilde{t}_k}{a_k}+\beta_m\right)=$$
$$\frac{1}{a_k}e\left(E_L\left(\tilde{t}_k,\alpha_m\frac{\varphi_k}{a_k}\right)\right)\prod_{\substack{(j,l)\neq(k,m)\\ 1\leqslant j\leqslant n, 1\leqslant l\leqslant N}} D_L\left(a_j\frac{z_k+\tilde{t}_k}{a_k}-z_j+a_j(\beta_m-\beta_l);\alpha_l\frac{\varphi_j}{a_j}\right).$$

Using the functional equation of $D_L(z;\varphi)$, (iii) in section 3 , it's easy to see that this expression is equal to

$$\frac{1}{a_k}\prod_{\substack{(j,l)\neq(k,m)\\ 1\leqslant j\leqslant n, 1\leqslant l\leqslant N}} D_L\left(\alpha_l\frac{\varphi_j}{a_j};a_j\frac{z_k+\tilde{t}_k}{a_k}-z_j+a_j(\beta_m-\beta_l)\right).$$

Hence, we get immediately our desired theorem 1.1. □

4.2. Proof of theorem 1.2:

In this section our tools are the same as in the previous subsection.
Let us fix the following parameters:
$a_1,\dots,a_n,\alpha_1,\dots,\alpha_N,m_1,\dots,m_n$ are positive integers, $x_1,\dots,x_n,\varphi_1,\dots,\varphi_n$ and $\beta_1,\dots,\beta_N$ are real numbers, such that

$$a_jx_k-a_kx_j\notin\mathbb{Z}a_j+\mathbb{Z}a_k \text{ and } \alpha_m\varphi_k\notin\mathbb{Z}.$$

To prove the theorem 1.2, we introduce the function G defined by

$$G(z)=\prod_{m=1}^{N}\prod_{k=1}^{n}\Big(\cot\pi\left(a_kz-x_k-a_k\beta_m\right)-\cot\pi\left(\alpha_m\varphi_k\right)\Big). \tag{4.1}$$

This function G satisfies the following nice properties:

(1) G is meromorphic function with simple poles. These poles are
$$z_{k,l}=\frac{x_k+t_k}{a_k}+\beta_l,\ t_k\in\mathbb{Z},\ 1\leqslant l\leqslant N,\ 1\leqslant k\leqslant n.$$

(2) G is periodic function with period 1.

We integrate G along the simple rectangular path

$$\gamma=[x+iy,x-iy,x+1-iy,x+1+iy,x+iy]\ ,$$

where x and y are chosen such that γ does not pass through any pole of G, and all poles $z_{k,l}$ of G have imaginary part $|\operatorname{Im}(z_{k,l})|<y$.
Using cotangent function, we have the contributions of the two vertical segments of γ cancel each other.
In other hand,

$$\lim_{y\to\infty}\cot(x\pm iy)=\mp i\ .$$

Thus we obtain

(4.2)
$$\int_\gamma f(z)\ dz = \prod_{k=1}^{n}\prod_{m=1}^{N}\left(i-\cot\left(\pi\alpha_m\varphi_k\right)\right) - \prod_{k=1}^{n}\prod_{m=1}^{N}\left(-i-\cot\left(\pi\alpha_m\varphi_k\right)\right)\ .$$

The residue of G at $z_{k,l}$ is equal to

(4.3) $\operatorname{Res}\left(G(z)dz, z = z_{k,l}\right) =$
$$\frac{1}{a_k}\sum_{t_k \bmod a_k}\prod_{\substack{(j,l)\neq(k,m)\\ 1\leqslant j\leqslant n\\ 1\leqslant l\leqslant N}}\left(\cot\pi\left(a_j\frac{x_k+t_k}{a_k}-x_j-a_j(\beta_l-\beta_m)\right)-\cot\pi\left(\alpha_j\varphi_j\right)\right).$$

On the other hand, by the Liouville residue theorem, we have

$$\frac{1}{2i}\int_\gamma G(z)\ dz = \sum_{\substack{1\leqslant k\leqslant n\\ 1\leqslant m\leqslant N}}\operatorname{Res}\Big(G(z)dz, z = z_{k,l}\Big).$$

The statement of Theorem 1.2 follows now from (4.2) and (4.3). □

4.3. Proof of theorem 1.4:

We deduce Theorem 1.4 from our main theorem 1.1 as follows. We write

$$\begin{aligned} a_j\frac{z_k+\tilde{t}_k}{a_k} &- z_j + a_j(\beta_m-\beta_l) \\ &= \left(a_j\frac{x_k+t_k}{a_k}-x_j\right)\tau - \left(a_j\frac{x'_k+t'_k}{a_k}-x'_j+a_j(\beta_l-\beta_m)\right), \end{aligned}$$

where

$$z_k = -x'_k + x_k\tau, \tilde{t}_k = -t'_k + t_k\tau$$
$$(x_k, x'_k), (t_k, t'_k) \in \mathbb{R}^2.$$

Assuming that $\alpha_m \frac{\varphi_j}{a_j} \in \mathbb{R}\backslash\mathbb{Z}$, from (3.4) we obtain

$$(4.4)\quad \frac{1}{2\pi i} \lim_{\mathrm{Im}(\tau)\to\infty} D_L\left(\alpha_l \frac{\varphi_j}{a_j}; a_j \frac{z_k+\tilde{t}_k}{a_k} - z_j + a_j(\beta_m-\beta_l)\right)$$
$$= \frac{e\left(\alpha_l \frac{\varphi_j}{a_j}\{a_j \frac{x_k+t}{a_k} - x_j\}\right)}{e\left(\alpha_l \frac{\varphi_j}{a_j}\right)-1} + \frac{e\left(-(a'_j \frac{x'_k+t'}{a'_k} - x'_j) + a_j(\beta_l-\beta_m)\right)}{e\left(-(a'_j \frac{x'_k+t'}{a'_k} - x'_j) + a_j(\beta_l-\beta_m)\right)-1}\delta_{0,\left\{a_j \frac{x_k+t}{a_k} - x_j\right\}}$$
$$= \frac{e\left(\alpha_l \frac{\varphi_j}{a_j}\{a_j \frac{x_k+t}{a_k} - x_j\}\right)}{e\left(\alpha_l \frac{\varphi_j}{a_j}\right)-1} - \frac{1}{e\left((a'_j \frac{x'_k+t'}{a'_k} - x'_j) + a_j(\beta_l-\beta_m)\right)-1}\delta_{0,\left\{a_j \frac{x_k+t}{a_k} - x_j\right\}}.$$

Then, from our theorem 1.1 we obtain

$$(4.5)\quad 0 = \sum_{k=1}^{n}\sum_{m=1}^{N}\frac{1}{a_k}\sum_{0\leqslant t_k,\tilde{t}'_k\leqslant a_k-1}\prod_{(j,l)\neq(k,m)}$$
$$\left(\frac{e\left(\alpha_l \frac{\varphi_j}{a_j}\{a_j \frac{x_k+t}{a_k} - x_j\}\right)}{e\left(\alpha_l \frac{\varphi_j}{a_j}\right)-1} - \frac{1}{e\left((a'_j \frac{x'_k+t'}{a'_k} - x'_j) + a_j(\beta_l-\beta_m)\right)-1}\delta_{0,\left\{a_j \frac{x_k+t}{a_k} - x_j\right\}}\right).$$

By hypothesis, we obtain

$$a_j x_k - a_k x_j \notin \mathbb{Z}a_j + \mathbb{Z}a_k \text{ and } \alpha_m \frac{\varphi_k}{a_k} \notin \mathbb{Z};$$

$\forall\ j,\ k,\ m$ such that $1 \leqslant j \neq k \leqslant n, 1 \leqslant m \leqslant N$.
Hence, we get

$$\left\{a_j \frac{x_k+t}{a_k} - x_j\right\} \neq 0,\ \forall\ 0 \leqslant t_k \leqslant a_k-1.$$

Therefore, we obtain

$$(4.6)\quad \sum_{k=1}^{n}\sum_{m=1}^{N}\frac{1}{a_k}\sum_{0\leqslant t_k,\tilde{t}'_k\leqslant a_k-1}\prod_{(j,l)\neq(k,m)}\frac{e\left(\alpha_l \frac{\varphi_j}{a_j}\{a_j \frac{x_k+t}{a_k} - x_j\}\right)}{e\left(\alpha_l \frac{\varphi_j}{a_j}\right)-1} = 0.$$

Hence, we arrive at the following result:

$$(4.7)\quad \sum_{k=1}^{n}\sum_{m=1}^{N}\sum_{t_k=0}^{a_k-1}\prod_{(j,l)\neq(k,m)}\frac{e\left(\alpha_l \frac{\varphi_j}{a_j}\{a_j \frac{x_k+t}{a_k} - x_j\}\right)}{e\left(\alpha_l \frac{\varphi_j}{a_j}\right)-1} = 0.$$

Now, we distinguish two cases.

Case 1: Let $N = 1$. We have

(4.8)
$$\sum_{k=1}^{n}\sum_{t_k=0}^{a_k-1}\prod_{1\leqslant j\neq k\leqslant n}\frac{e\left(\alpha_1\frac{\varphi_j}{a_j}\{a_j\frac{x_k+t}{a_k}-x_j\}\right)}{e\left(\alpha_1\frac{\varphi_j}{a_j}\right)-1}=0.$$

More precisely, we obtain

(4.9)
$$\sum_{k=1}^{n}\mathfrak{S}_k(A,\alpha,X,2\pi i\Phi)=0.$$

This is the desired result in Theorem 1.4 for $N = 1$.

Case 2: N is arbitrary. Then, we quote from the equality (4.7)

(4.10)
$$\sum_{k=1}^{n}\sum_{m=1}^{N}\sum_{t_k=0}^{a_k-1}\prod_{j\neq k}\prod_{l=1}^{N}\frac{e\left(\alpha_l\frac{\varphi_j}{a_j}\{a_j\frac{x_k+t}{a_k}-x_j\}\right)}{e\left(\alpha_l\frac{\varphi_j}{a_j}\right)-1}\times\prod_{l\neq m}\frac{1}{e\left(\alpha_l\frac{\varphi_j}{a_j}\right)-1}=0.$$

Then, we have

$$\sum_{k=1}^{n}\sum_{t_k=0}^{a_k-1}\prod_{j\neq k}\prod_{l=1}^{N}\frac{e\left(\alpha_l\frac{\varphi_j}{a_j}\{a_j\frac{x_k+t}{a_k}-x_j\}\right)}{e\left(\alpha_l\frac{\varphi_j}{a_j}\right)-1}\times\sum_{m=1}^{N}\prod_{1\leqslant l\neq m\leqslant N}\left(\cot\left(\alpha_l\frac{\varphi_k}{a_k}\right)-i\right)=0.$$

This gives the statement of our Theorem 1.4. □

Remark. —

(1) The main tool to obtain Theorem 1.4 is the theory of Jacobi modular forms D_L, contained in Theorem 1.1.

(2) An elementary proof of Theorem 1.4 not using modular forms would be of great interest.

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BIBLIOGRAPHY

[1] A. Bayad, "Jacobi forms in two variables: Multiple elliptic Dedekind sums, The Kummer-von Staudt Clausen Congruences for elliptic Bernoulli functions and values of Hecke L-functions", Preprint.

[2] ———, "Sommes de Dedekind elliptiques et formes de Jacobi", *Ann. Instit. Fourier* **51** Fasc. 1, (2001), p. 29-42.

[3] ———, "Applications aux sommes elliptiques multiples d'Apostol-Dedekind-Zagier", *C.R.A.S Paris, Ser. I* **339** fascicule 8, Série I, (2004), p. 539-532.

[4] ———, "Sommes elliptiques multiples d'Apostol-Dedekind-Zagier", *C.R.A.S Paris, Ser. I* **339** fascicule 7, Série I, (2004), p. 457-462.

[5] M. Beck, "Dedekind cotangent sums", *Acta Arithmetica* **109** (2003), no. 2, p. 109-130.

[6] B. C. Berndt, "Reciprocity theorems for Dedekind sums and generalizations", *Adv. in Math.* **23** (1977), no. 3, p. 285-316.

[7] B. C. Berndt & U. Dieter, "Sums involving the greatest integer function and Riemann-Stieltjes integration", *J. reine angew. Math.* **337** (1982), p. 208-220.

[8] U. Dieter, "Cotangent sums, a further generalization of Dedekind sums", *J. Number Th.* **18** (1984), p. 289-305.

[9] R. R. Hall, J. C. Wilson & D. Zagier, "Reciprocity formulae for general Dedekind-Rademacher sums", *Acta Arith.* **73** (1995), no. 4, p. 389-396.

[10] K. Katayama, "Barne's Double zeta function, the Dedekind Sum and Ramanujan's Formula", *Tokyo J. Math.* **27** (2004), no. 1, p. 41-56.

[11] ———, "Barne's Multiple function and Apostol's Generalized Dedekind Sum", *Tokyo J. Math.* **27** (2004), no. 1, p. 57-74.

[12] Y. Nagasaka, K. Ota & C. Sekine, "Generalizations of Dedekind sums and their reciprocity laws", *acta Arith* **106** (2003), no. 4, p. 355-378.

[13] K. Ota, "Derivatives of Dedekind sums and their reciprocity law", *Journal of Number Theory* **98** (2003), p. 280-309.

[14] S. N. M. Ruijsenaars, "On Barnes' Multiple Zeta and Gamma Functions", *Advances in Mathematics* **156** (2000), p. 107-132.

[15] R. Sczech, "Dedekindsummen mit elliptischen Funktionen", *Invent.math* **76** (1984), p. 523-551.

[16] A. Weil, *Elliptic functions according to Eisenstein and Kronecker* (Ergeb. der Math. 88), Springer-Verlag, 1976.

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