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ON THE RATIONAL APPROXIMATION TO THE THUE–MORSE–MAHLER NUMBERS

by Yann BUGEAUD

ABSTRACT. — Let $(t_k)_{k \geqslant 0}$ be the Thue–Morse sequence on $\{0, 1\}$ defined by $t_0 = 0$, $t_{2k} = t_k$ and $t_{2k+1} = 1 - t_k$ for $k \geqslant 0$. Let $b \geqslant 2$ be an integer. We establish that the irrationality exponent of the Thue–Morse–Mahler number $\sum_{k \geqslant 0} t_k b^{-k}$ is equal to 2.

RÉSUMÉ. — Soit $(t_k)_{k \geqslant 0}$ la suite de Thue–Morse définie sur $\{0, 1\}$ par $t_0 = 0$, $t_{2k} = t_k$ et $t_{2k+1} = 1 - t_k$ pour $k \geqslant 0$. Soit $b \geqslant 2$ un entier rationnel. Nous démontrons que l'exposant d'irrationalité du nombre de Thue–Morse–Mahler $\sum_{k \geqslant 0} t_k b^{-k}$ est égal à 2.

1. Introduction

Let ξ be an irrational, real number. The irrationality exponent $\mu(\xi)$ of ξ is the supremum of the real numbers μ such that the inequality

$$\left| \xi - \frac{p}{q} \right| < \frac{1}{q^\mu}$$

has infinitely many solutions in rational numbers p/q. It follows from the theory of continued fractions that $\mu(\xi)$ is always greater than or equal to 2, and an easy covering argument shows that $\mu(\xi)$ is equal to 2 for almost all real numbers ξ (with respect to the Lebesgue measure). Furthermore, Roth's theorem asserts that the irrationality exponent of every algebraic irrational number is equal to 2. It is in general a very difficult problem to determine the irrationality exponent of a given transcendental real number ξ. Apart from some numbers involving the exponential function or the Bessel function (see the end of Section 1 of [1]) and apart from more or

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less *ad hoc* constructions (see below), there do not seem to be examples of transcendental numbers ξ whose irrationality exponent is known. When they can be applied, the current techniques allow us only to get an upper bound for $\mu(\xi)$.

Clearly, the irrationality exponent of ξ can be read on its continued fraction expansion. But when ξ is defined by its expansion in some integer base $b \geqslant 2$, we do not generally get enough information to determine the exact value of $\mu(\xi)$. A combinatorial, and naïve, method is the following. Write

$$\xi = \lfloor \xi \rfloor + \sum_{k \geqslant 1} \frac{a_k}{b^k}, \tag{1.1}$$

where $a_k \in \{0, 1, \ldots, b-1\}$ for $k \geqslant 1$ and $\lfloor \cdot \rfloor$ denotes the integer part function. Assume that there are positive integers n, r, m such that $a_{n+h+ir} = a_{n+h}$ for $h = 1, \ldots, r$ and $i = 1, \ldots, m$. With such a triple (n, r, m) of positive integers, we associate the rational number ξ_n whose b-ary expansion is defined as follows: Truncate the expansion of ξ after the n-th digit a_n and complete by repeating infinitely many copies of the finite block $a_{n+1} \ldots a_{n+r}$. Clearly, ξ_n is a good rational approximation to ξ which satisfies

$$|\xi - \xi_n| < \frac{1}{b^{n+(m+1)r}},$$

since ξ and ξ_n have their first $n+(m+1)r$ digits in common. If we find infinitely many triples (n, r, m) as above for which $(m+1)r$ is 'large' compared to n, then we get a good lower bound for $\mu(\xi)$. If, in addition, these triples constitute a 'dense' sequence, in a suitable sense, then we can often deduce an upper bound for $\mu(\xi)$. In most cases, in particular when $\mu(\xi)$ is small, both bounds do not coincide. This is pointed out *e.g.*, in Lemma 4 of [15], where it is shown by this method that $\mu(\sum_{k\geqslant 1} 3^{-\lfloor \tau^k \rfloor}) = \tau$ for every real number τ at least equal to $(3+\sqrt{5})/2$, and that $\tau \leqslant \mu(\sum_{k\geqslant 1} 3^{-\lfloor \tau^k \rfloor}) \leqslant (2\tau-1)/(\tau-1)$ if $2 \leqslant \tau < (3+\sqrt{5})/2$.

There are however several examples of real numbers defined by their expansion in some integer base and whose continued fraction expansion is known, see [4, 17, 13, 7, 11], hence, whose irrationality exponent is determined. Among these examples are some automatic numbers [4, 11]. Recall that a real number ξ is an automatic number if there exists an integer $b \geqslant 2$ such that the b-ary expansion of ξ can be generated by a finite automaton; see [6] for precise definitions.

Adamczewski and Cassaigne [2] established that the irrationality exponent of every irrational automatic number is finite. Subsequently, Adamczewski and Rivoal [3] were able to bound from above the irrationality exponent of automatic numbers ξ constructed with Thue–Morse, Rudin–Shapiro, paperfolding or Baum–Sweet sequences, but their bounds, although quite small, are presumably not best possible. They gave two methods for producing rational approximations to the automatic numbers ξ considered in [3]. A first one is based on Padé approximants to the generating function of the automatic sequence. A second one is the naïve method described above: If some block of digits occurs at least twice near the beginning of the sequence, then the sequence is "close" to a periodic sequence, thus the associated real number is "close" to a rational number.

In this note we focus on the most classical family of automatic numbers, namely the Thue–Morse–Mahler numbers. Let

$$\mathbf{t} = t_0t_1t_2\ldots = 0110100110010110100101100110100110010110\ldots$$

denote the Thue–Morse word on $\{0,1\}$ defined by $t_0 = 0$, $t_{2k} = t_k$ and $t_{2k+1} = 1 - t_k$ for $k \geqslant 0$. Alternatively, $t_k = 0$ (resp. 1) if the number of 1's in the binary expansion of k is even (resp. is odd). Note also that, if σ is the morphism defined by $\sigma(0) = 01$ and $\sigma(1) = 10$, then $\mathbf{t}$ is precisely the fixed point of σ starting with 0.

Let $b \geqslant 2$ be an integer. In a fundamental paper, Mahler [16] established that the Thue–Morse–Mahler number

$$\xi_{\mathbf{t},b} = \sum_{k\geqslant 0} \frac{t_k}{b^k} = \frac{1}{b} + \frac{1}{b^2} + \frac{1}{b^4} + \frac{1}{b^7} + \frac{1}{b^8} + \ldots$$

is transcendental (see Dekking [14] for an alternative proof, reproduced in Section 13.4 of [6]). Adamczewski and Rivoal [3] showed that the irrationality exponent of $\xi_{\mathbf{t},b}$ is at most 4, improving the upper bound of 5 obtained in [2]. At the end of [3], they conjectured that the irrationality exponent of $\xi_{\mathbf{t},b}$ is at most 3. Let us add some comments: since $\mathbf{t}$ begins with 01101, it also begins with $\sigma^n(011)\sigma^n(01)$, for $n \geqslant 1$. Consequently, $\xi_{\mathbf{t},b}$ is close to the rational ρ_n whose b-ary expansion is purely periodic of period $\sigma^n(011)$, in the sense that $\xi_{\mathbf{t},b}$ and ρ_n have their first $5 \cdot 2^n$ digits in common. This is exactly the method followed in [2] to get that $\mu(\xi_{\mathbf{t},b}) \leqslant 5$. Note that this approach actually yields the better bound $\mu(\xi_{\mathbf{t},b}) \leqslant 4$, since the integers p_n and q_n defined in Section 6 of [2] have a large common divisor. Using another combinatorial property of $\mathbf{t}$, we can construct a second sequence of good rational approximations to $\xi_{\mathbf{t},b}$. Indeed, since $\mathbf{t}$ begins with 011, it also begins with $\sigma^n(0)\sigma^n(1)\sigma^n(1)$, for $n \geqslant 1$. Consequently, $\xi_{\mathbf{t},b}$ is close to the rational π_n whose b-ary expansion is ultimately periodic of period $\sigma^n(1)$ and preperiod $\sigma^n(0)$. In doing this, we find the rational approximations denoted by p_n/q_n in Section 4.2.1 of [3] and obtained by means of suitable Padé approximants. As noted in [3], this method gives the bound $\mu(\xi_{\mathbf{t},b}) \leqslant 4$. However, if we consider the union of the two sequences $(\rho_n)_{n\geqslant 1}$ and $(\pi_n)_{n\geqslant 1}$, a careful computation based on triangle inequalities implies the improved upper bound $\mu(\xi_{\mathbf{t},b}) \leqslant 3$.

The purpose of the present note is to establish a stronger result, namely that this exponent is equal to 2. Like in many papers on irrationality measures of classical numbers, our proof makes use of Padé approximants. Apparently, the Thue–Morse–Mahler numbers are the first examples of irrational numbers ξ defined by their expansion in an integer base and for which a method based on Padé approximants yields the exact value of $\mu(\xi)$.

2. Result

We establish the following theorem.

Theorem. — *For any integer $b \geqslant 2$, the irrationality exponent of the Thue–Morse–Mahler number $\xi_{\mathbf{t},b}$ is equal to 2.*

The strategy to prove the Theorem seems to be new. We make use of a result of Allouche, Peyrière, Wen, and Wen [5] on the non-vanishing of Hankel determinants of the Thue–Morse sequence to get many Padé approximants to the generating function of the Thue–Morse sequence. By means of these rational fractions we construct infinitely many good rational approximations to the Thue–Morse–Mahler number $\xi_{\mathbf{t},b}$, which form a sufficiently dense sequence to conclude that the irrationality exponent of $\xi_{\mathbf{t},b}$ cannot exceed 2. We believe that this general method will have further applications in Diophantine approximation.

The first 20000 partial quotients of the so-called Thue–Morse constant $\xi_{\mathbf{t},2}/2$ have been computed by Harry J. Smith, see [18]. Open Problem 9 on page 403 of [6] asks whether the Thue–Morse constant has bounded partial quotients. We have no idea how to solve this challenging question. Numerical experimentation performed by Maurice Mignotte tends to suggest that the answer should be negative and that the continued fraction expansion of $\xi_{\mathbf{t},2}/2$ shares most statistics with the continued fraction expansion of almost every real number (in the sense of the Lebesgue measure). Let us just mention that 1014303 of the first 2447729 partial quotients of $\xi_{\mathbf{t},2}/2$ are equal to 1, and that 45 exceed 100000.

Remark 2.1. — The Theorem can be made effective. This means that, for any $b \geqslant 2$ and any positive ε, we can compute explicitly a positive number $q_0(\varepsilon, b)$ such that

$$|\xi_{\mathbf{t},b} - p/q| > q^{-2-\varepsilon}, \quad \text{for every } q \geqslant q_0(\varepsilon, b).$$

This is justified at the end of the proof.

Remark 2.2. — For an integer $b \geqslant 2$ and an irrational real number ξ whose b-ary expansion is given by (1.1), let

$$p(n, \xi, b) = \mathrm{Card}\{a_k a_{k+1} \ldots a_{k+n-1} : k \geqslant 1\}$$

denote the number of distinct blocks of n letters occurring in the infinite word $a_1 a_2 a_3 \ldots$ It has been proved independently by several authors that $3(n-1) \leqslant p(n, \xi_{\mathbf{t},b}, b) \leqslant 4n$ for $n \geqslant 1$, see Exercise 10 on page 335 and Note 10.3 on page 341 of [6]. Our Theorem allows us to get some (very modest) information on the expansion of $\xi_{\mathbf{t},b}$ in other bases, namely that, for any integer $b' \geqslant 2$, we have

$$\lim_{n \to +\infty} p(n, \xi_{\mathbf{t},b}, b') - n = +\infty.$$

This follows from the fact that if a real number ξ is such that the sequence $(p(n, \xi, b') - n)_{n \geqslant 1}$ is bounded, then its irrationality exponent must exceed 2; see [1] for details (the key ingredient is a deep result of Berthé, Holton, and Zamboni [9] on the combinatorial structure of Sturmian sequences). Thus, the Thue–Morse–Mahler numbers are among the very few explicit examples of real numbers for which one can say something non trivial regarding their expansions in different integer bases.

Remark 2.3. — It was shown in [11] that every rational number greater than or equal to 2 is the irrationality exponent of some automatic number. This result motivates the study of the following question [3, 12]:

Is the irrationality exponent of an automatic number always rational?

Results from [12] and the Theorem speak in favour of a positive answer, but this question seems to be very difficult. It is likely that the proof of the Theorem can be adapted to determine the irrationality exponent of some other automatic numbers.

3. Auxiliary results

We begin this section by recalling several basic facts on Padé approximants. We refer the reader to [10, 8] for the proofs and for additional results. Let $f(z)$ be a power series in one variable with rational coefficients,

$$f(z) = \sum_{k \geqslant 0} c_k z^k, \quad c_k \in \mathbf{Q}.$$

Let p, q be non-negative integers. The Padé approximant $[p/q]_f(z)$ is any rational fraction $A(z)/B(z)$ in $\mathbf{Q}[[z]]$ such that

$$\deg(A) \leqslant p, \quad \deg(B) \leqslant q, \quad \text{and } \mathrm{ord}_{z=0}(B(z)f(z) - A(z)) \geqslant p + q + 1.$$

The pair (A, B) has no reason to be unique, but the fraction $A(z)/B(z)$ is unique.

For $k \geqslant 1$, let

$$H_k(f) := \begin{vmatrix} c_0 & c_1 & \cdots & c_{k-1} \\ c_1 & c_2 & \cdots & c_k \\ \vdots & \vdots & \ddots & \vdots \\ c_{k-1} & c_k & \cdots & c_{2k-2} \end{vmatrix}$$

be the Hankel determinant of order k associated to $f(z)$. If $H_k(f)$ is non-zero, then the Padé approximant $[k-1/k]_f(z)$ exists and we have

$$f(z) - [k-1/k]_f(z) = \frac{H_{k+1}(f)}{H_k(f)}\, z^{2k} + O(z^{2k+1}). \tag{3.1}$$

Our first auxiliary result states that the Hankel determinants of the Thue–Morse sequence do not vanish. It was proved by Allouche, Peyrière, Wen, and Wen [5].

Theorem APWW. — *Let $\Xi(z)$ be the generating function of the Thue–Morse sequence on $\{-1, 1\}$ starting with 1. Then, for every positive integer k, the Hankel determinant $H_k(\Xi)$ is non-zero and the Padé approximant $[k-1/k]_\Xi(z)$ exists.*

Proof. — This is Corollary 4.1 and Theorem 4.3 from [5]. □

As mentioned in the Introduction, a usual way to prove that a given number ξ is irrational is to construct an infinite sequence of good rational approximations to ξ. If this sequence appears to be sufficiently ‘dense’, we even get a bound for its irrationality exponent by an elementary use of triangle inequalities; see *e.g.*, Lemma 4.1 from [3] for a precise result.

Lemma 3.1. — *Let ξ be a real number. Assume that there exist a real number $\theta > 1$, positive real numbers c_1, c_2 and a sequence $(p_n/q_n)_{n \geqslant 1}$ of rational numbers such that*

$$q_n < q_{n+1} \leqslant q_n^\theta, \quad (n \geqslant 1), \tag{3.2}$$

and

$$\mathfrak{c}_1 q_n^2 \leqslant \left|\xi - \frac{p_n}{q_n}\right| \leqslant \frac{c_2}{q_n^2}, \quad (n \geqslant 1).$$

Then the irrationality exponent of ξ is at most equal to 2θ.

We stress that the rational numbers occurring in the statement of Lemma 3.1 are not assumed to be written in their lowest form.

Proof. — Let p/q be a reduced rational number whose denominator is sufficiently large. Let $n \geqslant 2$ be the integer determined by the inequalities

$$q_{n-1} < 2c_2 q \leqslant q_n. \tag{3.3}$$

We deduce from (3.2) and (3.3) that

$$q_n \leqslant (2c_2 q)^\theta. \tag{3.4}$$

If $p/q \neq p_n/q_n$, then the triangle inequality and (3.3) give

$$\begin{aligned} \left|\xi - \frac{p}{q}\right| &\geqslant \left|\frac{p}{q} - \frac{p_n}{q_n}\right| - \left|\xi - \frac{p_n}{q_n}\right| \\ &\geqslant \frac{1}{qq_n} - \frac{c_2}{q_n^2} \geqslant \frac{1}{2qq_n} \geqslant \frac{1}{2(2c_2)^\theta q^{1+\theta}}, \end{aligned} \tag{3.5}$$

by (3.4). If $p/q = p_n/q_n$, then we have

$$\left|\xi - \frac{p}{q}\right| = \left|\xi - \frac{p_n}{q_n}\right| \geqslant \frac{c_1}{q_n^2} \geqslant \frac{c_1}{(2c_2 q)^{2\theta}}, \tag{3.6}$$

again by (3.4). The combination of (3.5) and (3.6) yields that the irrationality exponent of ξ is at most equal to 2θ. □

Our last auxiliary result is an elementary lemma.

LEMMA 3.2. — *Let k and n_0 be positive integers. Let $(a_j)_{j\geqslant 1}$ be the increasing sequence composed of all the numbers of the form $h2^n$, where $n \geqslant n_0$ and h ranges over the odd integers in $[1, 2^k - 1]$. Then, there exists an integer j_0 such that*

$$a_{j+1} \leqslant (1 + 2^{-k+1}) a_j, \quad (j \geqslant j_0).$$

Proof. — Observe that every integer of the form $m2^n$ with $n \geqslant n_0 + k$ and $m \in \{2^{k-1}, 2^{k-1}+1, 2^{k-1}+2, \ldots, 2^k - 1, 2^k\}$ belongs to $(a_j)_{j\geqslant 1}$. Consequently, for j large enough, we have

$$\frac{a_{j+1}}{a_j} \leqslant \max_{2^{k-1} \leqslant h \leqslant 2^k} \frac{h+1}{h} = \frac{2^{k-1}+1}{2^{k-1}}.$$

This proves the lemma. □

4. Proof of the Theorem

Preliminaries

In the proof of the Theorem, it is much more convenient to work with the Thue–Morse sequence on $\{-1,1\}$ which begins with 1, that is, with the sequence

$$\mathbf{t}' = t'_0 t'_1 t'_2 \ldots = 1 \ -1 \ -1 \ 1 -1 \ 1 \ 1 \ -1 \ -1 \ 1 \ 1 \ -1 \ldots$$

Note that $t'_k = 1 - 2t_k$, for $k \geqslant 0$. Since, for $b \geqslant 2$, we have

$$\xi_{\mathbf{t}',b} := \sum_{k \geqslant 0} \frac{t'_k}{b^k} = \frac{b}{b-1} - 2\xi_{\mathbf{t},b},$$

the numbers $\xi_{\mathbf{t},b}$ and $\xi_{\mathbf{t}',b}$ have the same irrationality exponent. Let

$$\Xi(z) = \sum_{k \geqslant 0} t'_k z^k$$

denote the generating function of $\mathbf{t}'$. Since $t'_0 = 1$, $t'_{2k} = t'_k$ and $t'_{2k+1} = -t'_k$ for $k \geqslant 0$, we check easily that

$$\Xi(z) = (1-z)\Xi(z^2). \tag{4.1}$$

The fact that $\Xi(z)$ satisfies a functional equation plays a crucial role in Mahler's method [16] and in the proof of the Theorem.

First step: Construction of infinite sequences of good rational approximations to $\xi_{\mathbf{t}',b}$

Let k be an odd positive integer and m be a positive integer. It follows from Theorem APWW and (3.1) that there exist integer polynomials $P_{k,0}(z)$, $Q_{k,0}(z)$ of degree at most $k-1$ and k, respectively, and a non-zero rational number h_k such that

$$\Xi(z) - P_{k,0}(z)/Q_{k,0}(z) = h_k z^{2k} + O(z^{2k+1}).$$

Then, there exists a positive constant $C(k)$, depending only on k, such that

$$\left| \Xi(z^{2^m}) - \frac{P_{k,0}(z^{2^m})}{Q_{k,0}(z^{2^m})} - h_k z^{2^{m+1}k} \right| \leqslant C(k) z^{2^{m+1}k + 2^m}, \quad \text{for } 0 < z \leqslant 1/2. \tag{4.2}$$

An induction based on (4.1) gives that

$$\Xi(z) = \prod_{j=0}^{m-1} (1 - z^{2^j}) \Xi(z^{2^m}),$$

hence, by (4.2), we obtain

$$\begin{aligned}\Big|\Xi(z) - \prod_{j=0}^{m-1}(1-z^{2^j})\, P_{k,0}(z^{2^m})/Q_{k,0}(z^{2^m}) - h_k \prod_{j=0}^{m-1}(1-z^{2^j})\, z^{2^{m+1}k}\Big| \\ \leqslant C(k) z^{2^{m+1}k+2^m},\end{aligned} \tag{4.3}$$

for $0 < z \leqslant 1/2$.

For simplicity, set

$$P_{k,m}(z) = \prod_{j=0}^{m-1}(1-z^{2^j})\, P_{k,0}(z^{2^m})$$

and

$$Q_{k,m}(z) = Q_{k,0}(z^{2^m}).$$

Note that $P_{k,m}(z)/Q_{k,m}(z)$ is the Padé approximant $[2^m k - 1/2^m k]_\Xi(z)$ of $\Xi(z)$.

Let $m_0(k)$ be a positive integer such that

$$C(k) 2^{-2^m} \leqslant \frac{h_k \Xi(1/2)}{2}, \quad \text{for } m \geqslant m_0(k). \tag{4.4}$$

Assume that m is at least equal to $m_0(k)$. Since

$$\Xi(1/2) \leqslant \prod_{j=0}^{m-1}(1-2^{-2^j}) \leqslant \prod_{j=0}^{m-1}(1-z^{2^j}) \leqslant 1$$

holds for $0 < z \leqslant 1/2$, it follows from (4.3) and (4.4) that we have

$$\frac{h_k \Xi(1/2)}{2}\, z^{2^{m+1}k} \leqslant \frac{h_k \prod_{j=0}^{m-1}(1-z^{2^j})\, z^{2^{m+1}k}}{2} \leqslant \left|\Xi(z) - \frac{P_{k,m}(z)}{Q_{k,m}(z)}\right| \tag{4.5}$$

and

$$\left|\Xi(z) - \frac{P_{k,m}(z)}{Q_{k,m}(z)}\right| \leqslant \frac{3h_k \prod_{j=0}^{m-1}(1-z^{2^j})\, z^{2^{m+1}k}}{2} \leqslant \frac{3h_k}{2}\, z^{2^{m+1}k}, \tag{4.6}$$

for $0 < z \leqslant 1/2$.

Let $b \geqslant 2$ be an integer. Taking $z = 1/b$ in (4.5) and (4.6) and recalling that $\Xi(1/b) = \xi_{\mathbf{t}',b}$, we obtain the inequalities

$$\frac{h_k \Xi(1/2)}{2}\, b^{-2^{m+1}k} \leqslant \left|\xi_{\mathbf{t}',b} - \frac{P_{k,m}(1/b)}{Q_{k,m}(1/b)}\right| \leqslant \frac{3h_k}{2}\, b^{-2^{m+1}k}. \tag{4.7}$$

Define the integers

$$p_{k,m} = b^{2^m k} P_{k,m}(1/b)$$

and

$$q_{k,m} = b^{2^m k} Q_{k,m}(1/b).$$

Note that the polynomial $Q_{k,0}(z)$ (and, hence, the polynomial $Q_{k,m}(z)$) does not vanish at $z = 0$. This observation and (4.7) show that there exist positive real numbers $c_1(k), \dots, c_6(k)$, depending only on k, such that

$$c_1(k) b^{2^m k} \leqslant q_{k,m} \leqslant c_2(k) b^{2^m k}, \tag{4.8}$$

$$\frac{c_3(k)}{b^{2^{m+1}k}} \leqslant \left| \xi_{\mathbf{t}',b} - \frac{p_{k,m}}{q_{k,m}} \right| \leqslant \frac{c_4(k)}{b^{2^{m+1}k}}, \tag{4.9}$$

and, combining (4.8) and (4.9),

$$\frac{c_5(k)}{q_{k,m}^2} \leqslant \left| \xi_{\mathbf{t}',b} - \frac{p_{k,m}}{q_{k,m}} \right| \leqslant \frac{c_6(k)}{q_{k,m}^2}. \tag{4.10}$$

Consequently, for every odd positive integer k, we have constructed an infinite sequence of rational numbers $(p_{k,m}/q_{k,m})_{m \geqslant m_0(k)}$ that are close to $\xi_{\mathbf{t}',b}$.

Second step: Upper bound for the irrationality exponent of $\xi_{\mathbf{t}',b}$

Let $K \geqslant 2$ be an integer. Let $(V_{K,n})_{n \geqslant 1}$ be the sequence of positive integers composed of all the integers $q_{k,m}$ with k odd, $1 \leqslant k \leqslant 2^K - 1$, $m \geqslant m_0(k)$, ranged by increasing order. It follows from Lemma 3.2 and (4.8) that there exists an integer $n_0(K)$ such that

$$V_{K,n} < V_{K,n+1} \leqslant V_{K,n}^{1+2^{-K+2}}, \quad \text{for } n \geqslant n_0(K). \tag{4.11}$$

Furthermore, by (4.10), there are positive integers $U_{K,n}$ and positive constants $C_1(K)$, $C_2(K)$, depending only on K, such that

$$\frac{C_1(K)}{V_{K,n}^2} \leqslant \left| \xi_{\mathbf{t}',b} - \frac{U_{K,n}}{V_{K,n}} \right| \leqslant \frac{C_2(K)}{V_{K,n}^2}, \quad \text{for } n \geqslant n_0(K). \tag{4.12}$$

We then deduce from (4.11), (4.12) and Lemma 3.1 that the irrationality exponent of $\xi_{\mathbf{t}',b}$ is at most equal to $2 + 2^{-K+3}$. Since K is arbitrary and $\xi_{\mathbf{t}',b}$ is irrational, this exponent must be equal to 2. Recalling that $\xi_{\mathbf{t},b}$ and $\xi_{\mathbf{t}',b}$ have the same irrationality exponent, this concludes the proof of the Theorem.

This proof can be made effective, since, as shown in [10, 8], the coefficients of the polynomials $P_{k,0}(z)$, $Q_{k,0}(z)$ and the rational numbers h_k can be expressed as determinants of matrices whose entries are elements of the sequence $\mathbf{t}'$.

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