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EQUATIONS OF SOME WONDERFUL COMPACTIFICATIONS

by Pascal HIVERT

Abstract. — De Concini and Procesi have defined the wonderful compactification $\bar{X}$ of a symmetric space $X = G/G^\sigma$ where G is a complex semisimple adjoint group and G^σ the subgroup of fixed points of G by an involution σ. It is a closed subvariety of a Grassmannian of the Lie algebra $\mathfrak{g}$ of G. In this paper we prove that, when the rank of X is equal to the rank of G, the variety is defined by linear equations. The set of equations expresses the fact that the invariant alternate trilinear form w on $\mathfrak{g}$ vanishes on the (-1)-eigenspace of σ.

Résumé. — De Concini et Procesi ont défini la compactification magnifique minimale d'un espace symétrique $X = G/H$ où G est un groupe complexe semi-simple adjoint et H le sous-groupe des points fixes par une involution σ. C'est une sous-variété fermée d'une Grassmannienne des sous-espaces vectoriels de l'algèbre de Lie de G. Dans cet article, nous démontrons que, lorsque le rang de X est égal au rang de G, la variété est définie par des équations linéaires. Ces équations traduisent l'annulation de l'espace propre de σ de valeur propre -1 par la forme trilinéaire alternée invariante sur l'algèbre de Lie de G. L'article finit par des exemples lorsque le rang de G est deux.

1. Introduction

Throughout this paper, the Lie algebras, the vector spaces and the projective spaces are defined over the complex field $\mathbb{C}$. Let $\mathfrak{g}$ be a semisimple Lie algebra with adjoint group G, and κ be the Killing form on $\mathfrak{g}$. The trilinear alternate form $w\colon (x, y, z) \mapsto \kappa([x, y], z)$ is invariant under the adjoint action: it is an element of $(\bigwedge^3 \mathfrak{g}^\vee)^G$. We put $g = \dim \mathfrak{g}$.

Let σ be an involution of G, and $H = G^\sigma$ be the closed subgroup consisting of fixed points by σ. The *rank of the symmetric space* $X = G/H$ is the maximal dimension of the (-1)-eigenspace of σ acting on a σ-invariant

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Cartan subalgebra of $\mathfrak{g}$ (σ induces an involution on the Lie algebra $\mathfrak{g}$, denoted again by σ, moreover this involution preserves the Killing forms on $\mathfrak{g}$). By definition, the rank of the symmetric space is less than or equal to the rank of $\mathfrak{g}$.

In [2], the *minimal wonderful compactification* $\bar{X}$ of X is defined as the closure in the Grassmannian $\mathbb{G}(\dim \mathfrak{g}^\sigma, \mathfrak{g})$ of the G-orbit of the point $\mathfrak{g}^\sigma$, the Lie algebra of G^σ. The action of G in $\bar{X}$ has the following properties.

1. The variety $\bar{X}$ is a union of finitely many G-orbits.
2. The set $\bar{X} \smallsetminus G \cdot \mathfrak{g}^\sigma$ is a union of r hypersurfaces S_i, $i \in \{1, \dots, r\}$. As a consequence of Theorem 3.1 of [2], the integer r is equal to the rank of X.
3. The orbit closures are the intersections $S_J = \bigcap_{i \in J} S_i$ where J is a subset of $\{1, \dots, r\}$. By convention, we put $S_\varnothing = \bar{X}$.
4. $S_{J_1} \cap S_{J_2} = S_{J_1 \cup J_2}$ and $\operatorname{codim} S_J = \sharp J$.

We may ask how to define a set of equations of $\bar{X}$ in $\mathbb{G}(\dim \mathfrak{g}^\sigma, \mathfrak{g})$: we do not know any reference to this question in the literature. In this paper, we give an answer when the rank of X is equal to the rank of G denoted by l that is to say there exists $\mathfrak{h}$ a Cartan subalgebra such that $\sigma_{|_\mathfrak{h}} = -\mathrm{id}_\mathfrak{h}$. It follows that $\sigma(\Phi) = -\Phi$ where Φ is the root system of $(\mathfrak{g}, \mathfrak{h})$. Thus, σ is completely known, the symmetric space of maximal rank is unique up to isomorphism.

Theorem 1.1. — *If the rank of X is equal to l, $\bar{X}$ is defined in $\mathbb{G}(\frac{g-l}{2}, \mathfrak{g})$ by linear equations.*

Let us give a sketch of the proof. We assume in this paper that $\mathrm{rank} X = l$.

Definition 1.2. — *Let W be a vector subspace of $\mathfrak{g}$.*

(1) *The subspace W is a nullspace for $(\mathfrak{g}, w)$ if w vanishes on $W \times W \times W$.*
(2) *The subspace W is a maximal nullspace for $(\mathfrak{g}, w)$ if it has maximal dimension for property (1).*

We call Y the set of all maximal nullspaces. This a closed subset of a Grassmannian $\mathbb{G}(d', \mathfrak{g})$, where d' is the dimension of maximal nullspaces for $(\mathfrak{g}, w)$.

For an involution σ of $\mathfrak{g}$, the direct sum $\mathfrak{g} = \mathfrak{g}^\sigma \oplus \mathfrak{g}_{-1}$ where $\mathfrak{g}_{-1}$ is the (-1)-eigenspace is orthogonal with respect to κ; moreover the subspace $\mathfrak{g}_{-1}$ is a nullspace for $(\mathfrak{g}, w)$. Any Borel subalgebra is a nullspace, so the maximal dimension is greater than or equal to $d := \frac{g+l}{2}$.

We first prove that any maximal nullspace contains a Cartan subalgebra of $\mathfrak{g}$, and we deduce from this fact that $d' = d$. If W is a maximal nullspace which contains a Cartan subalgebra $\mathfrak{h}$, let Φ be the root system of $(\mathfrak{g},\mathfrak{h})$. We prove that for any $\alpha\in\Phi$, the vector space $\mathbb{C}x_\alpha\oplus\mathbb{C}x_{-\alpha}$, generated by a root vector of $\pm\alpha$ meets W along a line. We deduce a correspondence between the orbits of Y under G and the orbits of the parabolic subalgebras of $\mathfrak{g}$ (the corresponding parabolic subalgebra of W is $\mathfrak{p}=W+[W,W]$). The closed orbit consists of Borel subalgebras, and to prove the smoothness of Y, we analyze its tangent space over this orbit. This description corresponds to the wonderful compactification by the map $W\mapsto W^\perp$. We finish this paper with examples when $l=2$: $\mathfrak{sl}(3)$ and $\mathfrak{sp}(4)$.

For classical simple Lie algebras, Theorem 1.4 of chapter 4 in [8] gives the involutive automorphisms corresponding to the symmetric spaces of maximal rank. Hence, we can describe these symmetric spaces, we summarize it in the next table.

$\mathfrak{sl}(n)$	$\mathbb{P}(S^2\mathbb{C}^n)$
$\mathfrak{sp}(2n)$	$\mathrm{Hilb}_2\left(\mathrm{IG}(n,2n)\right)$
$\mathfrak{so}(2n)$	$\mathbb{G}(n,2n)/\sim$
$\mathfrak{so}(2n+1)$	$\mathbb{G}(n,2n+1)$

For the first line, $S^2\mathbb{C}^n$ is the space of quadrics on $\mathbb{C}^n$. For the second line, this is the Hilbert variety of length two subschemes of the isotropic Grassmannian. For the third line, the equivalence $\sim$ identifies a subspace and its orthogonal.

These symmetric spaces are (by definition) birationnally equivalent to their wonderful compactifications, but they have singular locus. For example, the variety of complete quadrics is the wonderful compactification of maximal rank for $\mathfrak{sl}(n)$: this is the blow up of $\mathbb{P}(S^2\mathbb{C}^n)$ along the singular locus.

2. Maximal nullspaces for $(\mathfrak{g},w)$

We follow the above-mentioned sketch.

Proposition 2.1. — *Every maximal nullspace contains a regular semi-simple element.*

Remark. — Let T be a maximal torus of G, and μ a one-parameter subgroup of T. We say that μ is *regular* if any μ-stable vector space W is T-stable. In particular, if $\mathfrak{h}$ is the Lie algebra of T, W is $\mathfrak{h}$-stable. See [3] for more details.

Let V be a maximal nullspace for $(\mathfrak{g}, w)$ and recall that $\dim V = d' \geqslant d$. Take μ a regular one-parameter subgroup and let $V_0 = \lim_{t\to 0} \mu(t) \cdot V$. The vector space V_0 is μ-stable, so $\mathfrak{h}$-stable, maximal for $(\mathfrak{g}, w)$.

Lemma 2.2. — *If V_0 contains a regular semisimple element, then so does V.*

Proof. — We consider the tautological vector bundle K over the Grassmannian $\mathbb{G}(d', \mathfrak{g})$, $p : K \to \mathbb{G}(d', \mathfrak{g})$ and $q : K \to \mathfrak{g}$ the two projections. Let $\mathfrak{g}_{rs}$ be the open set of regular semisimple elements of $\mathfrak{g}$. Since p is flat, $p\left(q^{-1}(\mathfrak{g}_{rs})\right)$ is an open set of $\mathbb{G}(d', \mathfrak{g})$ containing V_0, and so there exists $t_0 \in \mathbb{C}^*$ such that $\mu(t_0)V$ is included in $q\left(p^{-1}(\mathfrak{g}_{rs})\right)$. Finally, if $\mu(t_0)V$ contains a regular semisimple element, so does V. □

We prove Proposition 2.1 using a decreasing induction on

$$m = \sup_{\mathfrak{h}} \dim V \cap \mathfrak{h},$$

where $\mathfrak{h}$ ranges through all Cartan subalgebras.

Proof of Proposition 2.1. — The case $m = l$ is obvious.

Let $m < l$, and assume the result is true for all k such that $m < k \leqslant l$. Let $\mathfrak{h}$ be a Cartan subalgebra such that $\dim V \cap \mathfrak{h} = m$, T be a maximal torus of G such that $\mathfrak{h}$ is the Lie algebra of T, μ be a regular one-parameter subgroup of T, and Φ be the root system of $(\mathfrak{g}, \mathfrak{h})$. It follows that $V_0 = \lim_{t\to 0} \mu(t) \cdot V$ is $\mathfrak{h}$-stable, so we can choose to write it as the direct sum

$$V_0 = V_0 \cap \mathfrak{h} \oplus \bigoplus_{\alpha \in S} \mathbb{C}x_\alpha,$$

where S is a subset of Φ and x_α a non zero vector of the root space $\mathfrak{g}_\alpha$. Denoting $R = S \cap (-S)$, two cases appear.

i) $R = \varnothing$, so $\sharp S \leqslant \frac{g-l}{2}$, hence $l \geqslant \dim V_0 \cap \mathfrak{h} = \dim V_0 - \sharp S \geqslant l$, so this forces $\mathfrak{h} \subset V_0$; the conclusion follows from Lemma 2.2.

ii) For $\alpha \in R$, the linear form $w(x_\alpha, x_{-\alpha}, .)$ vanishes on V_0, so we have $V_0 \cap \mathfrak{h} \subset \ker \alpha$. The vector space $V_0 \cap \mathfrak{h} \oplus \mathbb{C}(x_\alpha + x_{-\alpha})$ is an abelian Lie algebra consisting of semisimple elements so is contained in a Cartan subalgebra $\mathfrak{h}_1$: $\dim V_0 \cap \mathfrak{h}_1 > \dim V_0 \cap \mathfrak{h}$. By induction, V_0 contains a regular semisimple element, hence so does V.

□

Corollary 2.3.

(a) *The maximal nullspace V contains a Cartan subalgebra.*

(b) *There exists a one-parameter subgroup μ such that $V_0 = \lim_{t\to 0} \mu(t) \cdot V$ is a Borel subalgebra.*

(c) $\dim V = d$.

Proof. — Let s be a regular semisimple element contained in V.

(a) The centralizer $\mathfrak{c}(s)$ is a Cartan subalgebra. let $\overline{\mathfrak{g}}$ be the quotient of $\mathfrak{g}$ by $\mathfrak{c}(s)$, π be the projection on $\overline{\mathfrak{g}}$. Since $\psi_s = w(s,.,.)$ is a non degenerate skewsymmetric bilinear form over $\overline{\mathfrak{g}}$ and $\pi(V)$ is an isotropic subspace,

$$\begin{aligned} \dim \pi(V) &\leqslant \frac{1}{2}\dim \overline{\mathfrak{g}} \\ \dim V - \dim V \cap \mathfrak{c}(s) &\leqslant \frac{1}{2}\left(\dim \mathfrak{g} - \dim \mathfrak{c}(s)\right) \\ \dim V \cap \mathfrak{c}(s) &\geqslant l \end{aligned}$$

and finally $\mathfrak{c}(s) \subset V$.

(b) Let T be a maximal torus of G with Lie algebra $\mathfrak{h} := \mathfrak{c}(s)$, Φ be the root system of $(\mathfrak{g}, \mathfrak{h})$, μ be a regular one-parameter subgroup of T. It follows that the limit subspace V_0 has a decomposition

$$V_0 = \mathfrak{h} \oplus \bigoplus_{\alpha \in S} \mathbb{C}x_\alpha,$$

where S and $-S$ form a partition of Φ. Now, for α, $\beta \in S$ such that $\alpha+\beta$ is a root, $w(x_\alpha, x_\beta, x_{-\alpha-\beta}) \neq 0$ proves that $\alpha+\beta \in S$, so we can choose a basis of Φ such that S is the set of positive roots.

(c) follows from (b) and $\dim V = \dim V_0$. □

Let V be a maximal nullspace of $(\mathfrak{g}, w)$ containing a Cartan subalgebra $\mathfrak{h}$, Φ be the root system of $(\mathfrak{g}, \mathfrak{h})$, choose $\alpha \in \Phi$, and let h_0 be an element of $\mathfrak{h}$ such that its centralizer is $\mathfrak{c}(h) = \mathfrak{h} \oplus \mathfrak{g}^\alpha \oplus \mathfrak{g}^{-\alpha}$. Using the argument in the proof of Corollary 2.3 (the first point), we have $\dim V \cap \mathfrak{c}(h) \geqslant l+1$. But $\mathfrak{h} \subset V$, so $\dim V \cap (\mathfrak{g}^\alpha \oplus \mathfrak{g}^{-\alpha}) \geqslant 1$. The linear form $w(x_\alpha, x_{-\alpha}, \cdot)$ is non zero on $\mathfrak{h}$, and so $\dim V \cap (\mathfrak{g}^\alpha \oplus \mathfrak{g}^{-\alpha}) = 1$.

Lemma 2.4. — *Let $V \in Y$. There exists a Cartan subalgebra $\mathfrak{h}$ such that*

$$V = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi^+} L_\alpha,$$

where Φ is the root system of $(\mathfrak{g}, \mathfrak{h})$ and L_α is a vector subspace of dimension 1 of $\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}$.

Remark. — If α, β are two positive roots such that $\alpha+\beta$ is a root, and if we denote by $v_{\alpha+\beta}$, v_α, v_β basis of $L_{\alpha+\beta}$, L_α, L_β, then $w(v_\alpha, v_\beta, v_{\alpha+\beta}) = 0$ shows $v_{\alpha+\beta}$ is defined, up to a scalar, by v_α, v_β. Let Δ be a root basis according to a Borel subalgebra. It is easy to compute that, up to conjugacy, we have two choices for L_α, $\alpha \in \Delta$: this is a root space or not.

3. Orbits of Y

The set Y of maximal nullspaces of $(\mathfrak{g}, w)$ is a closed subset of $\mathbb{G} := \mathbb{G}(d, \mathfrak{g})$, and is stable by the adjoint action of G. Thanks to Corollary 2.3, there is one closed orbit consisting of Borel subalgebras. In this section, we give a condition for two elements of Y to be conjugate.

Proposition 3.1.

(i) *The minimal parabolic subalgebra which contains $V \in Y$ is $\mathfrak{p}_V := V + [V, V]$.*

(ii) *If V_1 and V_2 are two elements of Y such that $\mathfrak{p}_{V_1} = \mathfrak{p}_{V_2}$, then V_1 and V_2 are conjugate under G.*

Proof.

(i) is obvious using Lemma 2.4.

(ii) Assume $\mathfrak{p}_{V_1} = \mathfrak{p}_{V_2}$. Up to conjugacy of V_2 under the adjoint group of $\mathfrak{p}_{V_1}$, assume the existence of a Cartan subalgebra $\mathfrak{h}$ contained in $V_1 \cap V_2$. Choosing a root system of $(\mathfrak{g}, \mathfrak{h})$, there are two Borel subalgebras $\mathfrak{b}_1$ and $\mathfrak{b}_2$ such that, for $i \in \{1, 2\}$

$$\mathfrak{p}_{V_i} = \mathfrak{b}_i \oplus \bigoplus_{\alpha \in S_i} \mathbb{C} x_{-\alpha}^{(i)}, \quad V_i = V_i \cap \mathfrak{b}_i \oplus \bigoplus_{\alpha \in S_i} \mathbb{C}(x_\alpha^{(i)} + x_{-\alpha}^{(i)}),$$

where S_i is the set of positive roots (roots of $\mathfrak{b}_i$) α such that V_i contains no roots vectors of $\pm\alpha$ and x_α^i a root vector of α such that $x_\alpha^{(i)} + x_{-\alpha}^{(i)} \in V_i$ for all α in S_i. There exists g in the adjoint group of $\mathfrak{p}_{V_1}$ such that $g \cdot \mathfrak{b}_1 = \mathfrak{b}_2$. Hence, for each α in S_1, there exists β in S_2 such that $g \cdot x_\alpha^{(1)}$ and $x_\beta^{(2)}$ are colinear. Let Δ_1 the root basis of Φ given by $\mathfrak{b}_1$. Up to conjugacy by an element of the maximal torus of G with Lie algebra equal to $\mathfrak{h}$, assume that $g \cdot (x_\alpha^{(1)} + x_{-\alpha}^{(1)}) \in V_2$ for $\alpha \in \Delta_1 \cap S_1$. The last remark of section 2 gives that $g \cdot (x_\alpha^{(1)} + x_{-\alpha}^{(1)}) \in V_2$ for $\alpha \in S_1$. Check that $g \cdot V_1 \cap \mathfrak{b}_1 \subset V_2 \cap \mathfrak{b}_2$ to conclude. $\square$

Remark. — The number of orbits in Y is equal to 2^l, the number of parabolic orbits. Indeed, Proposition 3.1 says that V_1 and V_2 are in the same orbit in Y if and only if $\mathfrak{p}_{V_1}$ and $\mathfrak{p}_{V_2}$ are conjugate. Conversely, for each parabolic subalgebra $\mathfrak{p}$, we can find an element of Y such that $\mathfrak{p}_V = \mathfrak{p}$.

Moreover, there is only one orbit with dimension equal to $\dim Y$, given by the parabolic subalgebra $\mathfrak{g}$,

$$Y = \overline{G \cdot V},$$

where V is the direct sum $\mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi^+} \mathbb{C}(x_\alpha + x_{-\alpha})$.

Given a Cartan subalgebra $\mathfrak{h}$ such that the restriction of the involution σ to $\mathfrak{h}$ is $-\mathrm{id}_{\mathfrak{h}}$, it follows that $\sigma(x_\alpha) = t_\alpha x_{-\alpha}$ with $t_\alpha^2 = 1$. Since $\mathfrak{g}^\sigma = \sum_{\alpha\in\Phi^+} \mathbb{C}(x_\alpha + t_\alpha x_{-\alpha})$, it is easy to see that

$$\mathfrak{g}_{-1} = (\mathfrak{g}^\sigma)^\perp = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Phi^+} \mathbb{C}(x_\alpha - t_\alpha x_{-\alpha})$$

(orthogonality being given by the Killing form). So, as sets, Y and the wonderful compactification are isomorphic (we identify $\mathbb{G}(d-l,\mathfrak{g})$ and $\mathbb{G}(d,\mathfrak{g})$ by the isomorphism $W \mapsto W^\perp$).

As a consequence, Y has dimension d. The next section shows that the equality is also true as a variety.

4. Equations of Y

Recall that the Grassmannian variety $\mathbb{G}$ has an exact sequence of locally free sheaves:

$$0 \longrightarrow K \longrightarrow \mathfrak{g}\otimes\mathcal{O}_{\mathbb{G}} \longrightarrow Q \longrightarrow 0, \tag{4.1}$$

where K is the tautological sheaf of rank d and Q the quotient sheaf of rank $\frac{g-l}{2}$. The datum $w \in \bigwedge^3 \mathfrak{g}^\vee$ gives a section $w_1 : \mathcal{O}_{\mathbb{G}} \to \bigwedge^3 K^\vee$ and by transposition, a morphism ${}^t w_1 : \bigwedge^3 K \to \mathcal{O}_{\mathbb{G}}$, whose image is an ideal defining Y as a scheme, denoted by I_Y.

Remark. — We describe this last morphism locally. Let $\Lambda \in Y$, take a base $x_1,\ldots,x_d$ of Λ and $y_1,\ldots,y_{n-d}$ a base of a complementary W of Λ. We can identify $\mathcal{U} = \mathrm{Hom}(\Lambda,W)$ with an affine open set of $\mathbb{G}$ by identifying $u\in\mathrm{Hom}(\Lambda,W)$ with the graph of u viewed in $\Lambda\oplus W = \mathfrak{g}$. Denote by $X_{i,j}$, with $1\leqslant i\leqslant d$ and $1\leqslant j\leqslant g-d$, the coordinates with respect to the previous basis. So ${}^t w_1\colon \bigwedge^3 \Lambda\otimes\mathcal{O}_{\mathbb{G},\Lambda} \to \mathcal{O}_{\mathbb{G},\Lambda}$ sends $x_{i_1}\bigwedge x_{i_2}\bigwedge x_{i_3}\otimes 1$ to

$$F_{i_1,i_2,i_3} := w\Big(x_{i_1}+\sum_j X_{i_1,j}\,y_j,\ x_{i_2}+\sum_j X_{i_2,j}\,y_j,\ x_{i_3}+\sum_j X_{i_3,j}\,y_j\Big). \tag{4.2}$$

The polynomials F_{i_1,i_2,i_3} for $1\leqslant i_1 < i_2 < i_3 \leqslant d$ span $I_{Y,\Lambda}$. To show that Y is isomorphic, as a scheme, to the wonderful compactification, we prove that Y is a smooth scheme. It is sufficient to show that Y is smooth on the minimal orbit, so we must analyze the stalk of Ω^1_Y, the sheaf of Kähler differentials, at a Borel subalgebra.

THEOREM 4.1. — *The scheme Y is smooth.*

Before proving the theorem, we need the following lemma.

Lemma 4.2. — *Let $\mathfrak{b}$ be a Borel subalgebra of $\mathfrak{g}$. The linear map D defined by*

$$\begin{array}{rcl} \bigwedge^3 \mathfrak{b} & \longrightarrow & \mathfrak{b} \otimes [\mathfrak{b}, \mathfrak{b}] \\ v_1 \wedge v_2 \wedge v_3 & \longmapsto & v_1 \otimes [v_2, v_3] + v_2 \otimes [v_3, v_1] + v_3 \otimes [v_1, v_2] \end{array}$$

has corank less than or equal to d.

Proof. — Let $\mathfrak{b} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_\alpha$ be a root space decomposition. For h and k in $\mathfrak{h}$, α and β in Φ^+, we have

$$\alpha(k) h \otimes x_\alpha = D(h \wedge k \wedge x_\alpha) + \alpha(h) k \otimes x_\alpha, \tag{4.3}$$

$$\alpha(h) x_\alpha \otimes x_\beta = D(h \wedge x_\alpha \wedge x_\beta) + \beta(h) x_\beta \otimes x_\alpha - h \otimes [x_\alpha, x_\beta], \tag{4.4}$$

$$\begin{aligned} x_{\alpha+\beta} \otimes x_{\alpha+\beta} = D(x_{\alpha+\beta} \wedge x_\alpha \wedge x_\beta) - x_\alpha \otimes [x_\beta, x_{\alpha+\beta}] \\ + x_\beta \otimes [x_\alpha, x_{\alpha+\beta}]. \end{aligned} \tag{4.5}$$

Let A be the subspace of coker D spanned by $h_\alpha \otimes x_\alpha$, where $\alpha(h_\alpha) = 2$ and $\alpha \in \Phi^+$. For suitable h and k, equalities (4.3) and (4.4) show that $h \otimes x_\alpha$ with $\alpha(h) = 0$ are in $\operatorname{Im} D$, and $x_\alpha \otimes x_\beta$ with $\alpha \neq \beta$ are in A, hence it follows from (4.5) that $x_\alpha \otimes x_\alpha \in A$ if α is not simple. Finally, coker $D \subset A$. So the number of generators is $\frac{g-l}{2} + l = d$. □

Proof of Theorem 4.1. — Let $\mathfrak{b}$ be a Borel subalgebra, $\mathfrak{h} \subset \mathfrak{b}$ be a Cartan subalgebra, Φ be the root system of $(\mathfrak{g}, \mathfrak{h})$, with positive roots given by $\mathfrak{b}$, and $\mathfrak{g} = \mathfrak{b} \oplus \mathfrak{n}^-$ be a root space decomposition with basis $x_1, \ldots, x_d$ for $\mathfrak{b}$ (positive root vectors and a basis of $\mathfrak{h}$), $y_1, \ldots, y_{n-d}$ for $\mathfrak{n}^-$ (negative root vectors) such that $\kappa(x_i, y_i) \neq 0$, for $i \in \{1, \ldots, d\}$. We use the following exact sequence on sheaves of differentials:

$$I_Y / I_Y^2 \longrightarrow \Omega_{\mathbb{G}} \otimes \mathcal{O}_Y \longrightarrow \Omega_Y \longrightarrow 0.$$

Locally, we can compute the differential of F_{i_1,i_2,i_3} in $\Omega_{\mathbb{G},\mathfrak{b}}$ (image of the first map in the sequence). The result is

$$\begin{aligned} dF_{i_1,i_2,i_3} = \sum_j w(y_j, x_{i_2}, x_{i_3})\, dX_{i_1,j} + \sum_j w(x_{i_1}, y_j, x_{i_3})\, dX_{i_2,j} \\ + \sum_j w(x_{i_1}, x_{i_2}, y_j) dX_{i_3,j}. \end{aligned} \tag{4.6}$$

But $\Omega_{\mathbb{G},\mathfrak{b}}$ is isomorphic to $(\mathfrak{g}/\mathfrak{b})^\vee \otimes \mathfrak{b}$, sending $dX_{i,j}$ to $y_i^\vee \otimes x_j$, where $y_i^\vee = \frac{1}{\kappa(x_i, y_i)} \kappa(x_i, \cdot)$ (the duality is relating to the Killing form κ, and we normalize to have $y_i^\vee(y_i) = 1$). For the first sum in (4.6), we have

$$\sum_j w(y_j, x_{i_2}, x_{i_3})\, y_j^\vee \otimes x_{i_1} = \kappa\left(\sum_j \frac{\kappa(y_j, [x_{i_2}, x_{i_3}])}{\kappa(y_j, x_j)} x_j\,, \cdot\right) \otimes x_{i_1} \tag{4.7}$$

$$= \kappa([x_{i_2}, x_{i_3}], \cdot) \otimes x_{i_1}. \tag{4.8}$$

The composition map $\bigwedge^3 \mathfrak{b} \otimes \mathcal{O}_{\mathbb{G},\mathfrak{b}} \to I_{Y,\mathfrak{b}}/I^2_{Y,\mathfrak{b}} \to \Omega_{\mathbb{G},\mathfrak{b}} \otimes \mathcal{O}_{Y,\mathfrak{b}}$ sends $x_{i_1} \bigwedge x_{i_2} \bigwedge x_{i_3}$ to

$$\kappa([x_{i_2}, x_{i_3}], \cdot) \otimes x_{i_1} + \kappa([x_{i_1}, x_{i_2}], \cdot) \otimes x_{i_3} + \kappa([x_{i_3}, x_{i_1}], \cdot) \otimes x_{i_2}.$$

Thanks to Lemma 4.2, we conclude that corank of $I_{Y,\mathfrak{b}}/I^2_{Y,\mathfrak{b}} \to \Omega_{\mathbb{G},\mathfrak{b}} \otimes \mathcal{O}_{Y,\mathfrak{b}}$ is less than of equal to d, so $\operatorname{rank} \Omega_{Y,\mathfrak{b}} \leqslant d$, and the result follows. □

A consequence of Theorem 4.1 is that Y is isomorphic to the wonderful compactification.

Theorem 4.3. — *The equations of Y in $\mathbb{G}$ are linear.*

Proof. — Recall that $\bigwedge^d K = \mathcal{O}_{\mathbb{G}}(-1)$, so $\operatorname{Hom}\left(\bigwedge^d K, \bigwedge^3 K\right) \simeq \bigwedge^{d-3} K^\vee$, and so $\bigwedge^3 K(1) \simeq \bigwedge^{d-3} K^\vee$. Moreover, from (4.1), we have $\bigwedge^{d-3} \mathfrak{g}^\vee \otimes \mathcal{O}_{\mathbb{G}} \twoheadrightarrow \bigwedge^{d-3} K^\vee$. This forces $\bigwedge^{d-3} K^\vee$ to be spanned by its sections, and so does it to $\bigwedge^3 K(1)$. Thanks to the morphism ${}^t w_1$, $I_Y(1)$ is spanned by its sections. □

We give a result on global sections of $I_Y(1)$ when $\mathfrak{g}$ is a simple Lie algebra. Extending $w\colon \bigwedge^3 \mathfrak{g} \to \mathbb{C}$ to $\bigwedge^{k+3} \mathfrak{g} \to \bigwedge^k \mathfrak{g}$ with k a positive integer, we build a $\mathfrak{g}$-invariant differential operator on $\bigwedge \mathfrak{g}$, denoted by δ^*, satisfying $(\delta^*)^2 = 0$. On the other side, by identifying $\mathfrak{g}$ and his dual by the Killing form, w can be seen as an element of $\bigwedge^3 \mathfrak{g}$, the morphism of $\mathfrak{g}$-module $\bigwedge^k \mathfrak{g} \overset{\wedge w}{\to} \bigwedge^{k+3} \mathfrak{g}$ with k a non negative integer defines another $\mathfrak{g}$-invariant differential operator on $\bigwedge \mathfrak{g}$, denoted by δ. Finally, We have built two complexes $(\bigwedge \mathfrak{g}, \delta)$ and $(\bigwedge \mathfrak{g}, \delta^*)$.

Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$, Φ be the root system of $(\mathfrak{g}, \mathfrak{h})$. We choose Φ^+ a set of positive roots of Φ, and denote by ρ is the half sum of positive roots. The $\mathfrak{g}$-module $\bigwedge \mathfrak{g}$ has highest weight 2ρ. The morphisms δ and δ^* vanish on all occurrences of $V_{2\rho}$ the irreducible representation of highest weight 2ρ in $\bigwedge \mathfrak{g}$, so the complexes $(\bigwedge \mathfrak{g}, \delta)$ and $(\bigwedge \mathfrak{g}, \delta^*)$ are not acyclic.

For k in $\{1, \dots, g\}$, we search the occurrences of the representation $V_{2\rho}$ in $\wedge^k \mathfrak{g}$.

(1) If $k < \sharp\Phi^+ = \frac{g-l}{2}$ or $k > g - \sharp\Phi^+ = \frac{g+l}{2}$, then there is no weight vector of weight 2ρ in $\wedge^k \mathfrak{g}$. So $V_{2\rho}$ does not appear in this case: we put $\overline{\wedge}^k \mathfrak{g} := \bigwedge^k \mathfrak{g}$.

(2) If $\frac{g-l}{2} \leqslant k \leqslant \frac{g+l}{2}$, $\bigwedge^{k-\frac{g-l}{2}} \mathfrak{h} \wedge \bigwedge_{\alpha\in\Phi^+} x_\alpha$ (x_α a root vector of α) is the weight space of weight 2ρ in $\bigwedge^k \mathfrak{g}$. Hence, $\bigwedge^{k-\frac{g-l}{2}} \mathfrak{h} \otimes V_{2\rho}$ represents all occurrences of $V_{2\rho}$ in $\bigwedge^k \mathfrak{g}$. Let $\overline{\wedge}^k \mathfrak{g}$ be a $\mathfrak{g}$-submodule of $\bigwedge^k \mathfrak{g}$ such that $\bigwedge^k \mathfrak{g} = \overline{\wedge}^k \mathfrak{g} \oplus \bigwedge^{k-g+d} \mathfrak{h} \otimes V_{2\rho}$.

Thanks to the fact that δ and δ^* preverse the weights, we can defined the restriction of δ, δ^* to $\overline{\wedge}\mathfrak{g}$. We have constructed $(\overline{\wedge}\mathfrak{g}, \delta)$ a subcomplex of $(\bigwedge \mathfrak{g}, \delta)$.

Lemma 4.4. — *Assume $\mathfrak{g}$ is simple. The sequences*

$$0 \longrightarrow \overline{\wedge}^{m'} \mathfrak{g} \xrightarrow{\delta} \overline{\wedge}^{m'+3} \mathfrak{g} \xrightarrow{\delta} \cdots \xrightarrow{\delta} \overline{\wedge}^{g-3} \mathfrak{g} \xrightarrow{\delta} \bigwedge^{g} \mathfrak{g} \longrightarrow 0,$$

$$0 \longrightarrow \mathbb{C} \xrightarrow{\delta} \overline{\wedge}^{3} \mathfrak{g} \xrightarrow{\delta} \cdots \xrightarrow{\delta} \overline{\wedge}^{m-3} \mathfrak{g} \xrightarrow{\delta} \overline{\wedge}^{m} \mathfrak{g} \longrightarrow 0,$$

with $m \in \{g-2, g-1, g\}$ and $m' \in \{0,1,2\}$, are exact.

Remarks. — We could write sequences with δ^* decreasing wedge powers of $\mathfrak{g}$, which gives other exact sequences for $\overline{\wedge}\mathfrak{g}$.

The complex $(\bigwedge \mathfrak{g}, \delta)$ is a direct sum of two complexes, the first $\overline{\wedge}\mathfrak{g}$ is acyclic, and the second given by $\bigwedge \mathfrak{h} \otimes V_{2\rho}$ with $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$, is trivial.

Assume for the moment this lemma. In the proof of Theorem 4.3, ${}^t w_1 : \bigwedge^3 K(1) \to \mathcal{O}_{\mathbb{G}}(1) \simeq \bigwedge^d K$ gives a $\mathfrak{g}$-invariant morphism on global sections $\bigwedge^{d-3} \mathfrak{g} = \mathrm{H}^0(\bigwedge^3 K(1)) \to \mathrm{H}^0(\mathcal{O}_{\mathbb{G}}(1)) = \bigwedge^d \mathfrak{g}$, it is just δ. We deduce the following proposition.

Proposition 4.5. — *If $\mathfrak{g}$ is simple, then $\mathrm{H}^0(I_Y(1))$ contains $\delta\Big(\bigwedge^{d-3} \mathfrak{g}\Big)$.*

This proposition shows that we can embed the wonderful compactification in a projective space with dimension smaller than $\mathbb{P}(\bigwedge^d \mathfrak{g})$.

Now we prove Lemma 4.4. The main idea is the study of $\zeta = \delta^*\delta + \delta\delta^*$ as a $\mathfrak{g}$-invariant differential operator. The multiplication by an element of $\mathfrak{g}$ and the derivation (action by an element of $\mathfrak{g}^\vee$) spans the ring of differential operators on $\bigwedge \mathfrak{g}$ identified to the Clifford algebra $\mathrm{Cliff}(\mathfrak{g} \oplus \mathfrak{g}^\vee, ev)$ where ev is the duality bracket. Recall that $\mathrm{Cliff}(\mathfrak{g} \oplus \mathfrak{g}^\vee, ev)$ has a $\mathbb{Z}/2\mathbb{Z}$-graduation, which allows us to put a structure of Lie superalgebra. Define a filtration (F^i) with F^i spanned by products of multiplications and at most i derivations. We recall two useful results:

1. $[F^i, F^j] \subset F^{i+j-1}$,
2. an element χ of F^i is zero if $\chi_{|_{\wedge^k \mathfrak{g}}} = 0$; for $k \leqslant i$, in other words, elements of F^i are completely known by the image of $\bigoplus_{k\leqslant i} \bigwedge^k \mathfrak{g}$.

For our case, $\delta \in F^0$ and $\delta^* \in F^3$ so $\zeta = [\delta, \delta^*] \in F^2$. The Casimir operator c and powers of the Euler operator e, $e^0 = \mathrm{id}$, e, e^2 (e is defined as $e_{|_{\wedge^i \mathfrak{g}}} = i \cdot \mathrm{id}$) are $\mathfrak{g}$-invariant differential operators in F^2. We need the following lemma to prove that ζ is a linear combination of c, id, e and e^2.

Lemma 4.6. — *Denote by $V_{a_1\omega_1+\cdots+a_l\omega_l}$ the irreducible representation of $\mathfrak{g}$ with highest weight $a_1\omega_1 + \cdots + a_l\omega_l$, where $\omega_1, \ldots, \omega_l$ are the fundamental weights according the notations of* [1]. *We have:*

(i) $\bigwedge^2 \mathfrak{sl}(n+1) = \mathfrak{sl}(n+1) \oplus V_{2\omega_1+\omega_{n-1}} \oplus V_{\omega_2+2\omega_n}$, *for* $n \geqslant 3$, *and* $\bigwedge^2 \mathfrak{sl}(3) = \mathfrak{sl}(3) \oplus V_{3\omega_1} \oplus V_{3\omega_2}$,
(ii) $\bigwedge^2 \mathfrak{sp}(2n) = \mathfrak{sp}(2n) \oplus V_{2\omega_1+\omega_2}$, *for* $n \geqslant 2$,
(iii) $\bigwedge^2 \mathfrak{so}(n) = \mathfrak{so}(n) \oplus V_{\omega_1+\omega_3}$, *for* $n \geqslant 6$,
(iv) $\bigwedge^2 \mathfrak{f}_4 = \mathfrak{f}_4 \oplus V_{\omega_2}$,
(v) $\bigwedge^2 \mathfrak{g}_2 = \mathfrak{g}_2 \oplus V_{3\omega_1}$,
(vi) $\bigwedge^2 \mathfrak{e}_6 = \mathfrak{e}_6 \oplus V_{\omega_4}$,
(vii) $\bigwedge^2 \mathfrak{e}_7 = \mathfrak{e}_7 \oplus V_{\omega_3}$,
(viii) $\bigwedge^2 \mathfrak{e}_8 = \mathfrak{e}_8 \oplus V_{\omega_8}$.

The proof of this lemma is given by computation with a program named *LiE* (see [7]).

Except for $\mathfrak{sl}(n+1)$, the $\mathfrak{g}$-module $\mathbb{C} \oplus \mathfrak{g} \oplus \bigwedge^2 \mathfrak{g}$ has four irreducible factors: c, id, e, e^2 form a basis of $\mathfrak{g}$-invariant differential operators of F^2, so ζ is a linear combination of c, id, e and e^2. For $\mathfrak{sl}(n+1)$, $n \geqslant 2$, remark that $\bigwedge^2 \mathfrak{sl}(n+1) = \mathfrak{sl}(n+1) \oplus W \oplus W^\vee$, with $W = V_{2\omega_1+\omega_{n-1}}$ or $W = V_{3\omega_1}$, and ζ, c, id, e, e^2 do not distinguish an irreducible representation and its dual. Considering $W \oplus W^\vee$ as one factor, $\mathbb{C} \oplus \mathfrak{g} \oplus \bigwedge^2 \mathfrak{g}$ has four factors. We can treat $\mathfrak{sl}(n+1)$ as the other simple Lie algebras.

There exist a scalar a and a polynomial P of degree less than or equal to 2 such that $\zeta - ac = P(e)$. Applying this expression on $1 \in \mathbb{C}$ and $w \in \bigwedge^3 \mathfrak{g}$, it follows that $P(0) = P(3) = \delta^*(w)$. But the isomorphism $\bigwedge^k \mathfrak{g} \simeq \bigwedge^{g-k} \mathfrak{g}$ shows that $P(g-3) = P(g) = P(3)$. Finally, P is constant, thus

$$\zeta = ac + \delta^*(w)\mathrm{id}. \tag{4.9}$$

If V_λ is an irreducible representation of highest weight λ, denote by the scalar c_λ the action of c on V_λ. So, applying (4.9) to the highest weight vector of $V_{2\rho}$, we have $0 = ac_{2\rho} + \delta^*(w)$, and so

$$\zeta = \delta^*(w)\left(\mathrm{id} - \frac{1}{c_{2\rho}} c\right).$$

Lemma 1 of chapter 2 in [5] gives that $c_\lambda < c_{2\rho}$, if $\lambda \neq 2\rho$ and 2ρ dominates the dominant weight λ. Moreover, an irreducible $\mathfrak{g}$-module V which appears in $\bigwedge \mathfrak{g}$ has highest weight dominated by 2ρ: the restriction of ζ to V is just the multiplication by a non-zero scalar, except for $V_{2\rho}$.

Proof of Lemma 4.4. — Let k be a positive integer, and V be an irreducible representation which appears in $\mathrm{Ker}(\delta) \cap \overline{\wedge}^k \mathfrak{g}$, $\zeta_{|V} = \lambda \mathrm{id}_V$ with $\lambda \neq 0$. This statement implies that if $x \in V$ then $x = \delta\left(\frac{1}{\lambda}\delta^*(x)\right) \in \mathrm{Im}(\delta)$. So all irreducible representations appearing in $\mathrm{Ker}(\delta) \cap \overline{\wedge}^k \mathfrak{g}$ are subsets of $\mathrm{Im}(\delta) \cap \overline{\wedge}^k \mathfrak{g}$, so $\mathrm{Ker}(\delta) \cap \overline{\wedge}^k \mathfrak{g} = \mathrm{Im}(\delta) \cap \overline{\wedge}^k \mathfrak{g}$. This forces the two sequences to be exact. $\square$

5. Correspondence between orbits of Y and sets of simple roots

The results on orbits of Y agree with the nice properties of the wonderful compactification. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ such that $\sigma_{|\mathfrak{h}} = -\mathrm{id}_{\mathfrak{h}}$. Let Φ be the root system of $(\mathfrak{g}, \mathfrak{h})$.

1. Denote by $\mathfrak{p}_i$ the parabolic subalgebra spanned by a Borel subalgebra $\mathfrak{b}$ such that $\mathfrak{h} \subset \mathfrak{b}$ and by $\mathfrak{g}_{-\alpha_j}$ where j runs over $\{\alpha_1, \dots, \alpha_{i-1}, \alpha_{i+1}, \dots, \alpha_l\}$, with $\Delta = \{\alpha_1, \dots, \alpha_l\}$ a root basis with respect to $\mathfrak{b}$. The orbit closures corresponding to the $\mathfrak{p}_i$'s are the l hypersurfaces S_{α_i}.
2. The orbit closures are $S_I = \cap_{\alpha \in I} S_\alpha$ where $I \subset \Delta$.

Now, we explain the correspondence between orbit closures and subsets of Δ. Denote by $\mathfrak{b}^-$ the Borel subalgebra spanned by $\mathfrak{h}$ and $x_{-\alpha}$ (with $\alpha \in \Delta$), $\mathfrak{n} = [\mathfrak{b}^-, \mathfrak{b}^-]$, N the adjoint group of $\mathfrak{n}$, and T the maximal torus of G with Lie algebra equal to $\mathfrak{h}$. Let

$$Y_{\mathfrak{h}} = \{U \in Y \text{ such that } U \cap \mathfrak{b}^- = \mathfrak{h}\}.$$

The last remark of Section 2 shows that

$$\begin{array}{cccc} p: & \mathbb{C}^\Delta & \longrightarrow & Y_{\mathfrak{h}} \\ & (t_\alpha)_{\alpha \in \Delta} & \longmapsto & \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi^+} (x_\alpha + t_\alpha x_{-\alpha}) \end{array}$$

where, for α, β in Φ such that $\alpha + \beta \in \Phi$,

$$t_{\alpha+\beta} = -\frac{w(x_{\alpha+\beta}, x_{-\alpha}, x_{-\beta})}{w(x_{-\alpha-\beta}, x_\alpha, x_\beta)} t_\alpha t_\beta,$$

is a T-equivariant isomorphism, so $\dim Y_{\mathfrak{h}} = l$ (the action of T on $\mathbb{C}^\Delta$ is defined by $e^h.(t_\alpha)_{\alpha \in \Delta} = (e^{\alpha(h)} t_\alpha)_{\alpha \in \Delta}$). We prove that there is a correspondence between G-orbit closures of Y and T-orbit closures of $\mathbb{C}^\Delta$, which are the $\mathbb{C}^I$, where I is a subset of Δ (the T-orbits have the form $(\mathbb{C}^*)^I$ with I a subset of Δ).

Proposition 5.1. — *The morphism*

$$\begin{array}{rccc} \psi : & N\times Y_{\mathfrak{h}} & \longrightarrow & Y \\ & (n,U) & \longmapsto & n.U \end{array}$$

is an open immersion.

Proof. — If n_1, n_2 are in N, and U_1, U_2 are two elements of $Y_{\mathfrak{h}}$ such that $n_1\cdot U_1 = n_2\cdot U_2$, then $n_2^{-1}n_1\cdot\mathfrak{h} = (n_2^{-1}n_1\cdot U_1)\cap\mathfrak{b}^- = U_2\cap\mathfrak{b}^- = \mathfrak{h}$, that is to say $n_2^{-1}n_1 = 1$ (the normalizer of a maximal torus contains no unipotent elements), so $U_1 = U_2$: ψ is injective. Moreover, $\dim N\times Y_{\mathfrak{h}} = \dim Y$ forces ψ to be dominant, so finally ψ is birational. We conclude with a corollary of the main theorem of Zariski: since ψ is birational, with finite fibres, then, because Y is smooth, ψ is an isomorphism between X and an open subset U of Y. □

Let O be an orbit of Y and $U\in O$. Recall that $\mathfrak{p}_U = U+[U,U]$ is a parabolic subalgebra of $\mathfrak{g}$, so there exists a subset S of Φ^+ such that $\mathfrak{p}_U$ is conjugate to $\mathfrak{p} = \mathfrak{b}\oplus\sum_{\alpha\in -S}\mathfrak{g}^\alpha$. Now we build an element V of $Y_{\mathfrak{h}}$ such that $\mathfrak{p} = V+[V,V]$:

$$V = \mathfrak{h}\oplus\bigoplus_{\alpha\in\Phi^+\smallsetminus S}\mathbb{C}x_\alpha\oplus\bigoplus_{\alpha\in S}\mathbb{C}(x_\alpha+x_{-\alpha}),$$

with $x_\alpha\in\mathfrak{g}^\alpha$ for $\alpha\in\Phi^+$ and $x_{-\alpha}\in\mathfrak{g}^{-\alpha}$ for $\alpha\in S$ such that $w(x_\alpha+x_{-\alpha}, x_\beta+x_{-\beta}, x_{\alpha+\beta}+x_{-\alpha-\beta}) = 0$ if α, β and $\alpha+\beta$ are in S (this is possible thanks to Lemma 2.4 and its following remark). Since $\mathfrak{p}_U$ and $\mathfrak{p}$ are conjugate, Proposition 3.1 implies that V and U are conjugate, so $O\cap Y_{\mathfrak{h}}\neq\emptyset$.

Proposition 5.2. — *There is a bijection between the T-orbit closures of $Y_{\mathfrak{h}}$ and the G-orbit closures of Y, defined as follows:*

$$\begin{array}{ccc} \{G-\text{orbit closure in } Y\} & \longleftrightarrow & \{T-\text{orbit closure in } Y_{\mathfrak{h}}\} \\ \overline{O} & \longmapsto & \overline{O}\cap Y_{\mathfrak{h}}. \end{array}$$

This map preserves intersections. Moreover, G-orbit closures are smooth.

Remark. — For a G-orbit closure $\overline{O}$, the set $\overline{O}\cap Y_{\mathfrak{h}}$ is a T-stable closed set, so is isomorphic to $(\mathbb{C})^I$ where I is a subset of Δ.

Proof. — Let $\overline{O}$ be a G-orbit closure of Y. The closed set $\overline{O}\cap\psi(N\times Y_{\mathfrak{h}})$ of $\psi(N\times Y_{\mathfrak{h}})$ is T-stable and N-stable, so there exists a subset I of Δ such that $\overline{O}\cap\psi(N\times Y_{\mathfrak{h}}) = \psi(N\times\mathbb{C}^I)$. But $\overline{O}\cap\psi(N\times Y_{\mathfrak{h}})$ is open in $\overline{O}$, so $\overline{O} = \overline{\psi(N\times\mathbb{C}^I)}$. This forces the map $\overline{O}\mapsto\overline{O}\cap Y_{\mathfrak{h}}$ to be an injection. Since Y has 2^l orbit closures and the cardinal of $\mathcal{P}(\Delta)$ is equal to 2^l, the bijection follows.

It is clear that $\overline{O} \cap \psi(N \times Y_{\mathfrak{h}}) \simeq N \times \mathbb{C}^I$ is smooth. If the singular set of $\overline{O}$ is non empty, it is a G-stable closed set, so meets $\overline{O} \cap \psi(N \times Y_{\mathfrak{h}})$, a contradiction. □

Remarks. — Let $\overline{O}$ be a G-orbit closure of Y. There exists $I \in \mathcal{P}(\Delta)$ such that $\overline{O} = \overline{\psi(N \times \mathbb{C}^I)}$.

(a) We have codim $\overline{O} = \sharp(\Delta \smallsetminus I)$.

(b) For $\alpha \in \Delta$, $\overline{\psi(N \times \mathbb{C}^{\Delta \smallsetminus \{\alpha\}})}$ is a G-stable closed sets of codimension one of Y, so is equal to S_α.

6. Examples

6.1. The case $\mathfrak{sl}(3)$

Let V be a vector space of dimension 3 and S^2V be the vector space of conics on V. The closure Z of the graph of the duality isomorphism in $\mathbb{P}(S^2V) \times \mathbb{P}(S^2V^\vee)$, defined as,

$$\begin{array}{ccc} \mathbb{P}(S^2V) & \cdots\longrightarrow & \mathbb{P}(S^2V^\vee) \\ q & \longmapsto & \wedge^2 q, \end{array}$$

is called the variety of *complete conics*, and the map $p : Z \to \mathbb{P}(S^2V)$ is known to be the blow up of $\mathbb{P}(S^2V)$ along the Veronese surface (conics of rank one on V). We refer to the appendix of [6] for more results. So, if J is the sheaf of ideals of the Veronese surface in $\mathbb{P}(S^2(V))$, $Z = \mathrm{Proj}(J)$ is embedded in $\mathrm{Proj}\left(\mathrm{H}^0(J(3))\right) = \mathbb{P}(\mathbb{C} \oplus V_{2\rho})$, where $V_{2\rho}$ is the irreducible $\mathfrak{sl}(3)$-module of dimension 27 which corresponds to the irreducible representation of highest weight 2ρ. We finish our description with the commutative diagram:

$$\begin{array}{ccc} Z = P \cap S & \longrightarrow & P := \mathbb{P}(\mathbb{C} \oplus V_{2\rho}) \\ \downarrow & & \downarrow \\ S := \mathbb{P}(S^2V) \times \mathbb{P}(S^2V^\vee) & \longrightarrow & \mathbb{P}(S^2V \otimes S^2V^\vee) \end{array}$$

Remark. — The variety Z can be defined as:

$$Z = \Big\{([q \otimes q']) \in \mathbb{P}(S^2V \otimes S^2V^\vee) \text{ such that } qq' \in \mathbb{C}\,\mathrm{Id}\Big\}.$$

We can easily find the orbit closures: one when $\operatorname{rank} q = 1$, an other when $\operatorname{rank} q' = 1$, and their intersection (the closed orbit).

Littelmann and Procesi in [2] show that Z is isomorphic to the wonderful compactification of $PGL(3)/PSO(3)$. In this part, we find equations defining the wonderful compactification in $\mathbb{G}(3, \mathfrak{sl}(3))$.

Let e_1, e_2, e_3 be a basis of V and consider the quadratic form $q = e_1^2 + e_2^2 + e_3^2$. The morphism $\sigma : PGL(3) \to PGL(3)$ which sends $[g]$ to $[q^{-1}\, {}^t g^{-1}\, q]$ is an involution, and $PGL(3)^\sigma = PSO(q)$. We have seen in Section 4 that the sheaf of equations of Y in $\mathbb{G}(5, \mathfrak{sl}(3))$ (and so $\bar{X}$ in $\mathbb{G}(3, \mathfrak{sl}(3))$) is the image $I_{\bar{X}}$ of $\bigwedge^3 K \to \mathcal{O}_{\mathbb{G}(5,\mathfrak{sl}(3))}$. So, thanks to Proposition 4.5, for $\mathfrak{sl}(3)$, $\mathrm{H}^0(I_{\bar{X}}(1))$ is a submodule of the $\mathfrak{sl}(3)$-module $\bigwedge^3 \mathfrak{sl}(3)$ which is isomorphic to $\bigwedge^2 \mathfrak{sl}(3) \oplus \mathbb{C} \oplus V_{2\rho}$, and contains $\bigwedge^2 \mathfrak{sl}(3)$. But two cases are impossible:

i. If $\mathrm{H}^0(I_{\bar{X}}(1)) = \bigwedge^2 \mathfrak{sl}(3) \oplus V_{2\rho}$, then $\bar{X}$ satisfies the equations of $\mathbb{P}(\mathbb{C})$, and so $\bar{X}$ is a point.

ii. If $\mathrm{H}^0(I_{\bar{X}}(1)) = \bigwedge^2 \mathfrak{sl}(3) \oplus \mathbb{C}$, then $\bar{X} \subset \mathbb{P}(V_{2\rho})$. But elements in the open orbit of $\bar{X} \subset \mathbb{P}(\bigwedge^3 \mathfrak{g})$ have non zero images by δ^*, so they can not be in $V_{2\rho} \subset \ker \delta^*$.

Hence, $\mathrm{H}^0(I_{\bar{X}}(1)) = \bigwedge^2 \mathfrak{sl}(3)$. In particular, $\bar{X}$ satisfies the equations of $\mathbb{P}(\mathbb{C} \oplus V_{2\rho}) \subset \mathbb{P}(\bigwedge^3 \mathfrak{sl}(3))$. We summarize the results obtained so far.

Lemma 6.1. — *The wonderful compactification is the intersection of* $\mathbb{P}(\mathbb{C} \oplus V_{2\rho})$ *and* $\mathbb{G}(3, \mathfrak{sl}(3))$ *in* $\mathbb{P}(\bigwedge^3 \mathfrak{sl}(3))$.

Since Z and $\bar{X}$ are identified to subvarieties of $\mathbb{P}(\mathbb{C} \oplus V_{2\rho})$, we can prove that the wonderful compactification is isomorphic to the variety of complete conics.

Proposition 6.2. — *There exists a* $PGL(3)$*-equivariant automorphism of* $\mathbb{P}(V_{2\rho} \oplus \mathbb{C})$ *which sends* $\bar{X}$ *to* Z.

Proof. — It is enough to find a $PGL(3)$-invariant isomorphism of $\mathbb{P}(\mathbb{C} \oplus V_{2\rho})$ which sends an element of the open orbit of $\bar{X}$ to an element of the open orbit of Z.

Now, $\mathfrak{sl}(3)$ being viewed as a submodule of $V \otimes V^\vee$, the composition of morphisms of $\mathfrak{sl}(3)$-modules denoted by Ψ,

$$\bigwedge^3 \mathfrak{sl}(3) \longrightarrow V \otimes V \otimes V \otimes V^\vee \otimes V^\vee \otimes V^\vee \longrightarrow V \otimes V \otimes V^\vee \otimes V^\vee \longrightarrow S^2 V \otimes S^2 V^\vee,$$

has a restriction to $\mathbb{C} \oplus V_{2\rho} \to \mathbb{C} \oplus V_{2\rho}$ which is an isomorphism.

As $q = e_1^2 + e_2^2 + e_3^2$ is a non-degenerate conic on V, it gives $q \otimes q^\vee = (e_1^2 + e_2^2 + e_3^2) \otimes ((e_1^\vee)^2 + (e_2^\vee)^2 + (e_3^\vee)^2)$ a point of the open orbit of Z, and $\mathfrak{so}(q)$, a point of the open orbit of $\bar{X}$, seen in $\bigwedge^3 \mathfrak{sl}(3)$ as $(e_1 \otimes e_2^\vee - e_2 \otimes e_1^\vee) \wedge (e_2 \otimes e_3^\vee - e_3 \otimes e_2^\vee) \wedge (e_1 \otimes e_3^\vee - e_3 \otimes e_1^\vee)$. The morphism Ψ sends the point of $\bar{X}$ to $q \otimes q^\vee + z^2$, $z = e_1 \otimes e_1^\vee + e_2 \otimes e_2^\vee + e_3 \otimes e_3^\vee$. Moreover, the component of $q \otimes q^\vee$ on the factor $\mathbb{C}$ in $V_{2\rho} \oplus \mathbb{C}$ is $1/2z^2$, so $\phi = \mathrm{id}_{V_{2\rho}} + \frac{1}{3}\mathrm{id}_{\mathbb{C}}$, the automorphism of $V_{2\rho} \oplus \mathbb{C}$ sends $q \otimes q^\vee + z^2$ to $q \otimes q^\vee$.

Finally, the composition map $\phi \circ \Psi$ is a $PGL(3)$-equivariant automorphism of $\mathbb{P}(V_{2\rho} \oplus \mathbb{C})$ which sends $\bar{X}$ to Z. $\square$

6.2. The case $\mathfrak{sp}(4)$

Let V be the irreducible representation of $\mathfrak{sp}(4)$ of dimension 4. We describe the elements of $\mathfrak{sp}(4)$ in block form in the decomposition $V = U \oplus U^\vee$ (U being an isotropic vector subspace of dimension 2):

$$\begin{pmatrix} u & v \\ w & -{}^t u \end{pmatrix}$$

with $u \in \mathrm{Hom}(U,U)$, and $v \in \mathrm{Hom}(U^\vee, U)$, $w \in \mathrm{Hom}(U, U^\vee)$ are symmetric. Let

$$S = \begin{pmatrix} \mathrm{id}_U & 0 \\ 0 & -\mathrm{id}_{U^\vee} \end{pmatrix},$$

then $\sigma\colon M \to SMS$ is an involution of $\mathfrak{sp}(4)$ and of its adjoint group $PSp(4)$. We have $\mathfrak{sp}(4)^\sigma \simeq U \otimes U^\vee \simeq \mathfrak{gl}(2)$, and the wonderful compactification of the corresponding symmetric space $PSp(4)/GL(2)$ is of rank 2. We use the variety Y introduced in Section 3 to describe its wonderful compactification.

Let W be the irreducible representation of $\mathfrak{so}(5)$ of dimension 5. Recall that $\bigwedge^2 W \simeq \mathfrak{so}(5) \simeq \mathfrak{sp}(4) \simeq S^2 V$, and $\bigwedge^2 V = W \oplus \mathbb{C}$. We denote $v\colon \mathbb{P}(V) \to \mathbb{P}(S^2 V)$ the Veronese surface embedding and $\mathcal{V} = v(\mathbb{P}(V))$, $\mathcal{V} \subset G(2,W)$ the Grassmannian variety of planes in W.

Theorem 6.3. — *The wonderful compatification of $PSp(4)/GL(2)$ with maximal rank is isomorphic to $\widetilde{G}$, the blow up of the Grassmannian $G(2,W)$ along the Veronese surface $\mathcal{V}$.*

Note that $\widetilde{G}$ is smooth and $\dim \widetilde{G} = \dim G(2,W) = 6$. The main idea is to embed our two smooth varieties in the same projective spaces $\mathbb{P}(V_{2\rho} \oplus \mathfrak{sp}(4))$, and then we find a $PSp(4)$-invariant automorphism of $\mathbb{P}(V_{2\rho} \oplus \mathfrak{sp}(4))$ which sends one of them to the other one.

Remark. — Lemma 4.4 gives the exact sequence of $\mathfrak{sp}(4)$-modules:

$$0 \longrightarrow \bigwedge^{9} \mathfrak{sp}(4) \xrightarrow{\delta} \bigwedge^{6} \mathfrak{sp}(4)/V_{2\rho} \xrightarrow{\delta} \bigwedge^{3} \mathfrak{sp}(4) \xrightarrow{\delta} \mathbb{C} \longrightarrow 0.$$

For the proof of Theorem 6.3, we need the decomposition into irreducible representations of each term:

$$\bigwedge^6 \mathfrak{sp}(4) = V_{2\rho} \oplus V_{4\omega_1} \oplus V_{3\omega_2} \oplus V_{2\omega_1+\omega_2} \oplus V_{2\omega_2} \oplus V_{\omega_2} \oplus \mathfrak{sp}(4),$$
$$\bigwedge^3 \mathfrak{sp}(4) = V_{4\omega_1} \oplus V_{3\omega_2} \oplus V_{2\omega_1+\omega_2} \oplus V_{2\omega_2} \oplus V_{\omega_2} \oplus \mathbb{C}.$$

Proof of Theorem 6.3.

For the wonderful compactification. Choose an element U in the open orbit of the variety Y and let u be a representative of U in $\bigwedge^6 \mathfrak{sp}(4)$. Clearly $\delta(u) = 0$, so thanks to the previous remark, u is an element of $V_{2\rho} \oplus \mathfrak{sp}(4)$ so $Y \subset \mathbb{P}(V_{2\rho} \oplus \mathfrak{sp}(4))$. If c is the Casimir element of $\mathfrak{sp}(4)$, a explicit computation shows that u and $c.u$ are independent, so u does not lie in one irreducible representation.

For the blow up $\widetilde{G}$. As $GL(W)$-module, $\mathrm{H}^0(\mathcal{O}_{G(2,W)}(3))$ is isomorphic to the irreducible representation with partition $(3,3)$ (see Proposition 3.14 in [9]). So using branching formulae in [4], we have the decomposition on irreducible $\mathfrak{sp}(4)$-modules $\mathrm{H}^0(\mathcal{O}_{G(2,W)}(3)) = V_{6\omega_1} \oplus V_{2\omega_1} \oplus V_{2\rho}$. Now, we use the exact sequence:

$$0 \longrightarrow I_{\mathcal{V}}(3) \longrightarrow \mathcal{O}_{G(2,W)}(3) \longrightarrow \mathcal{O}_{\mathcal{V}}(3) \longrightarrow 0,$$

where $I_{\mathcal{V}}$ is the sheaf of ideals which defines the Veronese surface in the Grassmannian $G(2,W)$, and so,

$$0 \longrightarrow \mathrm{H}^0(I_{\mathcal{V}}(3)) \longrightarrow \mathrm{H}^0(\mathcal{O}_{G(2,W)}(3)) \longrightarrow \mathrm{H}^0(\mathcal{O}_{\mathcal{V}}(3)) \longrightarrow \mathrm{H}^1(I_{\mathcal{V}}(3)) \longrightarrow \cdots$$

Recall that $\mathrm{H}^0(\mathcal{O}_{\mathcal{V}}(3)) \simeq \mathrm{H}^0(\mathcal{O}_{\mathbb{P}(V)}(6)) = S^6V = V_{6\omega_1}$, so $\mathrm{H}^0(I_{\mathcal{V}}(3)) = V_{2\rho} \oplus V_{2\omega_1}$. The pullback of $I_{\mathcal{V}}(3)$ is a very ample sheaf of $\widetilde{G}$. Indeed, denote by Q_1 the quotient sheaf of the Grassmannian $G(2,W)$. The morphism $S^2Q_1 \simeq S^2Q_1^\vee(2) \to I_{\mathcal{V}}(2)$ is surjective, so $I_{\mathcal{V}}(2)$ is spanned by its global sections. Since $\mathcal{O}_{G(2,W)}(1)$ is a very ample sheaf on $G(2,W)$, $I_{\mathcal{V}}(3)$ is a very ample sheaf on $\widetilde{G} = \mathrm{Proj}(I_{\mathcal{V}})$. To conclude, $\widetilde{G}$ is a subvariety of $\mathbb{P}\left(\mathrm{H}^0(I_{\mathcal{V}}(3))\right) = \mathbb{P}(V_{2\rho} \oplus \mathfrak{sp}(4))$.

The isomorphism. The Veronese surface $\mathcal{V}$ and $G(2,W)$ are $PSp(4)$-stable, so is $\widetilde{G}$. The fact $V \simeq U \oplus U^\vee$ induces that $W \simeq \bigwedge^2 U \oplus \bigwedge^2 U^\vee \oplus \mathfrak{sl}(2)$. It follows that $\bigwedge^2 U \oplus \bigwedge^2 U^\vee$ is an element of the open orbit of $G(2,W)$, which means that its intersection with its orthogonal is reduced to zero. Thus it defines a unique point $[x]$ in $\mathbb{P}(V_{2\rho} \oplus \mathfrak{sp}(4))$ which is invariant under the action of $GL(2)$.

Since $GL(2) \simeq G^\sigma$, $\mathfrak{gl}(2) \subset \mathfrak{sp}(4)$ is a point of the open orbit of $\bar{X}$, so $\mathfrak{gl}(2)^\perp$ is a point of the open orbit of Y, invariant under the action of $GL(2)$. Denote by $[y]$ this point in $\mathbb{P}(V_{2\rho} \oplus \mathfrak{sp}(4))$.

The two points x and y have components only on the $GL(2)$ trivial factor of $V_{2\rho} \oplus \mathfrak{sp}(4)$: there is one trivial factor in $\mathfrak{sp}(4)$; using Lemma 4.4, and decomposing each space into irreducible $GL(2)$-modules, we check that $V_{2\rho}$ has another one. Now, x and y have non zero components on these two trivial factors, if it were not the case, $\widetilde{G}$ or Y could be embedded in some smaller G-stable projective space, that is to say $\mathbb{P}(V_{2\rho})$ or $\mathbb{P}(\mathfrak{sp}(4))$. We can therefore find two non zero complex numbers α and β such that $\phi = \alpha \mathrm{id}_{V_{2\rho}} + \beta \mathrm{id}_{\mathfrak{sp}(4)}$ sends x to y. The morphism ϕ is a $PSp(4)$-equivariant automorphism of $\mathbb{P}(\mathfrak{sp}(4) \oplus V_{2\rho})$ and restricts to an isomorphism between $\widetilde{G}$ and Y. □

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