ANNALES

DE

L'INSTITUT FOURIER

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Tome 61, nº 5 (2011), p. 2183-2218.

<http://aif.cedram.org/item?id=AIF_2011__61_5_2183_0>

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Ann. Inst. Fourier, Grenoble
61, 5 (2011) 2183-2218

DECOMPOSITION OF REDUCTIVE REGULAR PREHOMOGENEOUS VECTOR SPACES

by Hubert RUBENTHALER

ABSTRACT. — Let (G, V) be a regular prehomogeneous vector space (abbreviated to PV), where G is a reductive algebraic group over $\mathbb{C}$. If $V = \oplus_{i=1}^n V_i$ is a decomposition of V into irreducible representations, then, in general, the PV's (G, V_i) are no longer regular. In this paper we introduce the notion of quasi-irreducible PV (abbreviated to Q-irreducible), and show first that for completely Q-reducible PV's, the Q-isotypic components are intrinsically defined, as in ordinary representation theory. We also show that, in an appropriate sense, any regular PV is a direct sum of Q-irreducible PV's. Finally we classify the Q-irreducible PV's of parabolic type.

RÉSUMÉ. — Soit (G, V) un espace préhomogène (en abrégé PV) régulier, où G est un groupe algébrique réductif, défini sur $\mathbb{C}$. Si $V = \oplus_{i=1}^n V_i$ est une décomposition de V en représentations irréductibles, alors, en général, les espaces préhomogènes (G, V_i) ne sont pas réguliers. Dans cet article nous introduisons la notion de PV quasi-irréductible (en abrégé Q-irréducible), et nous montrons d'abord que pour les PV complètement Q-réductibles, les composantes Q-isotypiques sont définies de manière intrinsèque, comme en théorie ordinaire des représentations. Nous montrons également que, dans un sens approprié, tout PV régulier est une somme directe de PV quasi-irréductibles. Finalement nous classifions les PV de type parabolique qui sont Q-irréductibles.

1. Introduction

Let us first recall that a prehomogeneous vector space (abbreviated to PV) is a triplet (G, ρ, V) where G is an algebraic group over $\mathbb{C}$, and ρ is a rational representation of G on the finite dimensional vector space V, such that G has a Zariski open orbit in V. The theory of PV's was created by Mikio Sato in the early 70's to provide generalizations of several kinds of

Keywords: reductive groups, prehomogeneous vector spaces, relative invariants, prehomogeneous vector spaces of parabolic type.
Math. classification: 11S90, 20G05, 17B20.

known local or global zeta functions satisfying a functional equation similar to that of the Mellin transform, the Riemann zeta function, the Epstein zeta function or the zeta function of a simple algebra [16].

For the basic results on PV's we refer the reader to [17] and to [4].

There are many papers concerned with local or global zeta functions of PV's and their functional equations. Among them let us mention [18], [14], [13], [1],[15] for example.

There are also many papers concerning the classification theory of PV's. Many of them are written by T. Kimura and his students. We refer to the bibliography of [4] and to [3], [6], [7], [5] for more details. The regular PV's of parabolic type were classified in [8].

In order to associate a zeta function to a reductive PV one needs a further condition on the PV, namely the so-called *regularity* condition (see Section 2.1) Therefore knowledge of the structure of the reductive regular PV's as well as their classification is of particular interest. Unfortunately if (G, V) is a non irreducible reductive regular PV, it can be seen in easy examples (see example 2.7) that the irreducible components of the representation (G, V), which are still prehomogeneous, are in general not regular. This makes understanding the structure of such PV's difficult. To get around this difficulty we introduce the notion of quasi-irreducible PV (abbreviated to Q-irreducible) and show that if (G, V) is a regular reductive PV, there exists a filtration of the space V:

$$\{0\} = U_{k+1} \subset U_k \subset \cdots \subset U_2 \subset U_1 = V,$$

and a filtration of the group G

$$G_k \subset G_{k-1} \subset \cdots \subset G_1 = G,$$

such that the $G_i's$ are reductive and the U_i and U_{i+1} are G_i-stable. Moreover (G_i, U_i) is a regular PV and $(G_i, U_i/U_{i+1})$ is completely Q-reducible, for $i = 1, \dots, k$. See Theorem 3.2 below for the precise statement.

Let us now describe the content of the paper.

It is worthwhile pointing out that usually the group G of a PV is supposed to be connected. For our purpose we do not make this hypothesis.

Therefore in Section 2.1 we begin by giving extensions of basic results to the case where the group is not connected.

In Section 2.2 we give the definition of Q-irreducible PV's and prove that, if G is reductive and if (G, V) is a regular PV which is completely Q-reducible, then the Q-isotypic components of (G, V) are intrinsically defined.

In Section 3 we give our structure theorem for reductive regular PV's which was already above.

In Section 4.1 we give a brief account of the theory of parabolic PV's, and in Section 4.2 we give the complete classification of regular Q-irreducible PV's.

Acknowledgement. I obtained the results of this paper a long time ago, but never published them. I would like to thank Tatsuo Kimura for the recent stimulating conversations about classification theory of PV's which convinced me to write them up. I would also like to thank the referee for his careful reading of the manuscript and for his pertinent remarks and suggestions.

2. Completely Q-reducible regular PV's

2.1. The regularity for non connected reductive groups

As said in the Introduction a prehomogeneous vector space is a triplet (G, ρ, V) where G is an algebraic group over $\mathbb{C}$, and ρ is a rational representation of G on the finite dimensional vector space V, such that G has a Zariski open orbit in V. The open orbit is usually denoted by Ω and $S = V \smallsetminus \Omega$ is the *singular set*. The elements in the open orbit are called *generic*. We often simply write (G, V) for a PV when we do not need to make the representation explicit. A *relative invariant* of the PV (G, V) is a rational function f on V, such that there exists a rational character χ of G, such that for all $x \in \Omega$ and all $g \in G$, one has $f(g.x) = \chi(g)f(x)$. The character χ determines f up to a multiplicative constant. The subgroups we shall consider in the sequel are isotropy subgroups. These will be reductive, but not necessarily connected. Therefore we need to extend slightly the basic results concerning the regularity.

PROPOSITION 2.1. — *Let (G, V) be a PV, where G is not necessarily connected and not necessarily reductive. Let G° be the connected component group of G. Denote by Ω the open orbit under G° and define $S = V \smallsetminus \Omega$. Let $S_1, \dots, S_k$ be the irreducible components of codimension one in S. Let $f_1, f_2, \dots, f_k$ be irreducible polynomials such that*

$$S_i = \{x \in V | f_i(x) = 0\}.$$

The f_i's are (as well known) the fundamental relative invariants of (G°, V).

Then:

(1) *Ω is also the open G-orbit.*
(2) *For any $g \in G$ and for any $i \in \{1,\dots,k\}$, there exists $\sigma^g(i) \in \{1,\dots,k\}$ and a non zero contant $c(i,g)$ such that for all $x \in V$, one has $f_i(g.x) = c(i,g)f_{\sigma^g(i)}(x)$. Therefore the group G acts by permutations on the set of lines $\{\mathbb{C}f_i, i = 1,\dots,k\}$.*
(3) *Let $I_1 \cup I_2 \cup \dots \cup I_r = \{1,2,\dots,k\}$ be the partition defined by the G-action on the lines $\mathbb{C}f_i$. Define $\varphi_j = \prod_{i\in I_j} f_i$. Then φ_j is a relative invariant under G. Any relative invariant φ under G can be uniquely written in the following way:*

$$\varphi = c\varphi_1^{m_1}\varphi_2^{m_2}\dots\varphi_r^{m_r}$$

where $m_j \in \mathbb{Z}$ and $c \in \mathbb{C}$.

Proof. — 1) Let Ω be the open G°-orbit of V. Let us prove first that for any $g \in G$ the set $g.\Omega$ is a G°-orbit. Let $u = g.x$ and $v = g.y$ ($x, y \in \Omega$) be two elements in $g.\Omega$. By definition there exists $h \in G^\circ$ such that $x = h.y$. Therefore

$$u = g.x = gh.y = ghg^{-1}g.y = h'g.y = h'.v$$

(where $h' = ghg^{-1} \in G^\circ$, because G° is a normal subgroup of G). As $g.\Omega$ is open, we have $g.\Omega = \Omega$, for all $g \in G$. Hence Ω is also the open G-orbit.

2) Denote by χ_i the G° character of f_i. For $g \in G$ and $x \in V$, define $\psi_i^g(x) = f_i(g.x)$. Then for $h \in G^\circ$ we have $\psi_i^g(h.x) = f_i(gh.x) = f_i(ghg^{-1}g.x) = \chi_i(ghg^{-1})\psi_i^g(x)$. Therefore ψ_i^g is an irreducible relative invariant of G°. Hence there exists $\sigma^g(i) \in \{1,\dots,k\}$ and a non zero contant $c(i,g)$ such that for all $x \in V$, one has $\psi_i^g(x) = f_i(g.x) = c(i,g)f_{\sigma^g(i)}(x)$.

3) Let φ_j as defined above. Let $g \in G$. One has $\varphi_j(g.x) = \prod_{i\in I_j} f_i(g.x) = (\prod_{i\in I_j} c(i,g))\varphi_j(x)$. Hence φ_j is a relative invariant under G, with character $\widetilde{\chi}_j(g) = (\prod_{i\in I_j} c(i,g))$. Let φ be a relative invariant under G. Let χ_φ be the corresponding G character. As φ is a relative invariant under G°, one has $\varphi = c\prod_{i=1}^k f_i^{n_i}$, where $c \in \mathbb{C}$ and where $n_i \in \mathbb{Z}$. We have, for $g \in G$ and $x \in \Omega$:

$$\varphi(g.x) = c\chi_\varphi(g)\prod_{i=1}^k f_i^{n_i}(x) = c\prod_{i=1}^k f_i^{n_i}(g.x) = c'\prod_{i=1}^k f_{\sigma^g(i)}^{n_i}(x) \quad (c' \in \mathbb{C}).$$

Therefore from the uniqueness of the decomposition for G° relative invariants, we obtain that for every $g \in G$ we have $n_{\sigma^g(i)} = n_i$. Hence the powers n_i of the f_i's in the same subset I_j, are the same, say m_j. This implies that

$$\varphi = c\varphi_1^{m_1}\varphi_2^{m_2}\dots\varphi_r^{m_r}.$$

□

Remark 2.2. — Of course all the f_i where $i \in I_j$ have the same degree.

DEFINITION 2.3. — *Let (G, V) be a PV where G is a reductive, non necessarily connected, algebraic group. The PV (G, V) is called regular if there exists a relative invariant f such that $\frac{df}{f} = \operatorname{gradlog}(f) : \Omega \longrightarrow V^*$ is generically surjective (i.e., has a Zariski dense image). Such a relative invariant is said to be nondegenerate.*

PROPOSITION 2.4 (Compare with [4], Th. 2.28), and [17], Remark 11 p. 64). — *Let G be a reductive algebraic group. Let G° be the connected component group of G and suppose that (G, V) is a PV.*

The following conditions are equivalent:

i) *(G, V) is regular.*
ii) *There exists a relative invariant f such that the Hessian $H_f(x) = \operatorname{Det}\left(\frac{\partial^2 f}{\partial x_i \partial x_j}(x)\right)$ is not identically zero*
iii) *The singular set S is a hypersurface.*
iv) *The open orbit $\Omega = V \setminus S$ is an affine variety.*
v) *Each generic isotropy subgroup is reductive.*
vi) *Each generic isotropy subalgebra is reductive.*

Suppose moreover that these conditions hold. Then any polynomial f satisfying $S = \{x \in V | f(x) = 0\}$ is a nondegenerate relative invariant of G°. In the notations of Proposition 2.1 the set of such polynomials which are relative invariants under G is the set of polynomials of the form

$$f = c\varphi_1^{m_1}\varphi_2^{m_2}\dots\varphi_r^{m_r}$$

where $m_j \in \mathbb{N}^*$ and $c \in \mathbb{C}^*$.

Proof. — We will of course use the same result which is known for connected reductive groups ([4], Th. 2.28. and [17]).

First of all we remark that by the same proof as in the case where the group is connected (see [17], Proposition 10 p. 62 and Remark 11 p. 64) we obtain i) $\Leftrightarrow$ ii).

i) $\Rightarrow$ iii): If (G, V) is regular, there exists a nondegenerate relative invariant f. This function is also a relative invariant of (G°, V), hence the singular set for the G° action is an hypersurface. But the singular set for G is the same as for G°, from Proposition 2.1. Assertion iii) is proved.

iii) $\Rightarrow$ iv): This is classical: the complementary set of a hypersurface is always an affine variety.

iv) $\Rightarrow$ v): From [4], Th. 2.28, we know that for $x \in \Omega$, the isotropy subgroup G_x° is reductive. Hence the isotropy subgroup G_x is reductive.

v) $\Rightarrow$ vi): As the Lie algebras of G° and of G are the same, this is obvious.

vi) $\Rightarrow$ i): Let $S_1, \dots, S_m$ be the irreducible components of S. They correspond to irreducible polynomials $f_1, \dots, f_m$ which are the fundamental relative invariants for G°. We know from [4], Th. 2.28 that if vi) is satisfied then (G°, V) is regular and therefore any polynomial f such that $S = \{x \in V | f(x) = 0\}$ is a nondegenerate relative invariant under G°. Among them the functions which are relative invariants under G are of the proposed form from Proposition 2.1. Hence (G, V) is regular and i) is true. □

Remark 2.5. — Under the assumptions of the preceding Proposition, the polynomial $f = f_1 f_2 \dots f_k = \varphi_1 \varphi_2 \dots \varphi_r$ is the unique polynomial of minimal degree which defines S. It is a relative invariant under G.

2.2. Quasi-irreducible PV's and complete Q-reducibility

The following result is often very useful.

PROPOSITION 2.6. — *Let (G, V) be a PV. Here we do not suppose that G is connected and we do not suppose that G is reductive. Suppose that $V = V_1 \oplus V_2$ where V_1 and V_2 are two non trivial G-invariant subspaces of V. Denote by p_1(resp. p_2) the projections on V_1(resp. V_2) defined by this decomposition.*

1) *The representations (G, V_1) and (G, V_2) are PV's. Moreover the open orbits in V_1(resp. V_2) are the projections of Ω_V i.e., $\Omega_{V_i} = p_i(\Omega_V), i = 1, 2$.*
2) *Let $x_0 + y_0$ be a generic element of (G, V), with $x_0 \in V_1$ and $y_0 \in V_2$. Let also G_{x_0} (resp. G_{y_0}) be the isotropy subgroup of x_0 (resp. y_0). Then (G_{y_0}, V_1) and (G_{x_0}, V_2) are PV's, and x_0 is generic in (G_{y_0}, V_1) and y_0 is generic in (G_{x_0}, V_2).*
3) *One has $G_{x_0} \cap G_{y_0} = G_{x_0+y_0}$. The open G_{y_0}-orbit in V_1 is equal to $\Omega_{V_1}(y_0) = \{x \in V_1, x + y_0 \in \Omega_V\}$ and the open G_{x_0}-orbit in V_2 is equal to $\Omega_{V_2}(x_0) = \{y \in V_2, x_0 + y \in \Omega_V\}$.*
4) *The subgroup $\widetilde{G}$ generated by G_{x_0} and G_{y_0} is open, and hence closed, therefore we have $\widetilde{G} = G$ if G is connected. More precisely the subset $G_{x_0}.G_{y_0}$ is open in G.*
5) *Suppose that G is reductive, and that (G, V) and (G, V_1) are regular. Then (G_{x_0}, V_2) is a regular reductive PV.*

Proof. — 1) As the projections p_1 and p_2 are open maps, the sets $\Omega_{V_i} = p_i(\Omega_V), i = 1, 2$ are open. Let x_1 and x_2 be two elements in Ω_{V_1}. From the definition there exists y_1 and y_2 in V_2 such that $x_1 + y_1$ and $x_2 + y_2$ belong to Ω_V. Therefore there exists $g \in G$ such that $g.(x_1 + y_1) = x_2 + y_2$. Hence $g.x_1 = x_2$. Hence two elements in Ω_{V_1} are conjugate. Conversely the conjugate of an element in Ω_{V_1} is still in Ω_{V_1}. This proves the first assertion for V_1. The argument for V_2 is the same.

2) Define $n = \dim V$, $n_1 = \dim V_1$, $n_2 = \dim V_2$. As (G, V) is prehomogeneous, we have $n = \dim G - \dim G_{x_0+y_0}$ and as (G, V_1) is also prehomogeneous we have $n_1 = \dim G - \dim G_{x_0}$. Therefore

$$\begin{aligned} n = n_1 + n_2 &= \dim G - \dim G_{x_0+y_0} \\ &= \dim G - \dim G_{x_0} + \dim G_{x_0} - \dim G_{x_0+y_0} \\ &= n_1 + \dim G_{x_0} - \dim G_{x_0+y_0}. \end{aligned}$$

Therefore $n_2 = \dim G_{x_0} - \dim G_{x_0+y_0}$ and as $G_{x_0+y_0} = (G_{x_0})_{y_0}$ is the isotropy subgroup of y_0 in G_{x_0}, the representation (G_{x_0}, V_2) is prehomogeneous, and y_0 is generic for this space.

3) The assertion $G_{x_0} \cap G_{y_0} = G_{x_0+y_0}$ is obvious. It is clear that $\Omega_{V_2}(x_0) = \{y \in V_2, x_0 + y \in \Omega_V\}$ is stable under G_{x_0}. Moreover if $y_1, y_2 \in \Omega_{V_2}(x_0)$, then $x_0 + y_1, x_0 + y_2 \in \Omega_V$ and there exists $g \in G$ such that $g(x_0 + y_1) = x_0 + y_2$, and hence $g \in G_{x_0}$ and $gy_1 = y_2$. This proves that the open G_{x_0}-orbit in V_2 is $\Omega_{V_2}(x_0)$. The proof for the space (G_{y_0}, V_2) is symmetric.

4) Consider the set $\mathcal{O} = (\Omega_{V_1}(y_0) \oplus \Omega_{V_2}(x_0) \cap \Omega_V$. This set is nonempty $(x_0 + y_0 \in \mathcal{O})$ and open. Let $x + y \in \mathcal{O}$. Then $x \in \Omega_{V_1}(y_0)$ and we know from the third assertion that there exists $g_1 \in G_{y_0}$ such that $g_1 x = x_0$. Hence $g_1(x + y) = x_0 + g_1 y$. As $x + y \in \Omega_V$, we have also $x_0 + g_1 y \in \Omega_V$. Hence $g_1 y \in \Omega_{V_2}(x_0)$. Then we know that there exists $g_2 \in G_{x_0}$ such that $g_2 g_1 y = y_0$. Hence $g_2 g_1(x + y) = x_0 + y_0$. Therefore the elements of $\mathcal{O}$ are conjugate under the set $G_{x_0}.G_{y_0}$. Hence $G_{x_0}.G_{y_0}/G_{x_0+y_0} \simeq \mathcal{O}$ is an open subset of $G/G_{x_0+y_0} \simeq \Omega_V$. This implies that $G_{x_0}.G_{y_0}$ is open in G. Therefore the group $\widetilde{G}$ generated by G_{x_0} and G_{y_0} is open and hence closed. If G is connected, then $\widetilde{G} = G$

5) From Proposition 2.4 we know that G_{x_0} is reductive and from assertion 2) we know that (G_{x_0}, V_2) is a PV. As $(G_{x_0})_{y_0} = G_{x_0+y_0}$, using again Proposition 2.4, we obtain that (G_{x_0}, V_2) is regular. □

Unfortunately the irreducible components of a reductive regular PV are in general not regular as shown by the following example.

Example 2.7. — Let $G = \mathbb{C}^* \times SL_n \times \mathbb{C}^*$, let $V = \mathbb{C}^n \times \mathbb{C}^n$ and define ρ as follows:

$$\rho(x, g, y)(v, w) = (xvg^{-1}, y^{-1}gw)$$

where $x, y \in \mathbb{C}^*$, $g \in SL_n$, where $v \in \mathbb{C}^n$ is considered as a row vector and where $w \in \mathbb{C}^n$ is considered as a column vector. A simple computation shows that if $v_0 = (1, 0, \dots, 0)$ and $w_0 = {}^t(1, 0, \dots, 0)$, then the isotropy subgroup is the set of triplets $(x, \begin{pmatrix} x & 0 \\ 0 & A \end{pmatrix}, x)$, where $A \in GL_{n-1}$, and such that $x.\mathrm{Det}A = 1$, and this proves that (G, ρ, V) is a regular PV. In fact the scalar product $Q(v, w) = v.w$ of v and w is the unique relative invariant. The irreducible components are $V_1 = \mathbb{C}^n \times \{0\}$ and $V_2 = \{0\} \times \mathbb{C}^n$, and the PV's $(G, \rho_{|V_i}, V_i)$ $(i = 1, 2)$ are obviously not regular.

The following lemma is also useful in the sequel.

Lemma 2.8. — *Let (G, V) be a PV where G is not necessarily connected and not necessarily reductive and suppose that $V = V_1 \oplus V_2$ where V_1 and V_2 are G-invariant subspaces.*

a) *Let f be a relative invariant of (G, V_1). Then the function $\tilde{f}$ defined by $\tilde{f}(x+y) = f(x)$ $(x \in V_1, Y \in V_2)$ is a relative invariant of (G, V) with the same character as f.*
b) *Let f be a relative invariant of (G, V) which is not identically zero on V_1, then for $x \in V_1, y \in V_2$, we have $f(x + y) = f(x)$.*

Proof. — a) Let χ_f be the character of f. For $g \in G$, we have:

$$\tilde{f}(g.x + g.y) = f(g.x) = \chi_f(g) f(x) = \chi_f(g) \tilde{f}(x + y).$$

b) For $x \in V_1, y \in V_2$ let us set $\tilde{f}(x + y) = f(x)$. From a) we know that $\tilde{f}$ is a relative invariant of (G, V) with the same character as f. Therefore there exists a constant $c \in \mathbb{C}$ such that $\tilde{f} = c.f$. But as $\tilde{f} = f$ on V_1, we have necessarily $c = 1$. □

Definition 2.9. — *Let G be a reductive group (not necessarily connected) and let (G, V) be a regular PV.*

a) *The prehomogeneous vector space (G, V) is called 1-irreducible if the singular set $S = V \setminus \Omega$ is an irreducible hypersurface. (According to Proposition 2.1, this is equivalent to the fact that there exists only one fundamental relative invariant under G°, up to constants).*
b) *The prehomogeneous vector space (G, V) is called almost irreducible if for any proper invariant subspace $U \subset V$, the prehomogeneous vector space (G, U) has no nontrivial relative invariant.*
c) *The prehomogeneous vector space (G, V) is called quasi-irreducible (abbreviated Q-irreducible) if for any proper invariant subspace $U \subset V$, the prehomogeneous vector space (G, U) is not regular.*

Remark 2.10. — We will see that for an important class of PV's, namely the PV's of parabolic type, the three notions of irreducibility introduced in the preceding definition are in fact equivalent. See Theorem 4.16 below. However this is not true for general PV's as shown by the example given in Remark 4.17.

Remark 2.11. — It is well known that if (G, V) is irreducible, than there exists at most one fundamental relative invariant. Therefore the irreducible regular PV's are 1-irreducible. The PV from Example 2.7 is 1-irreducible but not irreducible.

Proposition 2.12. — *Let (G, V) be a regular PV where G is reductive. Among the various definitions of irreducibility, we have the following implications:*

$$(G,V) \text{ is } 1-\textit{irreducible} \Rightarrow (G,V) \textit{ is almost irreducible} \Rightarrow (G,V) \textit{ is } Q-\textit{irreducible}.$$

Proof. — Suppose that (G, V) is not almost irreducible. Then it exists a proper invariant subspace $U \subset V$ such that (G, U) has a non trivial relative invariant f. Let W be a G-invariant supplementary subspace to U. Then according to Lemma 2.8 the function $\tilde{f}$ defined by $\tilde{f}(x + y) = f(x)$ ($x \in U, y \in W$) is a relative invariant on V depending only on x. Therefore the map $\frac{d\tilde{f}}{\tilde{f}}$ cannot be generically surjective. But as (G, V) is regular there exists a relative invariant φ such that $\frac{d\varphi}{\varphi}$ is generically surjective. This is not the case if $\varphi = c\tilde{f}^k$ ($c \in \mathbb{C}$). Therefore there exists another fundamental relative invariant, and hence (G, V) is not 1-irreducible.

Suppose now that (G, V) is not Q-irreducible. Then there exists a proper invariant subspace $U \subset V$ such that (G, U) is regular. Hence (G, U) has a non trivial relative invariant. Therefore (G, V) is not almost irreducible. $\square$

Proposition 2.13. — *Let (G, V) be a PV where G is reductive. Suppose that $V = \oplus_{i=1}^n V_i$ where each V_i is a G-invariant subspace such that (G, V_i) is regular. Let Ω and Ω_i be the open orbits of (G, V) and (G, V_i) respectively ($i = 1, \ldots, n$). Then (G, V) is regular and $\Omega = \oplus_{i=1}^n \Omega_i$. Moreover any polynomial relative invariant of (G, V) is a product of relative invariants of the spaces (G, V_i).*

Proof. — Let us make the usual identification $V^* = \oplus_{i=1}^n V_i^*$. Let f_i be a relatively invariant polynomial of (G, V_i) such that $\varphi_i = \frac{df_i}{f_i} : \Omega_i \longrightarrow V_i^*$ is generically surjective. Replacing eventually f_i by its square, we can suppose that $\partial^\circ(f_i) > 1$ ($\partial^\circ(f_i)$ denotes the degree of f_i). Define a relative invariant of (G,V) by:

$$f(x_1,x_2,\dots,x_n)=f_1(x_1)f(x_2)\dots f_n(x_n)\quad (x_i\in V_i).$$

Then $\varphi(x_1,x_2,\dots,x_n)=\dfrac{df(x_1,x_2,\dots,x_n)}{f(x_1,x_2,\dots,x_n)}=\varphi_1(x_1)\oplus\varphi_2(x_2)\oplus\dots\oplus\varphi_n(x_n)$. As the map $x_i\longrightarrow\varphi_i(x_i)$ is generically surjective from Ω_i to V_i^*, we see that φ is generically surjective from $\oplus_{i=1}^n\Omega_i$ to V^*. Then from Proposition 2.4 we obtain that (G,V) is regular. Moreover we have $\det d\varphi(x_1,x_2,\dots,x_n)=\prod_{i=1}^n\det d\varphi_i(x_i)$ and we know from [17] p. 63 that the Hessian H_f is given by

$$H_f(x)=(1-r)\det d\varphi(x).f(x)^k$$

where $r=\partial^\circ(f)$ and where $k=\dim V$. Hence $H_f\neq 0$ on $\oplus_{i=1}^n\Omega_i$. On the other hand it is known ([17], p. 70[(1)], [12] p. 22–23), that if $H_f\neq 0$ then $\Omega=\{x\mid f(x)H_f(x)\neq 0\}$. This implies that $\oplus_{i=1}^n\Omega_i\subset\Omega$. The reverse inclusion is a consequence of Proposition 2.6. The set $S_i=V_i\setminus\Omega_i$ is a hypersurface defined by an equation $P_i=0$ where P_i is a relatively invariant polynomial on V_i (Proposition 2.4). We will choose P_i of minimal degree among the polynomials defining S_i. Then $P_i=f_{i,1}\dots f_{i,l_i}$ where the $(f_{i,j})'s$ are irreducible relatively invariant polynomials under G° on V_i, which are algebraically independent. From Remark 2.5 we know that we can write $P_i=\varphi_{i,1}\dots\varphi_{i,m_i}$, where the $\varphi_{i,j}$'s are polynomials on V_i which are relatively invariant under G. As $\Omega=\oplus_{i=1}^n\Omega_i$ we obtain that

$$S=V\setminus\Omega=\Big\{(x_1,x_2,\dots,x_n)\in V\mid P(x_1,x_2,\dots,x_n)=\prod_{i=1}^n P_i(x_i)=0\Big\}.$$

Using again Proposition 2.1, we obtain that any G-relatively invariant polynomial on V is a product of polynomials of the form $\varphi_{i,j}^{\alpha_{i,j}}$, where $\alpha_{i,j}\in\mathbb{N}$. □

Definition 2.14. — *Let G be a reductive group (not necessarily connected) and let (G,V) be a PV. The PV (G,V) is called completely Q-reducible if there exists a decomposition $V=\oplus_{i=1}^n V_i$ where the V_i's are G-invariant subspaces such that (G,V_i) is Q-irreducible. The spaces V_i are then called Q-irreducible components of (G,V).*

(1) In the paper by M. Sato and T. Kimura, it is written that if (G,V) is a regular PV with G reductive, and if f is a relative invariant with $H_f\neq 0$, then $\Omega=\{x\mid H_f(x)\neq 0\}$, but analyzing their proof it is easy to see that in fact $\Omega=\{x\mid f(x)H_f(x)\neq 0\}$ (the first assertion would be wrong if $\partial^\circ f=2$).

Remark 2.15. — We know from Proposition 2.13 that a completely Q-irreducible PV is regular.

It is well known that for an ordinary finite dimensional completely reducible representation of a group G the equivalence classes occuring in any decomposition into irreducibles are uniquely determined, as well as the isotypic components. Our next aim is to prove analogous results for completely Q-reducible regular PV's where the irreducible components are replaced by the Q-irreducible components and the isotypic components are replaced by the Q-isotypic components.

THEOREM 2.16. — *Let (G, V) be a completely Q-reducible PV. Let $V = \oplus_{i=1}^{n} V_i$ be a decomposition of V into Q-irreducible components. Let $W \subset V$ be an invariant subspace such that (G, W) is regular. Then (G, W) is a completely Q-reducible PV. Moreover if W_j is a Q-irreducible component of (G, W), there exists an integer $\ell(j) \in \{1, 2, \dots, n\}$ such that the representation (G, W_j) is equivalent to $(G, V_{\ell(j)})$.*

The equivalence classes of the Q-irreducible components arising in (G, V) are uniquely determined.

Let δ be an equivalence class of one of the Q-irreducible components arising in $V = \oplus_{i=1}^{n} V_i$ (i.e., an equivalence class of one of the representations (G, V_j)). Let $I(\delta) = \{i \,|\, (G, V_i) \in \delta\}$ and let $m(\delta) = \mathrm{Card} I(\delta)$ be the multiplicity of δ. Let also $V(\delta) = \oplus_{i \in I(\delta)} V_i$ be the so-called Q-isotypic component of δ. Then $m(\delta)$ does not depend on the decomposition of V into Q-irreducible subspaces. Moreover if $U \subset V$ is an invariant subspace of type δ (this means that U is a direct sum of Q-irreducible invariant subspaces which are all of type δ), then U is a subspace of $V(\delta)$. In other words the Q-isotypic components are uniquely determined.

Proof. — Let $V_j = \oplus_{i=1}^{\ell(j)} U_j^i$ be a decomposition of V_j into irreducible components in the ordinary sense. As we are only interested in equivalence classes of representations we can assume that $W = (\oplus_{j \in A} V_j) \oplus (\oplus_{j \in A^c} \oplus_{i \in I_j} U_j^i)$, where A is a subset of $\{1, 2, \dots, n\}$ and where I_j is a proper subset of $\{1, 2, \dots, \ell(j)\}$. After renumbering, we can suppose that $I_j = \{1, 2, \dots, m(j)\}$ where $m(j) < \ell(j)$. Let us denote by x_j the variable in V_j and by x_j^i the variable in U_j^i. Hence $x_j = (x_j^1, x_j^2, \dots, x_j^{\ell(j)})$. Let $j_1, j_2, \dots, j_k$ be the elements of A, and $j_{k+1}, \dots, j_n$ be the elements in A^c.

Let f be a relative invariant of (G, W) such that $\dfrac{df}{f}$ is generically surjective. Then f is a function in the variables:

$$\Big(x_{j_1}, \dots, x_{j_k}; x_{j_{k+1}}^1, \dots, x_{j_{k+1}}^{m(j_{k+1})}; \dots; x_{j_n}^1, \dots, x_{j_n}^{m(j_n)}\Big).$$

We know from Proposition 2.13 that f is a product of relative invariants of the V_j's. Hence

$$\begin{aligned}&f(x_{j_1},\dots,x_{j_k};x^1_{j_{k+1}},\dots,x^{m(j_{k+1})}_{j_{k+1}};\dots;x^1_{j_n},\dots,x^{m(j_n)}_{j_n})\\&=f_{j_1}(x_{j_1})\dots f_{j_k}(x_{j_k})f_{j_{k+1}}(x^1_{j_{k+1}},\dots,x^{\ell_{j_{k+1}}}_{j_{k+1}})\dots f_{j_n}(x^1_{j_n},\dots,x^{\ell_{j_n}}_{j_n})\end{aligned}$$

where each f_{j_r} is a relative invariant of (G,V_{j_r}). Therefore:

$$\begin{array}{lcr} f_{j_{k+1}} & \text{depends only on the variables} & x^1_{j_{k+1}},\dots,x^{m(j_{k+1})}_{j_{k+1}}\\ \vdots & \vdots & \vdots\\ f_{j_n} & \text{depends only on the variables} & x^1_{j_n},\dots,x^{m(j_n)}_{j_n}.\end{array}$$

But as $\dfrac{df}{f}=\dfrac{df_{j_1}}{f_{j_1}}\oplus\dots\oplus\dfrac{df_{j_k}}{f_{j_k}}\oplus\dfrac{df_{j_{k+1}}}{f_{j_{k+1}}}\oplus\dots\oplus\dfrac{df_{j_n}}{f_{j_n}}$ is generically surjective, each $\dfrac{df_{j_r}}{f_{j_r}}$ must be generically surjective. For example $\dfrac{df_{j_{k+1}}}{f_{j_{k+1}}}$ will be generically surjective from an open set of $U^1_{j_{k+1}}\oplus\dots\oplus U^{m(j_{k+1})}_{j_{k+1}}$ to its dual. Therefore from Definition 2.3 we know that $(G,U^1_{j_{k+1}}\oplus\dots\oplus U^{m(j_{k+1})}_{j_{k+1}})$ would be regular. But this is impossible, since (G,V_{k+1}) is Q-irreducible. Hence $(G,W)\simeq(G,\oplus_{j\in A}V_j)$, and this shows that (G,W) is completely Q-reducible.

Let $W=\oplus_{j=1}^k W_j$ be a decomposition of W into Q-irreducible components. Then the same proof as above, applied to W_j instead of W shows that (G,W_j) is equivalent to $(G,\oplus_{k\in B}V_k)$, where $B\subset\{1,\dots,n\}$. But as (G,W_j) is Q-irreducible the set B is a single element. The same proof applied to V shows that any Q-irreducible component of V is equivalent to some V_i. Hence the equivalence classes of the Q-irreducible components are uniquely determined.

Let us now prove the assertion concerning the multiplicities. Let $V=\oplus_{k=1}^r U_k$ be another decomposition of V into Q-irreducible components. We can suppose that $r\leqslant n$. From above we know that $(G,U_1)\simeq(G,V_{i_1})$ where $i_1\in\{1,\dots,n\}$. Then by a classical argument $(G,\oplus_{k=2}^n U_k)\simeq(G,\oplus_{i\neq i_1}V_i)$. Then inductively one proves that $r=n$ and that there exists a permutation σ of $\{1,\dots,n\}$ such that $(G,U_i)\simeq(G,V_{\sigma(i)})$. Therefore the multiplicity does not depend on the decomposition into Q-irreducibles.

Let now $U\subset V$ be an invariant Q-irreducible subspace of type δ and define $V'=\oplus_{i\notin I(\delta)}V_i$. Let S be a G-invariant supplementary space of $U\cap V(\delta)$ in U. Hence we have:

$$U=U\cap V(\delta)\oplus S,\quad V=V(\delta)\oplus V'.$$

For $s \in S$ let us write $s = v_1 + v_2$ with $v_1 \in V(\delta)$ and $v_2 \in V'$. The linear mapping $\varphi : S \longrightarrow V'$ defined by $\varphi(s) = v_2$ is injective, because if $\varphi(s) = v_2 = 0$, then $s = v_1 \in U \cap V(\delta) \cap S = \{0\}$. Moreover φ is G-equivariant. Suppose that $U \cap V(\delta) = \{0\}$. If this is the case, we have $S = U$, and then $S' = \varphi(S)$ is a subspace of type δ of V'. This is not possible from the definition of V' and from what we have proved before. Therefore $U \cap V(\delta) \neq \{0\}$. Define $U' = U \cap V(\delta) \oplus S'$. As φ is G-equivariant, the subspace U' is invariant of type δ. Let f be a relative invariant of (G, U') such that $\dfrac{df}{f}$ is generically surjective. From Proposition 2.13 we know that $f(x, s') = \varphi_1(x)\varphi_2(s')$, where $x \in U \cap V(\delta), s' \in S'$, and where φ_1 and φ_2 are relative invariants of $(G, V(\delta))$ and (G, V') respectively. As $\dfrac{df}{f} = \dfrac{d\varphi_1}{\varphi_1} \oplus \dfrac{d\varphi_2}{\varphi_2}$, we obtain that $\dfrac{d\varphi_1}{\varphi_1}$ and $\dfrac{d\varphi_2}{\varphi_2}$ are generically surjective. This implies that $(G, U \cap V(\delta))$ is regular and this is possible if and only if $U \cap V(\delta) = U$, because (G, U) is Q-irreducible. Hence $U \subset V(\delta)$. $\square$

3. The decomposition theorem for reductive regular PV's

3.1. An example

Of course, reductive regular PV's are not necessarily completely Q-reducible as shown by the following example.

Example 3.1. — Let $n \geqslant 2$ be an integer and let $G = GL(n, \mathbb{C}) \times \mathbb{C}^*$ and $V = S(n, \mathbb{C}) \times \mathbb{C}^n$ where $S(n, \mathbb{C})$ is the space of complex n by n symmetric matrices. The action of G on V is given by

$$(g, a)(X, v) = (gX{}^tg, a{}^tg^{-1}v),\ \ g \in GL(n, \mathbb{C}), a \in \mathbb{C}^*, X \in S(n, \mathbb{C}), v \in \mathbb{C}^n.$$

The isotropy subgroup of (I_n, e_1), where I_n is the identity matrix and where e_1 is the first vector of the canonical basis of $\mathbb{C}^n$, is easily seen to be isomorphic to the orthogonal group $O(n-1)$. This proves that (G, V) is a reductive regular PV. The irreducible components are $S(n, \mathbb{C})$ and $\mathbb{C}^n$. As $(G, \mathbb{C}^n)$ is not regular, it follows that (G, V) is not completely Q-reducible.

3.2. Structure of reductive regular PV's

The following theorem shows the structure of reductive regular PV's.

THEOREM 3.2. — *Let (G, V) be a reductive regular PV and let x be a generic element of V. Denote by G_x the isotropy subgroup of x. There exist a sequence of subspaces $V_1, V_2, \dots, V_n$ such that $V = V_1 \oplus V_2 \oplus \cdots \oplus V_n$, a sequence of integers $i_1 = 1 < i_2 < \cdots < i_k \leqslant n$ and a sequence of reductive subgroups*

$$G_x = G_{k+1} \subset G_k \subset \cdots \subset G_1 = G$$

with the following properties:

1) *If $x = x_1 + x_2 + \cdots + x_n$ with $x_j \in V_j$, then*

$$G_{\ell+1} = (G_\ell)_{x_{i_\ell}+\cdots+x_{i_{\ell+1}-1}}.$$

2) *For $\ell \in \{1, \dots, k\}$ the space $V_{i_\ell} \oplus \cdots \oplus V_n$ is G_ℓ-invariant and $(G_\ell, V_{i_\ell} \oplus \cdots \oplus V_n)$ is a regular PV.*
3) *If $i_\ell \leqslant j \leqslant i_{\ell+1} - 1$, then V_j is G_ℓ-invariant , (G_ℓ, V_j) is a Q-irreducible PV and $(G_\ell, V_{i_\ell} \oplus \cdots \oplus V_{i_{\ell+1}-1})$ is a maximal completely Q-reducible PV in $V_{i_\ell} \oplus \cdots \oplus V_n$. Moreover $V_{i_{\ell+1}} \oplus \cdots \oplus V_n$ is $G_\ell - invariant$ but does not contain any subspace $U \neq \{0\}$ such that (G_ℓ, U) is regular.*

Proof. — The proof goes by induction on $\dim V$. There is nothing to prove if $\dim V = 1$. Suppose that the theorem is proved for all reductive regular PV's such that $\dim V \leqslant r$. Let then (G, V) be a reductive regular PV such that $\dim V = r + 1$. Let $V' \subset V$ be an invariant subspace such that (G, V') is completely Q-reducible and maximal in V for this property. Denote by

$$V' = V_1 \oplus V_2 \oplus \cdots \oplus V_{i_2-1}$$

a decomposition of V' into Q-irreducible components. Let V'' be an invariant supplement of V'. If $V'' = \{0\}$ the PV (G, V) is completely Q-reducible and the proof is finished. From the maximality of V' and Proposition 2.13, we know that (G, V'') does not contain any invariant subspace $U \neq \{0\}$ such that (G, U) is regular.

Let x be a generic element in V. Let us write:

$$x = x_1 + x_2 + \cdots + x_{i_2-1} + x'' \text{ where } x_j \in V_j \text{ and where } x'' \in V''.$$

Define $G_2 = G_{x_1+\cdots+x_{i_2-1}}$. From Proposition 2.4 we know that G_2 is reductive and from Proposition 2.6 5) we know that (G_2, V'') is regular. As $\dim V'' \leqslant r$, we know by induction that there exists a sequence of integers $i_2 < i_3 < \cdots < i_k \leqslant n$ and a sequence of reductive subgroups

$$G_x = (G_2)_{x''} = G_{k+1} \subset G_k \subset \cdots \subset G_2$$

which have the required properties for the triplet (G_2, V'', x''). Then the sequences $i_1 < i_2 < i_3 < \cdots < i_k \leqslant n$ and

$$G_x = (G_2)_{x''} = G_{k+1} \subset G_k \subset \cdots \subset G_2 \subset G_1 = G$$

have the required properties for the triplet (G, V, x). $\square$

Let us give three examples of the kind of decompositions arising in the preceding Theorem.

Example 3.3. — Let us return to Example 3.1. In the notations of the preceding Theorem, we take for G_2 the isotropy of $I_n \in S(n, \mathbb{C})$, namely $O(n, \mathbb{C}) \times \mathbb{C}^*$, and $V_1 = S(n, \mathbb{C})$ and $V_2 = V_{i_2} = \mathbb{C}^n$.

Example 3.4 (Example of the "descending chains" of F. Sato [14]*).* — Let $V_m = M(m+1, m)$ be the space of complex $(m+1) \times m$ matrices. Define $V = V_n \oplus V_{n-1} \oplus \cdots \oplus V_1$ and let $G = SO(n+1) \times GL(n) \times GL(n-1) \times \cdots \times GL(1)$. The group G acts by

$$(g_{n+1}, g_n, \ldots, g_1)(x_n, \ldots, x_1) = (g_{n+1} x_n g_n^{-1}, g_n x_{n-1} g_{n-1}^{-1}, \ldots, g_2 x_1 g_1^{-1})$$

where $g_{n+1} \in SO(n+1)$, $g_i \in GL(i)$, $x_i \in V_i$ for $i = 1, \ldots, n$. This representation is a regular PV and the fundamental relative invariants are given by

$$P_k(x_n, x_{n-1}, \ldots, x_1) = \det({}^t x_k\, {}^t x_{k-1} \ldots {}^t x_n x_n \ldots x_{k-1} x_k).$$

This PV is called the PV of descending chains of size n (see [14] for the details). It is then easily seen that (G, V_n) is a maximal Q-completely reducible subspace (in fact it is irreducible regular). Taking $x_n^0 = \begin{bmatrix} I_n \\ 0 \end{bmatrix}$ as regular element of (G, V_n), a simple computation shows that its isotropy subgroup $G_{x_n^0}$ is equal to $D(SO(n) \times SO(n)) \times GL(n-1) \times \cdots \times GL(1)$ where $D(SO(n) \times SO(n))$ stands for the diagonal subgroup of $SO(n) \times SO(n)$, the first factor being diagonally embedded in $SO(n+1)$. Therefore the regular PV $(G_{x_n^0}, V_{n-1} \oplus \cdots \oplus V_1)$ is essentially the PV of descending chains of size $n-1$. Therefore the sequence of completely Q-reducible spaces (under the successive isotropy subgroups) appearing in Theorem 3.2 is $V_n, V_{n-1}, \ldots, V_1$.

Example 3.5. — Let $G = GL(2) \times \mathrm{Spin}(10) \times \mathbb{C}^*$ where $\mathrm{Spin}(10)$ is the Spin group in dimension 10. Consider the representation $[\Lambda_1 \otimes \rho \otimes \mathrm{Id}] \oplus [\mathrm{Id} \otimes \mathrm{Spin} \otimes \square]$ of G where Λ_1 is the natural 2-dimensional representation of $GL(2)$, where ρ is the vector representation of $\mathrm{Spin}(10)$, where Spin is the half-spin representation of $\mathrm{Spin}(10)$, and where $\square$ is the natural representation by multiplication of $\mathbb{C}^*$ on $\mathbb{C}$.

This representation is a PV whose generic isotropy subgroup is isomorphic to the exceptional simple Lie group G_2 (see (42) p. 397 of [6]). Another argument to prove the prehomogeneity and the regularity is to remark that it corresponds to a PV of parabolic type in E_8 (see Section 4) and that the corresponding grading element is the semi-simple element of an $\mathfrak{sl}_2$-triple (see [8], case E_8^3 in Proposition 6.2.4 a) p. 134). The irreducible subspace V_2 corresponding to the Spin representation is not regular (Proposition 31 p. 121 in [17]). The irreducible subspace $V_1 \simeq \mathbb{C}^{20}$ of the representation $[\Lambda_1 \otimes \rho \otimes \mathrm{Id}]$ is well known to be regular. Its generic isotropy subgroup is locally isomorphic to $SO(2) \times SO(8) \times \mathbb{C}^*$ ([17], (15) p. 145) and the representation $(G_2 = SO(2) \times SO(8) \times \mathbb{C}^*, V_2)$ is regular by Proposition 2.6.

4. Classification of Q-irreducible reductive PV's of parabolic type

4.1. PV's of parabolic type

At this point a brief summary of the theory of PV's of Parabolic type is needed.

The PV's of parabolic type were introduced by the author in [9], [10] (see also [11] and [12])

Let $\mathfrak{g}$ be a simple complex Lie algebra. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ and denote by Σ the set of roots of $(\mathfrak{g}, \mathfrak{h})$. As usually, for $\alpha \in \Sigma$, we denote by H_α the corresponding co-root in $\mathfrak{h}$. We fix once and for all a system of simple roots Ψ for Σ. We denote by Σ^+ (resp. Σ^-) the corresponding set of positive (resp. negative) roots in Σ. Let θ be a subset of Ψ and let us make the standard construction of the parabolic subalgebra $\mathfrak{p}_\theta \subset \mathfrak{g}$ associated to θ. As usual we denote by $\langle\theta\rangle$ the set of all roots which are linear combinations of elements in θ, and put $\langle\theta\rangle^\pm = \langle\theta\rangle \cap \Sigma^\pm$.

Set

$$\mathfrak{h}_\theta = \theta^\perp = \{X \in \mathfrak{h} \,|\, \alpha(X) = 0 \ \ \forall \alpha \in \theta\}, \qquad \mathfrak{h}(\theta) = \sum_{\alpha\in\theta} \mathbb{C}H_\alpha$$

$$\mathfrak{l}_\theta = \mathfrak{z}_{\mathfrak{g}}(\mathfrak{h}_\theta) = \mathfrak{h} \oplus \sum_{\alpha\in\langle\theta\rangle} \mathfrak{g}^\alpha, \qquad \mathfrak{n}_\theta^\pm = \sum_{\alpha\in\Sigma^\pm\smallsetminus\langle\theta\rangle^\pm} \mathfrak{g}^\alpha.$$

Then $\mathfrak{p}_\theta = \mathfrak{l}_\theta \oplus \mathfrak{n}_\theta^+$ is called the standard parabolic subalgebra associated to θ. There is also a standard $\mathbb{Z}$-grading of $\mathfrak{g}$ related to these data. Define H_θ to be the unique element of $\mathfrak{h}_\theta$ satisfying the linear equations

$$\alpha(H_\theta) = 0 \quad \forall \alpha \in \theta \quad \text{and}$$

$$\alpha(H_\theta) = 2 \quad \forall \alpha \in \Psi \setminus \theta.$$

The above mentioned grading is just the grading obtained from the eigenspace decomposition of $\operatorname{ad} H_\theta$:

$$d_p(\theta) = \{X \in \mathfrak{g} \mid [H_\theta, X] = 2pX\}.$$

Then we obtain easily:

$$\mathfrak{g} = \oplus_{p\in\mathbb{Z}} d_p(\theta), \quad \mathfrak{l}_\theta = d_0(\theta), \quad \mathfrak{n}_\theta^+ = \sum_{p\geqslant 1} d_p(\theta), \quad \mathfrak{n}_\theta^- = \sum_{p\leqslant -1} d_p(\theta).$$

It is known that $(\mathfrak{l}_\theta, d_1(\theta))$ is a prehomogeneous vector space (in fact all the spaces $(\mathfrak{l}_\theta,\, d_p(\theta))$ with $p \neq 0$ are prehomogeneous, but there is no loss of generality if we only consider $(\mathfrak{l}_\theta,\, d_1(\theta))$). These spaces have been called prehomogeneous vector spaces of parabolic type ([9]). There are in general neither irreducible nor regular. But they are of particular interest, because in the parabolic context, the group (or more precisely its Lie algebra $\mathfrak{l}_\theta$) and the space (here $d_1(\theta)$) of the PV are embedded into a rich structure, namely the simple Lie algebra $\mathfrak{g}$. For example the derived representation of the PV is just the adjoint representation of $\mathfrak{l}_\theta$ on $d_1(\theta)$. Moreover the Lie algebra $\mathfrak{g}$ also contains the dual PV, namely $(\mathfrak{l}_\theta, d_{-1}(\theta))$.

It may be worthwhile noticing also that $d_1(\theta) = \sum_{\beta\in\sigma_1} \mathfrak{g}^\beta$, where σ_1 is the set of roots which belong to the set $(\Psi \setminus \theta) + \mathbb{Z}\theta$, where $\mathbb{Z}\theta$ is the $\mathbb{Z}$–span of θ.

As these PV's are in one to one correspondence with the subsets $\theta \subset \Psi$, we make the convention to describe them by the mean of the following weighted Dynkin diagram:

Definition 4.1. — *The diagram of the PV $(\mathfrak{l}_\theta, d_1(\theta))$ is the Dynkin diagram of* $(\mathfrak{g}, \mathfrak{h})$ *(or* Σ *), where the vertices corresponding to the simple roots of* $\Psi \setminus \theta$ *are circled (see an example below).*

This very simple classification by means of diagrams contains nevertheless some immediate and interesting information concerning the PV $(\mathfrak{l}_\theta, d_1(\theta))$ (for all these facts, see [9], [10] or [11]):

• The Dynkin diagram of $\mathfrak{l}'_\theta = [\mathfrak{l}_\theta, \mathfrak{l}_\theta]$ (*i.e.*, the semi-simple part of the Lie algebra of the group) is the Dynkin diagram of $\mathfrak{g}$ where we have removed the circled vertices and the edges connected to these vertices.

• In fact as a Lie algebra $\mathfrak{l}_\theta = \mathfrak{l}_\theta{}' \oplus \mathfrak{h}_\theta$ and $\dim \mathfrak{h}_\theta =$ the number of circled vertices.

• The number of irreducible components of the representation $(\mathfrak{l}_\theta,\, d_1(\theta))$ is also equal to the number of circled roots. More precisely, if α is a (simple) circled root, then any nonzero root vector $X_\alpha \in \mathfrak{g}^\alpha$ generates an irreducible

$\mathfrak{l}_\theta$–module V_α, and $d_1(\theta) = \oplus_{\alpha\in\Psi\setminus\theta} V_\alpha$ is the decomposition of $d_1(\theta)$ into irreducibles.

In fact the decomposition of the representation $(\mathfrak{l}_\theta, d_1(\theta))$ into irreducibles can also be described by using the eigenspace decomposition with respect to $\mathrm{ad}(\mathfrak{h}_\theta)$. Let me explain this. For each $\alpha \in \mathfrak{h}^*$, let $\overline{\alpha}$ be the restriction of α to $\mathfrak{h}_\theta$ and define

$$\mathfrak{g}^{\overline{\alpha}} = \{X \in \mathfrak{g} \,|\, \forall H \in \mathfrak{h}_\theta,\ [H, X] = \overline{\alpha}(H)X\}.$$

Then $\mathfrak{g}^{\overline{0}} = \mathfrak{l}_\theta$ and for $\alpha \in \Psi \setminus \theta$, we have $V_\alpha = \mathfrak{g}^{\overline{\alpha}}$. Hence we can write

$$d_1(\theta) = \oplus_{\alpha\in\Psi\setminus\theta} \mathfrak{g}^{\overline{\alpha}}.$$

Moreover one can notice (always for $\alpha \in \Psi \setminus \theta$) that $V_\alpha = \mathfrak{g}^{\overline{\alpha}} = \sum_{\beta\in\sigma_1^\alpha} \mathfrak{g}^\beta$, where σ_1^α is the set of roots which belong to $\alpha + \langle\theta\rangle$.

• Moreover one can directly read the highest weight of V_α from the diagram. The highest weight of V_α relatively to the Borel sub-algebra $\mathfrak{b}_\theta^- = \mathfrak{h} \oplus \sum_{\alpha\in\langle\theta\rangle^-} \mathfrak{g}^\alpha$ is $\tilde{\alpha} = \alpha_{|\mathfrak{h}(\theta)}$. Let ω_β $(\beta \in \theta)$ be the fundamental weights of $\mathfrak{l}'_\theta$ (*i.e.*, the dual basis of $(H_\beta)_{\beta\in\theta}$). For each circled root α (*i.e.*, for each $\alpha \in \Psi\setminus\theta$), let $J_\alpha = \{(\beta_i)\}$ be the set of roots in θ (= non-circled) which are connected to α in the diagram. From elementary diagram considerations we know that J_α may be empty and that there are always no more than 3 roots in J_α.

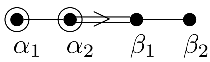
If $J_\alpha = \emptyset$, then V_α is the trivial one dimensional representation of $\mathfrak{l}_\theta$.

If $J_\alpha \neq \emptyset$, then $\tilde{\alpha} = \sum_{i\in J_\alpha} c_i\omega_{\beta_i}$ where $c_i = \alpha(H_{\beta_i})$ and $\alpha(H_{\beta_i})$ can be computed as follows:

$$(R)\begin{cases} \text{if } ||\alpha|| \leqslant ||\beta_i||, \text{then } \alpha(H_{\beta_i}) = -1\ ; \\ \text{if } ||\alpha|| > ||\beta_i|| \ \text{ and if } \alpha \text{ and } \beta_i \text{ are connected by } j \text{ arrows } (1 \leqslant j \leqslant 3), \\ \qquad\qquad\qquad \text{then } \alpha(H_{\beta_i}) = -j\ . \end{cases}$$

Let us illustrate this with an example.

Example 4.2. — Consider the following diagram:



The preceding diagram is the diagram of a PV of parabolic type inside $\mathfrak{g} \simeq F_4$. Here we have $\theta = \{\beta_1, \beta_2\}$ and $\Psi \setminus \theta = \{\alpha_1, \alpha_2\}$. The Lie algebra $\mathfrak{l}_\theta$ is isomorphic to $A_2 \oplus \mathfrak{h}_\theta$ where $\dim \mathfrak{h}_\theta$ = number of circled roots = 2. As $J_{\alpha_1} = \emptyset$, the representation of $\mathfrak{l}'_\theta$ on V_{α_1} is the trivial representation. Hence the action of $\mathfrak{l}_\theta$ on V_{α_1} reduces to the character of $\mathfrak{h}_\theta$ given by the restriction of the root α_1 to $\mathfrak{h}_\theta$. On the other hand we have $J_{\alpha_2} = \{\beta_1\}$. Therefore, applying the rules (R) above, we see that V_{α_2} is the irreducible A_2–module with highest weight $-2\omega_1$, where $\{\omega_1, \omega_2\}$ is the set of fundamental weights of A_2 corresponding to β_1 and β_2. Again the action of $\mathfrak{h}_\theta$ on V_{α_2} is scalar with eigenvalue the restriction of α_2 to $\mathfrak{h}_\theta$.

One can prove ([9]) that the PV of parabolic type $(\mathfrak{l}_\theta, d_1(\theta))$ is irreducible if and only if $\mathfrak{p}_\theta$ is a maximal parabolic subalgebra, *i.e.*, if and only if $\Psi \setminus \theta$ is reduced to a single root α_1.

The PV's of parabolic type which are irreducible and regular were classified by the list of the "weighted" Dynkin diagram of $\mathfrak{g}$, where the root α_1 in the discussion above is circled. This classification was announced first in [9] and then given explitly in [10] and [11] (see also the book [12]).

Remark 4.3. — Of course the irreducible regular PV's of parabolic type are Q-irreducible. Therefore in order to complete the classification of the Q-irreducible PV's of parabolic type, it is enough to consider only PV's which are reducible. This will be done in the rest of the paper.

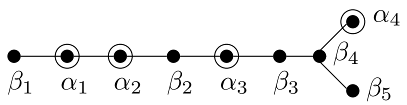
For further use we need also to introduce the notion of subdiagram of the weighted Dynkin diagram associated to (Ψ, θ). Let Γ be a subset of $\Psi \setminus \theta$, that is a subset of the circled roots. For $\alpha \in \Gamma$ define Ψ_α to be the connected component of $\theta \cup \{\alpha\}$ containing α. Define then

$$\Psi_\Gamma = \cup_{\alpha \in \Gamma} \Psi_\alpha \text{ and } \theta_\Gamma = \theta \cap \Psi_\Gamma.$$

Definition 4.4. — *The weighted Dynkin diagram associated to the pair* $(\Psi_\Gamma, \theta_\Gamma)$ *is called a subdiagram of the diagram associated to* (Ψ, θ).

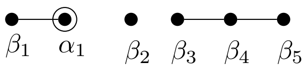
It can be noticed that a subdiagram is just a subset Γ of the circled roots together with the non-circled roots which are connected to a root in Γ (through a path in the non-circled roots). It may also be noticed that the subdiagrams of a connected diagram are not necessarily connected. Let us give an example.

Example 4.5. — Consider the following weighted diagram in D_9

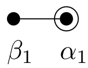
$D =$ 

where $\theta = \{\beta_1, \beta_2, \beta_3, \beta_4, \beta_5\}$ and $\Psi \setminus \theta = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$.

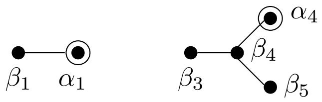
We have:

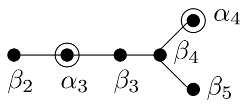
$\theta \cup \{\alpha_1\} =$ 

Therefore the irreducible subdiagram associated to $\{\alpha_1\}$ is given by:

$D_{\{\alpha_1\}} =$ 

Similarly the subdiagrams of D corresponding to $\Gamma = \{\alpha_1, \alpha_4\}$ and $\Gamma = \{\alpha_3, \alpha_4\}$ are respectively:

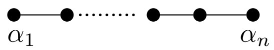
$D_{\{\alpha_1,\alpha_4\}} =$ 

$D_{\{\alpha_3,\alpha_4\}} =$ 

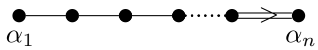
Definition 4.6. — *A weighted Dynkin diagram will be called regular (resp. Q-irreducible) if the corresponding PV of parabolic type is regular (resp. Q-irreducible).*

4.2. Classification of Q-irreducible reductive PV's of parabolic type

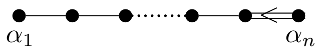
We adopt the following numbering of the roots for classical simple Lie algebras



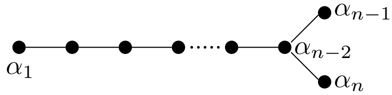
A_n



B_n



C_n



D_n

The classification of Q-irreducible PV's in the classical simple Lie algebras needs now some technical lemmas. If ω_i is the fundamental weight corresponding to the root α_i, we denote by $\Lambda_i(\mathfrak{g})$ the corresponding representation of $\mathfrak{g}$. If this representation can be lifted to a group G with Lie algebra $\mathfrak{g}$, we will denote by $\Lambda_i(G)$ the lifted representation of G. For example we will denote by $\Lambda_1(GL(n))$ (resp. by $\Lambda_n(GL(n))$) the natural representation of $GL(n)$ on $\mathbb{C}^n$ (resp. the dual of the natural representation of $GL(n)$ on $\mathbb{C}^n$).

LEMMA 4.7. — *Let G be a simple classical group. Let $d_1 = \dim \Lambda_1(G)$. Let $n \leqslant d_1$ and consider the PV $(G \times GL(n), \Lambda_1(G) \otimes \Lambda_n(GL(n)))$ (it is a PV because it is parabolic). Then either this PV is regular, or there exists a normal unipotent subgroup of the generic isotropy subgroup which is included in G.*

Proof. — If G is of type A_k then an obvious calculation shows the Lemma.

If G is of type B_k or D_k, then we know from table 1 in [10], that the given PV is always regular.

The same argument holds if G is of type C_k and if n is even.

If G is of type C_k and if n is odd, the space is not regular and the calculations made at p. 102 of [17] show the assertion concerning the normal unipotent subgroup. □

LEMMA 4.8. — *Let G be a reductive algebraic group and let Λ be a representation of G of dimension r. Let p and q be two integers such that $p < q$ and $r < q$. Suppose that the representation $[\Lambda_{p-1}(GL(p)) \otimes \Lambda_1(GL(q))] \oplus [\Lambda_{q-1}(GL(q)) \otimes \Lambda]$ of the group $GL(p) \times GL(q) \times G$ is prehomogeneous (this is automatically the case if $p \geqslant r$). Then:*

1) *If $p \neq r$, the preceding PV is not regular and there exists a non-trivial normal unipotent subgroup of the generic isotropy subgroup which is included in $GL(q)$.*
2) *If $p = r$, the preceding PV is regular and 1-irreducible (hence Q-irreducible from Proposition 2.12).*

Proof. — As G only acts through its representation $\Lambda(G)$, we can assume that $G \subset GL(r)$. The space of the representation is $M(q,p) \oplus M(r,q)$ (where $M(u,v)$ stands for the space of $u \times v$ matrices), and the group $GL(p) \times GL(q) \times G$ acts by

$$(g_1, g_2, g_3)(X, Y) = (g_2 X g_1^{-1}, g_3 Y g_2^{-1})$$

where $g_1 \in GL(p)$, $g_2 \in GL(q)$, $g_3 \in G$, $X \in M(q,p)$, $Y \in M(r,q)$.

As usually we denote by Ω the open orbit in $M(q,p) \oplus M(r,q)$.

• *Suppose first that $p < r$.*

As the representation is supposed to be prehomogeneous, we know from Proposition 2.6 that the open orbits of the components are matrices of maximal rank in $M(q,p)$ and $M(r,q)$ respectively. Let $X_0 = \begin{bmatrix} I_p \\ 0 \end{bmatrix} \in M(q,p)$ where I_p is the identity matrix of size p. An easy calculation shows that the isotropy subgroup of $(X_0, 0) \in M(q,p) \oplus M(r,q)$ is the set of matrices of the form:

$$\left(g_1, \begin{bmatrix} g_1 & B \\ 0 & D \end{bmatrix}, g_3\right),\ g_1 \in GL(p), D \in GL(q-p), B \in M(p,q-p), g_3 \in G.$$

It can also be easily seen that that the set $\mathcal{O}$ of matrices of the form

$$\begin{bmatrix} u \,|\, 0 \end{bmatrix} . \begin{bmatrix} g_1 & B \\ 0 & D \end{bmatrix}$$

where $g_1 \in GL(p), D \in GL(q-p), B \in M(p,q-p), u \in GL(r), \begin{bmatrix} u \,|\, 0 \end{bmatrix} \in M(r,q)$ contains a Zariski open subset of $M(r,q)$.

Therefore $\mathcal{O} \cap \{m \in M(r,q) \,|\, (X_0, m) \in \Omega\} \neq \emptyset$.

This implies that there exists a generic element of the form (X_0, Y_0) where $Y_0 = (y_0, 0)$ with $y_0 \in GL(r)$. Again a simple calculation shows that the isotropy subgroup of (X_0, Y_0) is the set of triplets of the form:

$$\left(g_1, \left[\begin{array}{cc|c} g_1 & B_1 & \\ 0 & D_1 & 0 \\ \hline 0 & D_2 & D_3 \end{array}\right], g_3\right)$$

where $g_1 \in GL(p), D_1 \in GL(r-p), D_2 \in M(q-r, r-p), D_3 \in GL(q-r), g_3 \in G \subset GL(r)$ and where

$$y_0 . \begin{bmatrix} g_1 & B_1 \\ 0 & D_1 \end{bmatrix} = g_3 . y_0 .$$

It is now clear that the set of triplets of the form

$$\left(I_p, \left[\begin{array}{cc|c} \multicolumn{2}{c|}{I_r} & 0 \\ \hline 0 & D_2 & I_{q-r} \end{array}\right], I_r\right)$$

is a unipotent normal subgroup of the (generic) isotropy subgroup of (X_0, Y_0).

- *Suppose that $p > r$.*

Let $X_0 = \begin{bmatrix} I_p \\ 0 \end{bmatrix} \in M(q,p)$ and let $Y_0 = \begin{bmatrix} I_r & 0 \end{bmatrix} \in M(r,q)$. The isotropy subgroup of (X_0, Y_0) is the set of triples of matrices of the form

$$\left(\begin{bmatrix} g_3 & 0 \\ C_1 & D_1 \end{bmatrix}, \left[\begin{array}{cc|c} g_3 & 0 & 0 \\ C_1 & D_1 & D_2 \\ \hline 0 & 0 & D_3 \end{array}\right], g_3\right),$$

where $g_3 \in G \subset GL(r), D_1 \in GL(p-r), C_1 \in M(p-r, r), D_3 \in GL(q-p), D_2 \in M(p-r, q-p)$.

A simple calculation of dimensions shows now that the representation is prehomogeneous and that (X_0, Y_0) is generic. Of course the set of triplets of the form

$$\left(I_p, \left[\begin{array}{c|c} I_p & \begin{array}{c} 0 \\ D_2 \end{array} \\ \hline 0 & I_{q-p} \end{array}\right], I_r\right)$$

is a unipotent normal subgroup of the (generic) isotropy subgroup of (X_0,Y_0).

• *Finally suppose that* $p = r$.

Let $X_0 = \begin{bmatrix} I_p \\ 0 \end{bmatrix} \in M(q,p)$ and let $Y_0 = \begin{bmatrix} I_p & 0 \end{bmatrix} \in M(p,q)$. The isotropy subgroup of (X_0, Y_0) is the set of triplets of the form

$$(*) \qquad \left(g_3, \begin{bmatrix} g_3 & 0 \\ 0 & D \end{bmatrix}, g_3\right)$$

where $g_3 \in G \subset GL(p)$, $D \in GL(q-p)$.

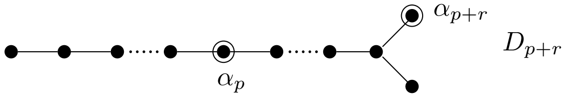
Again an easy computation of dimensions shows that this representation is prehomogeneous. As the generic isotropy subgroup is reductive, this PV is regular.

Let G_1 be the subgroup of $GL(p) \times GL(q) \times G$ generated by a generic isotropy subgroup and by the commutator subgroup $SL(p) \times SL(q) \times G'$. The characters of the relative invariants are exactly those characters which are trivial on G_1 (this is true for any PV). From $(*)$ it is easy to see that G/G_1 is always a one dimensional torus, hence there exists only one fundamental relative invariant. One can remark that this invariant is given by $f(X,Y) = \det(YX)$, $X \in M(p,q), Y \in M(q,p)$. $\square$

Lemma 4.9. — *Consider the representation*

$$[\Lambda_{p-1}(GL(p)) \otimes \Lambda_{r-1}(GL(r))] \oplus [\mathrm{Id}(GL(p) \otimes \Lambda_2(GL(r))]$$

of the group $GL(p) \times GL(r)$, with $r \geqslant 3$. Note that this representation is prehomogeneous since it is infinitesimally equivalent to the PV of parabolic type associated to the diagram



1) *If r is odd and if $p = r-1$, this space is regular and 1-irreducible (hence Q-irreducible from Proposition 2.12).*
2) *If r is odd and $p \leqslant r-2$, this space is not regular and there exists a non-trivial normal unipotent subgroup of the generic isotropy subgroup which is included in $SL(r)$.*

Proof. — The space of the representation is $V = M(r,p) \oplus \mathrm{Skew}(r)$, where $\mathrm{Skew}(r)$ denotes the spaces of skew-symmetric matrices of size r, and the action of the group $GL(p) \times GL(r)$ is given by

$$(g_1, g_2)(X, Y) = ({}^t g_2^{-1} X g_1^{-1}, g_2 Y {}^t g_2),$$

where $g_1 \in GL(p), g_2 \in GL(r), X \in M(r,p), Y \in \mathrm{Skew}(r)$. From the computations in [17], p. 75–76, we know that if $r = 2m+1$, there exists a generic element $Y_0 \in \mathrm{Skew}(r)$, such that the isotropy subgroup of $(0, Y_0) \in V$ is the set of pairs of the form

$$\left(g_1, \begin{bmatrix} A & B \\ 0 & D \end{bmatrix}\right)$$

where $g_1 \in GL(p)$, $A \in \mathrm{Sp}(m)$, $B \in M(2m,1)$, $D \in GL(1)$, and where $\mathrm{Sp}(m)$ denotes the symplectic group inside $GL(2m)$.

- *Suppose that* $p = r - 1$.

One shows easily that if $X_0 = \begin{bmatrix} I_{r-1} \\ 0 \end{bmatrix} \in M(r, r-1)$, the isotropy subgroup of (X_0, Y_0) is the set of pairs of matrices of the form

$$\left(g_1, \begin{bmatrix} {}^t g_1^{-1} & 0 \\ 0 & D \end{bmatrix}\right),$$

where $g_1 \in \mathrm{Sp}(m), D \in GL(1)$.

A simple calculation of dimensions proves then that (X_0, Y_0) is generic. As the preceding isotropy subgroup is reductive, this PV is regular. The normal subgroup G_1 of $GL(r-1) \times GL(r)$ generated by this isotropy subgroup and the commutator subgroup $SL(r-1) \times SL(r)$ is of codimension one. Therefore this PV is 1-irreducible. The fundamental relative invariant is given by $f(X,Y) = Pf({}^t X.Y.X)$ $(X \in M(r, r-1), Y \in \mathrm{Skew}(r))$, where $Pf(Z)$ denotes the Pfaffian of the skew-symmetric matrix Z.

- *Suppose that* $p \leqslant r - 2$.

Set $X_0 = \begin{bmatrix} I_p \\ 0 \end{bmatrix} \in M(r,p)$. Then the isotropy subgroup of (X_0, Y_0) is the set of pairs of matrices of the form

$$\left(g_1, \begin{bmatrix} {}^t g_1^{-1} & 0 & 0 \\ X & Y & B \\ 0 & 0 & D \end{bmatrix}\right),$$

where $X \in M(r-1-p, p), Y \in GL(r-1-p), D \in GL(1), B \in M(r-1-p, 1)$, and where

$$\begin{bmatrix} {}^t g_1^{-1} & 0 \\ X & Y \end{bmatrix} \in \mathrm{Sp}(m).$$

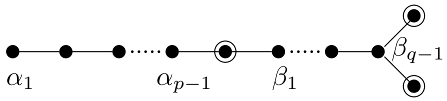
Then the set of matrices $\begin{bmatrix} I_p & 0 & 0 \\ 0 & I_{r-1-p} & B \\ 0 & 0 & 1 \end{bmatrix}$ is a normal unipotent subgroup in $SL(r)$. $\square$

Remark 4.10. — If r is odd and p is even ($p < r-2$) the function $(X, Y) \longmapsto Pf({}^tX.Y.X)$ (where Pf stands for the Pfaffian) is a non-trivial relative invariant of the PV considered in Lemma 4.9, which is non regular for these values of p and r. Hence the result from [9] which asserts that an irreducible PV of parabolic type is regular if and only if there exists a non-trivial relative invariant is no longer true if the representation is not irreducible.(See also Remark 4.12 for another example).

LEMMA 4.11. — *Let D_2 be the group $(\mathbb{C}^*)^2$ identified with the 2×2 diagonal matrices, and denote by Δ the natural representation of D_2 on $\mathbb{C}^2$. Consider the representation*

$$[\Lambda_{p-1}(GL(p)) \otimes \Lambda_1(SL(q)) \otimes \mathrm{Id}(D_2)] \oplus [\mathrm{Id}(GL(p)) \otimes \Lambda_{q-1}(SL(q)) \otimes \Delta]$$

of the group $GL(p) \times SL(q) \times D_2$. Note that this representation is prehomogeneous since it is infinitesimally equivalent to the PV of parabolic type associated to the diagram



D_{p+q+1}

1) *If $q > p$ and $p = 2$ this PV is regular and 1-irreducible (hence Q-irreducible from Proposition 2.12).*
2) *If $q > p$ and $p \neq 2$, then this PV is not regular and there exists a non-trivial normal unipotent subgroup of the generic isotropy subgroup which is included in $SL(q)$.*

Proof. — The space of the representation is $M(q, p) \oplus M(2, q)$ and the action of $GL(p) \times SL(q) \times D_2$ is given by

$$(g_1, g_2, g_3)(X, Y) = (g_2 X g_1^{-1}, g_3 Y g_2^{-1}),$$

where $g_1 \in GL(p), g_2 \in SL(q), g_3 \in D_2, X \in M(q, p), Y \in M(2, q)$.

• *Suppose that $q > p$ and $p = 2$.*

Let $X_0 = \begin{bmatrix} I_2 \\ 0 \end{bmatrix} \in M(q,2)$ and let $Y_0 = \begin{bmatrix} I_2 & 0 \end{bmatrix} \in M(2,q)$. A computation shows that the isotropy subgroup of (X_0, Y_0) is the set of triplets of the form $\left(d, \begin{bmatrix} d & 0 \\ 0 & g \end{bmatrix}, d\right)$, where $d \in D_2$ and $g \in GL(q-2)$. From the dimensions of the full group and of the isotropy subgroup, we see that (X_0, Y_0) is generic. Moreover as the isotropy subgroup is reductive, the PV is regular. The subgroup G_1 generated by the commutator subgroup ($\simeq SL(2) \times SL(q)$) and the generic isotropy is the subgroup of triples (g_1, g_2, g_3) with $\det g_1 = \det g_3$. Hence G/G_1 is one dimensional, therefore the PV is 1-irreducible. It is easy to see that the function $(X,Y) \longmapsto \det(YX)$ is the fundamental relative invariant.

• *Suppose that $q > p$ and $p > 2$.*

Let $X_0 = \begin{bmatrix} I_p \\ 0 \end{bmatrix} \in M(q,p)$ and let $Y_0 = \begin{bmatrix} I_2 & 0 \end{bmatrix} \in M(2,q)$. Then again one proves that (X_0, Y_0) is generic and one shows that its isotropy subgroup is the set of triples of the form

$$\left(\begin{bmatrix} d & 0 \\ 0 & D \end{bmatrix}, \begin{bmatrix} d & 0 & 0 \\ C & D & B \\ 0 & 0 & D' \end{bmatrix}, d\right),$$

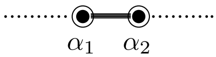
where $d \in D_2, D \in GL(p-2), B \in M(p-2, q-p), D' \in GL(q-p)$. The set of matrices of the form $\begin{bmatrix} I_2 & 0 & 0 \\ 0 & I_{p-2} & B \\ 0 & 0 & I_{q-p} \end{bmatrix}$ is a normal unipotent subgroup of $SL(q)$.

• *Suppose that $q > p$ and $p = 1$.*

Let $X_0 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \in M(q,1)$ and let $Y_0 = \begin{bmatrix} I_2 & 0 \end{bmatrix} \in M(2,q)$. It is easy to verify that (X_0, Y_0) is generic and that its isotropy subgroup is the set of triples of the form $\left(\lambda, \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ \gamma & -\gamma & D \end{bmatrix}, \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right)$, where $\lambda \in \mathbb{C}^*$, $D \in GL(q-2), \lambda^2 \det D = 1$, $\gamma \in M(q-2,1)$. The subset of matrices of the form $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \gamma & -\gamma & I_{q-2} \end{bmatrix}$ is a normal unipotent subgroup of $SL(q)$. $\square$

Remark 4.12. — If $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ is a vector in $\mathbb{C}^2$, let f_1 and f_2 be the two projections defined by $f_i\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = x_i, i = 1, 2$. Consider the *PV* from Lemma 4.11, with $q > p$ and $p = 1$. It is quite obvious that the mappings $(X, Y) \longmapsto f_i(Y.X)$ are relative invariants which are algebraically independant. This gives another example of a parabolic *PV* having nontrivial relative invariants and which is nonregular (see Remark 4.10).

LEMMA 4.13. — *Let $(\mathfrak{l}_\theta, d_1(\theta))$ be a PV of parabolic type in a simple Lie algebra $\mathfrak{g}$. Suppose that its diagram is of the following type*



where the boldface line stands for one or more edges in the Dynkin diagram. In other words, in the notation of Section 4.1, we suppose that $\Psi \setminus \theta$ contains two roots α_1 and α_2 (but possibly others) with $(\alpha_1|\alpha_2) \neq 0$. Let Ψ_1 be the connected component of $\Psi \setminus \{\alpha_2\}$ containing α_1 and let Ψ_2 be the connected component of $\Psi \setminus \{\alpha_1\}$ containing α_2. Set $\theta_1 = \theta \cap \Psi_1$ and $\theta_2 = \theta \cap \Psi_2$. Define

$$D^1(\theta) = \bigoplus_{\alpha \in \Psi_1 \setminus \theta_1} \mathfrak{g}^{\overline{\alpha}}, \; D^2(\theta) = \bigoplus_{\alpha \in \Psi_2 \setminus \theta_2} \mathfrak{g}^{\overline{\alpha}}.$$

(For the notations see Section 4.1, $D^1(\theta)$ (resp. $D^2(\theta)$) is just the sum of the irreducible components of $d_1(\theta)$ arising from the left of the root α_1 (resp. from the right of the root α_2). Then:

$$(L_\theta, d_1(\theta)) \text{ is regular } \iff (L_\theta, D^1(\theta)) \text{ and } (L_\theta, D^2(\theta)) \text{ are regular.}$$

Proof. — Suppose first that $(L_\theta, D^1(\theta))$ and $(L_\theta, D^2(\theta))$ are regular. Then, as $d_1(\theta) = D^1(\theta) \oplus D^2(\theta)$, we know from Proposition 2.13 that $(L_\theta, d_1(\theta))$ is regular.

Conversely suppose that $(L_\theta, D^1(\theta))$ is not regular (for example). Let $X_1 + X_2$ ($X_i \in D^i(\theta)$) be a generic element in $d_1(\theta)$. Then X_1 is generic in $(L_\theta, D^1(\theta))$ (see Proposition 2.6). From the hypothesis we know that the isotropy subgroup $(L_\theta)_{X_1}$ is not reductive (Proposition 2.4), hence $(L_\theta)_{X_1}$ contains a nontrivial normal unipotent subgroup U. The Lie algebra $\mathfrak{u}$ of U is a nonzero ideal in $(\mathfrak{l}_\theta)_{X_1}$. Let $\mathfrak{l}_1$(*resp.* $\mathfrak{l}_2$) be the semi-simple subalgebra of $\mathfrak{g}$ corresponding to θ_1(*resp.* θ_2). One has $\mathfrak{l}_\theta = \mathfrak{h}_\theta \oplus \mathfrak{l}_1 \oplus \mathfrak{l}_2$. From the hypothesis on α_1 and α_2, we have $[\mathfrak{l}_2, X_1] = \{0\}$. Therefore $(\mathfrak{l}_\theta)_{X_1} = (\mathfrak{h}_\theta \oplus \mathfrak{l}_1)_{X_1} \oplus \mathfrak{l}_2$, and hence $\mathfrak{u} \in (\mathfrak{h}_\theta \oplus \mathfrak{l}_1)_{X_1}$. But as $\mathfrak{u}$ is the Lie algebra of a unipotent subgroup we have $\mathfrak{u} \in \mathfrak{l}_1$. But as $[\mathfrak{l}_1, X_2] = \{0\}$, we obtain that

U stabilizes also X_2. As $(\mathfrak{l}_\theta)_{X_1+X_2} = (\mathfrak{l}_\theta)_{X_1} \cap (\mathfrak{l}_\theta)_{X_2}$, we see that $\mathfrak{u}$ is an ideal in $(\mathfrak{l}_\theta)_{X_1+X_2}$. Hence U is a normal subgroup of $(L_\theta)_{X_1+X_2}$. Therefore $(L_\theta, d_1(\theta))$ is not regular. $\square$

Theorem 4.14. — *The Q-irreducible PV's of parabolic type which are not irreducible regular are exactly the PV's from Table 1 at the end of the paper (where the numbers p_i are the number of roots in the connected components of θ).*

Proof. — A consequence of Lemma 4.13 is that the diagram of a Q-irreducible PV of parabolic type will never contain two circled roots which are connected by one or more edges. Therefore we will never consider such diagrams in this proof.

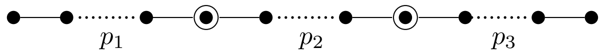
$\diamondsuit$ *Let us first consider the case of classical simple Lie algebras.*

As we do not consider irreducible PV's, we assume that $\mathrm{Card}(\Psi \setminus \theta) \geqslant 2$.

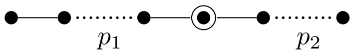
• The case A_n.

a) Suppose that $\mathrm{Card}(\Psi \setminus \theta) = 2$. Consider a diagram of the type:

(4.1)



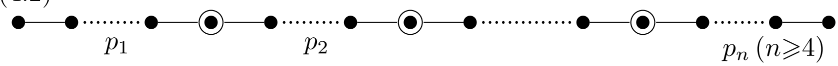
which is supposed to be Q-irreducible. If $p_1 \geqslant p_2$, Lemma 4.7 implies that either the subdiagram



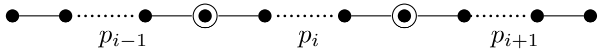
is regular or the generic isotropy subgroup contains a nontrivial normal unipotent subgroup which is included in $SL(p_1+1)$. Therefore, in the second case, this unipotent subgroup will be included in the generic isotropy of the diagram 4.1. Hence, in the second case the diagram 4.1, will not be regular. Therefore we have necessarily $p_1 < p_2$. The same arguments show that we have also $p_3 < p_2$. But then, from Lemma 4.8 we obtain that this PV is regular if and only if $p_3 = p_2$, and in this case it is 1-irreducible.

b) Suppose that $\mathrm{Card}(\Psi \setminus \theta) > 2$. Suppose that the following diagram is Q-irreducible:

(4.2)



As before Lemma 4.7 implies that $p_1 < p_2$ and $p_n < p_{n-1}$. By induction the same argument shows that there exists $i \in \{2, \ldots, n-1\}$ such that $p_{i-1} < p_i$ and $p_{i+1} < p_i$. If $p_{i-1} \neq p_{i+1}$ Lemma 4.8 implies that there exists a normal unipotent subgroup in the generic isotropy subgroup of the subdiagram



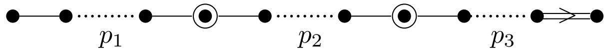
which is included in $SL(p_i)$. But this subgroup will still be included in the generic isotropy of the diagram 4.2, and hence the diagram 4.2 would not be regular.

If $p_{i-1} = p_{i+1}$ Lemma 4.8 implies that the subdiagram above is regular, hence diagram 4.2 is never Q-irreducible.

• The case B_n.

a) Suppose that $\mathrm{Card}(\Psi \setminus \theta) = 2$. Suppose that the diagram

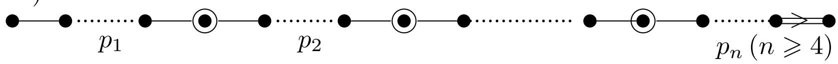
(4.3)



is Q-irreducible. As in the case A_n $a)$ before, Lemma 4.7 implies that $p_1 < p_2$ and $2p_3 + 1 < p_2 + 1$. Then Lemma 4.8 implies that the diagram 4.3 is Q-irreducible if and only if $2p_3 + 1 = p_1 + 1$, which is the condition in Table 1.

b) Suppose that $\mathrm{Card}(\Psi \setminus \theta) > 2$. Suppose that the following diagram is Q-irreducible:

(4.4)



Then as before Lemma 4.7 implies that $p_1 < p_2$ and $2p_n < p_{n-1}$. There are then two possibilities:

– either there exists $i \in \{2, \ldots, n-2\}$ such that $p_{i-1} < p_i$ and $p_{i+1} < p_i$,

– or $p_{n-2} < p_{n-1}$ and $2p_n < p_{n-1}$.

In both cases Lemma 4.8 implies that either diagramm 4.4 contains a regular subdiagram or it is not regular. We have showed that diagram 4.4 is never Q-irreducible.

• The cases C_n and D_n^1.

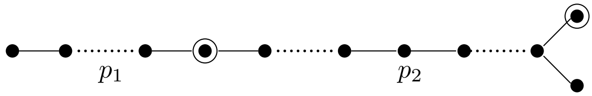
These cases can be treated in the same way as the cases A_n and B_n. It must be noticed that in the C_n case one cannot have a diagram where the root α_n is circled. This is because the subdiagram

would be regular (see the list of the irreductible regular PV's of parabolic type in [10] or in [12]).

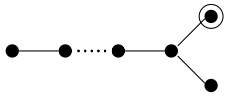
• The case D_n^2.

a) Suppose that $\mathrm{Card}(\Psi \setminus \theta) = 2$. Suppose that the diagram

(4.5)



is Q-irreducible. Then p_2 is even because if p_2 is odd the subdiagram



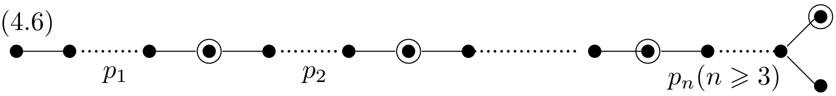
D_{p_2+1}

would be regular (see the list of the irreductible regular PV's of parabolic type in [10] or in [12]).

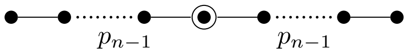
On the other hand from Lemma 4.7 we get that $p_2 > p_1$. Then Lemma 4.9 implies that only the case where $p_1 = p_2 - 1$ corresponds to a Q-irreducible PV.

b) Suppose that $\mathrm{Card}(\Psi \setminus \theta) > 2$. Suppose that the following diagram is Q-irreducible:

(4.6)



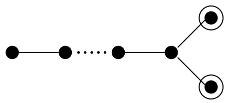
For the same reason as for the diagram 4.5, we necessarily have p_n even. Then from Lemma 4.7 we get $p_1 < p_2$ and from Lemma 4.9 we get $p_n \leqslant p_{n-1}$. If $p_n = p_{n-1}$ diagram 4.6 would contain the regular subdiagram



Hence $p_1 < p_2$ and $p_n < p_{n-1}$. There exists then $i \in \{2, \ldots, n\}$ such that $p_{i-1} < p_i$ and $p_{i+1} < p_i$. From Lemma 4.8 we obtain that either the diagram 4.6 is not Q-irreducible (if $p_{i-1} = p_{i+1}$), or non regular (if $p_{i-1} \neq p_{i+1}$). In any case diagram 4.6 is never Q-irreducible.

• The case D_n^3.

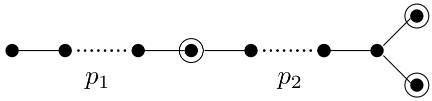
a) Suppose that $\mathrm{Card}(\Psi \setminus \theta) = 2$. It is easy to prove that the subdiagram



D_n

is regular if and only if $n = 3$, and then $D_3 = A_3$ and the corresponding diagram was already considered in the A_n case.

b) Suppose that $\mathrm{Card}(\Psi \setminus \theta) = 3$. Suppose that the following diagram is Q-irreducible.

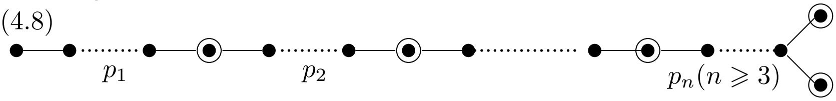
(4.7) 

We know from Lemma 4.11 that if $p_2 > p_1$ and $p_1 \neq 1$, diagram 4.7 is not regular. If $p_2 > p_1$ and $p_1 = 1$, the same Lemma implies that diagram 4.7 is Q-irreducible.

If $p_1 = p_2$, diagram 4.7 contains obviously an A_{n-2} regular irreducible subdiagram.

If $p_1 > p_2$, diagram 4.7 cannot be regular, as shown by Lemma 4.7.

c) Suppose that $\text{Card}(\Psi \setminus \theta) > 3$. The corresponding diagram is the following:

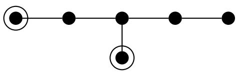
(4.8)


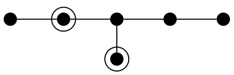
From Lemma 4.7 and Lemma 4.11 we deduce that if this diagram would be Q-irreducible, we would have $p_1 < p_2$ and $p_n < p_{n-1}$. Then, using the same method as in the A_n case, one proves that diagram 4.8 is never Q-irreducible.

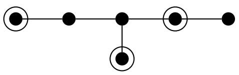
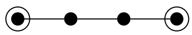
$\diamondsuit$ *Let us now consider the case of exceptional simple Lie algebras.*

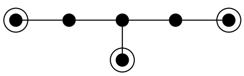
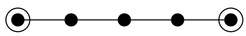
We only give the proof for E_6. The cases of E_7, E_8, F_4 and G_2 are analogous.

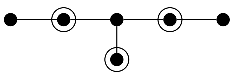
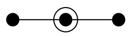
We begin by writing down all possible diagrams in which at least two roots are circled. The only (important) constraint comes from Lemma 4.13 which excludes diagrams having two circled roots which are connected. If a diagram contains a regular subdiagram, we will write the subdiagram on the same line. Taking into account the symmetry of the Dynkin diagram of E_6, the list is as follows:

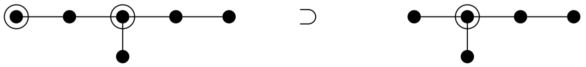
1) 

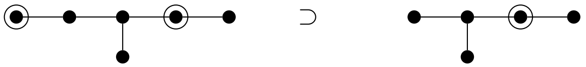
2) 

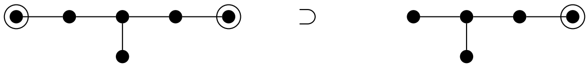
3)  $\supset$ 

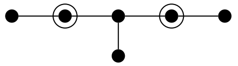
4)  $\supset$ 

5)  $\supset$ 

6) 

7) 

8) 

9) 

10) 

Let us consider the case 1) in the list above. The corresponding PV is infinitesimally equivalent to (G,V) where $G = GL(5)\times\mathbb{C}^*$, $V = M(5,1)\oplus \mathrm{Skew}(5)$ and the action is given by: $(g,a)(X,Y) = (agX, gY\,{}^tg)$ where $a\in\mathbb{C}^*, g\in GL(5), X\in M(5,1), Y\in\mathrm{Skew}(5)$.

Define $J = \begin{bmatrix} 0 & I_2 \\ -I_2 & 0\end{bmatrix} \in \mathrm{Skew}(4)$. Let then

$$X_0 = \begin{bmatrix} 0 \\ 1\end{bmatrix} \in M(5,1) \text{ and } Y_0 = \begin{bmatrix} J & 0 \\ 0 & 0\end{bmatrix} \in \mathrm{Skew}(5).$$

An easy computation shows that (X_0,Y_0) is generic and that its isotropy subgroup is the set of pairs of matrices of the form $\left(\begin{bmatrix} A & 0 \\ 0 & a\end{bmatrix}, a^{-1}\right)$, where $A\in\mathrm{Sp}(2), a\in\mathbb{C}^*$. Hence the PV is regular and one easily shows that the unique fundamental relative invariant is given by

$$f(X,Y) = Pf\left(\begin{bmatrix} Y & X \\ -({}^tX) & 0\end{bmatrix}\right).$$

For the cases 2) and 9) one computes a generic isotropy isotropy subgroup and one observes that it is not reductive. $\square$

Remark 4.15. — The exceptional Q-irreducible PV's arising in E_6, E_7 and E_8 are particular cases of families of Q-irreducible PV's which are not parabolic in general. More precisely the representations

$$(GL(n)\times\mathbb{C}^*, [\Lambda_1(GL(n))\otimes\square]\oplus[\Lambda_2(GL(n))\otimes\mathrm{Id}])(n\,\mathrm{odd})$$

and

$$(GL(n) \times GL(n-1), [\Lambda_1(GL(n)) \otimes \Lambda_1(GL(n-1))] \oplus [\Lambda_1(GL(n)) \otimes \mathrm{Id}])$$

are 1-irreducible PV's. (Here $\square$ denotes the one dimensional representation of $\mathbb{C}^*$ on $\mathbb{C}$ by multiplications). The first representation is an extension of the E_6 and E_8 cases, the second one is an extension of the E_7 case.

For the first representation the fundamental relative invariant is given by

$$f(X,Y) = Pf\left(\begin{bmatrix} Y & X \\ -({}^tX) & 0 \end{bmatrix}\right), (X \in M(n,1), Y \in \mathrm{Skew}(n))$$

and for the second representation it is given by

$$f(X,Y) = \det(\begin{bmatrix} X & Y \end{bmatrix}), (X \in M(n,n-1), Y \in M(n,1)).$$

Note that the first PV above is example 8) p. 95 of [3] and that the second PV is a particular case of the families of PV's studied in Section 4 of [7].

A consequence of the preceding classification is the following statement.

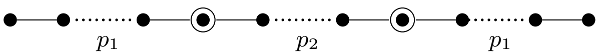
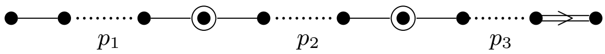
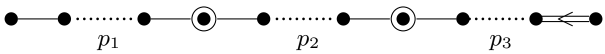
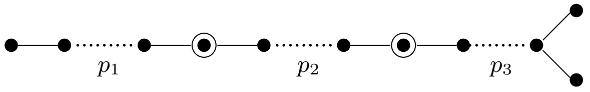
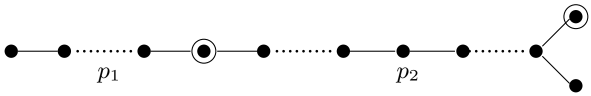
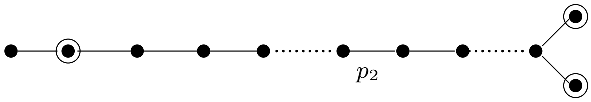
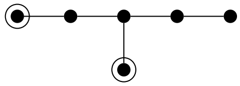
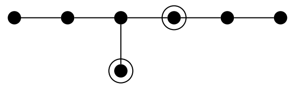
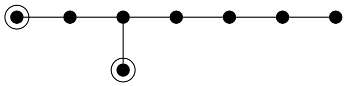
Theorem 4.16. — *The Q-irreducible PV's of parabolic type are 1-irreducible. In other words the three definitions of irreducibility given in Definition 2.9 are equivalent for PV's of parabolic type.*

Remark 4.17. — [2] Let $G = (GL(n))^{p+1}$, and let $V = (M(n))^p$. We denote by $g = (g_i)$ an element in G and by $v = (v_j)$ an element in V. Consider the representation of G on V defined by $(g_i).(v_j) = (g_j v_j g_{j+1}^{-1})$. Then (G, V) is a regular PV with p fundamental relative invariants given by $f_j(v) = \det(v_j)$. Let $N = pn^2 = \dim(V)$ and consider the castling transformation (see [17] or [4]) of (G,V) given by $(G \times GL(N-1), V \otimes \mathbb{C}^{N-1})$. It is known ([17] p. 67–68, and Remark 26 p. 73), that the regularity and the number of fundamental relative invariants does not change under castling transformation, therefore $(G \times GL(N-1), V \otimes \mathbb{C}^{N-1})$ is regular and has also p fundamental relative invariants. But it is easy to see that any proper $G \times GL(N-1)$-invariant subspace of $V \otimes \mathbb{C}^{N-1}$ is of the form $U \otimes \mathbb{C}^{N-1}$, with $\dim(U) < N-1$, and then $(G \times GL(N-1), U \otimes \mathbb{C}^{N-1})$ is a so-called *trivial PV*, which has no fundamental relative invariant. Hence $(G \times GL(N-1), V \otimes \mathbb{C}^{N-1})$ is a Q-irreducible PV which is not 1-irreducible. Therefore Theorem 4.16 is no longer true for non parabolic PV's.

[2] I would like to thank Tatsuo Kimura for providing me with this example.

Remark 4.18. — E. B. Dynkin has classified all $\mathfrak{sl}_2$-triples in the simple Lie algebras ([2]). He has proved that the semi-simple element of an $\mathfrak{sl}_2$-triple is always conjugate to an element H in a fixed Cartan subalgebra such that $\alpha(H) = 0, 1$ or 2 for every simple root α. For the exceptional Lie algebras he gave the list of all such elements (which correspond effectively to an $\mathfrak{sl}_2$-triple) in the form of weighted Dynkin diagrams where the simple root α has weight $\alpha(H) = 0, 1$ or 2. If the grading element H_θ of a given PV of parabolic type (see Section 4.1) is the semi-simple element of a $\mathfrak{sl}_2$-triple, then, from the definition, the weighted Dynkin diagram where the (non circled) roots in θ have weight 0, and the (circled) roots in $\Psi \setminus \theta$ have weight 2 appears in Dynkin's list. We have proved in [9] that an irreducible PV of parabolic type is regular if and only if the corresponding grading element H_θ is the semi-simple element of an $\mathfrak{sl}_2$-triple. As the weighted Dynkin diagrams corresponding to E_6, E_7, E_8 in Table 1 below do not appear in tables 18, 19, 20 of [2], such a result is no longer true for Q-irreducible PV's of parabolic type.

Table 1: non irreducible, Q-irreducible PV's of parabolic type

A_n $(p_2{>}p_1{\geqslant}0)$	
B_n $p_2{>}p_1,$ $2p_3{=}p_1, p_3{\geqslant}0)$	
C_n $p_2{>}p_1,$ $2p_3{=}p_1{+}1, p_3{>}0$ p_2 odd	
D_n^1 $p_2{>}p_1,$ $2p_3{=}p_1{+}1, p_3{\geqslant}2$ p_2 even	
D_n^2 $p_2{\geqslant}2,$ $p_1{=}p_2{-}1, p_2$ even	
D_n^3 $p_2{>}1$	
E_6	
E_7	
E_8	

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Manuscrit reçu le 7 avril 2010,
accepté le 27 juillet 2010.

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