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A GARSIDE PRESENTATION FOR ARTIN-TITS GROUPS OF TYPE $\widetilde{C}_n$

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ABSTRACT. — We prove that an Artin-Tits group of type $\widetilde{C}$ is the group of fractions of a Garside monoid, analogous to the known dual monoids associated with Artin-Tits groups of spherical type and obtained by the "generated group" method. This answers, in this particular case, a general question on Artin-Tits groups, gives a new presentation of an Artin-Tits group of type $\widetilde{C}$, and has consequences for the word problem, the computation of some centralizers or the triviality of the center. A key point of the proof is to show that this group is a group of fixed points in an Artin-Tits group of type $\widetilde{A}$ under an involution. Another important point is the study of the Hurwitz action of the usual braid group on the decomposition of a Coxeter element into a product of reflections.

RÉSUMÉ. — Nous prouvons qu'un groupe d'Artin-Tits de type $\widetilde{C}$ est le groupe des fractions d'un monoïde de Garside analogue des monoïdes duaux connus pour les groupes d'Artin-Tits de type sphérique et construit par la méthode dite du "groupe engendré". Ceci répond dans ce cas particulier à une question générale sur les groupes d'Artin-Tits, donne une nouvelle présentation d'un groupe d'Artin-Tits de type $\widetilde{C}$ et a plusieurs conséquences (problème du mot, calcul de centralisateurs, trivialité du centre). Un point clé de la preuve consiste à montrer que ce groupe est un groupe de points fixes sous une involution dans un groupe d'Artin-Tits de type $\widetilde{A}$. Un autre outil important est l'action de Hurwitz sur les décompositions d'un élément de Coxeter en produit de réflexions.

1. Introduction

The aim of this paper is to define a Garside structure on the Artin-Tits group of type $\widetilde{C}_n$. A Garside structure on a group means that this group is the group of fractions of a Garside monoid. A Garside monoid is a monoid

for which the two posets given by right or left divisibility have nice properties. In particular these two posets are lattices: there exist least common multiples and greatest common divisors, and moreover these two lattices have a common sublattice which generates the monoid and has a greatest element for both orders, the Garside element (see Definition 2.3 below). It is known that all Artin-Tits groups of spherical type have two nice Garside structures given respectively by the classical monoid, obtained by generating the Artin-Tits group by lifts of the simple reflections, and the dual monoid (see [2]), obtained by generating the Artin-Tits group by elements lifting all reflections which divide (see below beginning of Section 3) a given Coxeter element. In the case of non-spherical Artin-Tits groups the classical Artin-Tits monoid exists but is only locally Garside (*i.e.*, two elements have not always a common multiple, in particular there is no Garside element). An open question in general is the existence of a dual Garside structure for general Artin-Tits groups. Such a structure is known for type $\widetilde{A}$ and it has been conjectured by John Crisp and Jon McCammond that no Artin-Tits group of affine type other than type $\widetilde{A}$ and maybe $\widetilde{C}$ can have such a structure. A Garside structure provides normal forms for the elements of the group and is a tool for solving the word problem. It also allows to compute centralisers of periodic elements (roots of powers of the Garside element).

To get a dual Garside structure for an Artin-Tits group of type $\widetilde{C}_n$, we shall view this group as the group of fixed points under an involution in an Artin-Tits group of type $\widetilde{A}_{2n-1}$. In [10] this last group has been shown to be the group of fractions of several monoids only one of which, up to automorphism, is Garside, but unfortunately this one is not stable by the involution. On the other hand only one of these non-Garside monoids, up to automorphism, is stable by the involution. We show that by taking fixed points in this last monoid one gets a Garside structure for $\widetilde{C}_n$.

The paper is organised as follows. In Section 2 we introduce Garside monoids and give methods for getting a Garside monoid from a partially defined product on a subset called a germ. In Section 3 we recall and improve results from [10] on presentations of Artin-Tits groups of type $\widetilde{A}$. In Section 4 we get a Garside structure from the fixed points of an involution in a Garside group of type $\widetilde{A}$. In Section 5 we show that an Artin-Tits group of type $\widetilde{C}$ can be seen as a group of fixed points in an Artin-Tits group of type $\widetilde{A}$. In Section 6 we show that the Garside structure we have got in Section 4 can be obtained by the method of the "generated group" of [2, 0.4]. In Section 7 we prove that the group of fractions of our

Garside monoid is the Artin-Tits group of type $\widetilde{C}_n$ and we give a a dual presentation of this group similar to what has been done in [2], [3] and [10] for the other known dual monoids, where the generators are in one-to-one correspondence with a set of reflections in the Coxeter group. One of the intermediate results is that the Hurwitz action is transitive on the set of shortest decompositions of a Coxeter element of $W(\widetilde{C}_n)$ into a product of reflections. The analogous property is known for all finite Coxeter groups ([2, 2.1.4]), for all well-generated complex reflexion groups ([1, 7.5]), for Coxeter groups of type $\widetilde{A}$ ([10, 3.4]) and is conjectured to be true for all Coxeter groups. In Section 8 we deduce from the Garside structure the centralizer of a power of a lift of a Coxeter element in the Artin-Tits group.

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2. Germs, Garside groups

In this section we recall some definition and results on Garside monoids and groups.

DEFINITION 2.1 ([11]).

(i) *A germ of monoid is a set P endowed with a partially defined product $(x, y) \mapsto xy$, which has a unit, i.e., an element 1 such that $1p$ and $p1$ are defined and equal to p for any $p \in P$.*

(ii) *A germ is associative if for any a, b, c in P such that one of the products $a(bc)$ or $(ab)c$ is defined then the other one is also defined and both products are equal.*

(iii) *A germ is left (resp. right) cancellative if $ab = ac$ (resp. $ba = ca$) implies $b = c$. It is cancellative if it is cancellative on both sides.*

We say that an element a of a germ P left divides an element $b \in P$ if there exists $c \in P$ such that $b = ac$. Right divisibility is defined similarly. In an associative germ right and left divisibility are preorder relations. They are order relations if moreover the germ is cancellative and there is no invertible element different from 1.

DEFINITION 2.2. — *An associative germ is said to be left (resp. right) Noetherian if there is no strictly decreasing infinite sequence for left (resp. right) divisibility. It is called Noetherian if it is both left and right Noetherian.*

Note that in a Noetherian germ there is no non-trivial invertible element. Note also that to be left (resp. right) Noetherian is equivalent to the fact that there is no strictly increasing bounded infinite sequence for right (resp. left) divisibility.

A morphism from a germ to a monoid is a map which sends the (partial) product on P to the product in the monoid, and the unit of P to the unit of the monoid. The monoid $M(P)$ (resp. group $G(P)$) defined by a germ P is the monoid (resp. group) which has the universal property that it factorizes any morphism from P to a monoid (resp. group). In other words it is the monoid (resp. group) generated by P with only relations the relations given by equalities of products in P. It is known ([11, 3.5]) that P injects into $M(P)$ and is stable by left and right divisibility in $M(P)$. The following definition is the definition of Garsideness that we will use in the present paper (for small variations and generalizations of this definition see [9], [8] and [11]).

DEFINITION 2.3.

- *We say that a monoid is Garside if it is cancellative, Noetherian, if it is a lattice for both left and right divisibility and if there exists an element Δ (called a Garside element) whose sets of right and left divisors coincide and generate the monoid.*
- *A group is Garside if it is generated by a submonoid which is a Garside monoid.*

Note that here we do not assume the set of divisors of Δ to be finite. When this set is infinite what we call here a Garside monoid (resp. group) is what is usually called a quasi-Garside monoid (resp. group). A general reference for Garside monoids can be [11]. The following result is a combination of [11, 3.31, 5.4 and 5.5].

PROPOSITION 2.4. — *Let P be an associative Noetherian germ satisfying the following properties:*

  (i) *two elements of P have a least right common multiple in P;*
  (ii) *for all $m \in M(P)$, and a, b in P, if $am = bm$ or $ma = mb$, then $a = b$;*
  (iii) *the elements of P have a both left and right common multiple $\Delta \in P$.*

then $M(P)$ is a Garside monoid with Garside element Δ.

Conversely, in a Garside monoid M the divisors of the Garside element form a germ P satisfying the above properties and such that the canonical map $M(P) \to M$ is an isomorphism.

DEFINITION 2.5. — *A germ satisfying the assumptions of 2.4 is called a Garside germ.*

Given a Garside germ P, elements in $M(P)$ have normal forms: any element can be written uniquely $p_1 p_2 \cdots p_k$ with $p_i \in P$ such that p_i is the greatest element for left divisibility dividing $p_i p_{i+1} \cdots p_k$. In the (Garside) group of a Garside monoid M with Garside element Δ any element can be written as $\Delta^k x$ with $x \in M$ and $k \in \mathbb{Z}$.

We will use proposition 2.4 through its following corollary. Before stating this corollary we need:

DEFINITION 2.6. — *An automorphism of a germ P is a bijection $f \colon P \to P$ mapping the unit to the unit and such that ab is defined if and only if $f(a)f(b)$ is defined, in which case $f(ab) = f(a)f(b)$.*

COROLLARY 2.7. — *Let P be an associative and Noetherian germ having a (unique) both left and right common multiple Δ; assume that $M(P)$ is cancellative and that P has an automorphism σ such that any two elements of P have a unique minimal σ-stable common right multiple; then P^σ is a Garside germ.*

Proof. — It is clear that P^σ is an associative and Noetherian germ. We get the result by proving that P^σ satisfies the assumptions of Proposition 2.4. Since P is cancellative, if $x, y \in P^\sigma$ are such that x divides y in P then x divides y in P^σ. Hence the assumption of the corollary implies that any two elements of P^σ have an lcm in P^σ, whence (i).

The inclusion $P^\sigma \to M(P)$ extends to a morphism $M(P^\sigma) \to M(P)$. If $am = bm$ or $ma = mb$ as in (ii), taking the images in $M(P)$ we get $a = b$, since $M(P)$ is cancellative, whence (ii).

Unicity of Δ as the maximal element of P implies $\Delta \in P^\sigma$, hence Δ is a common right and left multiple of P^σ, whence (iii). $\qquad\square$

3. The monoids of type $\widetilde{A}_{2n-1}$

Before applying the previous results to Artin-Tits groups of type $\widetilde{C}_n$, we need to recall the presentation of an Artin-Tits group of type $\widetilde{A}$ given in [10] as group of fractions of the monoid $M(P)$ generated by an associative germ P.

We recall first a general method for constructing a germ. Given a group G generated as a monoid by a set S, *i.e.*, any element of G is a product of elements of S, without inverses, we let l_S be the length on G with respect

to S: the length of $g \in G$ is the length of a shortest expression of g as a product of elements of S. We say that $a \in G$ left divides $b \in G$, denoted by $a \preccurlyeq_G b$ if $l_S(a) + l_S(a^{-1}b) = l_S(b)$, and similarly for right divisibility. Starting with a balanced element δ (an element which has the same set of right and left divisors), we call D the set of (left or right) divisors of δ. Then D is a germ, the product of a and b in D being defined and equal to ab if $ab \in D$ and $l_S(ab) = l_S(a) + l_S(b)$. Associated to this germ we have a monoid $M(D)$ and a group $G(D)$. If D is a lattice then $M(D)$ is a Garside monoid with Garside element δ (result due to J. Michel, see [2, Theorem 0.5.2]). We call this construction the method of the "generated group". This construction starting with any finite Coxeter group, its set of Coxeter generators and taking the longest element for δ, gives the associated Artin-Tits monoid (or group). Starting with a finite Coxeter group with set of generators all reflections it gives the dual monoid if we take for δ any Coxeter element. Starting with a Coxeter element and all reflections in a Coxeter group of type $\widetilde{A}$ it gives the monoids $M(P)$ that we describe in this section. For these results see [2], [10] and [11].

We see a Coxeter group of type $\widetilde{A}$ as a subgroup of the periodic permutations of $\mathbb{Z}$. We need some notation.

DEFINITION 3.1.
  (i) *A permutation w of $\mathbb{Z}$ is said to be n-periodic if $w(i+n) = w(i)+n$ for any $i \in \mathbb{Z}$.*
  (ii) *A cycle is an n-periodic permutation which has precisely one orbit up to translation by n. We say that a cycle is finite if its orbits are finite. We call length of a cycle the cardinality of one of its orbits. We call support of a permutation w the union of its non-trivial orbits.*

Any n-periodic permutation of $\mathbb{Z}$ can be written uniquely as a product of disjoint cycles. There are two types of cycles: either all the orbits are finite, or all the non-trivial orbits are infinite. In the former case we will represent the cycle by one of its non-trivial orbits; in the latter case we will represent a cycle as $(a_1, a_2, \ldots, a_k)_{[h]}$, with all a_i distinct modulo n, meaning that the image of a_i is a_{i+1} for $0 \leqslant i < k$ and the image of a_k is $a_1 + nh$ (this cycle has $|h|$ non trivial orbits). To each n-periodic permutation w of $\mathbb{Z}$ we can associate its total shift $\frac{1}{n} \sum_{x=1}^{x=n} (w(x) - x)$. We recall the following facts:

PROPOSITION 3.2. — *The Coxeter group $W(\widetilde{A}_{n-1})$ of type $\widetilde{A}_{n-1}$ is isomorphic to the group of n-periodic permutations of $\mathbb{Z}$ with total shift equal*

to 0. The reflections of $W(\widetilde{A}_{n-1})$ correspond to the permutations (a,b) with a and b distinct modulo n. The simple reflections are the permutations $s_i = (i, i+1)$ for $i = 1, 2, \ldots, n$. The reflections s_i and s_j commute unless $i - j = \pm 1 \pmod{n}$ in which case their product has order 3.

In [10] a germ generating the Artin-Tits group of type $\widetilde{A}_{n-1}$ is defined for each partition of $\mathbb{Z}$ into two non-empty subsets X and Ξ stable by translation by n; such a partition corresponds to the choice of a Coxeter element. To recall this construction we need the following definitions. A graphical representation of these definitions will be given after Definition 3.6 We make the convention that Latin letters denote elements of X and Greek letters elements of Ξ.

DEFINITION 3.3. — We say that a cycle is positive and self non-crossing if it has one of the following forms: $(a_1, a_2, \ldots, a_k, \alpha_1, \alpha_2, \ldots, \alpha_l)$ or $(a_1, a_2, \ldots, a_k)_{[1]}$ or $(\alpha_1, \alpha_2, \ldots, \alpha_l)_{[-1]}$ with $a_i \in X$, $\alpha_j \in \Xi$ satisfying the conditions $a_1 < a_2 < \cdots < a_k < a_1 + n$ and $\alpha_1 > \alpha_2 > \cdots > \alpha_l > \alpha_1 - n$.

DEFINITION 3.4. — We say that two positive self non-crossing cycles are non-crossing if they satisfy one of the following:

(i) One of them is of the form $(a_1, a_2, \ldots, a_k)$ or $(a_1, a_2, \ldots, a_k)_{[1]}$ and the other one $(\alpha_1, \alpha_2, \ldots, \alpha_l)$ or $(\alpha_1, \alpha_2, \ldots, \alpha_l)_{[-1]}$.

(ii) One of them is of the form $(a_1, a_2, \ldots, a_k, \alpha_1, \alpha_2, \ldots, \alpha_l)$, and the other $(b_1, \ldots, b_q, \beta_1, \ldots, \beta_r)$ with $k, l, q, r > 0$, $a_k < b_i < a_1 + n$ for all i and $\alpha_l < \beta_i < \alpha_1 + n$ for all i.

(iii) One of them is of the form $(a_1, a_2, \ldots, a_k, \alpha_1, \alpha_2, \ldots, \alpha_l)$ with $k, l \geqslant 0$, and the other one $(b_1, \ldots, b_m)$ (resp. $(\beta_1, \ldots, \beta_m)$) with $a_j < b_i < a_{j+1}$ for some j and all i (resp. $\alpha_{j+1} < \beta_i < \alpha_j$ for some j and all i), where in this condition we put $a_{k+1} = a_1 + n$ and $\alpha_{l+1} = \alpha_1 - n$, (there is no condition on b_i if $k = 0$ and no condition on β_i if $l = 0$).

(iv) One of them is of the form $(a_1, a_2, \ldots, a_k)_{[1]}$ or $(\alpha_1, \alpha_2, \ldots, \alpha_k)_{[-1]}$ with $k > 0$ and the other one $(b_1, \ldots, b_m)$ (resp. $(\beta_1, \ldots, \beta_m)$) with $a_j < b_i < a_{j+1}$ for some j and all i (resp. $\alpha_{j+1} < \beta_i < \alpha_j$ for some j and all i), where in this condition we put $a_{k+1} = a_1 + n$ and $\alpha_{k+1} = \alpha_1 - n$.

DEFINITION 3.5. — We say that a periodic permutation of $\mathbb{Z}$ is positive and self non-crossing if it is the product of disjoint positive self non-crossing and pairwise non-crossing cycles.

Note that a finite cycle has total shift 0. Note also that a periodic positive self non-crossing permutation of $\mathbb{Z}$ has at most 2 infinite cycles, one in X

and one in Ξ and that, if it has total shift 0, it has either 0 or 2 infinite cycles.

DEFINITION 3.6. — *We call pseudo-cycle a positive self non-crossing n-periodic permutation of $\mathbb{Z}$ which is the product of two infinite disjoint cycles.*

A pseudo-cycle is a permutation $(a_1, \ldots, a_h)_{[1]}(\alpha_1, \alpha_2, \ldots, \alpha_l)_{[-1]}$ with the a_i in X, the α_i in Ξ and $a_1 < a_2 < \cdots < a_h < a_1 + n$, $\alpha_1 > \alpha_2 > \cdots > \alpha_n > \alpha_1 - n$. A pseudo-cycle has total shift 0.

We now give a graphical representation of the above definitions (see Figure 1 and Figure 2; in these figures we have taken $n = 9$, and modulo n there are 5 elements in X and 4 in Ξ). Consider two parallel oriented lines D and Δ in the plane, with same orientation. On D we put a discrete set of points in one-to-one ordered correspondence with X and on Δ we put a discrete set of points in one-to-one ordered correspondence with Ξ. Let $\mathcal{S}$ be the strip delimited in the plane by D and Δ. We associate to a permutation w of $\mathbb{Z}$ a union of oriented paths (one for each orbit of w) contained in the strip $\mathcal{S}$ joining i to $w(i)$ for all i, up to homotopy in the strip. A cycle is positive self non-crossing if it can be represented by a union of disjoint oriented paths such that each of them intersects X (resp. Ξ) along an increasing (resp. decreasing) or empty subsequence. Two positive self non-crossing cycles are non-crossing if they can be represented by two unions of paths which do not cross each other. Other figures (with $n = 10$) can be found in Section 6. Following [10] we now describe the germ P associated to the X and Ξ. The elements of P are the n-periodic positive self non-crossing permutations. The permutation c given by $x_i \mapsto x_{i+1}$ and $\xi_i \mapsto \xi_{i-1}$ where $X = (x_i)_{i \in \mathbb{Z}}$ and $\Xi = (\xi_i)_{i \in \mathbb{Z}}$ with $x_i < x_{i+1}$ and $\xi_i < \xi_{i+1}$, is a Coxeter element of $W(\widetilde{A}_{n-1})$ and the elements of P are precisely the left (or right) divisors of c (see [10, Proposition 2.19]). The germ is obtained by the method of the generated group described at the beginning of this section, hence the product of two elements w and w' of P is defined in P if the product ww' of the permutations is in P and if $l_{\widetilde{A}_{n-1}}(ww') = l_{\widetilde{A}_{n-1}}(w) + l_{\widetilde{A}_{n-1}}(w')$ where $l_{\widetilde{A}_{n-1}}$ is the length in $W(\widetilde{A}_{n-1})$ with respect to the generating set consisting of all reflections. We will give below an equivalent and more tractable condition.

By general results P is an associative, cancellative and Noetherian germ, the monoid $M(P)$ embeds in the group $G(P)$ and any element $G(P)$ can be written $a^{-1}b$ with a and b in $M(P)$. Since the length $l_{\widetilde{A}_{n-1}}$ is invariant by conjugation, right and left divisibility in P coincide.

The following is proved in [10, Theorem 4.1].
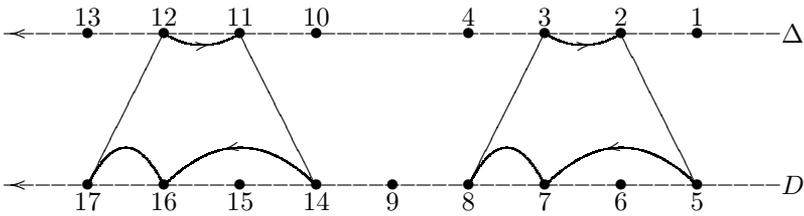
FIGURE 1: The 9-periodic cycle $(5, 7, 8, 3, 2)$
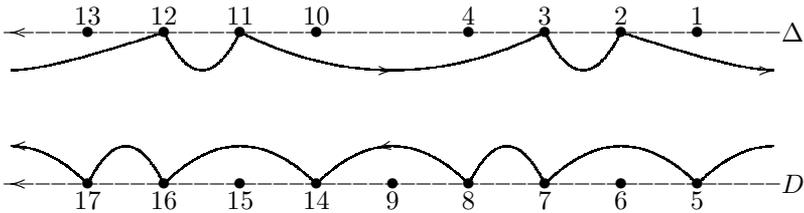
FIGURE 2: The 9-periodic pseudo-cycle $(5, 7, 8)_{[1]}(3, 2)_{[-1]}$

PROPOSITION 3.7. — *The group $G(P)$ is isomorphic to the Artin-Tits group of type $\widetilde{A}_{n-1}$.*

Note that by [10, Proposition 5.5] P is a Garside germ if and only if either X or Ξ contains exactly one element modulo n.

We now give a more tractable definition of divisibility in P. To each element $w \in P$ we associate the partition p_w of $\mathbb{Z}$ whose parts are the finite orbits of w and the union of the two infinite orbits of w if they exist. Such a partition is invariant by translation by n. We say that it is periodic. Moreover such a partition has at most one infinite part and this infinite part must meet both X and Ξ. Also such a partition is non-crossing in the following sense:

DEFINITION 3.8.

- *Two subsets A and B of $X \cup \Xi$ are non-crossing if for any $a, a' \in A$ and any $b, b' \in B$ there exist in the strip $\mathcal{S}$ a path γ from a to a' and a path δ from b to b' such that γ and δ have an empty intersection.*
- *We say that a partition is non-crossing if any two of its parts are non-crossing.*

By [10, Corollary 2.22] $w \mapsto p_w$ is a bijection from P onto the set of non-crossing periodic partitions of $\mathbb{Z}$ such that any infinite part meets both

X and Ξ. Such a partition can have at most one infinite part. We order P by divisibility and the set of partitions by refinement. Note that the largest element of P for the divisibility order is c and that p_c is the partition with only one part.

PROPOSITION 3.9. — *The bijection $w \mapsto p_w$ is an isomorphism of ordered sets.*

Proof. — Let $v, w \in P$. We write v and w as products $v = v_1 \cdots v_k$ and $w = w_1 \cdots w_l$ where each v_i and each w_i is either a a positive self non-crossing finite cycle (see Definition 3.3) or is a pseudo-cycle (see Definition 3.6) and the v_i (resp. the w_i) are pairwise non-crossing.

By [10, Lemma 2.20] a reflection of P divides w in $W(\widetilde{A}_n)$ if and only if its support is a subset of the support of w_i for some i. Now the following lemma (see [10, Corollary 2.9]) shows that any v_i can be written as a product $r_1 r_2 \cdots r_h$ in P of reflections whose union of supports is the support of v_i and such that the supports of r_i and r_{i+1} have a non-empty intersection.

LEMMA 3.10. — *The two following formulas give shortest decompositions of a finite cycle and of a pseudo-cycle respectively into products of reflections of $\widetilde{A}_{n-1}$.*

$$(a_1, a_2, \ldots, a_h) = (a_1, a_2)(a_2, a_3) \cdots (a_{h-1} a_h)$$

$$(a_1, \ldots, a_h)_{[1]}(\alpha_1, \alpha_2, \ldots, \alpha_l)_{[-1]} = (a_1, a_2)(a_2, a_3) \cdots$$
$$\cdots (a_{h-1}, a_h)(a_h, \alpha_1)(a_h, n + \alpha_1)(\alpha_1, \alpha_2)(\alpha_2, \alpha_3) \cdots (\alpha_{l-1}, \alpha_l).$$

Proof. — By [10, 2.8] we know that

$$l_{\widetilde{A}_{n-1}}((a_1, \ldots, a_h)_{[1]}(\alpha_1, \alpha_2, \ldots, \alpha_l)_{[-1]}) = h + l$$

and that

$$l_{\widetilde{A}_{n-1}}((a_1, a_2, \ldots, a_h)) = h - 1.$$

$\square$

Assume that $v = v_1 \cdots v_k$ divides w in P; fix i and let us we write $v_i = r_1 \cdots r_h$ as above. Then every r_j divides w so its support must be a subset of the support of some w_s. But since the supports of r_j and r_{j+1} have a non-empty intersection, they have to be included in the support of the same w_s, so that the whole support of v_i is a subset of the support of some w_s. This, being true for all i, means that p_v is finer than p_w.

Conversely, assume that p_v is finer than p_w. We prove by induction on $l_{\widetilde{A}_{n-1}}(v)$ that v divides w in P. If v is trivial the result is true. If v is not trivial, there exists a reflection s dividing v, so that the support of s is

included in the support of some v_i. Hence the support of s is also included in the support of some w_j, so that s divides w. Put $v = sv'$ and $w = sw'$; then v divides w if and only if v' divides w'. We will be done by induction if we prove that $p_{v'}$ is finer than $p_{w'}$. We have seen that s divides a cycle or a pseudo-cycle v_i of v and a cycle or pseudo-cycle w_j of w. Since disjoint cycles and pseudo-cycles commute we may assume that s divides v_1 and w_1. We have only to show that $p_{v'_1}$ is finer than $p_{w'_1}$ where $v_1 = sv'_1$ and $w_1 = sw'_1$. This will be a consequence of the following lemma:

LEMMA 3.11. — Let $u \in P$ be a finite cycle or a pseudo-cycle and let s be a reflection dividing u in P. We put $u = su'$.

- If $u = (a_1, a_2, \ldots, a_h) \in P$ is a finite cycle (with $a_i \in X \cup \Xi$) and $s = (a_1, a_j)$, then $u' = (a_1, a_2, \ldots, a_{j-1})(a_j, a_{j+1}, \ldots, a_h)$.
- If $u = (a_1, \ldots, a_h)_{[1]}(\alpha_1, \alpha_2, \ldots, \alpha_l)_{[-1]}$ with $a_i \in X$ and $\alpha_i \in \Xi$ is a pseudo-cycle and $s = (a_1, a_j)$ then $u' = (a_1, a_2, \ldots, a_{j-1})(a_j, a_{j+1}, \ldots, a_h)_{[1]}(\alpha_1, \alpha_2, \ldots, \alpha_l)_{[-1]}$.
- If $u = (a_1, \ldots, a_h)_{[1]}(\alpha_1, \alpha_2, \ldots, \alpha_l)_{[-1]}$ with $a_i \in X$ and $\alpha_i \in \Xi$ is a pseudo-cycle and $s = (\alpha_1, \alpha_j)$ then $u' = (a_1, a_2, \ldots, \ldots, a_h)_{[1]}(\alpha_1, \alpha_2, \ldots, \alpha_{j-1})(\alpha_j, \ldots, \alpha_l)_{[-1]}$.
- If $u = (a_1, \ldots, a_h)_{[1]}(\alpha_1, \alpha_2, \ldots, \alpha_l)_{[-1]}$ with $a_i \in X$ and $\alpha_i \in \Xi$ is a pseudo-cycle and $s = (a_1, \alpha_1)$ then $u' = (a_1, a_2, \ldots, \ldots, a_h, \alpha_1 + n, \alpha_2 + n, \ldots, \alpha_l + n)$.

This lemma is an easy computation (see also [10, Lemma 2.5]).

Applying the lemma with $u = v_1$ and with $u = w_1$ shows that $p_{v'_1}$ is finer than $p_{w'_1}$. □

4. Fixed points in $\widetilde{A}_{2n-1}$

It is well known that the Coxeter group of type $\widetilde{C}_n$ is the group of fixed points under the involution of $W(\widetilde{A}_{2n-1})$ which maps s_i to s_{2n-i}, with the notation of Proposition 3.2, the subscript i being taken modulo $2n$ (see e.g., [13]). This involution can be lifted to the Artin-Tits group using the same formula. We shall see in Section 5 that similarly the Artin-Tits group of type $\widetilde{C}_n$ is the group of fixed points under this lifted involution in the Artin-Tits group of type $\widetilde{A}_{2n-1}$.

To get a Garside germ for an Artin-Tits group of type $\widetilde{C}_n$ we shall start with a particular choice of X and Ξ in the construction of the previous section, compatible with this involution. We take X to be the the set of odd integers and Ξ to be the set of even integers. This corresponds to choosing the

Coxeter element $c = (2,3)(4,5)\cdots(2n,2n+1)(1,2)(3,4)\cdots(2n-1,2n) = s_2 s_4 \cdots s_{2n} s_1 s_3 \cdots s_{2n-1}$ in the previous section. Then the germ P has an automorphism σ coming from the map $i \mapsto 1-i$ which interchanges X and Ξ. This involution lifts to the Artin-Tits group the involution $s_i \mapsto s_{2n-i}$ of the Coxeter group. We still denote by σ this involution of $X \cup \Xi$.

THEOREM 4.1. — *The germ P^σ is Garside*

Proof. — We show that P and σ satisfy the assumptions of Corollary 2.7.

First P has a unique both right and left multiple of all its elements, which is c. Note that with our choice of X and Ξ, the element c seen as a $2n$-periodic permutation of $\mathbb{Z}$ is

$$i \longmapsto \begin{cases} i+2 & \text{for odd } i, \\ i-2 & \text{for even } i. \end{cases}$$

We have to show that any two elements v and w in P have a unique minimal σ-stable common multiple z. By Proposition 3.9, this amounts to show the existence of a unique minimal σ-stable non-crossing periodic partition of the form p_z coarser than p_v and p_w. Note that a σ-stable non-crossing partition has precisely 0, one or two infinite parts. Hence the condition for such a partition to be of the form p_z, *i.e.*, that any infinite part has a non empty intersection with both X and Ξ is equivalent to the condition for the partition to have at most one infinite part.

A σ-stable partition coarser than p_v and p_w is also coarser than the partitions ${}^\sigma p_v$ and ${}^\sigma p_w$. We claim that it is sufficient to show the existence of a unique minimal non-crossing partition coarser than these 4 partitions: such a partition will be periodic and σ-stable by unicity. Moreover since it is non-crossing and symmetric it will have precisely 0, one or two infinite parts. If it has zero or one infinite part it is of the form p_z and we are done. If it has two infinite parts these parts are interchanged by σ, in which case there is exactly one minimal coarser partition with one infinite part meeting both X and Ξ, obtained by putting together the two infinite parts. This partition is σ-stable and periodic. By Proposition 3.9 it corresponds to a minimal σ-stable common multiple of v and w and we are done also in this case.

The theorem is thus a consequence of the following lemma which ensures by induction the existence of a unique non-crossing partition coarser than any (finite) given number of partitions of $X \cup \Xi$.

LEMMA 4.2. — *There exists a unique minimal non-crossing partition coarser than any two given partitions of $X \cup \Xi$.*

Proof of the lemma. — Consider a graph whose vertices are the parts of the given partitions and such that there is an edge between two vertices if the corresponding parts cross each other (this includes the case where two parts have an intersection). Then the desired partition is the partition whose parts are the union of all the parts lying in the same connected component of our graph. $\qquad\square$

5. The Artin-Tits group of type $\widetilde{C}_n$

We denote by $G(\widetilde{A}_{2n-1})$ the Artin-Tits group of type $\widetilde{A}_{2n-1}$ and by $G(\widetilde{C}_n)$ the Artin-Tits group of type $\widetilde{C}_n$. They are the groups of fractions of the corresponding classical Artin-Tits monoids, respectively $M(\widetilde{A}_{2n-1})$ and $M(\widetilde{C}_n)$. The monoid $M(\widetilde{A}_{2n-1})$ has a presentation with generators $\mathbf{s}_1, \ldots, \mathbf{s}_{2n}$ and relations $\mathbf{s}_i \mathbf{s}_j = \mathbf{s}_j \mathbf{s}_i$ if $|i - j| \neq 1 \bmod 2n$ and $\mathbf{s}_i \mathbf{s}_{i+1} \mathbf{s}_i = \mathbf{s}_{i+1} \mathbf{s}_i \mathbf{s}_{i+1}$ for all i, where the indices are taken modulo $2n$; the monoid $M(\widetilde{C}_n)$ of type $\widetilde{C}_n$ has a presentation with generators $\boldsymbol{\sigma}_0, \ldots, \boldsymbol{\sigma}_n$ and relations the braid relations given by the Coxeter diagram

$$\underset{\boldsymbol{\sigma}_0}{\bigcirc}\!=\!\!\underset{\boldsymbol{\sigma}_1}{\bigcirc}\!-\!\underset{\boldsymbol{\sigma}_2}{\bigcirc}\cdots\underset{\boldsymbol{\sigma}_{n-2}}{\bigcirc}\!-\!\underset{\boldsymbol{\sigma}_{n-1}}{\bigcirc}\!=\!\!\underset{\boldsymbol{\sigma}_n}{\bigcirc}$$
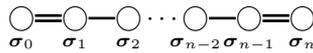
Like any Artin-Tits monoid, by [16], the monoids $M(\widetilde{A}_{2n-1})$ and $M(\widetilde{C}_n)$ embed in their respective groups $G(\widetilde{A}_{2n-1})$ and $G(\widetilde{C}_n)$.

By Proposition 3.7 the group $G(\widetilde{A}_{2n-1})$ is isomorphic to the group of fractions of the monoid $M(P)$ generated by the germ P defined in Section 4. This isomorphism maps the generator $\mathbf{s}_i$ of $M(\widetilde{A}_{2n-1})$ to the element $(i, i+1) \in P$, so that $M(\widetilde{A}_{2n-1})$ is a submonoid of $M(P)$. The involution σ considered in Section 4 restricts to $M(\widetilde{A}_{2n-1})$ in the diagram automorphism of $\widetilde{A}_{2n-1}$ which maps $\mathbf{s}_i$ to $\mathbf{s}_{2n-i}$ where the indices are taken modulo $2n$.

By [15, 4.4] the monoid of fixed points $M(\widetilde{A}_{2n-1})^\sigma$ is isomorphic to $M(\widetilde{C}_n)$. We will identify these two monoids. Under this identification we have $\boldsymbol{\sigma}_0 = \mathbf{s}_{2n}$, $\boldsymbol{\sigma}_1 = \mathbf{s}_1 \mathbf{s}_{2n-1}, \ldots, \boldsymbol{\sigma}_{n-1} = \mathbf{s}_{n-1} \mathbf{s}_{n+1}$, $\boldsymbol{\sigma}_n = \mathbf{s}_n$.

We summarise all these facts in the commutative diagram:

$$(5.1) \qquad \begin{array}{ccccc} M(\widetilde{C}_n) & \xrightarrow{\sim} & M(\widetilde{A}_{2n-1})^\sigma & \hookrightarrow & M(\widetilde{A}_{2n-1}) \\ & & \downarrow & & \downarrow \\ & & M(P)^\sigma & \hookrightarrow & M(P) \\ \downarrow & & \downarrow & & \downarrow \\ G(\widetilde{C}_n) & \longrightarrow & G(\widetilde{A}_{2n-1})^\sigma & \hookrightarrow & G(\widetilde{A}_{2n-1}) = G(P) \end{array}$$

The aim of this section is to prove the following theorem. I thank John Crisp and Dave Margalit for having pointed out that this theorem can be easily deduced from the results of Birman-Hilden and Maclachlan-Harvey.

THEOREM 5.1. — *The morphism* $G(\widetilde{C}_n) \to G(\widetilde{A}_{2n-1})^\sigma$ *of Diagram 5.1, which maps* $\boldsymbol{\sigma}_0$ *to* $\mathbf{s}_{2n}$, $\boldsymbol{\sigma}_i$ *to* $\mathbf{s}_i\mathbf{s}_{2n-i}$ *for* $i = 1, \ldots, n-1$ *and* $\boldsymbol{\sigma}_n$ *to* $\mathbf{s}_n$ *is an isomorphism.*

We need first to recall how the Artin-Tits groups of type $\widetilde{A}$ and $\widetilde{C}$ can be embedded into mapping class groups. Let E and F be two finite subsets of a 2-sphere S^2. We denote by $M(S^2, E, F)$ the subgroup of the mapping class group of the 2-sphere with punctures the points of $E \cup F$, represented by the diffeomorphisms which fix each point of F and stabilize E. The following result appears in [7, Section 2]:

PROPOSITION 5.2 (Charney-Crisp).
  (i) *Let* $E = \{P_1, \ldots, P_n\}$ *and* $F = \{P', P''\})$ *be two sets of points of* S^2; *the group* $G(\widetilde{A}_{n-1})$ *embeds into the group* $M(S^2, E, F)$. *This embedding maps the standard generator* $\mathbf{s}_i$ *of* $G(\widetilde{A}_{n-1})$ *to the positive braid twist exchanging* P_i *and* P_{i+1}, *for* $1 \leqslant i \leqslant n$, *where the indices are taken modulo* n.
  (ii) *Let* $E = \{P_1, \ldots, P_n\}$ *and* $F = \{P_0, Q', Q''\}$ *be subsets of* S^2 *then the group* $G(\widetilde{C}_n)$ *is isomorphic to* $M(S^2, E, F)$. *This isomorphism maps the standard generator (numbered as in the beginning of Section 5)* $\boldsymbol{\sigma}_i$ *of* $G(\widetilde{C}_n)$ *to the positive braid twist exchanging* P_i *and* P_{i+1} *for* $1 \leqslant i \leqslant n-1$ *and maps* $\boldsymbol{\sigma}_0$ (*resp.* $\boldsymbol{\sigma}_n$) *to the square of the positive braid twist exchanging* P_0 *and* P_1 (*resp.* P_n *and* Q'').

We need also the following result of Birman and Hilden (see [5] and [14]).

THEOREM 5.3 (Birman-Hilden). — *Let* E_1 *and* E_2 *be two finite sets of points of a 2-sphere* S^2, *let* $\pi\colon S^2 \to S^2/G$ *be a ramified covering realizing the quotient of* S^2 *by a finite group* G *of diffeomorphisms stabilizing* E_1 *and* E_2 *and let* $F \subset S^2/G$ *be the set of ramification points. Then the projection induces an isomorphism from the normalizer of* G *in the mapping class group* $M(S^2, E_1, E_2)$ *to the mapping class group* $M(S^2/G, \pi(E_1), \pi(E_2) \cup F)$.

Proof of Theorem 5.1. — Consider a set E consisting of $2n$ points $P_1, \ldots, P_{2n}$ regularly placed on the equator of a sphere S^2 and let P' and P'' be the north and south poles. We apply Proposition 5.2 (i), for embedding the group $G(\widetilde{A}_{2n-1})$ into the mapping class group $M(S^2, E, F)$

where $F = \{P', P''\}$. We then apply Theorem 5.3 with G the group of order 2 generated by the symmetry σ which exchanges P' and P'' and exchanges P_i and P_{2n-i} for all i. We can view S^2/G as a sphere so that $\pi \colon S^2 \to S^2/G$ is a ramified covering of a 2-sphere with 2 ramification points $A', A'' \in S^2/G$ which are antipodal on the equator. The points $\pi(P_i) = \pi(P_{2n-i})$ are regularly placed on one half of the equator and the point $P = \pi(P') = \pi(P'')$ is on the other half of this equator. We get an isomorphism $M(S^2, E, F)^\sigma \simeq M(S^2/G, \pi(E), \{P, A', A''\})$. Since $\pi(E)$ has cardinality n, we get by Proposition 5.2 (ii) that $M(S^2/G, \pi(E), \{P, A', A''\})$ is isomorphic to $G(\widetilde{C}_n)$. Since π maps the product of braid twists $\mathbf{s}_i\mathbf{s}_{n-i}$ to the braid twist $\boldsymbol{\sigma}_i$ for $i = 1, \ldots, n-1$ and maps the braid twist $\mathbf{s}_n$ (resp. $\mathbf{s}_{2n}$) to the square $\boldsymbol{\sigma}_0$ (resp. $\boldsymbol{\sigma}_n$) of the positive braid twist exchanging A' and $\pi(P_1)$ (resp. $\pi(P_n)$ and A''), we get that the isomorphism restricted to $G(\widetilde{A}_{2n-1})^\sigma$ is onto which means that $G(\widetilde{A}_{2n-1})^\sigma$ is in fact equal to $M(S^2, E, F)^\sigma$. Moreover we also see that the above isomorphism coincides with the morphism of Theorem 5.1, whence the result. $\square$

6. A generated group

Our aim in this section is to show that the germ P^σ is obtained from the Coxeter group of type $\widetilde{C}_n$ by the method of the generated group described at the beginning of Section 3. We denote the Coxeter group of type $\widetilde{C}_n$ by $W(\widetilde{C}_n)$ and see it as the group of fixed points under the involution $\sigma \colon s_i \mapsto s_{2n-i}$ in the Coxeter group $W(\widetilde{A}_{2n-1})$ of type $\widetilde{A}_{2n-1}$ (see the beginning of Section 4).

LEMMA 6.1. — *The reflections of $W(\widetilde{C}_n)$ are the σ-stable reflections of $W(\widetilde{A}_{2n-1})$ and the products $r.^\sigma r$ where r is a reflection of $W(\widetilde{A}_{2n-1})$ which is not σ-stable.*

Proof. — This is well known. We recall a proof. By [13], if σ is an automorphism of a Coxeter group permuting the simple reflections, the group of σ-fixed elements is a Coxeter group with simple reflections the longest elements of the parabolic subgroups generated by the σ-orbits of spherical type of simple reflections. The simple reflections of $W(\widetilde{C}_n)$ are thus s_n, s_{2n} and the products $s_i.^\sigma s_i = s_i s_{2n-i}$ for $1 \leqslant i \leqslant n-1$. An arbitrary reflection of $W(\widetilde{C}_n)$ is a conjugate under $W(\widetilde{C}_n)$ to a simple reflection, whence the result since this conjugation commutes with σ. $\square$

We put $c = s_2 s_4 \cdots s_{2n} s_1 s_3 \cdots s_{2n-1}$ as in Section 4. We denote the simple reflections of $W(\widetilde{C}_n)$ by $\sigma_0 = s_{2n}$, $\sigma_1 = s_1 s_{2n-1}$, $\sigma_2 = s_2 s_{2n-2}, \ldots,$

$\sigma_{n-1} = s_{n-1}s_{n+1}$ and $\sigma_n = s_n$. We have $c = \sigma_0\sigma_2\sigma_4\cdots\sigma_1\sigma_3\sigma_5\cdots$, hence c is both a σ-fixed Coxeter element of $W(\widetilde{A}_{2n-1})$ and a Coxeter element of $W(\widetilde{C}_n)$. We denote by $\preccurlyeq_{W(\widetilde{A}_{2n-1})}$ (resp. $\preccurlyeq_{W(\widetilde{C}_n)}$) the divisibility in $W(\widetilde{A}_{2n-1})$ (resp. $W(\widetilde{C}_n)$) with respect to the reflection length. Beware that the reflection length in $W(\widetilde{C}_n)$ is not the restriction of the reflection length in $W(\widetilde{A}_{2n-1})$, in particular the reflection length of c is $n+1$ in $W(\widetilde{C}_n)$ and $2n$ in $W(\widetilde{A}_{2n-1})$, since a Coxeter element in a Coxeter group of rank h has reflection length equal to h (see $e.g.$, [12]). Recall that divisibility for the reflection length in a Coxeter group can be defined by saying that w divides w' if for one, or for any, shortest decomposition of w into a product of reflections, there exists a shortest decomposition of w' as a product of reflections which begins with that decomposition of w. We now show that divisibility in $W(\widetilde{A}_{2n-1})$ restricts to divisibility in $W(\widetilde{C}_n)$ for the divisors of c. Recall that by the results of [10] P is the set of divisors of c in $W(\widetilde{A}_{2n-1})$, so that P^σ is the set of elements of $W(\widetilde{C}_n)$ which divide c in $W(\widetilde{A}_{2n-1})$.

PROPOSITION 6.2. — *Let w and w' be two elements of $W(\widetilde{C}_n)$; then $w \preccurlyeq_{W(\widetilde{C}_n)} w' \preccurlyeq_{W(\widetilde{C}_n)} c$ if and only if $w \preccurlyeq_{W(\widetilde{A}_{2n-1})} w' \preccurlyeq_{W(\widetilde{A}_{2n-1})} c$.*

Proof. — We first prove that divisibility in $W(\widetilde{C}_n)$ implies divisibility in $W(\widetilde{A}_{2n-1})$ for divisors of c in $W(\widetilde{C}_n)$.

We claim that in any decomposition of c into a product of reflections of $W(\widetilde{C}_n)$ there is at least one reflection conjugate to σ_0 and one reflection conjugate to σ_n. Indeed the quotient of $W(\widetilde{C}_n)$ by its commutator subgroup is the direct product of 3 groups of order 2, generated respectively by the image of σ_0, the image of σ_n and the image of any one of σ_1, ..., σ_{n-1} which have all same image since they are conjugate. Hence the image of c in this quotient must involve the images of σ_0 and σ_n. Whence the result since two conjugate reflections have same image in this quotient.

Let $c = r_0\cdots r_n$ be a decomposition of c into a product of $n+1$ reflections in $W(\widetilde{C}_n)$. Let k be the number of reflections in that decomposition which are conjugate to either σ_0 or σ_n. These k reflections are also reflections in $W(\widetilde{A}_{2n-1})$ and the other $n+1-k$ reflections are products of 2 reflections in $W(\widetilde{A}_{2n-1})$. So this product is the product of $k + 2(n+1-k) = 2n + 2 - k$ reflections in $W(\widetilde{A}_{2n-1})$. But c cannot be written with less than $2n$ reflections in $W(\widetilde{A}_{2n-1})$, whence $k \leqslant 2$. Since we know that $k \geqslant 2$, we have $k = 2$. We have proved that, replacing a reflection of $W(\widetilde{C}_n)$ by its image in the embedding $W(\widetilde{C}_n) \hookrightarrow W(\widetilde{A}_{2n-1})$, any shortest decomposition of c as a product of reflection of $W(\widetilde{C}_n)$ becomes a shortest decomposition of

c as a product of reflections of $W(\widetilde{A}_{2n-1})$. This gives that if $w \preccurlyeq_{W(\widetilde{C}_n)}$ $w' \preccurlyeq_{W(\widetilde{C}_n)} c$ then $w \preccurlyeq_{W(\widetilde{A}_{2n-1})} w' \preccurlyeq_{W(\widetilde{A}_{2n-1})} c$.

To prove the converse we first prove a formula for the reflection length $l_{W(\widetilde{C}_n)}$ in $W(\widetilde{C}_n)$ of the elements of P^σ. An element of P^σ can be written uniquely as a commuting product of elements of three types: either σ-stable non-crossing finite cycles, or products of a finite cycle and its image by σ, or σ-stable pseudo-cycles. There can be at most one pseudo-cycle and any two of these factors are non-crossing. In Figures 3, 4 and 5 we have given examples in $\widetilde{C}_5$, *i.e.*, the partitions are 10-periodic and σ comes from $i \mapsto 1 - i$ modulo 10.
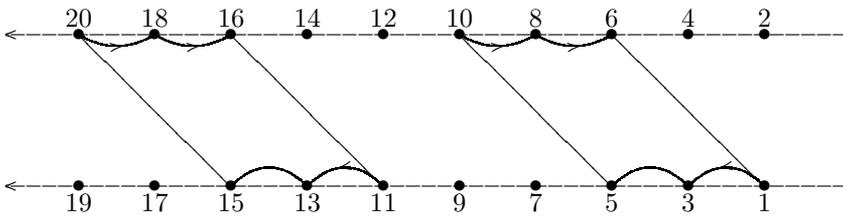

FIGURE 3: A σ-stable cycle in $\widetilde{A}_9$
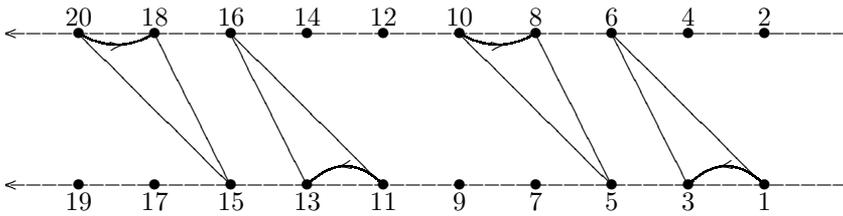
FIGURE 4: A non-crossing product of a cycle and its image by σ in $\widetilde{A}_9$
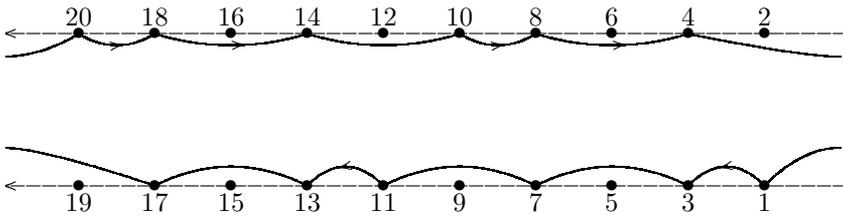

FIGURE 5: A σ-stable pseudo-cycle in $\widetilde{A}_9$

LEMMA 6.3. — *Let $w = w_1 \cdots w_r$ be a decomposition as above of $w \in P^\sigma$; then $l_{W(\widetilde{C}_n)}(w) = \sum_{i=1}^{i=r} l_{W(\widetilde{C}_n)}(w_i)$ and*

$$
l_{W(\widetilde{C}_n)}(w_i) = \begin{cases} k & \text{if } w_i \text{ is a } \sigma\text{-stable cycle of length } 2k \\ k-1 & \text{if } w_i \text{ is the product of a cycle of length } k \\ & \text{and its image by } \sigma \\ k+1 & \text{if } w_i \text{ is a pseudo-cycle whose support modulo } 2n \\ & \text{has cardinality } 2k. \end{cases}
$$

Proof. — We denote by $f(w)$ the formula of the lemma. We have to prove that $l_{W(\widetilde{C}_n)}(w) = f(w)$. First, it is clear that $f(w) = 0$ if and only if $w = 1$. We now prove:

LEMMA 6.4. — *For any $w \neq 1$ in P^σ and any reflection $\rho \in W(\widetilde{C}_n)$ such that $\rho \preccurlyeq_{W(\widetilde{A}_{2n-1})} w$ we have $f(\rho w) = f(w) - 1$.*

Proof. — Recall that σ is induced by the bijection $a \mapsto 1 - a$ of $\mathbb{Z}$ so that by Lemma 6.1 a reflection in $W(\widetilde{C}_n)$ is either equal to $(a, 2kn+1-a)$ for some a and k in $\mathbb{Z}$ or $(a, b)(1 - a, 1 - b)$ for some a and b in $\mathbb{Z}$ with $a + b \not\equiv 1 \bmod 2n$. Decompose w into a product $w = w_1 w_2 \ldots w_r$ as in lemma 6.3. Let ρ be a reflection of $W(\widetilde{C}_n)$ which divides w in $W(\widetilde{A}_{2n-1})$. By Proposition 3.9, if $\rho = (a, 2kn+1-a)$ for some a, then a and $2kn+1-a$ have to be in the support of the same w_i and if $\rho = (a, b)(1 - a, 1 - b)$, then a and b have to be in the support of the same w_i and $1 - a$ and $1 - b$ are then in the support of $^\sigma w_i$. Lemma 6.4 is then a consequence of the following formulas, obtained by an immediate computation. In each of these formulas the left hand side has the form ρw_i with w_i as above and one checks easily that f has value $f(w_i) - 1$ on the the right hand side.

$$
(a_1, 2kn+1-a_1)(a_1, a_2, \ldots, a_h, 2kn+1-a_1, 2kn+1-a_2, \ldots, 2kn+1-a_h)
$$
$$
= (a_1, a_2, \ldots, a_h)(1 - a_1, \ldots, 1 - a_h)
$$

$$
(a_1, 2kn + 1 - a_1)(a_1, \ldots, a_h)_{[1]}(1 - a_1, \ldots, 1 - a_h)_{[-1]}
$$
$$
= (a_1, \ldots, a_h, 2(k+1)n + 1 - a_1, \ldots, 2(k+1)n + 1 - a_h)
$$

$$
(a_1, b_1)(1 - a_1, 1 - b_1)(a_1, \ldots, a_h, b_1, \ldots, b_k)
$$
$$
\cdot (1 - a_1, \ldots, 1 - a_h, 1 - b_1, \ldots, 1 - b_k)
$$
$$
= (a_1, \ldots, a_h)(b_1, \ldots, b_k)(1 - a_1, \ldots, 1 - a_h)(1 - b_1, \ldots, 1 - b_k)
$$

$$(a_1, b_1)(1 - a_1, 1 - b_1)(a_1, \ldots, a_h, b_1, \ldots, b_k)_{[1]}$$
$$\cdot (1 - a_1, \ldots, 1 - a_h, 1 - b_1, \ldots, 1 - b_k)_{[-1]}$$
$$= (a_1, \ldots, a_h)(1 - a_1, \ldots, 1 - a_h)(b_1, \ldots, b_k)_{[1]}(1 - b_1, \ldots, 1 - b_k)_{[-1]}$$

$$(a_1, b_1)(1 - a_1, 1 - b_1)(a_1, \ldots, a_h, 1 - b_1, \ldots, 1 - b_k)_{[1]}$$
$$\cdot (1 - a_1, \ldots, 1 - a_h, b_1, \ldots, b_k)_{[-1]}$$
$$= (a_1, \ldots, a_h, 1 - a_1, \ldots, 1 - a_h)(b_1, \ldots, b_k, -2n + 1 - b_1, \ldots, -2n + 1 - b_k).$$

$\square$

We can now finish the proof of Lemma 6.3 by induction on $l_{W(\widetilde{C}_n)}(w)$. If $w = \rho_1 \rho_2 \cdots \rho_l$ is a decomposition of w as a product of reflections in $W(\widetilde{C}_n)$, with $l = l_{W(\widetilde{C}_n)}(w)$, then $\rho_1 \preccurlyeq_{W(\widetilde{C}_n)} w$, so that $\rho_1 \preccurlyeq_{W(\widetilde{A}_{2n-1})} w$ and by Lemma 6.4 we have $f(\rho_1 w) = f(w) - 1$. Since $f(\rho_1 w) = l_{W(\widetilde{C}_n)}(\rho_1 w)$ by induction and $l_{W(\widetilde{C}_n)}(\rho_1 w) = l_{W(\widetilde{C}_n)}(w) - 1$, we get the result. $\square$

We now end the proof of Proposition 6.2. First, since $l_{W(\widetilde{C}_n)} = f$ on P^σ, Lemma 6.4 shows that for any $w \in P^\sigma$ and any reflection $\rho \in W(\widetilde{C}_n)$ such that $\rho \preccurlyeq_{W(\widetilde{A}_{2n-1})} w$ we have $\rho \preccurlyeq_{W(\widetilde{C}_n)} w$.

Assume now that $w \preccurlyeq_{W(\widetilde{A}_{2n-1})} w' \preccurlyeq_{W(\widetilde{A}_{2n-1})} c$ with $w \neq 1$ and $w, w' \in W(\widetilde{C}_n)$, so that $w, w' \in P^\sigma$; choose a reflection ρ in $W(\widetilde{C}_n)$ such that $\rho \preccurlyeq_{W(\widetilde{C}_n)} w$, hence $\rho \preccurlyeq_{W(\widetilde{A}_{2n-1})} w \preccurlyeq_{W(\widetilde{A}_{2n-1})} w'$ and $\rho w \preccurlyeq_{W(\widetilde{A}_{2n-1})} \rho w'$, so that in particular $\rho \preccurlyeq_{W(\widetilde{C}_n)} w'$ by what we have just proved. We can assume by induction on $l_{W(\widetilde{A}_{2n-1})}(w)$ that $\rho w \preccurlyeq_{W(\widetilde{C}_n)} \rho w'$. Since $l_{W(\widetilde{C}_n)}(\rho w) = l_{W(\widetilde{C}_n)}(w) - 1$ and $l_{W(\widetilde{C}_n)}(\rho w') = l_{W(\widetilde{C}_n)}(w') - 1$, we get $l_{W(\widetilde{C}_n)}(w) + l_{W(\widetilde{C}_n)}(w^{-1} w') = 1 + l_{W(\widetilde{C}_n)}(\rho w) + l_{W(\widetilde{C}_n)}((\rho w^{-1})(\rho w')) = 1 + l_{W(\widetilde{C}_n)}(\rho w') = l_{W(\widetilde{C}_n)}(w')$ so that $w \preccurlyeq_{W(\widetilde{C}_n)} w'$. $\square$

COROLLARY 6.5.

(i) *The germ P^σ is in one-to-one correspondence with $\{w \in W(\widetilde{C}_n) \mid w \preccurlyeq_{W(\widetilde{C}_n)} c\}$ and this correspondence is compatible with divisibility.*

(ii) *The monoid $M(P^\sigma)$ has a presentation with set of generator $\underline{R}$ in one-to-one correspondence $\underline{r} \mapsto r$ with the set R of reflections $r \in W(\widetilde{C}_n)$ such that $r \preccurlyeq_{W(\widetilde{C}_n)} c$ and relations $\underline{r}_0 \underline{r}_1 \cdots \underline{r}_n = \underline{r}'_0 \underline{r}'_1 \cdots \underline{r}'_n$ for any two n-tuples of reflections such that $r_0 \cdots r_n = r'_0 \cdots r'_n = c$.*

Assertion (i) says that the germ P^σ is obtained from $W(\widetilde{C}_n)$ and the balanced element c by the method of the "generated group".

Proof. — Assertion (i) is precisely the content of Proposition 6.2.

Assertion (ii) is always true for a monoid obtained by the method of the generated group. We give a proof for the sake of completeness. If $w \in W(\widetilde{C}_n)$ is such that $w \preccurlyeq_{W(\widetilde{C}_n)} c$, we denote by $\underline{w}$ the corresponding element of the germ P^σ. For such a w there exist a sequence of reflections $r_0, \ldots, r_n$ and a non-negative integer $k \leqslant n$ such that $w = r_0 r_1 \cdots r_k$ and $c = r_0 \cdots r_n$, so that we have $\underline{w} = \underline{r}_0 \cdots \underline{r}_k$. A defining relation of $M(P^\sigma)$ such as $\underline{w}.\underline{w}' = \underline{ww}'$ can be written $\underline{r}_0 \cdots \underline{r}_n = \underline{c}$ if $w = r_0 \cdots r_k$, $w' = r_{k+1} \cdots r_l$ and $c = ww' r_{l+1} \cdots r_n$, using the cancellability of the germ. This means that we can reduce the set of generators to $\underline{R}$ and have the presentation given in (ii). □

DEFINITION 6.6. — *We call dual monoid of type $\widetilde{C}_n$ the monoid $M(P^\sigma)$.*

Remark 6.7. — Note that since all Coxeter elements are conjugate in $W(\widetilde{C}_n)$ changing the Coxeter element in the presentation of the dual monoid given in Corollary 6.5 (ii) leads to isomorphic monoids.

7. Hurwitz action; presentations

In this section we show that the Artin-Tits group of type $\widetilde{C}_n$ is a Garside group: more precisely it is the group of fractions $G(P^\sigma)$ of the Garside monoid $M(P^\sigma)$. We also get simpler presentations for the monoid and the group than in Corollary 6.5 (ii). For this we study the Hurwitz action on reduced decompositions of elements of P^σ. Recall if g is an element of some group G, the braid group with n strands acts on the set of decompositions of g into a product of n elements of G in the following way: if $(g_1, \ldots, g_n)$ is such that $g = g_1 g_2 \cdots g_n$, the action of the elementary braid $\mathbf{s}_i$ maps $(g_1, \ldots, g_n)$ onto $(g_1, \ldots, g_{i-1}, g_i g_{i+1} g_i^{-1}, g_i, g_{i+2}, \ldots, g_n)$. This action is called the Hurwitz action. Our first aim is to prove the following result where we call "reduced decomposition" of an element $w \in W(\widetilde{C}_n)$ a decomposition of w into a product of $l_{W(\widetilde{C}_n)}(w)$ reflections.

THEOREM 7.1. — *Let c be a Coxeter element of $W(\widetilde{C}_n)$ and w be a divisor of c; then the Hurwitz action is transitive on the set of reduced decompositions of w.*

Proof. — Note first that, since a conjugate of a reflection is a reflection, the Hurwitz action on reduced decompositions of an element w is well defined.

By induction on $l_{W(\widetilde{C}_n)}(w)$ it is sufficient to show that, starting with some fixed reduced decomposition of w, if ρ is a reflection which divides w we can get by the Hurwitz action a decomposition of w which begins with ρ.

Since by the Hurwitz action we can bring to the first place any reflection which appears in a reduced decomposition of w, it is enough to show that for any reflection ρ which divides w in $W(\widetilde{C}_n)$ we can get by the Hurwitz action a reduced decomposition of w involving ρ.

Since all Coxeter elements are conjugate in $W(\widetilde{C}_n)$ (see [6, §6, n°1, Lemme 1]) and since the Hurwitz action is compatible with conjugation, we may assume that $c = \sigma_0\sigma_2\sigma_4\cdots\sigma_1\sigma_3\sigma_5\cdots$ with the same notation as in Section 6. We view $W(\widetilde{C}_n)$ as $W(\widetilde{A}_{2n-1})^\sigma$ as before. We write $w = w_1\cdots w_r$ with w_i as in 6.3. We get a reduced decomposition of w by concatenation of reduced decompositions of the w_i. A reflection which divides w divides some w_i, so it is sufficient to show the result for $w = w_i$ which we now assume.

The element w is either a σ-stable finite cycle, or the product of a finite cycle and its image by σ or a σ-stable pseudo-cycle. In the first case w can be written $(a_1,\ldots,a_p,2kn+1-a_1,\ldots,2kn+1-a_p)$, with $k \in \mathbb{Z}$, all a_i odd and $a_1 < a_2 < \cdots < a_p < a_1 + 2n$, so is a Coxeter element of the Coxeter group of type C_p generated by $(a_i,a_{i+1})(2kn+1-a_i,2kn+1-a_{i+1})$ for $i = 1,\ldots p-1$ and $(a_1,2kn+1-a_1)$. In the second case w can be written $(a_1,\ldots,a_r,\ldots,a_p)(1-a_1,\ldots,1-a_r,\ldots,1-a_p)$ with a_i odd for $i \leqslant r$ and a_i even for $i > r$, and the inequalities $a_1 < a_2 < \cdots < a_r < a_1 + 2n$ and $a_{r+1} > \cdots > a_p > a_{r+1} - 2n$, so that w is a Coxeter element of the Coxeter group of type A_{p-1} generated by $(a_i,a_{i+1})(1-a_i,1-a_{i+1})$ for $i = 1,\ldots p-1$. In the third case w can be written $(a_1,a_3\ldots,a_{2p-1})_{[1]}(a_2,a_4,\ldots,a_{2p})_{[-1]}$ with the a_{2i-1} odd and $a_i + a_{2p+1-i} = 2n + 1$, so that w is a Coxeter element of the Coxeter group of type $\widetilde{C}_p$ generated by (a_1,a_{2p}), the products $(a_i,a_{i+1})(a_{2p-i},a_{2p+1-i})$ for $i = 1,\ldots,p-1$ and (a_p,a_{p+1}). Hence w is a Coxeter element of a Coxeter subgroup W of type B, C or $\widetilde{C}$; the reflections of W are reflections of $W(\widetilde{C}_n)$ and a reflection of $W(\widetilde{C}_n)$ divides w in $W(\widetilde{C}_n)$ if and only if it divides w in W. A reduced decomposition of w in $W(\widetilde{C}_n)$ is thus a reduced decomposition of w in that Coxeter subgroup.

Since we know that the Hurwitz action is transitive on the reduced decomposition of an element in groups of type A or C (see [2, Proposition 1.6.1]), the only case which remains is the case of a Coxeter element of a group of type $\widetilde{C}$: we have only to study the case where w is c itself. We start with the reduced decomposition of c given by $(\sigma_0,\sigma_2,\sigma_4,\ldots,\sigma_1,\sigma_3,\sigma_5,\ldots)$.

We remark that if we delete σ_0 (resp. σ_n) from this decomposition we get a reduced decomposition of a Coxeter element of the Coxeter group W' of type C_n generated by $\sigma_1, \sigma_2, \ldots, \sigma_n$ (resp. of the Coxeter group W'' of type C_n generated by $\sigma_0, \sigma_1, \ldots, \sigma_{n-1}$). Using the transitivity of the Hurwitz action for Coxeter groups of type C_n, we know that by the Hurwitz action we can make appear any reflection of W' or W'' in some reduced decomposition of c.

We remark that if $c = \rho_1 \cdots \rho_{n+1}$ is a reduced decomposition, the Hurwitz orbit of $(\rho_1, \ldots, \rho_{n+1})$ contains $(\rho_2, \ldots, \rho_{n+1}, c^{-1}\rho_1 c)$ so that by the Hurwitz action for any $n \in \mathbb{Z}$ and any $i \in \{1, \ldots, n+1\}$ we can get a reduced decomposition of c involving $c^n \rho_i c^{-n}$. So we will get our result if we prove that any reflection appearing in a reduced decomposition of c is conjugate by some power of c to a reflection of W' or of W''.

The action of c by conjugation on permutations of $\mathbb{Z}$ is induced by the translations $2i \mapsto 2i - 2$ and $2i + 1 \mapsto 2i + 3$ for all i. The reflections of W' are

$$\begin{cases} (a,b)(1-a,1-b) & \text{with } 1 \leqslant a < b \leqslant 2n \text{ and } a + b \neq 2n + 1, \\ (a, 2n+1-a) & \text{with } 1 \leqslant a \leqslant 2n. \end{cases}$$

The reflections of W'' are

$$\begin{cases} (a,b)(1-a,1-b) & \text{with } 1 - n \leqslant a < b \leqslant n \text{ and } a + b \neq 1, \\ (a, 1-a) & \text{with } 1 - n \leqslant a \leqslant n. \end{cases}$$

If a and b have same parity with $a < b \leqslant a + 2n$, we can conjugate $(a,b)(1-a)(1-b) \in P^\sigma$ by a power of c to $(1, b+1-a)(0, a-b)$ which is a reflection of W'. If a and b have different parities and $a+b \neq 1 \bmod 2n$ we can conjugate $(a,b)(1-a,1-b)$ by a power of c to $(\frac{a+b-1}{2}, \frac{a+b+1}{2})(\frac{1-a-b}{2}, \frac{3-a-b}{2})$ which is a reflection of W'. Last of all we can conjugate $(a, 2kn + 1 - a)$ where k is arbitrary to $(kn, kn+1)$ which is equal either to $(n, n+1)$ which is in W' or to $(0, 1)$ which is in W''. $\qquad\square$

Remark 7.2.

(i) One can conjecture that the Hurwitz action is transitive on the reduced decompositions of a Coxeter element in any Coxeter group.

(ii) We have seen in the proof of Theorem 7.1 that a divisor of a Coxeter element of $W(\widetilde{C}_n)$ is a Coxeter element of a Coxeter subgroup which is a direct product of groups of type A, C or $\widetilde{C}$ (there can be at most one component of type $\widetilde{C}$ and there cannot be at the same time a component of type C and a component of type $\widetilde{C}$).

Using Theorem 7.1 we can simplify the relations in the presentations of $M(P^\sigma)$ and $G(P^\sigma)$ given in Corollary 6.5 (ii). As in Corollary 6.5 (ii) and in its proof we denote by $\underline{w}$ the element of P^σ corresponding to an element $w \in W(\widetilde{C}_n)$ dividing c. From this presentation we deduce one of our main result, which is that the Artin-Tits group $G(\widetilde{C}_n)$ is isomorphic to $G(P^\sigma)$. We use the notations R and $\underline{R}$ as defined in Corollary 6.5.

THEOREM 7.3.

(i) *The monoid $M(P^\sigma)$ has the following monoid presentation by generators and relations:*

$$M(P^\sigma) = \langle \underline{R} \mid \underline{r}.\underline{t} = \underline{rtr}.\underline{r} \text{ if } r, t \in R \text{ and } rt \preccurlyeq_{W(\widetilde{C}_n)} c \rangle^+.$$

(ii) *The Artin-Tits group of type $\widetilde{C}_n$ is isomorphic to $G(P^\sigma)$; in particular it is the Garside group of the Garside monoid $M(P^\sigma)$ and has the group presentation $G(\widetilde{C}_n) = \langle \underline{R} \mid \underline{r}.\underline{t} = \underline{rtr}.\underline{r} \text{ if } r, t \in R \text{ and } rt \preccurlyeq_{W(\widetilde{C}_n)} c \rangle.*$

Proof. — By Corollary 6.5 (ii) the monoid $M(P^\sigma)$ has a presentation with set of generators $\underline{R}$ and relations given by the equalities between the lifts of any two reduced decompositions of c. We have to prove that one can pass from a reduced decomposition of c to another one by applying only relations of the form $r.t = (rtr).r$. But this is precisely the transitivity of the Hurwitz action, whence (i).

Since all the simple reflections of $W(\widetilde{C}_n)$ are in R, we have elements $\underline{\sigma}_i \in \underline{R}$ for $i = 0, \ldots, n$, where as in Section 6 the simple reflections of $W(\widetilde{C}_n) = W(\widetilde{A}_{2n-1})^\sigma$ are $\sigma_i = s_i s_{2n-i}$ for $i = 2, \ldots, n-1$, $\sigma_n = s_n$ and $\sigma_0 = s_{2n}$. The natural morphism $M(P^\sigma) \to M(P)^\sigma$ extends to a group morphism $f \colon G(P^\sigma) \to G(P)^\sigma$. We know that $G(P)$ is the group $G(\widetilde{A}_{2n-1})$ so that $G(P)^\sigma$ is the group $G(\widetilde{C}_n)$ by Theorem 5.1. The morphism f maps $\underline{\sigma}_i$ to $\boldsymbol{\sigma}_i$ for $i = 2, \ldots, n-1$ and maps $\underline{\sigma}_n$ to $\boldsymbol{\sigma}_n$ and $\underline{\sigma}_0$ to $\boldsymbol{\sigma}_0$.

We claim that $\underline{\sigma}_0, \ldots, \underline{\sigma}_n$ satisfy the braid relations of type $\widetilde{C}_n$. Indeed, for any pair of distinct simple reflections σ_i and σ_j, we have $\sigma_i \sigma_j \preccurlyeq_{W(\widetilde{C}_n)} c$, up to swapping i and j; if σ_i and σ_j, commute, then $\sigma_i \sigma_j \preccurlyeq_{W(\widetilde{C}_n)} c$ so that $\underline{\sigma}_i \underline{\sigma}_j = \underline{\sigma}_j \underline{\sigma}_i$ in P^σ; if σ_i and σ_j satisfy a braid relation of length 3 we have $\underline{\sigma}_j.\underline{\sigma}_i.\underline{\sigma}_j = \underline{\sigma}_j.\underline{\sigma_i \sigma_j \sigma_i}.\underline{\sigma}_i = \underline{\sigma_j \sigma_i \sigma_j \sigma_i \sigma_j}.\underline{\sigma}_j.\underline{\sigma}_i = \underline{\sigma}_i.\underline{\sigma}_j.\underline{\sigma}_i$; if σ_i and σ_j satisfy a braid relation of length 4 we have $\underline{\sigma}_i.\underline{\sigma}_j.\underline{\sigma}_i.\underline{\sigma}_j = \underline{\sigma}_i.\underline{\sigma}_j.\underline{\sigma_i \sigma_j \sigma_i}.\underline{\sigma}_i = \underline{\sigma}_i.\underline{\sigma_j \sigma_i \sigma_j \sigma_i \sigma_j}.\underline{\sigma}_j.\underline{\sigma}_i = \underline{\sigma_i \sigma_j \sigma_i \sigma_j \sigma_i \sigma_j \sigma_i}.\underline{\sigma}_i.\underline{\sigma}_j.\underline{\sigma}_i = \underline{\sigma}_j.\underline{\sigma}_i.\underline{\sigma}_j.\underline{\sigma}_i$. Hence there exists a morphism $g \colon G(\widetilde{C}_n) \to G(P^\sigma)$ which maps $\boldsymbol{\sigma}_i$ to $\underline{\sigma}_i$ for $i = 0, \ldots, n$. We have then $f \circ g = \text{Id}$ since $G(\widetilde{C}_n)$ is generated by $\{\boldsymbol{\sigma}_i, i = 0 \ldots, n\}$.

On the other hand the transitivity of the Hurwitz action and the presentation of $G(P^\sigma)$ imply that one can express any $\underline{r}$ where $r \preccurlyeq_{W(\widetilde{C}_n)} c$ as a conjugate of some $\underline{\sigma}_i$ by a product of elements of the form $\underline{\sigma}_j^{\pm 1}$, so that $G(P^\sigma)$ is generated by $\{\underline{\sigma}_i, i = 0, \dots, n\}$. This implies that g is surjective, so that f and g are isomorphisms. $\qquad\square$

8. Some consequences

Among consequences of our results we can recover the known fact that the center of $B(\widetilde{C}_n)$ is trivial. The Garside structure gives also a solution to the word problem in this group. As an illustration of the use of a Garside structure we compute the centralizer of powers of a Coxeter element in an Artin-Tits group of type $\widetilde{C}_n$.

PROPOSITION 8.1. — *With the notation of Section 5, let* $\mathbf{c} = \boldsymbol{\sigma}_0 \boldsymbol{\sigma}_1 \cdots \boldsymbol{\sigma}_n$; *then for any* $h \in \mathbb{Z} - \{0\}$, *the centralizer of* $\mathbf{c}^h$ *in the Artin-Tits group* $G(\widetilde{C}_n)$ *is isomorphic to the Artin-Tits group of type* $C_{\gcd(h,n)}$.

Proof. — It is a general result (see *e.g.*, [4, Proposition 2.26]) that the monoid of fixed points M^φ under an automorphism φ of a Garside monoid M fixing the Garside element Δ is a Garside monoid with same Garside element (this can also be seen as a particular case of Corollary 2.7): if P is the set of divisors of Δ in M then the set of divisors of Δ in M^φ is P^φ. In this situation the group of fixed points of φ in the group G of fractions of M is the group of fractions of M^φ. We apply this to $M = M(P^\sigma)$ and $G = G(P^\sigma)$, taking for φ the conjugation by $\mathbf{c}^h$. As in Section 4 we identify the elements of P^σ with the elements of $W(\widetilde{C}_n)$ corresponding to the $2n$-periodic, non-crossing σ-stable partitions of $X \cup \Xi$, where $X = 1 + 2\mathbb{Z}$ and $\Xi = 2\mathbb{Z}$. Recall that $\mathbf{c}$ is the permutation given by $x \mapsto x + 2$ for $x \in X$ and $\xi \mapsto \xi - 2$ for $\xi \in \Xi$.

Let p be a σ-stable non-crossing partition stable by the action of $\mathbf{c}^h$. If A is a part of p contained in X or in Ξ, then $A + 2kh$ is also a part of p for all $k \in \mathbb{Z}$. If A is a part of p which intersects both X and Ξ and is not $\mathbf{c}^h$-stable, then A crosses its image under the action of $\mathbf{c}^h$, which is impossible since two distinct parts of p are non-crossing. Hence any part A of p has to be invariant by the action of $\mathbf{c}^h$ which is equivalent to $A = A + 2kh$ for all $k \in \mathbb{Z}$. Hence a part of a partition p_w associated to an element $w \in (P^\sigma)^{\mathbf{c}^h}$ (see Proposition 3.9) is either an infinite part both $2n$ and $2h$ periodic (and there is at most one such part in p_w) or a finite part A contained in X or in Ξ and such that $A + 2kh$ and $A + 2kn$ are parts of p_w for any k. Conversely

if all the parts of a σ-stable non-crossing partition p_w are of one of these form then w is centralized by $\mathbf{c}^h$.

From the above study we see that the centralizer of $\mathbf{c}^h$ is equal to the centralizer of $\mathbf{c}^{\gcd(n,h)}$. Hence to prove the proposition we are reduced to the case where h divides n.

We first study the case $h = n$.

LEMMA 8.2. — *The centralizer of $\mathbf{c}^n$ is an Artin-Tits group of type C_n.*

Proof. — From the above discussion we see that an element $w \in (P^\sigma)^{\mathbf{c}^n}$ is a product of pairwise commuting and non-crossing elements of the form either $(2a_1 + 1, 2a_2 + 1, \ldots, 2a_r + 1)(-2a_1, -2a_2, \ldots, -2a_s, \ldots, -2a_r)$ with $a_1 < a_2 < \cdots < a_r < a_1 + n$, or $(2a_1 + 1, 2a_2 + 1, \ldots, 2a_r + 1)_{[1]}(-2a_1, -2a_2, \ldots, -2a_r)_{[-1]}$ with $0 < a_1 < a_2 < \cdots < a_r \leqslant n$. Hence all elements of $(P^\sigma)^{\mathbf{c}^n}$ can be written $x.^\sigma x$ where x is a permutation of the set X of odd integers and $^\sigma x$ is a permutation of the set Ξ of even integers. Since a permutation of X and a permutation of Ξ are always non-crossing, two such elements $x.^\sigma x$ and $y.^\sigma y$ in $(P^\sigma)^{\mathbf{c}^n}$ are non-crossing if and only if x and y are non-crossing. In [10, Proof of 5.10] we have described the germ for the dual presentation of an Artin-Tits group of type C_n as the set of n-periodic, non-crossing permutations of $\mathbb{Z}$. From this description we see that under the bijection $2k+1 \mapsto k$ from X to $\mathbb{Z}$, the map $x.^\sigma x \mapsto x$ defines an isomorphism of germs from $(P^\sigma)^{\mathbf{c}^n}$ to the dual germ for the Artin-Tits group of type C_n. $\qquad\square$

We deduce the proposition using the fact that the centralizer of $\mathbf{c}^h$ for a divisor h of n is the centralizer of $\mathbf{c}^h$ in the centralizer of $\mathbf{c}^n$ so that it is the centralizer of $\mathbf{c}^h$ in an Artin group of type C_n, which is known to be an Artin-Tits group of type C_h (see [4]). $\qquad\square$

BIBLIOGRAPHY

[1] D. BESSIS, "Complex reflection arrangements are $K(\pi, 1)$", math.GR/0610777.

[2] ———, "The dual braid monoid", *Ann. Sci. École Norm. Sup.* **36** (2003), p. 647-683.

[3] ———, "A dual braid monoid for the free group", *J. Algebra* **302** (2006), p. 55-69.

[4] D. BESSIS, F. DIGNE & J. MICHEL, "Springer theory in braid groups and the Birman-Ko-Lee monoid", *Pacific J. Math.* **205** (2003), no. 2, p. 287-309.

[5] J. BIRMAN & H. HILDEN, "On isotopies of homeomorphisms of Riemann surfaces", *Ann. of Math.* **97** (1973), p. 424-439.

[6] N. BOURBAKI, *Groupes et Algèbres de Lie, chapitres IV, V, VI*, Masson, Paris, 1981.

[7] R. CHARNEY & J. CRISP, "Automorphism groups of some affine and finite type Artin groups", *Math. Res. Lett.* **12** (2005), no. 2-3, p. 321-333.

[8] P. Dehornoy, "Groupes de Garside", *Ann. scient. Éc. Norm. Sup.* 4ᵉ *série* **35** (2002), p. 267-306.

[9] P. Dehornoy & L. Paris, "Gaussian groups and Garside groups, two generalisations of Artin groups", *Proc. London Math. Soc.* **79** (1999), no. 3, p. 569-604.

[10] F. Digne, "Présentations duales pour les groupes de tresses de type affine $\widetilde{A}$", *Comm. Math. Helvetici* **8** (2008), p. 23-47.

[11] F. Digne & J. Michel, "Garside and locally Garside categories", arXiv: math.GR/0612652.

[12] M. J. Dyer, "On minimal lengths of expressions of Coxeter group elements as product of reflections", *Proc. Amer. Math. Soc.* **129** (2001), no. 9, p. 2591-2595.

[13] J.-Y. Hée, "Système de racines sur un anneau commutatif totalement ordonné", *Geom. Dedicata* **37** (1991), p. 65-102.

[14] C. Maclachlan & W. Harvey, "On mapping class groups and Teichmüller spaces", *Proc. London Math. Soc.* **30** (1975), p. 496-512.

[15] J. Michel, "A note on words in braid monoids", *J. Algebra* **215** (1999), p. 366-377.

[16] L. Paris, "Artin monoids embed in their groups", *Comment. Math. Helv.* **77** (2002), no. 3, p. 609-637.

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