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ALGEBRAS OF DIFFERENTIABLE FUNCTIONS

by E. T. Y. LEE

1. Introduction.

Let $C_0 = C_0(\mathbf{R}^n)$ be the Banach space of all complex-valued continuous functions on $\mathbf{R}^n$ vanishing at infinity, supplied with the sup-norm $\|.\|$; and $\mathfrak{D}$ the dense subspace consisting of all infinitely differentiable functions with compact support. Denote by $\mathfrak{Q} = \mathfrak{Q}(\mathbf{R}^n)$ the linear space of all differential operators

$$A = \sum_{|\alpha| \leqslant k} a^\alpha D_\alpha$$

of constant (complex) coefficients. Here as usual

$$\alpha = (\alpha_1, \ldots, \alpha_n)$$

is a multi-index, with α_i nonnegative intergers, and

$$|\alpha| = \Sigma\alpha_i \ ; \quad D^\alpha = D_1^{\alpha_1} D_2^{\alpha_2} \ \ldots \ D_n^{\alpha_n}, \quad \text{where} \quad D_i^{\alpha_i} = \left(\frac{\partial}{\partial x_i}\right)^{\alpha_i}.$$

For an operator $A = \Sigma\alpha_\alpha D^\alpha$, the formal adjoint operator $\tilde{A}$ of A is

$$\tilde{A} = \Sigma\,(-1)^{|\alpha|} a_\alpha D^\alpha.$$

If $f \in C_0$, $A \in \mathfrak{Q}$, Af is defined in the distribution sense; thus we say Af is equal to a function $h \in C_0$ if and only if for every $\varphi \in \mathfrak{D}$,

$$\int (\tilde{A}\varphi) f = \int h\varphi.$$

Following de Leeuw and Mirkil ([1]), a subspace B of C_0 is said to be a *space of differentiable functions* if

$$B = C_0(\mathcal{A}) = \{f \in C_0 : Af \in C_0, \ \forall A \in \mathcal{A}\}$$

for some subset $\mathcal{A}$ of $\mathcal{Q}$. A space of differentiable functions which is invariant under all rotations in $\mathbf{R}^n$ (the precise sense of which will be defined in the next section) will be called a *rotating* space of differentiable functions. Examples of such rotating spaces are $C_0^N = C_0(\mathcal{Q}_N)$, where $\mathcal{Q}_N$ stands for all operators in $\mathcal{Q}$ of order not exceeding N; and also the space $C_0^\infty = C_0(\mathcal{Q}) = \cap C_0^N$. A space of differentiable functions which is not C_0^∞ and not any C_0^N will be called a *proper* space of differentiable functions. In [1], de Leeuw and Mirkil studied spaces of differentiable functions on $\mathbf{R}^2$, giving a complete classification of all the rotating ones. Other results are that each rotating space of differentiable functions is an algebra, under pointwise multiplication, and that except for C_0^∞, each is a Banach algebra, under a natural norm. In obtaining the classification theorem, an essential role is played by the fact that rotations in $\mathbf{R}^2$ form a commutative group. Even though this is no longer true for $\mathbf{R}^n$, $n > 2$, it is found that their results can also be extended very naturally to higher dimensions. The purpose of this work is to present these extensions.

Of necessity, then, we must repeat most of de Leeuw and Mirkil. In particular, we list here all the preliminary propositions that are relevant to us. For proofs and other details the reader is urged to refer to [1]. For a space of differentiable functions $B = C_0(\mathcal{A})$, we define

$$\mathcal{A}_B = \{A \in \mathcal{Q} : Af \in C_0, \ \forall f \in B\};$$

thus $\mathcal{A}_B$ is a subspace of $\mathcal{Q}$, $B = C_0(\mathcal{A}_B)$ and $\mathcal{A}_B \supset \mathcal{A}$. We endow the space $C_0(\mathcal{A})$ with the locally convex topology given by the semi-norms $\|f\|$, $\|Af\|$, $A \in \mathcal{A}$. Each $C_0(\mathcal{A})$ is a Frechet space, in which $\mathcal{D}$ is dense. If $C_0(\mathcal{A}_1) = C_0(\mathcal{A}_2)$, the topologies defined by $\mathcal{A}_1$ and $\mathcal{A}_2$ coincide. Thus, for any space of differentiable functions B, one may speak of the topology $\tau(B)$ of B, without reference to any $\mathcal{A}$ for which $B = C_0(\mathcal{A})$. The following simple proposition plays an important role in the classification of spaces of differentiable functions :

PROPOSITION 1.1. — *Let* $B_i = C_0(\mathcal{A}_i)$, $i = 1, 2$. *Then the following are equivalent:*

1° $B_1 \subset B_2$.

2° $\tau(B_1) \supset \tau(B_2)$, *when restricted to* $\mathcal{D}$.

3° *For each* $A \in \mathcal{A}_2$, *there exist* $A_1, \ldots, A_m \in \mathcal{A}_1$ *and* $K > 0$, *such that for all* $\varphi \in \mathcal{D}$,

$$\|A\varphi\| \leqslant K\left(\|\varphi\| + \sum_1^m \|A_i\varphi\|\right),$$

Sup-norm estimates of this kind contain useful information about the operators:

THEOREM 1.2. — *Let* $A_1, \ldots, A_m \in \mathcal{Q}$, *with orders* $\leqslant N$. *Suppose* $A \in \mathcal{Q}$ *and that there exists* $K > 0$ *such that*

$$|A\varphi(0)| \leqslant K\left(\|\varphi\| + \sum_1^m \|A_i\varphi\|\right), \qquad \varphi \in \mathcal{D},$$

then A *has order* $\leqslant N$; *and the homogeneous part of* A *of order* N *is a linear combination of the homogeneous parts of* $A_1, \ldots, A_m$ *of order* N.

As a direct consequence of this, we have, if A has order N, then there exists $f \in C_0^{N-1}$ such that $Af \notin C_0$. This also implies that $\mathcal{A}_{C_0^N} = \mathcal{Q}_N$.

Spaces of differentiable functions which lie between C_0^N and C_0^{N-1}, for some N, will be said to be *squeezed.* If $B = C_0(\mathcal{A})$ and $C_0^N \subset B \subset C_0^{N-1}$, then the linear space $[\mathcal{A}, \mathcal{Q}_{N-1}]$ spanned by $\mathcal{A}$ and $\mathcal{Q}_{N-1}$ is clearly contained in $\mathcal{A}_B$. On the other hand, each $A \in \mathcal{A}_B$ maps B continuously into C_0, so that A can be estimated by a finite number of operators $A_1, \ldots, A_m \in \mathcal{A} \subset \mathcal{Q}_{N-1}$. By Theorem 1.2, $A \in [\mathcal{A}, \mathcal{Q}_{N-1}]$. This gives Corollary 1.3 below, from which Corollary 1.4 can established as a consequence.

COROLLARY 1.3. — *If* $B = C_0(\mathcal{A})$ *and* $C_0^N \subset B \subset C_0^{N-1}$, *then*

$$\mathcal{A}_B = [\mathcal{A}, \mathcal{Q}_{N-1}].$$

COROLLARY 1.4. — *The map* $\mathcal{A} \to C_0(\mathcal{A})$ *is a one-one correspondence between subspaces* $\mathcal{A}$ *of* $\mathcal{Q}$ *satisfying*

$\mathcal{Q}_{N-1} \subset \mathcal{A} \subset \mathcal{Q}_N$ *and spaces of differentiable functions* B *with* $C_0^N \subset B \subset C_0^{N-1}$. *The inverse of this map is the map* $B \to \mathcal{A}_B$.

Let B be a space of differentiable functions, $C_0^N \subset B \subset C_0^{N-1}$, and let $A \in \mathcal{A}_B$. Leibnitz's rule shows there are A_i', $A_i'' \in \mathcal{Q}$, with orders less than that of A, such that

$$(1.1) \qquad A\varphi\psi = \varphi A\psi + \psi A\varphi + \Sigma A_i'\varphi A_i''\psi, \qquad \varphi, \psi \in \mathcal{D}.$$

Since $\mathcal{Q}_{N-1} \subset \mathcal{A}_B$, A_i', $A_i'' \in \mathcal{A}_B$ also. Since $\mathcal{Q}$ is dense in B, equation (1.1) extends to $\varphi, \psi \in B$. This implies that B is closed under pointwise multiplication. Since $\mathcal{A}_B \in \mathcal{Q}_N$, $\mathcal{A}_B$ is finite-dimensional, so that the sum of the semi-norms given by a basis for $\mathcal{A}_B$ is indeed a norm. One can also check that the multiplication is continuous on $B \times B$. Thus:

THEOREM 1.5. — *Every squeezed space of differentiable functions is a Banach algebra.*

In the next section we will show that every rotating space of differentiable functions is squeezed. Section 3 contains the main result, namely the classification of these rotating spaces on $\mathbf{R}^n$, $n > 2$. Section 4 contains extensions to spaces of differentiable functions on $\mathbf{C}^n$ invariant under the group $U(n)$ of unitary transformations in $\mathbf{C}^n$.

2. Squeezing of Rotating Spaces.

Let G be a group of invertible linear transformations in $\mathbf{R}^n$. For each $\sigma \in G$ and $f: \mathbf{R}^n \to \mathbf{C}$, we define $R_\sigma f$ by

$$(R_\sigma f)(x) = f(\sigma^{-1}x), \qquad x \in \mathbf{R}^n.$$

Each R_σ is an algebra isomorphism of, say, C_0; and the map $\sigma \to R_\sigma$ is a homomorphism of G. We also define the action of G on $\mathcal{Q}$: for $A \in \mathcal{Q}$, $\mathcal{R}_\sigma A$ is given by

$$(\mathcal{R}_\sigma A)\varphi = (R_\sigma \circ A \circ R_\sigma^{-1})\varphi, \qquad \varphi \in \mathcal{D}.$$

It can easily be checked that if $\sigma e_i = \Sigma c_{ji} e_j$, where $\{e_i\}$ is the standard basis of $\mathbf{R}^n$, then $\mathcal{R}_\sigma\left(\frac{\partial}{\partial x_i}\right) = \Sigma c_{ji} \frac{\partial}{\partial x_j}\cdot$ Since $\mathcal{R}_\sigma$

is linear and multiplicative, this shows that $\mathcal{R}_\sigma$ maps $\mathfrak{Q}$ into $\mathfrak{Q}$, and indeed is an algebra isomorphism of $\mathfrak{Q}$. Again $\sigma \to \mathcal{R}_\sigma$ is a homomorphism of G.

Let $\mathfrak{T}$ be the space of all polynomials in $x_1, \ldots, x_n$, with complex coefficients. If $A \in \mathfrak{Q}$, with $A = P(D)$, $P \in \mathfrak{T}$, the *characteristic polynomial* of A is defined to be the polynomial $\hat{A}$, with $\hat{A}(x) = P(ix)$, $x \in \mathbf{R}^n$. For each $\sigma \in O(n)$, the group of orthogonal transformations in $\mathbf{R}^n$, we have

$$\widehat{\mathcal{R}_\sigma A} = R_\sigma \hat{A}, \tag{2.1}$$

that is, the map $A \to \hat{A}$ commutes with $O(n)$. This is easy to check for the operators $\frac{\partial}{\partial x_i}$, $i = 1, \ldots, n$; and this is all we need to check, since $A \to \hat{A}$ is an algebra isomorphism of $\mathfrak{Q}$ and $\mathfrak{T}$.

A subspace of C_0 (or of $\mathfrak{Q}$) is said to be *invariant under* G if it is invariant under all R_σ, (or, respectively, $\mathcal{R}_\sigma$), $\sigma \in G$. We will restrict ourselves here to the group $G = SO(n)$ of all rotations in $\mathbf{R}^n$, although what follows in this section also holds for the group $O(n)$. We retain the terminology of de Leeuw and Mirkil: the word *rotating* is used to mean invariant under $SO(n)$. It is clear that if B is a space of differentiable functions, B is rotating if and only if $\mathcal{A}_B$ is rotating; and by (2.1), this occurs if and only if $\hat{\mathcal{A}}_B = \{\hat{A} : A \in \mathcal{A}_B\}$ is a rotating subspace of $\mathfrak{T}$. For any $P \in \mathfrak{T}$, let P^l denote the homogeneous part of P of degree l. It is simple to check that $(R_\sigma P)^l = R_\sigma P^l$; consequently if X is a rotating subsspace of $\mathfrak{T}$, then so is $X^l = \{P^l : P \in X\}$.

We call a set of differentiable operators $\{A_1, \ldots, A_m\}$ an *elliptic system* of operators of order N if each A_i has order not exceeding N and if the polynomials $\hat{A}_1^N, \ldots, \hat{A}_m^N$ possess no common zero in $\mathbf{R}^n$ except at the origin $x = 0$.

Lemma 2.1. — *Let* B *be a rotating space of differentiable functions. If* $\mathcal{A}_B$ *contains an operator of order* N, *then any basis of the space* $\mathcal{A}_B \cap \mathfrak{Q}_N$ *is an elliptic system.*

Proof. — If $\{A_1, \ldots, A_m\}$ is a basis of $\mathcal{A}_B \cap \mathfrak{Q}_N$, then $P_1 = \hat{A}_1, \ldots, P_m = \hat{A}_m$ form a basis of $X = \widehat{\mathcal{A}_B \cap \mathfrak{Q}_N}$, and $P_1^N, \ldots, P_m^N$ span X^N. If $x_0 \neq 0$ is a common zero of $\{P_i^N\}$, each $P \in X^N$ must vanish at x_0, and by homogeneity also at cx_0, $c > 0$. For each $x \in \mathbf{R}^n$, there exists $\sigma \in SO(n)$, $\sigma x = \frac{|x| x_0}{|x_0|}$, so that $P(x) = (R_\sigma P)\left(\frac{|x| x_0}{|x_0|}\right) = 0$, since X^N is rotating. Thus $X^N = \{0\}$, contrary to the assumption that $\mathcal{A}_B$ has an operator of order N.

Our purpose in this section is to show that any rotating space of differentiable functions $B \neq C_0^\infty$ is squeezed, and therefore, in view of Theorem 1.5, is a Banach algebra. This follows easily from the following.

THEOREM 2.2. — *Let $\{A_1, \ldots, A_m\}$ be an elliptic system of operators of order N. For each $A \in \mathcal{Q}$ with order $o(A) < N$, there exists $K > 0$, such that*

$$\|A\varphi\| \leqslant K\left(\|\varphi\| + \sum_1^m \|A_i\varphi\|\right) \qquad \varphi \in \mathcal{D}.$$

Let us first derive from it the squeezing theorem. If B is a space of differentiable functions such that $\mathcal{A}_B$ contains an elliptic system $\{A_1, \ldots, A_m\}$ of order N, then

$$B = C_0(\mathcal{A}_B) \subset C_0\{A_1, \ldots, A_m\} \subset C_0^{N-1},$$

by Theorem 2.2 and Proposition 1.1; thus $\mathcal{A}_B \supset \mathcal{Q}_{N-1}$. Now if $\mathcal{A}_B$ is rotating and infinite-dimensional, Proposition 2.1 shows $\mathcal{A}_B$ contains elliptic systems of arbitrarily high order, so that $\mathcal{A}_B = \mathcal{Q}$ and $B = C_0^\infty$. If $\mathcal{A}_B$ is rotating and $B \neq C_0^\infty$, let N be the smallest positive integer such that $\mathcal{A}_B \subset \mathcal{Q}_N$. The same argument then shows that $\mathcal{A}_B \supset \mathcal{Q}_{N-1}$. Thus:

THEOREM 2.3. — *Let B be a rotating space of differentiable functions. If $\mathcal{A}_B$ is infinite-dimensional, $B = C_0^\infty$. Otherwise there is a positive integer N such that $C_0^N \subset B \subset C_0^{N-1}$.*

Theorem 2.2 is a slight extension of a corresponding theorem of de Leeuw and Mirkil ([2]) concerning domination by a single elliptic operator. Our proof here will be also a mere modification of theirs. The proof relies on the following two theorems, both of which were proved in [2].

THEOREM 2.4. — *Let* $A, A_1, \ldots, A_m \in \mathfrak{Q}$ *have characteristic polynomials* $P, P_1, \ldots, P_m$. *Then there exists* $K > 0$ *such that*

$$\|A\varphi\| \leqslant K \sum_1^m \|A_i\varphi\|,$$

for all $\varphi \in \mathfrak{D}$, *if and only if*

$$P = \sum_1^m M_i P_i$$

for some Fourier-Stieltjes transforms $M_1, \ldots, M_m$.

THEOREM 2.5. — *If* f *is a homogeneous function on* $\mathbf{R}^n - \{0\}$ *of degree* $-k$, k *a positive integer, and is infinitely differentiable, then* f *is a Fourier transform near infinity. That is, there is a function in* $L^1(\mathbf{R}^n)$ *whose Fourier transform agrees with* f *outside some compact set.*

We now prove Theorem 2.2. We will show that if $P_1, \ldots, P_m \in \mathfrak{L}$ have maximum degree N and if $P_1^N, \ldots, P_m^N$ have no common non-zero zero in $\mathbf{R}^n$, then for any $P \in \mathfrak{L}$ with degree $\leqslant N - 1$, one can find Fourier-Stieltjes transforms M, $M_1, \ldots, M_m$ such that

$$P = M + \sum_1^m M_i P_i. \tag{2.2}$$

It suffices to prove (2.2) when P is homogeneous. First consider the case when all the $P_i's$ are homogeneous of degree N. Then $\Sigma|P_i|^2$ has no non-zero real zero, and we define

$$N_i(x) = \frac{P(x)\overline{P_i(x)}}{\Sigma|P_j(x)|^2}, \qquad x \neq 0. \tag{2.3}$$

Each N_i is a Fourier transform near infinity, by Theorem 2.5. Let ψ be a C^∞-function, zero near origin and 1 outside some appropriate compact set. Define $M_i = \psi N_i$; then $M_i \in C^\infty$ and agress with N_i near infinity. Each M_i is a Fourier transform. (For if $M_i = \hat{h}_i$ outside a compact set K, let $\varphi \in \mathfrak{D}$, $\varphi = 1$ on K. Then $\varphi(M_i - \hat{h}_i) = M_i - \hat{h}_i$. But φ and φM_i are Fourier transforms, being functions in $\mathfrak{D}$.)

From (2.3), $\Sigma M_i P_i = P$ near infinity, so that to obtain (2.3), we need only define $M = P - \Sigma M_i P_i$.

For the general case where the P_i's are not homogeneous, we have

$$\Sigma|P_i|^2 = \Sigma|P_i^N|^2\left(1 + \frac{Q}{\Sigma|P_i^N|^2}\right),$$

where Q is a polynomial of degree not exceeding $2N - 1$. $\dfrac{Q}{\Sigma|P_i^N|^2}$ is a Fourier transform near infinity, by Theorem 2.5 applied to each homogeneous part of Q. Let K be a compact neighborhood of the origin outside of which $\left|\dfrac{Q}{\Sigma|P_i^N|^2}\right| < 1$; choose a C^∞-function ψ, $0 \leqslant \psi \leqslant 1$, which is zero in a neighborhood of K and 1 near infinity. Define

$$M_i = \frac{\psi P\overline{P}_i}{\Sigma|P_j^N|^2}\,\frac{1}{1 + \psi(Q/\Sigma|P_j^N|^2)}; \tag{2.4}$$

then $M_i \in C^\infty$ and $\Sigma M_i P_i = P$ near infinity. Set

$$M = P - \Sigma M_i P_i,$$

then $M \in \mathfrak{D}$ and so is a Fourier transform.

It remains to show that the M_i's are Fourier-Stieltjes transforms. Now $\dfrac{\psi Q}{\Sigma|P_j^N|^2}$ and each $\dfrac{\psi P\overline{P}_i}{\Sigma|P_j^N|^2}$ are Fourier transforms, by our previous arguments; therefore it suffices to show that

$$\frac{1}{1 + \psi Q/\Sigma|P_j^N|^2}$$

is a Fourier-Stieltjes transform. This follows from a well known theorem in Banach algebras: If B is a commutative Banach algebra, $x \in B$, and if F is an analytic function defined on some open set in $\mathbf{C}$ including the spectrum of x, and $F(0) = 0$ in case B has no unit, then $F \circ \hat{x} = \hat{y}$ for some $y \in B$, where $\hat{x}$ denotes the Gelfand transform of x. In the present case $B = L^1(\mathbf{R}^n)$ and the Gelfand transform is simply the Fourier transform. Let $\frac{\psi Q}{\Sigma |P_j^N|^2} = \hat{f}$, $f \in L^1$, then the spectrum of f is contained in the open unit disk, since $|\hat{f}| < 1$. Taking $F(z) = \frac{z}{1+z}$, we see that $\frac{\hat{f}}{1+\hat{f}}$ is a Fourier transform, and therefore

$$\frac{1}{1+\hat{f}} = 1 - \frac{\hat{f}}{1+\hat{f}}$$

is a Fourier-Stieltjes transform, 1 being the transform of the unit mass at origin. This completes the proof of Theorem 2.2.

3. Classification of Rotating Spaces.

Let $\mathcal{O}_N$ be the space of homogeneous differential operators of order N; $\mathcal{P}_N$ the space of homogeneous polynomials in $\mathbf{R}^n$ of degree N. The mapping $\sigma \to \mathcal{R}_\sigma$ is a representation of the group $SO(n)$ in $\mathcal{O}_N$. Since a representation of a compact group is completely reducible, the space $\mathcal{O}_N$ can be expressed as a direct sum of a family of irreducible invariant subspaces :

$$\mathcal{O}_N = \oplus \{S_j : j \in I_N\}. \tag{3.1}$$

If $\{S_j : j \in I_N\}$ comprises all the irreductible subspaces of $\mathcal{O}_N$, then (3.1) is the unique decomposition of $\mathcal{O}_N$ into a direct sum of irreducible subspaces. (The converse is also true, as can be easily seen, since any representation of a compact group is equivalent to a unitary representation.) The following proposition is simply a re-wording of a theorem in [1] in a slightly different context.

Proposition 3.1. — *Suppose $\mathcal{O}_N = \oplus \{S_j : j \in I_N\}$ is the unique decomposition of $\mathcal{O}_N$ into a direct sum of irreducible rotating subspaces, then :*

1° *If $\mathcal{C} = \bigoplus_{j \in J} S_j$, J a proper subset of I_N, then $C_0(\mathcal{C})$ is a proper rotating space of differentiable functions and $C_0^N \subset C_0(\mathcal{C}) \subset C_0^{N-1}$.*

2° *If* $J_1 \neq J_2$ *and* $\mathcal{A}_i = \bigoplus_{j \in J_i} S_j$, $i=1, 2$, *then* $C_0(\mathcal{A}_1) \neq C_0(\mathcal{A}_2)$.

3° *If* B *is a proper rotating space of differentiable functions and* $C_0^N \subset B \subset C_0^{N-1}$, *then* $B = C_0(\mathcal{A})$ *for some* $\mathcal{A} = \bigoplus_{j \in J} S_j$, J *a proper subset of* I_N.

Proof. — 1° If $\mathcal{A} = \bigoplus_{j \in J} S_j$, then $\mathcal{A}$ contains an elliptic system of order N, since each S_j does. Thus $C_0(\mathcal{A}) \subset C_0^{N-1}$, as in the proof of Theorem 2.3; $C_0(\mathcal{A}) \supset C_0^N$ trivially, so that $C_0(\mathcal{A})$ is squeezed and $\mathcal{A}_{C_0(\mathcal{A})} = \mathcal{A} \oplus \mathcal{Q}_{N-1}$ by Corollary 1.3. This implies $C_0(\mathcal{A})$ is rotating, since $\mathcal{A}_{C_0(\mathcal{A})}$ is. It is easy to check that $C_0(\mathcal{A})$ is proper.

2° $C_0(\mathcal{A}_1) = C_0(\mathcal{A}_2)$ implies

$$\mathcal{A}_1 \oplus \mathcal{Q}_{N-1} = \mathcal{A}_{C_0(\mathcal{A}_1)} = \mathcal{A}_{C_0(\mathcal{A}_2)} = \mathcal{A}_2 \oplus \mathcal{Q}_{N-1},$$

so that $\mathcal{A}_1 = \mathcal{A}_2$, which is impossible since $J_1 \neq J_2$ and the spaces S_j, $j \in I_N$, are linearly independent.

3° (i) $\mathcal{A}_B$ is rotating and $\mathcal{Q}_{N-1} \subset \mathcal{A}_B \subset \mathcal{Q}_N$. Let $\mathcal{A} = \mathcal{A}_B \cap \mathcal{O}_N$. It is easy to check that $\mathcal{A}$ is a proper subspace of $\mathcal{O}_N$. $\mathcal{A}$ is rotating, so the complete reducibility of the representation $\sigma \to \mathcal{R}_\sigma$ applies; and by the uniqueness hypothesis, each irreducible rotating subspace of $\mathcal{A}$ must be some S_j, $j \in I_N$, so that $\mathcal{A} = \bigoplus_{j \in J} S_j$, for some proper subset J of I_N.

(ii) We must show that $B = C_0(\mathcal{A})$. By 1°, $C_0^N \subset C_0(\mathcal{A}) \subset C_0^{N-1}$, so that $\mathcal{A}_{C_0(\mathcal{A})} = \mathcal{A} \oplus \mathcal{Q}_{N-1}$. In view of Corollary 1.4, we must show that $\mathcal{A}_B = \mathcal{A} \oplus \mathcal{Q}_{N-1}$. $\mathcal{A}_B \supset \mathcal{A} \oplus \mathcal{Q}_{N-1}$ by definition of $\mathcal{A}$. If $A \in \mathcal{A}_B$, then $A^N = A - (A - A^N) \in \mathcal{A}_B + \mathcal{Q}_{N-1} = \mathcal{A}_B$; thus $A^N \in \mathcal{A}_B \cap \mathcal{O}_N = \mathcal{A}$ and $A = A^N + (A - A^N) \in \mathcal{A} \oplus \mathcal{Q}_{N-1}$. This proves $\mathcal{A}_B = \mathcal{A} \oplus \mathcal{Q}_{N-1}$.

It is clear, from the commutativity of the isomorphism $A \to \hat{A}$ with rotations, that we may just as well work with the representation $\sigma \to R_\sigma$ in $\mathcal{P}_N$. For any subspace X of $\mathcal{P}_N$, and any integer $k \geqslant 0$, denote by $r^{2k}X$ the space $\{r^{2k}P : P \in X\}$, where $r^2 = \Sigma x_i^2$. Let $\mathcal{H}_N$ stand for the subspace of $\mathcal{P}_N$ consisting of harmonic polynomials. $\mathcal{H}_N$ is rotating, since the Laplacien $\Delta = \Sigma D_i^2$ satisfies $\mathcal{R}_\sigma \Delta = \Delta$ for all rotations σ. Now it is known that for all $n \geqslant 2$, one has

$$\mathcal{P}_N = \mathcal{H}_N \oplus r^2 \mathcal{P}_{N-2}. \tag{3.2}$$

(See, for example, [4], p. 127, where it is shown that $\mathfrak{L}_N = \mathcal{H}_N + r^2\mathfrak{L}_{N-2}$. That the sum is direct is also known and quite simple to check.) As a consequence,

$$\mathfrak{L}_N = \bigoplus_{0 \leqslant 2m \leqslant N} r^{2m}\mathcal{H}_{N-2m}. \tag{3.3}$$

The following theorem, given by Brelot and Choquet ([3]), is precisely what we need here:

PROPOSITION 3.2. — *Let* $n \geqslant 3$.

1° *The action of* $SO(n)$ *on each* $\mathcal{H}_N$ *is irreducible.*

2° *Any irreducible rotating subspace of* $\mathfrak{L}_N$ *must be some* $r^{2m}\mathcal{H}_{N-2m}$. *In other words,* (3.3) *is the unique decomposition of* $\mathfrak{L}_N$ *into a direct sum of irreducible invariant subspaces.*

We sketch a proof of 2° here, since we will have to refer to the proof again in the next section. (The proof is somewhat different from that given in [3], but its essence is really contained there.) First, some simple remarks:

(i) $R_\sigma \circ \Delta = \Delta \circ R_\sigma$ on $\mathfrak{L}$.

(ii) For any $P \in \mathcal{H}_N$, $\Delta r^{2l+2}P = cr^{2l}P$, where $c > 0$ depends only on N, l and n.

(iii) If h_N denotes the dimension of $\mathcal{H}_N$, then h_N is strictly increasing with N, for each $n \geqslant 3$. (For $n = 2$, $h_N = 2$ for all $N \geqslant 1$.)

(i) is trivial, (ii) is obtained by direct computation, using Euler's identity for homogeneous polynomials; (iii) follows from an examination of formula (3.2).

LEMMA 3.3. — *Let* X *be an irreducible rotating subspace of* $\mathfrak{L}_{N+2l+2}$. *If* $\Delta X = r^{2l}\mathcal{H}_N$, *then* $X = r^{2l+2}\mathcal{H}_N$.

Proof. — $\Delta : X \to r^{2l}\mathcal{H}_N$ isomorphically, otherwise the null space of Δ in X is a proper rotating subspace, by (i). Let $P_1, \ldots, P_k$, $k = h_N$, be a basis of $\mathcal{H}_N$. Choose a basis $Q_1, \ldots, Q_k$, of X, such that $\Delta Q_i = cr^{2l}P_i$, where c is the positive constant given in (ii). Thus

$$\Delta(Q_i - r^{2l+2}P_i) = 0, \qquad i = 1, \ldots, h_N.$$

We wish to conclude that $Q_i = r^{2l+2}P_i$ for all i. If this is not the case, $\{Q_i - r^{2l+2}P_i\}_1^k$ would span a non-zero subspace Y of $\mathcal{H}_{N+2l+2}$. Now $\Delta R_\sigma Q_i = R_\sigma \Delta Q_i = cr^{2l}R_\sigma P_i$; thus if

$$R_\sigma P_i = \Sigma\alpha_{ji}(\sigma)P_j, \tag{3.4}$$

we have $\Delta(R_\sigma Q_i - \Sigma\alpha_{ji}(\sigma)Q_j) = 0$. Since Δ is injective on X, this implies

$$R_\sigma Q_i = \Sigma\alpha_{ji}(\sigma)Q_j. \tag{3.5}$$

Equations (3.4) and (3.5) imply that Y is rotating; thus $Y = \mathcal{H}_{N+2l+2}$, by the irreducibility of $\mathcal{H}_{N+2l+2}$. But this would imply $h_{N+2l+2} = h_N$, which is absurd by (iii). This proves the lemma.

Part 2° of Proposition 3.2 follows immediately. If X is a non-zero irreducible rotating subspace of $\mathfrak{L}_N$, let m be the non-negative integer such that $\Delta^m X \neq \{0\}$ but $\Delta^{m+1}X = \{0\}$. Then $\Delta^m X = \mathcal{H}_{N-2m}$, and repeated application of lemma 3.3 yields $X = r^{2m}\mathcal{H}_{N-2m}$.

Proposition 3.2, in conjunction with Proposition 3.1, gives us a complete classification of all rotating spaces of differentiable functions on $\mathbf{R}^n$, $n \geqslant 3$. Before summarizing the results, let us make some side remarks here. The spaces $\mathcal{H}_N$ are actually invariant under the group $O(n)$, instead of simply the subgroup $SO(n)$. This is clear since $\mathfrak{R}_\sigma\Delta = \Delta$ for all $\sigma \in O(n)$. For the case $n = 2$, the decomposition of $\mathfrak{L}_N$ into irreducible rotating subspaces is

$$\mathfrak{L}_N = \bigoplus_{m+n=N} [z^m\bar{z}^n], \tag{3.6}$$

where $z = x + iy$, $\bar{z} = x - iy$, and $[z^m\bar{z}^n]$ denotes the subspace of $\mathfrak{L}_N$ generated by the polynomial $z^m\bar{z}^n$. This can be obtained by noticing that any irreducible representation of a compact commutative group is one-dimensional. (See [1]). Note that a reflection of axis, say $y \to -y$, sends $z^m\bar{z}^n$ to $z^n\bar{z}^m$, so that $[z^m\bar{z}^n]$ is not invariant under $O(2)$ unless $m = n$. In fact, one sees that the irreducible invariant subspaces of $\mathfrak{L}_N$, under $O(2)$, are precisely the subspaces $[z^m\bar{z}^n, z^n\bar{z}^m] = r^{2m}\mathcal{H}_{N-2m}$ (if $m \leqslant n$). Thus, the decomposition of $\mathfrak{L}_N$ under $O(2)$ is also given by formula (3.3).

We summarize the results. Let $\mathfrak{K}_N$ denote the space of homogeneous differential operators in $\mathfrak{Q}$ of order N with harmonic characteristic polynomials; thus $\hat{\mathcal{K}}_N = \mathcal{H}_N \,.\, \Delta^m \mathcal{K}_N$ will mean the obvious thing: $\widehat{\Delta^m \mathcal{K}_N} = r^{2m} \mathcal{H}_N$. We have shown

THEOREM 3.4. — *Let*

$$\mathcal{F}_N(\mathbf{R}^n) = \{\Delta^i \mathcal{K}_{N-2i} : i = 0, 1, \ldots, [N/2]\}.$$

Then, for all $n \geqslant 2$,

1° *If* $\mathcal{A}$ *is a sum of a proper subset of* $\mathcal{F}_N(\mathbf{R}^n)$, *then* $C_0(\mathcal{A})$ *is a proper space of differentiable functions, invariant under* $O(n)$ *and squeezed between* C_0^N *and* C_0^{N-1}.

2° *Distinct subsets of* $\mathcal{F}_N(\mathbf{R}^n)$ *give distinct spaces of differentiable functions.*

3° *Any proper space of differentiable functions between* C_0^N *and* C_0^{N-1} *and invariant under* $O(n)$ *must be a* $C_0(\mathcal{A})$, *where* $\mathcal{A}$ *is a sum of some proper subset of* $\mathcal{F}_N(\mathbf{R}^n)$.

For $n \geqslant 3$, *the above description also gives all the spaces of differentiable functions invariant under* $SO(n)$.

Note that the number of distinct proper spaces of differentiable functions, between C_0^N and C_0^{N-1} and invariant under $O(n)$, is the same for all $n \geqslant 2$. $\mathcal{F}_1(\mathbf{R}^n)$ consists of only one element, so that there is no such proper invariant space between C_0^1 and C_0. Note, however, for $n = 2$, we do have two proper rotating spaces of differentiable functions between C_0^1 and C_0, from the results of de Leeuw and Mirkil. An example for $n = 3$: the two proper rotating spaces of differentiable functions between C_0^2 and C_0^1 are

$$C_0(\Delta); \qquad C_0\left(\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_3^2}, \frac{\partial^2}{\partial x_2^2} - \frac{\partial^2}{\partial x_3^2}, \frac{\partial^2}{\partial x_1 \partial x_2}, \frac{\partial^2}{\partial x_2 \partial x_3}, \frac{\partial^2}{\partial x_3 \partial x_1}\right).$$

The group $O(n)$ is the group of linear isometries of $\mathbf{R}^n$. The de Leeuw-Mirkil rotating spaces on $\mathbf{R}^2$ can be considered as the spaces of differentiable functions on $\mathbf{C}^1$ invariant under the linear isometries of $\mathbf{C}^1$, since these are precisely all the rotations on $\mathbf{R}^2$. Thus the next question to ask is: What are the spaces of differentiable functions on $\mathbf{C}^n$, $n \geqslant 2$, invariant under the linear isometries of $\mathbf{C}^n$?

4. Invariant Algebras of Differentiable Functions on $\mathbf{C}^n$.

We identify $\mathbf{C}^n$ with $\mathbf{R}^{2n}$ by

$$(z_1, \ldots, z_n) \longleftrightarrow (x_1, \ldots, x_n, y_1, \ldots, y_n),$$

where $z_j = x_j + iy_j$, $j = 1, \ldots, n$. For functions on $\mathbf{C}^n$, continuity and differentiation are defined in terms of corresponding functions on $\mathbf{R}^{2n}$. Naturally we define the spaces of differentiable functions in $C_0(\mathbf{C}^n)$ to be just the spaces of differentiable functions in $C_0(\mathbf{R}^{2n})$. The group of linear isometries of $\mathbf{C}^n$ is the group $U(n)$ of unitary transformations, which can be considered as a subgroup of $O(2n)$ when we identify $\mathbf{C}^n$ and $\mathbf{R}^{2n}$. Our problem here is simply to find all spaces of differentiable functions in $C_0(\mathbf{R}^{2n})$ which are invariant under this subgroup. One therefore expects a larger number of these spaces than there are such invariant under $O(2n)$. This is already seen to be the case when $n = 1$. In this section, whenever the word invariance is used, it will be understood to mean invariance under the group $U(n)$. The result of section 2, namely that invariant spaces of differentiable functions are squeezed, remains valid here, since one has only to modify trivially the proof of lemma 2.1, noting that the group $U(n)$ acts transitively on the unit sphere of $\mathbf{C}^n$.

The space $\mathfrak{Q} = \mathfrak{Q}(\mathbf{R}^{2n})$ is the space of all polynomials in $\frac{\partial}{\partial x_j}, \frac{\partial}{\partial y_j}$. With the notation

$$\frac{\partial}{\partial z_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right), \qquad \frac{\partial}{\partial \bar{z}_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\right),$$

$\mathfrak{Q}$ is the space of differential operators of the form

$$\Sigma a_{\alpha\beta}\left(\frac{\partial}{\partial z}\right)^{\alpha}\left(\frac{\partial}{\partial \bar{z}}\right)^{\beta},$$

where

$$\left(\frac{\partial}{\partial z}\right)^{\alpha} = \left(\frac{\partial}{\partial z_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial z_n}\right)^{\alpha_n}, \qquad \left(\frac{\partial}{\partial \bar{z}}\right)^{\beta} = \left(\frac{\partial}{\partial \bar{z}_1}\right)^{\beta_1} \cdots \left(\frac{\partial}{\partial \bar{z}_n}\right)^{\beta_n}.$$

Note the characteristic polynomial of $\frac{\partial}{\partial z_j}$ is a multiple of $\bar{z}_j$. The isomorphism $A \to \hat{A}$ takes operators $A \in \mathfrak{Q}$ into polynomials in $z_1, \ldots, z_n, \bar{z}_1, \ldots, \bar{z}_n$.

DEFINITION. — *$\mathfrak{P}$ is the space of polynomials in $2n$ variables $x_1, \ldots, x_n, y_1, \ldots, y_n$; $\mathfrak{P}_N$ the subspace consisting of polynomials homogeneous of degree N. For $P \in \mathfrak{P}$, let $\tilde{\mathfrak{P}}$ be the function on $\mathbf{C}^n$ given by*

$$\tilde{P}(z_1, \ldots, z_n) = P(z_1, \ldots, z_n, \bar{z}_1, \ldots, \bar{z}_n),$$

or in short, $\tilde{P}(z) = P(z, \bar{z})$. The collection $\{\tilde{P} : P \in \mathfrak{P}_N\}$ will be denoted by $\tilde{\mathfrak{P}}_N$.

The coefficients of the polynomial P can be expressed in terms of the derivatives at $z = 0$ of the function $\tilde{P}$. Thus $\tilde{P}(z) = 0$ for all $z \in \mathbf{C}^n$ implies P is the zero polynomial; in other words $\mathfrak{P}_N$ and $\tilde{\mathfrak{P}}_N$ are isomorphic. The map $A \to \hat{A}$ gives an isomorphism of $\mathcal{O}_N$ and $\tilde{\mathfrak{P}}_N$; thus the problem of classification of invariant spaces of differentiable functions on $\mathbf{C}^n$ amounts to finding all irreducible invariant subspaces of $\tilde{\mathfrak{P}}_N$.

DEFINITION. — *For any two non-negative integers p, q, let $\mathfrak{P}^{(p,q)}$ be the subspace of $\mathfrak{P}$ consisting of polynomials which are homogeneous of degree p in the variables $x_1, \ldots, x_n$ (as polynomials over $\mathbf{C}[y_1, \ldots, y_n]$), and homogeneous of degree q in $y_1, \ldots, y_n$ (as polynomials over $\mathbf{C}[x_1, \ldots, x_n]$). $\tilde{P}^{(p,q)}$ will stand for the space of all $\tilde{P}$ for which $P \in \mathfrak{P}^{(p,q)}$; a function $\tilde{P} \in \tilde{\mathfrak{P}}^{(p,q)}$ will sometimes be said to be of type (p, q). $\mathcal{H}^{(p,q)}$ will stand for the subspace of $\tilde{\mathfrak{P}}^{(p,q)}$ consisting of all $\tilde{P}$ which are harmonic.*

To avoid confusion with notations, let us note here that $\tilde{P} \in \mathcal{H}^{(p,q)}$ does not mean that P is harmonic; of course the polynomial P^*, given by

$$P^*(x_1, \ldots, x_n, y_1, \ldots, y_n) = \tilde{P}(x_1 + iy_1, \ldots, x_n + iy_n),$$

is harmonic. Note that if $P \in \mathfrak{P}_N$, then $P^* \in \mathfrak{P}_N$; but if $P \in \mathfrak{P}^{(p,q)}$, P^* does not have to be in $\mathfrak{P}^{(p,q)}$.

Clearly $\mathfrak{P}^{(p,q)} \subset \mathfrak{P}_{p+q}$. In fact, each $P \in \mathfrak{P}^{(p,q)}$ is of the form $\Sigma a_{\alpha\beta} x^\alpha y^\beta$, $|\alpha| = p$, $|\beta| = q$. One can easily check that if $\{P_i\}$ is a basis of $\mathfrak{L}^{(p,o)}$, $\{Q_j\}$ a basis of $\mathfrak{L}^{(o,q)}$, then $\{P_i Q_j\}$ is a basis of $\mathfrak{L}^{(p,q)}$. Thus

$$\dim \mathfrak{L}^{(p,q)} = \dim \mathfrak{L}^{(p,o)} . \dim \mathfrak{L}^{(o,q)}. \tag{4.1}$$

The distinct spaces $\mathfrak{L}^{(p,q)}$, $p + q = N$, are linearly independent. For let $\mathfrak{L}^{(p_i,q_i)}$, $i = 1, \ldots, k$, be these and $P_i \in \mathfrak{L}^{(p_i,q_i)}$ with $P_1 + \cdots + P_k = 0$. Replacing x by tx, $t > 0$, we have $t^{p_1} P_1 + \cdots + t^{p_k} P_k = 0$. Suppose p_1 is the smallest of the p_i's; divide the above result by t^{p_1} and then let $t \to 0$, we have $P_1 = 0$. Similarly the others. Since each $P \in \mathfrak{L}_N$ is of the form $\Sigma a_{\alpha\beta} x^\alpha y^\beta$, with $|\alpha| + |\beta| = N$, we conclude that

$$\mathfrak{L}_N = \bigoplus_{p+q=N} \mathfrak{L}^{(p,q)}. \tag{4.2}$$

One can verify directly that $\tilde{\mathfrak{L}}^{(p,q)}$, $\tilde{\mathcal{H}}^{(p,q)}$ are invariant under $U(n)$. Also, if $\tilde{P} \in \tilde{\mathfrak{L}}^{(p,q)}$, then

$$\left(\frac{\partial}{\partial z}\right)^\alpha \left(\frac{\partial}{\partial \bar{z}}\right)^\beta \tilde{P} \in \tilde{\mathfrak{L}}^{(p-|\alpha|, q-|\beta|)}.$$

The following fact can be proved in exactly the same manner as the corresponding fact (3.2):

$$\tilde{\mathfrak{L}}^{(p,q)} = \tilde{\mathcal{H}}^{(p,q)} \oplus r^2 \tilde{\mathfrak{L}}^{(p-1,q-1)}. \tag{4.3}$$

A direct computation from (4.1) and (4.3) shows that for $n \geqslant 2$,

$$\dim \tilde{\mathcal{H}}^{(p-1,q-1)} < \dim \tilde{\mathcal{H}}^{(p,q)}, \qquad p \geqslant 1, \quad q \geqslant 1. \tag{4.4}$$

With slight modifications, the proof of 1°, Proposition 3.2, also carries over. Thus, for $n \geqslant 2$, the action of $U(n)$ on each $\tilde{\mathcal{H}}^{(p,q)}$ is irreducible. A simple consequence of this, although we do not need it here, is that $\tilde{\mathcal{H}}^{(p,q)}$ has a basis consisting of functions of the form

$$\left(\sum_1^n a_i z_i\right)^p \left(\sum_1^n b_i \bar{z}_i\right)^q,$$

with $a_i, b_i \in \mathbf{C}$ such that $\sum_1^n a_i b_i = 0$.

Clearly $\tilde{\mathfrak{P}}^{(0,\mathrm{N})} = \tilde{\mathcal{H}}^{(0,\mathrm{N})}$, $\tilde{\mathfrak{P}}^{(\mathrm{N},0)} = \tilde{\mathcal{H}}^{(\mathrm{N},0)}$. These are the only $\tilde{\mathfrak{P}}^{(p,q)}$ which are irreducible; for if $p \neq 0$, $q \neq 0$, $\tilde{\mathcal{H}}^{(p,q)}$ is a proper subspace of $\tilde{\mathfrak{P}}^{(p,q)}$, since $\Delta z_1^p \bar{z}_1^q \neq 0$. Repeated application of (4.3) gives

$$\tilde{\mathfrak{P}}^{(p,q)} = \bigoplus_{0 \leqslant i \leqslant \min(p,q)} r^{2i} \tilde{\mathcal{H}}^{(p-i,q-i)}, \tag{4.5}$$

which, together with (4.2), yields a decomposition of $\tilde{\mathfrak{P}}_{\mathrm{N}}$ as a direct sum of irreducible subspaces:

$$\tilde{\mathfrak{P}}_{\mathrm{N}} = \bigoplus_{2i+j+k=\mathrm{N}} r^{2i} \tilde{\mathcal{H}}^{(p-i,q-i)}. \tag{4.6}$$

With the aid of (4.4), the proof of 2°, Proposition 3.2, can be carried out without change, showing that the only irreducible invariant subspaces of $\tilde{\mathfrak{P}}^{(p,q)}$ are the spaces $r^{2i}\tilde{\mathcal{H}}^{(p-i,q-i)}$, $i = 0, 1, \ldots, \min(p, q)$. That (4.6) indeed gives the unique decomposition of $\tilde{\mathfrak{P}}_{\mathrm{N}}$ as a direct sum of irreducible subspaces is shown by the following lemma.

Lemma 4.1. — *Any irreducible invariant subspace of* $\tilde{\mathfrak{P}}_{\mathrm{N}}$ *must be a subspace of some* $\tilde{\mathfrak{P}}^{(p,q)}$, $p + q = \mathrm{N}$.

Proof. — In other words, we want to show that all elements in an irreducible subspace X of $\tilde{\mathfrak{P}}_{\mathrm{N}}$ must be of a single type. (Clearly, the type must be the same for all elements in X, since X is irreducible.) If $\tilde{\mathrm{P}} \in \mathrm{X}$, $\tilde{\mathrm{P}} \neq 0$, then by (4.2), $\tilde{\mathrm{P}} = \sum_1^{\mathrm{N}+1} \tilde{\mathrm{P}}_j$, where $\tilde{\mathrm{P}}_j \in \tilde{\mathfrak{P}}^{(p_j,q_j)}$ and $\tilde{\mathfrak{P}}^{(p_j,q_j)}$, $j = 1, \ldots, \mathrm{N}+1$, are the distinct $\tilde{\mathfrak{P}}^{(p,q)}$'s of $\tilde{\mathfrak{P}}_{\mathrm{N}}$. Thus we wish to show that all except one of the $\tilde{\mathrm{P}}_j$'s must be zero. Suppose the contrary; in fact let us suppose that none of the $\tilde{\mathrm{P}}_j$'s are zero, since the following argument remains valid for other cases as well. Consider the unitary transformation

$$\sigma : z_j \to e^{-i\alpha} z_j, \qquad \alpha \in \mathbf{R},$$

then $\mathrm{R}_\sigma \tilde{\mathrm{P}} = \sum_j e^{i\alpha(p_j - q_j)} \tilde{\mathrm{P}}_j$. Now $p_j + q_j = p_k + q_k$, so that if $p_j - q_j = p_k - q_k$, we would have $(p_j, q_j) = (p_k, q_k)$.

Thus $R_\sigma\tilde{P} = \sum_1^{N+1} e^{i\alpha\lambda_j}\tilde{P}_j$, where the λ_j's are distinct integers. Hence for $|\alpha|$ sufficiently small, $e^{i\alpha\lambda_j} \neq e^{i\alpha\lambda_k}$ if $j \neq k$. We may also consider the maps $z_j \to e^{-i\mu\alpha}z_j$, with $\mu = 2, \ldots, N$. Altogether we see that X must contain the elements

$$\sum_{j=1}^{N+1} e^{i\mu\alpha\lambda_j}\tilde{P}_j, \qquad \mu = 0, 1, \ldots, N.$$

The determinant of the matrix $(e^{i\mu\alpha\lambda_j})_{\substack{j=1,\ldots,N+1\\ \mu=0,\ldots,N}}$ is just the Vandermonde determinant, and since $\{e^{i\alpha\lambda_j}\}$ are distinct, it is not zero. This implies that X must contain each $\tilde{P}_j$, $j = 1, \ldots, N$, which is impossible since, being irreducible, X cannot contain elements of different type.

Let us finally remark that for the case $n = 1$, there are no $\tilde{\mathcal{H}}^{(p,q)}$ with $p \neq 0$, $q \neq 0$. For any $\tilde{P} \in \tilde{\mathfrak{P}}^{(p,q)}$ is a multiple of $z^p\bar{z}^q = r^{2p}z^{p-q}$ (if $p > q$), so that $\tilde{\mathfrak{P}}^{(p,q)} = r^{2q}\tilde{\mathcal{H}}^{(p-q,0)}$ in this instance. In other words, each $\tilde{\mathfrak{P}}^{(p,q)}$ is irreducible, and all constituents in the sum (4.5) collapse into one. We now summarize the results. Let $\mathcal{K}^{(p,q)}$ denote the space of homogeneous differential operators A with $\hat{A} \in \tilde{\mathcal{H}}^{(p,q)}$. We have proved

THEOREM 4.2. — *Let*
$\mathcal{F}_N(\mathbf{C}^n) = \{\Delta^i\mathcal{K}^{(j,k)} : i, j, k$ *non-negative integers*, $2i + j + k = N\}$ *where, for* $n = 1$, *it is understood that the spaces* $\Delta^i\mathcal{K}^{(j,k)}$ *for which both* j *and* k *are non-zero are to be deleted from the list* $\mathcal{F}_N(\mathbf{C}^1)$. *Then for all* $n \geqslant 1$,

1° *If* $\mathcal{O}$ *is a sum of a proper subset of* $\mathcal{F}_N(\mathbf{C}^n)$, *then* $C_0(\mathcal{O})$ *is a proper invariant space of differentiable functions on* $\mathbf{C}^n$, *squeezed between* C_0^N *and* C_0^{N+1}.

2° *Distinct subsets of* $\mathcal{F}_N(\mathbf{C}^n)$ *give distinct spaces of differentiable functions.*

3° *Any proper space of differentiable functions on* $\mathbf{C}^n$, *between* C_0^N *and* C_0^{N-1} *and invariant under* $U(n)$, *must be a* $C_0(\mathcal{O})$, *where* $\mathcal{O}$ *is a sum of some proper subset of* $\mathcal{F}_N(\mathbf{C}^n)$.

Finally, some examples. $\mathcal{F}_1(\mathbf{C}^2)$ consists of two elements :

$$\mathcal{K}^{(0,1)} = \left[\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}\right]; \qquad \mathcal{K}^{(1,0)}.$$

$\mathcal{F}_2(\mathbf{C}^2)$ consists of four elements :

$$\mathcal{K}^{(0,2)} = \left[\frac{\partial^2}{\partial z_1^2}, \frac{\partial^2}{\partial z_1\,\partial z_2}, \frac{\partial^2}{\partial z_2^2}\right]; \qquad \mathcal{K}^{(2,0)}; \qquad \Delta\mathcal{K}^{(0,0)} = [\Delta];$$

$$\mathcal{K}^{(1,1)} = \left[\frac{\partial^2}{\partial z_1\,\partial \bar{z}_2}, \frac{\partial^2}{\partial z_2\,\partial \bar{z}_1}, \frac{\partial^2}{\partial z_1\,\partial \bar{z}_1} - \frac{\partial^2}{\partial z_2\,\partial \bar{z}_2}\right].$$

$\mathcal{F}_3(\mathbf{C}^2)$ consists of six elements :

$$\mathcal{K}^{(0,3)} = \left[\frac{\partial^3}{\partial z_1^3}, \frac{\partial^3}{\partial z_1^2\,\partial z_2}, \frac{\partial^3}{\partial z_1\,\partial z_2^2}, \frac{\partial^3}{\partial z_2^3}\right]; \qquad \mathcal{K}^{(3,0)};$$

$$\Delta\mathcal{K}^{(0,1)} = \left[\Delta\frac{\partial}{\partial z_1}, \Delta\frac{\partial}{\partial z_2}\right]; \qquad \Delta\mathcal{K}^{(1,0)};$$

$$\mathcal{K}^{(1,2)} = \left[\frac{\partial^3}{\partial z_1^2\,\partial \bar{z}_2}, \frac{\partial^3}{\partial z_2^2\,\partial \bar{z}_1}, \frac{\partial^3}{\partial z_1^2\,\partial \bar{z}_1} - 2\frac{\partial^3}{\partial z_1\,\partial z_2\,\partial \bar{z}_2}, \right.$$
$$\left. \frac{\partial^3}{\partial z_2^2\,\partial \bar{z}_2} - 2\frac{\partial^3}{\partial z_1\,\partial \bar{z}_1\,\partial z_2}\right]; \qquad \mathcal{K}^{(2,1)}.$$

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BIBLIOGRAPHY

[1] K. DE LEEUW and H. MIRKIL, Algebras of differentiable functions in the plane, *Ann. Inst. Fourier*, 13 (1963), 75-90.

[2] K. DE LEEUW and H. MIRKIL, A priori estimate for differential operators in L_∞ norm, *Ill. J. Math.*, 8 (1964), 112-124.

[3] M. BRELOT et G. CHOQUET, Polynômes harmoniques et polyharmoniques, Second Colloque sur les Équations aux Dérivées Partielles, Bruxelles (1954), 45-66.

[4] E. W. HOBSON, The theory of spherical and ellipsoidal harmonics, Cambridge (1931).

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