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A REMARK ON *CONSTRUIRE UN NOYAU DE LA FONCTORIALITÉ* BY LAFFORGUE

by Hervé JACQUET

Abstract. — Lafforgue has proposed a new approach to the principle of functoriality in a test case, namely, the case of automorphic induction from an idele class character of a quadratic extension. For technical reasons, he considers only the case of function fields and assumes the data is unramified. In this paper, we show that his method applies without these restrictions. The ground field is a number field or a function field and the data may be ramified.

Résumé. — Lafforque a proposé une nouvelle approche du problème de la fonctorialité de Langlands. Il traite le cas témoin de l'induction automorphe à partir d'un caractère des classes d'idèles pour une extension quadratique. Pour des raisons techniques, il se limite au cas des corps de fonctions et suppose les données non ramifiées. Dans cet article, nous montrons que sa methode s'applique sans restriction. Le corps de base est un corps de nombres ou un corps de fonctions et les données peuvent être ramifiées.

1. Introduction

In [2] Lafforgue proposes a new approach to the problem of establishing the principle of functoriality. To that end, he investigates a simple case. In more details, let F be a number field or a function field, E a quadratic extension of F. We denote by $F_{\mathbb{A}}$ the adèle ring of F and by $E_{\mathbb{A}}$ the adèle ring of E. We also denote by $F_{\mathbb{A}}^{\times}$ the group of idèles of F and by $E_{\mathbb{A}}^{\times}$ the group of idèles of E. Let χ be an idèle class character of E. We know how to associate to χ an irreducible representation $\pi(\chi)$ of $GL(2, F_{\mathbb{A}})$. The main property of the representation $\pi(\chi)$ is that the standard $L-$function attached to $\pi(\chi)$, namely $L(s, \pi(\chi))$, is equal to the $L-$function $L(s, \chi)$ attached to the idèle class character χ. The principle of functoriality asserts

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that $\pi(\chi)$ is automorphic. In [2] Lafforgue proves this anew in the simplest case: F is a function field, E is unramified at all places of F and χ is unramified at all places of E. In this note, we show that his method applies to a number field or a function field F, an **arbitrary** quadratic extension E and an **arbitrary** character χ. Our only contribution is to the local theory. We use the machinery of the local Weil representation to establish the required properties of the appropriate kernels. Since the definition of the local Weil representation does not depend on any assumption on the ramification of the data, we can free ourself from the restriction of [2]; the local quadratic extensions may be ramified and the character χ may be ramified. Furthermore, even in the special case studied in [2] where the data is unramified, our proof is much shorter. As for the global part, we have nothing new to say. We simply duplicate [2]. Indeed, following [2] step by step, we construct a family of functions $H(t,g)$ on $E_{\mathbb{A}}^{\times} \times GL(2,F_{\mathbb{A}})$. They have the following properties:

- For every $\delta \in E^{\times}$
$$H(\delta t, g) = H(t,g);$$
- For every $\gamma \in GL(2,F)$,
$$H(t,\gamma g) = H(t,g);$$
- For every idèle class character χ of E, the space spanned by the functions
$$g \mapsto \int_{E_{\mathbb{A}}^{\times}/E^{\times}} d^{\times}t\ \chi(t)\ H(t,g)$$
is invariant under right translation by $GL(2,F_{\mathbb{A}})$ and the corresponding representation of $GL(2,F_{\mathbb{A}})$ equivalent to $\pi(\chi)$.

More precisely, we let ψ be a non-trivial character of $F_{\mathbb{A}}/F$ and define two families of functions K and K^0 on $E_{\mathbb{A}}^{\times} \times GL(2,F_{\mathbb{A}})$ such that

$$K\left[t, \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right] = \psi(x)K(t,g),\ K^0\left[t, \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right] = K^0(t,g),$$

for all $x \in F_{\mathbb{A}}$. In turn, the functions K and K^0 are defined as products of local functions. Then H is defined by the formula

$$H(t,g) = \sum_{\delta \in E^{\times}, \alpha \in F^{\times}} K\left[\delta t, \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} g\right] + \sum_{\alpha \in F^{\times}} K^0\left[t, \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} g\right].$$

The invariance of the kernel H under an element $\delta \in E^{\times}$ is an issue but is not difficult. However, the invariance under an element $\gamma \in GL(2,F)$

is a serious issue. Indeed, a priori, we have only the following property of invariance:

$$H\left[t, \begin{pmatrix}\alpha & \beta\\ 0 & 1\end{pmatrix} g\right] = H(t,g),$$

for every $\alpha \in F^\times$, $\beta \in F$. To conclude we need to prove that

$$H\left[t, \begin{pmatrix}1 & 0\\ \beta & \alpha\end{pmatrix} g\right] = H(t,g),$$

for every every $\alpha \in F^\times$, $\beta \in F$. At this point we refer the reader to the proof of Theorem 1 below. The reader will see that the proof is closely related to the proof of the automorphy of $\pi(\chi)$ via the converse theorem, as in [1]. Yet, it is very different. In the converse theorem approach, the Poisson summation formula for E is used at the outset to establish the analytic properties of the functions $L(s,\chi)$. In the present approach, the Poisson summation formula is used at the very end.

In addition, in Section 6.1, we relate the above construction to the global Weil representation. This gives us an opportunity to check that certain expressions are absolutely convergent, a point which is somewhat glossed over in [2] — where the ground field is a function field and questions of convergence are of little concern. In Section 6.2 we conclude by proving that indeed $\pi(\chi)$ is automorphic. We also give a precise form of the Fourier expansion of the functions $g \mapsto \int_{E_{\mathbb{A}}^\times/E^\times} d^\times t\ \chi(t)\ H(t,g)$.

We hope that the present note will make the novel approach of [2] even more attractive.

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2. Notations

We let F be a number field or a function field and E a quadratic extension of F. For simplicity, we assume that if F is a function field the characteristic of F is odd. We denote by ι the Galois conjugation of E over F and by Tr and Nm the norm map and the trace map from E to F. Thus $\mathrm{Tr}(z) = z + \iota(z)$ and $\mathrm{Nm}(z) = z\iota(z)$. We let η be the quadratic idèle class character of F attached to E.

If x is a place of F we set as usual $E_x = E \otimes_F F_x$. The Galois conjugation ι induces a F_x automorphism of E_x that we still denote by ι. We also denote by Tr and Nm the norm map and the trace map from E_x to F_x. Thus, again, for $z \in E_x$, $\mathrm{Tr}(z) = z + \iota(z)$ and $\mathrm{Nm}(z) = z\iota(z)$. If x is split in E

we choose an isomorphism $E_x \simeq F_x \oplus F_x$. Then if $z = (z_1, z_2)$, we have $\iota(z) = (z_2, z_1)$, $\mathrm{Tr}(z) = z_1 + z_2$, $\mathrm{Nm}(z) = z_1 z_2$.

We choose a non-trivial additive character ψ of $F_{\mathbb{A}}/F$. Then $\psi(a) = \prod \psi_x(a_x)$, where ψ_x is a non-trivial additive character of F_x. For almost all finite x, the character ψ_x is normalized, that is, the largest ideal on which it is trivial is the ring of integers $\mathcal{O}_x$. Let dy be the Haar measure on F_x which is self-dual with respect to ψ_x. We denote by $L(s, 1_{F_x})$ the L factor at the place x (finite or infinite) of the Dedekind zeta function $L(s, 1_F)$ of the field F. Thus if x is finite and q_x is the cardinality of the residual field of F_x, then $L(s, 1_{F_x}) = (1 - q_x^{-s})^{-1}$. For $a \in F_x^\times$ we let $|a|_x$ be the absolute value of a if x is a finite or real place. If x is complex then we set $|a|_x = a\bar{a}$. In any case $\frac{dy}{|y|_x}$ is a Haar measure on $F_x^\times$. We then define a Haar measure on $F_x^\times$ by the formula

$$d^\times y = L(1, 1_{F_x}) \frac{dy}{|y|_x}.$$

If x is finite and the character ψ_x is normalized, then the ring of units $\mathcal{O}_x^\times$ has volume 1 for the measure $d^\times y$.

We define a character ψ_E of $E_{\mathbb{A}}/E$ by the formula

$$\psi_E(z) = \psi(\mathrm{Tr}(z)).$$

The local component of ψ_E at the place x is the character ψ_{E_x} of E_x defined by $\psi_{E_x}(z) = \psi_x(\mathrm{Tr}(z))$. The additive Haar measure dz on E_x is taken to be self-dual for ψ_{E_x}. Thus, if x is split and $z = (z_1, z_2)$, we have $\psi_{E_x}(z) = \psi_x(z_1)\psi_x(z_2)$, $dz = dz_1 dz_2$. We denote by $L(s, 1_{E_x})$ the L factor at the place x (finite or infinite) of the Dedekind zeta function $L(s, 1_E)$ of the field E. We define a Haar measure $d^\times z$ on $E_x^\times$ by the formula

$$d^\times z = L(1, 1_{E_x}) \, \frac{dz}{|\,\mathrm{Nm}(z)|_x}.$$

If x is finite and E_x is a field, then $L(s, 1_{E_x}) = (1 - q_{E_x}^{-s})^{-1}$ where q_{E_x} is the cardinality of the residual field of E_x. If x is split and $z = (z_1, z_2)$ then $d^\times z = d^\times z_1 d^\times z_2$. We let $\mathcal{O}_{E_x}$ the ring of integers in E_x. If ψ_x is normalized and E_x/F_x unramified then the volume of $\mathcal{O}_{E_x}^\times$ for $d^\times z$ is 1. If x is split then $\mathcal{O}_{E_x} \simeq \mathcal{O}_{F_x} \oplus \mathcal{O}_{F_x}$ and $\mathcal{O}_{E_x}^\times \simeq \mathcal{O}_{F_x}^\times \times \mathcal{O}_{F_x}^\times$. We denote by U_1 the unitary group in one variable, regarded as an algebraic group defined over F. Thus $U_1(F) = \{z \in E^\times : \mathrm{Nm}(z) = 1\}$ and $U_1(F_x) = \{z \in E_x^\times : \mathrm{Nm}(z) = 1\}$. In particular, if x is split, then $U_1(F_x) = \{(z_1, z_2) \in F_x^\times \times F_x^\times : z_1 z_2 = 1\}$. Let F_x^+ be the set of $x \in F_x^\times$ of the form $z = \mathrm{Nm}(h)$ for some $h \in E_x^\times$. Of course, if x is split, then $F_x^+ = F_x^\times$. We have an exact sequence of locally compact abelian groups

$$1 \longrightarrow U_1(F_x) \longrightarrow E_x^\times \xrightarrow{\mathrm{Nm}} F_x^+ \longrightarrow 1.$$

This defines a Haar measure du on $U_1(F_x)$; the quotient of the Haar measure on $E_x^\times$ by the measure du has for image the restriction to F_x^+ of the Haar measure on $F_x^\times$. In more concrete terms, we have the following integration formula. Let ϕ be a continuous function of compact support on $E_x^\times$. Let ϕ_0 be the function on $F_x^\times$ defined by

$$\phi_0(a) = \int_{U_1(F_x)} \phi(hu)du,$$

if $a = \mathrm{Nm}(h)$, and $\phi_0(a) = 0$ if a is not a norm. Note that indeed the right hand side does not depend on the choice of h. Then

$$\int_{F_x^\times} \phi_0(a)d^\times a = \int_{E_x^\times} \phi(h)d^\times h. \tag{2.1}$$

If x is split, the measure du is in fact given by the following formula:

$$\int_{U_1(F_x)} \phi(u)du = \int_{F_x^\times} \phi(t, t^{-1})d^\times t.$$

We define the Fourier transform $\mathrm{Ft}(\Phi)$ of a Schwartz-Bruhat function Φ on E_x by the formula

$$\mathrm{Ft}(\Phi)(z) = \int_{E_x} \Phi(u)\psi_{E_x}(uz)du.$$

Similarly, we define the Fourier transform of a Schwartz-Bruhat function Φ on $E_{\mathbb{A}}$ by the formula

$$\mathrm{Ft}(\Phi)(z) = \int_{E_{\mathbb{A}}} \Phi(u)\psi_E(uz)du,$$

where du is the product of the local Haar measures. We have then the identity (Poisson summation formula)

$$\sum_{\delta\in E} \Phi(\delta) = \sum_{\delta\in E} \mathrm{Ft}(\Phi)(\delta). \tag{2.2}$$

Let x be a place of F and χ be a character of $E_x^\times$ of absolute value 1. If x is a finite place, the local representation $\pi(\chi)$ attached to χ is an admissible irreducible representation of $GL(2, F_x)$ on some vector space V. If x is an infinite place, the reader may supplement [1] by the books of Wallach ([3], [4]) for the theory of the topological models of representations and Whittaker models. If x is an infinite place, then $\pi(\chi)$ is a unitary irreducible representation of $GL(2, F_x)$ on some Hilbert space H with norm $\| \bullet \|$. Then we let V be the space of C^∞ vectors in H. This is the space of vector v

such that the map $g \mapsto \pi(\chi)(g)v$ from $GL(2, F_x)$ to H is C^∞. For $v \in V$ and for X in the Lie algebra, or more generally the enveloping algebra $\mathfrak{U}_x$ of $GL(2, F_x)$, the vector $d\pi(\chi)(X)v$ is well defined and in V. We equip V with the topology defined by the semi-norms

$$v \mapsto \|d\pi(\chi)(X)v\|.$$

The space V is complete for this topology. The representation of $GL(2, F_x)$ on V is topologically irreducible. It is still noted $\pi(\chi)$. Let K_x be the standard maximal compact subgroup of $GL(2, F_x)$ ($K_x = O(2, \mathbb{R})$ or $K_x = U(2, \mathbb{R})$). Let H_{K_x} (resp. V_{K_x}) be the space of K_x-finite vectors in H (resp. V). Then $H_{K_x} = V_{K_x}$. The space H_{K_x} is invariant under the action of K_x and $\mathfrak{U}_x$. It is an algebraically irreducible $(\mathfrak{U}_x, K_x)-$module which we also denote by $\pi(\chi)$.

Finally, a Whittaker linear form for $\pi(\chi)$ is a linear form $\lambda \neq 0$ on V, continuous if x is infinite, and such that

$$\lambda\left(\pi(\chi)\begin{pmatrix}1 & u\\ 0 & 1\end{pmatrix}v\right) = \psi_x(u)\lambda(v),$$

for all $u \in F_x$ and $v \in V$. Such a form exists and is unique, within a scalar factor. The Whittaker model of $\pi(\chi)$ is the space $\mathcal{W}(\pi(\chi), \psi_x)$ spanned by the functions

$$g \mapsto \lambda(\pi(\chi)(g)v)$$

with $v \in V$. If x is finite the space is invariant under translations and the representation of $GL(2, F_x)$ on that space is equivalent to $\pi(\chi)$. If x is infinite we let $\mathcal{W}_{K_x}(\pi(\chi), \psi_x)$ be the space of K_x-finite vectors in $\mathcal{W}(\pi(\chi), \psi_x)$. This is a $(\mathfrak{U}_x, K_x)-$module equivalent to $\pi(\chi)$.

3. x inert

For each place x of F, we construct two families of functions on $E_x^\times \times GL(2, F_x)$. They will be used to construct the functions K and K^0. We first examine the case where x is **inert** in E, that is, E_x is a **field**. Thus we have a local quadratic extension E_x/F_x, **possibly ramified**. The Galois conjugation ι of E/F induces the Galois conjugation ι of E_x/F_x. We sometimes write $\overline{z}$ for $\iota(z)$. Recall we denote by Tr and Nm the trace and the norm of E_x to F_x. Recall the local character ψ_{E_x} is given by the formula $\psi_{E_x}(z) = \psi_x(\mathrm{Tr}(z))$. If ψ_x is normalized and E_x/F_x is unramified, then ψ_{E_x} is normalized. The local component η_x of η is the quadratic character of $F_x^\times$ associated with E_x/F_x.

3.1. Review of the Weil representation

We first recall the definition of the Weil constant $\lambda(E_x/F_x,\psi_x)$. It is defined by the following identity: for all $\Phi\in\mathcal{S}(E_x)$, the space of Schwartz-Bruhat functions on E_x,

$$\int_{E_x}\mathrm{Ft}(\Phi)(z)\psi_{E_x}(\mathrm{Nm}(z))dz=\lambda(E_x/F_x,\psi_x)\int_{E_x}\Phi(x)\psi_{E_x}(-\mathrm{Nm}(z))dz.$$

Then the Weil representation r of $SL(2,F_x)$ on the space $\mathcal{S}(E_x)$ is defined by the following conditions:

- $\left[r\begin{pmatrix}\alpha & 0\\ 0 & \alpha^{-1}\end{pmatrix}\Phi\right](z)=\eta_x(\alpha)|\alpha|_x\Phi(\alpha z);$
- $\left[r\begin{pmatrix}1 & u\\ 0 & 1\end{pmatrix}\Phi\right](z)=\psi_x(u\,\mathrm{Nm}(z))\Phi(z);$
- $\left[r\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\Phi\right](z)=\lambda(E_x/F_x,\psi_x)\,\mathrm{Ft}(\Phi)(\iota(z)).$

The constant $\lambda(E_x/F_x,\psi_x)$ is 1 if E_x/F_x is unramified and ψ_x is normalized; in general, if x is finite, it can be computed in terms of a Gauss sum. We refer the reader to the original article of Weil ([5]). [1] is also a convenient reference. We have also a linear representation ρ of $U_1(F_x)$ on $\mathcal{S}(E_x)$ defined by

$$(\rho(u)\Phi)(z)=\Phi(uz).$$

The representation r and the representation ρ commute to one another.

We fix a character χ of $E_x^\times$ of absolute value 1 and let $\mathcal{S}(\chi)$ be the space of $\Phi\in\mathcal{S}(E_x)$ such that

$$\Phi(zh)\chi(h)=\Phi(z),$$

for all $z\in E_x$ and all $h\in U_1(F_x)$. The space $\mathcal{S}(\chi)$ is invariant under the representation r. Let $GL(2,F_x)^+$, or simply G^+, be the subgroup of $g\in GL(2,F_x)$ such that $\det g$ is a norm of the quadratic extension E_x/F_x. The representation r on $\mathcal{S}(\chi)$ extends to a representation r_χ of G^+ on the same space such that

$$\left(r_\chi\begin{pmatrix}a & 0\\ 0 & 1\end{pmatrix}\Phi\right)(z)=|\,\mathrm{Nm}(h)|_x^{1/2}\chi(h)\Phi(zh),$$

if $a=\mathrm{Nm}(h)$. Because Φ is in $\mathcal{S}(\chi)$ the right hand side does not depend on the choice of h, so the left hand side is well defined.

Suppose first x is a finite place. Then the representation r_χ is admissible and (algebraically) irreducible. It induces a representation $\pi(\chi)$ of $GL(2,F_x)$ which is admissible and (algebraically) irreducible.

Now suppose x is real. The above formulas define a continuous representation r of $SL(2,F_x)$ on the topological vector space $\mathcal{S}(E_x)$. Moreover, each $\Phi \in \mathcal{S}(E_x)$ is a C^∞ vector in the sense that the map

$$g \mapsto r(g)\Phi$$

from $SL(2,F_x)$ to $\mathcal{S}(E_x)$ is C^∞. We can also use the same formulas to define a unitary representation (still noted r) on the Hilbert space $L^2(E_x, du)$. Let V be the space of C^∞ vectors for this representation. This is the space of functions $\Phi \in L^2$ such that the map

$$g \mapsto r(g)\Phi$$

from $SL(2,F_x)$ to L^2 is C^∞. Since the inclusion of $\mathcal{S}(E_x)$ into L^2 is continuous we have $V \supset \mathcal{S}(E_x)$. We claim that in fact $V = \mathcal{S}(E_x)$. Indeed, consider the following element of the Lie algebra

$$X_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

If Φ is in V then the function $dr(X_+)\Phi$ is well defined and in V. Now $dr(X_+)\Phi(z) = cz\overline{z}\Phi(z)$ for a suitable constant c. Thus V is stable under multiplication by the function $z \mapsto z\overline{z}$. On the other hand, if Φ is in V then the function $z \mapsto \mathrm{Ft}(\Phi)(\overline{z})$ is in V. It follows that V is stable under the differential operator $\frac{\partial^2}{\partial z \partial \overline{z}}$. From these properties, it follows that $V = \mathcal{S}(E_x)$. Moreover, the topology defined by the semi-norms

$$\Phi \mapsto \|dr(X)\Phi\|_2,$$

where $X \in \mathfrak{U}_x$, is equivalent to the Schwartz topology.

We consider the space $L^2(\chi)$ of (classes of) functions $\Phi \in L^2(E_x, du)$ such that $\Phi(zh)\chi(h) = \Phi(z)$ for all $z \in E_x$ and all $h \in U_1(F_x)$. It is invariant under the representation r. The above formulas define a continuous unitary representation r_χ of G^+ on the Hilbert space $L^2(\chi)$. The space of C^∞ vectors for this representation is the closed subspace $\mathcal{S}(\chi)$ of $\mathcal{S}(E_x)$. The unitary representation r_χ of G^+ on $L^2(\chi)$ is topologically irreducible. Therefore, the representation r_χ of G^+ on $\mathcal{S}(\chi)$ is topologically irreducible. We let $\pi(\chi)$ be the unitary representation of $GL_2(F_x)$ induced by r_χ in the unitary sense. It is topologically irreducible. Finally, the representation of $GL(2,F_x)$ induced by the representation of G^+ on $\mathcal{S}(\chi)$ is the space of C^∞ vectors for the unitary representation $\pi(\chi)$.

We now construct the Whittaker model of $\pi(\chi)$. We consider the linear form $\Phi \mapsto \Phi(1)$ on $\mathcal{S}(\chi)$ (or $\mathcal{S}(E_x)$). If x is a real place, it is continuous. In any case, it has the following invariance property: for every $y \in F_x$,

$$\left(r\begin{pmatrix}1 & y\\ 0 & 1\end{pmatrix}\Phi\right)(1) = \psi_x(y)\Phi(1).$$

For $\Phi \in \mathcal{S}(E_x)$ we define a function $W_{\Phi,\chi}$ on G^+ by

$$W_{\Phi,\chi}(g) = (r_\chi(g)\Phi_\chi)(1), \tag{3.1}$$

where Φ_χ is the function in $\mathcal{S}(\chi)$ defined by

$$\Phi_\chi(z) = \int_{U_1(F_x)} \Phi(zu)\chi(u)du. \tag{3.2}$$

We choose $\tau \in F_x^\times$ which is not a norm. Then

$$GL(2, F_x) = G^+ \cup G^+ \begin{pmatrix}\tau & 0\\ 0 & 1\end{pmatrix}.$$

We extend $W_{\Phi,\chi}$ to $GL(2, F_x)$ by demanding it be 0 on $G^+\left(\begin{smallmatrix}\tau & 0\\ 0 & 1\end{smallmatrix}\right)$. The Whittaker model $\mathcal{W}(\pi(\chi), \psi_x)$ of $\pi(\chi)$ is then the space spanned by the functions

$$g \mapsto W_{\Phi_1,\chi}(g) + W_{\Phi_2,\chi}\left[g\begin{pmatrix}\tau & 0\\ 0 & 1\end{pmatrix}\right],$$

with $\Phi_1, \Phi_2 \in \mathcal{S}(E_x)$. We note that, for $g \in SL(2, F_x)$,

$$r(g)(\Phi_\chi) = (r(g)\Phi)_\chi.$$

In particular, we have

$$\left(r\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\Phi_\chi\right)(z) = \lambda(E_x/F_x, \psi_x)\int_{U_1(F_x)} du\chi(u)\,\mathrm{Ft}(\Phi)(\iota(zu)).$$

We also summarize the formula defining the Whittaker function $W_{\Phi,\chi}$ as follows: if $\det g = 1$ and $a = \mathrm{Nm}(h)$ then

$$W_{\Phi,\chi}\left[\begin{pmatrix}a & 0\\ 0 & 1\end{pmatrix}g\right] = |\,\mathrm{Nm}(h)|_x^{1/2}\int_{U_1(F_x)} (r(g)\Phi)(hu)\chi(hu)du; \tag{3.3}$$

if $\det g = 1$ and a is not a norm, then

$$W_{\Phi,\chi}\left[\begin{pmatrix}a & 0\\ 0 & 1\end{pmatrix}g\right] = 0.$$

3.2. Construction of a kernel

We denote by $\widehat{E_x^\times}$ the dual group of $E_x^\times$, that is, the group of characters of $E_x^\times$ of absolute 1, endowed with the topology of uniform convergence on compact sets. It is a locally compact abelian group and we let $d\chi$ be the Haar measure on $\widehat{E_x^\times}$ dual to the Haar measure $d^\times z$ on $E_x^\times$. We let $\mathcal{C}\big(\widehat{E_x^\times}\big)$ be the space of functions P on $\widehat{E_x^\times}$ which are of the form

$$P(\chi) = \int_{E_x^\times} \chi^{-1}(u)\phi(u)d^\times u,$$

where ϕ is a function of compact support on $E_x^\times$, locally constant if x is finite and C^∞ if x is infinite. Thus the Fourier transform of P, that is, the function $\widetilde{P}$ on $E_x^\times$ defined by

$$\widetilde{P}(t) = \int_{\widehat{E_x^\times}} P(\chi)\chi(t)d\chi, \tag{3.4}$$

is equal to ϕ. For instance, if x is finite, ψ_x is normalized, E_x/F_x is unramified and we take for ϕ the characteristic function of $\mathcal{O}_{E_x}^\times$, then P is the characteristic function of the set of unramified characters.

Let $\Phi \in \mathcal{S}(E)$. We define a function $K_{x,P,\Phi}$ on $E_x^\times \times GL(2, F_x)$ as follows:

$$K_{x,P,\Phi}(t, g) = \int_{\widehat{E_x^\times}} d\chi \cdot P(\chi) \cdot \chi^{-1}(t) \cdot W_{\Phi,\chi}(g). \tag{3.5}$$

It follows from the definitions that, for $\det g_0 = 1$ and $t_0 \in E_x^\times$, we have

$$K_{x,P,\Phi}(tt_0, gg_0) = K_{x,P_0,\Phi_0}(t, g),$$

where $P_0(\chi) = P(\chi)\chi^{-1}(t_0)$ and $\Phi_0 = r(g_0)\Phi$. By construction, for every character χ, the function

$$\int_{E_x^\times} d^\times t \cdot \chi(t) \cdot K_{x,P,\Phi}(t, g) = P(\chi)W_{\Phi,\chi}(g)$$

belongs to the Whittaker model $\mathcal{W}(\pi(\chi), \psi_x)$ of the representation $\pi(\chi)$. It is supported on G^+.

We give another formula for the function $K_{x,P,\Phi}$. Assume $a = \mathrm{Nm}(h)$. Then

$$K_{x,P,\Phi}\left[t, \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right] \\ = |\,\mathrm{Nm}(h)|_x^{1/2} \int_{E_x^\times} d\chi\ P(\chi)\chi(t^{-1}) \int_{U_1(F_x)} \chi(hu)\Phi(hu)du.$$

Exchanging the order of integration, we get

$$\begin{aligned}&= |\operatorname{Nm}(h)|_x^{1/2} \int_{U_1(F_x)} \Phi(hu)du \int_{E_x^\times} d\chi\ P(\chi)\chi(hut^{-1}) \\ &= |\operatorname{Nm}(h)|_x^{1/2} \int_{U_1(F_x)} \widetilde{P}(hut^{-1})\Phi(hu)du.\end{aligned}$$

We now have the required formula. If $a = \operatorname{Nm}(h)$ and $\det g = 1$ then

$$K_{x,P,\Phi}\left[t, \begin{pmatrix} a & 0 \\ 0 & 1\end{pmatrix} g\right] = |\operatorname{Nm}(h)|_x^{1/2} \int_{U_1(F_x)} \widetilde{P}(hut^{-1})(r(g)\Phi)(hu)du; \tag{3.6}$$

on the other hand, the left hand side vanishes if $\det g = 1$ and a is not a norm.

We also need a formula for

$$K_{x,P,\Phi}\left[t^{-1}, \begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix}\begin{pmatrix} -a & 0 \\ 0 & 1\end{pmatrix}\right].$$

We have

$$\begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix}\begin{pmatrix} -a & 0 \\ 0 & 1\end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & 1\end{pmatrix}\begin{pmatrix} a^{-1} & 0 \\ 0 & a\end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix}.$$

Thus

$$K_{x,P,\Phi}\left[t^{-1}, \begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix}\begin{pmatrix} -a & 0 \\ 0 & 1\end{pmatrix} g\right] = K_{x,P,\Phi_a}\left[t^{-1}, \begin{pmatrix} a & 0 \\ 0 & 1\end{pmatrix}\right]$$

where we have set

$$\Phi_a = r\begin{pmatrix} a^{-1} & 0 \\ 0 & a\end{pmatrix} r\begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix}\Phi.$$

Explicitly,

$$\Phi_a(z) = \lambda(E_x/F_x, \psi_x)|a|_x^{-1}\eta_x(a)\operatorname{Ft}(\Phi)(a^{-1}\iota(z)).$$

Thus for $a = \operatorname{Nm}(h)$ and $u \in U_1(F_x)$, we find

$$\Phi_a(hu) = \lambda(E_x/F_x, \psi_x)|\operatorname{Nm}(h)|_x^{-1}\operatorname{Ft}(\Phi)(h^{-1}u^{-1}).$$

Hence, for $a = \operatorname{Nm}(h)$, we have

$$\begin{aligned}K_{x,P,\Phi_a}\left[t^{-1}, \begin{pmatrix} a & 0 \\ 0 & 1\end{pmatrix}\right] = \lambda(E_x/F_x, \psi_x)|\operatorname{Nm}(h)|_x^{-1/2} &\int_{\widehat{E_x^\times}} d\chi P(\chi)\chi(t) \\ &\times \int_{U_1(F_x)} \operatorname{Ft}(\Phi)(h^{-1}u^{-1})\chi(hu)du.\end{aligned}$$

Exchanging the order of integration, we find

$$\begin{aligned}&= \lambda(E_x/F_x,\psi_x)|\operatorname{Nm}(h)|_x^{-1/2}\\ &\quad\times\int_{U_1(F_x)}\operatorname{Ft}(\Phi)(h^{-1}u^{-1})du\int_{\widehat{E_x^\times}}d\chi P(\chi)\chi(thu)\\ &= \lambda(E_x/F_x,\psi_x)|\operatorname{Nm}(h)|_x^{-1/2}\\ &\qquad\qquad\times\int_{U_1(F_x)}\operatorname{Ft}(\Phi)(h^{-1}u^{-1})\widetilde{P}(hut)du.\end{aligned}$$

We now have the required formula. If $a=\operatorname{Nm}(h)$, then

$$\begin{aligned}(3.7)\quad & K_{x,P,\Phi}\left[t^{-1},\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}-a&0\\0&1\end{pmatrix}\right]\\ &= \lambda(E_x/F_x,\psi_x)|\operatorname{Nm}(h)|_x^{-1/2}\int_{U_1(F_x)}\operatorname{Ft}(\Phi)(h^{-1}u^{-1})\widetilde{P}(hut)du;\end{aligned}$$

on the other hand, if a is not a norm then the left hand side vanishes.

The key property of the kernel is given in the following Proposition.

Proposition 3.1. — *Let ω be a character of $F_x^\times$ of absolute value 1. Let $g\in GL(2,F_x)$.*

(i) *The function*

$$\Psi_1(t)=\omega^{-1}(\operatorname{Nm}(t))|\operatorname{Nm}(t)|_x^{-1/2}\int_{F_x^\times}K_{x,P,\Phi}\left[t,\begin{pmatrix}a&0\\0&1\end{pmatrix}g\right]\omega(a)d^\times a,$$

defined a priori on $E_x^\times$, extends by continuity to a Schwartz-Bruhat function on E_x.

(ii) *Similarly, the function*

$$\begin{aligned}\Psi_2(t)&=\omega(\operatorname{Nm}(t))|\operatorname{Nm}(t)|_x^{-1/2}\\ &\qquad\times\int_{F_x^\times}K_{x,P,\Phi}\left[t^{-1},\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}a&0\\0&1\end{pmatrix}g\right]\omega(a)d^\times a,\end{aligned}$$

defined a priori on $E_x^\times$, extends by continuity to a Schwartz-Bruhat function on E_x.

(iii) *Still denoting by Ψ_1 and Ψ_2 the extensions, we have*

$$\omega(-1)\lambda(E_x/F_x,\psi_x)\operatorname{Ft}(\Psi_1)=\Psi_2.$$

(iv) *Assume x is finite, the extension E_x/F_x unramified, the character ψ_x normalized and the character ω unramified. Assume further that P is the characteristic function of the set of unramified characters and Φ is the characteristic function of $\mathcal{O}_{E_x}$. Then $\omega(-1)=1$, $\lambda(E_x/F_x,\psi_x)=1$ and $\Psi_1=\Psi_2=\Phi$.*

Remark. — The factor $\omega(-1)$ does not appear in [2] because, in the kernel defining the function Ψ_2, the character ψ_x is replaced by the character ψ_x^{-1}. Here we use the same character for the kernels defining the functions Ψ_1 and Ψ_2. The factor $\lambda(E_x/F_x, \psi_x)$ does appear in [2]; however, because the extension E_x/F_x is assumed to be unramified, the factor takes the value ± 1.

Proof. — First we observe that we can write

$$g = \begin{pmatrix} \det g & 0 \\ 0 & 1 \end{pmatrix} g_0$$

with $\det g_0 = 1$. Making the change of variables $a \mapsto a \det g^{-1}$ in the two integrals of the Proposition, we see that it suffices to prove the Proposition with g replaced by g_0. We may even replace g_0 by 1, at the cost of replacing Φ by $r(g_0)\Phi$. So from now on we take $g = 1$.

Now we prove assertion (i) of the Proposition. We have, if $a = \mathrm{Nm}(h)$,

$$K_{x,P,\Phi}\left[t, \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right] = |\,\mathrm{Nm}(h)|_x^{1/2} \int_{U_1(F_x)} \widetilde{P}(hut^{-1})\Phi(hu)du.$$

On the other hand, the left hand side vanishes if a is not a norm. Applying our integration formula (2.1) we find

$$\Psi_1(t) = \omega(\mathrm{Nm}(t))^{-1} |\,\mathrm{Nm}(t)|_x^{-1/2} \times \int_{E_x^\times} |\,\mathrm{Nm}(h)|_x^{1/2} \omega(\mathrm{Nm}(h)) \widetilde{P}(ht^{-1})\Phi(h) d^\times h.$$

After changing h to ht we arrive at

$$\Psi_1(t) = \int_{E_x^\times} d^\times h \; |\,\mathrm{Nm}(h)|_x^{1/2} \omega(\mathrm{Nm}(h)) \widetilde{P}(h)\Phi(ht). \tag{3.8}$$

Since $\widetilde{P}$ is a smooth function of compact support on $E_x^\times$, the function

$$t \mapsto \int_{E_x^\times} d^\times h \; |\,\mathrm{Nm}(h)|_x^{1/2} \omega(\mathrm{Nm}(h)) \widetilde{P}(h)\Phi(ht)$$

is a Schwartz-Bruhat function equal to $\Psi_1(t)$ for $t \neq 0$ and assertion (i) follows. If we note again by Ψ_1 the continuous extension of Ψ_1 to E_x we have

$$\Psi_1(0) = \Phi(0) \int_{E_x^\times} d^\times h \; |\,\mathrm{Nm}(h)|_x^{1/2} \omega(\mathrm{Nm}(h)) \widetilde{P}(h). \tag{3.9}$$

Now we prove assertion (ii). Changing a to $-a$ in the formula defining Ψ_2 we find

$$\Psi_2(t) = \omega(-1)\omega(\mathrm{Nm}(t))|\,\mathrm{Nm}(t)|_x^{-1/2} \times \int_{F_x^\times} K_{x,P,\Phi}\left[t^{-1}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -a & 0 \\ 0 & 1 \end{pmatrix}\right] \omega(a) d^\times a.$$

Now, if $a = \mathrm{Nm}(h)$ then

$$K_{x,P,\Phi}\left[t^{-1}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -a & 0 \\ 0 & 1 \end{pmatrix}\right] = \lambda(E_x/F_x, \psi_x)|\,\mathrm{Nm}(h)|_x^{-1/2} \int_{U_1(F_x)} \mathrm{Ft}(\Phi)(h^{-1}u^{-1})\widetilde{P}(hut) du.$$

On the other hand, if a is not a norm then the left hand side vanishes. Applying once more our integration formula (2.1) we find

$$\Psi_2(t) = \omega(-1)\lambda(E_x/F_x, \psi_x)\omega(\mathrm{Nm}(t))|\,\mathrm{Nm}(t)|_x^{-1/2} \times \int_{E_x^\times} |\,\mathrm{Nm}(h)|_x^{-1/2}\omega(\mathrm{Nm}(h))\widetilde{P}(ht)\,\mathrm{Ft}(\Phi)(h^{-1})d^\times h.$$

Changing h to ht^{-1} we arrive at

$$\Psi_2(t) = \omega(-1)\lambda(E_x/F_x, \psi_x) \times \int_{E_x^\times} d^\times h|\,\mathrm{Nm}(h)|_x^{-1/2}\omega(\mathrm{Nm}(h))\widetilde{P}(h)\,\mathrm{Ft}(\Phi)(h^{-1}t). \tag{3.10}$$

Again, it is clear that Ψ_2 extends by continuity to E_x. The resulting function still noted Ψ_2 is a Schwartz-Bruhat function whose value at 0 is

$$\Psi_2(0) = \omega(-1)\lambda(E_x/F_x, \psi_x)\,\mathrm{Ft}(\Phi)(0) \times \int_{E_x^\times} d^\times h|\,\mathrm{Nm}(h)|_x^{-1/2}\omega(\mathrm{Nm}(h))\widetilde{P}(h). \tag{3.11}$$

For (iii), we observe that for $h \in E_x^\times$, the Fourier transform of the function

$$z \mapsto \Phi(hz)|\,\mathrm{Nm}(h)|_x^{1/2}$$

is the function

$$z \mapsto \mathrm{Ft}(\Phi)(h^{-1}z)|\,\mathrm{Nm}(h)|_x^{-1/2}$$

and our assertion follows.

We pass to assertion (iv). Under the given assumptions we have $\omega(-1)=1$, $\lambda(E_x/F_x,\psi_x)=1$ and $\mathrm{Ft}(\Phi)=\Phi$. Moreover, $\widetilde{P}$ is the characteristic function of $\mathcal{O}_{E_x}^{\times}$ and the measure of $\mathcal{O}_{E_x}^{\times}$ is 1. Our assertion follows from the formulas for Ψ_1 and Ψ_2. □

3.3. Complementary kernel

We denote by $U_1(F_x)^{\perp}$ the set of characters χ of $E_x^{\times}$ of absolute value 1 which are trivial on $U_1(F_x)$. Thus we may identify $U_1(F_x)^{\perp}$ to the dual of the quotient group $E_x^{\times}/U_1(F_x)$ and give to it the Haar measure $d\nu$ dual to the quotient Haar measure on $E_x^{\times}/U_1(F_x)$. We have then the Poisson summation formula

$$\int_{U_1(F_x)^{\perp}} P(\nu)d\nu = \int_{U_1(F_x)} \widetilde{P}(u)du,$$

and the more general formula

$$\int_{U_1(F_x)^{\perp}} P(\nu)\nu(t)d\nu = \int_{U_1(F_x)} \widetilde{P}(tu)du. \tag{3.12}$$

Let ν be a character in $U_1(F_x)^{\perp}$. We define a function $f_{\Phi,\nu}$ on G^+ by the following formula: if $a=\mathrm{Nm}(h)$ and $\det g=1$ then

$$f_{\Phi,\nu}\left[\begin{pmatrix} a & 0\\ 0 & 1\end{pmatrix} g\right] = \nu(h)|\,\mathrm{Nm}(h)|_x^{1/2}(r(g)\Phi)(0).$$

Again, the right hand side does not depend on the choice of h so the left hand side is well defined. We extend $f_{\Phi,\nu}$ to $GL(2,F_x)$ by demanding it be 0 on $G^+\left(\begin{smallmatrix} \tau & 0\\ 0 & 1\end{smallmatrix}\right)$.

Now we define a function $K^0_{x,P,\Phi}$ on $E_x^{\times}\times GL(2,F_x)$ by the formula

$$K^0_{x,P,\Phi}(t,g) = \int_{U_1(F_x)^{\perp}} d\nu\cdot P(\nu)\cdot\nu^{-1}(t)\cdot f_{\Phi,\nu}(g). \tag{3.13}$$

Note that the function is 0 outside $E_x^{\times}\times G^+$. Also, if $\det g_0=1$ and $t_0\in E_x^{\times}$ then

$$K^0_{x,P,\Phi}(t,gg_0) = K^0_{x,P_0,\Phi_0}(t,g),$$

where $P_0(\chi)=P(\chi)\chi^{-1}(t_0)$ and $\Phi_0=r(g_0)\Phi$.

As before, we need more explicit formulas for $K^0_{x,P,\Phi}$. So assume $a=\mathrm{Nm}(h)$ for some h and $\det g=1$. Then

$$K^0_{x,P,\Phi}\left[t,\begin{pmatrix} a & 0\\ 0 & 1\end{pmatrix} g\right] = (r(g)\Phi)(0)|\,\mathrm{Nm}(h)|_x^{1/2}\int_{U_1(F_x)^{\perp}} d\nu P(\nu)\nu(ht^{-1}).$$

Applying formula (3.12) we find

$$= (r(g)\Phi)(0)|\operatorname{Nm}(h)|_x^{1/2}\int_{U_1(F_x)} du\widetilde{P}(ht^{-1}u).$$

So we have our formula. If $a = \operatorname{Nm}(h)$ and $\det g = 1$ then

$$K^0_{x,P,\Phi}\left[t, \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} g\right] = (r(g)\Phi)(0)|\operatorname{Nm}(h)|_x^{1/2}\int_{U_1(F_x)} du\widetilde{P}(hut^{-1}); \tag{3.14}$$

on the other hand, if $\det g = 1$ and a is not a norm then the left hand side is 0.

We will also need a formula for

$$K^0_{x,P,\Phi}\left[t^{-1}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} -a & 0 \\ 0 & 1 \end{pmatrix}\right]$$

As for the kernel K this is also

$$K^0_{x,P,\Phi_a}\left[t^{-1}, \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right],$$

where we have set

$$\Phi_a = r\begin{pmatrix} a^{-1} & 0 \\ 0 & a \end{pmatrix} r\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\Phi.$$

Explicitly,

$$\Phi_a(0) = \lambda(E_x/F_x, \psi_x)|a|_x^{-1}\eta_x(a)\operatorname{Ft}(\Phi)(0).$$

Thus for $a = \operatorname{Nm}(h)$ we find

$$K^0_{x,P,\Phi_a}\left[t^{-1}, \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right] = \lambda(E_x/F_x, \psi_x)\operatorname{Ft}(\Phi)(0)|\operatorname{Nm}(h)|_x^{-1/2} \times \int_{U_1(F_x)^\perp} d\nu P(\nu)\nu(ht).$$

Applying formula (3.12) we find

$$= \lambda(E_x/F_x, \psi_x)\operatorname{Ft}(\Phi)(0)|\operatorname{Nm}(h)|_x^{-1/2} \times \int_{U_1(F_x)} \widetilde{P}(htu)du.$$

We now have the required formula. If $a = \operatorname{Nm}(h)$, then

$$K^0_{x,P,\Phi}\left[t^{-1}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} -a & 0 \\ 0 & 1 \end{pmatrix}\right] = \lambda(E_x/F_x, \psi_x)\operatorname{Ft}(\Phi)(0)|\operatorname{Nm}(h)|_x^{-1/2}\int_{U_1(F_x)} \widetilde{P}(hut)du; \tag{3.15}$$

on the other hand, if a is not a norm then the left hand side vanishes.

We can now state the key properties of the function $K^0_{x,P,\Phi}$.

Proposition 3.2. — *Let ω be a character of $F_x^\times$ of absolute value 1. Let $g \in GL(2, F_x)$. Let Ψ_1 and Ψ_2 be the functions defined in Proposition 3.1.*

(i) *The expression*

$$\omega(\mathrm{Nm}(t))^{-1}|\,\mathrm{Nm}(t)|^{-1/2}\int_{F_x^\times} K^0_{x,P,\Phi}\left[t, \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} g\right]\omega(a)d^\times a$$

is independent of t and equal to $\Psi_1(0)$.

(ii) *Similarly, the expression*

$$\omega(\mathrm{Nm}(t))|\,\mathrm{Nm}(t)|^{-1/2}\int_{F_x^\times} K^0_{x,P,\Phi}\left[t^{-1}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} g\right]\omega(a)d^\times a$$

is independent of t and equal to $\Psi_2(0)$.

Proof. — As before, it suffices to prove the Proposition when $g = 1$.

Now, in the integral for part (i), the integrand is 0 unless a is a norm. So assume $a = \mathrm{Nm}(h)$ for some h. Then

$$K^0_{x,P,\Phi}\left[t, \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right] = \Phi(0)|\,\mathrm{Nm}(h)|_x^{1/2}\int_{U_1(F_x)} du\widetilde{P}(hut^{-1}).$$

The expression for part (i) is thus

$$\omega(\mathrm{Nm}(t))^{-1}|\,\mathrm{Nm}(t)|^{-1/2} \times \Phi(0)\int_{E_x^\times}\widetilde{P}(ht^{-1})\omega(\mathrm{Nm}(h))|\,\mathrm{Nm}(h)|_x^{1/2}d^\times h.$$

Changing h to ht we get

$$\begin{aligned} &= \Phi(0) \int_{E_x^\times}\omega(\mathrm{Nm}(h))|\,\mathrm{Nm}(h)|_x^{1/2}\widetilde{P}(h)d^\times h \\ &= \Psi_1(0). \end{aligned}$$

For part (ii), consider the expression

$$\omega(\mathrm{Nm}(t))|\,\mathrm{Nm}(t)|_x^{-1/2}\int_{F_x^\times} K^0_{x,P,\Phi}\left[t^{-1}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right]\omega(a)d^\times a.$$

Changing a to $-a$ we find

$$\omega(-1)\omega(\mathrm{Nm}(t))|\,\mathrm{Nm}(t)|_x^{-1/2} \\ \times\int_{F_x^\times} K^0_{x,P,\Phi}\left[t^{-1}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} -a & 0 \\ 0 & 1 \end{pmatrix}\right]\omega(a)d^\times a.$$

The integrand vanishes unless $a=\mathrm{Nm}(h)$ for some h. Then

$$K^0_{x,P,\Phi}\left[t^{-1},\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}-a&0\\0&1\end{pmatrix}\right]$$
$$=\lambda(E_x/F_x,\psi_x)\,\mathrm{Ft}(\Phi)(0)|\,\mathrm{Nm}(h)|_x^{-1/2}\int_{U_1(F_x)}\widetilde{P}(hut)du.$$

Thus our expression is equal to

$$\omega(-1)\lambda(E_x/F_x,\psi_x)\,\mathrm{Ft}(\Phi)(0)\omega(\mathrm{Nm}(t))|\,\mathrm{Nm}(t)|_x^{-1/2}$$
$$\times\int_{E_x^\times}|\,\mathrm{Nm}(h)|_x^{-1/2}\omega(\mathrm{Nm}(h))\widetilde{P}(ht)d^\times h.$$

Changing h to ht^{-1} we arrive at

$$\begin{aligned}&=\omega(-1)\lambda(E_x/F_x,\psi_x)\,\mathrm{Ft}(\Phi)(0)\\&\qquad\times\int_{E_x^\times}|\,\mathrm{Nm}(h)|_x^{-1/2}\omega(\mathrm{Nm}(h))\widetilde{P}(h)d^\times h\\&=\Psi_2(0).\end{aligned}$$

□

Remark. — For $g\in G^+$ we could also write

$$f_{\Phi,\nu}(g)=\left(r_\nu(g)\Phi_0\right)(0)$$

where Φ_0 is the function in $\mathcal{S}(\nu)$ defined by

$$\Phi_0(z)=\frac{1}{\int_{U_1(F_x)}du}\int_{U_1(F_x)}\Phi(zu)du.$$

Let U be the space of functions on $GL(2,F_x)^+$ spanned by the functions $f_{\Phi,\nu}$ and their translates. It is invariant under right translations. If x is finite the representation of $GL(2,F_x)^+$ on U is equivalent to π_ν. If x is infinite, we have an analogous result for the space of K_x−finite vectors in U.

Furthermore, we can write ν in the form $\nu(t)=\chi_1(\mathrm{Nm}(t))=\chi_2(\mathrm{Nm}(t))$, where χ_1 and χ_2 are characters of $F_x^\times$ satisfying $\chi_2=\chi_1\eta_x$. If $g\in G^+$ and a_1a_2 is a norm we have

$$\begin{aligned}f_{\Phi,\nu}\left[\begin{pmatrix}a_1&x\\0&a_2\end{pmatrix}g\right]&=\left(r\begin{pmatrix}a_2^{-1}&0\\0&a_2\end{pmatrix}r_\nu\begin{pmatrix}a_1a_2&0\\0&1\end{pmatrix}r_\nu(g)\cdot\Phi_0\right)(0)\\&=\eta(a_2)|a_2|_x^{-1}\left(r_\nu\begin{pmatrix}a_1a_2&0\\0&1\end{pmatrix}r_\nu(g)\cdot\Phi_0\right)(0)\\&=\eta(a_2)|a_2|_x^{-1}\chi_1(a_1a_2)|a_1a_2|^{1/2}\left(r_\nu(g)\cdot\Phi_0\right)(0).\end{aligned}$$

So, finally,

$$f_{\Phi,\nu}\left[\begin{pmatrix} a_1 & x \\ 0 & a_2 \end{pmatrix} g\right] = \chi_1(a_1)\chi_2(a_2)\left|\frac{a_1}{a_2}\right|_x^{1/2} f_{\Phi,\nu}(g),$$

and $f_{\Phi,\nu}$ has a unique extension $f^1_{\Phi,\chi}$ to $Gl(2,F_x)$ satisfying the identity

$$f^1_{\Phi,\nu}\left[\begin{pmatrix} a_1 & x \\ 0 & a_2 \end{pmatrix} g\right] = \chi_1(a_1)\chi_2(a_2)\left|\frac{a_1}{a_2}\right|_x^{1/2} f^1_{\Phi,\nu}(g),$$

for all g, a_1, a_2, x. It has also a unique extension $f^2_{\Phi,\nu}$ satisfying the identity

$$f^2_{\Phi,\nu}\left[\begin{pmatrix} a_1 & x \\ 0 & a_2 \end{pmatrix} g\right] = \chi_2(a_1)\chi_1(a_2)\left|\frac{a_1}{a_2}\right|_x^{1/2} f^2_{\Phi,\nu}(g),$$

for all g, a_1, a_2, x. Since we extend $f_{\Phi,\nu}$ by 0 outside G^+ we find

$$f_{\Phi,\nu} = \frac{f^1_{\Phi,\nu} + f^2_{\Phi,\nu}}{2}. \tag{3.16}$$

In the notations of the next section, the function $f^1_{\Phi,\nu}$ (resp. $f^2_{\Phi,\nu}$) belongs to the space $\mathcal{B}(\chi_1,\chi_2)$ (resp. $\mathcal{B}(\chi_1,\chi_2)$). Let $\pi(\chi_1,\chi_2)$ (resp. $\pi(\chi_2,\chi_1)$) be the representation of $GL(2,F_x)$ on $\mathcal{B}(\chi_1,\chi_2)$ (resp. $\mathcal{B}(\chi_1,\chi_2)$) by right shifts. The representations $\pi(\chi_1,\chi_2)$ and $\pi(\chi_2,\chi_1)$ are equivalent. If x is finite it follows from formula (3.16) that they are equivalent to $\pi(\nu)$. If x is infinite, we have an analogous statement for the spaces of K_x-finite vectors.

4. x split

Suppose x is split in E. Recall we have chosen an isomorphism $E_x \simeq F_x \oplus F_x$. Moreover, the norm, the trace and the Galois conjugation are given by $\mathrm{Nm}(z_1,z_2) = z_1z_2$, $\mathrm{Tr}(z_1,z_2) = z_1 + z_2$ and $\iota(z_1,z_2) = (z_2,z_1)$, respectively. The character ψ_{E_x} is given by $\psi_{E_x}(z_1,z_2) = \psi_x(z_1)\psi_x(z_2)$. The self dual Haar measure on E_x is the product of the self-dual measure on F_x by itself. The Fourier transform of of a function Φ on E_x is thus given by

$$\mathrm{Ft}(\Phi)(z_1,z_2) = \int_{F_x\times F_x} \Phi(u_1,u_2)\psi_x(u_1z_1+u_2z_2)du_1du_2.$$

Finally, $\mathcal{O}_{E_x} \simeq \mathcal{O}_x \oplus \mathcal{O}_x$.

4.1. Review of the Weil representation

Let χ be a character of $E_x^\times$. With our identification $E_x \simeq F_x \oplus F_x$, we have $\chi = (\chi_1, \chi_2)$ where χ_1, χ_2 are characters of $F^\times$. In other words, $\chi(z_1, z_2) = \chi_1(z_1)\chi_2(z_2)$. Let $\mathcal{B}(\chi_1, \chi_2)$ be the space of functions $f : GL(2, F_x) \to \mathbb{C}$, invariant under a compact open subgroup if x is finite, C^∞ if x is infinite, such that

$$f\left[\begin{pmatrix} a_1 & x \\ 0 & a_2 \end{pmatrix} g\right] = \chi_1(a_1)\chi_2(a_2)\left|\frac{a_1}{a_2}\right|_x^{1/2} f(g),$$

for all g, a_1, a_2, x. If x is infinite the functions in $\mathcal{B}(\chi_1, \chi_2)$ are determined by their restrictions to K_x and the topology of $\mathcal{B}(\chi_1, \chi_2)$ is the topology induced by the topology of $C^\infty(K_x)$. The space is invariant by the operators of right translation and we denote by $\pi(\chi_1, \chi_2)$ the corresponding representation. Explicitly,

$$(\pi(\chi_1, \chi_2)(g)f)(h) = f(hg).$$

The space is irreducible, algebraically if x is finite and topologically if x is infinite. If x is infinite we can define a unitary representation on the space (of classes) of functions transforming as above which are square integrable on K_x. This representation is topologically irreducible and the topological space of C^∞ vectors of this representation is precisely the space $\mathcal{B}(\chi_1, \chi_2)$. By definition, the representation $\pi(\chi)$ is equivalent to $\pi(\chi_1, \chi_2)$ (and $\pi(\chi_2, \chi_1)$). On the space $\mathcal{B}(\chi_1, \chi_2)$, there exists a non-zero linear form λ, continuous if x is infinite, such that

$$\lambda\left(\pi(\chi_1, \chi_2)\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} f\right) = \psi_x(u)\lambda(f)$$

for all $u \in F_x$ and all $f \in \mathcal{B}(\chi_1, \chi_2)$. This form is unique within a scalar factor. Formally, we can define λ by the integral

$$\lambda(f) = \int_{F_x} f\left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}\right]\psi(-u)du.$$

However, this integral does not converge. It can be given a meaning by analytic continuation (see [1]). Indeed, for any $s \in \mathbb{C}$ define $\chi_{1,s}(z) = \chi_1(z)|z|_x^s$ and $\chi_{2,-s}(z) = \chi_2(z)|z|_x^{-s}$. We can define the space $\mathcal{B}(\chi_{1,s}, \chi_{2,-s})$ and for $f \in \mathcal{B}(\chi_{1,s}, \chi_{2,-s})$ the above integral converges if the real part of s is > 0. The integral has analytic continuation to the point $s = 0$. We can now define the Whittaker model $\mathcal{W}(\pi(\chi), \psi_x)$ as the space of functions W of the form

$$W(g) = \lambda(\pi(\chi)(g)f)$$

with $f \in \mathcal{B}(\chi_1, \chi_2)$.

However, we need to describe this space in terms of the Weil representation. The constant $\lambda(E_x/F_x, \psi_x)$ is 1, that is, we have the identity

$$\int_{E_x} \mathrm{Ft}(\Phi)(z)\psi_{E_x}(\mathrm{Nm}(z))dz = \int_{E_x} \Phi(x)\psi_{E_x}(-\mathrm{Nm}(z))dz,$$

for every $\Phi \in \mathcal{S}(E_x)$, the space of Schwartz-Bruhat functions on E_x. The Weil representation r of $SL(2, F_x)$ on $\mathcal{S}(E_x)$ is defined by the same formulas as in the inert case. Of course here $\eta_x = 1$. Explicitly, we have

- $\left[r\begin{pmatrix}\alpha & 0\\ 0 & \alpha^{-1}\end{pmatrix}\Phi\right](z_1, z_2) = |\alpha|_x\Phi(\alpha z_1, \alpha z_2);$
- $\left[r\begin{pmatrix}1 & u\\ 0 & 1\end{pmatrix}\Phi\right](z_1, z_2) = \psi_x(uz_1z_2)\Phi(z_1, z_2);$
- $\left[r\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\Phi\right](z_1, z_2) = \mathrm{Ft}(\Phi)(z_2, z_1).$

Let us denote by $\rho(g)$ the linear action of $g \in GL(2, F_x)$ on $\mathcal{S}(E_x)$, that is,

$$(\rho(g)\Phi)(z_1, z_2) = \Phi((z_1, z_2)g).$$

Let us also denote by Pt the partial Fourier transform defined by

$$\mathrm{Pt}(\Phi)(z_1, z_2) = \int_{F_x} \Phi(z_1, u)\psi_x(uz_2)du.$$

Then, for $g \in SL(2, F_x)$, we have

$$\mathrm{Pt}(r(g)\Phi) = \rho(g)\,\mathrm{Pt}(\Phi).$$

It is convenient to extend the representation r of $SL(2, F_x)$ to a representation r_{χ_1} of $GL(2, F_x)$ on the same space by the condition

$$r_{\chi_1}\begin{pmatrix}a & 0\\ 0 & 1\end{pmatrix}\Phi = \chi_1(a)|a|_x^{1/2}\rho\begin{pmatrix}a & 0\\ 0 & 1\end{pmatrix}\Phi$$

or, more explicitly,

$$r_{\chi_1}\begin{pmatrix}a & 0\\ 0 & 1\end{pmatrix}\Phi(z_1, z_2) = \chi_1(a)|a|_x^{1/2}\Phi(az_1, z_2).$$

Then

$$\mathrm{Pt}\left(r_{\chi_1}\begin{pmatrix}a & 0\\ 0 & 1\end{pmatrix}\Phi\right) = \chi_1(a)|a|_x^{1/2}\rho\begin{pmatrix}a & 0\\ 0 & 1\end{pmatrix}\mathrm{Pt}(\Phi).$$

However, the representation $\pi(\chi)$ is not equivalent to r_{χ_1}; it is equivalent to a quotient of r_{χ_1}.

Now let $\Phi \in \mathcal{S}(E_x)$ and set $\Phi_0 = \operatorname{Pt}\Phi$. Then the function f defined by the convergent integral

$$f(g) = \chi_1(g)|\det g|_x^{1/2} \int_{F_x^\times} \Phi_0((0,t)g)\chi_1\chi_2^{-1}(t)|t|_x d^\times t$$

is in the space $\mathcal{B}(\chi_1, \chi_2)$. Any function $f \in \mathcal{B}(\chi_1, \chi_2)$ can be obtained this way for a suitable Φ_0 (or Φ). Computing formally, we get

$$\begin{aligned}\lambda(f) = \chi_1(-1) \int_{F_x} \int_{F_x^\times} \Phi_0 &\left((0,t) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \right) \\ &\qquad\qquad \times \chi_1\chi_2^{-1}(t)|t|_x d^\times t \psi(-u) du \\ = \chi_1(-1) \int_{F_x} \int_{F_x^\times} &\Phi(t, tu)\chi_1\chi_2^{-1}(t)|t|_x d^\times t\psi(-u)du\end{aligned}$$

Exchanging the order of integration and changing u to ut^{-1} we arrive at

$$= \chi_1(-1) \int_{F_x^\times} \int_{F_x} \Phi_0(t, u)\psi(-ut^{-1})du\chi_1\chi_2^{-1}(t)d^\times t.$$

From Fourier inversion formula we get

$$\int_{F_x} \Phi_0(t, u)\psi(-ut^{-1})du = \Phi(t, t^{-1}).$$

Thus we have

$$\lambda(f) = \chi_1(-1) \int_{F_x^\times} \Phi(t, t^{-1})\chi_1\chi_2^{-1}(t)d^\times t.$$

Replacing the linear form λ by the scalar multiple $\chi_1(-1)\lambda$ we get

$$\lambda(f) = \int_{F_x^\times} \Phi(t, t^{-1})\chi_1\chi_2^{-1}(t)d^\times t.$$

Thus we can write

$$\lambda(f) = \int_{U_1(F_x)} \Phi(u)\chi(u)du.$$

More generally, for $\det g = 1$ and $a \in F^\times$, we obtain the following formula:

$$\begin{aligned}\lambda &\left(\pi(\chi_1, \chi_2) \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \pi(\chi_1, \chi_2)(g)f \right) \\ &\qquad\qquad = \chi_1(a)|a|_x^{1/2} \int_{U_1(F_x)} (r(g)\Phi)\left((a,1)u\right)\chi(u)du,\end{aligned}$$

where the product $(a, 1)u$ is computed in the algebra $F_x \oplus F_x$. Thus we have now a description of the Whittaker model $\mathcal{W}(\pi(\chi), \psi_x)$ analogous to the the one for the inert case. We state this as a Lemma. For $\Phi \in \mathcal{S}(E_x)$

we define $W_{\Phi,\chi}$ by the following formula. Let $a \in F_x^\times$ and g with $\det g = 1$. Then

$$W_{\Phi,\chi}\left[\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} g\right] = \chi_1(a)|a|_x^{1/2} \int_{U_1(F_x)} (r(g)\Phi)\left((a,1)u\right)\chi(u)du.$$

We can make the analogy with the inert case complete. Let h be any element of $E_x^\times$ such that $a = \mathrm{Nm}(h)$ and g with $\det g = 1$. Then

$$W_{\Phi,\chi}\left[\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} g\right] = \chi(h)|\,\mathrm{Nm}(h)|_x^{1/2} \int_{U_1(F_x)} (r(g)\Phi)(hu)\chi(u)du. \tag{4.1}$$

Lemma 4.1. — *The space $\mathcal{W}(\pi(\chi), \psi_x)$ is the space spanned by the functions $W_{\Phi,\chi}$.*

4.2. Construction of a kernel

We let $d\chi$ be the Haar measure on $\widehat{E_x^\times} = \widehat{F_x^\times} \times \widehat{F_x^\times}$ dual to the measure $d^\times z$. So if we write $\chi = (\chi_1, \chi_2)$ then $d\chi = d\chi_1 d\chi_2$. We denote by $\mathcal{C}\big(\widehat{E_x^\times}\big)$ the space of functions P on $\widehat{E_x^\times}$ of the form

$$P(\chi) = \int_{E_x^\times} \chi^{-1}(h)\phi(h)d^\times h,$$

where ϕ is a function of compact support on $E_x^\times$, locally constant if x is finite, C^∞ if x is infinite. Thus the function $\widetilde{P}$ defined by

$$\widetilde{P}(h) = \int_{\widehat{E_x^\times}} P(\chi)\chi(h)d\chi \tag{4.2}$$

is equal to ϕ. For instance, if E_x/F_x is unramified, ψ_x is normalized and we take for ϕ the the characteristic function of $\mathcal{O}_{E_x}^\times = \mathcal{O}_{F_x}^\times \times \mathcal{O}_{F_x}^\times$, then P is the characteristic function of the set of unramified characters, i.e., the set of characters χ of the form $\chi = (\chi_1, \chi_2)$ with χ_1, χ_2 unramified.

Let $P \in \mathcal{C}\big(\widehat{E_x^\times}\big)$ and $\Phi \in \mathcal{S}(E_x)$. We define a function $K_{x,P,\Phi}$ on $E_x^\times \times GL(2, F_x)$ by the following formula:

$$K_{x,P,\Phi}(t,g) = \int_{\widehat{E_x^\times}} d\chi \cdot P(\chi) \cdot \chi(t)^{-1} \cdot W_{\Phi,\chi}(g). \tag{4.3}$$

By construction, for every character χ, the function

$$\int_{E_x^\times} d^\times t \cdot \chi(t) \cdot K_{P,\Phi}(t,g) = P(\chi) W_{\Phi,\chi}(g)$$

belongs to the Whittaker model $\mathcal{W}(\pi(\chi), \psi_x)$ of the representation $\pi(\chi)$. Just as in the inert case, formulas (3.6) and (3.7) give alternate expressions for this kernel. Then we have the following Proposition which is the analog of Proposition 3.1 and is proved in the same way.

PROPOSITION 4.2. — *Let ω be a character of $F_x^\times$ of absolute value* 1. *Let $g \in GL(2, F_x)$.*

(i) *The function*

$$\Psi_1(t) = \omega^{-1}(\mathrm{Nm}(t))|\,\mathrm{Nm}(t)|^{-1/2} \times \int_{F_x^\times} K_{x,P,\Phi}\left[t, \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} g\right]\omega(a)d^\times a$$

defined a priori on $E_x^\times$ extends by continuity to a Schwartz-Bruhat function on E_x.

(ii) *Similarly, the function*

$$\Psi_2(t) = \omega(\mathrm{Nm}(t))|\,\mathrm{Nm}(t)|^{-1/2} \times \int_{F_x^\times} K_{x,P,\Phi}\left[t^{-1}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} g\right]\omega(a)d^\times a$$

defined a priori on $E_x^\times$ extends by continuity to a Schwartz-Bruhat function on E_x.

(iii) *Denoting again by Ψ_1 and Ψ_2 the extensions, we have*

$$\omega(-1)\,\mathrm{Ft}(\Psi_1) = \Psi_2.$$

(iv) *Assume x is finite, the character ψ_x normalized and the character ω unramified. Assume further that P is the characteristic function of the set of unramified characters. Finally, assume Φ is the characteristic function of $\mathcal{O}_{E_x}$. Then $\omega(-1) = 1$ and $\Psi_1 = \Psi_2 = \Phi$.*

4.3. Complementary kernel

Let $U_1(F_x)^\perp$ be the set of characters ν of $E_x^\times$ of absolute value 1 which are trivial on $U_1(F_x)$. We may identify the group $U_1(F_x)^\perp$ to the group dual to the quotient $E_x^\times/U_1(F_x)$ and let $d\nu$ be the Haar measure dual to the quotient Haar measure on the group $E_x^\times/U_1(F_x)$. We have then the Poisson summation formula

$$\int_{U_1(F_x)^\perp} P(\nu)d\nu = \int_{U_1(F_x)} \widetilde{P}(u)du.$$

In fact, a character $\nu \in U_1(F_x)^\perp$ is simply a character of the form $\nu = (\chi_1, \chi_1)$ where χ_1 is a character of $F_x^\times$. In other words, for $t = (t_1, t_2)$ we have $\nu(t) = \chi_1(t_1t_2) = \chi_1(\mathrm{Nm}(t))$. The above formula can also be written in the more concrete form

$$\int_{\widehat{F_x^\times}} P(\chi_1,\chi_1)d\chi_1 = \int_{F_x^\times} \widetilde{P}(t,t^{-1})d^\times t.$$

As in the inert case, we have the more general formula

$$\int_{U_1(F_x)^\perp} P(\nu)\nu(t)d\nu = \int_{U_1(F_x)} \widetilde{P}(tu)du. \tag{4.4}$$

For $\nu \in U_1(F_x)^\perp$ and $\Phi \in \mathcal{S}(E_x)$ we define a function $f_{\Phi,\nu}$ by the following formula: for $a \in F_x^\times$ and $\det g = 1$,

$$f_{\Phi,\nu}\left[t, \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} g\right] = \nu(\mathrm{Nm}(h))|\,\mathrm{Nm}(h)|_x^{1/2}\,(r(g)\Phi)(0),$$

where h is any element of $E_x^\times$ such that $\mathrm{Nm}(h) = a$. The character ν has the form $\nu = (\chi_1,\chi_1)$ and we check at once that

$$f_{\Phi,\nu}\left[\begin{pmatrix} a_1 & x \\ 0 & a_2 \end{pmatrix} g\right] = \chi_1(a_1)\chi_1(a_2)|a|_x^{1/2} f_{\Phi,\nu}(g),$$

for all a_1, a_2, x and $g \in GL(2,F_x)$. Thus $f_{\Phi,\nu}$ belongs to the space $\mathcal{B}(\chi_1,\chi_1)$ of the induced representation $\pi(\chi_1,\chi_1)$. By definition, $\pi(\nu)$ is equivalent to the representation $\pi(\chi_1,\chi_1)$.

We define a new function $K^0_{x,P,\Phi}$ on $E_x^\times \times GL(2,F_x)$ by the following formula:

$$K^0_{x,P,\Phi}(t,g) = \int_{U_1(F_x)^\perp} d\nu \cdot P(\nu)\cdot \nu^{-1}(t)\cdot f_{\Phi,\nu}(g). \tag{4.5}$$

As in the inert case, using formula (4.4), we find the alternate expressions (3.14) and (3.15) for this kernel.

Just as in the inert case, we have then the following Proposition.

Proposition 4.3. — *Let ω be a character of $F_x^\times$ of absolute value 1. Let $g \in GL(2,F_x)$. Let Ψ_1 and Ψ_2 be the functions defined in Proposition 4.2.*

(i) *The expression*

$$\omega^{-1}(\mathrm{Nm}(t))|_x\,\mathrm{Nm}(t)|^{-1/2}\int_{F_x^\times} K^0_{x,P,\Phi}\left[t, \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} g\right]\omega(a)d^\times a$$

is independent of t and equal to $\Psi_1(0)$.

(ii) *Similarly, the expression*

$$\omega(\mathrm{Nm}(t))|\,\mathrm{Nm}(t)|_x^{-1/2} \times \int_{F_x^\times} K^0_{x,P,\Phi}\left[t^{-1}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} g\right]\omega(a)d^\times a$$

is independent of t *and equal to* $\Psi_2(0)$.

5. Construction of global kernels

5.1. Main results

We go back to the quadratic extension E/F of number fields or function fields. We give to $F_{\mathbb{A}}^{\times}$ the Haar measure $d^{\times}y$ product of the local Haar measure $d^{\times}y_x$. We define similarly a Haar measure $d^{\times}z$ on $E_{\mathbb{A}}^{\times}$. We let du be the Haar measure on $U_1(F_{\mathbb{A}})$ which is the product of the local Haar measures. As in the local situation, we have then the following integration formula. Let ϕ be a continuous function of compact support on $F_{\mathbb{A}}^{\times}$. Define a function ϕ_0 on $F_{\mathbb{A}}^{\times}$ by

$$\phi_0(a) = \int_{U_1(F_{\mathbb{A}})} \phi(hu)du$$

if $a = \mathrm{Nm}(h)$ and $\phi_0(a) = 0$ if a is not a norm. Then

$$\int_{F_{\mathbb{A}}\times} \phi_0(a)d^{\times}a = \int_{E_{\mathbb{A}}^{\times}} \phi(h)d^{\times}h. \tag{5.1}$$

As usual, for $a \in F_{\mathbb{A}}^{\times}$, we set $|a| = \prod_x |a_x|_x$.

We let (P_x) be a family of functions $P_x \in \mathcal{C}\big(\widehat{E_x^{\times}}\big)$. For almost all finite x, we assume that the function P_x is the characteristic function of the set of unramified characters of $E_x^{\times}$. Thus we can define a function P on $\widehat{E_{\mathbb{A}}^{\times}}$ by the formula $P(\chi) = \prod_x P_x(\chi_x)$ and a function $\widetilde{P}$ on $E_{\mathbb{A}}^{\times}$ by $\widetilde{P}(a) = \prod_x \widetilde{P_x}(a_x)$. Then

$$\int_{E_{\mathbb{A}}^{\times}} \widetilde{P}(h)\chi^{-1}(h)d^{\times}h = P(\chi).$$

We let Φ be a Schwartz-Bruhat function on $E_{\mathbb{A}}$ which is a product $\Phi = \prod \Phi_x$. For almost all finite x, the place x is unramified in E, the character ψ_x is normalized and the function Φ_x is the characteristic function of $\mathcal{O}_{E_x}$ in E_x. Then we define two functions on $E_{\mathbb{A}}^{\times} \times GL(2, F_{\mathbb{A}})$ by the following formulas:

$$K_{P,\Phi}(t, g) = \prod_x K_{x,P_x,\Phi_x}(t_x, g_x), \tag{5.2}$$

$$K_{P,\Phi}^0(t, g) = \prod_x K_{x,P_x,\Phi_x}^0(t_x, g_x). \tag{5.3}$$

With our choice of data, for a given g and t, almost all factors are equal to 1 in the two infinite products.

We set

$$\lambda = \prod_x \lambda(E_x/F_x, \psi_x),$$

the product being over all places x of F, or, what amounts to the same, over all inert places of F. In [5] Weil proves that, in fact, $\lambda = 1$, but we will not need this fact. Then we have the following Theorem.

Theorem 5.1. — *The product of the expression*

$$\sum_{\delta\in E^\times,\gamma\in F^\times} K_{P,\Phi}\left[\delta t, \begin{pmatrix}\gamma & 0\\ 0 & 1\end{pmatrix} g\right)] + \sum_{\gamma\in F^\times} K^0_{P,\Phi}\left[t, \begin{pmatrix}\gamma & 0\\ 0 & 1\end{pmatrix} g\right]$$

by λ is equal to the expression

$$\sum_{\delta\in E^\times,\gamma\in F^\times} K_{P,\Phi}\left[\delta t, \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\gamma & 0\\ 0 & 1\end{pmatrix} g\right] + \sum_{\gamma\in F^\times} K^0_{P,\Phi}\left[t, \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\gamma & 0\\ 0 & 1\end{pmatrix} g\right].$$

Proof. — At the cost of changing P we may assume $t = 1$. We may also change δ to δ^{-1} in the second expression. Thus we have to prove that the product of

$$\sum_{\delta\in E^\times,\gamma\in F^\times} K_{P,\Phi}\left[\delta, \begin{pmatrix}\gamma & 0\\ 0 & 1\end{pmatrix} g\right] + \sum_{\gamma\in F^\times} K^0_{P,\Phi}\left[1, \begin{pmatrix}\gamma & 0\\ 0 & 1\end{pmatrix} g\right]$$

by λ is equal to

$$\sum_{\delta\in E^\times,\gamma\in F^\times} K_{P,\Phi}\left[\delta^{-1}, \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\gamma & 0\\ 0 & 1\end{pmatrix} g\right] + \sum_{\gamma\in F^\times} K^0_{P,\Phi}\left[1, \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\gamma & 0\\ 0 & 1\end{pmatrix} g\right].$$

This is equivalent to proving that for every idèle class-character ω (of absolute value 1) the product of

$$\int_{F_{\mathbb{A}}^\times} \omega(a) d^\times a \left(\sum_{\delta\in E^\times} K_{P,\Phi}\left[\delta, \begin{pmatrix}a & 0\\ 0 & 1\end{pmatrix} g\right]\right) + \int_{F_{\mathbb{A}}^\times} \omega(a) d^\times a\ K_{P,\Phi}\left[1, \begin{pmatrix}a & 0\\ 0 & 1\end{pmatrix} g\right]$$

by λ is equal to

$$\int_{F_{\mathbb{A}}^{\times}} \omega(a)d^{\times}a \left(\sum_{\delta\in E^{\times}} K_{P,\Phi}\left[\delta^{-1}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} g\right]\right) + \int_{F_{\mathbb{A}}^{\times}} \omega(a)d^{\times}a\ K^0_{P,\Phi}\left[1, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} g\right].$$

For a given g and a given ω, we have at each place x the element g_x and the character ω_x to which we have associated Schwartz-Bruhat functions $\Psi_{1,x}$ and $\Psi_{2,x}$ (Proposition 3.1 and Proposition 4.2). For almost all finite x, $\Psi_{1,x}$ and $\Psi_{2,x}$ are equal to the characteristic function of $\mathcal{O}_{E_x}$. Thus the products $\Psi_1 = \prod_x \Psi_{1,x}$ and $\Psi_2 = \prod_x \Psi_{2,x}$ are Schwartz-Bruhat functions. Since $\omega(-1) = 1$, it follows from Proposition 3.1 and Proposition 4.2 that

$$\Psi_2 = \lambda\,\mathrm{Ft}(\Psi_1).$$

By Poisson summation formula (equation (2.2)) we have

$$\sum_{\delta\in E} \Psi_2(\delta) = \lambda \sum_{\delta\in E} \Psi_1(\delta). \tag{5.4}$$

We have

$$\begin{aligned}\int_{F_{\mathbb{A}}^{\times}} \omega(a)\, d^{\times}a \sum_{\delta\in E^{\times}} K_{P,\Phi}\left[\delta, \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} g\right] &= \sum_{\delta\in E^{\times}} \int_{F_{\mathbb{A}}^{\times}} \omega(a)d^{\times}a\ K_{P,\Phi}\left(\delta, \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} g\right) \\ &= \sum_{\delta\in E^{\times}} \omega(\mathrm{Nm}(\delta))|\,\mathrm{Nm}(\delta)|^{1/2}\Psi_1(\delta) \\ &= \sum_{\delta\in E^{\times}} \Psi_1(\delta),\end{aligned}$$

the last equality because the absolute value of an element of $F^{\times}$ is 1. We also have

$$\int_{F_{\mathbb{A}}^{\times}} \omega(a)d^{\times}aK^0_{P,\Phi}\left[1, \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} g\right] = \Psi_1(0).$$

Similarly,

$$\int_{F_{\mathbb{A}}^\times} \omega(a)d^\times a \sum_{\delta\in E^\times} K_{P,\Phi}\left[\delta^{-1}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} g\right]$$
$$= \sum_{\delta\in E^\times} \int_{F_{\mathbb{A}}^\times} \omega(a)d^\times a K_{P,\Phi}\left[\delta^{-1}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} g\right]$$
$$= \sum_{\delta\in E^\times} \Psi_2(\delta).$$

Similarly,

$$\int_{F_{\mathbb{A}}^\times} \omega(a)d^\times a K^0_{P,\Phi}\left[1, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} g\right] = \Psi_2(0).$$

Our assertion follows now from formula (5.4). However, we need to show that our expressions are absolutely convergent. This will be done in the next section. □

We can now state the second Theorem. We set

$$H_{P,\Phi}(t,g) = \sum_{\delta\in E^\times, \gamma\in F^\times} K_{P,\Phi}\left[\delta t, \begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix} g\right] + \sum_{\gamma\in F^\times} K^0_{P,\Phi}\left[t, \begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix} g\right].$$

Theorem 5.2. — *The function $H_{P,\Phi}(t,g)$ is invariant under $E^\times \times GL(2,F)$ on the left.*

Proof. — As a function of g it is clear that it is invariant on the left by the subgroup

$$\left\{\begin{pmatrix} \alpha & \beta \\ 0 & 1 \end{pmatrix} : \alpha\in F^\times, \beta\in F\right\}.$$

The identity just established proves that it is invariant on the left under the subgroup

$$\left\{\begin{pmatrix} 1 & 0 \\ \beta & \alpha \end{pmatrix} : \alpha\in F^\times, \beta\in F\right\}.$$

Since these two subgroups generate $GL(2,F)$ the invariance under $GL(2,F)$ follows.

For the invariance under $E^\times$ it suffices to prove that

$$\sum_{\gamma\in F^\times} K^0_{P,\Phi}\left[\delta, \begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix} g\right]$$

does not depend on $\delta \in E^\times$. It amounts to the same to prove that for every ω the integral

$$\int_{F_{\mathbb{A}}^\times} \omega(a)d^\times a K^0_{P,\Phi}\left[\delta, \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} g\right]$$

does not depend on δ. Indeed, it is equal to $\omega(\mathrm{Nm}(\delta))|\mathrm{Nm}(\delta)|^{1/2}\Psi_1(0) = \Psi_1(0)$. □

Remark. — For $t = 1$, the equality of Theorem 1 can be reformulated as the equality

$$H_{P,\Phi}\left[1, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} g\right] = \lambda H_{P,\Phi}(1, g).$$

Since the function $g \mapsto H_{P,\Phi}(1, g)$ is invariant under $GL(2, F)$ on the left (and not identically 0 for a suitable choice of the data) we see again that $\lambda = 1$.

6. Global theory

6.1. Relation with the global Weil representation

We need to justify our formal computations. First we summarize our construction. Let x be any place of F and $a \in F_x^\times$. If a has the form $a = \mathrm{Nm}(h)$ and $g_x \in SL(2, F_x)$ we have

$$K_{x,P_x,\Phi_x}\left[t, \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} g_x\right] = |\mathrm{Nm}(h)|_x^{1/2} \int_{U_1(F_x)} \widetilde{P_x}(hut^{-1})(r(g_x)\Phi_x)(hu)du.$$

If a is not a norm (which may happen only if x is inert), then the left hand side vanishes. Let $\Phi = \prod_x \Phi_x$ and $\widetilde{P} = \prod_x \widetilde{P}_x$. Following Weil we can define a representation r of $SL(2, F_\mathbb{A})$ on the space $\mathcal{S}(E_\mathbb{A})$ of Schwartz-Bruhat functions as the tensor product of the local Weil representations. Let $a \in F_\mathbb{A}^\times$ and $g \in SL(2, F_\mathbb{A})$. If $a = \mathrm{Nm}(h)$ then

$$K_{P,\Phi}\left[t, \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} g\right] = |a|^{1/2} \int_{U_1(F_\mathbb{A})} \widetilde{P}(ht^{-1}u)(r(g)\Phi)(hu)du,$$

where, we recall, du is the Haar measure on the group $U_1(F_\mathbb{A})$ product of the local measures. If a is not a norm, then the left-hand side vanishes.

We have a similar formula for $K^0_{P,\Phi}$. Let x be any place of F and $a \in F_x^\times$. If $a = \mathrm{Nm}(h)$, and $g_x \in SL(2, F_x)$ then

$$K^0_{x,P_x,\Phi_x}\left[t, \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} g_x\right] = |a|_x^{1/2} \int_{U_1(F_x)} \widetilde{P}(ht^{-1}u)du\ (r(g_x)\Phi_x)(0).$$

If a is not a norm, then the left hand side vanishes. Let $a \in F_\mathbb{A}^\times$ and $g \in SL(2, F_\mathbb{A})$. If $a = \mathrm{Nm}(h)$ then

$$K^0_{P,\Phi}\left[t, \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} g\right] = |a|^{1/2} \int_{U_1(F_\mathbb{A})} \widetilde{P}(ht^{-1}u)du\ (r(g)\Phi)(0).$$

If a is not a norm, then the left hand side vanishes.

Next, consider the formula defining

$$H_{P,\Phi}\left[t, \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} g\right]$$

where $\det g = 1$. Each term in the formula is 0 unless there is $\gamma \in F^\times$ such that γa is a norm at every place of F. It is so if and only if $\eta(a) = 1$. Assuming it is the case, we can write $a = \gamma_0 a_0$ with $\gamma_0 \in F^\times$ and $a_0 = \mathrm{Nm}(h)$ for some h. Then

$$H_{P,\Phi}\left[t, \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} g\right] = H_{P,\Phi}\left[t, \begin{pmatrix} a_0 & 0 \\ 0 & 1 \end{pmatrix} g\right].$$

Thus we are reduced to studying the formula for

$$H_{P,\Phi}\left[t, \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} g\right]$$

where $a = \mathrm{Nm}(h)$ for some h and $\det g = 1$. In the sums over $\gamma \in F^\times$ defining $H_{P,\Phi}$, γ must be a norm at every place, hence must be the norm of an element of $E^\times$. Thus we may write γ in the form $\gamma = \mathrm{Nm}(\epsilon)$ with $\epsilon \in E^\times / U_1(F)$. Then

$$\begin{aligned} H_{P,\Phi}\left[t, \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} g\right] = |a|^{1/2} & \sum_{\delta \in E^\times, \epsilon \in E^\times / U_1(F)} \\ & \times \int_{U_1(F_{\mathbb{A}})} \widetilde{P}(h\delta^{-1}\epsilon t^{-1} u)(r(g)\Phi)(h\epsilon u) du \\ & + |a|^{1/2} \sum_{\epsilon \in E^\times / U_1(F)} \int_{U_1(F_{\mathbb{A}})} \widetilde{P}(h\epsilon t^{-1} u)(r(g)\Phi)(0) du. \end{aligned}$$

In the first sum we change ϵ to $\epsilon\delta$ and then δ to $\delta\epsilon^{-1}$. We can then combine the two sums into one to arrive at the following formula:

$$|\,\mathrm{Nm}(h)|^{1/2} \sum_{\delta \in E, \epsilon \in E^\times / U_1(F)} \int_{U_1(F_{\mathbb{A}})} \widetilde{P}(h\epsilon t^{-1} u)(r(g)\Phi)(h\delta u) du.$$

This may also be written in the following form:

$$|\,\mathrm{Nm}(h)|^{1/2} \int_{U_1(F_{\mathbb{A}})/U_1(F)} du \left(\sum_{\epsilon \in E^\times} \widetilde{P}(h\epsilon t^{-1} u) \right) \left(\sum_{\delta \in E} (r(g)\Phi)(h\delta u) \right).$$

This expression is absolutely convergent. Indeed, there is $\Phi_0 \in \mathcal{S}(E_{\mathbb{A}})$, $\Phi_0 \geqslant 0$, such that

$$|(r(g)\Phi)(z)| \leqslant \Phi_0(z).$$

Moreover, from Poisson summation formula, we get

$$\sum_{\delta\in E}\Phi_0(t\delta)\leqslant C(1+|\operatorname{Nm}(t)|^{-1}).$$

Thus

$$\sum_{\delta\in E}|(r(g)\Phi)(h\delta u)|$$

is bounded above independently of u. On the other hand, since $\widetilde{P}$ has compact support and $U_1(F_{\mathbb{A}})/U_1(F)$ is compact, the sum over ϵ is finite and

$$\int_{U_1(F_{\mathbb{A}})/U_1(F)} du\left(\sum_{\epsilon\in E^\times}|\widetilde{P}|(h\epsilon t^{-1}u)\right)<+\infty.$$

Our assertion follows. Thus the formula defining $H_{P,\Phi}$ is an absolutely convergent series.

Let us observe also that the function μ defined by

$$\mu(h)=\sum_{\epsilon\in E^\times}\widetilde{P}(h\epsilon)$$

is a smooth function of compact support on $E_{\mathbb{A}}^\times/E^\times$. Suppose $\det g=1$ and $\eta(a)=1$. Let $\gamma\in F^\times$ such that $\gamma a=\operatorname{Nm}(h)$ for some h. Then

$$\begin{aligned}&H_{P,\Phi}\left[t,\begin{pmatrix}a&0\\0&1\end{pmatrix}g\right]\\&\qquad=|\operatorname{Nm}(h)|^{1/2}\int_{U_1(F_{\mathbb{A}})/U_1(F)}du\mu(ht^{-1}u)\left(\sum_{\delta\in E}(r(g)\Phi)(h\delta u)\right);\end{aligned}$$

if $\eta(a)=-1$ and $\det g=1$ then the left hand side is 0. By the theory of the global Weil representation, it is then easy to check directly the invariance properties of the function $H_{P,\Phi}$ on this formula. In particular, from $\lambda=1$, we get

$$r\begin{pmatrix}0&1\\-1&0\end{pmatrix}\Phi(z)=\operatorname{Ft}(\Phi)(\iota(z))$$

and the invariance under $\left(\begin{smallmatrix}0&1\\-1&0\end{smallmatrix}\right)$ follows from Poisson summation formula.

Next, from formula (5.1), we get, for $\det g=1$,

$$\begin{aligned}&\int_{F_{\mathbb{A}}^\times}\omega(a)d^\times aK_{P,\Phi}\left[t,\begin{pmatrix}a&0\\0&1\end{pmatrix}g\right]\\&\qquad=\int_{E_{\mathbb{A}}^\times}|\operatorname{Nm}(h)|^{1/2}\omega(\operatorname{Nm}(h))\widetilde{P}(ht^{-1})(r(g)\Phi)(h)d^\times h.\end{aligned}$$

Similarly,

$$\int_{F_{\mathbb{A}}^{\times}} \omega(a) d^{\times} a K^{0}_{P,\Phi}\left[t, \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} g\right]$$
$$= (r(g)\Phi)(0) \int_{E_{\mathbb{A}}^{\times}} |\operatorname{Nm}(h)|^{1/2} \omega(\operatorname{Nm}(h)) \widetilde{P}(ht^{-1}) d^{\times} h.$$

Since $\widetilde{P}$ has compact support, both integrals are absolutely convergent. We need also to check that

$$\sum_{\delta \in E^{\times}} \int_{F_{\mathbb{A}}^{\times}} \omega(a) d^{\times} a K_{P,\Phi}\left[\delta, \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} g\right]$$

is absolutely convergent. Computing formally, we get

$$\sum_{\delta \in E^{\times}} \int_{E_{\mathbb{A}}^{\times}} |\operatorname{Nm}(h)|^{1/2} \omega(\operatorname{Nm}(h)) \widetilde{P}(h\delta^{-1}) (r(g)\Phi)(h) d^{\times} h.$$

Changing h to $h\delta$ this becomes

$$= \int_{E_{\mathbb{A}}^{\times}} |\operatorname{Nm}(h)|^{1/2} \omega(\operatorname{Nm}(h)) \widetilde{P}(h) \left(\sum_{\delta \in E^{\times}} (r(g)\Phi)(h\delta) \right) d^{\times} h.$$

There is $\Phi_0 \in \mathcal{S}(E_{\mathbb{A}})$, $\Phi_0 \geqslant 0$, such that for h in the support of $\widetilde{P}$,

$$|(r(g)\Phi)(hz)| \leqslant \Phi_0(z).$$

Thus

$$\sum_{\delta \in E^{\times}} |(r(g)\Phi)(h\delta)| \leqslant \sum_{\delta \in E^{\times}} \Phi_0(\delta) < +\infty.$$

On the other hand,

$$\int_{E_{\mathbb{A}}^{\times}} |\operatorname{Nm}(h)|^{1/2} |\widetilde{P}(h)| d^{\times} h < +\infty.$$

Thus our expression is indeed absolutely convergent.

6.2. Conclusion

Fix a character χ of $E_{\mathbb{A}}^{\times}/E^{\times}$ of absolute value 1. Set

$$\phi_{P,\Phi}(g) = \int_{E_{\mathbb{A}}^{\times}/E^{\times}} d^{\times} t\, \chi(t)\, H_{P,\Phi}(t, g).$$

We first compute the Fourier expansion of $\phi_{P,\Phi}$. We have

$$\phi_{P,\Phi}(g) = \sum_{\gamma\in F^\times} \int_{E_{\mathbb{A}}^\times} d^\times t \chi(t) K_{P,\Phi}\left[t, \begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix} g\right] \tag{6.1}$$

$$+ \int_{E_{\mathbb{A}}^\times/E^\times} d^\times t \chi(t) \sum_{\gamma\in F^\times} K^0_{P,\Phi}\left[t, \begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix} g\right] \tag{6.2}$$

We have

$$\int_{E_{\mathbb{A}^\times}} K_{P,\Phi}(t,g)\chi(t) d^\times t = P(\chi) W_{\Phi,\chi}(g),$$

where $W_{\Phi,\chi}$ is the product of the functions W_{Φ_x,χ_x}. Thus the first term (6.1) is simply

$$P(\chi) \sum_{\gamma\in F^\times} W_{\Phi,\chi}\left[\begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix} g\right].$$

Let us compute the second term (6.2) for

$$g = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} g_0$$

where $\det g_0 = 1$. We find

$$\int_{E_{\mathbb{A}}^\times/E^\times} d^\times t \chi(t) \sum_{\epsilon,\gamma} K^0_{P,\Phi}\left[t, \begin{pmatrix} \mathrm{Nm}(\epsilon)\gamma a & 0 \\ 0 & 1 \end{pmatrix} g_0\right]$$

where the sums are for $\epsilon \in E^\times/U_1(F)$, $\gamma \in F^\times/\mathrm{Nm}(E^\times)$ with the extra condition that $\gamma a = \mathrm{Nm}(h)$ for some h. The sum over γ has at most one term. Assuming the sum is not empty, we find

$$(r(g_0)\Phi)(0)|\mathrm{Nm}(h)|^{1/2} \int_{E_{\mathbb{A}}^\times/E^\times} d^\times t \chi(t) \sum_{\epsilon} \int_{U_1(F_{\mathbb{A}})} \widetilde{P}(h\epsilon t^{-1} u) du.$$

Formal manipulations bring this to the form

$$(r(g_0)\Phi)(0)|\mathrm{Nm}(h)|^{1/2}\chi(h) \int_{E_{\mathbb{A}}^\times} \widetilde{P}(t^{-1})\chi(t) d^\times t \int_{U_1(F_{\mathbb{A}})/U_1(F)} \chi(u) du.$$

or

$$P(\chi)\ (r(g_0)\Phi)(0)\ |\mathrm{Nm}(h)|^{1/2}\ \chi(h)\ \int_{U_1(F_{\mathbb{A}})/U_1(F)} \chi(u) du\ .$$

The integral over $U_1(F_{\mathbb{A}})/U_1(F)$ is 0 unless the restriction of χ to $U_1(F_{\mathbb{A}})$ is trivial. Assuming this to be the case we have $\chi = \chi_1 \circ \mathrm{Nm} = \chi_2 \circ \mathrm{Nm}$ where χ_1, χ_2 are characters of $F_{\mathbb{A}}^\times/F^\times$ and $\chi_2 = \chi_1 \eta$. Then the function $f_{\Phi,\chi} = \prod f_{\Phi_x,\chi_x}$ is defined and the above expression is

$$P(\chi) \int_{U_1(F_{\mathbb{A}})/U_1(F)} du \sum_{\gamma} f_{\Phi,\chi}\left[\begin{pmatrix} \gamma a & 0 \\ 0 & 1 \end{pmatrix} g_0\right],$$

where the sum is for $\gamma \in F^\times / \operatorname{Nm}(E^\times)$ with the extra condition that $\gamma a = \operatorname{Nm}(h)$ for some h. The sum is not empty if an only if $\eta(a) = 1$ and then it has one term. Thus, in all cases,

$$\sum_\gamma f_{\Phi,\chi}\left[\begin{pmatrix}\gamma a & 0\\ 0 & 1\end{pmatrix} g_0\right] = |a|^{1/2}\ \chi_1(a)\frac{1+\eta(a)}{2} f_{\Phi,\chi}(g_0)$$
$$= |a|^{1/2}\ \frac{\chi_1(a)+\chi_2(a)}{2} f_{\Phi,\chi}(g_0).$$

By definition, the function $f_{\Phi,\chi}$ is a function on the group G^+ of $g \in GL(2,F_{\mathbb{A}})$ such that $\det g$ is a norm at each place. The function is further extended by 0 outside G^+. It has a unique extension $f^1_{\Phi,\chi}$ to $GL(2,F_{\mathbb{A}})$ satisfying

$$f^1_{\Phi,\chi}\left[\begin{pmatrix}a_1 & x\\ 0 & a_2\end{pmatrix} g\right] = \left|\frac{a_1}{a_2}\right|^{1/2} \chi_1(a_1)\chi_2(a_2) f^1_{\Phi,\chi}(g),$$

for all g, a_1, a_2, x; it has also a unique extension $f^2_{\Phi,\chi}$ satisfying

$$f^1_{\Phi,\chi}\left[\begin{pmatrix}a_1 & x\\ 0 & a_2\end{pmatrix} g\right] = \left|\frac{a_1}{a_2}\right|^{1/2} \chi_2(a_1)\chi_1(a_2) f^2_{\Phi,\chi}(g),$$

for all g, a_1, a_2, x. Then

$$|a|^{1/2}\ \frac{\chi_1(a)+\chi_2(a)}{2} f_{\Phi,\chi}(g_0)$$
$$= \frac{1}{2}\left(f^1_{\Phi,\chi}\left[\begin{pmatrix}a & 0\\ 0 & 1\end{pmatrix} g_0\right] + f^2_{\Phi,\chi}\left[\begin{pmatrix}a & 0\\ 0 & 1\end{pmatrix} g_0\right]\right).$$

Finally, we see that

$$\int_{E^\times_{\mathbb{A}}/E^\times} d^\times t \chi(t) \sum_{\gamma\in F^\times} K^0_{P,\Phi}\left[t, \begin{pmatrix}\gamma & 0\\ 0 & 1\end{pmatrix} g\right]$$
$$= P(\chi)\frac{\int_{U_1(F_{\mathbb{A}})/U_1(F)} du}{2} \left(f^1_{\Phi,\chi}(g) + f^2_{\Phi,\chi}(g)\right).$$

We have thus obtained the Fourier expansion of $\phi_{P,\Phi}$.

$$\phi_{P,\Phi}(g) = P(\chi)\sum_{\gamma\in F^\times} W_{\Phi,\chi}\left[\begin{pmatrix}\gamma & 0\\ 0 & 1\end{pmatrix} g\right]$$
$$+ P(\chi)\frac{\int_{U_1(F_{\mathbb{A}})/U_1(F)} \chi(u)du}{2} \left(f^1_{\Phi,\chi}(g) + f^2_{\Phi,\chi}(g)\right).$$

From this formula, it is clear than one can choose the data so that $\phi_{P,\Phi}$ is not identically 0.

Assume F is a number field. One can see that the function $\phi_{P,\Phi}$ is slowly increasing. In more detail, for s real, $s > 0$, denote by s_∞ the idèle of F such that $(s_\infty)_x = 1$ for x finite , $(s_\infty)_x = s$ for x infinite. Let Ω be a compact set of $GL(2, F_{\mathbb{A}})$ and $C > 0$. Let $\mathfrak{S}(C, \Omega)$ be the set of matrices g of the form

$$g = \begin{pmatrix} s_\infty & 0 \\ 0 & 1 \end{pmatrix} \omega, \ s \geqslant C, \omega \in \Omega.$$

The assertion $\phi_{P,\Phi}$ is slowly increasing means that, for every Ω and C, there exist D and n such that

$$|\phi_{P,\Phi}(g)| \leqslant Ds^n$$

for all $g \in \mathfrak{S}(C, \Omega)$. An estimate of this type is obtained in [1] for the first term. It is easy to obtain an estimate of this type for the second term.

Let U be the space spanned by the functions $\phi_{P,\Phi}$ and their translates. Let also $\mathcal{W}(\pi(\chi), \psi)$ be the space spanned by the functions $W_{\Phi,\chi}$ and their translates. If x is a finite place we have the representation $\pi(\chi_x)$ of $GL(2, F_x)$. If F is a function field one defines the representation $\pi(\chi)$ of $GL(2, F_{\mathbb{A}})$ as the tensor product of the local representations $\pi(\chi_x)$. The representation of $GL(2, F_{\mathbb{A}})$ on $\mathcal{W}(\pi(\chi), \psi)$ is then equivalent to $\pi(\chi)$. If F is a number field and x is an infinite place, we have a representation of $(\mathfrak{U}_x, K_x)$, that is, a $(\mathfrak{U}_x, K_x)-$module noted $\pi(\chi_x)$. Let ∞ be the set of infinite places. Set $K_\infty = \prod_{x \in \infty} K_x$ and let $\mathfrak{U}_\infty$ be the enveloping algebra of $\prod_{x \in \infty} GL(2, F_x)$. Let also $F_{\mathbb{A}}^\infty$ be the ring of finite adèles. The tensor product $\pi(\chi)$ of the local $\pi(\chi_x)$ is now a $GL(2, F_{\mathbb{A}}^\infty)-$module and a $(\mathfrak{U}_\infty, K_\infty)-$module. The space $\mathcal{W}_{K_\infty}(\pi(\chi), \psi)$ of $K_\infty-$finite vectors in $\mathcal{W}(\pi(\chi), \psi)$ is invariant under $GL(2, F_{\mathbb{A}}^\infty)$ and $(\mathfrak{U}_x, K_x)$. It is equivalent to $\pi(\chi)$, as a $GL(2, F_{\mathbb{A}}^\infty)-$module and a $(\mathfrak{U}_\infty, K_\infty)-$module. If ϕ is in U, it has a Fourier expansion

$$\phi(g) = \sum_{\gamma \in F^\times} W \left[\begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix} g \right] + \phi^0(g)$$

where $W \in \mathcal{W}(\pi(\chi), \psi)$ and

$$\phi^0(g) = \int_{F_{\mathbb{A}}/F} \phi \left[\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} g \right] du.$$

We claim that $W = 0$ implies $\phi = 0$. This is clear if $\phi^0 = 0$. If $\phi^0 \neq 0$ we find $\phi = \phi^0$. Since ϕ is invariant under $GL(2, F)$ and ϕ^0 invariant under the matrices $\left(\begin{smallmatrix} 1 & u \\ 0 & 1 \end{smallmatrix}\right)$ with $u \in F_{\mathbb{A}}$ this would imply ϕ^0 is invariant under $SL(2, F_{\mathbb{A}})$. This contradicts the explicit expression we have found for ϕ^0. Our assertion follows.

Thus we have an injective map $W \mapsto \phi$. As a consequence, in the function field case, the space U is invariant under $GL(2, F_{\mathbb{A}})$ and the representation of $GL(2, F_{\mathbb{A}})$ on U is equivalent to $\pi(\chi)$. Similarly, in the number field case, the space U_{K_∞} spanned by the functions ϕ with $W \in \mathcal{W}_{K_\infty}(\pi(\chi), \psi)$ is then invariant under $GL(2, F_{\mathbb{A}}^\infty)$ and $(K_\infty, \mathfrak{U}_\infty)$. As a $GL(2, F_{\mathbb{A}}^\infty)-$module and a $(\mathfrak{U}_\infty, K_\infty)-$module, the space V_{K_∞} is equivalent to $\pi(\chi)$. Thus we have proved that $\pi(\chi)$ is automorphic.

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