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Nikolai NIKOLSKI

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IN A SHADOW OF THE RH: CYCLIC VECTORS OF HARDY SPACES ON THE HILBERT MULTIDISC

by Nikolai NIKOLSKI

Abstract. — Completeness of a dilation system $(\varphi(nx))_{n\geqslant 1}$ on the standard Lebesgue space $L^2(0,1)$ is considered for 2-periodic functions φ. We show that the problem is equivalent to an open question on cyclic vectors of the Hardy space $H^2(\mathbb{D}_2^\infty)$ on the Hilbert multidisc $\mathbb{D}_2^\infty$. Several simple sufficient conditions are exhibited, which include however practically all previously known results (Wintner; Kozlov; Neuwirth, Ginsberg, and Newman; Hedenmalm, Lindquist, and Seip). For instance, each of the following conditions implies cyclicity of a function $f \in H^2(\mathbb{D}_2^\infty)$: 1) $f^{1+\epsilon} \in H^2(\mathbb{D}_2^\infty)$, $f^{-\epsilon} \in H^2(\mathbb{D}_2^\infty)$; 2) $Re(f(z)) \geqslant 0$, $z \in \mathbb{D}_2^\infty$; 3) $f \in Hol((1+\epsilon)\mathbb{D}_2^\infty)$ and $f(z) \neq 0$ on $\mathbb{D}_2^\infty$. The Riemann Hypothesis on zeros of the Euler ζ-function is known to be equivalent to a completeness of a similar but non-periodic dilation system (due to Nyman).

Résumé. — Il s'agit du problème de la complétude d'un système de dilatations $(\varphi(nx))_{n\geqslant 1}$ dans l'espace de Lebesgue $L^2(0,1)$ où φ est une fonction impaire 2-périodique. Sans utiliser les séries de Dirichlet, on montre que le problème est équivalent à une question ouverte sur les vecteurs cycliques dans l'espace de Hardy $H^2(\mathbb{D}_2^\infty)$ du multidisque $\mathbb{D}_2^\infty$ de Hilbert. Quelques conditions suffisantes de cyclicité sont établies, ce qui néanmoins inclut pratiquement tous les résultats précédents du sujet (ceux de Wintner; Kozlov; Neuwirth, Ginsberg, and Newman; Hedenmalm, Lindquist, and Seip). Par exemple, chacune des conditions suivantes entraîne la cyclicité d'une fonction f dans $H^2(\mathbb{D}_2^\infty)$: 1) $f^{1+\epsilon} \in H^2(\mathbb{D}_2^\infty)$, $f^{-\epsilon} \in H^2(\mathbb{D}_2^\infty)$; 2) $Re(f(z)) \geqslant 0$, $z \in \mathbb{D}_2^\infty$; 3) $f \in Hol((1+\epsilon)\mathbb{D}_2^\infty)$ et $f(z) \neq 0$ sur $\mathbb{D}_2^\infty$. L'Hypothèse de Riemann sur les zéros de la fonction ζ d'Euler est équivalente à un problème semblable de la complétude des dilatations (B.Nyman).

1. Introduction

Let $H^2 = H^2(\mathbb{D})$ be the Hardy space of the disc,

$$H^2 = \Big\{ f = \sum_{k=0}^{\infty} \hat{f}(n) z^n : \sum_{k=0}^{\infty} \Big|\hat{f}(n)\Big|^2 < \infty \Big\},$$

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and

$$H_0^2 = \Big\{ f \in H^2 : \hat{f}(0) = f(0) = 0 \Big\}.$$

Given $n \in \mathbb{N}$, where $\mathbb{N} = \{1, 2, \dots\}$ is regarded as a subsemigroup of a multiplicative group of positive reals $\mathbb{R}_+ = (0, \infty)$, we define an operator T_n on H_0^2 by

$$T_n f(z) = f(z^n), f \in H_0^2.$$

Clearly, T_n is an isometry on H_0^2 and $n \longmapsto T_n$ is a representation of $\mathbb{N}$ on H^2. We are interested in *cyclic vectors* of the semigroup (T_n), i.e. in functions $f \in H_0^2$ such that

$$\operatorname{span}_{H_0^2}(T_n f : n \in \mathbb{N}) = H_0^2,$$

where span_X means the closed (in X) linear hull. This interest is stimulated by the relations of invariant subspaces of the semigroup (T_n) with the zeros of the Riemann zeta function, see [22], [7], [6], [2], [3], [14]. We briefly describe below these links and a history of the problem. In fact, the problem of a description of the lattice of (closed) (T_n)-invariant subspaces

$$\operatorname{Lat}(T_n) = \{ E \subset H_0^2 : T_n E \subset E, \forall n \},$$

and in particular, (T_n)-cyclic vectors, is interesting and challenging in its own. Obviously, the function $f(z) = z$ is (T_n)-cyclic, and functions $f(z) = z^k$, $k > 1$, are not. It is less obvious that a function $f(z) = z(\lambda - z^k)$, $k \geqslant 0$, is cyclic for $|\lambda| \geqslant 1$ and not cyclic for $|\lambda| < 1$. And it is more involved that the functions $f_N = z(\lambda - z)^N$, $\lambda > 1$, are cyclic for small values of N ($N < \log 2 / \log(1 + \frac{1}{|\lambda|})$), and are not for N large enough ($N \geqslant \lambda$); for example, for $\lambda = 3$, f_2 is cyclic but f_3 is not.

We explain these and many other effects using a unitary transformation $f \longmapsto Uf$, which is a kind of dual eigenfunction representation of $f \in H_0^2$, generated by the eigenvector bundle of (T_n^*). In order to find eigenfunctions of (T_n^*), we denote consecutive prime numbers by

$$p_1 = 2, \quad p_2 = 3, \quad p_3 = 5, \quad \dots, \quad p_s, \quad \dots$$

and identify a number $n \in \mathbb{N}$ with an infinite sequence of nonnegative integers

$$\alpha(n) = (\alpha_1, \alpha_2, \alpha_3, \dots)$$

coming from the canonical representation of $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_s^{\alpha_s}$ (α_t are eventually zero (as $t \longrightarrow \infty$), so that the sequence α always has a finite support).

In fact, $\alpha : n \longmapsto \alpha(n)$ is a bijection from $\mathbb{N}$ onto the set

$$\mathbb{Z}_+(\infty) = \bigcup_{k\geqslant 1} \mathbb{Z}_+^k$$

of all finitely supported sequences of nonnegative integers. It is also a semi-group homomorphism from the multiplicative $\mathbb{N}$ to the additive $\mathbb{Z}_+(\infty)$.

Lemma. — *A nonzero element $f = \sum_{k=1}^{\infty} a_n z^n \in H_0^2$ is an eigenvector of (T_n^*), i.e. there exists a sequence (λ_n) of complex numbers such that*

$$T_n^* f = \lambda_n f, \forall n,$$

if and only if there is a sequence $\zeta = (\zeta_1, \zeta_2, \dots, \zeta_k, \dots)$ of complex numbers and a constant $c \neq 0$ such that $|\zeta_k| < 1$ ($\forall k$), $\sum_{k\geqslant 1} |\zeta_k|^2 < \infty$ and

$$a_n = c\zeta^{\alpha(n)}, \forall n,$$

where $\zeta^{\alpha(n)} = \zeta_1^{\alpha_1}\zeta_2^{\alpha_2}\dots\zeta_s^{\alpha_s}\dots$. In this case, $\lambda_n = \zeta^{\alpha(n)}$ for every $n \in \mathbb{N}$.

Proof. — Let f be an eigenvector of (T_n^*). By the definition,

$$\lambda_n(f, g) = (T_n^* f, g) = (f, g(z^n))$$

for every n and every $g \in H_0^2$, which implies $\lambda_n a_k = a_{nk}$ for every n and k. By induction, $a_n = a_1\zeta_1^{\alpha_1}\zeta_2^{\alpha_2}\dots\zeta_s^{\alpha_s}$, where (using the above notation) $n = p_1^{\alpha_1}p_2^{\alpha_2}\dots p_s^{\alpha_s}$ and $\zeta_j = \lambda_{p_j}$. The formula for λ_n follows from $T_n = T_{p_1}^{\alpha_1}T_{p_2}^{\alpha_2}\dots T_{p_s}^{\alpha_s}$. It remains to note (taking $c = 1$) that

$$\infty > \sum_{n\geqslant 1} |a_n|^2 = \sum_{\alpha\in\mathbb{Z}_+(\infty)} |\zeta^\alpha|^2 = \sup_{n\in\mathbb{N}} \sum_{(\alpha_1,\dots,\alpha_n)\in\mathbb{Z}_+^n} |\zeta_1^{\alpha_1}|^2 \dots |\zeta_n^{\alpha_n}|^2$$
$$= \sup_{n\in\mathbb{N}} \prod_{k=1}^{n} \Big(\frac{1}{1-|\zeta_k|^2}\Big) = \prod_{k\geqslant 1} \Big(\frac{1}{1-|\zeta_k|^2}\Big).$$

□

Having a (anti)*holomorphic vector bundle of eigenvectors* $f_{\overline{\zeta}} = \sum_{n\geqslant 1} \overline{\zeta}^{\alpha(n)} z^n$, we in a standard way can represent elements of H_0^2 as holomorphic functions, so that the action of T_n becomes a multiplication by independent variables:

$$U : f \longrightarrow Uf(\zeta) = (f, f_{\overline{\zeta}}),$$

where ζ runs over an infinite dimensional multidisc described in Lemma above. Formalizing this idea, we define the following transform, which was previously used by H. Bohr in studies of Dirichlet series, see [8]. The *Bohr transform* of $f \in H_0^2$ is

$$Uf(\zeta) = \sum_{n \geqslant 1} \hat{f}(n)\zeta^{\alpha(n)},$$

where $\zeta = (\zeta_1, \zeta_2, \dots) \in \mathbb{D}_2^\infty$ and $\zeta^\alpha = \zeta_1^{\alpha_1}\zeta_2^{\alpha_2}\dots\zeta_s^{\alpha_s}\dots$, and $\mathbb{D}_2^\infty$ stands for *the Hilbert multidisc*

$$\mathbb{D}_2^\infty = \{\zeta = (\zeta_1, \zeta_2, \dots) : \zeta \in l^2, |\zeta_j| < 1 \text{ for every } j \geqslant 1\}.$$

It is essentially known (see [8], [12], [14]) that U is well defined as a unitary transformation from H_0^2 onto the Hardy space of the Hilbert multidisc,

$$U : H_0^2 \longrightarrow H^2(\mathbb{D}_2^\infty),$$

$$H^2(\mathbb{D}_2^\infty) =: \Big\{ F = \sum_{\alpha \in \mathbb{Z}_+(\infty)} c_\alpha(F)\zeta^\alpha : \|F\|_2^2 = \sum_{\alpha \in \mathbb{Z}_+(\infty)} |c_\alpha(F)|^2 < \infty \Big\}.$$

For the readers convenience, we give a short proof in Lemma 2.1 below. Moreover, the following intertwining property holds

$$(UT_nU^{-1})f(\zeta) = \zeta^{\alpha(n)}f(\zeta)$$

for every n and every $\zeta \in \mathbb{D}_2^\infty$. It follows that

$$\mathrm{Lat}(T_n) = U^{-1}\,\mathrm{Lat}(M_\zeta),$$

which means that a (closed) subspace $E \subset H_0^2$ is T_n-invariant for every $n \in \mathbb{N}$ if and only if UE is ζ_k-invariant for every $k \in \mathbb{N}$ ($f \in UE \Rightarrow \zeta_k f \in UE$ for every $k \in \mathbb{N}$). In particular, *a function $f \in H_0^2$ is (T_n)-cyclic if and only if Uf is cyclic in $H^2(\mathbb{D}_2^\infty)$ with respect to multiplications by independent variables (the shift operators on $\mathbb{D}_2^\infty$)*, i.e. if and only if

$$H^2(\mathbb{D}_2^\infty) = \mathrm{span}\left(\zeta^\alpha Uf : \alpha \in \mathbb{Z}_+(\infty)\right) = \mathrm{clos}(Uf \cdot H^\infty(\mathbb{D}_2^\infty)),$$

where span and clos mean, respectively, the closed liner hull and the closure for the norm topology in $H^2(\mathbb{D}_2^\infty)$, and

$$H^\infty(\mathbb{D}_2^\infty) = \{F \in H^2(\mathbb{D}_2^\infty) : F \text{ is bounded on } \mathbb{D}_2^\infty\}.$$

The first result of Section 3 (known already to A. Beurling, [7]) says that the condition $Uf(\zeta) \neq 0$ for all $\zeta \in \mathbb{D}_2^\infty$ is necessary for (T_n)-cyclicity of a function $f \in H_0^2$. We show that it becomes sufficient if one supposes that the Fourier spectrum of f is finitely generated in $\mathbb{N}$ and $\hat{f}(n) = O(\frac{1}{n^\epsilon})$ for some $\epsilon > 0$. This generalizes a result by Neuwirth, Ginsberg, and Newman who showed it for a polynomial f, [12]. It is also shown that if a function $\zeta \longmapsto Uf(\zeta)$, $\zeta = (\zeta_1, \dots, \zeta_n, \dots)$, depends on a finite number of variables ζ_k only, say, on $(\zeta_1, \dots, \zeta_N) \in \mathbb{D}^N$, then f is (T_n)-cyclic if and only if Uf is cyclic in $H^2(\mathbb{D}^N)$. This provides us with some curious examples of cyclic/non-cyclic functions, see Section 4 below. Moreover, it is shown that each of the following conditions on a function $f \in H_0^2$ implies that f is (T_n)-cyclic:

1) $Uf \in H^2(\mathbb{D}_2^\infty), \frac{1}{Uf} \in H^\infty(\mathbb{D}_2^\infty)$ (is contained also in [14], with a different proof);

2) there exist $\epsilon > 0$ and $\delta > 0$ such that $Uf \in H^{2+\epsilon}(\mathbb{D}_2^\infty)$, $\frac{1}{(Uf)^\delta} \in H^2(\mathbb{D}_2^\infty)$;

3) $\mathrm{Re}(Uf(\zeta)) \geqslant 0$ for every $\zeta \in \mathbb{D}_2^\infty$ (and $f \neq 0$).

4) Uf depends on a finitely many variables ζ_i, $i = 1, \dots, N$, $Uf \in \mathrm{Hol}((1+\epsilon)\mathbb{D}^N)$ and $Uf(\zeta) \neq 0$ for $\zeta \in \mathbb{D}^N$ (is contained in [12] for the case of a polynomial f).

5) $Uf = Uf_1 \cdot Uf_2 \cdot Uf_3 \cdot Uf_4$, where f_i satisfies condition i) above, $i = 1, 2, 3, 4$.

Here $\mathrm{Hol}(\Omega)$ stands for the space of all holomorphic functions on an open set Ω, $\Omega \subset \mathbb{C}^N$. By the way, concerning claims 1)–3) above, it is of interest to notice that the condition $f \in H^\infty(\mathbb{D})$ does not imply $Uf \in H^\infty(\mathbb{D}_2^\infty)$, see Remark to Point (8), Section 2. We also consider criteria of (T_n)-cyclicity for polynomials $p = \sum_{n=1}^N a_k z^k$ of low degrees in more explicit terms of the coefficients a_k.

All proofs below are short and elementary. In particular, we have an easy proof for the fact that a function $f = \sum_{k=1}^\infty \hat{f}(n) z^n$ with $\hat{f}(1) = 1$ and $\sum_{k=2}^\infty |\hat{f}(n)| \leqslant 1$ is cyclic (this is one of important examples in [14], where the proof is different of ours). Similarly, Wintner's result [28] establishing (T_n)-cyclicity of functions $f = \sum_{n \geqslant 1} \frac{z^n}{n^s}$, $Re(s) > 1/2$, is a simple consequence of proposition 2) above. Also, the author is aware of the existence of a huge litterature on the function theory in polydiscs and other domains in $\mathbb{C}^n$, but he found almost no information on cyclic functions. We quote however several papers on polydisc invariant subspace theory, [1], [10], [20], [25], [26], [12], [24], [28]. A few of more specific comments as well as historical remarks are collected below.

An Abridged Story of an Invariant Subspace Approach to the RH

B. Nyman, in his thesis of 1950 [22], established the following equivalence. *Let $\rho(x) = x - [x]$ for $x \in \mathbb{R}$ (fractional part of x),*

$$\varphi(x) = \rho(1/x) \text{ for } x \in (0, 1).$$

Then the following are equivalent.

(1) *All zeros of the Euler ζ-function $\zeta(s) = \sum_{n\geqslant 1} \frac{1}{n^s}$, after a holomorphic extension to $\{s \in \mathbb{C} : \mathrm{Re}(s) > 0, s \neq 1\}$, are on the line* $\mathrm{Re}(s) = \frac{1}{2}$.

(2) $\mathrm{span}_{L^2(0,1)}(\varphi(tx) : t \geqslant 1) = L^2(0,1)$

(3) $\chi_{(0,1)} \in \mathrm{span}_{L^2(0,\infty)}(\varphi(tx) : t \geqslant 1)$, *where χ_A stands for the characteristic function of A.*

For our days, the proof is quite simple. Indeed, after the Fourier-Mellin transform, the closure in the left hand side of (2) becomes a translation invariant subspace of the Hardy space $H^2(\mathbb{C}_+)$, whose characteristic inner function, say θ, is expressed in terms of the ζ-function (via the known representation

$$\frac{1}{s-1} - \frac{\zeta(s)}{s} = \int_0^\infty \rho(t) t^{-s-1}\, dt$$

for $\mathrm{Re}(s) > 0$); identifying the zeros of θ with $\{s : \zeta(s + \frac{1}{2}) = 0, \mathrm{Re}(s) > 0\}$ and observing the absence of singular inner factors in θ (since ζ is holomorphic on $\mathrm{Re}(s) = 1/2$), one obtains the result. □

In fact, Nyman has used an approximation by linear combinations $f(x) = \sum_j a_j \varphi(t_j x)$ satisfying $f(1) = \sum_j a_j / t_j = 0$. But since the functional $L : f \longmapsto f(1)$ is discontinuous on $L^2(0,1)$, it is easy to see that the above form of Nyman's result is equivalent to the original one (indeed, if the set X of all linear combinations (as in (2) above) is dense in $L^2(0,1)$, then the restriction $L|X$ is unbounded, and hence $Ker(L|X)$ is dense in X, and so in $L^2(0,1)$). This question is also discussed in [3].

More than 40 years later, L. Báez-Duarte [2] showed (with a more involved reasoning) that Nyman's (1)–(3) are equivalent to

(4) $\chi_{(0,1)} \in \mathrm{span}_{L^2(0,\infty)}(\varphi(nx) : n \in \mathbb{N})$.

For more details about Nyman's approach to the RH, we refer to an interesting survey by M.Balazard [5]. The above form of Nyman's result suggests the following general dilation completeness problem.

Dilation Completeness Problem. To characterize functions $f \in L^p(0,\infty)$ such that

$$\mathrm{span}_{L^p(0,1)}(f(nx) : n \in \mathbb{N}) = L^p(0,1).$$

Extending the problem, we can also ask the same question but replacing $(0,1)$ by another subset $E \subset (0,\infty)$ and a subsemigroup trajectory $(D_n f : n \in \mathbb{N}), D_n f(x) = f(nx)$, by just a family of dilations $(f(tx) : t \in T)$ with a set $T \subset (0,\infty)$ given in advance.

It is also of interest to describe all (D_n)-invariant subspaces of $L^p(0,\infty)$, i.e., closed subspaces $X \subset L^p(0,\infty)$ such that $D_n X \subset X$ for every $n \geqslant 1$.

In particular, to know when the characteristic function $\chi_{(0,1)}$ belongs to such a subspace. Following B. Nyman and L. Báez-Duarte, the Riemann Hypothesis corresponds to the case $f = \varphi$, $T = \mathbb{N}$, $p = 2$.

An interesting partial case of the DCP is the following *Periodic DCP*, which was initiated much earlier by A. Wintner (1944) and (independently) A. Beurling (1945).

Periodic Dilation Completeness Problem. Given positive reals $a > 0$, $b > 0$, to characterize a-periodic functions f on $(0, \infty)$, $f \in L^p(0, a)$, such that

(2′) $\mathrm{span}_{L^p(0,b)}(f(nx) : n \in \mathbb{N}) = L^p(0, b)$.

A. Wintner 1944 [28] showed that a sequence $\{f(kx) : k \in \mathbb{N}\}$ is complete in $L^2(0, 1)$ for $f = \rho(x/2)$ (so, $a = 2$, $b = 1$), and — more general — this is the case for

$$f_s = \sum_{n \geqslant 1} n^{-s} \sin(n\pi x)$$

with $\mathrm{Re}(s) > 1/2$. This result is a simple consequence of Theorem 3.3 below.

A. Beurling 1945 [7] observed that the condition $Uf(\zeta) \neq 0$ for $\zeta \in \mathbb{D}_2^\infty$ is necessary for $(T_n f)_{n\geqslant 1}$ to be complete in H_0^2 (in fact, Beurling used the language of dilations D_n). J. Neuwirth, J. Ginsberg, and D. Newman 1970 [12] have established that a polynomial f, $\deg(f) \leqslant N$, is (T_n)-cyclic if and only if $Uf(\zeta) \neq 0$ for $\zeta \in \mathbb{D}^N$. H. Hedenmalm, P. Lidquist, and K. Seip, [14], proved point 1) above, as well as the (T_n)-cyclicity of $f = z + \sum_{n\geqslant 2} a_k z^k$ satisfying $\sum_{n\geqslant 2} |a_k| \leqslant 1$ and a number of results on Riesz bases in H_0^2 of the form $(T_n f)_{n\geqslant 0}$. Their statements are given in a language of multipliers of a space of Dirichlet series $\sum_{n\geqslant 1} \frac{a_n}{n^s}$, $\sum_{n\geqslant 1} |a_n|^2 < \infty$. All results on the PDCP listed above are simple partial cases of results proved in this paper. The link between the Wintner–Beurling version of the PDCP and the semigroup (T_n) is very simple: taking an orthonormal basis $\{\sqrt{2}\sin(\pi kx)\}_{k\geqslant 1}$ in $L^2(0, 1)$ and defining a unitary operator $V : H_0^2 \longrightarrow L^2(0, 1)$ by $(Vz^k)(x) = \sqrt{2}\sin(\pi kx)$, $k = 1, 2, \ldots$, one can easily see that a function $\varphi \in L^2(0, 1)$,

$$\varphi = \sum_{k\geqslant 1} a_k \sqrt{2}\sin(\pi kx),$$

(developed in its Fourier series) satisfies $\varphi(nx) = (VT_nV^{-1}\varphi)(x)$, and hence it obeis completeness property (2') if and only if a function

$$F = V^{-1}\varphi = \sum_{k\geqslant 1} a_k z^k$$

is (T_n)-cyclic in H_0^2.

Therefore, tempting to approach to the Riemann Hypothesis by Nyman-Wintner-Beurling approximations, one can hope on Hardy space techniques in $\mathbb{D}_2^\infty$ and specific arithmetic properties of the Fourier coefficients $a_k = a_k(\varphi)$ for the function we need to treat, namely for $\varphi(x) = \rho(1/x)$ (as it was the case for Wintner's results [28], where $k \longmapsto a_k$ was a contractive semicharacter of the multiplicative semigroup $\mathbb{N}$, see Section 3 for details). Do not forget, however, that in the RH we deal mostly with the DCP, and not with the PDCP. It is also worth mentioning that the completeness problem for *all dilations* $f(tx)$, $t > 0$, is usually settled with classical techniques (say, Plancherel-Mellin theorem for L^2-approximation, and the Wiener Tauberian Theorem for L^1-approximation), whereas the *discrete semigroup* (T_n) requires more sophisticated tools.

Acknowledgements

I am indebted to my former colleague at the University Bordeaux 1 Michel Balazard, who initiated my interest to the semigroup (T_n) in the beginning of the 1990s, asking me several interesting questions. Most of the results presented here was obtained shortly after but have waited for more than 15 years in order to be taped. I am also grateful to the referee for a careful reading the manuscript.

2. Semigroup (T_n) and the Hardy space on the Hilbert multidisc

This Section deals with several elementary properties of functions in $H^2(\mathbb{D}_2^\infty)$, some of which were observed already by D.Hilbert [16]. We would like to avoid technical subtleties, unnecessary for our goals and related to analysis of functions of infinitely many variables. This is why we do not touch more general H^p spaces ($p \neq 2$), for which even the question on the natural domain of analyticity is not completely obvious. See a few of remarks below.

The following lemma is mostly known, see [14]. The spaces H_0^2 and $H^2(\mathbb{D}_2^\infty)$, as well as the mapping U are defined in the Introduction.

Lemma 2.1. — 1. *For every $f \in H^2_0$ and $\zeta \in \mathbb{D}^\infty_2$, the series $Uf(\zeta) = \sum_{n\geqslant 1} \hat{f}(n)\zeta^{\alpha(n)}$ is absolutely convergent, and one has*

$$|Uf(\zeta)| \leqslant \|f\|_2 \prod_{k\geqslant 1} \Big(\frac{1}{1-|\zeta_k|^2}\Big)^{1/2}.$$

2. *U is a unitary mapping from H^2_0 onto $H^2(\mathbb{D}^\infty_2)$, which maps an orthonormal basis $(z^n)_{n\geqslant 1}$ of H^2_0 on the orthonormal basis $(\zeta^\alpha)_{\alpha\in\mathbb{Z}_+(\infty)}$ of $H^2(\mathbb{D}^\infty_2)$.*

3. *For every $f \in H^2_0$, $\zeta \in \mathbb{D}^\infty_2$ and $n \in \mathbb{N}$,*

$$(UT_n f)(\zeta) = \zeta^{\alpha(n)}(Uf)(\zeta)$$

where $\alpha(n) = (\alpha_1(n), \dots, \alpha_k(n), \dots)$ is defined in the Introduction.

Proof. — 1. By Cauchy-Schwarz,

$$\begin{aligned}|Uf(\zeta)|^2 \leqslant \|f\|_2^2 \sum_{\alpha\geqslant 0} |\zeta^\alpha|^2 &= \|f\|_2^2 \sup_{n\in\mathbb{N}} \sum_{(\alpha_1,\dots,\alpha_n)\in\mathbb{Z}^n_+} |\zeta_1^{\alpha_1}|^2 \dots |\zeta_n^{\alpha_n}|^2 \\ &= \|f\|_2^2 \sup_{n\in\mathbb{N}} \prod_{k=1}^{n} \Big(\frac{1}{1-|\zeta_k|^2}\Big) \\ &= \|f\|_2^2 \prod_{k\geqslant 1} \Big(\frac{1}{1-|\zeta_k|^2}\Big).\end{aligned}$$

2. By definition, $Uf(\zeta) = \sum_{n\geqslant 1} \hat{f}(n)\zeta^{\alpha(n)}$, and hence $Uz^n = \zeta^{\alpha(n)}$, $n \in \mathbb{N}$. Since α is a bijection from $\mathbb{N}$ to $\mathbb{Z}_+(\infty)$, the result follows.

3. Since α is a homomorphisme,

$$(UT_n z^k)(\zeta) = (Uz^{kn})(\zeta) = \zeta^{\alpha(kn)} = \zeta^{\alpha(n)}\zeta^{\alpha(k)} = \zeta^{\alpha(n)}(Uz^k)(\zeta)$$

for every $k \in \mathbb{N}$. Since $f \longmapsto Uf(\zeta)$ is linear and bounded on H^2_0 (see point 1 above), we get the result. □

Corollary 2.2. — *Let E be a (closed) subspace of H^2_0.*

$$E \in \mathrm{Lat}(T_n) \Leftrightarrow UE \in \mathrm{Lat}(M_\zeta).$$

A function $f \in H^2_0$ is (T_n)-cyclic if and only if Uf is M_ζ-cyclic in $H^2(\mathbb{D}^\infty_2)$. □

Now, since the problem of (T_n)-invariant subspaces is reduced to the Hardy space $H^2(\mathbb{D}^\infty_2)$, we need to recall some properties of the latter. Since the author is not able to localise references (if existed), we are giving a list of these properties with short proofs (where the statement is not completely obvious). In fact, all properties are natural analogues of the corresponding properties of $H^2(\mathbb{D})$ and $H^2(\mathbb{D}^n)$.

2.1. A few general properties of $H^2(\mathbb{D}_2^\infty)$ functions. **(1)** A polynomial p is a finite linear combination of monomials: $p = \sum_{\alpha\in\sigma} c_\alpha \zeta^\alpha$, where $c_\alpha \in \mathbb{C}$ and σ is finite, $\sigma \subset \mathbb{Z}_+(\infty)$. *The set of polynomials $\mathcal{P}$ is norm dense in $H^2(\mathbb{D}_2^\infty)$.* □

(2) Let $F \in H^2(\mathbb{D}_2^\infty)$. Denote

$$F_{(n)}(\zeta) = F(\zeta_1, \dots, \zeta_n, 0, 0, \dots), \zeta \in \mathbb{D}_2^\infty$$

and $\mathbb{Z}_+(n) = \{\alpha \in \mathbb{Z}_+(\infty) : \alpha_j = 0 \text{ for } j > n\}$; since $F_{(n)}$ does not depend on ζ_j, $j > n$, we sometimes will also write $F_{(n)}(\zeta) = F_{(n)}(\zeta_1, \dots, \zeta_n)$. Then,

$$F_{(n)} = \sum_{\alpha\in\mathbb{Z}_+(n)} c_\alpha(F)\zeta^\alpha,$$

$$\|F_{(n)}\|_2^2 = \sum_{\alpha\in\mathbb{Z}_+(n)} |c_\alpha(F)|^2, \quad \lim_n \|F - F_{(n)}\|_2 = 0, \|F\|_2 = \sup_{n\geqslant 1} \|F_{(n)}\|_2,$$

and also,

$$\|F_{(n)}\|_2^2 = \sup_{0<r<1} \int_{\mathbb{T}^n} |F_{(n)}(r\zeta_1, \dots, r\zeta_n)|^2 \, dm(\zeta_1) \dots dm(\zeta_n),$$

$$\|F\|_2^2 = \sup_{n\geqslant 1} \sup_{0<r<1} \int_{\mathbb{T}^n} |F_{(n)}(r\zeta_1, \dots, r\zeta_n)|^2 \, dm(\zeta_1) \dots dm(\zeta_n)$$

where $\mathbb{T}^n = \mathbb{T} \times \cdots \times \mathbb{T}$, $\mathbb{T}$ being the unit circle $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$.

Conversely, if (f_n) is a sequence of functions $f_n \in H^2(\mathbb{D}^n)$, such that $f_{n+1}(\zeta, 0) = f_n(\zeta)$ for $\zeta \in \mathbb{D}^n$ and $\sup_n \|f_n\|_2 < \infty$, then there exists a (unique) function $F \in H^2(\mathbb{D}_2^\infty)$ such that $f_n = F_{(n)}$ for every $n \geqslant 1$.

Indeed, all claims are easily verified, including the last one, where we simply define F by $\hat{F}(\alpha) = \hat{f}_n(\alpha)$ for $\alpha \in \mathbb{Z}_+(n)$. In principle, this last assertion is a partial case of results of D. Hilbert [16]. □

(3) *We have $H^\infty(\mathbb{D}_2^\infty) \subset H^2(\mathbb{D}_2^\infty)$. More precisely,*

$$\|F\|_2 \leqslant \|F\|_\infty$$

for $F \in H^\infty(\mathbb{D}_2^\infty)$, where $\|F\|_\infty = \sup\{|F(\zeta)| : \zeta \in \mathbb{D}_2^\infty\}$.

Indeed, it follows from (2) that $\|F_{(n)}\|_2 \leqslant \|F\|_\infty$ for every n, and hence $\|F\|_2 \leqslant \|F\|_\infty$. □

(4) *The reproducing kernel for $H^2(\mathbb{D}_2^\infty)$.* Let

$$k_\lambda(\zeta) = \sum_{\alpha\in\mathbb{Z}_+(\infty)} \overline{\lambda}^\alpha \zeta^\alpha,$$

where $\lambda, \zeta \in \mathbb{D}_2^\infty$ and $\overline{\lambda} = (\overline{\lambda}_1, \dots, \overline{\lambda}_n, \dots)$. Then $k_\lambda \in H^2(\mathbb{D}_2^\infty)$, $\|k_\lambda\|_2^2 = \prod_{n\geqslant 1} \frac{1}{1-|\lambda_n|^2}$ and, for every $F \in H^2(\mathbb{D}_2^\infty)$,

$$F(\lambda) = (F, k_\lambda).$$

Indeed, the reproducing property of k_λ is obvious by the definition of the scalar product in $H^2(\mathbb{D}_2^\infty)$, $(F,G) = \sum_{\alpha\in\mathbb{Z}_+(\infty)} c_\alpha(F)\overline{c}_\alpha(G)$. The norm of k_λ is computed as in Lemma 2.1. □

Remark. — It is curious to mention that $k_\lambda \in H^\infty(\mathbb{D}_2^\infty)$ *if and only if* $\sum_{n\geqslant 1} |\lambda_n| < \infty$. Moreover, since

$$\sum_{\alpha\in\mathbb{Z}_+^n} |\lambda_1^{\alpha_1}\lambda_2^{\alpha_2}\dots\lambda_n^{\alpha_n}| = \lim_{r\longrightarrow 1} k_\lambda\Big(\frac{r\lambda_1}{|\lambda_1|}, \dots, \frac{r\lambda_n}{|\lambda_n|}, 0, 0, \dots\Big),$$

we get

$$\sup_{\zeta\in\mathbb{D}_2^\infty} |k_\lambda(\zeta)| = \sup_n \sup_{\zeta\in\mathbb{D}^n} \Big| \sum_{\alpha\in\mathbb{Z}_+(n)} \overline{\lambda}^\alpha \zeta^\alpha \Big| = \sup_n \sum_{\alpha\in\mathbb{Z}_+^n} |\lambda_1^{\alpha_1}\lambda_2^{\alpha_2}\dots\lambda_n^{\alpha_n}| =$$
$$= \sup_n \prod_{k=1}^n \frac{1}{1-|\lambda_k|} = \prod_{k=1}^\infty \frac{1}{1-|\lambda_k|}. \quad \square$$

(5) *If $F \in H^\infty(\mathbb{D}_2^\infty)$ then $F_{(n)} \in H^\infty(\mathbb{D}_2^\infty)$($F_{(n)}$ is defined in (2)) and*

$$\|F_{(n)}\|_\infty \leqslant \|F_{(n+1)}\|_\infty \leqslant \|F\|_\infty, \qquad \lim_n F_{(n)}(\lambda) = F(\lambda)$$

for every $\lambda \in \mathbb{D}_2^\infty$ (the latter is true for every $F \in H^2(\mathbb{D}_2^\infty)$).

Indeed, it is clear from the definition of $F_{(n)}$. In fact, $F_{(n)}$ can be seen as a restriction of F on $\mathbb{D}^n$. The claimed convergence property follows from (2). □

(6) *Clearly,*

$$H^2(\mathbb{D}_2^\infty) = \operatorname{span}_{H^2}(k_\lambda : \lambda \in \mathbb{D}_2^\infty)$$

and hence, a sequence (F_n) converges weakly in $H^2(\mathbb{D}_2^\infty)$ to a function $F \in H^2(\mathbb{D}_2^\infty)$ if and only if $\lim_n F_n(\lambda) = F(\lambda)$ for every $\lambda \in \mathbb{D}_2^\infty$ and $\sup_n \|F_n\|_2 < \infty$. □

(7) *For every $\varphi \in H^\infty(\mathbb{D}_2^\infty)$, there exists a sequence of polynomials $(p_n)_{n\geqslant 1}$ such that $\lim_n p_n(\zeta) = \varphi(\zeta)$ for $\zeta \in \mathbb{D}_2^\infty$ and $\|p_n\|_\infty \leqslant \|\varphi\|_\infty$ for every $n \geqslant 1$.*

Indeed, first, we can take a sequence $(\varphi_{(n)})$ from (5); next, we pass to Fejer polynomials p_n for $\varphi_{(n)}$ of sufficiently large degree $N = N(n)$. Since the Hilbert multidisc $\mathbb{D}_2^\infty$ is not locally compact (and we do not know any kind of Vitali's theorem), let us enter in some details. Precisely, we exploit a property of Fejer approximation maps on $\mathbb{D}^n$, $f \longmapsto (f - \Phi_{N,n} * f)$,

$N = 1, 2, \ldots$, that their restrictions to a compact set $\Delta\overline{\mathbb{D}}^n$, $0 < \Delta < 1$ (denote them by $\Phi_{N,n,\Delta} : f \longmapsto (f - \Phi_{N,n} * f)|\Delta\mathbb{D}^n$) tend to zero *for the operator norm* $H^2(\mathbb{D}^n) \longrightarrow H^\infty(\Delta\mathbb{D}^n)$, for every Δ, i.e.

$$\lim_{N\to\infty} \|\Phi_{N,n,\Delta}\| = \lim_{N\to\infty} \sup_{\|f\|_2 \leqslant 1} \|f - \Phi_{N,n} * f\|_{H^\infty(\Delta\mathbb{D}^n)} = 0.$$

Using this property, we obtain for $p_n(\zeta) = \Phi_{N,n} * \varphi_{(n)}(\zeta)$,

$$\begin{aligned} |\varphi(\zeta) - p_n(\zeta)| &\leqslant |\varphi(\zeta) - \varphi_{(n)}(\zeta)| + |\varphi_{(n)}(\zeta) - \Phi_{N,n} * \varphi_{(n)}(\zeta)| \\ &\leqslant \|\varphi - \varphi_{(n)}\|_2 \|k_\zeta\|_2 + \|\Phi_{N,n,\Delta}\| \cdot \|\varphi_{(n)}\|_2, \end{aligned}$$

where $\Delta = \Delta(\zeta) = \max_j |\zeta_j| < 1$. Now, it is clear that there exists a sequence $N = N(n) \longrightarrow \infty$ such that $\lim_n |\varphi(\zeta) - p_n(\zeta)| = 0$ for every $\zeta \in \mathbb{D}_2^\infty$ (and even uniformly on the sets $\{\zeta \in \mathbb{D}_2^\infty : \|k_\zeta\| \leqslant A, \Delta(\zeta) \leqslant \Delta < 1\}$). □

Remark. — It is curious to note that $F \in H^2(\mathbb{D}_2^\infty)$ *does not imply* $F_r \in H^\infty(\mathbb{D}_2^\infty)$ for $0 < r < 1$ (as it is the case on $\mathbb{D}^n$). For example, if $\lambda \in \mathbb{D}_2^\infty$ but $\sum |\lambda_j| = \infty$, then $(k_\lambda)_r = k_{r\lambda} \notin H^\infty(\mathbb{D}_2^\infty)$ (see Remark to (4), Section 2).

(8) *The space of multipliers of* $H^2(\mathbb{D}_2^\infty)$*,* $Mult(H^2(\mathbb{D}_2^\infty)) =: \{\varphi : F \in H^2(\mathbb{D}_2^\infty) \Rightarrow \varphi F \in H^2(\mathbb{D}_2^\infty)\}$*, is* $H^\infty(\mathbb{D}_2^\infty)$.

Indeed, (2) implies that every $\varphi \in H^\infty(\mathbb{D}_2^\infty)$ is a multiplier and $\|\varphi\|_{Mult} \leqslant \|\varphi\|_\infty$. The converse is a well-known common place — every multiplier φ is bounded (whatever the basic space is) —

$$|\varphi(\lambda)| = |(\varphi^n \cdot 1)(\lambda)|^{1/n} \leqslant (\|\varphi\|_{Mult}^n \|1\|_2 \|k_\lambda\|_2)^{1/n}$$

for every $\lambda \in \mathbb{D}_2^\infty$, which implies $\varphi \in H^\infty(\mathbb{D}_2^\infty)$ and $\|\varphi\|_\infty \leqslant \|\varphi\|_{Mult}$. □

Remark. — It is curious to note that $UH^\infty(\mathbb{D})$ *is NOT contained in* $H^\infty(\mathbb{D}_2^\infty)$. To see that, one can use, for example, the fact that the set P of prime numbers *is not a Sidon set* in $\mathbb{Z}$ (a set $A \subset \mathbb{Z}$ is Sidon if every L^∞ function f with the Fourier spectrum in A is in the Wiener class $\sum |\hat{f}(k)| < \infty$). Indeed, for every Sidon set S, there exists a constant $c(S)$ such that the length of any arithmetic progression in S, $A \subset S$, is bounded by $c(S)$: $|A| \leqslant c(S)$; see, for example, [19]. To the contrary, the famous Green-Tao theorem claims that P contains arbitrarily long arithmetic progressions, [13]. Since P is not Sidon, there exists an $H^\infty(\mathbb{D})$ function $f = \sum_{p\in P} a_p z^p$ such that $\sum_{p\in P} |a_p| = \infty$. Since $Uf(\zeta) = \sum_{p\in P} a_p \zeta_p$ for $\zeta \in \mathbb{D}_2^\infty$, we obviously have $Uf \notin H^\infty(\mathbb{D}_2^\infty)$.

(9) *Let* $F \in E$ *and* $E \in \mathrm{Lat}(M_\zeta)$*. Then,* $F \cdot H^\infty(\mathbb{D}_2^\infty) \subset E$.

Indeed, let $\varphi \in H^\infty(\mathbb{D}_2^\infty)$ and (p_k) be a sequence of polynomials mentioned in (7). Then $p_k F \in E$ and $(p_k F)_k$ weakly converges to φF: $\|p_k F\|_2 \leqslant \|p_k\|_\infty \|F\|_2 \leqslant \|\varphi\|_\infty \|F\|_2$ (see (8)) and $\lim_k p_k(\zeta)F(\zeta) = \varphi(\zeta)F(\zeta)$ for every $\zeta \in \mathbb{D}_2^\infty$ (see (6)). Hence, $\varphi F \in E$. $\square$

(10) Let $S \subset \mathbb{N}$ and, for $\alpha \in \mathbb{Z}_+(\infty)$, the symbol $\alpha\chi_S$ means a product of two functions on $\mathbb{N}$: $\alpha\chi_S = (\alpha_1\chi_S(1), \dots, \alpha_k\chi_S(k), \dots)$. Further, let $H^2(\mathbb{D}_2^S)$ be a subspace of $H^2(\mathbb{D}_2^\infty)$ defined by

$$\begin{aligned} H^2(\mathbb{D}_2^S) &= \Big\{F = \sum_{\alpha \in \chi_S \mathbb{Z}_+(\infty)} c_\alpha(F)\zeta^\alpha : \sum_{\alpha \in \chi_S \mathbb{Z}_+(\infty)} |c_\alpha(F)|^2 < \infty\Big\} \\ &= \Big\{F \in H^2(\mathbb{D}_2^\infty) : \sigma(F) \subset \mathbb{Z}_+^S\Big\} \end{aligned}$$

where $\sigma(F)$ stands for the *Fourier spectrum*

$$\sigma(F) = \{\alpha \in \mathbb{Z}_+(\infty) : c_\alpha(F) \neq 0\}$$

of a function $F \in H^2(\mathbb{D}_2^\infty)$, and $\mathbb{Z}_+^S = \{\alpha \in \mathbb{Z}_+(\infty) : \alpha_j = 0 \text{ for } j \in S'\}$, $S' = \mathbb{N} \setminus S$. One can say that $H^2(\mathbb{D}_2^S)$ consists of functions depending on variables ζ_j, $j \in S$ only. We write $H^2(\mathbb{D}^n)$ for $H^2(\mathbb{D}_2^{\{1,2,\dots,n\}})$ and $\mathbb{Z}_+(n)$ for $\mathbb{Z}_+^S$ with $S = \{1, 2, \dots, n\}$.

Now, if $S \bigcap S' = \emptyset$, the subspaces $H^2(\mathbb{D}_2^S)$ and $H^2(\mathbb{D}_2^{S'})$ are independent in the following sense:

i) $H^2(\mathbb{D}_2^S) \perp H^2(\mathbb{D}_2^{S'})$;

ii) $F \in H^2(\mathbb{D}_2^S)$, $G \in H^2(\mathbb{D}_2^{S'})$ imply $FG \in H^2(\mathbb{D}_2^{S''})$, where $S'' = S \cup S'$, and $\|FG\|_2^2 = \|F\|_2^2 \|G\|_2^2$;

iii) moreover, $H^2(\mathbb{D}_2^S) \otimes H^2(\mathbb{D}_2^{S'}) = H^2(\mathbb{D}_2^{S''})$, where $S'' = S \cup S'$ and

$$H^2(\mathbb{D}_2^S) \otimes H^2(\mathbb{D}_2^{S'}) = \operatorname{span}\Big(\sum_{\text{finite}} F_j G_j : F_j \in H^2(\mathbb{D}_2^S), G_j \in H^2(\mathbb{D}_2^{S'})\Big)$$

Indeed, i) is obvious; ii) follows from the Fubini theorem and property (2) above; iii) is clear since polynomials depending on variables S are densely contained in $H^2(\mathbb{D}_2^S)$, and similarly for $H^2(\mathbb{D}_2^{S'})$ and $H^2(\mathbb{D}_2^{S''})$. $\square$

(11) *Let $S \subset \mathbb{N}$, $S' = \mathbb{N} \setminus S$ be the complement of S, and E a (closed) subspace of $H^2(\mathbb{D}_2^S)$. Then*

$$E \otimes H^2(\mathbb{D}_2^{S'}) = H^2(\mathbb{D}_2^\infty)$$

if and only if $E = H^2(\mathbb{D}_2^S)$.

Indeed, the sufficiency follows from (10)-iii. For the necessity, we first observe that $\sigma(F) \subset \mathbb{Z}_+^S$ for every function $F \in E$, and — on the other hand — the product Fp with every polynomial $p \in H^2(\mathbb{D}_2^{S'})$ can be written as $Fp = Fp(0) + F(p - p(0))$, where $F(p - p(0))$ is orthogonal to each ζ^α

with $\mathrm{supp}(\alpha) \subset S$, and hence $F(p - p(0)) \perp H^2(\mathbb{D}_2^S)$. Now, if we assume $E \otimes H^2(\mathbb{D}_2^{S'}) = H^2(\mathbb{D}_2^\infty)$ and take an element $G \in H^2(\mathbb{D}_2^S)$ and a sequence $(\Phi_k)_k$ from

$$\Big\{ \sum_{\text{finite}} F_j p_j : F_j \in E, \quad p_j \text{ are polynomials in } H^2(\mathbb{D}_2^{S'}) \Big\}$$

such that $\lim_k \|\Phi_k - G\|_2 = 0$, then $\Phi_k = G_k + G'_k$, *where* $G_k \in E$ *and* $\|\Phi_k - G\|_2^2 = \|G_k - G\|_2^2 + \|G'_k\|_2^2$. This means that $\lim_k \|G_k - G\|_2 = 0$, and hence $G \in E$. Therefore, $E = H^2(\mathbb{D}_2^S)$. $\square$

(12) *L^p-norms on $H^2(\mathbb{D}_2^\infty)$ and powers of $H^2(\mathbb{D}_2^\infty)$ functions.*

i) Let $F \in H^2(\mathbb{D}_2^\infty)$ and $F(\zeta) \neq 0$ for $\zeta \in \mathbb{D}_2^\infty$. Then for every t, $0 < t \leqslant 1$, there exist functions $F^t \in H^2(\mathbb{D}_2^\infty)$ satisfying $F^t F^s = F^{t+s}$, $(F^t)^s = F^{st}$, $F^1 = F$.

Indeed, a family of functions $(F_{(n)})^t$ with these properties obviously exists on $\mathbb{D}^n$. Since $|a^t - b^t| \leqslant C|a-b|^t$ for every $a, b \in \mathbb{C}$, we get

$$\|(F_{(n)})^t - (F_{(m)})^t\|_2 \leqslant C\|F_{(n)} - F_{(m)}\|_2^t.$$

The last expression tends to 0 as $n, m \longrightarrow \infty$. So, property (2) implies that there exists $F^t \in H^2(\mathbb{D}_2^\infty)$ such that $(F_{(n)})^t = (F^t)_{(n)}$ for every n and t. The result follows. $\square$

ii) For a function $F \in H^2(\mathbb{D}_2^\infty)$ and $2 \leqslant p < \infty$, *we define the H^p norm of F* (finite or not) as follows

$$\|F\|_p^p = \sup_{n \geqslant 1} \sup_{0<r<1} \int_{\mathbb{T}^n} \Big| F_{(n)}(r\zeta_1, \dots, r\zeta_n)|^p \, dm(\zeta_1) \dots dm(\zeta_n).$$

Further, given r, $0 < r < 1$, we set $F_r(\zeta) = F(r\zeta)$ and $F_{(n)r}(\zeta) = F_{(n)}(r\zeta)$ for all $\zeta \in \mathbb{D}_2^\infty (F_{(n)}$ are defined in (2) above). Then,

$\|F_r\|_p \leqslant \|F\|_p$, $\|F_{(n)r}\|_p \leqslant \|F_{(n)}\|_p \leqslant \|F\|_p$ for every $F \in H^p(\mathbb{D}_2^\infty)$, n and r, $0 < r < 1$. Moreover, $2 \leqslant p \leqslant q \Rightarrow \|F\|_q \geqslant \|F\|_p$.

Indeed, it is clear by the definition of H^p and the properties of the Poisson means in $\mathbb{D}^n$. $\square$

iii) Let $G \in H^2(\mathbb{D}_2^\infty)$ and $G(\zeta) \neq 0$ for $\zeta \in \mathbb{D}_2^\infty$. Then for every t, $0 < t \leqslant 1$, $\|G^t\|_{2/t} < \infty$ and there exists a sequence of polynomials $(p_n)_n$ such that $\lim_n \|G^t - p_n\|_{2/t} = 0$.

In order to find polynomials p_n, we can use obvious properties that for every $G \in H^2(\mathbb{D}_2^\infty)$ we have $\lim_n \|G_{(n)} - G\|_2 = 0$ and $\lim_{r \to 1} \|G_{(n)r} - G_{(n)}\|_2 = 0$ (for every n), and hence there exists a sequence $r(n) \longrightarrow 1$ such that $\lim_n \|G_{(n)r(n)} - G\|_2 = 0$. But $G_{(n)r(n)}(\zeta) \neq 0$ for $\zeta \in \mathbb{D}_2^\infty$, and using an above inequality $|a^t - b^t| \leqslant C|a-b|^t$, we obtain

$$\|G^t_{(n)r(n)} - G^t\|_{2/t}^{2/t} \leqslant C^{2/t} \|G_{(n)r(n)} - G\|_2^2 \longrightarrow 0 \text{ (as } n \longrightarrow \infty).$$

Approximating $G^t_{(n)r(n)}$ by polynomials p_n uniformly on $\overline{\mathbb{D}}^n$, we obtain the result. □

iv) For a general function F on $\mathbb{D}^\infty_2$, which is a sum of an absolutely convergent power series in $\mathbb{D}^\infty_2$, we define its roots of a natural degree. Namely, given $N \in \mathbb{N}$, we say that $F^{1/N}$ exists if there is an absolutely convergent power series in $\mathbb{D}^\infty_2$, whose sum — $F^{1/N}$ — satisfies $(F^{1/N})^N = F$.

Remark. — A problem for defining H^p spaces of infinitely many variables is, in particular, in the choice of a natural domain where the space should be defined. For example, for $1 \leqslant p \leqslant 2$, given an absolutely convergent power series F, the Hausdorff–Young inequality says that

$$\|\hat{F}_{(n)}\|_{l^{p'}(\mathbb{Z}^n_+)} \leqslant \|F_{(n)}\|_{H^p(\mathbb{D}^n)} \leqslant \|F\|_{H^p} = \sup_n \|F_{(n)}\|_{H^p(\mathbb{D}^n)}$$

for every n. If the latter quantity is finite, then an absolutely convergent representation $F(\zeta) = \sum_{\alpha\in\mathbb{Z}_+(\infty)} c_\alpha \zeta^\alpha$ exists for

$$\zeta \in \mathbb{D}^\infty_p = \Big\{\zeta : \sum_{j\geqslant 1} |\zeta_j|^p < \infty, |\zeta_j| < 1(\forall j)\Big\}.$$

However, an extension to $\mathbb{D}^\infty_2$ is still unclear. In the opposite direction, when $p > 2$, it seems that $\mathbb{D}^\infty_2$ could be a natural domain for $H^p(\mathbb{D}^\infty_2)$, but we prefer do not enter in a discussion. However, in the limit case as $p \longrightarrow \infty$, one can consider the corresponding "multi-disc algebras", and even if for the $\|\cdot\|_\infty$ closure of polynomials the natural domain, perhaps, is still $\mathbb{D}^\infty_2$, for the corresponding Wiener algebra

$$W = \Big\{F(\zeta) = \sum_{\alpha\in\mathbb{Z}_+(\infty)} c_\alpha\zeta^\alpha : \sum_{\alpha\in\mathbb{Z}_+(\infty)} |c_\alpha| < \infty\Big\}$$

the natural domain is definitly much larger, namely it is $\mathbb{D}^\infty_\infty = \{\zeta \in \mathbb{C}^\infty : |\zeta_j| < 1, \forall j\}$.

3. Some invariant subspaces and cyclic vectors of (T_n)

Here we obtain some necessary/sufficient conditions for a function $f \in H^2(\mathbb{D})$ to be (T_n)-cyclic, or equivalently, $Uf = F \in H^2(\mathbb{D}^\infty_2)$ to be M_ζ-cyclic. In order to place the question properly, recall that a partial case of the problem of M_ζ-cyclicity is a problem of cyclic function in $H^2(\mathbb{D}^n)$, a quite famous question which is still open at least from the time of Rudin's book [24] (1969). In fact, Theorem 3.3 below is nothing but a multidisc analogue of facts proved for the disc $\mathbb{D}$ already in [21], Section 2.1., and even for a variety of function spaces (including weighted Hardy and Bergman spaces). We start with two lemmas.

Lemma 3.1. — *Let $F \in H^2(\mathbb{D}_2^\infty)$. If F is M_ζ-cyclic then $F(\zeta) \neq 0$ for $\zeta \in \mathbb{D}_2^\infty$.*

Proof. — Obvious since Lemma 2.1. □

We need a bit more notation. Let $J \subset \mathbb{N}$ and

$$\mathbb{N}_J = \{p^\alpha = p_1^{\alpha_1} \dots p_m^{\alpha_m} : (\alpha_1, \dots, \alpha_m, \dots) \in \mathbb{Z}_+^J\}$$

be a subsemigroup of $\mathbb{N}$ generated by prime numbers $p_j, j \in J$. More general, we consider subsets $\Sigma \subset \mathbb{Z}_+(\infty)$ being a positive part of a subgroup $\Sigma' \subset \mathbb{Z}^\infty$, namely, $\Sigma = \Sigma' \cap \mathbb{Z}_+(\infty)$; we call them "semi-subgroups". Such a semi-subgroup Σ can be characterized by the following property: if $\alpha \in \Sigma$, $\beta \in \mathbb{Z}_+(\infty)$ and $\alpha + \beta \in \Sigma$ then $\beta \in \Sigma$. At the level of $\mathbb{N}$, for the corresponding (multiplicative) subgroups $\sigma = \{p^\alpha : \alpha \in \Sigma\}$ $(p = (p_1, p_2, \dots))$, the latter property says $n \in \sigma$, $m \in \mathbb{N}$ and $nm \in \sigma \Rightarrow m \in \sigma$. For instance, semigroups $\mathbb{Z}_+^J$ and every one-generated semigroup $\Sigma = \{n\alpha : n \in \mathbb{Z}_+\}$, $\alpha \in \mathbb{Z}_+(\infty)$ are semi-subgroups in $\mathbb{Z}_+(\infty)$, and their counterparts in $\mathbb{N}$ are $\mathbb{N}_J$ and $\sigma = \{p^{n\alpha} : n \in \mathbb{Z}_+\}$.

The following lemma is stated for semi-subgroups $\mathbb{N}_J$ *but it is also true for an arbitrary semi-subgroup* $\sigma \subset \mathbb{N}$. The leading partial case is for $J = \{1, \dots, m\}$, $m \in \mathbb{N}$, when $H^2(\mathbb{D}_2^J) = H^2(\mathbb{D}^m)$.

Lemma 3.2. — *Let $f \in H_0^2(\mathbb{D})$ having the Fourier spectrum in a semi-subgroup $\mathbb{N}_J$, $\sigma(f) \subset \mathbb{N}_J$, for a subset $J \subset \mathbb{N}$. Then, f is (T_n)-cyclic if and only if Uf is $(M_{\zeta^\alpha})_{\alpha \in \mathbb{Z}_+^J}$-cyclic in $H^2(\mathbb{D}_2^J)$.*

Proof. — Assume Uf is $(M_{\zeta^\alpha})_{\alpha \in \mathbb{Z}_+^J}$-cyclic in $H^2(\mathbb{D}_2^J)$, and let $\mathcal{E}$ be an M_ζ-invariant subspace generated by Uf in $H^2(\mathbb{D}_2^\infty)$. Clearly,

$$\mathcal{E} \supset H^2(\mathbb{D}_2^J) \otimes H^2(\mathbb{D}_2^{J'}) = H^2(\mathbb{D}_2^\infty)$$

(see (11), Section 2), i.e. Uf is M_ζ-cyclic.

Conversely, assume that Uf is M_ζ-cyclic in $H^2(\mathbb{D}_2^\infty)$. We employ a reasoning similar to those of (11), Section 2. Namely, let P be an orthogonal projection in $H^2(\mathbb{D}_2^\infty)$ on the subspace $H^2(\mathbb{D}_2^J)$. Notice, that if a monomial $\zeta^\alpha = \zeta_1^{\alpha_1} \dots \zeta^{\alpha_n}$ has the spectrum $\sigma(\zeta^\alpha) = \{\alpha_1, \dots \alpha_n, 0, 0, \dots\}$, which is not in $\mathbb{Z}_+^J$ (i.e., there is a "new variable" in ζ^α), then

$$\zeta^\alpha H^2(\mathbb{D}_2^J) \perp H^2(\mathbb{D}_2^J).$$

It follows that, for every polynomial q, we have $P(qUf) = (Pq)Uf$. Let now $G \in H^2(\mathbb{D}_2^J)$ and (q_k) be a sequence of polynomials such that $\lim_k \|G - q_k Uf\|_2 = 0$. Then, $\lim_k \|G - (Pq_k)Uf\|_2 = 0$ and $Pq_k \in H^2(\mathbb{D}_2^J)$. This means that Uf is $(M_{\zeta^\alpha})_{\alpha\in\mathbb{Z}_+^J}$-cyclic in $H^2(\mathbb{D}_2^J)$. □

Theorem 3.3. — *Let F be a function on $\mathbb{D}_2^\infty$ such that $F(\zeta) \neq 0$ for $\zeta \in \mathbb{D}_2^\infty$.*

(1) If $F \in H^2(\mathbb{D}_2^\infty)$ and $1/F \in H^\infty(\mathbb{D}_2^\infty)$ then F is M_ζ-cyclic in $H^2(\mathbb{D}_2^\infty)$.

(2) If $F \in H^2(\mathbb{D}_2^\infty)$ and $\mathrm{Re}(F(\zeta)) \geqslant 0$ for $\zeta \in \mathbb{D}_2^\infty$ then F is M_ζ-cyclic in $H^2(\mathbb{D}_2^\infty)$.

(3) If there exist $\epsilon > 0$ and $N \geqslant 1$ such that $F^{1+\epsilon} \in H^2(\mathbb{D}_2^\infty)$ and $1/F^{1/N} \in H^2(\mathbb{D}_2^\infty)$, then F is cyclic in $H^2(\mathbb{D}_2^\infty)$.

Proof. — Let E be a (closed) M_ζ-invariant subspace of $H^2(\mathbb{D}_2^\infty)$ generated by F. For cyclicity, it suffices to prove that $1 \in E$.

(1) By (9), Section 2, we have $1 = F \cdot \frac{1}{F} \in E$.

(2) By (9), Section 2, we have $\frac{F}{F+\epsilon} \in E$ for every $\epsilon > 0$. Moreover, $|\frac{F(\zeta)}{F(\zeta)+\epsilon}| \leqslant 1$ and $\lim_{\epsilon\longrightarrow 0} \frac{F(\zeta)}{F(\zeta)+\epsilon} = 1$ for every $\zeta \in \mathbb{D}_2^\infty$. By (6) and (3) of Section 2, we obtain $1 \in E$.

(3) Without loss of generality, we can assume that $1/\epsilon \in \mathbb{N}$. Let $\gamma = \frac{\epsilon/N}{1+\epsilon}$, $q = \frac{2(1+\epsilon)}{\epsilon}$. Then, by (12) of Section 2, we have $1/F^\gamma \in H^q(\mathbb{D}_2^\infty)$. Let p_k be polynomials found in (12) iii, Section 2, so that $\lim_k \|\frac{1}{F^\gamma} - p_k\|_q = 0$. By Hölder, since $\frac{1}{2(1+\epsilon)} + \frac{1}{q} = \frac{1}{2}$, we obtain

$$\|F^{1-\gamma} - p_k F\|_2 = \Big\|F(\frac{1}{F^\gamma} - p_k)\Big\|_2 \leqslant \|F^{1+\epsilon}\|_2^{1/1+\epsilon}\Big\|\frac{1}{F^\gamma} - p_k\Big\|_q.$$

This implies $F^{1-\gamma} \in E$. Repeating the preceding step for

$$\|F^{1-2\gamma} - p_k F^{1-\gamma}\|_2 = \|F^{1-\gamma}(\frac{1}{F^\gamma} - p_k)\|_2,$$

we get $F^{1-2\gamma} \in E$, etc. — so that, in $1/\gamma$ steps we obtain $1 \in E$, and hence F is cyclic. □

Below, in theorem 3.4, we show that for functions $f \in H_0^2(\mathbb{D})$ having the Fourier support in $\mathbb{N}_J$ for a finite J, the trivial necessary condition of Lemma 3.1, (together with a slight smoothness condition), is, in fact, sufficient for f to be a (T_n)-cyclic element. We give a separated proof for the case of a polynomial since, in this case, the proof is much easier.

Theorem 3.4. — *(1) Let $f = \sum_{k=1}^n \hat{f}(k)z^k$ be a polynomial on $\mathbb{D}$. Then f is (T_n)-cyclic if and only if $Uf(\zeta) \neq 0$ for $\zeta \in \mathbb{D}_2^\infty$ (in fact, for $\zeta \in \mathbb{D}^J$, $J = \bigcup_k \mathrm{supp}(\alpha(k))$ where the union is taken over all k with $\hat{f}(k) \neq 0$).*

(2) Let J be a finite set in $\mathbb{N}$ and $f \in H^2_0(\mathbb{D})$ having the Fourier spectrum in $\mathbb{N}_J$, $\sigma(f) \subset \mathbb{N}_J$, such that $\hat{f}(k) = o(k^{-\epsilon})$ for $k \longrightarrow \infty$ for a positive ϵ. Then f is (T_n)-cyclic if and only if $Uf(\zeta) \neq 0$ for $\zeta \in \mathbb{D}^\infty_2$ (in fact, for $\zeta \in \mathbb{D}^J$).

(3) The same as in (2) but for functions f with $Uf \in \mathrm{Hol}(\Delta\mathbb{D}^J)$, $\Delta > 1$.

(4) Let $n > 1$ be an integer and $f \in H^2_0$ with $\sigma(f) \subset \{n^k : k \in \mathbb{Z}_+\}$. Let further $\varphi = \sum_{k \geqslant 0} \hat{f}(n^k) z^k$. Then f is (T_n)-cyclic if and only if φ is a Beurling inner function. Moreover, if E is a (M_ζ)-invariant subspace of $H^2(\mathbb{D}^\infty_2)$ generated by $F = Uf$, then

$$E = I(\zeta^{\alpha(n)})H^2 \otimes H^2(\mathbb{D}_2^{\mathbb{N}\setminus\{n\}}),$$

where I is the inner part of φ.

Proof. — (1) Necessity is a general fact (Lemma 3.1), we need to prove the sufficiency. Assume $Uf(\zeta) \neq 0$ for $\zeta \in \mathbb{D}^m$, $m = \pi(n)$, $\pi(n)$ being the number of primes less or equals to n. By Lemma 3.2, it suffices to check that $F = Uf$ is $(M_{\zeta_j})_{1 \leqslant j \leqslant m}$ cyclic in $H^2(\mathbb{D}^m_2)$. Let E be an invariant subspace generated by F. Since $1/F_r \in H^\infty(\mathbb{D}^m)$ for every r, $0 < r < 1$, we have $F/F_r \in E$ and $\lim_{r \longrightarrow 1} \frac{F(\zeta)}{F_r(\zeta)} = 1$ for every $\zeta \in \mathbb{D}^m$. Moreover,

$$\left|\frac{F(\zeta)}{F_r(\zeta)}\right| \leqslant 2^{\deg(F)} \text{ for every } 0 < r < 1 \text{ and } \zeta \in \mathbb{D}^m$$

the latter is a slightly improved inequality from [12]; see also [11] (for the readers convenience, we give a short proof of this inequality in Remark 3.5 after the Theorem). Hence, $\sup_{0<r<1} \|\frac{F}{F_r}\|_2 < \infty$ and, therefore, $\lim_{r \to 1} \frac{F}{F_r} = 1$ weakly in $H^2(\mathbb{D}^m)$. This implies $1 \in E$, and the result follows.

(2)–(3) As before, the necessity is obvious. The proof of the sufficiency makes use of Theorem 3.3(3). For notation simplicity we argue for the case $J = \{1, \ldots, m\}$ only; the general case is similar. Let $F = Uf$, $F(\zeta) \neq 0$ for $\zeta \in \mathbb{D}^m$. We will check that $F \in H^\infty(\mathbb{D}^m)$ and $1/F \in H^\delta(\mathbb{D}^m)$ for a positive δ. Indeed, the condition $\hat{f}(k) = o(k^{-\epsilon})$ means that

$$\hat{f}(p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_s^{\alpha_m}) = o(p_1^{-\epsilon\alpha_1} p^{-\epsilon\alpha_2} \ldots p_s^{-\epsilon\alpha_m}) \text{ as } \alpha_j \longrightarrow \infty \text{ for } 1 \leqslant j \leqslant m.$$

This implies that the function

$$F(\zeta) = \sum_{\alpha \in \mathbb{Z}_+(m)} \hat{f}(p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_s^{\alpha_m}) \zeta^\alpha$$

is holomorphic at least for $|\zeta_j| < 2^\epsilon$, $1 \leqslant j \leqslant m$. In particular, $F \in H^\infty(\mathbb{D}^m)$.

For the inverse $1/F$, we use a Lojaciewicz theorem ([18], see also [17]) on the zero set of an analytic function of several variables. In particular, the theorem tells that given an analytic function $G \neq 0$ of m variables, there exist constants $N > 0$, $0 < C < \infty$ such that

$$|G(\zeta)| \geqslant C(\mathrm{dist}(\zeta, Z(G)))^N,$$

where $Z(G) = \{z : G(z) = 0\}$ is the zero set of G(we suppose that all this happens on a compact set in $\mathbb{C}^m$, in whose open neighborhood G is analytic).

First, we use Lemma 3.6 (below) on the zero set of F and obtain $Z(F) = A \times \mathbb{D}^\sigma$, where $A \subset \mathbb{T}^{\sigma'}$ is a finite union of analytic manifolds of the dimension strictly less than $d = \mathrm{card}(\sigma')$ (the notation is taken from Lemma 3.6). Next, we apply Lojaciewicz's theorem for F getting, for $\zeta \in \mathbb{D}^m$,

$$|F(\zeta)| \geqslant C(\mathrm{dist}(\zeta, Z(F)))^N = C(\mathrm{dist}(\zeta_{\sigma'}, A))^N.$$

Moreover, there exists a constant $c > 0$ such that

$$\mathrm{dist}(r\zeta_{\sigma'}, A) \geqslant c \cdot \mathrm{dist}(\zeta_{\sigma'}, A)$$

for every $0 < r < 1$ and $\zeta_{\sigma'} \in \mathbb{T}^{\sigma'}$ (indeed, all distances in $\mathbb{C}^{\sigma'}$ being equivalent, consider $\|z\|_\infty = \max |z_i|$; then, for $z, \zeta \in \mathbb{T}^{\sigma'}$, we have $\|z-\zeta\| \leqslant \|z - rz\| + \|rz - \zeta\| = 1 - r\|z\| + \|rz - \zeta\| \leqslant 2\|\zeta - rz\|$). This implies ($\lambda_m$ stands for Lebesgue measure on $\mathbb{T}^m$)

$$\begin{aligned}\int_{\mathbb{T}^m} \frac{d\lambda_m(\zeta)}{|F(r\zeta)|^\delta} &\leqslant \frac{1}{C^\delta c^{\delta N}} \int_{\mathbb{T}^m} \frac{d\lambda_m(\zeta)}{\big(\mathrm{dist}(\zeta_{\sigma'}, A)\big)^{N\delta}} \\ &= \frac{1}{C^\delta c^{\delta N}} \int_{\mathbb{T}^{\sigma'}} \frac{d\lambda_d(\zeta_{\sigma'})}{\big(\mathrm{dist}(\zeta_{\sigma'}, A)\big)^{N\delta}} < \infty\end{aligned}$$

if $N\delta < 1$ and $d = \mathrm{card}(\sigma')$; indeed, in a neighborhood of any point of A and with a choice of a convenient parametrization for an analytic manifold A (i.e., up to a convenient C^∞ diffeomorphism of $\mathbb{R}^d$), one has $\zeta_{\sigma'} = x = (x_1, \ldots, x_d)$, $A \subset H = \{x = (x_1, \ldots, x_d) : x_1 = 0\}$, and hence $\mathrm{dist}(\zeta_{\sigma'}, A) \geqslant \mathit{const} \cdot \mathrm{dist}(x, H) = |x_1|$, so that the last integral is bounded by

$$\mathit{const} \cdot \int_{|x_j|<1, \forall j} \frac{dx_1 \ldots dx_d}{|x_1|^{N\delta}} < \infty.$$

It follows that $1/F \in H^\delta(\mathbb{D}^m)$, and we are done.

(4) Let $\sigma = \{n^k : k \in \mathbb{Z}_+\}$ and $H^2(\sigma) = \{g \in H^2(\mathbb{D}) : \sigma(g) \subset \sigma\}$. Then, the restriction $T_n|H^2(\sigma)$ is obviously unitarily equivalent to the shift operator $SG = zG$ on the Hardy space $H^2(\mathbb{D})$. Now, making use of a remark just before Lemma 3.2 (applied for $\sigma = \{n^k : k \in \mathbb{Z}_+\}$), we obtain the claim on the cyclicity of f (the V. I. Smirnov (1934)–A. Beurling (1949) criterion for cyclicity is also used: φ is z-cyclic in H^2 if and only if φ is outer). Using property (11) of Section 2 (and Smirnov–Beurling's description of simply generated z-invariant subspaces), we get the formula for E. □

Remark. — Let F be a polynomial such that $F(\zeta) \neq 0$ for $\zeta \in \mathbb{D}_2^\infty$. Here we give a short proof of the inequality

$$\left|\frac{F(\zeta)}{F_r(\zeta)}\right| \leqslant 2^{\deg(F)}$$

for $0 \leqslant r \leqslant 1$ and $\zeta \in \mathbb{D}_2^\infty$; different versions can be found in [12] and [11], where the degree $\deg(F)$ is replaced by larger values. Of course, it is sufficient to restrict ourselves to $\zeta \in \mathbb{D}^m$, where m is the number of variables involved into F.

For a polynomial of 1 variable, $F(z) = A(z-z_1)\dots(z-z_d)$ with $|z_j| \geqslant 1$, one has $\left|\frac{z-z_j}{rz-z_j}\right| \leqslant 2$ for every $z \in \mathbb{D}$ (and hence, for $z \in \overline{\mathbb{D}}$) and $0 \leqslant r \leqslant 1$; indeed,

$$\left|\frac{z-z_j}{rz-z_j}\right| = \left|1 + \frac{z-rz}{rz-z_j}\right| \leqslant 2.$$

The claimed inequality follows. In several variables, we fix $\zeta = (\zeta_1, \dots, \zeta_m) \in \mathbb{D}^m$ and pass to a polynomial $P_\zeta(z) = F(z\zeta)$, $z \in \mathbb{D}$. Applying the 1-dimensional result, we get $\left|\frac{P_\zeta(z)}{P_\zeta(rz)}\right| \leqslant 2^{\deg(F)}$ for $z \in \overline{\mathbb{D}}$. It remains to set $z = 1$.

Lemma 3.5. — *Let F be a function holomorphic on $(1+\epsilon)\mathbb{D}^m$ such that $F(z) \neq 0$ for every $z \in \mathbb{D}^m$, and $Z(F) = \{z \in \overline{\mathbb{D}}^m : F(z) = 0\}$. Then there exist subsets $\sigma \subset \{1, 2, \dots, m\}$ (maybe, empty) and $A \subset \mathbb{T}^{\sigma'}$ such that*

$$Z(F) = A \times \mathbb{D}^\sigma$$

where $A \times \mathbb{D}^\sigma = \{z = (z_1, \dots, z_m) : (z_i)_{i\in\sigma} \in \mathbb{D}^\sigma, (z_i)_{i\in\sigma'} \in A\}$, σ' stands for the complementary set $\sigma' = \{1, 2, \dots, m\} \setminus \sigma$. Moreover, A is a finite union of analytic manifolds of real dimensions strictly less than $\mathrm{card}(\sigma')$.

Proof. — Assume that $Z(F)$ is not contained in $\mathbb{T}^m$, and let $i \in \{1, \dots, m\}$ and $z \in Z(F)$ such that $|z_i| < 1$. Writing $z = (z', z_i)$, we show that $\{z'\} \times \mathbb{D} \subset Z(F)$. Indeed, functions $\varphi_r(w) = F(rz', w)$, $0 < r < 1$, are holomorphic and non vanishing in $\mathbb{D}$. The limit function $\varphi(w) = \lim_{r\to 1} \varphi_r(w)$

is vanishing at $w = z_i \in \mathbb{D}$, and hence $\varphi = 0$. This implies the inclusion claimed. Setting

$$\sigma = \{i : \text{ there exists } z \in Z(F) \text{ such that } |z_i| < 1\},$$

we get the formula claimed in the Lemma. Take an arbitrary $z_\sigma \in \mathbb{D}^\sigma$. Since A can be regarded as the zero set of the restriction F_0 of the function $z_{\sigma'} \longmapsto F(z_{\sigma'}, z_\sigma)$ onto the torus $\mathbb{T}^{\sigma'}$, we can use a geometric part of Lojaciewicz's theorem (see, [18], [17]): being holomorphic on a neighborhood of $\mathbb{T}^{\sigma'}$, function F_0 has a zero set consisting of a finite union of analytic manifolds of the dimensions strictly less than $\mathrm{card}(\sigma')$, unless $F_0 = 0$. The latter property is impossible since it entails that $F = 0$. □

Corollary 3.6. — *Let $f \in H^2_0$, and there exist f_j ($j = 1, 2, 3, 4$) in H^2_0 such that $Uf = Uf_1 \cdot Uf_2 \cdot Uf_3 \cdot Uf_4$, where f_i for $i = 1, 2, 3$ satisfy condition (i) of Theorem 3.3 and f_4 satisfies (3) of Theorem 3.4. Then f is (T_n) cyclic.*

Indeed, let E_{Uf} be an (M_ζ) invariant subspace of $H^2(\mathbb{D}^\infty_2)$ generated by Uf. Then, it follows from the proof of Theorem 3.3 that $Uf_2 \cdot Uf_3 \cdot Uf_4 \in E_{Uf}$, and next $g =: Uf_3 \cdot Uf_4 \in E_{Uf}$. As to the function g, it is clear that $g^{1+\epsilon} \in H^2(\mathbb{D}^\infty_2)$ (since $(Uf_3)^{1+\epsilon} \in H^2(\mathbb{D}^\infty_2)$) and it follows from the proofs of Theorem 3.3 and 3.4 (and Hölder inequality) that $1/g^{\delta'} \in H^2(\mathbb{D}^\infty_2)$ for some $0 < \delta' < 1/N$ (N is from condition (3) of Theorem 3.3). Hence, $1/g^{1/N'} \in H^2(\mathbb{D}^\infty_2)$ for an integer $N' > N$, and by Theorem 3.3, $1 \in E_{Uf}$, which completes the proof. □

We finish the Section showing that the reproducing kernels k_λ are (M_ζ)-cyclic in $H^2(\mathbb{D}^\infty_2)$. In a different language and with a different proof, this fact is proved in [14]. Wintner's functions mentioned in the Introduction correspond to a special choice of $\lambda \in \mathbb{D}^\infty_2$.

Corollary 3.7. — *Every reproducing kernel k_λ, $\lambda \in \mathbb{D}^\infty_2$, is a (M_ζ)-cyclic vector in $H^2(\mathbb{D}^\infty_2)$, or, equivalently, any function $f_\lambda = \sum_{n\geqslant 1} \lambda^{\alpha(n)} z^n$ is (T_n)-cyclic in $H^2(\mathbb{D})$.*

Indeed,

$$k_\lambda(\zeta) = \sum_{\alpha \in \mathbb{Z}_+(\infty)} \overline{\lambda}^\alpha \zeta^\alpha = \prod_{s\geqslant 1} F_{\lambda_s}(\zeta_s)$$

where $F_a(z) = (1 - \overline{a}z)^{-1}$ ($a, z \in \mathbb{D}$) (a computation similar to 2.3(4)). On the other hand, $\|F_a\|^p_{H^p(\mathbb{T})} = 1 + |pa/2|^2(1 + o(1))$ as $a \longrightarrow 0$ ($\forall p < \infty$), and hence

$$\|k_\lambda\|^p_p = \prod_{s\geqslant 1} \|F_{\lambda_s}\|^p_{H^p(\mathbb{T})} < \infty \text{ for every } \lambda,\ \ \lambda = (\lambda_s) \in \mathbb{D}^\infty_2;$$

Similarly, $\|1/k_\lambda\|_2^2 = \prod_{s\geqslant 1} \|1 - \overline{\lambda}_s\zeta_s\|^2_{H^2(\mathbb{T})} < \infty$. Now, the cyclicity of k_λ follows from Theorem 3.3. □

Corollary 3.8 (Wintner, 1944). — *Every function $f_a = \sum_{k\geqslant 1} k^{-a}z^k$, $\mathrm{Re}(a) > 1/2$ is (T_n)-cyclic.*

Indeed, $Uf_a = k_\lambda$ where $\lambda = (\lambda_s)_{s\geqslant 1}$, $\lambda_s = p_s^{-a}$ (p_s are primes). □

4. A few of examples

It is worth mentioning that the language of the U-transforms for (T_n)-cyclicity criteria for $f \in H_0^2$ is very natural but not as transparent (efficient) as it seems to be. Speaking on "efficiency" we mean that given a function

$$f = \sum_{k\geqslant 1} a_k z^k$$

in H_0^2, we want to know its cyclicity properties just in terms of the Taylor (Fourier) coefficients a_k, and not of an extension of f to $\mathbb{D}_2^\infty$. We show some hidden effects on several examples. But we start with a corollary of Theorem 3.3, which is essentially contained in [14], with a different proof.

Corollary 4.1. — *(1) If $|a_1| \geqslant \sum_{k\geqslant 2}|a_k|$, then $f = \sum_{k\geqslant 1} a_k z^k$ is (T_n)-cyclic ($f \neq 0$).*

(2) Conversely, suppose f is cyclic and its Fourier spectrum $\sigma(f)$ is contained in the set P of prime numbers. Then $|a_1| \geqslant \sum_{k\geqslant 2}|a_k|$.

Indeed, for (1), we assume, without loss of generality, $a_1 > 0$. Then, $\mathrm{Re}(Uf(\zeta)) = a_1 + \mathrm{Re}\left(\sum_{n\geqslant 2} a_n\zeta^{\alpha(n)}\right) \geqslant 0$ for every $\zeta \in \mathbb{D}_2^\infty$, and Theorem 3.3(2) implies the cyclicity of f. Conversely, if $f \in H_0^2$ is cyclic, then $Uf(\zeta) \neq 0$ for $\zeta \in \mathbb{D}_2^\infty$. If we suppose $|a_1| < \sum_{n\geqslant 2}|a_n|$, then there exists $z = (z_1, \dots, z_N, 0, 0, \dots)$ with $|z_i| < 1$ such that $Uf(z) = a_1 + \sum_{k=1}^N a_{p_k} z_k = 0$. Contradiction. □

4.1. Some cyclic polynomials and functions. (1) Let

$$f = a_1 z + a_2 z^2 + a_3 z^3$$

be a polynomial, $\deg(f) \leqslant 3$. Then, by Corollary 4.1, *f is (T_n)-cyclic if and only if $|a_1| \geqslant |a_2| + |a_3|$.* □

(2) Let

$$f = a_1 z + a_2 z^2 + a_3 z^3 + a_4 z^4$$

be a polynomial, $\deg(f) \leqslant 4$. Then, by Theorem 3.4(1), f is (T_n)-cyclic if and only if $Uf(\zeta) \neq 0$ for $\zeta \in \mathbb{D}^2$, where

$$Uf(\zeta_1, \zeta_2) = a_1 + a_2\zeta_1 + a_4\zeta_1^2 + a_3\zeta_2 = q(\zeta_1) + a_3\zeta_2.$$

So, *a necessary and sufficient condition for cyclicity is*

$$q(\mathbb{D}) \cap a_3\mathbb{D} = \emptyset$$

Condition $|a_1| \geqslant |a_2| + |a_3| + |a_4|$ is, of course, sufficient but not necessary.

For example, for $a_3 = 0$, necessary and sufficient condition for cyclicity is $q(z) \neq 0$ for $|z| < 1$ what is the case for every $q(z) = (z - z_1)(z - z_2)$ with $|z_j| \geqslant 1$; and if $|z_1 z_2| < 1 + |z_1 + z_2|$(as for $z_1 = z_2 = t$, $1 \leqslant t \leqslant 2$), we get a cyclic polynomial with $|a_1| < |a_2| + |a_3| + |a_4|$.

For $a_3 \neq 0$, the condition of cyclicity $\min_{|z|\leqslant 1} |q(z)| \geqslant |a_3|$ is also compatible with $|a_1| < |a_2| + |a_3| + |a_4|$. □

(3) For a polynomial f of $\deg(f) \leqslant 5$, the condition $Uf(\zeta) \neq 0$ for $\zeta \in \mathbb{D}^3$ is also transparent:

$$q(\mathbb{D}) \cap (|a_3| + |a_5|)\mathbb{D} = \emptyset,$$

where $f = a_1 z + a_2 z^2 + a_3 z^3 + a_4 z^4 + a_5 z^5$ and $q(\zeta) = a_1 + a_2\zeta + a_4\zeta^2$. □

For a generic polynomial f of $\deg(f) \geqslant 6$, it is hardly possible to express $Uf \neq 0$ in reasonable terms explicitely depending on coefficients. We consider, however, a special case.

(4) *Let* $f = z(\lambda - z)^N$ *where* $|\lambda| > 1$ *and* $N \in \mathbb{N}$. *We show that*

$$N < \frac{\log 2}{\log(1 + \frac{1}{|\lambda|})} \Rightarrow f \text{ is cyclic in } H_0^2;$$

$$N > |\lambda| \Rightarrow f \text{ is NOT cyclic in } H_0^2.$$

If $N \geqslant 3$, *the case* $N = |\lambda|$ *is also non-cyclic.*

Indeed, since $f = z\sum_{n=0}^{N}\binom{N}{n}\lambda^{N-n}(-z)^n$ we have

$$Uf(\zeta) = \lambda^N + \sum_{n=1}^{N}\binom{N}{n}\lambda^{N-n}(-1)^n\zeta^{\alpha(n+1)}.$$

In particular, for $\zeta \in \mathbb{D}_2^\infty$ (of course, for the notation simplicity, we assume that λ is real and $\lambda > 1$),

$$|Uf(\zeta)| \geqslant \lambda^N - \sum_{n=1}^{N}\binom{N}{n}\lambda^{N-n} = 2\lambda^N - \sum_{n=0}^{N}\binom{N}{n}\lambda^{N-n} = 2\lambda^N - (\lambda+1)^N.$$

Therefore, if $2\lambda^N > (\lambda + 1)^N$, i.e. if

$$N < \frac{\log 2}{\log(1 + \frac{1}{\lambda})},$$

then f is a cyclic function. On the other hand, $Uf(0) = \lambda^N$, and assuming $s \in \mathbb{N}$ such that $2^s \leqslant N+1 < 2^{s+1}$, we obtain for $0 < t < 1$,

$$Uf(t,0,0,\dots) = \lambda^N + \sum_{k=1}^{s} \binom{N}{2^k - 1} \lambda^{N-2^k+1}(-1)^{2^k-1} t^k < \lambda^N - N\lambda^{N-1}t^s.$$

Therefore, if $N > \lambda$, there exists $0 < t_0 < 1$ making Uf negative $Uf(t_0, 0, \dots) < 0$, and by continuity, we find t, $0 < t < t_0$ such that $Uf(t,0,\dots) = 0$. Hence, f is not cyclic in H_0^2. In fact, taking into account the term $k = 2$ of the expansion for $Uf(t,0,\dots)$ (assuming $N \geqslant 3$), we obtain the same conclusion also for $\lambda = N$. □

In particular, for $\lambda = 3$, the function f is cyclic iff $N \leqslant 2$, for $\lambda = 4$ — iff $N \leqslant 3$, for $\lambda = 5$ — cyclic for $N \leqslant 3$ but non-cyclic for $N \geqslant 5$, etc.

(5) Cyclic functions depending on a combination of variables. *Let $\varphi \in H^2(\mathbb{D})$, $\alpha \in \mathbb{Z}_+(\infty)$ and $F(\zeta) = \varphi(\zeta^\alpha)$ for $\zeta \in \mathbb{D}_2^\infty$. Then, F is cyclic in $H^2(\mathbb{D}_2^\infty)$ if and only if φ is an outer function. In particular, whatever is $n \in \mathbb{N} \setminus \{1\}$, a function $f = \sum_{k\geqslant 0} a_k z^{n^k}$ is (T_n) cyclic if and only if $\varphi = \sum_{k\geqslant 0} a_k z^k$ is outer.*

Example. — Polynomials

$$p_1 = a_0 z + a_1 z^2 + a_2 z^4 + a_3 z^8 + a_4 z^{16}$$

and

$$p_2 = a_0 z + a_1 z^{12} + a_2 z^{144} + a_3 z^{1728} + a_4 z^{20736}$$

are (T_n)-cyclic or not simultaneously (and if and only if $\varphi = \sum_{k=0}^{4} a_k z^k$ is outer, i.e. all roots are in $\mathbb{C} \setminus \mathbb{D}$), but — in general — this is not the case for

$$p_3 = a_0 z + a_1 z^2 + a_2 z^3 + a_3 z^4 + a_4 z^5.$$

Indeed, the claim immediately follows from Theorem 3.4(3) and property (11), Section 2. For $\varphi = (z-1)^2$, we have $p_3 = z(z-1)^2$, and it is easy to see that Up_3 vanish' on $\mathbb{D}^2$(so, p_3 is not (T_n)-cyclic, whereas p_1 and p_2 are). □

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Nikolai NIKOLSKI
Université de Bordeaux 1
UFR de Mathématiques et Informatique
351 cours de la Libération
33405 Talence
France

Steklov Institute of Mathematics
27 Fontanka
191023, St.Petersburg
Russia
nikolski@math.u-bordeaux1.fr