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ON SEMISIMPLE CLASSES AND SEMISIMPLE CHARACTERS IN FINITE REDUCTIVE GROUPS

by Olivier BRUNAT

Abstract. — In this article, we study the elements with disconnected centralizer in the Brauer complex associated to a simple algebraic group $\mathbf{G}$ defined over a finite field with corresponding Frobenius map F and derive the number of F-stable semisimple classes of $\mathbf{G}$ with disconnected centralizer when the order of the fundamental group has prime order. We also discuss extendibility of semisimple characters of the fixed point subgroup $\mathbf{G}^F$ to their inertia group in the full automorphism group. As a consequence, we prove that "twisted" and "untwisted" simple groups of type E_6 are "good" in defining characteristic, which is a contribution to the general program initialized by Isaacs, Malle and Navarro to prove the McKay Conjecture in representation theory of finite groups.

Résumé. — Dans cet article, on étudie les éléments de centralisateur non connexe du complexe de Brauer associé à un groupe algébrique simple $\mathbf{G}$ défini sur un corps fini. On déduit alors, lorsque le groupe fondamental est d'ordre premier, le nombre de classes de conjugaison semi-simples rationnelles de $\mathbf{G}$ dont les représentants ont un centralisateur non connexe. On étudie également l'extensibilité des caractères semisimples du groupe des points fixes $\mathbf{G}^F$ à leur groupe d'inertie dans le groupe des automorphismes de $\mathbf{G}^F$, où F est l'endomorphisme de Frobenius de $\mathbf{G}$ relatif à la structure rationnelle. Comme conséquence, on montre qu'un groupe fini simple de type E_6 vérifie la condition inductive de McKay en caractéristique naturelle. Ce travail s'inscrit dans le programme général initialisé par Isaacs, Malle et Navarro pour prouver la conjecture de McKay en théorie des représentations des groupes finis.

1. Introduction

This article is concerned with the semisimple characters of finite reductive groups. A finite reductive group is the fixed-point subgroup $\mathbf{G}^F$ of a connected reductive group $\mathbf{G}$ defined over the finite field $\mathbb{F}_q$ of characteristic $p > 0$, where $F : \mathbf{G} \to \mathbf{G}$ is the Frobenius map corresponding to this

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$\mathbb{F}_q$-structure. The semisimple characters of $\mathbf{G}^F$ are the constituents of the duals of Gelfand-Graev characters (for the Alvis-Curtis duality) and play an important role in the ordinary representation theory of $\mathbf{G}^F$, because, apart from a few exceptions, they are the p'-characters of $\mathbf{G}^F$ (that is the irreducible characters of $\mathbf{G}^F$ whose degree is prime to p). In the following, we will write $\mathrm{Irr}_s(\mathbf{G}^F)$ for the set of semisimple characters of $\mathbf{G}^F$. One of the aims of this work is to study these characters, compute their number, understand the action of the automorphism group of $\mathbf{G}^F$ on $\mathrm{Irr}_s(\mathbf{G}^F)$, and determine the extendibility of $\chi \in \mathrm{Irr}_s(\mathbf{G}^F)$ to its inertia group in the full automorphism group. These questions are crucial, for example in order to prove that $\mathbf{G}^F$ satisfies the inductive McKay condition at the prime p.

Using Deligne-Lusztig theory [10], Lusztig has shown that the irreducible characters of $\mathbf{G}^F$ can be partitioned into series (the so-called rational Lusztig series) labelled by the semisimple classes of $\mathbf{G}^{*F^*}$, where $(\mathbf{G}^*, F^*)$ denotes a pair dual to $(\mathbf{G}, F)$. If such a series is labelled by a semisimple class of $\mathbf{G}^{*F^*}$ with representative s, then it contains $|A_{\mathbf{G}^*}(s)^{F^*}|$ semisimple characters, where $A_{\mathbf{G}^*}(s) = \mathrm{C}_{\mathbf{G}^*}(s)/\,\mathrm{C}^\circ_{\mathbf{G}^*}(s)$ is the component group of s. In fact, $\mathrm{Irr}_s(\mathbf{G}^F)$ can be parametrized in a natural way by pairs (s, ξ) where s runs over a set of representatives of the semisimple classes of $\mathbf{G}^{*F^*}$ and $\xi \in \mathrm{Irr}(A_{\mathbf{G}^*}(s)^{F^*})$. So, in order to understand $\mathrm{Irr}_s(\mathbf{G}^F)$, we have to particularly consider the semisimple classes of $\mathbf{G}^{*F^*}$ and their component groups.

In [7], we explicitly computed the number of semisimple classes of $\mathbf{G}^{*F^*}$ when $\mathbf{G}^*$ is simple and p is a good prime for $\mathbf{G}^*$. For that, we used the theory of Gelfand-Graev characters for connected reductive groups with disconnected center, developed by Digne-Lehrer-Michel in [12] and [13]. This method gives a lot of information on the F^*-stable semisimple $\mathbf{G}^*$-classes of $\mathbf{G}^*$ with disconnected centralizer (by the centralizer of an F^*-stable class, we mean the centralizer of a fixed F^*-stable representative), which allows us to prove the inductive McKay condition in defining characteristic for simple groups coming from simple algebraic groups with fundamental group of order 2; see [5, Theorem 1.1]. However, we cannot derive from [7] all information that we need, especially phenomena appearing only in the algebraic group, as for example the description of the F^*-stable semisimple classes with disconnected centralizer such that the fixed-point subgroup $A_{\mathbf{G}^*}(s)^{F^*}$ is trivial (s is any F^*-stable representative); see Remark 2.14.

In this work, we will consider the Brauer complex, initially introduced by J. Humphreys in [18] for describing p-modular representation theory of $\mathbf{G}^F$. When $\mathbf{G}$ is a simple simply-connected group, Deriziotis proved in [11] that the F-stable semisimple classes of $\mathbf{G}$ (and thus, the semisimple classes of $\mathbf{G}^F$) are parametrized by the faces of the Brauer complex of maximal dimension. We generalize here some results to any simple algebraic groups using an approach of Bonnafé [2].

Moreover, we are interested in the problem of the extendibility of semisimple characters of $\mathbf{G}^F$ to their inertia groups in $\mathrm{Aut}(\mathbf{G}^F)$. Digne-Michel [15] and Malle [21] developed a theory of Deligne-Lusztig characters for finite disconnected reductive groups. Using this theory, Sorlin [23] constructed extensions of Gelfand-Graev characters of $\mathbf{G}^F$ to $\mathbf{G}^F \rtimes \langle \sigma \rangle$, where σ is a quasi-central semisimple or unipotent automorphism of $\mathbf{G}$. We will use results of [23] in order to prove that, under certain assumptions, the semisimple characters of $\mathbf{G}^F$ are extendible to their inertia groups in the full automorphism group.

Finally, recall that the McKay Conjecture asserts that for any finite group G, if $\mathrm{Irr}_{p'}(G)$ denotes the set of p'-characters of G, then $|\mathrm{Irr}_{p'}(G)| = |\mathrm{Irr}_{p'}(\mathrm{N}_G(P))|$, where P is a fixed p-Sylow subgroup of G. In [20], Isaacs-Malle-Navarro proved a reduction theorem of this conjecture to finite simple groups. They showed that if every simple group satisfies a refined property, the so-called inductive McKay condition (see [20, §10] for more details), then the McKay Conjecture holds for all finite groups. As an application of our results, we will prove that "twisted" and "untwisted" finite simple groups of type E_6 satisfy the inductive McKay condition in defining characteristic.

The paper is organized as follows. In Section 2, we introduce the Brauer complex of $\mathbf{G}$ and describe the faces containing points with disconnected centralizer; see Theorem 2.5. Then we compute the number of F-stable semisimple classes of $\mathbf{G}$ with disconnected centralizer when $\mathbf{G}$ is not of type D_{2n} and has fundamental group of prime order; see Proposition 2.11. Note that this result requires no condition on q. Furthermore, if $\mathbf{G}$ is not of type D_{2n}, we describe the F-stable points of the Brauer complex in the case that F acts trivially on the center of $\mathbf{G}$; see Propositions 2.16 and 2.17. As first consequences we prove that if p is odd, then the McKay Conjecture holds for $\mathbf{G}^F$ at the prime p, where $\mathbf{G}$ is a simple and simply-connected group of type D_{2n+1}; see Remark 2.18. It also holds for $p = 2$ (resp. $p = 3$) when $\mathbf{G}$ is a simple and simply-connected group of type E_6 (resp. E_7); see Remark 2.15. In Section 3, we recall the construction of semisimple characters and give the action of automorphisms of $\mathbf{G}^F$ on $\mathrm{Irr}_s(\mathbf{G}^F)$. Then we discuss extendibility in a special case of these characters to their inertia groups; see Proposition 3.13. Finally, in Section 4 we prove the inductive McKay condition in defining characteristic (for $p > 3$) for "untwisted" simple groups of type E_6 in Theorem 4.9, and for the "twisted" version in Theorem 4.10. Note that these methods also prove that the inductive McKay condition is satisfied at the prime p by simple groups of Lie type of type A_{2n} such that $2n+1$ is a prime number distinct from p; see Proposition 4.12.

It should be noted that recently, after the submission of this paper, using a work of Maslowski [22], Späth showed that all finite simple groups of Lie type are "good" for the defining characteristic; see [24].

2. Semisimple classes with disconnected centralizers

2.1. Notation

Throughout this paper, $\mathbf{G}$ denotes a simple algebraic group over $\overline{\mathbb{F}}_p$. We fix a maximal torus $\mathbf{T}$ contained in a Borel subgroup $\mathbf{B}$ of $\mathbf{G}$. Let Φ be the root system of $\mathbf{G}$ relative to $\mathbf{T}$. We write Φ^+ and Δ for the set of positive roots and the set of simple roots of Φ corresponding to $\mathbf{B}$. We denote by $X(\mathbf{T})$ and $Y(\mathbf{T})$ the groups of characters and cocharacters of $\mathbf{T}$ and write $\langle\,,\rangle : X(\mathbf{T}) \times Y(\mathbf{T}) \to \mathbb{Z}$ for the duality pairing between $X(\mathbf{T})$ and $Y(\mathbf{T})$. For $\alpha \in \Phi$, we denote by $\alpha^\vee$ the coroot of α and write $\Phi^\vee = \{\alpha^\vee \mid \alpha \in \Phi\}$. We define $V = \mathbb{Q} \otimes_{\mathbb{Z}} Y(\mathbf{T})$ and $V^* = \mathbb{Q} \otimes_{\mathbb{Z}} X(\mathbf{T})$, and extend $\langle\,,\rangle$ to a nondegenerate bilinear form

$$\langle\,,\rangle : V^* \times V \to \mathbb{Q}.$$

Let $\mathbf{G}_{\mathrm{sc}}$ (resp. $\mathbf{G}_{\mathrm{ad}}$) be the simply-connected version (resp. adjoint version) of $\mathbf{G}$. We denote by $\pi_{\mathrm{sc}} : \mathbf{G}_{\mathrm{sc}} \to \mathbf{G}$ and $\pi_{\mathrm{ad}} : \mathbf{G} \to \mathbf{G}_{\mathrm{ad}}$ corresponding isogenies. Write $\mathbf{T}_{\mathrm{sc}} = \pi_{\mathrm{sc}}^{-1}(\mathbf{T})$ and $\mathbf{T}_{\mathrm{ad}} = \pi_{\mathrm{ad}}(\mathbf{T})$. Then $\mathbf{T}_{\mathrm{sc}}$ and $\mathbf{T}_{\mathrm{ad}}$ are maximal tori of $\mathbf{G}_{\mathrm{sc}}$ and $\mathbf{G}_{\mathrm{ad}}$, and the surjective homomorphisms $\pi_{\mathrm{sc}} : \mathbf{T}_{\mathrm{sc}} \to \mathbf{T}$ and $\pi_{\mathrm{ad}} : \mathbf{T} \to \mathbf{T}_{\mathrm{ad}}$ induce injective homomorphisms $\pi_{\mathrm{sc},X} : X(\mathbf{T}) \to X(\mathbf{T}_{\mathrm{sc}}), \chi \to \chi \circ \pi_{\mathrm{sc}}$ and $\pi_{\mathrm{ad},X} : X(\mathbf{T}_{\mathrm{ad}}) \to X(\mathbf{T}), \chi \to \chi \circ \pi_{\mathrm{ad}}$. Using $\pi_{\mathrm{sc},X}$ and $\pi_{\mathrm{ad},X}$, we identify $X(\mathbf{T})$ with a subgroup of $X(\mathbf{T}_{\mathrm{sc}})$ containing $X(\mathbf{T}_{\mathrm{ad}})$, such that the root systems $\pi_{\mathrm{sc}}(\Phi)$ and $\pi_{\mathrm{ad}}^{-1}(\Phi)$ of $\mathbf{G}_{\mathrm{sc}}$ and $\mathbf{G}_{\mathrm{ad}}$ are identified with Φ. Similarly, using the injective morphisms $\pi_{\mathrm{sc},Y} : Y(\mathbf{T}_{\mathrm{sc}}) \to Y(\mathbf{T}), \gamma \to \pi_{\mathrm{sc}} \circ \gamma$ and $\pi_{\mathrm{ad},Y} : Y(\mathbf{T}) \to Y(\mathbf{T}_{\mathrm{ad}}), \gamma \to \pi_{\mathrm{ad}} \circ \gamma$ induced by π_{sc} and π_{ad}, the group $Y(\mathbf{T})$ is viewed as a subgroup of $Y(\mathbf{T}_{\mathrm{ad}})$ containing $Y(\mathbf{T}_{\mathrm{sc}})$ such that $\pi_{\mathrm{sc},Y}^{-1}(\Phi^\vee)$ and $\pi_{\mathrm{ad},Y}(\Phi^\vee)$ are identified with $\Phi^\vee$. Note that $V = \mathbb{Q} \otimes_{\mathbb{Z}} Y(\mathbf{T}_{\mathrm{sc}}) = \mathbb{Q} \otimes_{\mathbb{Z}} Y(\mathbf{T}_{\mathrm{ad}})$, $V^* = \mathbb{Q} \otimes_{\mathbb{Z}} X(\mathbf{T}_{\mathrm{sc}}) =$

$\mathbb{Q}\otimes_{\mathbb{Z}} X(\mathbf{T}_{\mathrm{ad}})$, and the linear maps $\pi_{\mathrm{sc},X}$ and $\pi_{\mathrm{sc},Y}$ (resp. $\pi_{\mathrm{ad},X}$ and $\pi_{\mathrm{ad},Y}$) are adjoint maps with respect to $\langle\,,\,\rangle$. We define the group of weights by

$$\Lambda = \{\lambda \in V^* \mid \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z} \quad \forall\, \alpha \in \Phi\}. \tag{2.1}$$

Recall that $X(\mathbf{T}_{\mathrm{ad}}) = \mathbb{Z}\Phi$ and $X(\mathbf{T}_{\mathrm{sc}}) = \Lambda$, and the fundamental group of Φ is the finite group $\Lambda/\mathbb{Z}\Phi = X(\mathbf{T}_{\mathrm{sc}})/X(\mathbf{T}_{\mathrm{ad}})$. Now, we denote by $(\varpi_\alpha^\vee)_{\alpha\in\Delta}$ and $(\varpi_\alpha)_{\alpha\in\Delta}$ the dual bases with respect to $\langle\,,\,\rangle$ of Δ and $\Delta^\vee = \{\alpha^\vee \mid \alpha \in \Delta\}$, respectively. Since $X(\mathbf{T}_{\mathrm{ad}}) = \mathbb{Z}\Phi$ and $X(\mathbf{T}_{\mathrm{sc}}) = \langle\, \varpi_\alpha,\, \alpha \in \Delta\,\rangle_{\mathbb{Z}}$, we deduce that

$$Y(\mathbf{T}_{\mathrm{sc}}) = \bigoplus_{\alpha\in\Delta} \mathbb{Z}\alpha^\vee \quad \text{and} \quad Y(\mathbf{T}_{\mathrm{ad}}) = \bigoplus_{\alpha\in\Delta} \mathbb{Z}\,\varpi_\alpha^\vee. \tag{2.2}$$

We denote by $W = \mathrm{N}_{\mathbf{G}}(\mathbf{T})/\mathbf{T}$ the Weyl group of $\mathbf{G}$. Then W acts on $X(\mathbf{T})$ and on $Y(\mathbf{T})$ by

$$ {}^w\chi(t) = \chi(t^w) \quad \text{and} \quad \gamma^w(t) = \gamma(t)^w, \tag{2.3}$$

for $\gamma \in Y(\mathbf{T})$, $\chi \in X(\mathbf{T})$ and $t \in \mathbf{T}$. In particular, we have $\langle\, {}^w\chi, \gamma\,\rangle = \langle\, \chi, \gamma^w\,\rangle$ and $W(\Phi) = \Phi$ and $W(\Phi^\vee) = \Phi^\vee$. Recall from [9, §1.9] that $W = \langle\, s_\alpha \mid \alpha \in \Phi\,\rangle$, and for $\chi \in X(\mathbf{T})$ and $\gamma \in Y(\mathbf{T})$, we have $s_\alpha(\chi) = \chi - \langle\, \chi, \alpha^\vee\,\rangle\alpha$ and $s_\alpha(\gamma) = \gamma - \langle\, \alpha, \gamma\,\rangle\alpha^\vee$.

2.2. Semisimple classes with disconnected centralizers

We denote by $\mathbb{Q}_{p'}$ the additive subgroup of $\mathbb{Q}$ of rational numbers of the form a/b with b not divisible by p, and write $\tilde{\imath} : \mathbb{Q} \to \overline{\mathbb{F}}_p^\times$ for the composition of the morphisms $\mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to \mathbb{Q}_{p'}/\mathbb{Z} \to \overline{\mathbb{F}}_p^\times$, where the second morphism is obtained by killing the p-torsion subgroup of $\mathbb{Q}/\mathbb{Z}$, and the last morphism is a fixed isomorphism between $\mathbb{Q}_{p'}/\mathbb{Z}$ and $\overline{\mathbb{F}}_p^\times$ as in [9, 3.1.3]. We set $\tilde{\imath}_{\mathbf{T}} : \mathbb{Q}\otimes_{\mathbb{Z}} Y(\mathbf{T}) \to \mathbf{T}$, $r\otimes\gamma \to \gamma(\tilde{\imath}(r))$, which induces a bijective morphism $\mathbb{Q}_{p'}/\mathbb{Z}\otimes_{\mathbb{Z}} Y(\mathbf{T}) \to \mathbf{T}$; see [9, 3.1.2, 3.1.3]. Furthermore, the surjective homomorphism from $\mathbb{Q}_{p'}\otimes_{\mathbb{Z}} Y(\mathbf{T})$ onto $\mathbb{Q}_{p'}/\mathbb{Z}\otimes_{\mathbb{Z}} Y(\mathbf{T})$ with kernel $Y(\mathbf{T})$ induces the isomorphism of groups

$$\mathbf{T} \simeq \mathbb{Q}_{p'}/\mathbb{Z}\otimes_{\mathbb{Z}} Y(\mathbf{T}) \simeq \mathbb{Q}_{p'}\otimes_{\mathbb{Z}} Y(\mathbf{T})/Y(\mathbf{T}). \tag{2.4}$$

Note that the action of W on $Y(\mathbf{T})$ defined in Equation (2.3) can be naturally extended to $\mathbb{Q}_{p'}/\mathbb{Z}\otimes_{\mathbb{Z}} Y(\mathbf{T})$ and to $(\mathbb{Q}_{p'}\otimes_{\mathbb{Z}} Y(\mathbf{T}))/Y(\mathbf{T})$ and is compatible with the isomorphisms of Equation (2.4). We define $\overline{W}_a = Y(\mathbf{T}) \rtimes W$. Note that $\overline{W}_a$ acts on V as a group of affine transformations by

$$(\gamma\cdot w)(\lambda\otimes\gamma') = \lambda\otimes\gamma'^w + \gamma,$$

for γ, $\gamma' \in Y(\mathbf{T})$, $\lambda \in \mathbb{Q}$ and $w \in W$. So, we have

$$\left(\mathbb{Q}_{p'} \otimes_{\mathbb{Z}} Y(\mathbf{T})/Y(\mathbf{T})\right)/W = \mathbb{Q}_{p'} \otimes_{\mathbb{Z}} Y(\mathbf{T})/\overline{W}_a.$$

Since the set of semisimple classes $s(\mathbf{G})$ of $\mathbf{G}$ is in bijection with the set $\mathbf{T}/W$ of W-orbits on $\mathbf{T}$ (see [9, 3.7.1]), we deduce that $s(\mathbf{G})$ is in bijection with

$$(\mathbb{Q}_{p'} \otimes_{\mathbb{Z}} Y(\mathbf{T}))/\overline{W}_a, \tag{2.5}$$

Now, we write α_0 for the highest root of Φ (with respect to the height defined by Δ) and put

$$\alpha_0 = \sum_{\alpha \in \Delta} n_\alpha\, \alpha, \tag{2.6}$$

with $n_\alpha \in \mathbb{N}^*$. We define the affine Weyl group W_a of V as the subgroup of affine transformations of V generated by s_α (for $\alpha \in \Delta$) and $s_{\alpha_0,1} = s_{\alpha_0} + \alpha_0^\vee$. Then by [4, p.174], the alcove

$$\mathscr{C} = \{\lambda \in V \mid \langle \alpha, \lambda \rangle \geqslant 0\ \forall \alpha \in \Delta,\ \langle \alpha_0, \lambda \rangle \leqslant 1\}$$

is a fundamental domain for the action of W_a on V. Recall that $W_a = Y(\mathbf{T}_{\mathrm{sc}}) \rtimes W$ (see [4, VI§2, Prop 1]). In particular, $W_a \leqslant \overline{W}_a$, which implies that every $\overline{W}_a$-orbit of V contains at least one element of $\mathscr{C}$. We write

$$\widetilde{\Delta} = \Delta \cup \{-\alpha_0\}$$

for the affine Dynkin diagram of $\mathbf{G}$ and $\widetilde{\Delta}_{\min} = \{\alpha \in \widetilde{\Delta} \mid n_\alpha = 1\}$, with the convention that $n_{-\alpha_0} = 1$. For $\alpha \in \widetilde{\Delta}_{\min}$, we set $z_\alpha = w_\alpha w_0$, where w_0 and w_α are the longest elements of W and $W_{\Delta\setminus\{\alpha\}}$, respectively (note that $w_{-\alpha_0} = w_0$ and $z_{-\alpha_0} = 1$). Then z_α induces an automorphism of the extended Dynkin diagram $\widetilde{\Delta}$. We define

$$\mathcal{A} = \{z_\alpha \mid \alpha \in \widetilde{\Delta}_{\min}\}.$$

Recall that $\mathcal{A}$ is isomorphic to the center $\mathrm{Z}(\mathbf{G}_{\mathrm{sc}})$ as follows. By [4, Corollaire p.177], we have $\mathrm{Z}(\mathbf{G}_{\mathrm{sc}}) = \{\tilde{\iota}_{\mathbf{T}}(\varpi_\alpha^\vee) \mid \alpha \in \widetilde{\Delta}_{\min}\}$. Moreover, for $z \in \mathrm{Z}(\mathbf{G}_{\mathrm{sc}})$, there is a unique element $w_z \in W$ (obtained as the projection on W of any element ω of W_a satisfying $\omega(\mathscr{C}) = y + \mathscr{C}$ for $y \in \mathbb{Q}_{p'} \otimes Y(\mathbf{T})$ such that $\tilde{\iota}_{\mathbf{T}}(y) = z$; note that w_z is well-defined because it does not depend on y). Then [4, Proposition 6 p.176] implies that $z_\alpha = w_{\tilde{\iota}_{\mathbf{T}}(\varpi_\alpha^\vee)}$ for $\alpha \in \widetilde{\Delta}_{\min}$. Furthermore, the map

$$\mathrm{Z}(\mathbf{G}_{\mathrm{sc}}) \to \mathcal{A}, \quad \tilde{\iota}_{\mathbf{T}}(\varpi_\alpha^\vee) \mapsto z_\alpha, \tag{2.7}$$

is an isomorphism of groups; see [4, VI.§2.3].

Now, we write $\widehat{W}_a = Y(\mathbf{T}_{\mathrm{ad}}) \rtimes W$ and $\Gamma_{\mathscr{C}}$ for the subgroup of $\widehat{W}_a$ which stabilizes $\mathscr{C}$. Then in [4, VI,§2 Prop 6], the following result is proven.

Proposition 2.1. — *The group of automorphisms of $\widetilde{\Delta}$ induced by elements of W is $\mathcal{A}$. For $\alpha \in \widetilde{\Delta}_{\min}$, we set $f_\alpha = z_\alpha + \varpi_\alpha^\vee$. Then $f_\alpha \in \widehat{W}_a$ and we have*

$$\Gamma_{\mathscr{C}} = \{f_\alpha \mid \alpha \in \widetilde{\Delta}_{\min}\}.$$

Moreover, the map

$$\varpi^\vee : \mathcal{A} \to Y(\mathbf{T}_{\mathrm{ad}})/Y(\mathbf{T}_{\mathrm{sc}}),\, z_\alpha \mapsto \varpi_\alpha^\vee + Y(\mathbf{T}_{\mathrm{sc}}),$$

is an isomorphism of groups.

Note that, by composition of $\varpi^\vee$ with the isomorphism defined in Equation (2.7), we can identified the quotient $Y(\mathbf{T}_{\mathrm{ad}})/Y(\mathbf{T}_{\mathrm{sc}})$ with $\mathrm{Z}(\mathbf{G}_{\mathrm{sc}})$.

For $\lambda \in V$, we will denote by $(\lambda_\alpha)_{\alpha\in\widetilde{\Delta}}$ its affine coordinates, that is, the unique family of rational numbers such that $\sum_{\alpha\in\widetilde{\Delta}} \lambda_\alpha = 1$ and $\lambda = \sum_{\alpha\in\widetilde{\Delta}} \frac{\lambda_\alpha}{n_\alpha}\varpi_\alpha^\vee$, where n_α are the integers defined in Equation (2.6) and $\varpi_{-\alpha_0}^\vee = 0$. Note that $\lambda \in \mathscr{C}$ if and only if $\lambda_\alpha \geqslant 0$ for every $\alpha \in \widetilde{\Delta}$; see [4, Corollaire p.175].

Now, following [2], we define the subgroup $\mathcal{A}_{\mathbf{G}}$ of $\mathcal{A}$ to be the inverse image of $Y(\mathbf{T})/Y(\mathbf{T}_{\mathrm{sc}})$ under $\varpi^\vee$. We also define

$$\widetilde{\Delta}_{\min,\mathbf{G}} = \{\alpha \in \widetilde{\Delta}_{\min} \mid z_\alpha \in \mathcal{A}_{\mathbf{G}}\} \quad \text{and} \quad \Gamma_{\mathbf{G}} = \{f_\alpha \mid \alpha \in \widetilde{\Delta}_{\min,\mathbf{G}}\}. \tag{2.8}$$

Then Bonnafé proves in [2, (3.10), Cor. 3.12, Prop. 3.13]

Theorem 2.2. —

(1) *For $\alpha \in \widetilde{\Delta}_{\min}$ and $\lambda = (\lambda_\beta)_{\beta\in\widetilde{\Delta}} \in \mathscr{C}$, we have $f_\alpha(\lambda)_\beta = \lambda_{z_\alpha^{-1}(\beta)}$ for $\beta \in \widetilde{\Delta}$.*

(2) *The points λ, $\mu \in \mathscr{C}$ are in the same $\overline{W}_a$-orbit if and only if there is $z \in \mathcal{A}_{\mathbf{G}}$ such that $z(\lambda) - \mu \in Y(\mathbf{T})$.*

(3) *Let $[s]_{\mathbf{G}} \in s(\mathbf{G})$ be a semisimple class with representative $s \in \mathbf{G}$ corresponding to a $\overline{W}_a$-orbit $[\lambda]$ (here, λ denotes a representative in $\mathscr{C}$) on $\mathscr{C}$. Then $I_\lambda = \{\alpha \in \widetilde{\Delta} \mid \lambda_\alpha = 0\}$ is a basis of the root system of $\mathrm{C}_{\mathbf{G}}(s)^\circ$ and the component group $A_{\mathbf{G}}(s) = \mathrm{C}_{\mathbf{G}}(s)/\mathrm{C}_{\mathbf{G}}(s)^\circ$ is isomorphic to*

$$A_{\mathbf{G}}(\lambda) = \{z \in \mathcal{A}_{\mathbf{G}} \mid \forall \alpha \in \widetilde{\Delta},\, \lambda_{z(\alpha)} = \lambda_\alpha\}.$$

For $\alpha \in \widetilde{\Delta}_{\min}$, we write

$$V_\alpha = \{v \in V \mid f_\alpha(v) = v\} \tag{2.9}$$

for the affine subspace of V of the invariants under f_α.

LEMMA 2.3. — *Let $\alpha, \alpha' \in \widetilde{\Delta}_{\min}$. If $\langle z_\alpha \rangle = \langle z_{\alpha'} \rangle$, then $V_\alpha = V_{\alpha'}$. Moreover, if Ω_α denotes the set of $\langle z_\alpha \rangle$-orbits on $\widetilde{\Delta}$, then*

$$\dim(V_\alpha) = |\Omega_\alpha| - 1.$$

In Table 2.1, we explicitly give the dimension of V_α for every irreducible extended affine Dynkin diagram. For the notation, we use the labelling of Bourbaki [4, Planche I-IX]. *In particular, note that $z_{\alpha_i}(-\alpha_0) = \alpha_i$.*

Type	$\lvert\mathcal{A}\rvert$	$\mathcal{A}$	z_α	$\dim(V_\alpha)$
A_n	$n+1$	$\langle z_{\alpha_1} \rangle$	$d \mid n+1, \quad z_{\alpha_1}^d$	$d-1$
B_n	2	$\langle z_{\alpha_1} \rangle$	z_{α_1}	$n-1$
C_n	2	$\langle z_{\alpha_n} \rangle$	z_{α_n}	$\lfloor \frac{n}{2} \rfloor$
D_{2n+1}	4	$\langle z_{\alpha_n} \rangle$	$z_{\alpha_n},\ z_{\alpha_{n-1}} = z_{\alpha_n}^{-1}$	$n-1$
			$z_{\alpha_1} = z_{\alpha_n}^2$	$2n-1$
D_{2n}	4	$\langle z_{\alpha_{n-1}} \rangle \times \langle z_{\alpha_n} \rangle$	$z_{\alpha_n},\ z_{\alpha_{n-1}}$	n
			$z_{\alpha_1} = z_{\alpha_n} z_{\alpha_{n-1}}$	$2n-2$
E_6	3	$\langle z_{\alpha_1} \rangle$	$z_{\alpha_1},\ z_{\alpha_1}^{-1}$	2
E_7	2	$\langle z_{\alpha_1} \rangle$	z_{α_1}	4
E_8	1	$\{1\}$	1	8
G_2	1	$\{1\}$	1	2
F_4	1	$\{1\}$	1	4

Table 2.1. The dimension of the invariant subspace

2.3. Fixed-points under a Frobenius map

Now, we suppose that $\mathbf{G}$ is defined over the finite field $\mathbb{F}_q$ (for q a p-power) and we denote by $F : \mathbf{G} \to \mathbf{G}$ the corresponding Frobenius map. We assume that F is the composition of a split Frobenius with a graph automorphism coming from a symmetry ρ of Δ (which could be trivial). We suppose that the maximal torus $\mathbf{T}$ of $\mathbf{G}$ is chosen to be F-stable and contained in an F-stable Borel subgroup $\mathbf{B}$. Moreover, F induces Frobenius maps on $\mathbf{G}_{\rm sc}$ and $\mathbf{G}_{\rm ad}$ (also denoted by F) such that F and the isogenies $\pi_{\rm sc}$ and $\pi_{\rm ad}$ commute. We can define an F-action on $X(\mathbf{T})$ and $Y(\mathbf{T})$ by setting

$$F(\chi)(t)=\chi(F(t)) \quad \text{and} \quad F(\gamma)(x)=F(\gamma(x)),$$

for $\chi\in X(\mathbf{T})$, $\gamma\in Y(\mathbf{T})$, $t\in\mathbf{T}$ and $x\in\overline{\mathbb{F}}_p^{\times}$. The F-action on $Y(\mathbf{T})$ extends naturally to $\mathbb{Q}_{p'}/\mathbb{Z}\otimes_{\mathbb{Z}}Y(\mathbf{T})$ and to $(\mathbb{Q}_{p'}\otimes_{\mathbb{Z}}Y(\mathbf{T}))/Y(\mathbf{T})$ and is compatible with the isomorphisms of Equation (2.4). Since the set of F-stable semisimple classes of $\mathbf{G}$ is in bijection with the set $(\mathbf{T}/W)^F$; see [9, 3.7.2], it follows that it is in bijection with the set $(\mathbb{Q}_{p'}\otimes_{\mathbb{Z}}Y(\mathbf{T})/\overline{W}_a)^F$.

We put $\overline{W}_{a,q}=F^{-1}(Y(\mathbf{T}))\rtimes W$. Then the map

$$F:\overline{W}_{a,q}\to\overline{W}_a,\ yw\mapsto F(y)F(w),$$

is an isomorphism of groups. Note that $\overline{W}_a$ is a subgroup of $\overline{W}_{a,q}$. Furthermore, for $v\in V$ and $g\in\overline{W}_{a,q}$, we have

$$F(g)(F(v))=F(g(v)). \tag{2.10}$$

We define $W_{a,q}=F^{-1}(Y(\mathbf{T}_{\rm sc}))\rtimes W$ and $\widehat{W}_{a,q}=F^{-1}(Y(\mathbf{T}_{\rm ad}))\rtimes W$. In particular, we have $W_{a,q}\leqslant\overline{W}_{a,q}\leqslant\widehat{W}_{a,q}$. Moreover, Equation (2.10) implies that the set

$$\mathscr{C}_q=\{v\in V\mid F(v)\in\mathscr{C}\}$$

is a fundamental region for the $W_{a,q}$-action on V. We denote by $h_q:V\to V$ the homothety with origin 0 and ratio $\frac{1}{q}$. Recall that F acts on V by $h_q^{-1}\cdot F_0$, where $F_0:V\to V$ is a linear transformation defined by $F_0(\alpha)=\rho^{-1}(\alpha)$ for all $\alpha\in\Delta$. Note that V can be regarded as a euclidean space on which the elements of $\widehat{W}_a$ and F_0 act as isometries. In the following, we will denote by d_0 the associated metric. Since $\langle F_0(\chi),\gamma\rangle=\langle\chi,F_0(\gamma)\rangle$, we deduce that $F_0(\varpi_\alpha^\vee)=\varpi_{\rho^{-1}(\alpha)}^\vee$ for every $\alpha\in\Delta$. For $\alpha\in\widetilde{\Delta}$, we set

$$\varpi_{\alpha,q}^\vee=F^{-1}(\varpi_\alpha^\vee)=\frac{1}{q}\cdot\varpi_{\rho(\alpha)}^\vee.$$

Note that $n_{\rho^{-1}(\alpha)}=n_\alpha$. Hence, $\widetilde{\Delta}_{\min}$ is ρ^{-1}-stable. For $\alpha\in\widetilde{\Delta}_{\min}$, we define

$$f_{\alpha,q}=z_{\rho(\alpha)}+\varpi_{\alpha,q}^\vee.$$

Lemma 2.4. — *Let $\alpha \in \widetilde{\Delta}_{\min}$. Then the following diagram is commutative.*

$$\begin{tikzcd} \mathscr{C} \arrow[r, "f_\alpha"] \arrow[d, "F^{-1}"] & \mathscr{C} \arrow[d, "F^{-1}"] \\ \mathscr{C}_q \arrow[r, "f_{\alpha,q}"] & \mathscr{C}_q \end{tikzcd}$$

Proof. — For every $\alpha \in \widetilde{\Delta}_{\min}$, we set $I_\alpha = \Delta\backslash\{\alpha\}$ and $\Phi_\alpha = W_{I_\alpha}(I_\alpha)$ the corresponding root subsystem with basis I_α. Since $f_{-\alpha_0} = f_{-\alpha_0,q} = 1$, the lemma is trivially true for $-\alpha_0$. Fix now $\alpha \in \Delta$ with $n_\alpha = 1$. For every $x \in V$, we have

$$F^{-1} f_\alpha F(x) = \frac{1}{q} \cdot w^\vee_{\rho(\alpha)} + F_0^{-1} z_\alpha F_0(x).$$

Moreover, we have $\rho(I_\alpha) = I_{\rho_\alpha}$ and $\Phi_{\rho(\alpha)} = \rho(\Phi_\alpha)$. If w_α and $w_{\rho(\alpha)}$ are the longest elements of W_{I_α} and $W_{I_{\rho(\alpha)}}$, then $w_{\rho(\alpha)} = \rho(w_\alpha)$. Indeed, for every $\beta \in \Delta$, we have $s_{\rho(\beta)} = F_0^{-1} s_\beta F_0$ which implies that $\rho(w_\alpha) = F_0^{-1} w_\alpha F_0$. Now, if $\beta \in \Phi^+_{\rho(\alpha)}$, then $\rho(w_\alpha)(\beta) = F_0^{-1}(-F_0(\beta)) = -\beta$, as required. So, we have $F_0^{-1} z_\alpha F_0 = z_{\rho(\alpha)}$, and the result follows. □

Note that $W_{a,q}$ is the affine Weyl group generated by s_α for $\alpha \in \Delta$ and by the affine reflection $s_{\alpha_0,1/q} = s_{\alpha_0} + \frac{1}{q}\alpha_0^\vee$. We denote by $\mathcal{H}_q$ the collection of all hyperplanes defined by the affine reflections of $W_{a,q}$. Moreover, W_a is a subgroup of $W_{a,q}$. It follows that $\mathscr{C}$ is a union of certain transforms of $\mathscr{C}_q$ under $W_{a,q}$. We write E_q for the set of elements $\omega \in W_{a,q}$ such that $\omega(\mathscr{C}_q) \subseteq \mathscr{C}$ and define

$$\Omega_q = \{\omega(\mathscr{C}_q) \mid \omega \in E_q\}. \tag{2.11}$$

We now can prove

Theorem 2.5. — *Let $\alpha \in \widetilde{\Delta}_{\min}$. We define*

$$M_{\alpha,q} = \{\omega \in E_q \mid f_\alpha\left(\omega(\mathscr{C}_q)\right) = \omega(\mathscr{C}_q)\}.$$

Let V_α be the subspace of invariants as in Equation (2.9). If V_α is contained in some hyperplane of $\mathcal{H}_q$, then $|M_{\alpha,q}| = 0$. Otherwise, we have $|M_{\alpha,q}| = q^{\dim V_\alpha}$.

Proof. — Since $\widehat{W}_a$ is a subgroup of $\widehat{W}_{a,q}$, it follows that $f_\alpha \in \widehat{W}_{a,q}$. In particular, f_α permutes the elements of $\mathcal{H}_q$ and also the set of alcoves $\omega(\mathscr{C}_q)$ for $\omega \in W_{a,q}$. Hence, f_α permutes the elements of Ω_q (because for

$\omega \in E_q$, we have $f_\alpha(\omega(\mathscr{C}_q)) \subseteq \mathscr{C})$. We denote by $r_\alpha \in W_{a,q}$ the element such that

$$r_\alpha(\mathscr{C}_q) = f_\alpha(\mathscr{C}_q). \tag{2.12}$$

Let $\omega \in E_q$. Then we have

$$\begin{aligned}\omega(\mathscr{C}_q) &= f_\alpha(\omega(\mathscr{C}_q))\\ &= f_\alpha \omega f_\alpha^{-1}\,(f_\alpha(\mathscr{C}_q))\\ &= f_\alpha\omega f_\alpha^{-1} r_\alpha(\mathscr{C}_q),\end{aligned}$$

Furthermore, the group $\widehat{W}_{a,q}$ normalizes $W_{a,q}$, which implies that $f_\alpha\omega f_\alpha^{-1} \in W_{a,q}$. In particular, we have $f_\alpha\omega f_\alpha^{-1}r_\alpha \in W_{a,q}$. However, by [4, VI.§2.1], the group $W_{a,q}$ acts simply-transitively on the set of alcoves $\{\omega(\mathscr{C}_q) \mid \omega \in W_{a,q}\}$. It follows that $f_\alpha\omega f_\alpha^{-1}r_\alpha = \omega$, which implies

$$\omega^{-1} f_\alpha \omega = r_\alpha^{-1} f_\alpha \tag{2.13}$$

Note that $r_\alpha^{-1} f_\alpha(\mathscr{C}_q) = \mathscr{C}_q$ and $r_\alpha^{-1} f_\alpha \in \widehat{W}_{a,q}$. Proposition 2.1 implies that there is $\widetilde{\alpha} \in \widetilde{\Delta}_{\min}$ such that $f_{\widetilde{\alpha},q} = r_\alpha^{-1} f_\alpha$. We define

$$m : \widetilde{\Delta}_{\min} \to \widetilde{\Delta}_{\min},\ \alpha \mapsto \widetilde{\alpha}. \tag{2.14}$$

Hence, the following diagram is commutative

$$\begin{tikzcd} \mathscr{C}_q \arrow[r, "\omega"] \arrow[d, "f_{\widetilde{\alpha},q}"] & \omega(\mathscr{C}_q) \arrow[d, "f_\alpha"] \\ \mathscr{C}_q \arrow[r, "\omega"] & \omega(\mathscr{C}_q) \end{tikzcd}$$

For $A \subseteq V$ and $f : V \to V$, we denote by A^f the subset of elements of A invariant under f. So, Equation (2.13) implies that $\omega : \mathscr{C}_q^{f_{\widetilde{\alpha},q}} \to \omega(\mathscr{C}_q)^{f_\alpha}$ is bijective. Thus the sets $\mathscr{C}_q^{f_{\widetilde{\alpha},q}}$ and $\omega(\mathscr{C}_q)^{f_\alpha}$ are isometric. Let $x \in V_\alpha$. Suppose that x lies in the interior of some $\omega(\mathscr{C}_q)$ for some $\omega \in E_q$. Then $x = f_\alpha(x)$ lies in the interior of $f(\omega(\mathscr{C}_q))$, which implies that $f_\alpha(\omega(\mathscr{C}_q)) = \omega(\mathscr{C}_q)$. Conversely, if $\omega(\mathscr{C}_q)$ is f_α-invariant, then the interior of $\omega(\mathscr{C}_q)$ contains elements of V_α (for example, the isobarycentre of the simplex $\omega(\mathscr{C}_q)$, because f_α is an affine map). It follows that $M_{\alpha,q} = \emptyset$ if and only if V_α is contained in some hyperplane of $\mathcal{H}_q$.

Suppose that V_α is not contained in some hyperplane of $\mathcal{H}_q$. For $H \in \mathcal{H}_q$, consider the affine space $V_\alpha \cap H$. Then $\dim(V_\alpha \cap H) < \dim V_\alpha$ (otherwise,

V_α would be contained in H). This implies that

$$\mathrm{Vol}(\mathscr{C}^{f_\alpha}) = \mathrm{Vol}\left(\mathscr{C}^{f_\alpha}\setminus \bigcup_{H\in H_q,\, H\cap\mathscr{C}\neq\emptyset} H\right). \tag{2.15}$$

Furthermore, we have

$$\mathscr{C}^{f_\alpha}\setminus \bigcup_{H\in H_q,\, H\cap\mathscr{C}\neq\emptyset} H = \bigsqcup_{\omega\in M_{\alpha,q}} \omega(\overset{\circ}{\mathscr{C}_q})^{f_\alpha}. \tag{2.16}$$

Since $\mathrm{Vol}(\omega(\mathscr{C}_q)^{f_\alpha}) = \mathrm{Vol}(\omega(\overset{\circ}{\mathscr{C}_q})^{f_\alpha})$ for every $\omega\in M_{\alpha,q}$, Equations (2.15) and (2.16) imply that

$$\mathrm{Vol}(\mathscr{C}^{f_\alpha}) = \sum_{\omega\in M_{\alpha,q}} \mathrm{Vol}\left(\omega(\mathscr{C}_q)^{f_\alpha}\right). \tag{2.17}$$

Moreover, the sets $\mathscr{C}_q^{f_{\widetilde{\alpha},q}}$ and $\omega(\mathscr{C}_q)^{f_\alpha}$ are isometric. Thus they have the same volume, and Equation (2.17) implies that

$$|M_{\alpha,q}| = \frac{\mathrm{Vol}\left(\mathscr{C}^{f_\alpha}\right)}{\mathrm{Vol}\left(\mathscr{C}_q^{f_{\widetilde{\alpha},q}}\right)}. \tag{2.18}$$

Thanks to Lemma 2.4, we have $F(\mathscr{C}_q^{f_{\widetilde{\alpha},q}}) = \mathscr{C}^{f_{\widetilde{\alpha}}}$. By Equation (2.13) and Lemma 2.4, f_α and $f_{\widetilde{\alpha}}$ are conjugate (in the group of affine transformations of V). Hence, z_α and $z_{\widetilde{\alpha}}$ have the same order. If $\mathbf{G}$ is not of type D_{2n}, then $\mathcal{A}$ is cyclic. This implies that $\langle\, z_\alpha\,\rangle = \langle\, z_{\widetilde{\alpha}}\rangle$ and thanks to Lemma 2.3, we have $V_\alpha = V_{\widetilde{\alpha}}$. Hence, we have $\mathscr{C}^{f_\alpha} = \mathscr{C}^{f_{\widetilde{\alpha}}}$. If $\mathbf{G}$ is of type D_{2n}, then the invariant subspaces of f_α and $f_{\widetilde{\alpha}}$ have the same dimension (because f_α and $f_{\widetilde{\alpha}}$ are conjugate). Table 2.1 implies that there is $i\geqslant 0$ such that $\widetilde{\alpha}=\rho^i(\alpha)$. Then $f_{\widetilde{\alpha}} = F_0^{-i} f_\alpha F_0^i$, and $\mathscr{C}^{f_{\widetilde{\alpha}}}$ and $\mathscr{C}^{f_\alpha}$ are isometric. We have proven that

$$\mathrm{Vol}\left(F(\mathscr{C}_q^{f_{\widetilde{\alpha},q}})\right) = \mathrm{Vol}\left(\mathscr{C}^{f_\alpha}\right).$$

Since $F = h_q^{-1}F_0$, we deduce that $\mathrm{Vol}\left(F(\mathscr{C}_q^{f_{\widetilde{\alpha},q}})\right) = q^{\dim(V_\alpha)}\,\mathrm{Vol}\left(\mathscr{C}_q^{f_{\widetilde{\alpha},q}}\right)$, and it follows that $\mathrm{Vol}\left(\mathscr{C}_q^{f_{\widetilde{\alpha},q}}\right) = \frac{1}{q^{\dim(V_\alpha)}}\,\mathrm{Vol}(\mathscr{C}^{f_\alpha})$. Now, the result follows from Equation (2.18). □

Remark 2.6. — If V_α is contained in some hyperplane H of $\mathcal{H}_q$, then H is not a wall of $\mathscr{C}$. Indeed, the walls of $\mathscr{C}$ are the hyperplanes $H_\beta = \{\lambda\in V \mid \langle\,\beta,\lambda\,\rangle = m_\beta\}$ for $\beta\in\widetilde{\Delta}$, where $m_\beta = 0$ for $\beta\in\Delta$ and $m_{-\alpha_0} = -1$.

Let $\beta \in \widetilde{\Delta}$. We then define the element λ^α by setting $\lambda^\alpha_{z^i_\alpha(\beta)} = \frac{1}{|\langle z_\alpha \rangle \cdot \beta|}$ for all $i \geqslant 0$ and $\lambda^\alpha_\gamma = 0$ for $\gamma \notin \langle z_\alpha \rangle \cdot \beta$. Note that $\lambda^\alpha \in \mathscr{C}$ and $f_\alpha(\lambda^\alpha) = \lambda^\alpha$. Moreover, $\langle \beta, \lambda^\alpha \rangle = m_\beta + \frac{1}{n_\beta |\langle z_\alpha \rangle \cdot \beta|} \neq m_\beta$, which implies that $\lambda^\alpha \notin H_\beta$.

Lemma 2.7. — *Let $\lambda \in \omega(\mathscr{C}_q)$ for some $\omega \in E_q$. We suppose that $\lambda \notin \widetilde{\Delta}_{\min}$. Then, λ lies in an F-stable $\overline{W}_a$-orbit if and only if there is $\alpha \in \widetilde{\Delta}_{\min}$ with $z_\alpha \in A_{\mathbf{G}}$ such that $F(\lambda) = F(\omega) f_\alpha(\lambda)$.*

Proof. — Since $F(\omega(\mathscr{C}_q))$ is an alcove for W_a, there is a unique $v \in W_a$ such that $vF(\omega(\mathscr{C}_q)) = \mathscr{C}$. In particular, $vF(\lambda) \in \mathscr{C}$. If $F(\lambda)$ and λ lie in the same $\overline{W}_a$-orbit, then $vF(\lambda)$ and λ too. Then, by Theorem 2.2(2), there is $z_\alpha \in \mathcal{A}_{\mathbf{G}}$ (for $\alpha \in \widetilde{\Delta}_{\min}$) such that $vF(\lambda) - z_\alpha(\lambda) \in Y(\mathbf{T})$. Since $\varpi^\vee_\alpha \in Y(\mathbf{T})$, we deduce that $vF(\lambda) - f_\alpha(\lambda) \in Y(\mathbf{T})$. But $vF(\lambda)$ and $f_\alpha(\lambda)$ lie in $\mathscr{C}$. We denote by $(\mu_\beta)_{\beta \in \widetilde{\Delta}}$ and $(\lambda'_\beta)_{\beta \in \widetilde{\Delta}}$ the affine coordinate of $vF(\lambda)$ and $f_\alpha(\lambda)$. We have

$$vF(\lambda) - f_\alpha(\lambda) = \sum_{\beta \in \Delta} \frac{\mu_\beta - \lambda'_\beta}{n_\beta} \varpi^\vee_\beta.$$

We have $0 \leqslant \mu_\beta \leqslant 1$ and $0 \leqslant \lambda'_\beta \leqslant 1$ for all $\beta \in \widetilde{\Delta}$ (because $vF(\lambda)$ and $f_\alpha(\lambda)$ lie in $\mathscr{C}$). Thus, for every $\beta \in \Delta$, we have

$$-\frac{1}{n_\beta} \leqslant \frac{\mu_\beta - \lambda'_\beta}{n_\beta} \leqslant \frac{1}{n_\beta}$$

Since $Y(\mathbf{T}) \leqslant Y(\mathbf{T}_{\mathrm{ad}})$, we deduce from Equation (2.2) that $\mu_\beta = \lambda'_\beta$ if $\beta \notin \widetilde{\Delta}_{\min}$. Suppose that $\beta \in \widetilde{\Delta}_{\min}$. Then $n_\beta = 1$ and $\mu_\beta - \lambda'_\beta \in \{-1, 0, 1\}$. If $\mu_\beta - \lambda'_\beta \neq 0$, then we can assume that $\mu_\beta = 1$ and $\lambda'_\beta = 0$. Then $\mu_{\beta'} = 0$ for all $\beta' \neq \beta$, and $vF(\lambda) = \varpi^\vee_\beta$. Similarly, we deduce that $f_\alpha(\lambda) = \varpi^\vee_{\beta'}$ for some $\beta' \in \widetilde{\Delta}_{\min}$ (because $f_\alpha(\lambda) \neq vF(\lambda)$, which implies that there is $\beta' \in \widetilde{\Delta}_{\min}$ with $\lambda'_{\beta'} = 1$). It follows that $\lambda = \varpi^\vee_{z_\alpha(\beta')} \in \widetilde{\Delta}_{\min}$.

So, if we assume that $\lambda \notin \widetilde{\Delta}_{\min}$, we deduce that $vF(\lambda) = f_\alpha(\lambda)$. Moreover, $vF(\omega(\mathscr{C}_q)) = \mathscr{C}$ implies that $F(\omega)(\mathscr{C}) = v^{-1}(\mathscr{C})$. Since $F(\omega) \in W_a$, it follows from [4, VI.§2.1] that $F(\omega) = v^{-1}$. Conversely, if $F(\lambda) = F(\omega) f_\alpha(\lambda)$, then $F(\omega^{-1})F(\lambda)$ and $f_\alpha(\lambda)$ are in $\mathscr{C}$. So, we have

$$F(\omega^{-1})F(\lambda) - z_\alpha(\lambda) = -\varpi^\vee_\alpha \in Y(\mathbf{T}),$$

and the result comes from Theorem 2.2(2). □

Lemma 2.8. — *Suppose that $\mathcal{A}$ is cyclic. For $\alpha \in \widetilde{\Delta}_{\min}$, $\alpha \neq -\alpha_0$ and $\omega \in M_{\alpha,q}$, if $\lambda \in \omega(\mathscr{C}_q)$ lies in an F-stable $\overline{W}_a$-orbit, then $\lambda \in V_\alpha$.*

Proof. — Suppose there is $\varpi_\beta^\vee \in \omega(\mathscr{C}_q)$ with $\beta \in \widetilde{\Delta}_{\min}$. Then $f_\alpha(\varpi_\beta^\vee) = \varpi_\beta^\vee$. Moreover, $\varpi_\beta^\vee = f_\beta(0)$ and $f_\alpha(0) = \varpi_\alpha^\vee$. It follows that

$$f_\beta(\varpi_\alpha^\vee) = f_\beta f_\alpha(0) = f_\alpha f_\beta(0) = f_\alpha(\varpi_\beta^\vee) = \varpi_\beta^\vee = f_\beta(0),$$

which implies that $\varpi_\alpha^\vee = 0$, i.e. $\alpha = -\alpha_0$.

So, this proves that $\lambda \notin \widetilde{\Delta}_{\min}$ and by Lemma 2.7, there is $\beta \in \widetilde{\Delta}_{\min}$ such that $\omega F^{-1} f_\beta(\lambda) = \lambda$. Then λ is a fixed-point of the map $\omega F^{-1} f_\beta : \mathscr{C} \to \omega(\mathscr{C}_q)$, which is a contraction map with rapport $1/q$ with respect to the metric d_0. By the contraction mapping theorem, λ is the unique fixed-point of this map. We write $\widetilde{\alpha} = m(\alpha)$ where m is the map defined in Equation (2.14). Then, using Lemma 2.4 and Equation (2.13), we deduce that

$$\begin{aligned} \omega F^{-1} f_\beta f_{\widetilde{\alpha}} &= \omega F^{-1} f_{\widetilde{\alpha}} f_\beta \\ &= \omega f_{\widetilde{\alpha},q} F^{-1} f_\beta \\ &= f_\alpha \omega F^{-1} f_\beta. \end{aligned}$$

In particular, we have $\omega F^{-1} f_\beta(\mathscr{C}^{f_{\widetilde{\alpha}}}) = \omega(\mathscr{C}_q)^{f_\alpha}$. Moreover, in the proof of Theorem 2.5 we have seen that $\mathcal{A}$ cyclic implies that $\mathscr{C}^{f_{\widetilde{\alpha}}} = \mathscr{C}^{f_\alpha}$. Hence, it follows that

$$\omega F^{-1} f_\beta(\mathscr{C}^{f_\alpha}) \subseteq \mathscr{C}^{f_\alpha}.$$

So, λ is the limit of the sequence defined by $x_0 \in \mathscr{C}^{f_\alpha}$ and $x_k = \omega F^{-1} f_\beta(x_{k-1})$ for $k \geqslant 1$. Since $\mathscr{C}^{f_\alpha}$ is closed, we deduce that $\lambda \in \mathscr{C}^{f_\alpha}$, as required. □

The proof of Lemma 2.8 shows that the map $\omega F^{-1} f_\alpha : \mathscr{C} \to \omega(\mathscr{C}_q)$ for $\omega \in E_q$ and $\alpha \in \widetilde{\Delta}_{\min}$ has a unique fixed-point, denoted by $\lambda_{\omega,\alpha}$ in the following. Moreover, we define

$$S_{q,\alpha} = \{\lambda_{\omega,\alpha} \,|\, \omega \in E_q\}. \tag{2.19}$$

To simplify, we will write $S_{q,-\alpha_0} = S_q$ and $\lambda_{\omega,-\alpha_0} = \lambda_\omega$.

Lemma 2.9. — *Let $\alpha \in \widetilde{\Delta}_{\min}$. If $m : \widetilde{\Delta}_{\min} \to \widetilde{\Delta}_{\min}$ denotes the map defined in Equation (2.14), then $\varpi_\alpha^\vee \in S_q$ if and only if $m(\alpha) = \alpha$. Moreover, if $\varpi_\alpha^\vee \in S_q$, then for $\omega \in E_q$ and $\beta \in \widetilde{\Delta}_{\min}$ we have*

$$f_\alpha(\lambda_{\omega,\beta}) = \lambda_{\omega',\beta},$$

where $\omega' = f_\alpha \omega f_\alpha^{-1} r_\alpha$ and r_α is defined in Equation (2.13).

Proof. — We have

$$\begin{aligned} r_\alpha F^{-1}(\varpi_\alpha^\vee) &= r_\alpha F^{-1} f_\alpha(0) \\ &= r_\alpha f_{m(\alpha),q} F^{-1}(0) \\ &= f_{m(\alpha)}(0) \\ &= \varpi_{m(\alpha)}^\vee. \end{aligned}$$

Hence, $m(\alpha) = \alpha$ if and only if $\varpi_\alpha^\vee \in S_q$. For $\omega \in E_q$, we denote by $\omega' \in E_q$ the element such that $f_\alpha(\omega(\mathscr{C}_q)) = \omega(\mathscr{C}_q)$. We have

$$f_\alpha \omega f_\alpha^{-1} r_\alpha(\mathscr{C}_q) = \omega'(\mathscr{C}_q).$$

Since $f_\alpha \omega f_\alpha^{-1} r_\alpha$ and ω' lie in $W_{a,q}$, we deduce from [4, VI.§2.1] that $w' = f_\alpha \omega f_\alpha^{-1} r_\alpha$. Now, if $m(\alpha) = \alpha$, for $\beta \in \widetilde{\Delta}_{\min}$, we deduce that

$$\begin{aligned} f_\alpha(\lambda_{\omega,\beta}) &= f_\alpha \omega F^{-1} f_\beta(\lambda_{\omega,\beta}) \\ &= f_\alpha \omega f_\alpha^{-1} r_\alpha f_{m(\alpha),q} F^{-1} f_\beta(\lambda_{\omega,\beta}) \\ &= w' F^{-1} f_\beta\left(f_{m(\alpha)}(\lambda_{\omega,\beta})\right) \\ &= w' F^{-1} f_\beta\left(f_\alpha(\lambda_{\omega,\beta})\right). \end{aligned}$$

By unicity, we deduce that $f_\alpha(\lambda_{\omega,\beta}) = \lambda_{\omega',\beta}$. □

Remark 2.10. — In [9, 3.8], it is proved that $S_q \subset \mathbb{Q}_{p'} \otimes_{\mathbb{Z}} Y(\mathbf{T}_{\mathrm{sc}})$. Since $Y(\mathbf{T}_{\mathrm{sc}}) \leqslant Y(\mathbf{T})$, the $\overline{W}_a$-orbits on S_q parametrize some F-stable semisimple classes of $\mathbf{G}$.

Proposition 2.11. — *Assume there is $\alpha \in \widetilde{\Delta}_{\min,\mathbf{G}}$, where $\widetilde{\Delta}_{\min,\mathbf{G}}$ is the set defined in Equation (2.8), such that $\mathcal{A}_{\mathbf{G}} = \langle z_\alpha \rangle$ has prime order and assume that $\mathbf{G}$ is not of type D_{2n}. If V_α is not contained in some hyperplane of $\mathcal{H}_q$, then the number of F-stable semisimple classes of $\mathbf{G}$ with disconnected centralizer is $q^{\dim(V_\alpha)}$. Otherwise, there are no F-stable classes with disconnected centralizer.*

Proof. — Since $\mathcal{A}_{\mathbf{G}}$ has prime order, the $\overline{W}_a$-orbit of $\lambda \in \mathscr{C}$ corresponding to (see Equation (2.5)) a semisimple class of $\mathbf{G}$ with disconnected centralizer lies in V_α, by Theorem 2.2(3). First suppose that V_α is not contained in some hyperplane of $\mathcal{H}_q$. Then, $M_{\alpha,q}$ is non-empty. In the proof of Lemma 2.8, we showed that if $\lambda \in \omega(\mathscr{C}_q)$ with $\omega \in M_{\alpha,q}$ is such that its $\overline{W}_a$-orbit is F-stable, then $\lambda = \lambda_{\omega,\beta}$ for some $\beta \in \widetilde{\Delta}_{\min,\mathbf{G}}$. Moreover, by Lemma 2.8, we have $\lambda \in V_\alpha$. Since $\mathcal{A}_{\mathbf{G}}$ is generated by z_α, we have $V_\alpha \subseteq V_\beta$. This implies that $f_\beta(\lambda) = \lambda$. Thus, $\lambda = \lambda_{\omega,\beta}$ for all $\beta \in \widetilde{\Delta}_{\min,\mathbf{G}}$ and λ is the unique element of $\omega(\mathscr{C}_q)$ lying in an F-stable $\overline{W}_a$-orbit. So, in particular, $\lambda = \lambda_\omega$. Moreover, Theorem 2.2(2) implies that for $\omega' \in M_{\alpha,q}$ such that

$\omega \neq \omega'$, the elements λ_ω and $\lambda_{\omega'}$ are not $\overline{W}_a$-conjugate. Then there are $|M_{\alpha,q}|$ such elements. By Remark 2.10, these points lie in $\mathbb{Q}_{p'} \otimes_{\mathbb{Z}} Y(\mathbf{T}) \cap \mathscr{C}$ and correspond to the F-stable semisimple classes of $\mathbf{G}$ with disconnected centralizer. The claim then follows from Theorem 2.5.

Suppose now that V_α is contained in some hyperplane of $\mathcal{H}_q$, and assume there is $\lambda \in V_\alpha$ with F-stable $\overline{W}_a$-orbit. Then $\lambda \notin \widetilde{\Delta}_{\min}$ (because $\alpha \neq -\alpha_0$ and no element of $\widetilde{\Delta}_{\min}$ is fixed by f_α). Hence, Lemma 2.7 implies that $F(\lambda) = F(\omega)(\lambda)$ with $\omega \in E_q$ such that $\lambda \in \omega(\mathscr{C}_q)$, i.e., $\lambda = \omega F^{-1}(\lambda)$. Then, $\lambda \in S_q$. However, by [9, 3.8.2], λ lies in a unique element of Ω_q, which contradicts the fact that λ lies in a hyperplane which is not a wall of $\mathscr{C}$ (see Remark 2.6). ☐

Lemma 2.12. — *If p does not divide $|\mathcal{A}_{\mathbf{G}}|$, then for every $\alpha \in \widetilde{\Delta}_{\min,\mathbf{G}}$, the invariant subspace V_α is contained in no hyperplane of $\mathcal{H}_q$.*

Proof. — As we remarked at the end of the proof of Proposition 2.11, if there is $\lambda \in S_q$ such that $f_\alpha(\lambda) = \lambda$ for $\alpha \in \widetilde{\Delta}_{\min,\mathbf{G}}$, then V_α is contained in no hyperplane of $\mathcal{H}_q$. Suppose there is $\lambda \in \mathscr{C}$ such that:

(1) We have $\lambda \in \mathbb{Q}_{p'} \otimes_{\mathbb{Z}} Y(\mathbf{T}_{\mathrm{sc}}) \cap \mathscr{C}$.
(2) The type I_λ is not equal to the type of I_μ for every $\mu \neq \lambda$.
(3) We have $f_\alpha(\lambda) = \lambda$.

Let λ be such an element. Property (1) implies that there is a semisimple class $[s]_{\mathbf{G}_{\mathrm{sc}}}$ in $\mathbf{G}_{\mathrm{sc}}$ corresponding to λ. Then $F(s)$ is a semisimple element whose centralizer is of the same type as the one of s (because F is an isogeny). Let μ be the point of $\mathscr{C}$ which is W_a-conjugate to $F(\lambda)$, i.e, μ corresponds to the class of $F(s)$ in the identification given in Equation (2.5). By Theorem 2.2(3), we have $I_\mu = I_\lambda$ and it follows from Property (2) that $\lambda = \mu$. Hence, the W_a-orbit of λ is F-stable. We conclude using Property (3).

Suppose that p is a prime that satisfies the condition in Table 2.2. For types B_n, C_{2n}, D_{2n}, E_6 and E_7, the corresponding elements given in Table 2.2 satisfy Properties (1), (2), and (3). The result follows in these cases. Furthermore, for the type C_{2n+1}, if we denotes by λ_n the corresponding element of Table 2.2, we have

$$F(\lambda_n) = (q - \epsilon_0) \cdot \lambda_n + \epsilon_0 \cdot \lambda_n,$$

where $\epsilon_0 \in \{-1, 1\}$ is such that $q \equiv \epsilon_0 \mod 4$. Put $\delta = 0$ if $\epsilon_0 = 1$ and $\delta = 1$ otherwise, and define $r_n = t_n w_0^\delta$, with t_n the translation of vector $(q - \epsilon_0) \cdot \lambda_n \in Y(\mathbf{T}_{\mathrm{sc}})$ and w_0 is the longest element of W. By [4, Ch. VI, §4.8], w_0 acts on V as -1, which implies that $w_0(\lambda_n) = -\lambda_n$. Thus, we have $r_n \in W_a$ and $F(\lambda_n) = r_n(\lambda_n)$. It follows that $\lambda_n \in S_q$ and we conclude as above.

Note that if $\mathbf{G}$ is of type A_n, there is no element $\lambda \in \mathscr{C}$ which satisfies the above properties (1), (2), (3). So, in order to prove the result for the type A_n, we will more precisely describe the hyperplanes of $\mathcal{H}_q$. For this, we write $\Delta = \{\alpha_i \,|\, 1 \leqslant i \leqslant n\}$ as in [4, Planche I] and recall that the positive roots are the elements $\alpha_{ij} = \sum_{i \leqslant r < j} \alpha_r$ for $1 \leqslant i < j \leqslant n+1$. Hence, the hyperplanes of $\mathcal{H}_q$ are given by

$$H_{ij,k} = \left\{ \lambda \in V \,|\, \langle\, \alpha_{ij}, \lambda \,\rangle = \frac{k}{q} \right\},$$

for $1 \leqslant i < j \leqslant n+1$ and $k \in \mathbb{Z}$. In particular, an element $\lambda = (\lambda_i)_{1 \leqslant i \leqslant n}$ lies in $H_{ij,k}$ if and only if

$$\sum_{i \leqslant r < j} \lambda_r = \frac{k}{q}. \tag{2.20}$$

Let $z_\alpha \in \mathcal{A}$. Write d for the order of z_α and define the element $\lambda_d \in \mathscr{C}$ such that $\lambda_{d, z_{\alpha_1}^i} = 1/d$ for all $i \geqslant 0$ and $\lambda_{d,\beta} = 0$ for $\beta \in \widetilde{\Delta}$ which is not in $\langle\, z_\alpha \,\rangle \cdot \alpha_1$. In particular, thanks to Equation (2.20), $\lambda_d \in H_{ij,k}$ for some $1 \leqslant i < j \leqslant n+1$ and $1 \leqslant k \leqslant (q-1)$ (we indeed can suppose that $H_{ij,k}$ is not a wall of $\mathscr{C}$ by Remark 2.6) if and only if $qn_d = kd$, where $n_d = |\{\alpha_r \,|\, i \leqslant r < j\} \cap \langle\, z_\alpha \,\rangle \cdot \alpha_1|$. Since $q > k \geqslant 0$, we deduce that p divides d. It follows that if p does not divide $|\mathcal{A}_{\mathbf{G}}|$, then V_α is not contained in some hyperplane, as required.

Finally, if $\mathbf{G}$ is of type D_{2n+1}, then we show using equations of hyperplanes derived from [4, Planche III] and an argument similar to type A_n, that the f_α-stable element $\frac{1}{4}(\varpi_1^\vee + \varpi_{2n}^\vee + \varpi_{2n+1}^\vee)$ lies in no hyperplane of $\mathcal{H}_q$ which are not walls of $\mathscr{C}$. □

Corollary 2.13. — *In table 2.3, we give the number $n(q)$ of F-stable semisimple classes with disconnected centralizer for simple algebraic groups such that $\mathcal{A}$ has prime order. If s is a representative of such a class, we write $A_{\mathbf{G}}(s) = \mathrm{C}_{\mathbf{G}}(s)/\,\mathrm{C}_{\mathbf{G}}(s)^\circ$ for the component group of $\mathrm{C}_{\mathbf{G}}(s)$.*

Proof. — This is a direct consequence of Proposition 2.11 and Table 2.1. The condition on p comes from Lemma 2.12. □

Remark 2.14. — Recall that the Lang-Steinberg theorem implies that the number $|s(\mathbf{G}^F)|$ of semisimple classes of the finite fixed-point subgroup

Type		λ	Type of I_λ
B_n	$p \neq 2$	$\frac{1}{2} \cdot \varpi_2^\vee$	$B_{n-2} \times A_1 \times A_1$
C_{2n}	$p \neq 2$	$\frac{1}{2} \cdot \varpi_n^\vee$	$C_n \times C_n$
C_{2n+1}	$p \neq 2$	$\frac{1}{2} \cdot \varpi_{2n+1}^\vee$	A_{2n}
D_{2n}	$p \neq 2$	$\frac{1}{2} \cdot \varpi_n^\vee$	$D_n \times D_n$
E_6	$p \neq 3$	$\frac{1}{3} \cdot \varpi_4^\vee$	$A_2 \times A_2 \times A_2$
E_7	$p \neq 2$	$\frac{1}{2} \cdot \varpi_2^\vee$	A_7

Table 2.2. Some invariant elements

$\mathbf{G}^F$ is given by

$$|s(\mathbf{G}^F)| = \sum_{\substack{[s]_{\mathbf{G}} \in s(\mathbf{G})^F \\ F(s)=s}} |A_{\mathbf{G}}(s)^F|, \tag{2.21}$$

where the sum is over the set of F-stable semisimple classes of $\mathbf{G}$ and the representative s is chosen to be F-stable, which is possible by the Lang-Steinberg theorem. Suppose that $\mathcal{A}_{\mathbf{G}}$ has prime order. Then every semisimple element s with disconnected centralizer has a component group $A_{\mathbf{G}}(s)$ isomorphic to $\mathcal{A}_{\mathbf{G}}$ and so to a subgroup H of $\mathrm{Z}(\mathbf{G}_{\mathrm{sc}})$ of order $|\mathcal{A}_{\mathbf{G}}|$ (using the isomorphism of Equation (2.7)), such that the actions of F on $A_{\mathbf{G}}(s)$ and on H are equivalent. In particular, we deduce that the actions of F on the groups $A_{\mathbf{G}}(s)$ for all s such that $A_{\mathbf{G}}(s)$ is not trivial, are equivalent. We denote by c_1 (resp. c_2) a set of representatives of the F-stable semisimple classes of $\mathbf{G}$ with connected centralizer (resp. a disconnected centralizer) and we suppose that the elements of c_1 and c_2 are chosen to be F-stable. Then Equation (2.21) gives

$$|s(\mathbf{G}^F)| = |c_1| + |H^F| \cdot |c_2|. \tag{2.22}$$

In [7], we computed $|s(\mathbf{G}^F)|$ for every simple algebraic group $\mathbf{G}$ defined over $\mathbb{F}_q$; see [7, Table 1]. Moreover, it is well-known [9, 3.7.6] that

$$q^{|\Delta|} = |c_1| + |c_2|. \tag{2.23}$$

In particular, if $|H^F|$ is not trivial (this condition is for example related in [7, Table 1]), we can deduce $n(q) = |c_2|$ from Equations (2.22), (2.23) and [7, Table 1]. We retrieve the results of Table 2.3. But, when H^F is trivial (for example, for $q \equiv 2 \mod 3$ for $\mathbf{G}$ of type E_6), we get in Equation (2.22) no new information, and $n(q)$ cannot be computed using [7].

Remark 2.15. — In [7], the prime p is supposed to be a good prime for $\mathbf{G}$, i.e. p does not divide any of the numbers n_α (for $\alpha \in \Delta$) defined in Equation (2.6). Note that Proposition 2.11 applies for any prime p which does not divide $|\mathcal{A}_{\mathbf{G}}|$; see Lemma 2.12. In particular, we deduce from Equations (2.22), (2.23), and Table 2.3 that

$$|s\left({}^{\epsilon}E_{6,\mathrm{ad}}(2^f)\right)| = 2^{6f} + 2^{2f+1} \quad \text{and} \quad |s\left(E_{7,\mathrm{ad}}(3^f)\right)| = 3^{7f} + 3^{4f},$$

where $\epsilon = 1$ if F is a split Frobenius map and $\epsilon = -1$ otherwise. Moreover, thanks to [7, Prop. 4.1 and Prop. 5.9], the ordinary McKay Conjecture holds in defining characteristic for these groups.

Type			$n(q)$	$\lvert A_{\mathbf{G}}(s)\rvert$
$A_{n,\mathrm{ad}}$	$n+1$ prime	$p \neq n+1$	1	$n+1$
$B_{n,\mathrm{ad}}$		$p \neq 2$	q^{n-1}	2
$C_{n,\mathrm{ad}}$		$p \neq 2$	$q^{\lfloor \frac{n}{2} \rfloor}$	2
$E_{6,\mathrm{ad}}$		$p \neq 3$	q^2	3
$E_{7,\mathrm{ad}}$		$p \neq 2$	q^4	2

Table 2.3. Number of semisimple classes with disconnected centralizer

Now, we define

$$\Theta_q = \bigsqcup_{\alpha \in \widetilde{\Delta}_{\min,\mathbf{G}}} S_{q,\alpha}.$$

Proposition 2.16. — *Suppose that F_0 is trivial and that*

$$q \equiv 1 \mod |\mathcal{A}_{\mathbf{G}}|.$$

Then the $\Gamma_{\mathbf{G}}$-orbits of Θ_q correspond to the F-stable semisimple classes of $\mathbf{G}$ in the bijection given in Equation (2.5). Moreover, if $\mathcal{A}_{\mathbf{G}}$ is cyclic and $\mathbf{G}$ is not of type D_{2n}, and if for $\alpha \in \widetilde{\Delta}_{\min,\mathbf{G}}$, the invariant space V_α is not contained in some hyperplane of $\mathcal{H}_q$, then we have

$$|\Theta_{q,\alpha}| = q^{\dim(V_\alpha)},$$

where $\Theta_{q,\alpha}$ denotes the set of $\Gamma_{\mathbf{G}}$-orbits of Θ_q whose representatives are contained in V_α.

Proof. — We have $|\mathcal{A}_{\mathbf{G}}| = |Y(\mathbf{T})/Y(\mathbf{T}_{\mathrm{sc}})|$. In particular, $|\mathcal{A}_{\mathbf{G}}|$ is the product of the elementary divisors of $Y(\mathbf{T}_{\mathrm{sc}}) \leqslant Y(\mathbf{T})$ (viewed as $\mathbb{Z}$-modules). Hence, it follows that $|\mathcal{A}_{\mathbf{G}}| \cdot Y(\mathbf{T}) \leqslant Y(\mathbf{T}_{\mathrm{sc}})$ and $|\mathcal{A}_{\mathbf{G}}| \cdot F^{-1}(Y(\mathbf{T})) \leqslant F^{-1}(Y(\mathbf{T}_{\mathrm{sc}}))$. For $\alpha \in \widetilde{\Delta}_{\min}$, we have $\varpi^{\vee}_{\alpha,q} = \frac{1}{q} \cdot \varpi^{\vee}_{\alpha}$ (because F_0 is trivial) and $\varpi^{\vee}_{\alpha} = (q-1)\varpi^{\vee}_{\alpha,q} + \varpi^{\vee}_{\alpha,q}$. Denote by r_α the translation of vector $(q-1)\varpi^{\vee}_{\alpha,q}$. So, $\varpi^{\vee}_{\alpha} = q\varpi^{\vee}_{\alpha,q} = r_\alpha(\varpi^{\vee}_{\alpha,q})$. Since $|\mathcal{A}_{\mathbf{G}}|$ divides $q-1$, the translation r_α lies in $F^{-1}(Y(\mathbf{T}))$, and we have $r_\alpha(\mathscr{C}_q) \subset \mathscr{C}$. Thus, $f_\alpha(\mathscr{C}_q)) = r_\alpha(\mathscr{C}_q)$. Furthermore, we have

$$r_\alpha F^{-1}(\varpi^{\vee}_{\alpha}) = \varpi^{\vee}_{\alpha}.$$

This proves that $\varpi^{\vee}_{\alpha} \in S_\alpha$, and by Lemma 2.9, we deduce that $m(\alpha) = \alpha$ and that $\Gamma_{\mathbf{G}}$ acts on Θ_q. Now, since $\varpi^{\vee}_{\alpha} \in \Theta_q$, Lemma 2.7 implies that the elements of Θ_q are the elements of $\mathscr{C}$ whose $\overline{W}_q$-orbit is F-stable. Moreover, by Theorem 2.2(2), two elements of Θ_q are $\overline{W}_a$-conjugate if and only if they lie in the same $\Gamma_{\mathbf{G}}$-orbit. As we have seen in the proof of Theorem 2.5, $\Gamma_{\mathbf{G}}$ acts on the set Ω_q. We denote by S a system of representatives of $\Omega_q/\Gamma_{\mathbf{G}}$. Then we have

$$\Theta_q/\Gamma_{\mathbf{G}} = \bigsqcup_{\omega(\mathscr{C}_q)\in S} \{\Gamma_{\mathbf{G}} \cdot \lambda_{\omega,\beta} \,|\, \beta \in \widetilde{\Delta}_{\min,\mathbf{G}}\}.$$

Let $\omega(\mathscr{C}_q) \in S$. Then we have $|\{\Gamma_{\mathbf{G}} \cdot \lambda_{\omega,\beta} \,|\, \beta \in \widetilde{\Delta}_{\min,\mathbf{G}}\}| = |S_{\omega,q}|$, where $S_{\omega,q} = \{\lambda_{\omega,\beta} \,|\, \beta \in \widetilde{\Delta}_{\min,\mathbf{G}}\}$. Furthermore, the proof of Lemma 2.9 implies that $\lambda_{\omega,\beta} = \lambda_{\omega,\beta'}$ if and only if there is $f \in \Gamma_{\mathbf{G}}$ with $f(\omega(\mathscr{C}_q)) = \omega(\mathscr{C}_q)$ and β' is the element of $\widetilde{\Delta}_{\min,\mathbf{G}}$ such that $f_{\beta'} = f f_\alpha$. This proves that

$$|S_{\omega,q}| = |\Gamma_{\mathbf{G}}|/|\operatorname{Stab}_{\Gamma_{\mathbf{G}}}(\omega(\mathscr{C}_q))| = |\Gamma_{\mathbf{G}} \cdot \omega(\mathscr{C}_q)|.$$

It follows that

$$|\Theta_q/\Gamma_{\mathbf{G}}| = \sum_{\omega(\mathscr{C}_q)\in S} |\Gamma_{\mathbf{G}} \cdot \omega(\mathscr{C}_q)| = |E_q| = q^{|\Delta|}.$$

The last equality comes from [11, §3]. However, the number of F-stable semisimple classes of $\mathbf{G}$ is $q^{|\Delta|}$; see [9, 3.6.7]. Hence, there are at most $q^{|\Delta|}$ orbits under $\overline{W}_a$ in $\mathbb{Q}_{p'} \otimes_{\mathbb{Z}} Y(\mathbf{T})$ which are F-stable. It follows that the elements of $\Theta_q/\Gamma_{\mathbf{G}}$ have to contain points in $\mathbb{Q}_{p'} \otimes_{\mathbb{Z}} Y(\mathbf{T}) \cap \mathscr{C}$, and correspond to the F-stable semisimple classes of $\mathbf{G}$.

Now, we remark that $\Gamma_{\mathbf{G}}$ stabilizes $\Theta_q \cap V_\alpha$. Hence, we have

$$\Theta_{q,\alpha} = \Theta_q \cap V_\alpha/\Gamma_{\mathbf{G}}.$$

We set $S_\alpha = \{\omega(\mathscr{C}_q) \in S \,|\, \omega \in M_{\alpha,q}\}$. Note that the elements of S_α do not depend on the choice of representatives S of $\Omega_q/\Gamma_{\mathbf{G}}$. Indeed, if $\omega \in M_{\alpha,q}$ and $\omega' \in E_q$ is such that there is $f \in \Gamma_{\mathbf{G}}$ with $\omega'(\mathscr{C}_q) = f(\omega(\mathscr{C}_q))$, then

$\omega' \in M_{\alpha,q}$ because f and f_α commute. Moreover, for $\omega \in M_{\alpha,q}$, every $\lambda_{\omega,\beta}$ with $\beta \in \widetilde{\Delta}_{\min,\mathbf{G}}$ lies in V_α (by Lemma 2.8, because $\mathcal{A}_{\mathbf{G}}$ is cyclic). Therefore, we have

$$\Theta_{q,\alpha} = \bigsqcup_{\omega(\mathscr{C}_q)\in S_\alpha} \{\Gamma_{\mathbf{G}} \cdot \lambda_{\omega,\beta} \,|\, \beta \in \widetilde{\Delta}_{\min,\mathbf{G}}\}.$$

As above, we conclude that $|\Theta_{q,\alpha}| = |M_{\alpha,q}|$, and the result comes from Theorem 2.5. □

Proposition 2.17. — *Suppose that F_0 is not trivial and that $q \equiv -1 \mod |\mathcal{A}_{\mathbf{G}}|$. Moreover, assume that $\mathbf{G}$ is not of type D_{2n}. Then the conclusion of Proposition 2.16 holds.*

Proof. — Let $\alpha \in \widetilde{\Delta}_{\min}$. For every $\beta \in \Delta$, we define $\widetilde{s}_\beta = s_\beta - \frac{\delta_{\alpha\beta}}{q} \cdot \alpha^\vee$. Write $z_\alpha = \prod_{\beta\in I} s_\beta$ for some index subset I of Δ and define

$$\widetilde{z}_\alpha = \prod_{\beta\in I} \widetilde{s}_\beta \in W_{a,q}.$$

Note that $\widetilde{z}_\alpha(\mathscr{C}_q) = \mathscr{C}_q - \frac{1}{q}\cdot\varpi_\alpha^\vee$ (by Proposition 2.1) and $\widetilde{z}_\alpha(\mathscr{C}_q - \frac{1}{q}\cdot\varpi_\alpha^\vee) = \mathscr{C}_q - \frac{2}{q}\cdot\varpi_\alpha^\vee$. Furthermore, we have $F_0 z_\alpha F_0^{-1} = z_{\rho^{-1}(\alpha)}$ (see the proof of Lemma 2.4). Moreover, ρ acts on $\mathcal{A}$ by $x \mapsto x^{-1}$ (because $\mathcal{A}$ is cyclic; see [4, Planche I-IX]). This implies that $f_{\rho^{-1}(\alpha)} f_\alpha = \mathrm{Id}$. Hence

$$F_0 z_\alpha F_0^{-1}(\frac{1}{q}\cdot\varpi_\alpha^\vee) = -\frac{1}{q}\cdot\varpi_{\rho(\alpha)}^\vee + f_{\rho^{-1}(\alpha)} f_\alpha(0),$$

and we deduce that $z_\alpha(F_0^{-1}(\frac{1}{q}\cdot\varpi_\alpha^\vee)) = -\frac{1}{q}\cdot\varpi_\alpha^\vee$. Note that $\widetilde{z}_\alpha(-\frac{1}{q}\cdot\varpi_\alpha^\vee) = -\frac{1}{q}\cdot\varpi_\alpha^\vee$. Since $|\mathcal{A}_{\mathbf{G}}|$ divides $(q+1)$ the translation t of vector $(q+1)\cdot\frac{1}{q}\varpi_\alpha^\vee$ lies in W_a. We set $r_\alpha = t\widetilde{z}_\alpha z_\alpha \in W_a$. Then $r_\alpha(\mathscr{C}_q) \subset \mathscr{C}$ and $\varpi_\alpha^\vee$ lies in $r_\alpha(\mathscr{C}_q)$. Thus $f_\alpha(\mathscr{C}_q) = r_\alpha(\mathscr{C}_q)$. Moreover, we have

$$r_\alpha F_0^{-1}(\frac{1}{q}\cdot\varpi_\alpha^\vee) = \varpi_\alpha^\vee.$$

This proves that $\varpi_\alpha^\vee \in S_\alpha$ and we conclude as in the proof of Proposition 2.16. □

Remark 2.18. — Suppose that $\mathbf{G}$ is adjoint of type D_{2n+1}, that p is odd and that F^* acts trivially on $\mathrm{Z}(\mathbf{G}^*)$ (that is, $q \equiv 1 \mod 4$ if F is split, and $q \equiv -1 \mod 4$ otherwise). Thanks to Propositions 2.16 and 2.17, we have

α	$-\alpha_0$	α_1	α_{2n}	α_{2n+1}
$\lvert\Theta_{q,\alpha}\rvert$	q^{2n+1}	q^{2n-1}	q^{n-1}	q^{n-1}

Note that we have in fact $\Theta_{q,\alpha_{2n}} = \Theta_{q,\alpha_{2n+1}}$. We denote by $c_d(q)$ a set of representatives (chosen to be F-stable) of F-stable semisimple classes of $\mathbf{G}$

whose centralizer component group has order d. Since $\mathbf{G}$ is of adjoint type, we have $\mathcal{A}_{\mathbf{G}} = \mathcal{A} \simeq \mathbb{Z}_4$ and $d|4$. Furthermore, since $V_{\alpha_{2n}} \subseteq V_{\alpha_1} \subseteq V_{-\alpha_0}$, we have

(2.24)

$$|c_4(q)| = q^{n-1}, \quad |c_2(q)| = q^{2n-1} - q^{n-1} \quad \text{and} \quad |c_1(q)| = q^{2n+1} - q^{2n-1}.$$

Therefore, Equation (2.21) implies that

$$|s(\mathbf{G}^F)| = q^{2n+1} + q^{2n-1} + 2q^{n-1},$$

and we retrieve the result of [7, Table 1]. Now, using [7, Prop. 4.1] (note that $\mathbf{G}^*$ is simply-connected), if we denote by $\mathrm{Irr}_{p'}(\mathbf{G}^{*F^*})$ the set of irreducible p'-characters of $\mathbf{G}^{*F^*}$, then we deduce that

$$|\operatorname{Irr}_{p'}(\mathbf{G}^{*F^*})| = q^{2n+1} + 3q^{2n-1} + 12q^{n-1}.$$

Comparing with [7, Prop. 5.10], this proves that the ordinary McKay Conjecture holds for $\mathbf{G}^{*F^*}$ at the prime p.

3. Semisimple characters

3.1. Stable semisimple and regular characters

In this section we keep the notation of Section 2.3 and suppose that the Frobenius map $F : \mathbf{G} \to \mathbf{G}$ is split (i.e the map $F_0 : V \to V$ defined on p.1679 is trivial). Moreover, we assume that p is a good prime for $\mathbf{G}$. For every $\alpha \in \Phi$, we write $\mathbf{X}_\alpha$ for the corresponding one-dimensional subgroup of $\mathbf{G}$ normalized by $\mathbf{T}$ and choose an isomorphism $x_\alpha : \overline{\mathbb{F}}_p \to \mathbf{X}_\alpha$ in such a way that $F(x_\alpha(u)) = x_\alpha(u^q)$. Let ρ be a symmetry of the Dynkin diagram. Then we write $\sigma : \mathbf{G} \to \mathbf{G}$ for the graph automorphism on $\mathbf{G}$ defined for all $\alpha \in \Phi$ and $u \in \overline{\mathbb{F}}_p$ by $\sigma(x_\alpha(u)) = x_{\rho(\alpha)}(\gamma_\alpha u)$ where $\gamma_\alpha = \pm 1$ is chosen such that $\gamma_{\pm\alpha} = 1$ for all $\alpha \in \Delta$; see [8, 12.2.3]. Note that F and σ commute. We denote by $\mathbf{U}$ the unipotent radical of $\mathbf{B}$. Recall that $\mathbf{B} = \mathbf{U} \rtimes \mathbf{T}$ and that $\mathbf{U} = \prod_{\alpha \in \Phi^+} \mathbf{X}_\alpha$. Note that the product in the last equation is the inner product of $\mathbf{G}$. Now, we define the normal subgroup

$$\mathbf{U}_0 = \prod_{\alpha \in \Phi^+ \setminus \Delta} \mathbf{X}_\alpha \subseteq \mathbf{U}$$

and the quotient $\mathbf{U}_1 = \mathbf{U}/\mathbf{U}_0$ (with canonical projection map $\pi_{\mathbf{U}_0} : \mathbf{U} \to \mathbf{U}_1$). Then we have $\mathbf{U}_1 \simeq \prod_{\alpha \in \Delta} \mathbf{X}_\alpha$ (as direct product), and $\mathbf{U}_0$ is F-stable and connected, which implies

$$\mathbf{U}_1^F \simeq \prod_{\alpha \in \Delta} \mathbf{X}_\alpha^F, \tag{3.1}$$

(as direct product), because $\mathbf{X}_\alpha$ is F-stable for every $\alpha \in \Delta$. Fix $u_1 \in \mathbf{U}^F$ such that $\pi_{\mathbf{U}_0}(u_1)_\alpha \neq 1$ for all $\alpha \in \Delta$ (such an element is regular) and recall that $A_{\mathbf{G}}(u_1) = \mathrm{Z}(\mathbf{G})$, because p is a good prime for $\mathbf{G}$; see [14, 14.15, 14.18]. Then by the Lang-Steinberg theorem, we can parametrize the $\mathbf{G}^F$-classes of regular elements by $H^1(F, \mathrm{Z}(\mathbf{G}))$ [14, 14.24]. For $z \in H^1(F, \mathrm{Z}(\mathbf{G}))$, we denote by $\mathcal{U}_z$ the corresponding class of $\mathbf{G}^F$. Furthermore, a linear character $\phi \in \mathrm{Irr}(\mathbf{U}^F)$ is regular if it has $\mathbf{U}_0^F$ in its kernel, and if the induced character on $\mathbf{U}_1^F$ (also denoted by the same symbol) satisfies $\mathrm{Res}^{\mathbf{U}_1^F}_{\mathbf{X}_\alpha^F}(\phi) \neq 1_{\mathbf{X}_\alpha^F}$ for all $\alpha \in \Delta$. By [14, 14.28], we also can parametrize the $\mathbf{T}^F$-orbits of regular characters of $\mathbf{U}^F$ by $H^1(F, \mathrm{Z}(\mathbf{G}))$. For this, we fix a regular character ϕ_1 of $\mathbf{U}^F$. Then, for every $z \in H^1(F, \mathrm{Z}(\mathbf{G}))$, the regular character $\phi_z = {}^{t_z}\phi_1$, where t_z is an element of $\mathbf{T}$ such that $t_z^{-1}F(t_z) \in z$, is a representative of the $\mathbf{T}^F$-orbit corresponding to z.

Now, for $z \in H^1(F, \mathrm{Z}(\mathbf{G}))$, we define the corresponding Gelfand-Graev character by setting

$$\Gamma_z = \mathrm{Ind}_{\mathbf{U}^F}^{\mathbf{G}^F}(\phi_z).$$

We denote by $D_{\mathbf{G}^F}$ the duality of Alvis-Curtis and define $\mathrm{Irr}_r(\mathbf{G}^F) = \{\chi \in \mathrm{Irr}(\mathbf{G}^F) \mid \exists z \in H^1(F, \mathrm{Z}(\mathbf{G})),\ \langle \chi, \Gamma_z \rangle \neq 0\}$ and

$$\mathrm{Irr}_s(\mathbf{G}^F) = \{\epsilon_\chi D_{\mathbf{G}^F}(\chi) \mid \chi \in \mathrm{Irr}_r(\mathbf{G}^F)\}, \tag{3.2}$$

where ϵ_χ is a sign chosen to be such that $\epsilon_\chi D_{\mathbf{G}^F}(\chi) \in \mathrm{Irr}(\mathbf{G}^F)$. The elements of $\mathrm{Irr}_r(\mathbf{G}^F)$ (resp. of $\mathrm{Irr}_s(\mathbf{G}^F)$ are the so-called regular characters (resp. semisimple characters) of $\mathbf{G}^F$. In order to describe more precisely the sets $\mathrm{Irr}_s(\mathbf{G}^F)$ and $\mathrm{Irr}_r(\mathbf{G}^F)$, we first introduce further notation. We choose a σ- and F-stable torus $\mathbf{T}_0$ containing $\mathrm{Z}(\mathbf{G})$ and we consider the connected reductive group

$$\widetilde{\mathbf{G}} = \mathbf{T}_0 \times_{\mathrm{Z}(\mathbf{G})} \mathbf{G}, \tag{3.3}$$

where $\mathrm{Z}(\mathbf{G})$ acts on $\mathbf{G}$ and on $\mathbf{T}_0$ by translation. We extend σ and F to $\widetilde{\mathbf{G}}$. Note that $\widetilde{\mathbf{G}}$ has connected center and the derived subgroup of $\widetilde{\mathbf{G}}$ contains $\mathbf{G}$. Furthermore, $\widetilde{\mathbf{T}} = \mathbf{T}_0\mathbf{T}$ is an F-stable maximal torus of $\widetilde{\mathbf{G}}$ contained in the F-stable Borel subgroup $\widetilde{\mathbf{B}} = \mathbf{U} \rtimes \widetilde{\mathbf{T}}$ of $\widetilde{\mathbf{G}}$. Moreover, we write $(\mathbf{G}^*, F^*)$ and $(\widetilde{\mathbf{G}}^*, F^*)$ for pairs dual to $(\mathbf{G}, F)$ and $(\widetilde{\mathbf{G}}, F)$, respectively. Then the embedding $i : \mathbf{G} \to \widetilde{\mathbf{G}}$ induces a surjective homomorphism $i^* : \widetilde{\mathbf{G}}^* \to \mathbf{G}^*$. Now, we write $\mathcal{T}$ and $\widetilde{\mathcal{T}}$ for a set of F^*-stable representatives of $s(\mathbf{G}^*)^{F^*}$ and $s(\widetilde{\mathbf{G}}^*)^{F^*}$. Note that $s(\widetilde{\mathbf{G}})^{F^*} = s(\widetilde{\mathbf{G}}^{F^*})$ because the center of $\widetilde{\mathbf{G}}$ is connected, and $\widetilde{\mathcal{T}}$ is then a system of representatives of the semisimple classes of $\widetilde{\mathbf{G}}^{F^*}$. Furthermore, for $s \in \mathcal{T}$ and $a \in H^1(F^*, A_{\mathbf{G}^*}(s))$, we choose $g_a \in \mathbf{G}^*$ such that the F^*-class of $g_a^{-1}F^*(g_a)$ in $A_{\mathbf{G}^*}(s)$ is a and write

$s_a = g_a s g_a^{-1} \in \mathbf{G}^{*F^*}$. Then the elements s_a for $a \in H^1(F^*, A_{\mathbf{G}^*}(s))$ are a system of representatives of the $\mathbf{G}^{*F^*}$ classes of $[s]_{\mathbf{G}^*} \cap \mathbf{G}^{*F^*}$. In particular, the set

$$\mathcal{S} = \bigsqcup_{s \in \mathcal{T}} \left\{ s_a \,|\, a \in H^1(F^*, A_{\mathbf{G}^*}(s)) \right\} \tag{3.4}$$

is a set of representatives of $s(\mathbf{G}^{*F^*})$. Note that the elements of $\widetilde{\mathcal{T}}$ are chosen such that, if $s \in \mathcal{S}$, there is $\widetilde{s} \in \widetilde{\mathcal{T}}$ with $i^*(\widetilde{s}) = s$. Now, for any semisimple element $s \in \mathbf{G}^{*F^*}$ and $\widetilde{s} \in \widetilde{\mathbf{G}}^{*F^*}$, we denote by $\mathcal{E}(\mathbf{G}^F, s) \subseteq \mathrm{Irr}(\mathbf{G}^F)$ and $\mathcal{E}(\widetilde{\mathbf{G}}^F, \widetilde{s}) \subseteq \mathrm{Irr}(\widetilde{\mathbf{G}}^F)$ the corresponding rational Lusztig series. Recall that $\mathcal{E}(\mathbf{G}^F, s)$ consists of the irreducible constituents of Deligne-Lusztig characters $R_{\mathbf{T}^*}^{\mathbf{G}}(s)$, where $\mathbf{T}^*$ is any F^*-stable maximal torus of $\mathbf{G}^*$ containing s, and we have

$$\mathrm{Irr}(\mathbf{G}^F) = \bigsqcup_{s \in \mathcal{S}} \mathcal{E}(\mathbf{G}^F, s) \quad \text{and} \quad \mathrm{Irr}(\widetilde{\mathbf{G}}^F) = \bigsqcup_{\widetilde{s} \in \widetilde{\mathcal{T}}} \mathcal{E}(\widetilde{\mathbf{G}}^F, \widetilde{s}).$$

For $\widetilde{s} \in \widetilde{\mathcal{T}}$, let $W(\widetilde{s}) \subseteq W$ be the Weyl group of $\mathrm{C}_{\widetilde{\mathbf{G}}^*}(\widetilde{s})$. Denote by $\widetilde{\mathbf{T}}_1^*$ an F-stable maximal torus of $\mathrm{C}_{\widetilde{\mathbf{G}}^*}(\widetilde{s})$ containing s and for $w \in W(s)$ we write $\widetilde{\mathbf{T}}_w^*$ for the F^*-stable maximal torus of $\mathrm{C}_{\widetilde{\mathbf{G}}^*}(\widetilde{s})$ obtained by twisting $\widetilde{\mathbf{T}}_1^*$ by $w \in W$. We define

$$\rho_{\widetilde{s}} = \frac{\varepsilon_{\widetilde{\mathbf{G}}} \varepsilon_{\mathrm{C}_{\widetilde{\mathbf{G}}^*}(\widetilde{s})}}{|W(\widetilde{s})|} \sum_{w \in W(\widetilde{s})} R_{\widetilde{\mathbf{T}}_w}^{\widetilde{\mathbf{G}}}(\widetilde{s}), \tag{3.5}$$

$$\chi_{\widetilde{s}} = \frac{\varepsilon_{\widetilde{\mathbf{G}}} \varepsilon_{\mathrm{C}_{\widetilde{\mathbf{G}}^*}(\widetilde{s})}}{|W(\widetilde{s})|} \sum_{w \in W^\circ(\widetilde{s})} \varepsilon(w) R_{\widetilde{\mathbf{T}}_w}^{\widetilde{\mathbf{G}}}(\widetilde{s}), \tag{3.6}$$

where ε is the sign character of W and $\varepsilon_{\widetilde{\mathbf{G}}} = (-1)^{\mathrm{rk}_{\mathbb{F}_q}(\widetilde{\mathbf{G}})}$. Here, $\mathrm{rk}_{\mathbb{F}_q}(\widetilde{\mathbf{G}})$ denotes the $\mathbb{F}_q$-rank of $\widetilde{\mathbf{G}}$; see [14, 8.3]. Then we have $\mathrm{Irr}_s(\widetilde{\mathbf{G}}^F) = \{\rho_{\widetilde{s}} \,|\, \widetilde{s} \in \widetilde{\mathcal{T}}\}$ and $\mathrm{Irr}_r(\widetilde{\mathbf{G}}^F) = \chi_{\widetilde{s}} \,|\, \widetilde{s} \in \widetilde{\mathcal{T}}\}$. Let $s \in \mathcal{T}$. Write $\widetilde{s} \in \widetilde{\mathcal{T}}$ such that $i^*(\widetilde{s}) = s$ and define

$$\chi_s = \mathrm{Res}_{\mathbf{G}^F}^{\widetilde{\mathbf{G}}^F}(\chi_{\widetilde{s}}) \quad \text{and} \quad \rho_s = \mathrm{Res}_{\mathbf{G}^F}^{\widetilde{\mathbf{G}}^F}(\rho_{\widetilde{s}}). \tag{3.7}$$

Furthermore, for $s \in \mathcal{S}$, we recall that there is a surjective group homomorphism [3, (8.4)]

$$\hat{\omega}_s^0 : H^1(F, \mathrm{Z}(\mathbf{G})) \to \mathrm{Irr}\left(A_{\mathbf{G}^*}(s)^{F^*}\right).$$

We now can recall the following result [3, Proposition 15.3, Corollaire 15.14].

THEOREM 3.1. — *For every $s \in \mathcal{S}$, we have $\langle \Gamma_1, \chi_s \rangle_{\mathbf{G}^F} = 1$. We write $\chi_{s,1}$ for the common constituent and put $\rho_{s,1} = \varepsilon_{\mathbf{G}} \varepsilon_{C^\circ_{\mathbf{G}^*}(s)} D_{\mathbf{G}}(\chi_{s,1})$. Moreover, for $\xi \in \operatorname{Irr}(A_{\mathbf{G}^*}(s)^{F^*})$, we define*

$$\chi_{s,\xi} = {}^{t_z}\chi_{s,1} \quad \text{and} \quad \rho_{s,\xi} = {}^{t_z}\rho_{s,1},$$

where z is any elements of $H^1(F, \mathcal{Z}(\mathbf{G}))$ such that $\hat{\omega}^0_s(z) = \xi$ and $t_z \in \mathbf{T}$ with $t_z^{-1}F(t_z) \in z$. Then

(1) *For $z \in H^1(F, \mathrm{Z}(\mathbf{G}))$ and $\xi \in \operatorname{Irr}(A_{\mathbf{G}^*}(s)^{F^*})$, the character $\chi_{s,\xi}$ (resp. $\rho_{s,\xi}$) is an irreducible constituent of Γ_z (resp. of $D_{\mathbf{G}}(\Gamma_z)$), if and only if $\xi = \hat{\omega}^0_s(z)$.*

(2) *We have*

$$\chi_s = \sum_{\xi \in \operatorname{Irr}(A_{\mathbf{G}^*}(s)^{F^*})} \chi_{s,\xi} \quad \text{and} \quad \rho_s = \sum_{\xi \in \operatorname{Irr}(A_{\mathbf{G}^*}(s)^{F^*})} \rho_{s,\xi}.$$

(3) *We have $\operatorname{Irr}_s(\mathbf{G}^F) = \{\rho_{s,\xi} \mid s \in \mathcal{S},\ \xi \in \operatorname{Irr}(A_{\mathbf{G}^*}(s)^{F^*})\}$ and $\operatorname{Irr}_r(\mathbf{G}^F) = \{\chi_{s,\xi} \mid s \in \mathcal{S},\ \xi \in \operatorname{Irr}(A_{\mathbf{G}^*}(s)^{F^*})\}$.*

CONVENTION 3.2. — *The regular character $\phi_1 \in \operatorname{Irr}(\mathbf{U}^F)$ will be chosen to be σ-stable. This choice is possible by* [5, 3.1] *(note that Lemma 3.1 of* [5] *is stated for a Frobenius map F' which commutes with F. But the argument is still valuable for a graph automorphism commuting with F).*

PROPOSITION 3.3. — *Assume that $\phi_1 \in \operatorname{Irr}(\mathbf{U}^F)$ is chosen as in Convention 3.2. For every $z \in H^1(F, \mathrm{Z}(\mathbf{G}))$, we have*

$${}^\sigma\Gamma_z = \Gamma_{\sigma(z)} \quad \text{and} \quad {}^\sigma D_{\mathbf{G}^F}(\Gamma_z) = D_{\mathbf{G}^F}(\Gamma_{\sigma(z)}).$$

Moreover, the operation of $\langle \sigma \rangle$ on the set of constituents of Γ_1 and of $\langle \sigma^{*-1} \rangle$ on $s(\mathbf{G}^{*F^*})$ commute, and for $s \in \mathcal{S}$, if the $\mathbf{G}^{*F^*}$-class of s is σ^*-stable, where $\sigma^* : \mathbf{G}^* \to \mathbf{G}^*$ denotes the automorphism of $\mathbf{G}^*$ obtained in dualizing σ, then for every $z \in H^1(F, \mathrm{Z}(\mathbf{G}))$, we have

$${}^\sigma\chi_{s,\hat{\omega}^0_s(z)} = \chi_{s,\hat{\omega}^0_s(\sigma(z))} \quad \text{and} \quad {}^\sigma\rho_{s,\hat{\omega}^0_s(z)} = \rho_{s,\hat{\omega}^0_s(\sigma(z))}.$$

Proof. — For $s \in \mathcal{S}$, we have

$$ {}^\sigma\mathcal{E}(\mathbf{G}^F, s) = \mathcal{E}(\mathbf{G}^F, \sigma^{*-1}(s)). \tag{3.8}$$

The proof is similar to [6, Proposition 1.1] (because F and σ commute). In particular, one has ${}^\sigma\chi_s = \chi_{\sigma^{*-1}(s)}$ and ${}^\sigma\rho_s = \rho_{\sigma^{*-1}(s)}$. Since ϕ_1 is σ-stable, it follows that ${}^\sigma\Gamma_1 = \Gamma_1$. This implies that if s is σ^*-stable, then $\chi_{s,1}$ and $\rho_{s,1}$ are σ-stable. We conclude as in the proof of [5, Theorem 3.6]. □

Remark 3.4. — Note that Proposition 3.3 shows that $H^1(F,\mathrm{Z}(\mathbf{G}))^\sigma$ parametrizes the σ-stable Gelfand-Graev characters of $\mathbf{G}^F$.

Lemma 3.5. — *Suppose that $H^1(F,\mathrm{Z}(\mathbf{G}))$ has prime order. Then every σ-stable regular (resp. semisimple) character of $\mathbf{G}^F$ is a constituent of some σ-stable Gelfand-Graev character (resp. dual of Gelfand-Graev character) of $\mathbf{G}^F$.*

Proof. — Let χ be a σ-stable regular character of $\mathbf{G}^F$. Thanks to Theorem 3.1(3), there is $s \in \mathcal{S}$ and $\xi \in \mathrm{Irr}(A_{\mathbf{G}^*}(s)^{F^*})$ such that $\chi = \chi_{s,\xi}$ and Proposition 3.3 implies that the $\mathbf{G}^{*F^*}$-class of s is σ^*-stable. Let $z \in H^1(F,\mathrm{Z}(\mathbf{G}))$ be any element such that $\hat{\omega}_s^0(z) = \xi$. In particular, $\chi \in \Gamma_z$ by Theorem 3.1(1). Since $H^1(F,\mathrm{Z}(\mathbf{G}))$ has prime order, we deduce that $\ker(\hat{\omega}_s^0)$ is is either trivial or equal to $H^1(F,\mathrm{Z}(\mathbf{G}))$. If $\ker(\hat{\omega}_s^0) = H^1(F,\mathrm{Z}(\mathbf{G}))$, then $\xi = \hat{\omega}_s^0(1)$ and $\chi \in \Gamma_1$, which is σ-stable with our choice in Convention 3.2. Suppose now that $\ker(\hat{\omega}_s^0)$ is trivial. By Proposition 3.3, we also have ${}^\sigma\chi_{s,\xi} = \chi_{s,\hat{\omega}_s^0(\sigma(z))}$. It follows from Theorem 3.1(1) that χ is σ-stable, if and only if $\hat{\omega}_s^0(\sigma(z)) = \hat{\omega}_s^0(z)$, which is equivalent to $z^{-1}\sigma(z) \in \ker(\hat{\omega}_s^0)$. Then $\sigma(z) = z$ and Proposition 3.3 implies that ${}^\sigma\Gamma_z = \Gamma_z$, as required. □

Remark 3.6. — Note that in general, if χ is a σ-stable regular character of $\mathbf{G}^F$, then χ is not necessarily a constituent of some σ-stable Gelfand-Graev character of $\mathbf{G}^F$. For example, consider a simple simply-connected group $\mathbf{G}$ of type A_3 defined over $\mathbb{F}_q$ (with $q \equiv 1 \mod 4$) and suppose that the corresponding Frobenius map F is split. We denote by α_1, α_2 and α_3 the simple roots of $\mathbf{G}$ (relative to an F and σ-stable maximal torus $\mathbf{T}$ of $\mathbf{G}$) that we label as in [4, Planche I]. Write σ for the non-trivial graph automorphism of A_3. The condition on q implies that F acts trivially on $\mathrm{Z}(\mathbf{G})$. Hence, $H^1(F,\mathrm{Z}(\mathbf{G})) = \mathrm{Z}(\mathbf{G})$. Write z_0 for a generator of $\mathrm{Z}(\mathbf{G})$. With the choice of Convention 3.2, we write $\Gamma_i = \Gamma_{z_0^i}$ (with $0 \leqslant i \leqslant 3$) for the 4 Gelfand-Graev characters of $\mathbf{G}^F$. Furthermore, we will denote by $\varpi_i^\vee$ the fundamental weight corresponding to α_i. Recall that $\mathbf{G}^* = \mathbf{G}_{\mathrm{ad}}$. Define $\lambda = \frac{1}{2}(\varpi_1^\vee + \varpi_3^\vee)$ and write $s = \tilde{\iota}_{\mathbf{T}}(\lambda) \in \mathbf{T}_{\mathrm{ad}}$. Then $F(s) = s$ and $\sigma(s) = s$. Moreover, λ is stable under $f_{z_0^2}$, but not under f_{z_0}. Thus, by Theorem 2.2, we have $A_{\mathbf{G}^*}(\lambda) = \langle z_0^2 \rangle$, and $A_{\mathbf{G}^*}(s)^{F^*} \simeq \mathbb{Z}_2$. Denote by 1 and η the irreducible characters of $A_{\mathbf{G}^*}(s)^{F^*}$, and by $\chi_{s,1}$ and $\chi_{s,\eta}$ the corresponding regular characters of $\mathbf{G}^F$ as in Theorem 3.1(1). Since σ acts as $x \to x^{-1}$ on $\mathrm{Z}(\mathbf{G})$, we have

$$\hat{\omega}_s^0(1) = \hat{\omega}_s^0(z_0^2) = 1 \quad \text{and} \quad \hat{\omega}_s^0(z_0) = \hat{\omega}_s^0(z_0^3) = \eta,$$

and Proposition 3.3 implies that $\chi_{s,1}$ and $\chi_{s,\eta}$ are σ-stable. Moreover, thanks to Theorem 3.1(1), the Gelfand-Graev characters which have $\chi_{s,1}$ (resp. $\chi_{s,\eta}$) as constituent are Γ_0 and Γ_2 (resp. Γ_1 and Γ_3). However, by Proposition 3.3, we have

$$^{\sigma}\Gamma_0 = \Gamma_0, \quad {}^{\sigma}\Gamma_2 = \Gamma_2 \quad \text{and} \quad {}^{\sigma}\Gamma_1 = \Gamma_3,$$

and $\chi_{s,\eta}$ is a σ-stable regular character of $\mathbf{G}^F$ which is constituent of no σ-stable Gelfand-Graev characters of $\mathbf{G}^F$, as claimed.

3.2. Disconnected reductive groups

By Clifford theory, an irreducible character χ of $\mathbf{G}^F$ is σ-stable, if and only if it extends to the group $\mathbf{G}^F \rtimes \langle \sigma \rangle$. Moreover, if $E(\chi)$ denotes an extension of χ, then Gallagher's theorem [19, 6.17] implies that every extension of χ is obtained by tensoring $E(\chi)$ with a linear character of $\mathbf{G}^F \rtimes \langle \sigma \rangle$ trivial on $\mathbf{G}^F$. So, in order to obtain information about the sets $\mathrm{Irr}_r(\mathbf{G}^F)^{\sigma}$ and $\mathrm{Irr}_s(\mathbf{G}^F)^{\sigma}$, we aim to understand the extensions of these characters to $\mathbf{G}^F \rtimes \langle \sigma \rangle$. For this, we will consider the group

$$\mathbf{H} = \mathbf{G} \rtimes \langle \sigma \rangle.$$

We extend F to a Frobenius map on $\mathbf{H}$ by setting $F(\sigma) = \sigma$ (to simplify notation, the extended map will also be denoted by F). Note that $\mathbf{H}$ is a disconnected reductive group defined over $\mathbb{F}_q$ (the rational structure is given by F), and $\mathbf{H}^{\circ} = \mathbf{G}$. Moreover, σ is a rational quasi-central element in the sense of [15, 1.15]. Now, for $i \geqslant 0$, we define a scalar product on the space of class functions on the coset $\mathbf{G}^F \cdot \sigma^i$, by setting

$$\langle \chi, \chi' \rangle_{\mathbf{G}^F \cdot \sigma^i} = \frac{1}{|\mathbf{G}^F|} \sum_{g \in \mathbf{G}^F} \chi(g\sigma^i)\overline{\chi'(g\sigma^i)}.$$

Recall that in [15, 4.10], Digne and Michel define a duality involution $D_{\mathbf{G}^F,\sigma^i}$ for $i \geqslant 0$ on the set of class functions defined over the coset $\mathbf{G}^F \cdot \sigma^i$, and prove in [15, 4.13] that if $\chi \in \mathrm{Irr}(\mathbf{H}^F)$ restricts to an irreducible character on $\mathbf{G}^F$, then the class function $D_{\mathbf{H}^F}(\chi)$ defined for all $g \in \mathbf{G}^F$ and $i \geqslant 0$ by

$$D_{\mathbf{H}^F}(\chi)(g\sigma^i) = D_{\mathbf{G}^F,\sigma^i}(\chi|_{\mathbf{G}^F \cdot \sigma^i})(g\sigma^i), \tag{3.9}$$

is (up to a sign) an irreducible character of $\mathbf{H}^F$.

We suppose that $\phi_1 \in \mathrm{Irr}(\mathbf{U}^F)$ is chosen as in Convention 3.2. In particular, ϕ_1 is σ-stable and linear. Thus, ϕ_1 extends to a linear character $E(\phi_1)$ of $\mathbf{U}^F \rtimes \langle \sigma \rangle$ by setting

$$E(\phi_1)(u\sigma) = \phi_1(u) \quad \forall u \in \mathbf{U}^F. \tag{3.10}$$

This extension is the so-called canonical extension of ϕ_1. We define

$$E(\Gamma_1) = \mathrm{Ind}_{\mathbf{U}^F \rtimes \langle \sigma \rangle}^{\mathbf{H}^F}(E(\phi_1)). \tag{3.11}$$

Note that, as direct consequence of Mackey's theorem [19, (5.6) p.74], we have

$$\mathrm{Res}_{\mathbf{G}^F}^{\mathbf{H}^F}(E(\Gamma_1)) = \Gamma_1.$$

Hence, $E(\Gamma_1)$ extends Γ_1. We write $\Gamma_{1,\sigma} = \mathrm{Res}_{\mathbf{G}^F \cdot \sigma}(E(\Gamma_1))$.

Write C_1 for the set of irreducible constituents of Γ_1 and for $\chi \in C_1^\sigma$, denote by $E(\chi)$ the constituent of $E(\Gamma_1)$ that extends χ. Define

$$\Psi_1 = \sum_{\chi \in C_1^\sigma} D_{\mathbf{H}^F}(E(\chi)). \tag{3.12}$$

Lemma 3.7. — *We suppose that Convention 3.2 holds. Then we have*

$$\mathrm{Res}_{\mathbf{G}^F \cdot \sigma}(\Psi_1) = D_{\mathbf{G}^F,\sigma}(\Gamma_{1,\sigma}),$$

and $\langle \Psi_1, \Psi_1 \rangle_{\mathbf{H}^F} = \langle \Gamma_{1,\sigma}, \Gamma_{1,\sigma} \rangle_{\mathbf{G}^F \cdot \sigma}$. *In particular,*

$$\langle \Gamma_{1,\sigma}, \Gamma_{1,\sigma} \rangle_{\mathbf{G}^F \cdot \sigma} = \left| s(\mathbf{G}^{*F^*})^{\sigma^*} \right|.$$

In [23], Sorlin develops a theory of Gelfand-Graev characters for disconnected groups when σ is semisimple or unipotent. These characters are extensions of some σ-stable Gelfand-Graev characters of $\mathbf{G}^F$ to $\mathbf{H}^F$; see [23, §5]. In particular, the following result is proven [23, 8.3].

Theorem 3.8. — *Suppose that σ is a unipotent or a semisimple element of $\mathbf{H}^F$ and that $H^1(F, \mathrm{Z}(\mathbf{G}^\sigma))$ is trivial. Then $\mathbf{H}^F$ has a unique Gelfand-Graev character Γ and we have*

$$\langle \Gamma_\sigma, \Gamma_\sigma \rangle_{\mathbf{G}^F \cdot \sigma} = |\, \mathrm{Z}(\mathbf{G}^\sigma)^{\circ F}| q^l,$$

where l is the semisimple rank of $\mathbf{G}^\sigma$ and Γ_σ denotes the restriction of Γ to the coset $\mathbf{G}^F \cdot \sigma$.

Remark 3.9. — The character $E(\Gamma_1)$ defined in Equation (3.11) is a Gelfand-Graev character of $\mathbf{H}^F$ in the sense of [23], because the linear character $E(\phi_1)$ defined in Equation (3.10) is regular [23, Définition 4.1]). Note that by [8, 12.2.3] the graph automorphism σ that we consider here always satisfies the condition (RS) defined in [23, Notation 2.1].

3.3. A result of extendibility

Let n be a positive integer. The map $F' = F^n$ is a Frobenius map of $\mathbf{G}$, which gives a rational structure over $\mathbb{F}_{q^n}$. Note that F and σ commute with F'. Then restrictions of these endomorphisms to $\mathbf{G}^{F'}$ induce automorphisms of $\mathbf{G}^{F'}$, denoted by the same symbol in the following. Note that, viewed as an automorphism of $\mathbf{G}^{F'}$, the automorphism F has order n. We write $A = \langle F, \sigma \rangle$ and

$$N_{F'/F} : \mathbf{U}_1^{F'} \to \mathbf{U}_1^F,\ u \mapsto uF(u)\dots F^{n-1}(u), \tag{3.13}$$

for the norm map of $\mathbf{U}_1$, where $\mathbf{U}_1$ is the group defined before Equation (3.1), and we set $N^*_{F'/F} : \mathrm{Irr}(\mathbf{U}_1^F) \to \mathrm{Irr}(\mathbf{U}_1^{F'})$, $\phi \mapsto \phi \circ N_{F'/F}$. Since F and σ commute, we have

$$\sigma \circ N_{F'/F} = N_{F'/F} \circ \sigma. \tag{3.14}$$

Lemma 3.10. — *If ϕ is a σ-stable character of $\mathbf{U}_1^F$, then the character $N^*_{F'/F}(\phi)$ is stable under F and σ.*

Proof. — Since $\mathbf{U}_1$ is abelian and connected, the map $N_{F'/F}$ is surjective [5, §2.4], and $N^*_{F'/F}$ is a bijection between $\mathrm{Irr}(\mathbf{U}_1^F)$ and $\mathrm{Irr}(\mathbf{U}_1^{F'})^F$. Moreover, for every $\phi \in \mathrm{Irr}(\mathbf{U}_1^F)^\sigma$, Equation (3.14) implies that $N_{F'/F}(\phi)$ is σ-stable, as required. □

Remark 3.11. — If $\phi \in \mathrm{Irr}(\mathbf{U}^F)$ is regular and σ-stable, then the corresponding character of $\mathbf{U}_1^F$ is σ-stable. Applying Lemma 3.10 to this character, we obtain a character of $\mathrm{Irr}(\mathbf{U}_1^{F'})$ stable under F and σ. Denote by $\widetilde{\phi}$ the corresponding character of $\mathbf{U}^{F'}$ (with $\mathbf{U}_0^{F'}$ in its kernel). Then $\widetilde{\phi}$ is a regular character of $\mathbf{U}^{F'}$ stable under F and σ. Thus, $\widetilde{\phi}$ extends to $\mathbf{U}^{F'} \rtimes \langle \sigma \rangle$. Now, it follows from Equation (3.10) that $E(\widetilde{\phi})$ is F-stable.

Convention 3.12. — *The character ϕ_1 of $\mathbf{U}^{F'}$ used to parametrize the Gelfand-Graev characters of $\mathbf{G}^{F'}$ is chosen to be σ and F-stable. This is possible by Remark 3.11.*

Proposition 3.13. — *Assume that $\phi_1 \in \mathrm{Irr}(\mathbf{U}^{F'})$ is chosen as in Convention 3.12. Suppose that σ is semisimple and that the characteristic p is a good prime of $(\mathbf{G}^\sigma)^\circ$. If $H^1(F', \mathrm{Z}(\mathbf{G}^\sigma))$ is trivial, then the constituents of Ψ_1 are F-stable.*

Proof. — Denote by $\mathcal{U}_\sigma$ the set of regular elements of $\mathbf{H}$ which are $\mathbf{G}$-conjugate to an element of the coset $\mathbf{U} \cdot \sigma$. In [23, §8], Sorlin defines a family of subsets $(\mathcal{U}_z)_{z\in H^1(F',\mathrm{Z}(\mathbf{G}))}$ of $\mathcal{U}_\sigma^F$ which form a partition of $\mathcal{U}_\sigma^F$ (see [23, 8.1]). Furthermore, we define

$$\gamma_u(g) = \begin{cases} |\mathbf{G}^{F'}|/|\mathcal{U}_\sigma^F| & \text{if } g \in \mathcal{U}_\sigma^F \\ 0 & \text{otherwise} \end{cases}.$$

Remark 3.9 and the proof of [23, Théorème 8.4] imply that

$$D_{\mathbf{G}^F,\sigma}(\Gamma_{1,\sigma}) = \gamma_u, \tag{3.15}$$

because $H^1(F',\mathrm{Z}(\mathbf{G}^\sigma))$ is trivial. Recall that the irreducible characters of $\langle\,\sigma\,\rangle$ are described as follows. We fix a primitive $|\langle\,\sigma\,\rangle|$-complex root of unity σ_0, and recall that the linear characters of $\langle\,\sigma\,\rangle$ are the morphisms $\varepsilon_i : \langle\,\sigma\,\rangle \to \mathbb{C}^\times$ such that $\varepsilon_i(\sigma) = \sigma_0^i$. Let $\rho_{s,1}$ be a σ-stable constituent of $D_{\mathbf{G}^{F'}}(\Gamma_1)$. Then the set $\mathrm{Irr}(\mathbf{H}^{F'},\rho_{s,1})$ of extensions of $\rho_{s,1}$ to $\mathbf{H}^{F'}$ consists of the characters

$$\rho_{s,1,i} = E(\rho_{s,1}) \otimes \varepsilon_i \in \mathrm{Irr}(\mathbf{H}^{F'}),$$

for any $i \geqslant 0$, where $E(\rho_{s,1})$ denotes an extension of $\rho_{s,1}$ to $\mathbf{H}^{F'}$ (such extensions exist by [19, 11.22]). Now, [23, Proposition 8.1] implies that $\mathcal{U}_\sigma^{F'}$ is an $\mathbf{H}^{F'}$-class (because p is good for $(\mathbf{G}^\sigma)^\circ$, the group $H^1(F',\mathrm{Z}(\mathbf{G}^\sigma))$ is trivial and σ is semisimple). Hence, by Lemma 3.7 and Equation (3.15), for any $h \in \mathcal{U}_\sigma^{F'}$, we have

$$\begin{aligned}\rho_{s,1,i}(h) &= \sigma_0^i \rho_{s,1,i}(h) \\ &= \sigma_0^i \langle\,\gamma_u,\rho_{s,1}\,\rangle_{\mathbf{G}^{F'}\cdot\sigma} \\ &= \sigma_0^i\, \varepsilon_{\mathbf{G}}\, \varepsilon_{\mathrm{C}^\circ_{\mathbf{G}^*}(s)}.\end{aligned}$$

In particular, since σ_0 has order $|\langle\,\sigma\,\rangle|$, we deduce that

$$\rho_{s,1,i} = \rho_{s,1,j} \iff \rho_{s,1,i}(h) = \rho_{s,1,j}(h) \quad \text{for } h \in \mathcal{U}_\sigma^{F'}. \tag{3.16}$$

Suppose now that $\rho_{s,1}$ is F-stable. Then ${}^F\rho_{s,1,i} = \rho_{s,1,j}$ for some $j \geqslant 0$, because $\mathrm{Irr}(\mathbf{H}^{F'},\rho_{s,1})$ is F-stable, and for $h \in \mathcal{U}_\sigma^{F'}$, we have

$$\begin{aligned}\rho_{s,1,j}(h) &= {}^F\rho_{s,1,i}(h) \\ &= \rho_{s,1,i}(F(h)) \\ &= \rho_{s,1,i}(h),\end{aligned}$$

because $F(h) \in \mathcal{U}_\sigma^{F'}$. Therefore, Equation (3.16) implies that ${}^F\rho_{s,1,i} = \rho_{s,1,i}$. □

Remark 3.14. — In fact, in the proof of Proposition 3.13 we proved that every extension to $\mathbf{H}^{F'}$ of an F- and σ-stable constituent of $D_{\mathbf{G}^F}(\Gamma_1)$ is F-stable.

4. Application to finite groups of type E_6

4.1. Preliminaries

In this section, $\mathbf{G}$ denotes a simple simply-connected group of type E_6 over $\overline{\mathbb{F}}_p$. We suppose that p is a good prime for $\mathbf{G}$ (i.e., $p \neq 2, 3$). Let $\mathbf{T}$ be a maximal torus of $\mathbf{G}$ contained in a Borel subgroup $\mathbf{B}$ of $\mathbf{G}$. We denote by Φ the root system of $\mathbf{G}$ relative to $\mathbf{T}$, and by Φ^+ and Δ the sets of positive roots and simple roots corresponding to $\mathbf{B}$. For $\alpha \in \Phi$, we write $\mathbf{X}_\alpha$ for the corresponding root subgroup and choose an isomorphism $x_\alpha : \overline{\mathbb{F}}_p \to \mathbf{X}_\alpha$. Since $\mathbf{G}$ is simple, we have $\mathbf{G} = \langle\, x_\alpha(u) \,|\, \alpha \in \Phi,\, u \in \overline{\mathbb{F}}_p \,\rangle$. For $\alpha \in \Phi$ and $t \in \overline{\mathbb{F}}_p$, we set $n_\alpha(t) = x_\alpha(t)x_{-\alpha}(-t^{-1})x_\alpha(t)$. Recall that the Weyl group W of $\mathbf{G}$ is generated by the coset $n_\alpha(1)\cdot\mathbf{T}$ for all $\alpha \in \Phi$. Moreover, for $\alpha \in \Phi$ one has $\alpha^\vee(t) = n_\alpha(t)n_\alpha(1)^{-1}$ for all $t \in \overline{\mathbb{F}}_p$. Then, we have $\mathbf{T} = \langle\, \alpha^\vee(t) \,|\, \alpha \in \Phi,\, t \in \overline{\mathbb{F}}_p^\times \,\rangle$ and $\mathbf{B} = \langle\, \mathbf{T},\, x_\alpha(u),\, \alpha \in \Phi^+,\, u \in \overline{\mathbb{F}}_p \,\rangle$; see [17, 1.12.1].

We write $\Delta = \{\alpha_1, \ldots, \alpha_6\}$ as in [4, Planche V] and denote by ρ the symmetry of Δ of order 2. As in §3.1, we define the corresponding graph automorphism $\sigma : \mathbf{G} \to \mathbf{G}$ and a split Frobenius map by setting $F(x_\alpha(u)) = x_\alpha(u^p)$ for $\alpha \in \Phi$ and $u \in \overline{\mathbb{F}}_p$, which commute with σ and defines an $\mathbb{F}_p$-structure on $\mathbf{G}$. Note that $\mathbf{T}$ and $\mathbf{B}$ are stable under F and σ. Moreover, by [4, Planche V], we have

$$\widetilde{\iota}_{\mathbf{T}}(\omega_{\alpha_1}^\vee) = \alpha_1^\vee(\xi)\alpha_3^\vee(\xi^2)\alpha_5^\vee(\xi)\alpha_6^\vee(\xi^2),$$

where $\xi \in \overline{\mathbb{F}}_p$ has order 3 (such an element exists because $p \neq 3$). Define

$$\mathbf{T}_0 = \{\alpha_1^\vee(t)\alpha_3^\vee(t^2)\alpha_5^\vee(t)\alpha_6^\vee(t^2) \,|\, t \in \overline{\mathbb{F}}_p^\times\}. \tag{4.1}$$

Then $\mathbf{T}_0$ is a subtorus of $\mathbf{T}$ which contains $\mathrm{Z}(\mathbf{G}) = \langle\, \widetilde{\iota}_{\mathbf{T}}(\omega_{\alpha_1}^\vee) \,\rangle$, and is stable under σ and F. In the following, we will use this torus for the construction of $\widetilde{\mathbf{G}}$ as in Equation (3.3).

Recall that $\mathbf{G}^\sigma$ is a simple group of type F_4 (by [17, 1.15.2]) and $\mathbf{T}^\sigma$ is a maximal F-stable torus of $\mathbf{G}^\sigma$, contained in the F-stable Borel subgroup $\mathbf{B}^\sigma$ of $\mathbf{G}^\sigma$ (see [17, 4.1.4(c)]). In particular, $\mathrm{Z}(\mathbf{G}^\sigma)$ is trivial (for example, by Proposition 2.1, because $Y(\mathbf{T}^\sigma_{\mathrm{ad}})/Y(\mathbf{T}^\sigma_{\mathrm{sc}})$ is trivial).

Lemma 4.1. — *With the above notation, the group $(\widetilde{\mathbf{G}}^\sigma)^\circ$ is a simple group of type F_4.*

Proof. — The automorphism σ of $\widetilde{\mathbf{G}}$ stabilizes $\widetilde{\mathbf{T}}$ and $\widetilde{\mathbf{B}}$. Then it is quasi-semisimple (see [15, 1.1]) and by [16, 0.1], the group $\widetilde{\mathbf{G}}^\sigma$ is reductive and the root system of $(\widetilde{\mathbf{G}}^\sigma)^\circ$ only depends on Φ (the root system of $\widetilde{\mathbf{G}}$) and on σ. Hence, $(\widetilde{\mathbf{G}}^\sigma)^\circ$ and $\mathbf{G}^\sigma$ have the same type, i.e., an irreducible root system of type F_4. Furthermore, by [15, 1.8], $\mathbf{T}' = (\widetilde{\mathbf{T}}^\sigma)^\circ$ is a maximal torus of $(\widetilde{\mathbf{G}}^\sigma)^\circ$. Now the exact sequence

$$0 \to (X(\mathbf{T}') \cap \mathbb{Q}\Phi)/\mathbb{Z}\Phi \to X(\mathbf{T}')/\mathbb{Z}\Phi \to X(\mathbf{T}')/(X(\mathbf{T}') \cap \mathbb{Q}\Phi) \tag{4.2}$$

induces an exact sequence for the p'-torsion subgroups of these groups. Since $X(\mathbf{T}')/(X(\mathbf{T}') \cap \mathbb{Q}\Phi)$ has no torsion, we deduce that

$$((X(\mathbf{T}') \cap \mathbb{Q}\Phi)/\mathbb{Z}\Phi)_{p'} \simeq (X(\mathbf{T}')/\mathbb{Z}\Phi)_{p'} \tag{4.3}$$

However, the group $X(\mathbf{T}')\cap\mathbb{Q}\Phi$ is a subgroup of the weight lattice Λ. So, it follows that $((X(\mathbf{T}') \cap \mathbb{Q}\Phi)/\mathbb{Z}\Phi)_{p'}$ is a subgroup of $X(\mathbf{T}_{\mathrm{sc}}^\sigma)/X(\mathbf{T}_{\mathrm{ad}}^\sigma) = \{1\}$ (because $\mathbf{G}^\sigma$ is a simple group of type F_4, which implies that its fundamental group is trivial). It follows from Equation (4.3) and [3, 4.1] that $\mathrm{Z}((\widetilde{\mathbf{G}}^\sigma)^\circ)$ is connected. Denote by $\chi_0 : \widetilde{\mathbf{T}} \to \overline{\mathbb{F}}_p^\times$ the character of $\widetilde{\mathbf{T}}$ induced by the character $\mathbf{T}_0 \to \overline{\mathbb{F}}_p^\times$, $t \mapsto t^3$ in $X(\mathbf{T}_0)$ (the character χ_0 is well-defined because it is trivial on $\mathrm{Z}(\mathbf{G})$). Note that $X(\widetilde{\mathbf{T}}) = \langle\, \alpha,\, \alpha \in \Delta;\, \chi_0 \,\rangle$. Moreover, ${}^\sigma\chi_0 = -\chi_0$ implies that

$$\mathrm{Rk}_{\mathbb{Z}}\left((1-\sigma)X(\widetilde{\mathbf{T}})\right) = \mathrm{Rk}_{\mathbb{Z}}\left((1-\sigma)X(\mathbf{T})\right) + 1.$$

Now, the proof of [15, 1.28] implies that

$$\begin{aligned}\dim(\mathbf{T}') &= \mathrm{Rk}_{\mathbb{Z}}\left(X(\widetilde{\mathbf{T}})/(1-\sigma)X(\widetilde{\mathbf{T}})\right)\\ &= \mathrm{Rk}_{\mathbb{Z}}\left(X(\widetilde{\mathbf{T}})\right) - \mathrm{Rk}_{\mathbb{Z}}\left((1-\sigma)X(\widetilde{\mathbf{T}})\right)\\ &= \mathrm{Rk}_{\mathbb{Z}}(X(\mathbf{T})) - \mathrm{Rk}_{\mathbb{Z}}\left((1-\sigma)X(\mathbf{T})\right)\\ &= \dim(\mathbf{T}^\sigma).\end{aligned}$$

Hence, if Φ_σ denotes the root system of $\mathbf{G}^\sigma$, we deduce from [25, 8.1.3] that

$$\begin{aligned}\dim((\widetilde{\mathbf{G}}^\sigma)^\circ) &= \dim(\mathbf{T}') + |\Phi_\sigma|\\ &= \dim(\mathbf{T}^\sigma) + |\Phi_\sigma|\\ &= \dim(\mathbf{G}^\sigma).\end{aligned}$$

The result follows. □

Let n be a positive integer, $F' = F^n$ and $A = \langle\, F, \sigma \,\rangle$ as in §3.3. We consider the finite group $\mathbf{G}^{F'}$.

Lemma 4.2. — *The subgroups of A are $\langle\, \sigma^i F^j \,\rangle$ (for $i \in \{1,3\}$ and a divisor j of n) and $\langle\, \sigma \,\rangle \times \langle\, F^j \,\rangle$ (for a divisor j of n).*

Proof. — Note that the elements of order 2 of A are σ, $F^{n/2}$ and $\sigma F^{n/2}$ if n is even and σ otherwise. Let H be a subgroup of A. Then $H = H_2 \times H_{2'}$ with H_2 the 2-Sylow subgroup of H and $H_{2'} \leqslant \langle F \rangle$. Then H is cyclic if and only if H_2 is cyclic if and only H_2 contains a unique element of order 2. The result comes from the fact that if H_2 contains more than one element of order 2, then $\sigma \in H_2$. □

Denote by $\mathrm{Irr}_l(\mathbf{U}^{F'})$ the set of linear characters of $\mathbf{U}^{F'}$ and by $\mathrm{Irr}_s(\mathbf{B}^F)$ the set of irreducible characters χ of $\mathbf{B}^{F'}$ such that $\mathrm{Res}^{\mathbf{B}^{F'}}_{\mathbf{U}^{F'}}(\chi)$ has constituents in $\mathrm{Irr}_l(\mathbf{U}^{F'})$. The characters of $\mathrm{Irr}_s(\mathbf{B}^{F'})$ can be described as follows. The $\widetilde{\mathbf{T}}^{F'}$-orbits of $\mathrm{Irr}_l(\mathbf{U}^{F'})$ are parametrized by the subsets of Δ. For $J \subset \Delta$, we denote by ω_J the corresponding $\widetilde{\mathbf{T}}^{F'}$-orbit, and write $\mathbf{L}_J$ for the standard Levi subgroup (which is F-stable) with set of simple roots J. Note that ω_J corresponds to the regular characters of $\mathrm{Irr}(\mathbf{U}_J^{F'})$.

Convention 4.3. — *Write* $A_J = \mathrm{Stab}_A(\omega_J)$. *Then by* [5, Lemma 3.1] *and Remark 3.11 there is ϕ_J in ω_J an A_J-stable character. Moreover,* $\mathrm{Stab}_A(\phi_J) = A_J$ *and if $\tau \in A$, then* $\mathrm{Stab}_A({}^\tau\phi_J) = \mathrm{Stab}_A(\phi_J)$, *because A is abelian. Let Ω be an A-orbit of* $\mathrm{Irr}_l(\mathbf{U}^{F'})/\widetilde{\mathbf{T}}^{F'}$ *and we fix $\omega_J \in \Omega$. In the following, we will fix a A_J-stable character $\phi_J \in \omega_J$, and if $J' \subseteq \Delta$ is such that there is $\tau \in A$ with $\omega_{J'} = {}^\tau\omega_J$, then we choose $\phi_{J'} = {}^\tau\phi_J$ as representative for $\omega_{J'}$. Note that $\phi_{J'}$ is well-defined (because it does not depend on the choice of $\tau \in A$ with ${}^\tau\omega_J = \omega_{J'}$) and is $A_{J'}$-stable. This choice is compatible with Convention 3.12.*

Now, for $z \in H^1(F', \mathrm{Z}(\mathbf{L}_J))$, we choose $t_z \in \mathbf{T}$ such that $t_z^{-1}F'(t_z) \in z$ and define

$$\phi_{J,z} = {}^{t_z}\phi_J. \tag{4.4}$$

Then the family $(\phi_{J,z})_{J\subseteq\Delta,\, z\in H^1(F',\mathrm{Z}(\mathbf{L}_J))}$ is a system of representatives of the $\mathbf{T}^{F'}$-orbits of $\mathrm{Irr}_l(\mathbf{U}^{F'})$. Moreover, if we write $\mathrm{Z}_J = \mathrm{Z}(\mathbf{L}_J)$, then for every $z \in H^1(F', \mathrm{Z}(\mathbf{L}_J))$, we have $\mathrm{Stab}_{\mathbf{T}^{F'}}(\phi_{J,z}) = \mathrm{Z}_J^{F'}$. Now, for $J \subseteq \Delta$, $z \in H^1(F', \mathrm{Z}(\mathbf{L}_J))$ and $\psi \in \mathrm{Irr}(\mathrm{Z}_J^{F'})$, we define

$$\chi_{J,z,\psi} = \mathrm{Ind}^{\mathbf{B}^{F'}}_{\mathbf{U}^F \rtimes \mathrm{Z}_J^{F'}}(\hat{\phi}_{J,z} \otimes \psi), \tag{4.5}$$

where $\hat{\phi}_{J,z}$ is the extension of $\phi_{J,z}$ to $\mathbf{U}^{F'} \rtimes \mathrm{Z}_J^{F'}$ defined by $\hat{\phi}_{J,z}(ut) = \phi_{J,z}(u)$ for all $u \in \mathbf{U}^{F'}$ and $t \in \mathrm{Z}_J^{F'}$.

Lemma 4.4. — *Assume that Convention 4.3 holds. For $\tau \in A$, $J \subseteq \Delta$, $z \in H^1(F', \mathrm{Z}(\mathbf{L}_J))$ and $\psi \in \mathrm{Irr}(\mathrm{Z}_J^{F'})$, we have*

$${}^\tau\chi_{J,z,\psi} = \chi_{\tau(J),\tau(z),{}^\tau\psi}.$$

Proof. — Using the induction formula [19, 5.1], we have

$$ {}^{\tau}\chi_{J,z,\psi} = \mathrm{Ind}_{\mathbf{U}^{F'}\rtimes\tau(\mathrm{Z}_J^{F'})}^{\mathbf{B}^{F'}}({}^{\tau}\hat{\phi}_{J,z}\otimes{}^{\tau}\psi). \tag{4.6}$$

Since τ and F' commute, we have $\tau(\mathrm{Z}_J^{F'}) = \tau(\mathrm{Z}_J)^{F'} = \mathrm{Z}_{\tau(J)}^{F'}$, because $\tau(\mathbf{L}_J) = \mathbf{L}_{\tau(J)}$. Moreover, the choices in Convention 4.3 imply that ${}^{\tau}\phi_{J,z}$ is $\mathbf{T}^{F'}$-conjugate to $\phi_{\tau(J),\tau(z)}$, and the result follows. □

4.2. Equivariant bijections

We define $B = D \rtimes A$, where D is the group of outer diagonal automorphism of $\mathbf{G}^{F'}$ induced by the inner automorphisms of $\widetilde{\mathbf{T}}^{F'}/\mathbf{T}^{F'}$. We denote by $\mathcal{O}_D$ and $\mathcal{O}'_D$ the set of D-orbits of $\mathrm{Irr}_s(\mathbf{G}^{F'})$ and $\mathrm{Irr}_s(\mathbf{B}^{F'})$. The group D has order 1 or 3. For $i \in \{1,3\}$, we write $\mathcal{O}_{D,i}$ and $\mathcal{O}'_{D,i}$ for the subset of elements of $\mathcal{O}_D$ and $\mathcal{O}'_D$ of size i. For $\nu \in \mathrm{Irr}(\mathrm{Z}(\mathbf{G}^{F'})$, we denote by $\mathrm{Irr}(\mathbf{G}^{F'}|\nu)$ the set of irreducible characters of $\mathbf{G}^{F'}$ lying over ν, that is $\chi \in \mathrm{Irr}(\mathbf{G}^{F'}|\nu)$ if and only if $\chi|_{\mathrm{Z}(\mathbf{G}^{F'})} = \chi(1)\cdot\nu$. We recall that $\mathrm{Z}(\mathbf{B}) = \mathrm{Z}(\mathbf{G})$, and for $\nu \in \mathrm{Irr}(\mathrm{Z}(\mathbf{G}^{F'}))$ we set

$$\mathrm{Irr}_s(\mathbf{G}^{F'}|\nu) = \mathrm{Irr}(\mathbf{G}^{F'}|\nu) \cap \mathrm{Irr}_s(\mathbf{G}^{F'})$$

and

$$\mathrm{Irr}_s(\mathbf{B}^{F'}|\nu) = \mathrm{Irr}(\mathbf{B}^{F'}|\nu) \cap \mathrm{Irr}_s(\mathbf{B}^{F'}),$$

and denote by $\mathcal{O}_{D,\nu}$ and $\mathcal{O}'_{D,\nu}$ the set of D-orbits of $\mathrm{Irr}(\mathbf{G}^{F'}|\nu)$ and $\mathrm{Irr}(\mathbf{B}^{F'}|\nu)$, respectively. For $i \in \{1,3\}$, we write $\mathcal{O}_{D,\nu,i}$ (resp. $\mathcal{O}'_{D,\nu,i}$) for the subset of elements of $\mathcal{O}_{D,\nu}$ (resp. of $\mathcal{O}'_{D,\nu}$) of size i.

Remark 4.5. — By Theorem 3.1(2), every D-orbit of $\mathrm{Irr}_s(\mathbf{G}^{F'})$ is the set of constituents of some ρ_s with $s \in \mathcal{S}$. We denote by δ_s the D-orbit corresponding to $s \in \mathcal{S}$. Note that $|\delta_s| = |A_{\mathbf{G}^*}(s)^{F'}|$. For $J \subseteq \Delta$ and $\psi \in \mathrm{Irr}(\mathrm{Z}_J^{F'})$, we define

$$\delta_{J,\psi} = \{\chi_{J,z,\psi} \mid z \in H^1(F', \mathrm{Z}(\mathbf{L}_J))\}. \tag{4.7}$$

Then the D-orbits of $\mathrm{Irr}_s(\mathbf{B}^{F'})$ are the sets $\delta_{J,\psi}$ with $J \subseteq \Delta$ and $\psi \in \mathrm{Irr}(\mathrm{Z}_J^{F'})$. Moreover, we have $|\delta_{J,\psi}| = |H^1(F', \mathrm{Z}(\mathbf{L}_J))|$.

Lemma 4.6. — *Let $\nu \in \mathrm{Irr}(\mathrm{Z}(\mathbf{G}^{F'})$. Write $A_\nu = \mathrm{Stab}_A(\nu)$ and suppose that $\mathcal{O}_{D,\nu,k}$ and $\mathcal{O}'_{D,\nu,k}$ for $k \in \{1,3\}$ are A_ν-equivalent. Then $\mathrm{Irr}_s(\mathbf{G}^{F'}|\nu)$ and $\mathrm{Irr}_s(\mathbf{B}^{F'}|\nu)$ are $D \rtimes A_\nu$-equivalent.*

Proof. — We choose A_ν-equivariant bijections $f_1 : \mathcal{O}_{D,\nu,1} \to \mathcal{O}'_{D,\nu,1}$ and $f_3 : \mathcal{O}_{D,\nu,3} \to \mathcal{O}'_{D,\nu,3}$. We define $\Psi_\nu : \mathrm{Irr}_s(\mathbf{G}^{F'}|\nu) \to \mathrm{Irr}_s(\mathbf{B}^{F'}|\nu)$ as follows. Let $\delta_s \in \mathcal{O}_{D,\nu}$. If $\delta_s \in \mathcal{O}_{D,\nu,k}$ for $k \in \{1,3\}$, then by Remark 4.5 there is $J \subseteq \Delta$ and $\psi \in \mathrm{Irr}(\mathrm{Z}_J^{F'})$ such that $f_k(\delta_s) = \delta_{J,\psi}$. Then we set

$$\Psi_\nu(\rho_{s,z}) = \chi_{J,z,\psi}.$$

Note that, if $H^1(F', \mathrm{Z}(\mathbf{L}_J))$ is not trivial, then $H^1(F', \mathrm{Z}(\mathbf{L}_J))$ and $H^1(F', \mathrm{Z}(\mathbf{G}))$ are identified by the map $h_J^1 : H^1(F', \mathrm{Z}(\mathbf{G})) \to H^1(F', \mathrm{Z}(\mathbf{L}_J))$ defined in [14, 14.31] (which is an isomorphism in this case). Hence, the map Ψ_ν is well-defined and is an $D \rtimes A_\nu$-equivariant bijection by Theorem 3.1, Lemma 4.4, Proposition 3.3 and [5, 3.6]. $\square$

Theorem 4.7. — *Suppose that* $\mathbf{G}$ *is a simple simply-connected group of type* E_6 *defined over* $\mathbb{F}_q$ *with corresponding Frobenius* F'. *We suppose that* F' *is split. With the above notation, if* $\nu \in \mathrm{Irr}(\mathrm{Z}(\mathbf{G})^{F'})$, *then the sets* $\mathrm{Irr}_s(\mathbf{G}^{F'}|\nu)$ *and* $\mathrm{Irr}_s(\mathbf{B}^{F'}|\nu)$ *are* $D \rtimes A_\nu$*-equivalent.*

Proof. — Write $q = p^n$, F the split Frobenius map of $\mathbf{G}$ over $\mathbb{F}_p$ which stabilizes $\mathbf{T}$ and $\mathbf{B}$, and σ the graph automorphism of $\mathbf{G}$ with respect to $\mathbf{T}$ and $\mathbf{B}$, as above. Recall that $F' = F^n$. We will prove that $\mathrm{Irr}_s(\mathbf{G}^{F'}|\nu)$ and $\mathrm{Irr}_s(\mathbf{B}^{F'}|\nu)$ are $D \rtimes A_\nu$-equivalent, using Lemma 4.6. In order to prove that $\mathcal{O}_{D,\nu,k}$ and $\mathcal{O}'_{D,\nu,k}$ for $k \in \{1,3\}$ are A_ν-equivalent, we use [19, 13.23].

First, we suppose that $\mathrm{Z}(\mathbf{G})^F = 1$ and $\mathrm{Z}(\mathbf{G})^{F'} = \mathrm{Z}(\mathbf{G})$, that is $p \not\equiv 1 \mod 3$ and $q \equiv 1 \mod 3$. We write $\mathrm{Irr}(\mathrm{Z}(\mathbf{G})^{F'}) = \{1_Z, \varepsilon, \varepsilon^2\}$. Note that n is even. Moreover, we have

$$A_{1_Z} = A \quad \text{and} \quad A_\varepsilon = A_{\varepsilon^2} = \langle\, \sigma F \,\rangle.$$

Suppose that $k = 3$ and $\nu = 1_Z$. Let $H \leqslant A$ (the subgroups of A are described in Lemma 4.2). By Lemma [5, 5.7] and Equation (3.8) (which is valid for any element of A), we deduce that $|\mathcal{O}_{D,1_Z,3}^H|$ is equal to the number of H^*-stable classes of $s(\mathbf{G}^*)$ with disconnected centralizer, where H^* denotes the subgroup of automorphisms of $\mathbf{G}^*$ induced by elements of H. If $H = \langle\, \sigma \,\rangle \times \langle\, F^j \,\rangle$ or $H = \langle\, \sigma^i F^j \,\rangle$, then we write $d = \mathrm{Ord}(F^j)$. We claim that $|\mathcal{O}_{D,1_Z,3}^H| = p^{2n/d}$. Indeed, Theorem 2.2(3) implies that every semisimple class of $\mathbf{G}$ with disconnected centralizer is σ-stable, and we conclude with Table 2.3.

Now, in the proof of [5, 5.9], it is shown that $|\delta'_{J,\psi}| = 3$, if and only if $\mathrm{Z}_J^{F'} = \mathbf{T}_J^{F'} \times H_J^{F'}$, where $\mathbf{T}_J$ is a torus of rank $|\Delta| - |J|$ and $H_J^{F'}$ is isomorphic to $H^1(F', \mathrm{Z}(\mathbf{L}_J))$. Moreover, the elements of $\delta'_{J,\psi\otimes\varepsilon^m}$ lie over ε^m. So, $\mathcal{O}'_{D,\varepsilon^m,3}$ consists of the orbits $\delta'_{J,\psi\otimes\varepsilon^m}$ with $|H^1(F', \mathrm{Z}(\mathbf{L}_J))| = 3$. By [1, Lemme 2.16, Table 2.17], a subset $J \subseteq \Delta$ parametrizes such an orbit if and only if it contains $\{\alpha_1,\alpha_3,\alpha_5,\alpha_6\}$. Furthermore, by [17, Lemma 4.4.7], the group $\mathbf{G}^\sigma$ is a simple group of type F_4 with root system $\Phi_\sigma=\{\widetilde{\alpha}\,|\,\alpha\in\Phi\}$ where $\widetilde{\alpha}=\frac{1}{2}(\alpha+\rho(\alpha))$, and the set of simple roots of $\mathbf{G}^\sigma$ with respect to $\mathbf{T}^\sigma$ and $\mathbf{B}^\sigma$ is $\Delta_\sigma=\{\widetilde{\alpha}_1,\ldots,\widetilde{\alpha}_4\}$. Note that the labelling is as in [4, Planche VIII]. In particular, $\widetilde{\alpha}_1=\alpha_2$ and $\widetilde{\alpha}_2=\alpha_4$ are the long roots of Δ_σ. Moreover, the root subgroup corresponding to $\widetilde{\alpha}\in\Phi_\sigma$ is $\widetilde{\mathbf{X}}^\sigma_{\widetilde{\alpha}}$, where $\widetilde{\mathbf{X}}_{\widetilde{\alpha}}=\mathbf{X}_\alpha$ if $\alpha=\rho(\alpha)$ and $\widetilde{\mathbf{X}}_{\widetilde{\alpha}}=\mathbf{X}_\alpha\cdot\mathbf{X}_{\rho(\alpha)}$ if $\alpha\neq\rho(\alpha)$; see the proof of [17, Lemma 4.4.7]. We associate to $J\subseteq\Delta$ the subset $\widetilde{J}\subseteq\Delta_\sigma$ such that, if a σ-orbit of Δ lies in J, the corresponding root of Δ_σ lies in $\widetilde{J}$. Write $\Phi_{\widetilde{J}}=\Phi_\sigma\cap\mathbb{Z}\widetilde{J}$. Then we have

$$\mathbf{L}_J=\langle\,\mathbf{T},\widetilde{\mathbf{X}}_{\widetilde{\alpha}},\,\widetilde{\alpha}\in\Phi_{\widetilde{J}}\rangle\quad\text{and}\quad\mathbf{L}_{\widetilde{J}}=\langle\,\mathbf{T}^\sigma,\widetilde{\mathbf{X}}^\sigma_{\widetilde{\alpha}},\,\widetilde{\alpha}\in\Phi_{\widetilde{J}}\rangle,$$

where $\mathbf{L}_{\widetilde{J}}$ is the standard Levi subgroup of $\mathbf{G}^\sigma$ with respect to $\mathbf{T}^\sigma$ corresponding to $\widetilde{J}$, because $\mathbf{T}^\sigma$ is connected. Note that $\mathbf{L}_J^\sigma=\mathbf{L}_{\widetilde{J}}$ and $D(\mathbf{L}_J^\sigma)=D(\mathbf{L}_J)^\sigma$, where $D(\mathbf{L}_J)$ denotes the derived subgroup of $\mathbf{L}_J$. Furthermore, we have $\mathbf{T}_J=\mathrm{Rad}(\mathbf{L}_J)$, which implies that $\mathbf{T}_J$ is σ-stable. Since $\mathbf{L}_J^\sigma$ is connected (as a Levi subgroup), we deduce from [15, 1.31] and [25, 2.2.1] that $\mathbf{L}_J^\sigma=\mathbf{T}_J^\sigma\,D(\mathbf{L}_J)^\sigma$. This product is direct, because σ fixes no nontrivial element of the center of $D(\mathbf{L}_J)$. It follows that $\mathbf{T}_J^\sigma$ is connected (as group isomorphic to the connected quotient $\mathbf{L}_J^\sigma/D(\mathbf{L}_J)^\sigma$). In particular, $\mathbf{T}_J^\sigma$ is a subgroup of $\mathrm{Rad}(\mathbf{L}_J^\sigma)$. Moreover, since $D(\mathbf{L}_J)^\sigma=D(\mathbf{L}_J^\sigma)$, we deduce from [25, 2.3.3] that $\dim(\mathbf{T}_J^\sigma)=\dim(\mathrm{Rad}(\mathbf{L}_J^\sigma))$, and [25, 1.8.2] implies that

$$\mathbf{T}_J^\sigma=\mathrm{Rad}(\mathbf{L}_J^\sigma).\tag{4.8}$$

Let $\delta'_{J,\psi\otimes 1_Z}\in\mathcal{O}'^{\langle\,\sigma\,\rangle\times\langle\,F^j\,\rangle}_{D,1_Z,3}$. Then J is σ-stable and contains $\{\alpha_1,\alpha_3,\alpha_5,\alpha_6\}$, and ${}^\sigma\psi=\psi$. Moreover, we have $(\mathbf{T}_J^{F'})^{\langle\,\sigma,F^j\,\rangle}=(\mathbf{T}_J^\sigma)^{F^j}$ and Equation (4.8) implies that the set $\mathcal{O}'^{\langle\,\sigma\,\rangle\times\langle\,F^j\,\rangle}_{D,1_Z,3}$ is in bijection with the set of characters $\widetilde{\chi}_{\widetilde{J},\widetilde{\psi}}$ of $\mathrm{Irr}_s((\mathbf{B}^\sigma)^{F^j})$ such that $\widetilde{J}$ contains $\{\widetilde{\alpha}_3,\widetilde{\alpha}_4\}$. Therefore, [7, Lemma 5.4] implies that

$$\left|\mathcal{O}'^{\langle\,\sigma\,\rangle\times\langle\,F^j\,\rangle}_{D,1_Z,3}\right|=p^{2n/d}.\tag{4.9}$$

Similarly, since $(\sigma^iF^j)^n=F'$ (because n is even), for a σ^i-stable $J\subseteq\Delta$, we have $(\mathbf{T}_J^{F'})^{\sigma^iF^j}=\mathbf{T}_J^{\sigma^iF^j}$, and we deduce that $\mathcal{O}'^{\langle\,\sigma^iF^j\,\rangle}_{D,1_Z,3}$ is in bijection with the D-orbits of size 3 of $\mathrm{Irr}_s(\mathbf{B}^{\sigma^iF^j},1_Z)$. As above, [1, 2.17] and [7, Lemma 5.4] implies that $|\mathcal{O}'^{\langle\,\sigma^iF^j\,\rangle}_{D,1_Z,3}|=p^{2n/d}$. This discussion proves that, if $\nu=1_Z$, then for every $H\leqslant A$ we have

$$|\mathcal{O}^H_{D,1_Z,3}|=|\mathcal{O}'^H_{D,1_Z,3}|.\tag{4.10}$$

Now, if $\nu = \varepsilon^m$ with $m = \pm 1$, then $A_\nu = \langle \sigma F \rangle$ and for every $H = \langle \sigma^i F^{n/d} \rangle \leqslant A_\nu$, the same argument shows that

$$|\mathcal{O}^H_{D,\varepsilon^m,3}| = p^{2n/d} = |\mathcal{O}'^H_{D,\varepsilon^m,3}|. \tag{4.11}$$

So, this proves that for every $\nu \in \mathrm{Irr}(\mathrm{Z}(\mathbf{G})^{F'})$, the sets $\mathcal{O}_{D,\nu,3}$ and $\mathcal{O}'_{D,\nu,3}$ are A_ν-equivalent.

Suppose now that $k = 1$. Note that $s(\mathbf{G}^{*F'^*})$ and $\mathcal{O}_D$ are A-equivalent. Let d be a divisor of n. Then, we have $|s(\mathbf{G}^{*F'^*})^{\langle \sigma^{*i} F^{*n/d} \rangle}| = |s(\mathbf{G}^{*F^{*n/d}})|$, because the set of representatives $\mathcal{T}$ of F'-stable semisimple classes of $\mathbf{G}^*$ can be chosen such that if the class of $t \in \mathcal{T}$ is $\sigma^{*i} F^{*n/d}$-stable, then $\sigma^{*i} F^{*n/d}(t) = t$. We then conclude using the fact that a power of $\sigma^{*i} F^{*n/d}$ equals F' and with Equation (3.4). So, by [7, 1.1], we deduce that

$$\left| s(\mathbf{G}^{*F'^*})^{\langle \sigma^{*i} F^{*n/d} \rangle} \right| = \begin{cases} p^{n|\Delta|/d} + 2p^{2n/d} & \text{if } \mathrm{Z}(\mathbf{G})^{\sigma^i F^{n/d}} = \mathrm{Z}(\mathbf{G}) \\ p^{n|\Delta|/d} & \text{otherwise} \end{cases}. \tag{4.12}$$

Furthermore, if $t \in \mathcal{T}$ is chosen $F^{*n/d}$-stable when the class of t in $\mathbf{G}^*$ is $F^{*n/d}$-stable, then we have $|s(\mathbf{G}^{*F'^*})^{\langle \sigma^* \rangle \times \langle F^{*n/d} \rangle}| = |s(\mathbf{G}^{*F^{n/d}})^{\sigma^*})|$. Thanks to Theorem 3.8, we deduce that

$$\left| s(\mathbf{G}^{*F'^*})^{\langle \sigma^* \rangle \times \langle F^{*n/d} \rangle} \right| = p^{n|\Delta_\sigma|/d}. \tag{4.13}$$

Now, using the case $k = 3$, the fact that $\mathcal{O}_D = \mathcal{O}_{D,1} \sqcup \mathcal{O}_{D,3}$ and Equations (4.12) and (4.13), we deduce that

$$\left| \mathcal{O}_{D,1}^{\langle \sigma^{*i} F^{*n/d} \rangle} \right| = p^{n|\Delta|/d} - p^{2n/d} \quad \text{and} \quad \left| \mathcal{O}_{D,1}^{\langle \sigma^* \rangle \times \langle F^{*n/d} \rangle} \right| = p^{n|\Delta_\sigma|/d} - p^{2n/d}. \tag{4.14}$$

Moreover, note that the argument at the beginning of the proof shows that the set $\mathcal{O}'^{\langle \sigma \rangle \times \langle F^{n/d} \rangle}_D$ is in bijection with $\mathrm{Irr}_s((\mathbf{B}^\sigma)^{F^{n/d}})$, which has $p^{d|\Delta_\sigma|}$ elements by [6, Proposition 3]. The set $\mathcal{O}'^{\langle \sigma^i F^{n/d} \rangle}_D$ is in bijection with $\mathrm{Irr}_s(\mathbf{B}^{\sigma^i F^{n/d}})$, which has $p^{n|\Delta|/d} + 2p^{2n/d}$ elements if $\sigma^i F^{n/d}$ acts trivially on $\mathrm{Z}(\mathbf{G})$, and $p^{d|\Delta|}$ elements otherwise. Since $\mathcal{O}'_D = \mathcal{O}'_{D,1} \sqcup \mathcal{O}'_{D,3}$, we deduce from Equations (4.9), (4.10) and (4.11) that

$$\left| \mathcal{O}'^{\langle \sigma^i F^{n/d} \rangle}_{D,1} \right| = p^{n|\Delta|/d} - p^{2n/d} \quad \text{and} \quad \left| \mathcal{O}'^{\langle \sigma \rangle \times \langle F^{n/d} \rangle}_{D,1} \right| = p^{n|\Delta_\sigma|/d} - p^{2n/d}. \tag{4.15}$$

Equations (4.14) and (4.15) prove that the sets $\mathcal{O}_{D,1}$ and $\mathcal{O}'_{D,1}$ are A-equivalent. So, by [7, Theorem 1.1], the sets $\mathrm{Irr}_s(\mathbf{G}^{F'}, \nu)$ and $\mathrm{Irr}_s(\mathbf{B}^{F'}, \nu)$ are in bijection. Since $\mathcal{O}_{D,\nu,3}$ and $\mathcal{O}'_{D,\nu,3}$ are in bijection by Equations (4.10) and (4.11), we deduce that $\mathcal{O}_{D,\nu,1}$ and $\mathcal{O}'_{D,\nu,1}$ have the same cardinal. Moreover, if H is not a subgroup of $\langle \sigma F \rangle$, then every H-stable character of $\mathrm{Irr}_s(\mathbf{G}^{F'})$ (resp. $\mathrm{Irr}_s(\mathbf{B}^{F'})$) lies over 1_Z. Now, let $H = \langle \sigma^i F^{n/d} \rangle$ be a subgroup of $\langle \sigma F \rangle$. We consider the norm map $N_{F'/\sigma^i F^{n/d}} : \mathrm{Z}(\mathbf{G}) \to \mathrm{Z}(\mathbf{G})$, which is well-defined because $(\sigma^i F^{n/d})^n = F'^n$. By [5, Lemma 5.8, Lemma 5.9], if $N_{F'/\sigma^i F^{n/d}}$ is surjective, then every set $\mathcal{O}'_{D,\nu,1}$ and $\mathcal{O}'_{D,\nu,1}$ contains $\frac{1}{3}(p^{n|\Delta|/d} - p^{2n/d})$ characters invariant under $\sigma^i F^{n/d}$. Otherwise (i.e., when $N_{F'/\sigma^i F^{n/d}}$ is trivial), the $\sigma^i F^{n/d}$-stable characters of $\mathrm{Irr}_s(\mathbf{G}^{F'})$ and $\mathrm{Irr}_s(\mathbf{B}^{F'})$ lie over 1_Z. So, this proves that, for every subgroup H of A_ν, we have

$$|\mathcal{O}^H_{D,\nu,1}| = |\mathcal{O}'^H_{D,\nu,1}|.$$

Therefore, the sets $\mathcal{O}_{D,\nu,1}$ and $\mathcal{O}'_{D,\nu,1}$ are A_ν-equivalent, as required.

Now, we suppose that $\mathrm{Z}(\mathbf{G})^F$ and $\mathrm{Z}(\mathbf{G})^{F'}$ are trivial, that is, $p \not\equiv 1 \mod 3$ and $q \not\equiv 1 \mod 3$. Then D is trivial and we conclude using the argument of Equations (4.12) and (4.13), and the discussion following Equation (4.14).

Finally, suppose that $\mathrm{Z}(\mathbf{G})^F = \mathrm{Z}(\mathbf{G})$, i.e. $p \equiv q \equiv 1 \mod 3$. Then $\mathrm{Z}(\mathbf{G})^{F'} = \mathrm{Z}(\mathbf{G})$. If n is odd, we remark that if X is a $\langle \sigma F^{n/d} \rangle$-set and an $\langle F' \rangle$-set, and if x is fixed by $\sigma F^{n/d}(x)$ and F', then $F^{n/d}(x) = x$ and $\sigma(x) = x$. In particular, we can compare $|\mathrm{Irr}_s(\mathbf{G}^{F'})^{\sigma F^{n/d}}|$ with the number of semisimple classes of $\mathbf{G}^{F^{n/d}}$ fixed by σ (there are $p^{n|\Delta_\sigma|/d}$ such classes by Lemma 3.7 and Theorem 3.8), and $|\mathrm{Irr}_s(\mathbf{B}^{F'})^{\langle \sigma F^{n/d} \rangle}|$ with $|\mathrm{Irr}_s((\mathbf{B}^\sigma)^{F^d})| = p^{n|\Delta_\sigma|/d}$. If n is even, then $(\sigma^i F^j) = F'$ and we are in the same situation as the first case of the proof. Using the same argument as above we deduce the numbers of Table 4.1. We set $a_{F^{n/d}} = b_{F^{n/d}} = p^{n|\Delta|/d} - p^{2n/d}$ when the norm map $N_{F'/F^{n/d}}$ is surjective, and $a_{F^{n/d}} = p^{n|\Delta|/d} - 3p^{2n/d}$ and $b_{F^{n/d}} = 0$ otherwise.

It follows that $\mathcal{O}_{D,\nu,k}$ and $\mathcal{O}'_{D,\nu,k}$ are A_ν-equivalent. This proves the claim. □

4.3. Inductive McKay condition

Lemma 4.8. — *Let $H = GC$ be a finite central product with $Z = G \cap C$. Suppose that C is abelian and that $\tau \in \mathrm{Aut}(H)$ acts on G and C. For $\chi \in \mathrm{Irr}(G)^\tau$, write $\nu = \mathrm{Res}^G_Z(\chi)$. If ν extends to a τ-stable character of C, then we have*

$$|\mathrm{Irr}(H|\chi)^\tau| = |(C/Z)^\tau|.$$

H	n	$\langle\,\sigma\,\rangle\times\langle\,F^{n/d}\,\rangle$	$\langle\,F^{n/d}\,\rangle$	$\langle\,\sigma F^{n/d}\,\rangle$
$\lvert\mathcal{O}^H_{D,1_Z,3}\rvert$		$p^{2n/d}$	$p^{2n/d}$	$p^{2n/d}$
$\lvert\mathcal{O}^H_{D,\varepsilon,3}\rvert$, $\varepsilon\neq 1_Z$		0	$p^{2n/d}$	0
$\lvert\mathcal{O}^H_{D,1_Z,1}\rvert$	odd	$p^{n\lvert\Delta_\sigma\rvert/d}-p^{2n/d}$	$a_{F^{n/d}}$	$p^{n\lvert\Delta_\sigma\rvert/d}-p^{2n/d}$
$\lvert\mathcal{O}^H_{D,1_Z,1}\rvert$	even	$p^{n\lvert\Delta_\sigma\rvert/d}-p^{2n/d}$	$a_{F^{n/d}}$	$p^{n\lvert\Delta\rvert/d}-p^{2n/d}$
$\lvert\mathcal{O}^H_{D,\varepsilon,1}\rvert$, $\varepsilon\neq 1_Z$		0	$b_{F^{n/d}}$	0
$\lvert\mathcal{O}'^H_{D,1_Z,3}\rvert$		$p^{2n/d}$	$p^{2n/d}$	$p^{2n/d}$
$\lvert\mathcal{O}'^H_{D,\varepsilon,3}\rvert$, $\varepsilon\neq 1_Z$		0	$p^{2n/d}$	0
$\lvert\mathcal{O}'^H_{D,1_Z,1}\rvert$	odd	$p^{n\lvert\Delta_\sigma\rvert/d}-p^{2n/d}$	$a_{F^{n/d}}$	$p^{n\lvert\Delta_\sigma\rvert/d}-p^{2n/d}$
$\lvert\mathcal{O}'^H_{D,1_Z,1}\rvert$	even	$p^{n\lvert\Delta_\sigma\rvert/d}-p^{2n/d}$	$a_{F^{n/d}}$	$p^{n\lvert\Delta\rvert/d}-p^{2n/d}$
$\lvert\mathcal{O}'^H_{D,\varepsilon,1}\rvert$, $\varepsilon\neq 1_Z$		0	$b_{F^{n/d}}$	0

Table 4.1. Case when $\mathrm{Z}(\mathbf{G})^F=\mathrm{Z}(\mathbf{G})$

Proof. — Recall that, for $\theta\in\mathrm{Irr}(H)$, if we write $\gamma=\mathrm{Res}^H_Z(\theta)$, then there are unique $\theta_1\in\mathrm{Irr}(G|\gamma)$ and $\theta_2\in\mathrm{Irr}(C|\gamma)$ such that $\theta=\theta_1\centerdot\theta_2$, where $\centerdot$ is the dot product, defined by $\theta_1\centerdot\theta_2(g_1g_2)=\theta_1(g_1)\theta_2(g_2)$ for all $g_1\in G$ and $g_2\in C$. Let $\chi\in\mathrm{Irr}(G)^\tau$. Write $\nu=\mathrm{Res}^H_Z(\chi)$. Then we have

$$\mathrm{Irr}(H|\chi)=\{\chi\centerdot\theta\,|\,\theta\in\mathrm{Irr}(C|\nu)\}.$$

Since χ is τ-stable, so is ν and $\mathrm{Irr}(C|\nu)$ is τ-stable. Furthermore, $\chi\centerdot\theta$ is τ-stable if and only if θ is τ-stable. In particular, the set $\mathrm{Irr}(H|\chi)^\tau$ is parametrized by $\mathrm{Irr}(C|\nu)^\tau$. Moreover, C is abelian, then ν extends to a linear character of C (denoted by the same symbol) [19, 5.4] and by assumption, ν can be supposed to be τ-stable. It follows that the map $g_\nu:\mathrm{Irr}(C|1_Z)\to\mathrm{Irr}(C|\nu)$, $\theta\mapsto\theta\nu$ is a bijection such that $\mathrm{Irr}(C|\nu)^\tau=g_\nu(\mathrm{Irr}(C|1_Z)^\tau)$ (because if ν is τ-stable, so is ν^{-1}). But we can identify $\mathrm{Irr}(C|1_Z)$ with $\mathrm{Irr}(C/Z)$ and the action of τ on these sets is compatible. Hence, we have $|\mathrm{Irr}(C|1_Z)^\tau|=|\mathrm{Irr}(C/Z)^\tau|$ and the result follows from [19, 6.32]. $\square$

We recall that if N is a normal subgroup of G, then we can associate to every G-invariant irreducible character of N an element $[\chi]_{G/N}$ of the second cohomology group $H^2(G/N,\mathbb{C}^\times)$ of G/N.

Theorem 4.9. — *Let $p > 3$ be a prime number and q a p-power. Then the finite simple group $E_6(q)$ is "good" for the prime p.*

Proof. — Let X be a simple group of type E_6 with parameter $q = p^n$. In order to prove that X is "good" for p, we will show that X satisfies the properties (1)–(8) of [20, §10]. Let $\mathbf{G}$ be a simple simply-connected group of type E_6 defined over $\mathbb{F}_q$ and with split Frobenius map $F' : \mathbf{G} \to \mathbf{G}$. Recall that $X = \mathbf{G}^{F'}/\mathrm{Z}(\mathbf{G})^{F'}$ and $\mathbf{G}^{F'}$ is the universal cover of X. Moreover, the subgroup $\mathbf{U}^F$ is a p-Sylow subgroup of $\mathbf{G}^{F'}$ with normalizer $\mathbf{B}^{F'}$. We set $M = \mathbf{B}^{F'}$ and we will show that M has the required properties. By [6, Lemma 5], we have $\mathrm{Irr}_s(\mathbf{G}^{F'}) = \mathrm{Irr}_{p'}(\mathbf{G}^{F'})$ and $\mathrm{Irr}_s(\mathbf{B}^{F'}) = \mathrm{Irr}_{p'}(\mathbf{B}^{F'})$. Let $\nu \in \mathrm{Irr}(\mathrm{Z}(\mathbf{G})^{F'})$ and $\mathcal{A}_\nu = D \rtimes A_\nu$ as above. By Theorem 4.7, there is an $\mathcal{A}_\nu$-equivariant bijection $\Phi_\nu : \mathrm{Irr}_{p'}(\mathbf{G}^{F'}|\nu) \to \mathrm{Irr}_{p'}(\mathbf{B}^{F'}|\nu)$. Thus, Properties (1)–(4) of [20, §10] are satisfied. Moreover, by Lemma 4.6, it is sufficient to prove the result for $\chi = \rho_{s,1}$ and $\Psi_\nu(\chi) = \chi_{J,1,\psi}$, where (J, ψ) corresponds to s. For such a χ, we define

$$G_\chi = G \rtimes \mathrm{Stab}_{A_\nu}(\chi) \quad \text{and} \quad G'_\chi = G' \rtimes \mathrm{Stab}_{A_\nu}(\Phi_\nu(\chi)),$$

where $G = \mathbf{G}^{F'}$ (resp. $G' = \mathbf{B}^{F'}$) if χ is not D-stable, and $G = \widetilde{\mathbf{G}}^{F'}$ (resp., $G' = \widetilde{\mathbf{B}}^{F'}$) otherwise. Recall that $\widetilde{\mathbf{G}}$ denotes the connected reductive group defined in Equation (3.3) with the convention of §4.1. We have $\mathrm{Z}(\mathbf{G}^{F'}) \leqslant \mathrm{Z}(G_\chi)$ and $\mathrm{Stab}_{D\rtimes A}(\chi)$ is induced by the conjugation action of $\mathrm{N}_{G_\chi}(\mathbf{U}^{F'})$. So, the property (5) of [20, §10] holds. We set $C = \mathrm{Z}(G)$. Then $C = \mathrm{C}_{G_\chi}(\mathbf{G}^{F'})$ and the property (6) of [20, §10] is true.

In order to prove the properties (7) and (8) of [20, §10], we will first show that χ and $\Phi_\nu(\chi)$ extend to G_χ and G'_χ, respectively. If χ is not D-stable, then $\mathrm{Stab}_A(\chi) \leqslant \langle F, \sigma \rangle$. If $\mathrm{Stab}_A(\chi)$ is cyclic, then χ extends to G_χ by [19, 11.22]. Otherwise, $\mathrm{Stab}_A(\chi) = \langle \sigma, F^{n/d} \rangle$ for some divisor d of n, and by Proposition 3.13, the extensions of χ to $\mathbf{G}^{F'} \rtimes \sigma$ are $F^{n/d}$-stable, and thus extend to G_χ by [19, 11.22]. So, χ is extendible to G_χ.

Suppose now that D is not trivial and χ is D-stable. In particular, χ is extendible to $\widetilde{\mathbf{G}}^{F'}$. Write $A_\chi = \mathrm{Stab}_A(\chi)$. We will prove that χ extends to an A_χ-stable character of $\widetilde{\mathbf{G}}^{F'}$. First, we suppose that $\sigma \in A_\chi$ and we write $H = GC$. Recall that H is a central product and that $C = \mathrm{Z}(\widetilde{\mathbf{G}}^{F'})$. By [3, §6.B], H has index $|D|$ in $\widetilde{\mathbf{G}}^{F'}$. Note that χ is over 1_Z. Since 1_C lies in $\mathrm{Irr}(C|1_Z)$, we deduce from Lemma 4.8 that the character $\chi \centerdot 1_C$ is A_χ-stable. Moreover, Gallagher's theorem implies that the elements of $E = \mathrm{Irr}(\widetilde{\mathbf{G}}^{F'}|\chi \centerdot 1_C)$ extend χ and $|E| = |D| = 3$. Moreover, E is A_χ-stable. Denote by $\rho : A_\chi \to \mathfrak{S}_E$ the homomorphism of groups induced by this operation. Suppose that $\sigma \in A_\chi$. Note that σ does not act trivially on E (because if σ fixes a character of E, then the action of σ on E is equivalent to the action of σ on $\widetilde{\mathbf{G}}^{F'}/H$, which is not trivial). So, $\rho(\sigma)\neq 1$ and $\rho(\sigma)$ has order 2. Thus, σ has to fix a character of E. Suppose now that there is $F^i\in A_\chi$. Since σ and F^i commute, we deduce that $\rho(F^i)$ centralizes $\rho(\sigma)$. But $\mathfrak{S}_E\simeq\mathfrak{S}_3$ and the centralizer of $\rho(\sigma)$ in $\mathfrak{S}_E$ has order 2. Thus, there is an A_χ-stable character $\widetilde{\chi}$ in E. By [15, 1.31], we have $\widetilde{\mathbf{G}}^\sigma=\operatorname{Rad}(\widetilde{\mathbf{G}})^\sigma D(\mathbf{G})^\sigma$, because σ has order 2 and $|\operatorname{Z}(D(\widetilde{\mathbf{G}}))|=3$. Since $\operatorname{Rad}(\widetilde{\mathbf{G}})=\mathbf{T}_0$ (cf. Equation (4.1)) has dimension 1, we deduce that σ acts by inversion on $\mathbf{T}_0$ and $|\mathbf{T}_0^\sigma|=2$. Hence, $\widetilde{\mathbf{G}}^\sigma$ is disconnected with connected component a simple group of type F_4 (by Lemma 4.1) of index 2. We deduce that $|\operatorname{Z}(\widetilde{\mathbf{G}}^\sigma)|=2$. Denote by $\widetilde{\Psi}$ the character of $\widetilde{\mathbf{G}}$ constructed in Equation (3.12). The character $\widetilde{\chi}$ extends to a character $E(\widetilde{\chi})$ of $\widetilde{\mathbf{G}}^{F'}\rtimes\langle\sigma\rangle$, and $\langle\widetilde{\Psi},E(\widetilde{\chi})\rangle_{\widetilde{\mathbf{G}}^{F'}\cdot\sigma}=\pm1$. Moreover, by [23, Proposition 7.2] $\widetilde{\Psi}$ has non-zero values only on $\widetilde{\mathcal{U}}_\sigma^{F'}$ (see the proof of Proposition 3.13 for the definition), which is the union of the two classes $\widetilde{\mathcal{U}}_1$ and $\widetilde{\mathcal{U}}_{-1}$ of $\widetilde{\mathbf{G}}^{F'}\rtimes\langle\sigma\rangle$ (by [23, Proposition 8.1], because σ is semisimple). Then $E(\widetilde{\chi})$ has a non-zero value on at least $\widetilde{\mathcal{U}}_1$ or $\widetilde{\mathcal{U}}_{-1}$. But F^i fixes these two classes (because there is an F^i-stable element in $\widetilde{\mathcal{U}}_\sigma^{F'}$. So, its $\widetilde{\mathbf{G}}^{F'}\rtimes\langle\sigma\rangle$-class is F^i-stable and the other class has to be also F^i-stable). Then the argument of Proposition 3.13 shows that $E(\widetilde{\chi})$ is F^i-stable. It follows that $E(\widetilde{\chi})$ extends to $(\widetilde{\mathbf{G}}^{F'}\rtimes\langle\sigma\rangle)\rtimes\langle F^i\rangle$ by [19, 11.22]. This proves that χ extends to G_χ.

Now, we suppose that $\sigma\notin A_\chi$. Then $A_\chi=\langle\sigma^iF^{n/d}\rangle$ for some $d\neq n$. Moreover, as we have seen in the proof of Theorem 4.6, we have $F'=(\sigma^iF^{n/d})^n$ (if not, σ has to fix χ). Then, there is a semisimple element s of $\mathbf{G}^{*F'^*}$ such that $\chi=\rho_{s,1}$. Since $A_{\mathbf{G}^*}(s)^{F'^*}$ is trivial, by the Lang-Steinberg theorem, we can suppose that s is chosen to be $\sigma^{*i}F^{*n/d}$-stable. Let $\widetilde{s}$ be an $\sigma^{*i}F^{*n/d}$-stable semisimple element of $\widetilde{\mathbf{G}}^*$ such that $i^*(\widetilde{s})=s$ (such elements exist by the Lang-Steinberg theorem, because $\ker(i^*)$ is connected and $\sigma^{*i}F^{*n/d}$-stable). Note that $\widetilde{s}\in\widetilde{\mathbf{G}}^{*F'^*}$, and since $\widetilde{s}$ is $\sigma^{*i}F^{*n/d}$-stable, the character $\rho_{\widetilde{s}}$ is $\sigma^iF^{n/d}$-stable. Moreover, $\rho_{\widetilde{s}}$ extends χ. Since G_χ is a cyclic extension of $\widetilde{\mathbf{G}}^{F'}$, we deduce from [19, 11.22] that χ extends to G_χ, as required.

Write $\chi'=\Phi_\nu(\chi)$. Then there are $J\subseteq\Delta$, $z\in H^1(F',\operatorname{Z}_J)$ and $\psi\in\operatorname{Irr}(\mathbf{T}_J)^{F'}$ with $\operatorname{Res}^{\mathbf{T}_J^{F'}}_{\operatorname{Z}(\mathbf{G})^{F'}}(\psi)=\nu$, such that $\chi'=\chi_{J,z,\psi}$. Suppose that χ' is D-stable. Then $z=1$ and $\mathbf{T}_J$ is connected. Write $\widetilde{\mathbf{T}}_J=\operatorname{C}_{\widetilde{\mathbf{T}}^{F'}}(\phi_J)$, where ϕ_J is chosen as in Convention 4.3. Note that $\widetilde{\mathbf{T}}_J$ is a torus because the center of $\widetilde{\mathbf{G}}$ is connected. Then $\mathbf{T}_J$ is a subtorus of $\widetilde{\mathbf{T}}_J$ and by [14, 0.5], there is an F'-stable subtorus $\mathbf{T}'$ of $\widetilde{\mathbf{T}}_J$ such that $\widetilde{\mathbf{T}}_J=\mathbf{T}_J\cdot\mathbf{T}'$ (as direct product). By [19, 6.17] and the construction of Equation (4.5) applied to $\mathbf{B}$ and $\widetilde{\mathbf{B}}$, we deduce that

$$\mathrm{Irr}(\mathbf{B}^{F'}|\chi_{J,1,\psi}) = \{\widetilde{\chi}_{J,1,\psi\otimes\mu}\,|\,\mu\in\mathrm{Irr}(\mathbf{T}'^{F'})\},$$

where $\widetilde{\chi}_{J,1,\psi\otimes\mu}$ denotes the character of $\widetilde{\mathbf{B}}^{F'}$ defined in Equation (4.5). Note that if $\mathbf{T}_J$ is $\sigma^i F^j$-stable (for any i, j), then $\mathbf{T}'$ is $\sigma^i F^j$-stable. Write $A_{\chi'} = \mathrm{Stab}_A(\chi')$. It follows that the character $\widetilde{\chi}_{J,1,\psi\otimes 1_{\mathbf{T}'^{F'}}} \in \mathrm{Irr}(\widetilde{\mathbf{B}}^{F'})$ is an $A_{\chi'}$-stable extension of χ'. For $\tau\in A_{\chi'}$, ϕ_J and ψ are τ-stable. Write $\widetilde{\psi} = \psi\otimes 1_{\mathbf{T}'^{F'}}$. Then $\widetilde{\mathbf{T}}_J^{F'}$ and $\widehat{\phi}_J\otimes\widetilde{\psi}$ are $A_{\chi'}$-stable. Hence, as linear character, $\widehat{\phi}_J\otimes\widetilde{\psi}$ extends to a linear character $E(\widehat{\phi}_J\otimes\widetilde{\psi})$ of $(\mathbf{U}^{F'}\rtimes\widetilde{\mathbf{T}}_J^{F'})\rtimes A_{\chi'}$, and by [19, (5.6) p.74], we have

$$\widetilde{\chi}_{J,1,\widetilde{\psi}} = \mathrm{Res}^{\widetilde{\mathbf{B}}^{F'}\rtimes A_{\chi'}}_{\widetilde{\mathbf{B}}^{F'}}\left(\mathrm{Ind}^{\widetilde{\mathbf{B}}^{F'}\rtimes A_{\chi'}}_{(\mathbf{U}^{F'}\rtimes\widetilde{\mathbf{T}}_J^{F'})\rtimes A_{\chi'}} E(\widehat{\phi}_J\otimes\widetilde{\psi})\right). \tag{4.16}$$

Hence, χ' extends to G'_χ. Suppose that χ' is not D-stable. If $A_{\chi'}$ is cyclic, then χ' extends to G'_χ by [19, 11.22]. Otherwise, $z=1$ and we can show that χ' extends to $A_{\chi'}$, because we can construct an extension of $\chi_{J,1,\psi}$ to $\mathbf{B}^{F'}\rtimes A_{\chi'}$ as in Equation (4.16).

We now will prove the properties (7) and (8) of [20, §10]. If χ is not D-stable, then $C = \mathrm{Z}(\mathbf{G})^{F'}$ and we choose $\gamma=\nu$. Then $\chi\centerdot\gamma=\chi$ and $\Phi_\nu(\chi)\centerdot\gamma = \Phi_\nu(\chi)$ and by the preceding discussion, these characters extend to G_χ and G'_χ. Then by [19, 11.7], we have

$$[\chi]_{G_\chi/\mathbf{G}^{F'}} = [\Phi_\nu(\chi)]_{G'_\chi/\mathbf{B}^{F'}},$$

and Properties (7) and (8) of [20, §10] are proven in this case.

Suppose now that D is not trivial and that χ is D-stable. If $\nu = 1_Z$ then we set $\gamma = 1_C$, which is G_χ and G'_χ-stable. Moreover, as we have seen, the characters $\chi\centerdot 1_C$ and $\Phi_\nu(\chi)\centerdot 1_C$ extend to G_χ and G'_χ. If $\nu\neq 1_Z$, then $G_\chi = \langle\, F''\,\rangle$ with $F'' = \sigma^i F^{n/d}$ for some $d\neq 1$ and $F''^n = F'$. Moreover, since $\nu\neq 1$ is F''-stable, it follows that F'' acts trivially on $\mathrm{Z}(\mathbf{G})^{F'}$. Note that $\Phi_\nu(\chi)$ is D-stable and extends to G'_χ. Denote by $\widetilde{\chi}'$ such an extension and write

$$\gamma = \mathrm{Res}^{G'_\chi}_{\mathrm{Z}(\widetilde{\mathbf{G}})^{F'}}(\widetilde{\chi}').$$

Then γ is G'_χ-stable and G_χ-stable and lies over ν. Furthermore, by [19, 6.17], we have

$$\left|\mathrm{Irr}(\widetilde{\mathbf{G}}^{F'}|\chi)^{F''}\right| = \sum_{\theta\in\mathrm{Irr}(\mathbf{G}^F\,C|\chi)^{F''}}\left|\mathrm{Irr}(\widetilde{\mathbf{G}}^{F'}|\theta)^{F''}\right|. \tag{4.17}$$

As we have seen previously, $\chi=\rho_{s,1}$ for some F''-stable semisimple element $s\in\mathbf{G}^{*F'^*}$ and there is an F''-stable semisimple element $\widetilde{s}\in\widetilde{\mathbf{G}}^{*F'^*}$ such that $i^*(\widetilde{s}) = s$. Then by [3, (15.9)], we have $\mathrm{Irr}(\widetilde{\mathbf{G}}^{F'}|\chi) = \{\rho_{\widetilde{sz},1} \mid z \in C\}$ and [3, Lemma 8.3] implies that this set is in bijection with C because $A_{\mathbf{G}^*}(s)^{F'}$ is trivial. Moreover, since $\widetilde{s}$ is F''-stable, these operations are F''-equivalent and $|\,\mathrm{Irr}(\widetilde{\mathbf{G}}^{F'},|\chi)^{F''}| = |C^{F''}|$. Now, Lemma 4.8 implies that $|\,\mathrm{Irr}(\mathbf{G}^F\,|\chi)^{F''}| = |C^{F''}|/3$, because F'' acts trivially on $\mathrm{Z}(\mathbf{G}^F) = \mathbf{G}^F \cap C$. Since $|\,\mathrm{Irr}(\widetilde{\mathbf{G}}^{F'}|\theta)^{F''}| \leqslant 3$ (because $\mathbf{G}^F C$ has index 3 in $\widetilde{\mathbf{G}}^{F'}$ and by [19, 6.17]), we deduce from Equation (4.17) that $|\,\mathrm{Irr}(\widetilde{\mathbf{G}}^{F'}|\theta)^{F''}|$ has to be equal to 3. But $\chi\cdot\gamma \in \mathrm{Irr}(\mathbf{G}^F\,C|\chi)^{F''}$. Thus, $\chi\cdot\gamma$ extends to an F''-stable character of $\widetilde{\mathbf{G}}^{F'}$ and [19, 11.22] implies that $\chi\cdot\gamma$ extends to G_χ. We conclude using [19, 11.7] that

$$[\chi\centerdot\gamma]_{G_\chi/\mathbf{G}^{F'}C} = [\Phi_\nu(\chi)\centerdot\gamma]_{\mathbf{G}'_\chi/\mathbf{G}^{F'}\,C}.$$

Hence, the properties (7) and (8) of [20, §10] hold, as required. □

THEOREM 4.10. — *Let $p > 3$ be a prime number and q a p-power. Then the finite simple group ${}^2E_6(q)$ is "good" for the prime p.*

Proof. — We set $F' = \sigma F^n$. Then $\mathbf{G}^{F'}$ is the universal cover of ${}^2E_6(q)$, and the outer automorphism group of $\mathbf{G}^{F'}$ is $D \rtimes \langle\, F\,\rangle$, where F acts on $\mathbf{G}^{F'}$ as an automorphism of order $2n$. For $\omega \in \Delta/\rho$, we set $\mathbf{X}_\omega = \prod_{\alpha\in\omega}\mathbf{X}_\alpha$. Note that $\mathbf{X}_\omega$ is a subgroup of $\mathbf{U}$, because $\mathbf{G}$ is of type E_6 and if the roots α and $\rho(\alpha)$ are distinct, then they are orthogonal. Moreover, by [9, §8.1] we have $\mathbf{U}_1^{F'} \simeq \prod_{\omega\in\Delta/\rho}\mathbf{X}_\omega^{F'}$ (as direct product) and can construct the Gelfand-Graev characters, the regular characters and the semisimple characters of $\mathbf{G}^{F'}$ as in §3.1. Note that the analogue of Theorem 3.1 is valid (see [3, Proposition 15.3, Corollaire 15.14]) and the regular character ϕ_1 of $\mathrm{Irr}_l(\mathbf{U}^{F'})$ can be chosen F-stable by [19, 6.32]. In particular, if s is an F^*-stable element of $\mathbf{G}^{*F'^*}$, then for every $z \in H^1(F', \mathrm{Z}(\mathbf{G}))$, we have

$${}^{F^i}\rho_{s,\,\widehat{\omega}_s^0(z)} = \rho_{s,\,\widehat{\omega}_s^0(F^i(z))}.$$

For every $J \subseteq \Delta/\rho$, we write ω_J for the corresponding $\widetilde{\mathbf{T}}^{F'}$-orbit in $\mathrm{Irr}_l(\mathbf{U}^{F'})$ and we choose a representative $\phi_J \in \omega_J$ as in Convention 4.3. Then we can define $\chi_{J,z,\psi} \in \mathrm{Irr}_s(\mathbf{B}^{F'})$ as in Equation (4.5), which satisfies Lemma 4.4.

Let E be an F- and F'-set, and let $x \in E$ be such that $F^k(x) = x$ and $F'(x) = x$. Denote by d the order of F^k (viewed as an automorphism of $\mathbf{G}^{F'}$). If d is even (resp. odd), then this is equivalent to $\sigma(x) = x$ and $F^k(x) = x$ (resp. $\sigma F^{n/d}(x) = x$). Moreover, note that if $\mathrm{Z}(\mathbf{G})^{F'} = \mathrm{Z}(\mathbf{G})$, then $\mathrm{Z}(\mathbf{G})^F = \{1\}$ and n is odd. Using these facts, we can prove in a similar way as in the proof of Theorem 4.7, that for every $\nu \in \mathrm{Irr}(\mathrm{Z}(\mathbf{G})^{F'})$ and $A_\nu = \mathrm{Stab}_{\langle\, F\,\rangle}(\nu)$, there is an $D \rtimes A_\nu$-equivariant bijection between

$\mathrm{Irr}_s(\mathbf{G}^{F'},\nu)$ and $\mathrm{Irr}_s(\mathbf{B}^{F'})$. Finally, we prove that the properties (5)-(8) of [20, §10] are satisfied as in the proof of Theorem 4.9. □

Remark 4.11. — This method is not sufficient to show the statement for $p\in\{2,3\}$. Indeed, we need the assumption that p is a good prime for $\mathbf{G}$ to apply the "relative" version of the McKay Conjecture proved in [7, Theorem 1.1], and [23, Proposition 8.1(ii)].

PROPOSITION 4.12. — *Suppose that $X=\mathrm{PSL}_\ell(p^n)$ or $X=\mathrm{PSU}_\ell(p^n)$ such that p is odd and ℓ is an odd prime number that not divides p. Then X is "good" for p.*

Proof. — We set $\widetilde{\mathbf{G}}=\mathrm{GL}_\ell(\overline{\mathbb{F}}_p)$ and denote by F the standard Frobenius map of $\widetilde{\mathbf{G}}$ which acts by raising all entries of a matrix to the p-power. Write $F'=F^n$ and σ for the non-trivial graph automorphism with respect to the F-stable Borel subgroup of lower triangular matrices and the F-stable maximal torus $\widetilde{\mathbf{T}}$ of diagonal matrices of $\widetilde{\mathbf{G}}$. We set $\mathbf{G}=\mathrm{SL}_\ell(\overline{\mathbb{F}}_p)$, $\mathbf{T}=\widetilde{\mathbf{T}}\cap\mathbf{G}$ and $\mathbf{B}=\widetilde{\mathbf{B}}\cap\mathbf{G}$. Note that $\widetilde{\mathbf{G}}^\sigma=\mathrm{GO}_\ell(\overline{\mathbb{F}}_p)$ and $\mathbf{G}^\sigma=\mathrm{SO}_\ell(\overline{\mathbb{F}}_p)$, which implies that $\mathrm{Z}(\mathbf{G}^\sigma)=\{1\}$ and $\mathrm{Z}(\widetilde{\mathbf{G}}^\sigma)$ has order 2. Denote by $\Delta=\{\alpha_1,\ldots,\alpha_{\ell-1}\}$ the set of simple roots of $\mathbf{G}$ with respect to $\mathbf{T}$ and $\mathbf{B}$. We set $\widetilde{\alpha}_i=\frac{1}{2}(\alpha_i+\alpha_{\ell-i})$ and $\Delta_\sigma=\{\widetilde{\alpha}_i\,|\,1\leqslant 1\leqslant(\ell-1)/2\}$. Then it is proven in [17, Lemma 4.4.7] that Δ_σ is the set of simple roots of $\mathbf{G}^\sigma$ with respect to $\mathbf{T}^\sigma$ and $\mathbf{B}^\sigma$. Moreover, if we set $\widetilde{\mathbf{X}}_{\widetilde{\alpha}_i}=\mathbf{X}_{\alpha_i}.\mathbf{X}_{\alpha_{\ell-i}}$ for $1\leqslant i\leqslant(\ell-3)/2$ and $\widetilde{\mathbf{X}}_{\widetilde{\alpha}_{(\ell-1)/2}}=\langle\,\mathbf{X}_{\alpha_{(\ell-1)/2}},\mathbf{X}_{\alpha_{(\ell+1)/2}}\,\rangle$, then the group $\widetilde{\mathbf{X}}^\sigma_{\widetilde{\alpha}_i}$ is the root subgroup corresponding to $\widetilde{\alpha}_i$; see the proof of [17, Lemma 4.4.7]. Write $\widetilde{J}$ for the subset of Δ_σ associated to a σ-stable subset J of Δ. Since $\mathrm{Z}(\mathbf{G})^\tau=\{1\}$, it follows from the argument of the proof of Theorem 4.7 that $\mathbf{T}_J^\sigma$ (here, $\mathbf{T}_J$ denotes the radical of $\mathbf{L}_J$ as before) is the radical of the Levi subgroup $\mathbf{L}_{\widetilde{J}}$ of $\mathbf{G}^\sigma$.

Therefore, by a similar argument to Lemma 4.6 and Theorem 4.7, we show that for $\nu\in\mathrm{Irr}(\mathrm{Z}(\mathbf{G})^{F'})$, there is an A_ν-equivariant bijection between $\mathrm{Irr}_s(\mathbf{G}^{F'}|\nu)$ and $\mathrm{Irr}_s(\mathbf{B}^{F'}|\nu)$.

Now, we suppose that $\chi\in\mathrm{Irr}(\mathbf{G}^{F'})$ is $\widetilde{\mathbf{T}}^{F'}$-stable and we denote by A_χ its inertia subgroup in $\mathrm{Aut}(\widetilde{\mathbf{G}}^{F'})$. As above, we write $E=\mathrm{Irr}(\widetilde{\mathbf{G}}^{F'}|\chi\centerdot 1_C)$, where $C=\mathrm{Z}(\widetilde{\mathbf{G}}^{F'})$. Then $|E|=\ell$ and $\chi\centerdot 1_C$ is A_χ-stable. So, A_χ acts on E. Suppose that $\sigma\in A_\chi$. Since σ has order 2 and that ℓ is odd, σ fixes a character $\widetilde{\chi}$ of E. Thus, by Clifford theory and by [19, 6.32], the actions of σ on E and on $\widetilde{\mathbf{G}}^{F'}/\mathbf{G}^FC$ are equivalent. Since σ acts by inversion on this group and has no non-trivial fixed point (because ℓ is odd), we deduce that $\widetilde{\chi}$ is the unique σ-stable character of E. Let $\tau\in A_\chi$. Then τ and σ commute and $\tau(\widetilde{\chi})$ is a σ-stable character of E. By unicity, $\tau(\widetilde{\chi})=\widetilde{\chi}$, which proves that E has an A_χ-stable element. Finally, we conclude with a similar argument to the proof of Theorem 4.9. The proof for a twisted Frobenius map is similar and the claim is proven. □

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