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Invariant meromorphic functions on Stein spaces

INVARIANT MEROMORPHIC FUNCTIONS
ON STEIN SPACES

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ABSTRACT. — In this paper we develop fundamental tools and methods to study meromorphic functions in an equivariant setup. As our main result we construct quotients of Rosenlicht-type for Stein spaces acted upon holomorphically by complex-reductive Lie groups and their algebraic subgroups. In particular, we show that in this setup invariant meromorphic functions separate orbits in general position. Applications to almost homogeneous spaces and principal orbit types are given. Furthermore, we use the main result to investigate the relation between holomorphic and meromorphic invariants for reductive group actions. As one important step in our proof we obtain a weak equivariant analogue of Narasimhan's embedding theorem for Stein spaces.

RÉSUMÉ. — Dans ce travail nous développons des outils et des méthodes fondamentaux afin d'étudier les fonctions méromorphes invariantes sur les espaces de Stein X munis d'une action holomorphe d'un groupe complexe-réductif G. Nous construisons des quotients à la Rosenlicht pour l'action d'un sous-groupe algébrique de G sur X. En particulier on montre que dans cette situation les fonctions méromorphes invariantes sous ce sous-groupe algébrique séparent ses orbites en position générale. Nous donnons aussi des applications concernant les espaces presque homogènes et les types d'orbite principaux. De plus, le résultat principal est utilisé afin de clarifier la relation entre les invariants holomorphes voire méromorphes de G. Une étape importante de notre preuve consiste à montrer un analogue faible équivariant du théorème de Narasimhan sur les plongements propres des espaces de Stein.

1. Introduction

One of the fundamental results relating invariant theory and the geometry of algebraic group actions is Rosenlicht's Theorem [25, Thm. 2]: for

any action of a linear algebraic group on an algebraic variety there exists
a finite set of invariant rational functions that separate orbits in general
position. Moreover, there exists a rational quotient, i.e., a Zariski-open
invariant subset on which the action admits a geometric quotient. It is
the purpose of this paper to study meromorphic functions invariant under
holomorphic group actions and to construct quotients of Rosenlicht-type
in the analytic category.

Examples of non-algebraic holomorphic actions of $\mathbb{C}^*$ on projective sur-
faces with nowhere Hausdorff orbit space show that even in the compact
analytic case an analogue of Rosenlicht's Theorem does not hold without
further assumptions. If a complex-reductive group acts *meromorphically* on
a compact Kähler space (and more generally a compact complex space in
class $\mathscr{C}$), existence of meromorphic quotients was shown by Lieberman [16]
and Fujiki [5].

As a natural starting point in the non-compact case we consider group
actions on spaces with rich function theory such as Stein spaces. Actions of
reductive groups and their subgroups on these spaces are known to possess
many features of algebraic group actions. However, while the holomorphic
invariant theory in this setup is well understood, cf. [11], invariant mero-
morphic functions until now have been less studied.

In this paper we develop fundamental tools to study meromorphic func-
tions in an equivariant setup. We use these tools to prove the following
result, which provides a natural generalisation of Rosenlicht's Theorem to
Stein spaces with actions of complex-reductive groups.

MAIN THEOREM. — *Let $H < G$ be an algebraic subgroup of a complex-
reductive Lie group G and let X be a Stein G-space. Then, there exist an
H-invariant Zariski-open dense subset Ω in X and a holomorphic map
$p\colon \Omega \to Q$ to a Stein space Q such that*

(1) *the map p is a geometric quotient for the H-action on Ω,*
(2) *the map p is universal with respect to H-stable analytic subsets of
Ω,*
(3) *the map p is a submersion and realises Ω as a topological fibre
bundle over Q,*
(4) *the map p extends to a weakly meromorphic map (in the sense of
Stoll) from X to Q,*
(5) *for every H-invariant meromorphic function $f \in \mathscr{M}_X(X)^H$, there
exists a unique meromorphic function $\bar{f} \in \mathscr{M}_Q(Q)$ such that $f|_\Omega =
\bar{f} \circ p$, i.e., the map $p\colon \Omega \to Q$ induces an isomorphism between
$\mathscr{M}_X(X)^H$ and $\mathscr{M}_Q(Q)$, and*

(6) the H-invariant meromorphic functions on X separate the H-orbits in Ω.

The idea of proof is to first establish a weak equivariant analogue of Remmert's and Narasimhan's embedding theorem for Stein spaces [19]. More precisely, given a G-irreducible Stein G-space we prove the existence of a G-equivariant holomorphic map into a finite-dimensional G-representation space V that is a proper embedding when restricted to a big Zariski-open G-invariant subset, see Proposition 5.2. Since the G-action on V is algebraic, we may then apply Rosenlicht's Theorem to this linear action. Subsequently, a careful comparison of algebraic and holomorphic geometric quotients allows us to carry over the existence of a Rosenlicht-type quotient from V to X.

The geometric quotient constructed in this paper provides us with a new and effective tool to investigate invariant meromorphic functions on Stein spaces. In the following we shortly describe two typical applications of our main result.

Given a Stein G-space we show that every invariant meromorphic function is a quotient of two invariant holomorphic functions precisely if the generic fibre of the natural invariant-theoretic quotient $\pi \colon X \to X /\!\!/ G$ contains a dense orbit, see Theorem 3.5. An important class of examples for this situation consists of representation spaces of semisimple groups G.

An important fundamental result of Richardson [24] states that in every connected Stein G-manifold there exists an open and dense subset on which all isotropy groups are conjugate in G. Under further assumptions on the group action we use the Main Theorem to sharpen Richardson's result by showing that there exists a Zariski-open subset on which the conjugacy class of stabiliser groups is constant, see Proposition 3.11. In particular, for every effective torus action on a Stein manifold we find a Zariski-open subset that is a principal fibre bundle over the meromorphic quotient, cf. Remark 3.14.

This paper is organised as follows. In Section 2 we introduce the necessary background on actions of complex-reductive groups and on related notions of quotient spaces. Furthermore, we shortly discuss the main technical tools used in this paper. In Sections 3 and 4 we give applications of the Main Theorem as well as some examples which illustrate that the result does not hold for non-algebraic subgroups of G. In Section 5 we establish the Weak Equivariant Embedding Theorem, before we prove the Main Theorem in the final Sections 6 and 7.

2. Preliminaries: Definitions and Tools

In the following, all *complex spaces* are assumed to be reduced and to have countable topology. If $\mathscr{F}$ is a sheaf on a complex space X, and $U \subset X$ is an open subset, then $\mathscr{F}(U)$ denotes the set of sections of $\mathscr{F}$ over U. By definition, analytic subsets of complex spaces are closed. Furthermore, an *algebraic group* is by definition linear algebraic, i.e., a closed algebraic subgroup of some $\mathrm{GL}_N(\mathbb{C})$.

2.1. Actions of Lie groups

If L is a real Lie group, then a *complex L-space Z* is a complex space with a real-analytic action $\alpha \colon L \times Z \to Z$ such that all the maps $\alpha_g \colon Z \to Z$, $z \mapsto \alpha(g, z) =: g \bullet z$ are holomorphic. If L is a complex Lie group, a *holomorphic L-space Z* is a complex L-space such that the action map $\alpha \colon L \times Z \to Z$ is holomorphic. If X is at the same time a Stein space and a holomorphic L-space, we shortly say that X is a *Stein L-space*. A complex L-space is called *L-irreducible* if L acts transitively on the set of irreducible components of X. Note that in this case X is automatically pure-dimensional. If the set X/L of L-orbits can be endowed with the structure of a complex space such that the quotient map $p \colon X \to X/L$ is holomorphic, then X is L-irreducible if and only if X/L is irreducible. In particular, under this condition $\mathscr{M}_{X/L}(X/L)$ and $\mathscr{M}_X(X)^L$ are fields.

2.2. Geometric quotients

One of the main tasks in the proof of the Main Theorem is the construction of a geometric quotient for actions of complex Lie groups on complex spaces in the sense of the following definition.

DEFINITION 2.1. — *Let L be a complex Lie group and let X be a holomorphic L-space. A geometric quotient for the action of L on X is a holomorphic map $p \colon X \to Q$ onto a complex space Q such that*

   (1) *for all $x \in X$, we have $p^{-1}(p(x)) = L \bullet x$,*
   (2) *Q has the quotient topology with respect to p,*
   (3) *$(p_*\mathscr{O}_X)^L = \mathscr{O}_Q$.*

If a geometric quotient $p\colon X \to Q$ for the action of L exists, we can identify Q with the set of L-orbits in X and we will often write X/L instead of Q. The map $p\colon X \to Q$ has the following universality property: for any G-invariant holomorphic map $\phi\colon X \to Y$ into a complex space Y there exists a uniquely defined holomorphic map $\overline{\phi}\colon Q \to Y$ such that $\phi = \overline{\phi} \circ p$. We call a geometric quotient $p\colon X \to Q$ *universal with respect to invariant analytic subsets* if for every such set $A \subset X$ the restriction $p|_A\colon A \to p(A)$ is a geometric quotient for the L-action on A. Note that $p(A)$ is always an analytic subset of Q, see Lemma 6.1.

Remark 2.2. — We also use the corresponding concepts in the algebraic category. Note that in this case item (2) of Definition 2.1 requires the quotient to have the quotient Zariski-topology with respect to the map p.

The following general existence result for geometric quotients in the algebraic category by Rosenlicht is the starting point of this paper.

Theorem 2.3 (Thm. 2 of [25]). — *Let H be a linear algebraic group and X an H-irreducible algebraic H-variety. Then, there exists an H-invariant Zariski-open dense subset U of X that admits a geometric quotient. Furthermore, this quotient fulfills $\mathbb{C}(U/H) = \mathbb{C}(X)^H$.*

2.3. Analytic Hilbert quotients and slice-type stratification

Let G be a complex-reductive Lie group and X a holomorphic G-space. A complex space Y together with a G-invariant surjective holomorphic map $\pi\colon X \to Y$ is called an *analytic Hilbert quotient* of X by the action of G if

   (1) π is a locally Stein map, and
   (2) $(\pi_* \mathcal{O}_X)^G = \mathcal{O}_Y$ holds.

An analytic Hilbert quotient of a holomorphic G-space X is unique up to biholomorphism once it exists, and we will denote it by $X/\!/G$. This is the natural analogue of the concept of *good quotient* or *categorical quotient* in Algebraic Geometry, cf. [2, Ch. 3]. Moreover, if X is an algebraic G-variety with a good quotient $\pi\colon X \to X/\!/G$, then the associated map $\pi^h\colon X^h \to (X/\!/G)^h$ is an analytic Hilbert quotient, see [18]. If X is a Stein G-space, then the analytic Hilbert quotient $\pi\colon X \to X/\!/G$ exists, see [26], [11]. It has the following properties, cf. [12]:

   (1) Given a G-invariant holomorphic map $\phi\colon X \to Z$ to a complex space Z, there exists a unique holomorphic map $\bar{\phi}\colon X/\!/G \to Z$ such that $\phi = \bar{\phi} \circ \pi$.

(2) For every Stein subspace A of $X/\!\!/G$ the inverse image $\pi^{-1}(A)$ is a Stein subspace of X.

(3) If A_1 and A_2 are G-invariant analytic (in particular, closed) subsets of X, then we have $\pi(A_1) \cap \pi(A_2) = \pi(A_1 \cap A_2)$.

(4) For a G-invariant closed complex subspace A of X, which is defined by a G-invariant sheaf $\mathscr{I}_A$ of ideals, the image sheaf $(\pi_* \mathscr{I}_A)^G$ endows the image $\pi(A)$ in $X/\!\!/G$ with the structure of a closed complex subspace of $X/\!\!/G$. Moreover, the restriction of π to A is an analytic Hilbert quotient for the action of G on A.

It follows that two points $x, x' \in X$ have the same image in $X/\!\!/G$ if and only if $\overline{G \bullet x} \cap \overline{G \bullet x'} \neq \emptyset$. For each $q \in X/\!\!/G$, the fibre $\pi^{-1}(q)$ contains a unique closed G-orbit $G \bullet x$. The orbit $G \bullet x$ is affine (see [26, Prop. 2.3 and 2.5]) and hence, the stabiliser G_x of x in G is a complex-reductive Lie group by a result of Matsushima.

Let G be a complex-reductive Lie group and let X be a holomorphic G-space with analytic Hilbert quotient. There exist two related important stratifications of the quotient $X/\!\!/G$. The main reference for these stratifications in the algebraic case is [17, Sect. III.2]. In the following we are going to use the notion of *orbit type* and of *slice type* as defined in [10, Sect. 4]: If X is a G-irreducible Stein G-space and $q \in X/\!\!/G$, then there exists a unique closed G-orbit $G \bullet x$ in the fibre $\pi^{-1}(q)$. We define the slice type of q to be the type of the G_x-representation on the Zariski tangent space $T_x X$, i.e., the isomorphism class of the G-vector bundle $G \times_{G_x} (T_x X)$. Analogously, we denote by $\mathrm{Type}(G \bullet x)$ the orbit type of x, i.e., the conjugacy class (G_x) in G of the isotropy subgroup G_x of G at x. Using the holomorphic slice theorem and the corresponding results in the algebraic category one obtains the following result.

PROPOSITION 2.4. — *Let X be a G-irreducible holomorphic G-space with analytic Hilbert quotient $\pi\colon X \to X/\!\!/G$. The decomposition of $X/\!\!/G$ according to slice types defines a complex analytic stratification of the quotient $X/\!\!/G$. In particular, in $X/\!\!/G$ there exists a maximal, Zariski-open stratum $S_{\max}$ of the slice-type stratification. The orbit-type of closed orbits is constant on this stratum. Furthermore, the restriction of π to $X_{\max} := \pi^{-1}(S_{\max})$ realises $X_{\max}$ as a holomorphic fibre bundle over $S_{\max}$.*

2.4. (Weakly) meromorphic maps and functions

Recall from [22] and [4, sect. 4.6] that there is a natural correspondence between meromorphic functions on a (pure-dimensional) complex space X

and so-called meromorphic graphs, i.e., graphs of meromorphic maps from X to $\mathbb{P}_1 = \mathbb{C} \cup \{\infty\}$ that do not map any irreducible component of X to ∞. For a meromorphic function f on X we denote by P_f the *pole variety* of f. It is a nowhere dense analytic subset of X, and the smallest subset of X such that f is holomorphic on $X \setminus P_f$. We set $\mathrm{dom}(f) := X \setminus P_f$ and we call it the *domain of definition* of f.

Suppose now that a complex Lie group L acts on a complex space X. Then we have an induced action of L on the algebra $\mathscr{M}_X(X)$ of meromorphic functions as follows. Let f be a meromorphic function on X with graph $\Gamma_f \subset X \times \mathbb{P}_1$. The group L acts on $X \times \mathbb{P}_1$ by the L-action on the first factor. Given $g \in L$, we define a new meromorphic graph $\Gamma_{g \bullet f} := g \bullet \Gamma_f \subset X \times \mathbb{P}_1$ and hence a meromorphic function $g \bullet f$ on X. In this way we obtain an action of L on $\mathscr{M}_X(X)$ by algebra homomorphisms. A meromorphic function $f \in \mathscr{M}_X(X)$ is L-invariant if and only if its graph Γ_f is an L-invariant analytic subset of $X \times \mathbb{P}_1$. In this case the pole variety of f is an L-invariant analytic subset of X.

The following definition is taken from [27] and [28]. It is useful when considering maps into (non-compactifiable) non-compact target spaces, as we will do in the following.

DEFINITION 2.5. — *Let X be a complex space and let A be a nowhere dense analytic subset of X. Let Y be a complex space. Then, a holomorphic map $\phi\colon X \setminus A \to Y$ is called* weakly meromorphic, *if for any point $p_0 \in A$ and any one-dimensional complex submanifold C of X with $C \cap A = \overline{C} \cap A = \{p_0\}$ there exists at most one point $q_0 \in Y$ with the following property: there exists a sequence $(p_n)_{n \in \mathbb{N}} \subset C \setminus A$ with $\lim_{n \to \infty} p_n = p_0$ such that q_0 is the accumulation point of $\big(\phi(p_n)\big)_{n \in \mathbb{N}}$.*

Example 2.6. — A meromorphic map is in particular weakly meromorphic, see [27, Satz 3.3].

3. Applications

In the following sections we give applications of the Main Theorem to almost-homogeneous spaces (Section 3.1), to the problem of realising meromorphic invariants as quotients of holomorphic invariants (Section 3.2), to holomorphically convex spaces with actions of compact groups (Section 3.3), to the existence of principal orbits types (Section 3.4), and to actions of unipotent groups (Section 3.5).

3.1. Characterising almost-homogeneous spaces

Using Rosenlicht's Theorem, we see that an algebraic variety is almost-homogeneous for a group G acting on X if and only if every G-invariant rational function on X is constant. As a corollary of our main result we obtain the corresponding result in the complex-analytic category:

PROPOSITION 3.1. — *Let X be a Stein G-space and let H be an algebraic subgroup of G. Then H has an open orbit in X if and only if $\mathscr{M}_X(X)^H = \mathbb{C}$.*

Remark 3.2. — Note that this can also be proven without using the Main Theorem: in case $\mathscr{M}_X(X)^H = \mathbb{C}$, clearly also $\mathscr{O}_X(X)^G = \mathbb{C}$. A result of Snow [26, Cor. 5.6] implies that X naturally carries the structure of an affine algebraic G-variety. Consequently, X is almost-homogeneous by Rosenlicht's Theorem, see Theorem 2.3.

3.2. The connection between holomorphic and meromorphic invariants

Let X be a G-irreducible Stein G-space with analytic Hilbert quotient $\pi\colon X \to X/\!\!/G$. Then clearly the field of invariant meromorphic functions $\mathscr{M}_X(X)^G$ contains the field $\mathscr{M}_{X/\!\!/G}(X/\!\!/G)$ of meromorphic functions on $X/\!\!/G$ via the pull-back morphism

$$(3.1) \qquad \pi^*\colon \mathscr{M}_{X/\!\!/G}(X/\!\!/G) \hookrightarrow \mathscr{M}_X(X)^G.$$

Using our main result, in this section we describe the image of π^* in $\mathscr{M}_X(X)^G$ and we characterise those spaces for which π^* is an isomorphism. Furthermore, we give examples why in general this cannot be expected. It follows that in most situations the information encoded in the Rosenlicht-type quotient constructed in the Main Theorem cannot be recovered from the analytic Hilbert quotient $X/\!\!/G$.

The following result characterises the image of π^* in $\mathscr{M}_X(X)^G$.

PROPOSITION 3.3. — *Let X be a G-irreducible Stein G-space with analytic Hilbert quotient $\pi\colon X \to X/\!\!/G$. A function $f \in \mathscr{M}_X(X)^G$ is contained in $\mathrm{Im}(\pi^*)$ if and only if there exist $p, q \in \mathscr{O}_X(X)^G$, $q \neq 0$, such that $f = p/q$.*

Proof. — If $f = p/q$ for $p, q \in \mathscr{O}_X(X)^G$, $q \not\equiv 0$, then by the universal properties of the analytic Hilbert quotient there exist holomorphic functions $\bar{p}$ and $\bar{q}$ on $X/\!\!/G$ such $p = \pi^*\bar{p}$ and $q = \pi^*\bar{q}$, respectively. Consequently, f is

the pull-back of the meromorphic function $\bar{p}/\bar{q} \in \mathscr{M}_{X/\!/G}(X/\!/G)$. Conversely, assume that $f = \pi^*\bar{f}$ for some $\bar{f} \in \mathscr{M}_{X/\!/G}(X/\!/G)$. Since $X/\!/G$ is Stein, the Poincaré problem on $X/\!/G$ is universally solvable, see for example [6, Ch. 4, §2, Thm. 4]; i.e., there exist holomorphic functions $\bar{p}, \bar{q} \in \mathscr{O}_{X/\!/G}(X/\!/G)$, $\bar{q} \not\equiv 0$, such that $\bar{f} = \bar{p}/\bar{q}$. Then, $f = \pi^*\bar{p}/\pi^*\bar{q}$ is a quotient of holomorphic invariants. $\qquad\square$

The following example shows that the inclusion (3.1) is strict in general.

Example 3.4. — Consider the action of $\mathbb{C}^*$ on $\mathbb{C}^2$ by scalar multiplication. Then, the meromorphic function $f(z, w) = z/w$ is $\mathbb{C}^*$-invariant. However, the analytic Hilbert quotient is $\pi\colon \mathbb{C}^2 \to \{\text{point}\}$, so that f is not the pull-back of a meromorphic function via π. In order to construct an example for a semisimple group action from this one, let $T < G = \mathrm{SL}_2(\mathbb{C})$ be the maximal torus of diagonal matrices. Consider the diagonal action of T on the product $G \times \mathbb{C}^2$, where $T \cong \mathbb{C}^*$ acts on $\mathbb{C}^2$ by scalar multiplication as above. The group G acts holomorphically on the quotient $X := G \times_T \mathbb{C}^2$ by this T-action. By construction we have $X/\!/G \cong \mathbb{C}^2/\!/T = \{\text{point}\}$. Furthermore, the G-invariant rational function on X defined by $f[g, (z, w)] = z/w$ is not a pull-back via the quotient map $X \to \{\text{point}\}$.

The following result characterises those Stein G-spaces for which any meromorphic invariant is a quotient of two holomorphic invariants, i.e., those spaces for which π^* is an isomorphism.

THEOREM 3.5. — *Let X be a G-irreducible Stein G-space with analytic Hilbert quotient $\pi\colon X \to X/\!/G$. Then, the following are equivalent:*

a) *There exists a non-empty open subset $U \subset X/\!/G$ such that for all $q \in U$ the fibre $\pi^{-1}(q)$ contains a dense G-orbit.*
b) *There exists a non-empty Zariski-open subset $U \subset X/\!/G$ such that for all $q \in U$ the fibre $\pi^{-1}(q)$ contains a dense G-orbit.*
c) *The pull-back map π^* establishes an isomorphism between the fields $\mathscr{M}_{X/\!/G}(X/\!/G)$ and $\mathscr{M}_X(X)^G$.*

Before we prove Theorem 3.5 we list a few typical situations where it can be applied.

COROLLARY 3.6. — *Let X be a G-irreducible Stein G-space with analytic Hilbert quotient $\pi\colon X \to X/\!/G$. Assume that there exists a point $x \in X$ such that $\pi^{-1}(\pi(x)) = G \bullet x$. Then, the pull-back π^* establishes an isomorphism between $\mathscr{M}_{X/\!/G}(X/\!/G)$ and $\mathscr{M}_X(X)^G$.*

Proof. — Under the hypotheses of the corollary, there exists a non-empty Zariski-open subset $U \subset X/\!/G$ such that for all $q \in U$ the fibre $\pi^{-1}(q)$ consists of a single G-orbit. Hence, Theorem 3.5 applies.   □

COROLLARY 3.7. — *Let G be a semisimple algebraic group and let X be a G-irreducible affine algebraic G-variety with factorial coordinate ring. Then, $\mathscr{M}_{X/\!/G}(X/\!/G)$ and $\mathscr{M}_X(X)^G$ are isomorphic via π^*.*

Remark 3.8. — The assumptions of Corollary 3.7 are in particular fulfilled for $X = V$ a representation space of a semisimple group.

Proof of Corollary 3.7. — Under the hypotheses of the corollary the generic fibre of the morphism $\pi\colon X \to \mathrm{Spec}\big(\mathbb{C}[X]^G\big)$ to the invariant-theoretic quotient contains a dense G-orbit, see for example [20, §3.2]. Since the corresponding map $\pi^h\colon X^h \to \big(\mathrm{Spec}(\mathbb{C}[X]^G)\big)^h$ of complex spaces is the analytic Hilbert quotient of the Stein space X^h, Theorem 3.5 applies.   □

In the remainder of the present section we prove Theorem 3.5.

Proof of Theorem 3.5. — The implication b) $\Rightarrow$ a) is clear. As a second step we prove a) $\Rightarrow$ b). Let $S_{\max} \subset X/\!/G$ be the maximal slice-type stratum, cf. Section 2.3, and let $X_{\max} = \pi^{-1}(S_{\max})$. Since $S_{\max}$ is dense in $X/\!/G$, there exists a point $q \in U \cap S_{\max}$. Recall from Proposition 2.4 that the map $\pi|_{X_{\max}}$ realises $X_{\max}$ as a holomorphic fibre bundle over $S_{\max}$ with typical fibre $\pi^{-1}(q)$. It therefore follows from the assumption in a) that for every $q' \in S_{\max}$ the fibre $\pi^{-1}(q')$ contains a dense G-orbit.

Next we prove c) $\Rightarrow$ b). Suppose on the contrary that the fibre $\pi^{-1}(q)$ does not contain a dense G-orbit for any $q \in S_{\max}$. Let Ω be a Zariski-open subset with geometric quotient whose existence is guaranteed by the Main Theorem. We may assume that Ω is contained in $X_{\max}$. Since Ω is Zariski-open and dense in $X_{\max}$, for generic $q \in S_{\max}$ the intersection $\Omega \cap \pi^{-1}(q)$ is Zariski-open and dense in $\pi^{-1}(q)$. By assumption, this intersection therefore contains two distinct G-orbits $G \bullet x_1 \neq G \bullet x_2$. By part (5) of the Main Theorem, there exist an $f \in \mathscr{M}_X(X)^G$ whose values at x_1 and x_2 are well-defined and distinct. Consequently, f is not contained in $\mathrm{Im}(\pi^*)$, a contradiction.

Finally, we prove implication b) $\Rightarrow$ c). We first study the local geometry of the quotient map $\pi\colon X \to X/\!/G$ under the hypotheses of b). As in the previous paragraph, let Ω be a Zariski-open subset with geometric quotient $p\colon \Omega \to Q$. Without loss of generality we may assume that $\Omega \subset X_{\max}$. Let $x_0 \in \Omega$ be chosen such that $p(x_0)$ and $\pi(x_0)$ are smooth points of Q and $X/\!/G$, respectively, and such that $G \bullet x_0$ is dense in $\pi^{-1}\big(\pi(x_0)\big)$. Since $\pi|_{X_{\max}}\colon X_{\max} \to S_{\max}$ is a holomorphic fibre bundle there exists a

local holomorphic section $\sigma\colon U \to X$ of $\pi|_{X_{\max}}$ through x_0, defined on a neighbourhood U of $\pi(x_0)$ in $X/\!/G$. Since $x_0 \in \Omega$, we may assume that $\sigma(U) \subset \Omega$. The situation is sketched in Figure 3.1.
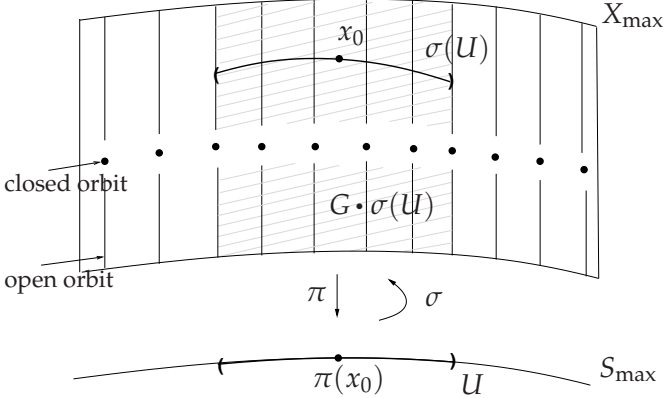

Figure 3.1. *The local geometry of the quotient map*

Note that $\widetilde{U} := G \bullet \sigma(U) \subset \Omega$ is open and that $\pi|_{\widetilde{U}}\colon \widetilde{U} \to U$ parametrises the G-orbits in $\widetilde{U}$ set-theoretically. Since Q carries the quotient topology with respect to p, the set $p(\widetilde{U})$ is an open neighbourhood of $p(x)$ in Q. By the choice of x_0, shrinking U if necessary, we may assume that both U and $\widetilde{U}$ are smooth. Since π is a G-invariant holomorphic map, its restriction to Ω induces a uniquely defined holomorphic map $\overline{\pi}\colon Q \to X/\!/G$ such that the diagram

$$
\begin{array}{ccc}
X & \longleftarrow & \Omega \\
\pi \downarrow & & \downarrow p \\
X/\!/G & \xleftarrow{\ \overline{\pi}\ } & Q
\end{array}
$$

commutes. By construction, the induced map $\overline{\pi}|_{p(\widetilde{U})}\colon p(\widetilde{U}) \to U$ is a holomorphic bijection between complex manifolds, hence biholomorphic.

Let now $f \in \mathscr{M}_X(X)^G$ be given, and let $\Gamma_f \subset X \times \mathbb{P}_1$ be its graph. It is our aim to show that f descends to a meromorphic function on $X/\!/G$. The idea of the proof is to show that the image $\pi(P_f)$ of the pole variety P_f of f is nowhere dense in $X/\!/G$ in order to apply Proposition 5.2 of [8]. To this end, we consider the restricted functions $f|_{\widetilde{U}} \in \mathscr{M}_X(\widetilde{U})^G$ as well as $f|_{\pi^{-1}(U)} \in \mathscr{M}_X(\pi^{-1}(U))^G$.

SUBCLAIM. — *There exists $\bar{f} \in \mathscr{M}_{X/\!/G}(U)$ such that $f|_{\pi^{-1}(U)} = \pi^*\bar{f}$.*

Proof of the subclaim. — Denote the graph of $f|_{\pi^{-1}(U)}$ by Γ', and set $\widetilde{\Gamma} := \Gamma' \cap (\widetilde{U} \times \mathbb{P}_1)$. Note that $\widetilde{\Gamma}$ coincides with the graph of $f|_{\widetilde{U}}$. We define $\Pi := \pi|_{\pi^{-1}(U)} \times \mathrm{id}_{\mathbb{P}_1} \colon \pi^{-1}(U) \times \mathbb{P}_1 \to U \times \mathbb{P}_1$ and $P := p|_{\widetilde{U}} \times \mathrm{id}_{\mathbb{P}_1} \colon \widetilde{U} \times \mathbb{P}_1 \to p(\widetilde{U}) \times \mathbb{P}_1$. The map Π is an analytic Hilbert quotient, and P is a geometric quotient for the respective G-actions on $\pi^{-1}(U) \times \mathbb{P}_1$ and $\widetilde{U} \times \mathbb{P}_1$. With this notation we summarise our setup in the following commutative diagram

$$
\begin{array}{ccc}
\Gamma' \subset \pi^{-1}(U) \times \mathbb{P}_1 & \longleftarrow\!\!\!\!\!\supset\; \widetilde{U} \times \mathbb{P}_1 \supset \widetilde{\Gamma} \\[2pt]
\Pi \downarrow & & \downarrow P \\[2pt]
U \times \mathbb{P}_1 & \underset{\cong}{\overset{\overline{\Pi}}{\longleftarrow}} & p(\widetilde{U}) \times \mathbb{P}_1.
\end{array}
$$

The function $f|_{\widetilde{U}}$ descends to a meromorphic function on $p(\widetilde{U}) \subset Q$ with graph $P(\widetilde{\Gamma})$, cf. the proof of part (1) of Proposition 7.2. Furthermore, we note that $\overline{\Pi}\big(P(\widetilde{\Gamma})\big)$ is analytic, hence closed in $U \times \mathbb{P}_1$. From this and the fact that $\widetilde{\Gamma}$ is Zariski-open, hence dense in the G-irreducible space Γ' it follows that

$$\Pi(\Gamma') = \overline{\Pi(\widetilde{\Gamma})} = \overline{\overline{\Pi}\big(P(\widetilde{\Gamma})\big)} = \overline{\Pi}\big(P(\widetilde{\Gamma})\big).$$

Here, $\overline{\;\cdot\;}$ denotes the topological closure in $U \times \mathbb{P}_1$. Consequently, $\Pi(\Gamma')$ is a meromorphic graph over U associated with a meromorphic function $\bar{f} \in \mathscr{M}_{X /\!\!/ G}(U)$ fulfilling $f|_{\pi^{-1}(U)} = \pi^* \bar{f}$. $\qquad\square$

Finally, we consider the pole variety P_f of f and its image $\pi(P_f)$ in $X /\!\!/ G$. Note that $\pi(P_f) \cap U$ coincides with the image under π of the pole variety of $f|_{\pi^{-1}(U)}$. The subclaim implies that $f|_{\pi^{-1}(U)} = \pi^* \bar{f}$ for some $\bar{f} \in \mathscr{M}_{X /\!\!/ G}(U)$. Consequently, $\pi(P_f) \cap U$ is nowhere dense in U. Since $\pi(P_f)$ is analytic in the irreducible complex space $X /\!\!/ G$, we deduce that $\pi(P_f)$ is nowhere dense in $X /\!\!/ G$. By Proposition 5.2 in [8], this implies that the image of Γ_f under the map $\pi \times \mathrm{id}_{\mathbb{P}_1} \colon X \times \mathbb{P}_1 \to X /\!\!/ G \times \mathbb{P}_1$ is a meromorphic graph over $X /\!\!/ G$ and that f descends to $X /\!\!/ G$. This completes the proof of b) $\Rightarrow$ c). $\qquad\square$

3.3. Holomorphically convex K-spaces

In this section we consider applications of the main result to spaces without actions of complex groups, e.g. bounded domains.

PROPOSITION 3.9. — *Let X be a K-irreducible complex K-space of dimension n and set*

$$m := \max_{x \in X}\{\dim_{\mathbb{C}}\big(T_x(K \bullet x) + J \cdot T_x(K \bullet x)\big)\}.$$

Suppose that there exist a Stein K-space Y and an equivariant surjection $\varphi\colon X \to Y$ which is injective outside a nowhere dense K-invariant analytic set $A \subset X$. Then, there exist $(n-m)$ analytically independent K-invariant meromorphic functions on X.

Remark 3.10. — Proposition 3.9 in particular applies to strongly pseudoconvex (also called 1-convex) complex spaces or, more generally, to holomorphically convex spaces whose Remmert reduction is a proper modification.

Proof of Proposition 3.9. — Let $G = K^{\mathbb{C}}$ be the complexification of K. According to the main result of [11] there exist a Stein G-space $Y^{\mathbb{C}}$ and a K-equivariant open embedding $\imath\colon Y \to Y^{\mathbb{C}}$ such that $Y^{\mathbb{C}} = G \bullet \imath(Y)$. Then, applying our Main Theorem we obtain a G-invariant Zariski-open subset $\Omega \subset Y^{\mathbb{C}}$ such that the geometric quotient $p\colon \Omega \to Q = \Omega/G$ exists.

Since φ is surjective and injective outside A, we have $\dim Y^{\mathbb{C}} = n$. Moreover, the maximal dimension of the G-orbits in $Y^{\mathbb{C}}$ is m; hence, $\dim \Omega/G = n - m$. By part (5) of the Main Theorem the G-invariant meromorphic functions separate the G-orbits in Ω. This implies there exist at least $n-m$ analytically independent G-invariant meromorphic functions on $Y^{\mathbb{C}}$. Restricting these to Y and pulling them back to X via φ yields the desired K-invariant meromorphic functions on X. $\square$

3.4. Actions with reductive generic stabiliser

For G-connected Stein G-manifolds Richardson [24] proves the existence of a *principal orbit type* in the following sense: in every such manifold there exists an open and dense subset U such that the stabiliser groups of points in U are conjugate in G. Here, we sharpen his result in the case of reductive stabiliser groups and draw a few consequences.

PROPOSITION 3.11. — *Let G be a complex-reductive Lie group and let X be a G-connected Stein G-manifold. Assume that the principal orbit type is reductive. Then,*

(1) *in the statement of the Main Theorem the set Ω can be chosen in such a way that for all $x, y \in \Omega$ there exists a $g \in G$ with $G_y = gG_xg^{-1}$. In particular, there exists a G-invariant Zariski-open dense subset of X consisting of orbits of principal orbit type;*

(2) *additionally, Ω can be chosen such that $p\colon \Omega \to Q$ is an analytic Hilbert quotient. In particular, Ω is Stein and $p\colon \Omega \to Q$ is a holomorphic fibre bundle with typical fibre G/G_x.*

Remark 3.12. — (1) The assumption on the stabiliser groups is equivalent to the requirement that the normaliser $N_G(G_x)$ be reductive, see [21, proof of Thm. 1.2(ii)].

(2) The assumption on the stabiliser groups is automatically fulfilled by any commutative reductive group acting on a connected Stein manifold.

As a special case of Proposition 3.11 we explicitly note the result for the case of generically free actions.

COROLLARY 3.13. — *Let G be a complex-reductive Lie group and let X be a G-connected Stein G-manifold. Assume that the action is generically free. Then, in the statement of the Main Theorem the set Ω can be chosen such that $p\colon \Omega \to Q$ is a G-principal fibre bundle.*

Remark 3.14. — The additional assumption of Corollary 3.13 is automatically fulfilled by any commutative reductive group acting effectively on a connected Stein manifold, as can be seen using the holomorphic Slice Theorem.

Proof of Proposition 3.11. — (1) Let $\phi\colon X \to V$ be a map from X to a G-representation space V of the form guaranteed by the Weak Equivariant Embedding Theorem, Proposition 5.2. Let Y be the algebraic Zariski-closure of $\phi(X)$ in V, cf. Section 6.3. Then, for any $y \in Y$, we let $G_y = L_y \ltimes U_y$ be the Levi decomposition of the stabiliser G_y. By a further result of Richardson [23, Thm. 9.3.1], there exists a G-invariant smooth Zariski-open subset W of Y such that $L_{y'}$ is conjugate to L_y in G for all $y, y' \in W$ and such that $(U_y)_{y \in W}$ is an algebraic family of algebraic subgroups of G in the sense of [23, Def. 6.2.1]. Owing to the definition of Y, the image $\phi(X)$ intersects W non-trivially. By the assumption on the principal orbit type there exists a point $y_0 \in W \cap \phi(X_{\max})$ such that $G_{y_0} = L_{y_0}$ and $U_{y_0} = \{e\}$. Since the $(U_y)_{y \in W}$ form an algebraic family, we may assume that the number of connected components of U_y is constant for all $y \in W$. Consequently, since $y_0 \in W$, the group U_y is zero-dimensional and connected for all $y \in W$, therefore trivial. It follows that G_y is conjugate to $G_{y'}$ in G for all $y, y' \in W$. Consequently, the same is true for any pair of points in the Zariski-open G-invariant subset $\phi^{-1}(W) \cap X_{\max}$ of X. Intersecting the set guaranteed by the Main Theorem with $\phi^{-1}(W) \cap X_{\max}$ we arrive at the desired result.

(2) We have seen that in the affine algebraic variety Y there exists a Zariski-open and dense subset W such that G_x is reductive for all $x \in W$. Hence, the existence of a Zariski-open subset Ω such that $p\colon \Omega \to Q$ is an analytic Hilbert quotient is a consequence of Lemma 3.15 below. The

fact that the quotient map is a holomorphic fibre bundle then follows by combining part (1) with the holomorphic Slice Theorem, cf. Proposition 2.4 and [17, Cor. 3.2.5]. □

LEMMA 3.15. — *Let X be an algebraic G-variety. Assume that for $x \in X$ in general position the stabiliser G_x is reductive. Then, there exists a non-empty Zariski-open affine G-invariant subset U in X such that the action of G on U admits a good geometric quotient. The associated map $U^h \to (U/G)^h$ of complex spaces is an analytic Hilbert quotient.*

Proof. — By a result of Reichstein and Vonessen [21] there exists a birational G-equivariant map $\phi\colon X \dashrightarrow Y$ to an affine G-variety Y with the following property: if $\pi_Y\colon Y \to Y /\!\!/ G$ denotes the categorical good quotient, then the set $Y^{st} := \{y \in Y \mid \pi_Y^{-1}(\pi_Y(y)) = G \bullet y\}$ is non-empty. Without loss of generality, we may assume that Y^{st} is affine and that $\phi^{-1}|_{Y^{st}}\colon Y^{st} \to X$ is an isomorphism onto its image. This shows the first claim. The second claim is a consequence of [18]. □

3.5. Unipotent groups

In this section we discuss consequences of our main result for actions of unipotent groups.

PROPOSITION 3.16. — *Let $H < G$ be a unipotent algebraic subgroup of a complex-reductive Lie group and let X be a Stein G-space. Then, every H-orbit in X is closed. Furthermore, the topological quotient X/H is generically Hausdorff and there exists a H-invariant Zariski-open dense subset U of X such that the restriction of the topological quotient $p\colon X \to X/H$ to U is a geometric quotient in the category of complex spaces.*

Proof. — Assuming that every H-orbit is closed, the remaining statements follow directly from the Main Theorem. So, let $H \bullet x \subset X$ be any H-orbit. Then, clearly $\overline{H \bullet x} \subset \overline{G \bullet x}$. However, the fibre $\pi^{-1}(\pi(x)) \supset \overline{G \bullet x}$ of the analytic Hilbert quotient $\pi\colon X \to X /\!\!/ G$ carries a natural affine algebraic structure with respect to which the G-action is algebraic, see [26, Cor. 5.6]. Since $H < G$ is algebraic by assumption, by the corresponding result in the affine algebraic case (which is proven for example in [3, Appendix]) the orbit $H \bullet x$ is closed in $\pi^{-1}(\pi(x))$ and hence in X. □

Example 3.17. — There exists a domain of holomorphy D in $\mathbb{C}^2$ endowed with a free holomorphic action of $H = \mathbb{C}$ such that the topological

closure of every H-orbit is a real hypersurface in D, cf. [14, Sect. 7-8]. In particular, there is no open $\mathbb{C}$-invariant subset Ω in D such that $\Omega/\mathbb{C}$ is Hausdorff. Hence, we cannot expect holomorphic actions of algebraic groups to have any of the properties stated in the Main Theorem if they do not extend to holomorphic actions of some complex-reductive group.

4. Examples

In the analytic setup the question of existence of a Rosenlicht quotient consists of the following two parts:

(1) Does there exist a Zariski-open subset on which the action of G admits a geometric quotient?
(2) Do the invariant meromorphic functions separate the G-orbits in general position?

In the following we are going to describe examples showing that the assumptions made in the Main Theorem are indeed necessary to obtain a positive answer to both questions.

In the Main Theorem it is assumed that the group H under discussion is an algebraic subgroup of a reductive group acting on X. The following examples show that this algebraicity assumption is indeed necessary in order to obtain a geometric quotient. All these examples deal with actions of discrete groups, which we denote by Γ instead of H.

Example 4.1. — Let $G = \mathbb{C}^* \times \mathbb{C}^*$ and let $\Gamma \cong \mathbb{Z}^2$ be the discrete subgroup generated by the elements e and $e^{-\pi}$. Furthermore, consider the subgroup $M = \mathrm{diag}(\mathbb{C}^*)$. Then, the quotient $X = G/M$ is isomorphic to $\mathbb{C}^*$ via the map $[(z, w)] \mapsto z/w$, and the induced $\mathbb{Z}^2$-action is given by $(m_1, m_2) \bullet z = e^{m_1 + m_2\pi}$. For this action there does not even exist an open subset of $\mathbb{C}^*$ that admits a Hausdorff topological quotient.

Note that the action of the ambient reductive group G is not effective in the above example. The next example shows that even if the G-action is effective geometric quotients for non-algebraic subgroups might not exist.

Example 4.2. — We consider the action of the discrete subgroup $\Gamma :=$ $\mathrm{SL}_2(\mathbb{Z})$ of $G := \mathrm{SL}_2(\mathbb{C})$ on the homogeneous Stein manifold $X = G/T$, where T is the maximal torus of diagonal matrices in G. We claim that there does not exist a Γ-invariant Zariski-open subset U of X such that the quotient U/Γ is Hausdorff. First, we consider an explicit realisation of this action. Taking a regular element $\xi \in \mathfrak{t} = \mathrm{Lie}(T)$ we have $\mathrm{Ad}(G)\xi \cong G/T$.

Note that the ring of invariants $\mathbb{C}[\mathfrak{g}]^G$ for the adjoint action of G on its Lie algebra $\mathfrak{g}$ is equal to $\mathbb{C}[\det]$. An element $\xi \in \mathfrak{g}$ is regular if and only if $\det(\xi) \neq 0$. Therefore, G/T can be identified with

$$\left\{ (x, y, z) \in \mathbb{C}^3 \, ; \, \det \begin{pmatrix} x & y \\ z & -x \end{pmatrix} = -1 \right\}.$$

The Γ-action on $X = G/T$ in this realization is induced by conjugation. As an auxiliary tool, we are going to consider the induced action of $\mathbb{C}$ and of the discrete subgroup $\mathbb{Z} < \mathbb{C}$ on X given by the embedding of $\mathbb{C}$ into $\mathrm{SL}_2(\mathbb{C})$ as upper triangular matrices. Explicitly, for $t \in \mathbb{C}$ we obtain

$$(4.1) \qquad t \bullet (x, y, z) = (x + tz, y - 2tx - t^2 z, z).$$

Let now U be any Γ-invariant Zariski-open subset of X. Since there are no Γ-invariant analytic hypersurfaces in G (see [1], or [15, §2.2, Ex.2] for an elementary proof) there are no such hypersurfaces in X either. Consequently, the complement $A := X \setminus U$ has pure codimension two in X, i.e., A is a discrete set of points. The map $p \colon X \to \mathbb{C}$, $p(x, y, z) = z$ is $\mathbb{C}$-invariant, the fibre $p^{-1}(a)$ for $a \neq 0$ consists of a single orbit. However, the fiber $p^{-1}(0)$ is the union of the two $\mathbb{C}$-orbits $\mathbb{C} \bullet (1, 0, 0) = \{(1, y, 0) \, ; \, y \in \mathbb{C}\}$ and $\mathbb{C} \bullet (-1, 0, 0) = \{(-1, y, 0) \, ; \, y \in \mathbb{C}\}$, which cannot be separated by $\mathbb{C}$-invariant open neighbourhoods. Hence, $X/\mathbb{C}$ is not Hausdorff.

We show by direct calculation that U/Γ is not Hausdorff, either. Since A consists of isolated points, there exists $y_0 \in \mathbb{C} \setminus \mathbb{Q}$ such that both $(1, y_0, 0)$ and $(-1, y_0, 0)$ are contained in U. Using that y_0 is irrational, one checks by direct computation that these two points lie in different Γ-orbits. We are going to show that the orbits $\Gamma \bullet (1, y_0, 0)$ and $\Gamma \bullet (-1, y_0, 0)$ cannot be separated by invariant open sets. To this end, let V be any open Γ-invariant neighbourhood of $(1, y_0, 0)$ in U. Since A is discrete, for all integers $m \gg 0$ the points $p_m := (1, y_0, -2/m)$ are contained in V. Using (4.1) we compute

$$m \bullet p_m = \left(1 + m \cdot \frac{-2}{m}, y_0 - 2m - m^2 \cdot \frac{-2}{m}, \frac{-2}{m} \right) = \left(-1, y_0, \frac{-2}{m} \right) \in V.$$

If W is any open neighbourhood of $(-1, y_0, 0)$ in U, then by the above computation $m \bullet p_m \in W$ for $m \gg 0$. Since the $\mathbb{Z}$-orbits of the p_m are contained in the corresponding Γ-orbits, every Γ-invariant neighbourhood V of $(1, y_0, 0)$ thus intersects every open neighbourhood W of $(-1, y_0, 0)$, so U/Γ cannot be Hausdorff.

As a concluding remark, note that by removing the fibre $p^{-1}(0)$ from X we obtain a Zariski-open $\mathbb{C}$-invariant subset of X on which the $\mathbb{C}$-action admits a Hausdorff quotient, in accordance with Rosenlicht's Theorem and with the main result of this paper.

In the case of non-algebraic subgroups acting on Stein manifolds the invariant meromorphic functions do not necessarily separate generic orbits, even if a meromorphic quotient exists. This is exemplified in the following.

Example 4.3. — We consider the subgroup $\Gamma := \mathrm{SL}_2(\mathbb{Z})$ of $\mathrm{SL}_2(\mathbb{C})$ acting on the Stein manifold $X = \mathrm{SL}_2(\mathbb{C})$ by $\gamma \bullet g = g\gamma^{-1}$. Then, the action is proper and free, and hence the geometric quotient X/Γ exists. Let U be any Γ-invariant analytically Zariski-open subset of X. Then U/Γ exists and is biholomorphic to the image of U in the quotient X/Γ. Using that there are no Γ-invariant hypersurfaces in $\mathrm{SL}_2(\mathbb{C})$ we see that the complement of U in X has no codimension-one components. It follows that every Γ-invariant meromorphic function on U extends to a Γ-invariant meromorphic function on the whole of X by Levi's Theorem. However, because of the non-existence of invariant hypersurfaces, the pole variety of every Γ-invariant meromorphic function on X is empty. Hence, every such function is holomorphic, and therefore constant.

5. A weak equivariant embedding theorem

In the following let X be a G-irreducible Stein G-space for a complex-reductive Lie group G. The main technical ingredient in the proof of our main result is an equivariant version of the following result of Remmert and Narasimhan:

THEOREM 5.1 ([19]). — *Let X be a finite-dimensional Stein space. Then there exist a finite dimensional complex vector space V and a proper injective holomorphic map $\phi\colon X \to V$ that is an immersion on $X \setminus X_{sing}$.*

In this paper a holomorphic map $\phi\colon X \to Y$ from a Stein space X into an affine variety Y is called a *Narasimhan map* if ϕ is proper injective and if $\phi|_{X \setminus X_{\mathrm{sing}}}$ is an immersion.

In the following we will investigate to what extend there exists an equivariant version of this fundamental result.

Suppose that the Stein G-space X admits an equivariant Narasimhan map $\phi\colon X \to V$ into a finite-dimensional G-representation space V. In this situation the stratification of $X/\!/G$ into orbit-types is necessarily finite. In contrast, Heinzner [10, Sect. 3] has given an example of a Stein $\mathbb{C}^*$-manifold that contains a sequence of points $\{x_n\}$ lying in closed $\mathbb{C}^*$-orbits and having isotropy groups $\mathbb{C}^*_{x_n} = \mathbb{Z}_{p_n}$, where $\{p_n\}$ is a sequence of prime numbers such that $\lim_{n\to\infty} p_n = \infty$. Hence, the first guess for an equivariant version of Theorem 5.1 does not lead to the desired result.

The problems encountered in the above example are caused be the appearance of "too many" different isotropy groups and slice-representations for our given action. However, recall from Proposition 2.4 that there exists a maximal, Zariski-open stratum $S_{\max}$ in $X/\!/G$ over which the type of the slice representation is constant. Using the methods of [10] we prove the following equivariant version of Theorem 5.1.

PROPOSITION 5.2 (Weak Equivariant Embedding Theorem). — *Let X be a G-irreducible Stein G-space with associated analytic Hilbert quotient $\pi\colon X \to X/\!/G$. Let $S_{\max}$ be the maximal stratum of the slice-type stratification, and $X_{\max} := \pi^{-1}(S_{\max})$. Then there exists a finite-dimensional G-module V with analytic Hilbert quotient $\pi_V\colon V \to V/\!/G$, and a holomorphic G-equivariant map $\phi\colon X \to V$ with the following properties:*

(1) *The induced holomorphic map $\overline{\phi}\colon X/\!/G \to V/\!/G$ is a Narasimhan map and*

(2) *the induced holomorphic map*

$$\phi|_{X_{\max}}\colon X_{\max} \to V_{\max} := V \setminus \pi_V^{-1}\big(\overline{\phi}(S_{\max}^c)\big)$$

is a closed embedding.

The main technical part in the proof of Proposition 5.2 is contained in the following lemma, the proof of which is adapted from [10, Sect.4, Lemma 1]. For the reader's convenience we describe the arguments here in some detail. In the following we write $A_{\max} := A \cap S_{\max}$ for any analytic subset $A \subset X/\!/G$.

LEMMA 5.3. — *Let X be a G-irreducible Stein G-space and let $\pi\colon X \to X/\!/G$ be its analytic Hilbert quotient. For any analytic subset A of $X/\!/G$ that intersects the maximal slice-type stratum $S_{\max}$ non-trivially the following holds:*

(1) *There exists an analytic subset A' of A fulfilling $\dim(A'_{\max}) < \dim(A_{\max})$, a complex G-module V_1, and an equivariant holomorphic map $\phi\colon X \to V_1$ that is an immersion along $\pi^{-1}(A_{\max} \setminus A'_{\max})$.*

(2) *There exists an analytic subset A'' of A fulfilling $\dim(A''_{\max}) < \dim(A_{\max})$, a complex G-module V_2, and an equivariant holomorphic map $\psi\colon X \to V_2$ whose restriction to every closed G-orbit in $\pi^{-1}(A_{\max} \setminus A''_{\max})$ is a proper embedding.*

Proof. — Let $A \subset X/\!/G$ be an analytic subset such that $A_{\max} \neq \emptyset$. Removing the irreducible components which are not of maximal dimension, we may assume without loss of generality that $A_{\max}$ is pure-dimensional. We denote the irreducible components of $A_{\max}$ by A_i, $i \in I$, and choose

for every $i \in I$ a point $p_i \in A_i \setminus \bigcup_{j \neq i} A_j$. For each i, let $x_i \in \pi^{-1}(p_i)$ be a point lying in the unique closed G-orbit in this fibre.

The slice type of every point x_i, $i \in I$, is equal to a fixed model $G \times_H W$. This model admits an equivariant holomorphic embedding into a G-module V_1. By the holomorphic Slice Theorem ([26]), for each $i \in I$ we can choose a small neighbourhood U_i of p_i in X such that $\pi^{-1}(U_i)$ has a G-equivariant holomorphic embedding into a saturated open subset of the G-module V_1. Application of [10, Sect. 1, Prop. 1] to the induced map $\dot{\bigcup}_{i \in I} \pi^{-1}(U_i) \to V_1$ yields a G-equivariant holomorphic map $\phi \colon X \to V_1$ that is an immersion along $\dot{\bigcup}_{i \in I} \pi^{-1}(p_i)$. The set

$$R := \{x \in X \mid \phi \text{ is not an immersion in } x\}$$

is a G-invariant analytic subset of X. It follows that $A' := \pi(R) \cap A$ is an analytic subset of A. Since the map ϕ is an immersion at every point in $\bigcup_{i \in I} \pi^{-1}(p_i)$, we conclude that $\dim(A'_{\max}) < \dim(A_{\max})$, as desired.

Let us now prove the second claim. Since by definition the slice type is constant on $S_{\max}$, all orbits $G \bullet x_i$ have the same orbit type (H), where H is a complex-reductive subgroup of G. Since there is a proper equivariant embedding of G/H into some G-module V_2, there exists a proper holomorphic map $\dot{\bigcup}_{i \in I} G \bullet x_i \to V_2$. By [10, Sect. 1, Bemerkung 2] this map extends to a G-equivariant holomorphic map $\psi \colon X \to V_2$ such that $\psi|_{G \bullet x_i}$ is a proper embedding for each $i \in I$.

Let $\pi_{V_2} \colon V_2 \to V_2 /\!\!/ G$ be the analytic Hilbert quotient. For $q \in V_2 /\!\!/ G$ we set $\mathrm{Type}(q) := \mathrm{Type}(G \bullet q)$ and define

$$C := \big\{ q \in V_2 /\!\!/ G \mid \mathrm{Type}(q) < H \big\} \subset V_2 /\!\!/ G.$$

Then $\Omega := V_2 \setminus \pi_{V_2}^{-1}(C)$ is an algebraically Zariski-open G-saturated subset of V_2, see [17, Ch. III]. Note that we have $\psi(x_i) \in \Omega$ for all $i \in I$. Let $\overline{\psi} \colon X /\!\!/ G \to V_2 /\!\!/ G$ be the induced map and set $A'' := \overline{\psi}^{-1}(C)$, which is an analytic subset of $X /\!\!/ G$. We want to show that for every $x \in X$ such that $G \bullet x$ is closed with $\pi(x) \in A_{\max} \setminus A''_{\max}$ the restriction of ψ to $G \bullet x$ is a proper embedding into V_2.

For this suppose that $\psi(G \bullet x) = G \bullet \psi(x)$ is not closed. Then there is a unique closed G-orbit $G \bullet v \subset \Omega$ in the closure of $G \bullet \psi(x)$ and for this orbit we have $\mathrm{Type}(G \bullet v) < \mathrm{Type}\big(G \bullet \psi(x)\big) \leqslant \mathrm{Type}(G \bullet x) = (H)$, a contradiction. Consequently, $G \bullet \psi(x)$ must be closed in V and, since it lies in Ω, we have $\mathrm{Type}\big(G \bullet \psi(x)\big) = (H)$. Therefore, $\psi \colon G \bullet x \to G \bullet \psi(x)$ is an isomorphism, hence $\psi|_{G \bullet x}$ is a proper embedding. Finally, since all the x_i are contained in $A_{\max} \setminus A''_{\max}$, clearly $\dim(A''_{\max}) < \dim(A_{\max})$, as claimed. $\qquad\square$

Now we are in the position to give the proof of Proposition 5.2.

Proof of Proposition 5.2. — Let X be a G-irreducible Stein G-space with associated analytic Hilbert quotient $\pi\colon X \to X/\!/G$. Let $\varphi_0\colon X/\!/G \to V_0$ be a Narasimhan map and $\phi_0\colon X \to V_0$ the lifted map $\phi_0 := \varphi_0 \circ \pi$. By a repeated application of the first part of Lemma 5.3 we obtain an equivariant holomorphic map $\phi_1\colon X \to V_1$ to a complex G-module V_1 that is an immersion at every point in $\pi^{-1}(S_{\max})$. Additionally, by a repeated application of the second part of the same lemma we obtain an equivariant holomorphic map $\psi\colon X \to V_2$ into a complex G-module whose restriction to every closed orbit in $\pi^{-1}(S_{\max})$ is a closed embedding. Let $V := V_0 \oplus V_1 \oplus V_2$ and let $\phi\colon X \to V$ be the product map.

Let $\pi_V\colon V \to V/\!/G$ denote the invariant-theoretic quotient by the G-action and let $\overline{\phi}\colon X/\!/G \to V/\!/G$ be the induced map. Since $\overline{\phi}$ is proper (we started with φ_0 which was assumed to be a Narasimhan map), the image of $S_{\max}^c := (X/\!/G) \setminus S_{\max}$ under $\overline{\phi}$ is an analytic subset of $V/\!/G$. The restriction $\phi|_{X_{\max}}\colon X_{\max} \to V \setminus \pi_V^{-1}(\overline{\phi}(S_{\max}^c)) =: V_{\max}$ is an immersion and a closed embedding when restricted to any closed orbit in $X_{\max}$. By [10, Sect. 2, Prop. 2] the restriction of ϕ to every fibre of π is a closed embedding. Hence, ϕ is an injective immersion, since $\overline{\phi}$ separates the points of $S_{\max}$. It therefore remains to check that $\phi|_{X_{\max}}$ is proper, which can be done the same way as in the last paragraph in the proof of [9, Thm. 9.6].   □

6. Constructing geometric quotients

We continue to consider the action of a complex-reductive group G on a Stein space X as well as the induced action of an algebraic subgroup H of G. In this section we prove the existence of an H-invariant Zariski-open dense subset Ω of X that admits a geometric quotient $p\colon \Omega \to Q$ with the properties listed in parts (1) – (3) of the Main Theorem.

The idea of proof is to use the Weak Equivariant Embedding Theorem established above in order to reduce to an algebraic situation. Then classical results on algebraic transformation groups and especially Rosenlicht's theorem will allow us to show the existence of geometric quotients for algebraic subgroups $H \subset G$.

In order to avoid the corresponding technical difficulties we show in Section 6.1 that it is sufficient to treat the H-irreducible case. In Section 6.2 we then discuss the universality properties of (algebraic) geometric quotients before we prove the existence of geometric H-quotients in the final subsection.

6.1. Reduction to H-irreducible Stein G-spaces

Let X be a Stein G-space, and H an algebraic subgroup of G. Suppose that the Main Theorem is proven under the additional asumption that X is H-irreducible.

Let $X = \bigcup_{i=1}^{m} X_i$ be the decomposition of X into its H-irreducible components. Then, we may apply the Main Theorem to each of the components X_i and obtain H-invariant Zariski-open dense subsets $\Omega_i \subset X_i$ with geometric quotients $\Omega_i \to \Omega_i/H$. Note that we can choose Ω_i to be contained in $X_i \backslash \bigcup_{k \neq i} X_k$. It follows that the disjoint union of the sets Ω_i is H-invariant, Zariski-open and dense in X. Furthermore, it admits a geometric quotient $\dot{\bigcup}_{i=1}^{m} \Omega_i \to \dot{\bigcup}_{i=1}^{m}(\Omega_i/H)$ by the H-action with the properties listed in the Main Theorem.

6.2. Universality of geometric quotients

First, we discuss the universality properties of geometric quotients with respect to invariant analytic subsets.

LEMMA 6.1. — *Let L be a complex Lie group and let X be an L-irreducible holomorphic L-space admitting a geometric quotient $p\colon X \to X/L$. If $A \subset X$ is an L-invariant analytic subset of X, then $p(A)$ is an analytic subset of X/L.*

Proof. — Since the geometric quotient $p\colon X \to X/L$ exists, we conclude from [13, §3, Satz 7] that all L-orbits are analytic in X and have the same dimension. Hence, the corollary in Section 3.7 of [4] applies to show that $p(A)$ is locally analytic in X/L. Since the image of an L-invariant closed set under p is again closed, the set $p(A)$ is analytic in X/L. □

More can be said if the quotient map $p\colon X \to X/L$ is assumed to be a submersion:

LEMMA 6.2. — *Let L be a complex Lie group and let X be an L-irreducible holomorphic L-space admitting a geometric quotient $p\colon X \to X/L$. Suppose that p is a submersion. Then, p is universal with respect to L-invariant analytic sets of X.*

Proof. — Given an L-invariant analytic subset A of X, we must show that the map $p|_A\colon A \to p(A)$ fulfills properties (1) – (3) of Definition 2.1. We already know from Lemma 6.1 that $p(A)$ is an analytic subset of X/L. The fibres of $p|_A$ are G-orbits, since the same is true for p. Hence, it remains

to show that $p(A)$ carries the quotient topology with respect to $p|_A$ and that the structure sheaf of $p(A)$ as a reduced complex subspace of X/L is isomorphic to the sheaf of L-invariant holomorphic functions on A.

Using that X/L carries the quotient topology with respect to p one checks directly that the same is true for A and $p|_A$: Let $U \subset A$ be an L-invariant open subset. By definition of the subspace topology on A, there exists an open subset $\widetilde{U}$ in X such that $\widetilde{U} \cap A = U$. Then, $\widehat{U} := G \bullet \widetilde{U}$ is G-invariant, open, and still fulfills $\widehat{U} \cap A = U$. Consequently, $p(\widehat{U})$ is open in X/L and we have $p|_A(U) = p(\widehat{U}) \cap p(A)$. Hence, $p|_A(U)$ is open in $p(A)$.

It remains to consider the structure sheaves. For this let $U \subset A$ be an L-invariant open subset and let $f \in \mathscr{O}(U)^L$. Then there is a continuous function $\bar{f} \colon p(U) \to \mathbb{C}$ with $f = p^* \bar{f}$ and we must show $\bar{f} \in \mathscr{O}_{p(A)}\big(p(U)\big)$. Since this assertion is local, we may apply [4, Thm. in §2.18] to p and obtain, after possibly shrinking U, a commutative diagram

$$\begin{array}{ccc} \widetilde{U} & \xrightarrow{\;\;\psi\;\;} & D \times \widetilde{V} \\ & {\scriptstyle p}\searrow \quad \swarrow {\scriptstyle \pi_{\widetilde{V}}} & \\ & \widetilde{V}, & \end{array}$$

where $\widetilde{U}$ is an open subset of X such that $U = \widetilde{U} \cap A$, where $\widetilde{V}$ is an open subset of X/L such that $p(\widetilde{U}) \subset \widetilde{V}$, and where D is a domain in $\mathbb{C}^N$ for $N = \dim X - \dim X/L$. In this picture we have $\psi(A) = D \times p(A)$ and $f \circ \psi^{-1}(z, y) = \bar{f}(y)$ for $y \in p(A)$. Hence, $\bar{f}$ is indeed holomorphic on $p(U)$, as was to be shown. $\qquad\square$

Remark 6.3. — Lemma 6.2 can be used to prove the following observation which might be of independent interest: Let H be an algebraic group and let X be an algebraic H-irreducible H-variety admitting an (algebraic) geometric quotient $p \colon X \to X/H$ such that p is a submersion. If $A \subset X$ is an H-invariant analytic subset, then $p|_A \colon A \to p(A)$ is a (holomorphic) geometric H-quotient.

6.3. Existence of geometric quotients for algebraic subgroups of G

Let G be complex-reductive and let X be an H-irreducible Stein G-space where H is an algebraic subgroup of G. Note that X is also G-irreducible. We want to combine the results of the previous two sections in order to

prove that geometric quotients for the H-action exist on Zariski-open dense subsets of X.

We introduce some notation in order to prepare the proof of statements (1)–(3) of the Main Theorem. Let $\phi\colon X \to V$ be the weak equivariant embedding constructed in Proposition 5.2 and let $S_{\max}$ be the maximal slice-type stratum in $X/\!\!/G$. Since the induced map $\overline{\phi}\colon X/\!\!/G \to V/\!\!/G$ is in particular proper, $\overline{\phi}(S_{\max}^c)$ is an analytic subset of $V/\!\!/G$ and the set $V_{\max} = V \setminus \pi_V^{-1}\big(\overline{\phi}(S_{\max}^c)\big)$ is analytically Zariski-open in V. Let Y be the algebraic Zariski-closure of $\phi(X)$ in V. The preimage $X_{\max} = \pi^{-1}(S_{\max})$ is analytically Zariski-dense in X, hence the algebraic Zariski-closure of $\phi(X_{\max})$ coincides with Y. Note furthermore that Y is H-irreducible.

In summary, we have found a G-equivariant map $\phi\colon X \to Y$ into an H-irreducible affine variety Y which is a proper embedding from $X_{\max}$ into $Y_{\max} := Y \cap V_{\max}$. Since the G-action on Y is algebraic we may now apply classical results on algebraic transformation groups and transport them to X via ϕ.

By Rosenlicht's Theorem, see Theorem 2.3, there exists an algebraically Zariski-open H-irreducible subset Ω_Y of Y that admits an algebraic geometric quotient $p_Y\colon \Omega_Y \to \Omega_Y/H$ by the H-action. In the next step we will shrink Ω_Y in order to improve the properties of p_Y and of Ω_Y/H. Note that the set where p_Y is a submersion is an H-invariant algebraically Zariski-open subset of Ω_Y; shrinking Ω_Y we may assume that p_Y is a submersion. Repeating this procedure if necessary we may also assume that Ω_Y and Ω_Y/H are smooth and that Ω_Y/H is affine. Finally, using [29, Cor. 5.1] we may furthermore suppose that $p_Y\colon \Omega_Y \to \Omega_Y/H$ is a topological fibre bundle with respect to the complex topologies of Ω_Y and Ω_Y/H. Note that we still have $\phi(X_{\max})\cap\Omega_Y \neq \emptyset$ since Ω_Y is algebraically Zariski-dense in Y.

After these preparations we are now in the position to prove the existence of a geometric H-quotient on a dense Zariski-open subset of X:

PROPOSITION 6.4. — *Let X be an H-irreducible Stein G-space where H is an algebraic subgroup of G. Then there exist a Zariski-open dense H-invariant subset Ω of X and a holomorphic map $p\colon \Omega \to Q$ to an irreducible Stein space Q that is a geometric quotient for the H-action on Ω and additionally possesses the properties listed under (2) and (3) in the Main Theorem.*

Proof. — We use the notation introduced above. We first define the desired set Ω. To this end, let $\pi\colon Y \to Y/\!\!/G$ denote the Hilbert quotient of the G-action on Y and note that, since $Y/\!\!/G$ is an affine variety and thus a Stein space, we may find a non-constant function $f \in \mathscr{O}_{Y/\!\!/G}(Y/\!\!/G)$ which

vanishes on $\overline{\phi}(S_{\max}^c)$. Consequently, $(Y/\!/G) \setminus \{f = 0\}$ is an analytically Zariski-open Stein subset of $Y/\!/G$. Let U be its inverse image under π in Y and define

$$\Omega := \varphi^{-1}(U \cap \Omega_Y) \subset X.$$

By construction, Ω is an H-irreducible analytically Zariski-open dense subset of X contained in $X_{\max}$. Moreover, we define

(6.1) $p := p_Y \circ (\phi|_\Omega) \colon \Omega \to Q := p_Y\big(\phi(\Omega)\big) \subset \Omega_Y/H.$

Since $\phi \colon X_{\max} \to Y_{\max}$ is a closed embedding, its image is an analytic subset of $Y_{\max}$ biholomorphic to $X_{\max}$. Since the analytically Zariski-open subset $\Omega \subset X$ is contained in $X_{\max}$, the image $\phi(\Omega)$ equals the analytic subset $\phi(X_{\max}) \cap U \cap \Omega_Y$ of $U \cap \Omega_Y$ and $\phi|_\Omega \colon \Omega \to \phi(\Omega)$ is biholomorphic.

By construction the map $p_Y \colon \Omega_Y \to \Omega_Y/H$ is a submersion, hence Lemma 6.2 applies to show that $p_Y\big(\phi(\Omega)\big) = Q$ is an analytic subset of $p_Y(U \cap \Omega_Y)$ and that $p \colon \Omega \to Q$ is a geometric quotient for the H-action on Ω. Moreover, $p \colon \Omega \to Q$ is a submersion and a topological fibre bundle; both properties are inherited from p_Y. Together with another application of Lemma 6.2 this shows the properties listed under (2) and (3) in the Main Theorem.

The proof is completed by showing that Q is a Stein space: Since Q is an analytic subset of $p_Y(U \cap \Omega_Y)$, for this it suffices to prove that $p_Y(U \cap \Omega_Y)$ is a Stein open subset of Ω_Y/H. Recall that $U = \pi^{-1}(\{f \neq 0\}) = \{\pi^* f \neq 0\}$ by definition. Since $(\pi^* f)|_{\Omega_Y}$ is H-invariant, there is a function $\bar{f} \in \mathscr{O}_{\Omega_Y/H}(\Omega_Y/H)$ such that $\pi^* f|_{\Omega_Y} = p_Y^* \bar{f}$. It follows that $p_Y(U \cap \Omega_Y) = \{\bar{f} \neq 0\} \subset \Omega_Y/H$, and consequently $p_Y(U \cap \Omega_Y)$ is a Stein open subset of the Stein space Ω_Y/H. $\square$

Thus, we have shown the existence of a geometric H-quotient with the properties listed in parts (1) – (3) of the Main Theorem under the assumption that X is H-irreducible. Combining this with the observation noted in Section 6.1, parts (1), (2), and (3) of the Main Theorem are proven.

7. Pushing down meromorphic functions

In this section we will prove that the geometric quotient $p \colon \Omega \to Q$ constructed in the previous section additionally has the properties stated in parts (4)–(6) of the Main Theorem, thus completing its proof. Before we do this in Section 7.2, we give a criterion in terms of meromorphic functions for a densely defined holomorphic map to extend to a weakly meromorphic map.

7.1. Meromorphic functions and weakly meromorphic maps

The following lemma is concerned with the relation of Definition 2.5 to meromorphic functions. It will be used in the proof of part (4) of the Main Theorem in the next subsection.

LEMMA 7.1. — *Let X be a complex space, $A \subset X$ a nowhere dense analytic subset, $Y \subset U \subset \mathbb{C}^N$ an analytic subset of an analytically Zariski-open subset U in $\mathbb{C}^N$, and $z_1, \ldots, z_N$ linear coordinates on $\mathbb{C}^N$. Let $\phi \colon X \setminus A \to \mathbb{C}^N$ be a holomorphic map with $\phi(X \setminus A) \subset Y$. Assume that $\phi^*(z_j)$ extends to a meromorphic function on X for all $j = 1, \ldots, N$. Then, $\phi \colon X \setminus A \to Y$ is weakly meromorphic.*

Proof. — We consider the compactification $\mathbb{O}^N = (\mathbb{P}_1)^N$ of $\mathbb{C}^N$, the so-called Osgood space. By assumption, $\varphi_j := \phi^*(z_j)$ is a meromorphic function on X for all $j = 1, \ldots N$. Hence, by [22, Satz 33] the map $\phi \colon X \dashrightarrow \mathbb{O}^N$ is a meromorphic map in the sense of Remmert and therefore in particular also weakly holomorphic. Since $\phi(X \setminus A) \subset Y$, the map φ is still weakly meromorphic after restricting the range to Y, see [27, Satz 3.13].   □

7.2. Completing the proof of the Main Theorem

For the reader's convenience we recall parts (4)–(6) of the Main Theorem in the following

PROPOSITION 7.2. — *Let $H < G$ be an algebraic subgroup of a complex-reductive Lie group G and let X be an H-irreducible Stein G-space. Let $p \colon \Omega \to \Omega/H$ be the geometric quotient constructed in Proposition 6.4. Then,*

(4) *the quotient map $p \colon \Omega \to \Omega/H$ extends as a weakly meromorphic map to X,*

(5) *for every $f \in \mathscr{M}_X(X)^H$ there exists a unique $\bar{f} \in \mathscr{M}_{\Omega/H}(\Omega/H)$ such that $f|_\Omega = p^* \bar{f}$, and*

(6) *the H-invariant meromorphic functions on X separate H-orbits in Ω.*

Remark 7.3. —   Since $p \colon \Omega \to \Omega/H$ is an open holomorphic map, the pull-back p^* from $\mathscr{M}_{\Omega/H}(\Omega/H)$ to $\mathscr{M}_\Omega(\Omega)^H$ is well-defined and, as the proof of Proposition 7.2 will show, an isomorphism.

Proof of Proposition 7.2. — In order to prove the first claim recall from Section 6.3 that by construction Ω/H is an analytic subset of an analytically Zariski-open subset of an affine variety Q and that $p_Y \colon \Omega_Y \to Q$ is the algebraic geometric quotient whose existence is guaranteed by Rosenlicht's theorem. In particular the quotient map p_Y extends as a rational map to Y. Let us choose an embedding of Q into a finite dimensional complex vector space V and let $(z_1, \ldots, z_N)$ be linear coordinates in V. These induce invariant rational functions $p_Y^* z_j$, $j \in \{1, \ldots, N\}$, on Y. Since $\phi^{-1}(A)$ is a nowhere dense analytic set in X for every nowhere dense algebraic set $A \subset Y$, the pull-back $\phi^* p_Y^* z_j = p^* z_j$ is a meromorphic function on X for every j, see [7, Chapter 6.3.3]. Thus by Lemma 7.1 the map $p = (p^* z_1, \ldots, p^* z_N)$ extends as a weakly meromorphic map to X.

For the second claim let $f \in \mathscr{M}_X(X)^H$ be given. By abuse of notation we denote by f also its restriction to Ω and thus have $f \in \mathscr{M}_\Omega(\Omega)^H$. Recall that f is holomorphic on $\mathrm{dom}\, f = \Omega \setminus P_f$. Applying Lemma 6.2 we see that $\overline{P_f} := p(P_f)$ is a nowhere dense analytic subset of Ω/H. Since $P := p \times \mathrm{id}_{\mathbb{P}_1} \colon \Omega \times \mathbb{P}_1 \to (\Omega/H) \times \mathbb{P}_1$ is a geometric quotient for the H-action on $\Omega \times \mathbb{P}_1$, Lemma 6.2 implies that $P(\Gamma_f) =: \overline{\Gamma_f}$ is an analytic subset of $(\Omega/H) \times \mathbb{P}_1$. We will prove that it is a meromorphic graph.

We summarise our setup in the following diagram.

$$
\begin{array}{ccc}
\Omega \times \mathbb{P}_1 \longleftarrow\!\!\!\supset \Gamma_f & \xrightarrow{\; p_\Omega \;} & \Omega \\
\Big\downarrow{\scriptstyle P} \qquad \Big\downarrow{\scriptstyle P|_{\Gamma_f}} & & \Big\downarrow{\scriptstyle p} \\
(\Omega/H) \times \mathbb{P}_1 \longleftarrow\!\!\!\supset \overline{\Gamma_f} & \xrightarrow{\; p_{\Omega/H} \;} & \Omega/H.
\end{array}
$$

First, we note that $p_{\Omega/H}^{-1}(\overline{P_f}) = P\big(p_\Omega^{-1}(P_f)\big)$ is a nowhere dense analytic subset of $\overline{\Gamma_f}$. Moreover, the restriction of f to $\mathrm{dom}(f)$ is holomorphic, and hence there exists a uniquely defined holomorphic function $\bar{f}$ on the open subset $p(\mathrm{dom}\, f) = (\Omega/H) \setminus \overline{P_f}$ such that $p^* \bar{f} = f|_{\mathrm{dom}\, f}$. It follows from the construction that over $p(\mathrm{dom}\, f)$ the graph of $\bar{f}$ coincides with $\overline{\Gamma_f}$. In summary we have shown that $\overline{\Gamma_f}$ is a meromorphic graph over Ω/H. Consequently, there exists a meromorphic function $\bar{f} \in \mathscr{M}_{\Omega/H}(\Omega/H)$ such that $\Gamma_{\bar{f}} = \overline{\Gamma_f}$. By construction this function fulfills $p^* \bar{f} = f \in \mathscr{M}_\Omega(\Omega)$.

Finally, for the proof of property (6) let $H \bullet x$ and $H \bullet y$ be two orbits in Ω. Then we have $p^* z_j(x) \neq p^* z_j(y)$ for some $j \in \{1, \ldots, N\}$. By the same argument as above $p^* z_j$ yields an H-invariant meromorphic function on X which separates $H \bullet x$ and $H \bullet y$. $\qquad\square$

Applying the observation from Section 6.1 in order to remove the assumption of H–irreducibility finally completes the proof of the Main Theorem.

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