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QUASI-PERIODIC AND PERIODIC SOLUTIONS OF THE TODA LATTICE VIA THE HYPERELLIPTIC SIGMA FUNCTION

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Abstract. — A lattice model with exponential interaction, was proposed and integrated by M. Toda in the 1960s; it was then extensively studied as one of the completely integrable (differential-difference) equations by algebro-geometric methods, which produced both quasi-periodic solutions in terms of theta functions of hyperelliptic curves and periodic solutions defined on suitable Jacobians by the Lax-pair method. In this work, we revisit Toda's original approach to give solutions of the Toda lattice in terms of hyperelliptic Kleinian ("sigma") functions for arbitrary genus. We then show that periodic solutions of the Toda lattice correspond to the zeros of Kiepert-Brioschi's division polynomials, and note these are related to solutions of Poncelet's closure problem. The hyperelliptic curve of our approach is related in a non-trivial way to the one given by the Lax pair.

Résumé. — M. Toda a donné la définition et l'intégration au moyen les fonctions elliptiques de Jacobi d'un réseau dont les noeuds réagissent réciproquement exponentiellement. La hiérarchie de Toda des équations (différentielles-différences) ont été beaucoup étudiées via les fonctions thêta hyperelliptiques ; une matrice de Lax donne l'intégration dans le cas périodique. Dans ce travail, utilisant la méthode de Toda et les formules d'addition qu'on vienne d'établir pour les fonctions ("sigma") de Klein hyperelliptiques de n'importe quel genre, nous donnons la solution du réseau quasi-périodique qui est donc aussi une solution de la fermeture de Poncelet. Les coefficients de la matrice de Lax peuvent être écrits comme fonctions rationnelles des coordonnées affines de la courbe hyperelliptique que nous utilisons pour la solution.

1. Introduction

The Toda lattice is an "algebraically completely integrable system". As such, it admits classes of solutions parametrized by Jacobi varieties of compact Riemann surfaces (or algebraic curves), the "algebro-geometric solitons" [21]. One advantage of the algebro-geometric solution is that the time

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flows become linear on a hyperelliptic Jacobian, and the difference operator is translation on the Jacobian. These are the objects of concern here.

On the other hand, recent work [17, 38] on Kleinian σ-functions has focused on addition formulae on a certain stratification of the Jacobian; given the relevance of this stratification in terms of the orbits of the Toda lattice [2, 1, 12, 30], our program is to study the explicit relationship between the two, with the result of an exchange of knowledge.

In this first paper we identify and study the (quasi-)periodic solutions of the Toda lattice equation in terms of the hyperelliptic σ-functions. Our approach is somewhat different from the one that exists in the literature, and in particular it gives us a different 'spectral curve' and algebraic conditions for periodicity. To give a sketch of our method and results, we first say a few words about the history and the significance of the σ-function, especially in the field of differential/difference equations. In the 19th century, it was the study of Riemann-surface theory that led to the associated algebraic (meromorphic) or analytic (θ-) functions and the (differential) equations that they satisfy, apart some exceptional cases. In keeping with such philosophy, Baker in [6] discovered the KdV hierarchy and KP equation[(1)] for every hyperelliptic curve of genus g as an application of Kleinian σ-functions (1903) and posed this challenge:

> These equations put a *problem*: To obtain a theory of differential equations which shall shew from them why, if we assume
>
> $$\wp_{\lambda\mu}(u) = -\partial^2 \log \sigma(u)/\partial u_\lambda \partial u_\mu,$$
>
> the function $\sigma(u)$ has the properties which *a priori* we know it to possess, and how far the forms of the equations are essential to these properties.

Baker extensively studied the differential equations satisfied by the σ-function. In the late 1960s and early 1970s, on the contrary, many authors started from the ("completely integrable") differential equations, and arrived at a spectral curve which is a Riemann surface and carries a "Baker-Akhiezer" eigenfunction for the (linearizing) "Lax-pair" equations. Mumford in his book on theta functions [43] demonstrated the close relationship between the differential equations and the algebraic approach, together with the Abelian function theory of the 19th century based on Jacobi's

[(1)] KP appears in the case when the affine model of the hyperelliptic curve has two points at infinity [33].

theory; he gave an explicit dictionary between the θ-function for hyperelliptic curves and three polynomials in one variable, sometimes called the Mumford triplet (whose definition he attributes to Jacobi), which parametrize the Jacobi variety, cf. Remark 5.7.

Recently the Kleinian σ-function was reexamined, generalized and studied by several authors [9, 15]. These authors showed that sigma is more efficient than Riemann's theta function (in fact, the σ-function approaches the Schur polynomials in the limit when the curve becomes rational) to solve differential equations. This is our point of view.

We add, as suggested by the referee, a comment on the geometric significance of sigma: for more detail we refer to the aforementioned studies. Since sigma vanishes on the (higher) Abel images of the curve X_g, it is better suited to be expanded in the abelian variables which correspond to the hyperosculating flows (Section 3) to the image of X_g, which are the flows of the KP hierarchy. It is in these variables that sigma equals a Schur polynomial[(2)] up to higher-order terms [44]. In fact expansions and computer-algebra work have enabled guesses and proofs of the addition formulas which generalize (2.4) and are essential to solving integrable equations. Lastly, sigma (unlike Riemann's theta, cf. [43, II.5]) is invariant under the action of the modular group on the period matrix of X_g.

We also comment on the fact that our addition formulae, which are key to the solution as explained in the next paragraph, are particular to hyperelliptic curves. Of course addition theorems hold for general curves, cf. e.g. Theorems 9.1 and 10.1 in [16] for the trigonal case; however, they will not be as expedient as the hyperelliptic ones, for example an expression in terms of algebraic functions on the curve is as yet unwieldy, as perhaps can be expected, since the non-hyperelliptic Neumann systems have polynomial Hamiltonians of a very complicated type [46, 47, 48]. The basic reason why the hyperelliptic case simplifies, and in fact was the one for which Baker was able to study explicitly the σ-function, is the hyperelliptic involution which lifts to the Jacobian in such a way that it acts on sigma only by a sign.

In this article, we consider the solutions of the, one- and two-(time-)dimensional, Toda one-(space-)dimensional lattice. Toda gave an exact solution for the Toda lattice equation using an identity of elliptic-function theory [50]. For an arbitrary hyperelliptic curve of genus g, we construct a meromorphic function on the Jacobian of the curve that obeys the Toda

[(2)] Schur polynomials were used by Sato to define his τ-function and originally construct the universal solution to the KP hierarchy. The relationship of sigma with tau is pursued in [14].

lattice equation linearly in time over the Jacobian, by using an identity of hyperelliptic abelian functions (see Theorems 5.4 and 5.5 for the one- and two-(time-)dimensional Toda lattice equations respectively.). Remark 5.7 shows that the two-(time-)dimensional Toda equation is equivalent to a relation which generates Fay's trisecant formula and Baker's derivation of the KdV hierarchy and the KP equation for every genus. Despite this equivalence, we stress again here the geometric nature of sigma compared to theta. That nature gives a dictionary between Abelian and algebraic functions of the curve, specifically, the affine coordinates of its planar representation, as well as algebraic coefficients for the relations in the ring of differential operators that act on the Jacobian (more details are given in Remarks 4.13 and 5.7(2)(b)). Weierstrass recognised this fact in his study of hyperelliptic θ-functions [52]; in fact, he defined the al and Al functions (named after Abel) which can be used to give solutions of finite-dimensional Hamiltonian systems [36].

We obtain quasi-periodic(3) (hyperelliptic) solutions of the Toda lattice, since *a priori* the discrete variable has no rational relation with the period lattice of the curve. Finding periodic solutions is equivalent to giving a torsion point of the curve, which satisfies a "division polynomial", known as Kiepert's or Brioschi's polynomial. Using a zero of the division polynomial (if it exists), we construct a hyperelliptic solution of Toda of genus g with period $N(> g)$ in Theorems 6.3 and 6.11. We illustrate the $g = 1$ case as an example and in that case point out that the relation between the division polynomial and Toda lattice is the same as Poncelet's closure (Appendix).

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(3) As customary in the literature on the Toda lattice, cf. the monographs [3] and [51], the word "periodic" does not refer to the time, but rather the space variable of the lattice sites.

2. Genus-one case

In this Section, we demonstrate how one gets an elliptic solution of the Toda lattice [50, 27, 34]. Let X_1 be an elliptic curve given by

$$X_1 :\ y^2 = x^3 + \lambda_2 x^2 + \lambda_1 x + \lambda_0$$

$$= (x - e_1)(x - e_2)(x - e_3), \tag{2.1}$$

where the e's are distinct complex numbers and $\lambda_2 = -(e_1 + e_2 + e_3) = 0$.

The Weierstrass elliptic σ function associated with the curve X_1 is connected with the Weierstrass $\wp$ and ζ functions by

$$\wp(u) = -\frac{d^2}{du^2}\log\sigma(u), \quad \zeta(u) = \frac{d}{du}\log\sigma(u). \tag{2.2}$$

The Jacobian of the curve X_1 is given by $\mathcal{J}_1 = \mathbb{C}/(\mathbb{Z}\omega' + \mathbb{Z}\omega'')$ using the periods (ω', ω''), complete integrals of the first kind. The abelian coordinate u, actually defined on the universal cover of the Jacobian $\mathcal{J}_1$ but used as customary up to congruence when no ambiguity arises, is given by

$$u = \int_\infty^{(x,y)} \frac{dx}{2y}, \tag{2.3}$$

with $x(u) = \wp(u)$, $2y(u) = \wp'(u)$ and ∞ the infinity point of X_1.

The key to obtain a $\wp$-function solution of the Toda lattice is the addition formula,

$$\wp(u) - \wp(v) = -\frac{\sigma(u+v)\sigma(u-v)}{[\sigma(v)\sigma(u)]^2}. \tag{2.4}$$

By differentiating the logarithm of (2.4) with respect to u twice, we have

$$-\frac{d^2}{du^2}\log[\wp(u) - \wp(v)] = \wp(u+v) - 2\wp(u) + \wp(u-v). \tag{2.5}$$

Thus for a constant number u_0 and any integer n, by letting $u = nu_0 + t + t_0$, $v = u_0$,

$$\begin{aligned} -\frac{d^2}{dt^2}\log[\wp(nu_0 + t + t_0) - \wp(u_0)] &= [\wp((n+1)u_0 + t + t_0) - \wp(u_0)] \\ &\quad - 2[\wp(nu_0 + t + t_0) - \wp(u_0)] + [\wp((n-1)u_0 + t + t_0) - \wp(u_0)]. \end{aligned} \tag{2.6}$$

Which, by letting $V_n(t) := -\wp(nu_0 + t + t_0)$, $V_c := -\wp(u_0)$ and $q_n := -\log[V_n(t) - V_c]$, becomes

$$-\frac{d^2}{dt^2}q_n = \mathrm{e}^{-q_{n+1}} - 2\mathrm{e}^{-q_n} + \mathrm{e}^{-q_{n-1}} \qquad (n \in \mathbb{Z}). \tag{2.7}$$

This can be identified with the continuous Toda lattice equation, where q_n are the interaction potentials. Indeed, by letting $q_n = Q_n - Q_{n-1}$, where Q_n

is the displacement of the n-th particle and obeys the nonlinear differential equation of the exponential lattice [50],

$$-\frac{d^2}{dt^2}Q_n = \mathrm{e}^{Q_n - Q_{n+1}} - \mathrm{e}^{Q_{n-1}-Q_n} \qquad (n \in \mathbb{Z}). \tag{2.8}$$

To provide the connection between Toda's original solution for $e_1, e_2, e_3 \in \mathbb{R}$ [50] and this elliptic solution derived from the addition formula (2.4) with (2.2), we observe that, by letting $t_0 = -\omega''$,

$$\wp(t + nu_0 - \omega'') - \wp(t + nu_0) = (e_1 - e_3)\ \mathrm{ns}^2((e_1 - e_3)^{1/2}(t + nu_0)) + e_3,$$

where $\mathrm{ns}(u) = k\mathrm{sn}(u + \imath K')$ and the modulus of the Jacobi elliptic functions is $k^2 = (e_2 - e_3)/(e_1 - e_3)$ [53, 22·351].

3. Hyperelliptic curve X_g and sigma functions

In this Section, we give background information on the hyperelliptic θ-functions and the σ-function, a generalization to higher genus of the Weierstrass elliptic σ function.

3.1. Geometrical setting for hyperelliptic curves

Let X_g be a hyperelliptic curve defined by

$$X_g \ :\ y^2 = f(x) := x^{2g+1} + \lambda_{2g}x^{2g} + \cdots + \lambda_0$$

where λ_j's are complex numbers, together with a smooth point ∞ at infinity. Let the affine ring related to X_g be $R_g := \mathbb{C}[x,y]/(y^2 - f(x))$. We fix a basis of holomorphic one-forms

$$\nu^{\mathrm{I}}{}_i = \frac{x^{i-1}dx}{2y} \qquad (i = 1, \dots, g).$$

We also fix a homology basis for the curve X_g so that

$$\mathrm{H}_1(X_g, \mathbb{Z}) = \bigoplus_{j=1}^{g} \mathbb{Z}\alpha_j \oplus \bigoplus_{j=1}^{g} \mathbb{Z}\beta_j,$$

with the intersections given by $[\alpha_i, \alpha_j] = 0$, $[\beta_i, \beta_j] = 0$ and $[\alpha_i, \beta_j] = -[\beta_i, \alpha_j] = \delta_{ij}$. We consider the half-period matrix $\omega = \begin{bmatrix}\omega' \\ \omega''\end{bmatrix}$ of X_g with respect to the given basis where

$$\omega' = \frac{1}{2}\left[\oint_{\alpha_j} \nu^{\mathrm{I}}{}_i\right], \quad \omega'' = \frac{1}{2}\left[\oint_{\beta_j} \nu^{\mathrm{I}}{}_i\right],$$

Let Λ be the lattice in $\mathbb{C}^g$ spanned by the column vectors of $2\omega'$ and $2\omega''$. The Jacobian variety of X_g is denoted by $\mathcal{J}_g$ and is identified with $\mathbb{C}^g/\Lambda$. For a non-negative integer k, we define the Abel map from the k-th symmetric product $\mathrm{S}^k X_g$ of the curve X_g to $\mathbb{C}^g$ by first choosing any (suitable) path of integration(4):

$$w : \mathrm{S}^k X_g \to \mathbb{C}^g, \quad w((x_1,y_1),\dots,(x_k,y_k)) = \sum_{i=1}^k \int_\infty^{(x_i,y_i)} \begin{pmatrix} {\nu^{\mathrm{I}}}_1 \\ \vdots \\ {\nu^{\mathrm{I}}}_g \end{pmatrix}.$$

By letting the map κ be the natural projection,

$$\kappa : \mathbb{C}^g \longrightarrow \mathcal{J}_g,$$

the image of $\kappa \circ w$ is denoted by $\mathcal{W}_k = \kappa \circ w(\mathrm{S}^k X_g)$. The mapping $\kappa \circ w$ is surjective when $k \geqslant g$ by Abel's theorem, and is injective when $k = g$ if we restrict the map to the pre-image of the complement of a specific connected Zariski-closed subset of dimension at most $g-2$ in $\mathcal{J}_g$, by Jacobi's theorem.

3.2. Sigma function and its derivatives

We define differentials of the second kind,

$${\nu^{\mathrm{II}}}_j = \frac{1}{2y} \sum_{k=j}^{2g-j} (k+1-j)\lambda_{k+1+j} x^k dx, \quad (j = 1,\dots,g)$$

and (half of) complete hyperelliptic integrals of the second kind

$$\eta' = \frac{1}{2}\left[\oint_{\alpha_j} {\nu^{\mathrm{II}}}_i\right], \quad \eta'' = \frac{1}{2}\left[\oint_{\beta_j} {\nu^{\mathrm{II}}}_i\right].$$

For this basis of a $2g$-dimensional space of meromorphic differentials, the half-periods $\omega', \omega'', \eta', \eta''$ satisfy the generalized Legendre relation

$$\mathfrak{M} \begin{pmatrix} 0 & -1_g \\ 1_g & 0 \end{pmatrix} \mathfrak{M}^T = \frac{\imath\pi}{2} \begin{pmatrix} 0 & -1_g \\ 1_g & 0 \end{pmatrix}. \tag{3.1}$$

where $\mathfrak{M} = \begin{pmatrix} \omega' & \omega'' \\ \eta' & \eta'' \end{pmatrix}$. Let $\mathbb{T} = {\omega'}^{-1}\omega''$. The theta function on $\mathbb{C}^g$ with "modulus" $\mathbb{T}$ and characteristics $\mathbb{T}a + b$ for $a, b \in \mathbb{C}^g$ is given by

$$\theta\begin{bmatrix} a \\ b \end{bmatrix}(z;\mathbb{T}) = \sum_{n\in\mathbb{Z}^g} \exp\left[2\pi i \left\{\frac{1}{2}\, {}^t(n+a)\mathbb{T}(n+a) + {}^t(n+a)(z+b)\right\}\right].$$

(4) The results presented below are independent of the particular choice.

The σ-function ([4], p.336, [9]), an analytic function on the space $\mathbb{C}^g$ and a theta-series having modular invariance of a given weight with respect to $\mathfrak{M}$, is given by the formula

$$\sigma(u) = \gamma_0 \exp\left\{-\frac{1}{2}\,{}^t u \eta' {\omega'}^{-1} u\right\} \theta\begin{bmatrix}\delta''\\ \delta'\end{bmatrix}\left(\frac{1}{2}{\omega'}^{-1}u\,;\,\mathbb{T}\right),$$

where δ' and δ'' are half-integer characteristics giving the vector of Riemann constants with basepoint at ∞ and γ_0 is a certain non-zero constant. The σ-function vanishes exactly on $\kappa^{-1}(\mathcal{W}_{g-1})$ (see for example [4, XI.206]). The Kleinian $\wp$ and ζ functions are defined by

$$\wp_{ij} = -\frac{\partial^2}{\partial u_i \partial u_j}\log\sigma(u), \quad \zeta_i = \frac{\partial}{\partial u_i}\log\sigma(u).$$

Let $\{\phi_i(x,y)\}$ be an ordered set of $\mathbb{C}\cup\{\infty\}$-valued functions over X_g defined by

$$\phi_i(x,y) = \begin{cases} x^i & \text{for } i \leqslant g,\\ x^{\lfloor (i-g)/2\rfloor + g} & \text{for } i > g \text{ and } i-g \text{ even},\\ x^{\lfloor (i-g)/2\rfloor} y & \text{for } i > g \text{ and } i-g \text{ odd}. \end{cases} \tag{3.2}$$

We note that $\{\phi_i(x,y)\}$ give a basis of the (infinite-dimensional) $\mathbb{C}$ vector space R_g.

Following [45, 37], we introduce a multi-index $\natural^n$. For n with $1 \leqslant n < g$, we let $\natural^n$ be the set of positive integers i such that $n+1 \leqslant i \leqslant g$ with $i \equiv n+1 \bmod 2$. Namely,

$$\natural^n = \begin{cases} n+1, n+3, \dots, g-1 & \text{for } g-n \equiv 0 \bmod 2\\ n+1, n+3, \dots, g & \text{for } g-n \equiv 1 \bmod 2 \end{cases}$$

and partial derivatives over the multi-index $\natural^n$

$$\sigma_{\natural^n} = \left(\prod_{i\in\natural^n}\frac{\partial}{\partial u_i}\right)\sigma(u).$$

For $n \geqslant g$, we define $\natural^n$ as empty and $\sigma_{\natural^n}$ as σ itself. The first few examples are given in Table 1, where we let $\sharp$ denote $\natural^1$ and $\flat$ denote $\natural^2$.

For $u \in \mathbb{C}^g$, we denote by u' and u'' the unique vectors in $\mathbb{R}^g$ such that

$$u = 2\,{}^t\omega' u' + 2\,{}^t\omega'' u''.$$

We define

$$\begin{aligned} L(u,v) &= {}^t u(2\,{}^t\eta' v' + 2\,{}^t\eta'' v''),\\ \chi(\ell) &= \exp\left\{2\pi i\left({}^t\ell'\delta'' - {}^t\ell''\delta' + \tfrac{1}{2}{}^t\ell'\ell''\right)\right\} \ (\in\{1,-1\}) \end{aligned}$$

Table 3.1

genus	$\sigma_\sharp \equiv \sigma_{\natural^1}$	$\sigma_\flat \equiv \sigma_{\natural^2}$	$\sigma_{\natural^3}$	$\sigma_{\natural^4}$	$\sigma_{\natural^5}$	$\sigma_{\natural^6}$	$\sigma_{\natural^7}$	$\sigma_{\natural^8}$	$\cdots$
1	σ	σ	σ	σ	σ	σ	σ	σ	$\cdots$
2	σ_2	σ	σ	σ	σ	σ	σ	σ	$\cdots$
3	σ_2	σ_3	σ	σ	σ	σ	σ	σ	$\cdots$
4	σ_{24}	σ_3	σ_4	σ	σ	σ	σ	σ	$\cdots$
5	σ_{24}	σ_{35}	σ_4	σ_5	σ	σ	σ	σ	$\cdots$
6	σ_{246}	σ_{35}	σ_{46}	σ_5	σ_6	σ	σ	σ	$\cdots$
7	σ_{246}	σ_{357}	σ_{46}	σ_{57}	σ_6	σ_7	σ	σ	$\cdots$
8	σ_{2468}	σ_{357}	σ_{468}	σ_{57}	σ_{68}	σ_7	σ_8	σ	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

for u, $v \in \mathbb{C}^g$ and for ℓ $(= 2\,{}^t\omega'\ell' + 2\,{}^t\omega''\ell'') \in \Lambda$. Then $\sigma_{\natural^n}(u)$ for $u \in \kappa^{-1}(\mathcal{W}_1)$ satisfies the translational relation ([45], Lemma 7.3):

$$\sigma_{\natural^n}(u+\ell) = \chi(\ell)\sigma_{\natural^n}(u)\exp L(u+\tfrac{1}{2}\ell,\ell) \quad \text{for } u \in \kappa^{-1}(\mathcal{W}_1). \tag{3.3}$$

Further for $n \leqq g$, we note that $\sigma_{\natural^n}(-u) = (-1)^{ng+\frac{1}{2}n(n-1)}\sigma_{\natural^n}(u)$ for $u \in \kappa^{-1}(\mathcal{W}_n)$, especially,

$$\begin{cases} \sigma_\flat(-u) = -\sigma_\flat(u) & \text{for } u \in \kappa^{-1}(\mathcal{W}_2) \\ \sigma_\sharp(-u) = (-1)^g\sigma_\sharp(u) & \text{for } u \in \kappa^{-1}(\mathcal{W}_1) \end{cases} \tag{3.4}$$

by Proposition 7.5 in [45].

4. The addition formulae

In this Section, we give the addition formulae of the hyperelliptic σ-functions which are the generalization of the addition formula (2.4). These are essential to construct the hyperelliptic solution of the Toda lattice.

4.1. Generalized Frobenius-Stickelberger formula

We first recall the generalized Frobenius-Stickelberger formula which gives a generalization of the addition formula (2.4).

Definition 4.1. — For a positive integer $n \geqslant 1$ and $(x_1, y_1), \dots, (x_n, y_n)$ in X_g, we define the Frobenius-Stickelberger determinant [38],

$$\Psi_n((x_1, y_1), \dots, (x_n, y_n)) := \begin{vmatrix} 1 & \phi_1(x_1, y_1) & \cdots & \phi_{n-2}(x_1, y_1) & \phi_{n-1}(x_1, y_1) \\ 1 & \phi_1(x_2, y_2) & \cdots & \phi_{n-2}(x_2, y_2) & \phi_{n-1}(x_2, y_2) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & \phi_1(x_{n-1}, y_{n-1}) & \cdots & \phi_{n-2}(x_{n-1}, y_{n-1}) & \phi_{n-1}(x_{n-1}, y_{n-1}) \\ 1 & \phi_1(x_n, y_n) & \cdots & \phi_{n-2}(x_n, y_n) & \phi_{n-1}(x_n, y_n) \end{vmatrix}.$$

where $\phi_i(x_j, y_j)$'s are the monomials defined in (3.2).

Then we have the following theorem (Theorem 8.2 in [45]):

Proposition 4.2. — *For a positive integer $n > 1$, let $(x_1,y_1),\dots,(x_n,y_n)$ in X_g, and $u^{(1)}, \dots, u^{(n)}$ in $\kappa^{-1}(\mathcal{W}_1)$ be points such that $u^{(i)} = w((x_i, y_i))$. Then the following relation holds:*

$$\frac{\sigma_{\natural^n}(\sum_{i=1}^n u^{(i)}) \prod_{i<j} \sigma_\flat(u^{(i)} - u^{(j)})}{\prod_{i=1}^n \sigma_\sharp(u^{(i)})^n} = \epsilon_n \Psi_n((x_1, y_1), \dots, (x_n, y_n)), \tag{4.1}$$

where $\epsilon_n = (-1)^{g+n(n+1)/2}$ for $n \leqslant g$ and $\epsilon_n = (-1)^{(2n-g)(g-1)/2}$ for $n \geqslant g+1$.

4.2. The algebraic addition formula

We first describe linear equivalence of divisors on the curve X_g (which will result on the addition law on the Jacobian) by algebraic formulas [38]:

Definition 4.3. — *For given $P_1, \dots, P_n \in X_g$, we define*

$$\mu_n(P; P_1, \dots, P_n) = \lim_{Q_i \to P_i} \frac{\Psi_{n+1}(P, Q_1, \dots, Q_n)}{\Psi_n(Q_1, \dots, Q_n)}$$

for distinct Q_i's (the order in which the limits are taken is irrelevant).

Proposition 4.4. — *For given $P_1, \dots, P_n \in X_g$, we find $Q_1, \dots, Q_\ell$ with $\ell = g$ for $n \geqslant g$ and $\ell = n$ otherwise, such that*

$$P_1 + P_2 + \cdots + P_n + Q_1 + Q_2 + \cdots + Q_\ell - (n+\ell)\infty \sim 0$$

by taking the zero-divisor of $\mu_n(P; P_1, \dots, P_n)$. For each $Q_i = (x_i, y_i)$, by letting $-Q_i = (x_i, -y_i)$, we have the addition property,

$$P_1 + P_2 + \cdots + P_n - n\infty \sim (-Q_1) + (-Q_2) + \cdots + (-Q_\ell) - (-\ell)\infty.$$

As usual, the symbol of addition in the (free abelian) divisor group is used as well for addition up to linear equivalence.

Remark 4.5. — We should note that the hyperelliptic involution ι : $(x, y) \mapsto (x, -y)$ induces the $[-1]$-action on $\mathcal{J}_g$, defined by $u \mapsto -u$.

4.3. The analytic addition formula

We have the following addition formula for the hyperelliptic σ functions (Theorem 5.1 in [17]):

THEOREM 4.6. — *Assume that (m, n) is a pair of positive integers. Let (x_i, y_i) $(i = 1, \ldots, m)$, (x'_j, y'_j) $(j = 1, \ldots, n)$ in X_g and $u \in \kappa^{-1}(\mathcal{W}_m)$, $v \in \kappa^{-1}(\mathcal{W}_n)$ be points such that $u = w((x_1, y_1), \ldots, (x_m, y_m))$ and $v = w((x'_1, y'_1), \ldots, (x'_n, y'_n))$. Then the following relation holds:*

(4.2)

$$\frac{\sigma_{\natural^{m+n}}(u+v)\sigma_{\natural^{m+n}}(u-v)}{\sigma_{\natural^m}(u)^2\sigma_{\natural^n}(v)^2}$$
$$= \delta(g,m,n)\frac{\prod_{i=0}^{1}\Psi_{m+n}((x_1,y_1),\ldots,(x_m,y_m),(x'_1,(-1)^i y'_1),\ldots,(x'_n,(-1)^i y'_n))}{(\Psi_m((x_1,y_1),\ldots,(x_m,y_m))\Psi_n((x'_1,y'_1),\ldots,(x'_n,y'_n)))^2}$$
$$\times \prod_{i=1}^{m}\prod_{j=1}^{n}\frac{1}{\Psi_2((x_i,y_i),(x'_j,y'_j))}$$

where $\delta(g, m, n) = (-1)^{gn+\frac{1}{2}n(n-1)+mn}$.

Theorem 4.6 with $m = g$ and $n = 2$ leads to the following Corollary:

COROLLARY 4.7. — *Let $(x_i, y_i) \in X_g$ $(i = 1, \ldots, g)$, $(x'_j, y'_j) \in X_g$ $(j = 1, 2)$, $u \in \mathbb{C}^g$, $v := v^{[1]} + v^{[2]} \in \kappa^{-1}(\mathcal{W}_2)$, and $v^{[j]} \in \kappa^{-1}(\mathcal{W}_1)$ $(j = 1, 2)$ be points such that $u = w((x_1, y_1), \ldots, (x_g, y_g))$ and $v^{[j]} = w((x'_j, y'_j))$, $(j = 1, 2)$. Then the following relation holds:*

$$\frac{\sigma(u+v)\sigma(u-v)}{\sigma(u)^2\sigma_\flat(v)^2} = -\Xi(u,v), \tag{4.3}$$

where $\Xi(u, v)$ is equal to

$$F(x'_1)F(x'_2)\left(\sum_{i=1}^{g}\frac{y_i}{(x_i-x'_1)(x_i-x'_2)F'(x_i)}\right)^2 - F(x'_1)F(x'_2)\left(\sum_{i=1}^{2}\frac{(-1)^i y'_i}{(x'_1-x'_2)F(x'_i)}\right)^2,$$

and $F(x) := (x-x_1)(x-x_2)\cdots(x-x_g) \equiv \mu_g((x,y);(x_1,y_1),\dots,(x_g,y_g))$ *and* $F'(x) := \partial F(x)/\partial x$.

Proof. — By letting $\Delta(x_1,x_2,\dots,x_\ell)$ be the Vandermonde determinant, i.e.,

$$\Delta(x_1,x_2,\dots,x_\ell) = \begin{vmatrix} 1 & x_1 & \cdots & x_1^{\ell-1} \\ 1 & x_2 & \cdots & x_2^{\ell-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x_\ell & \cdots & x_\ell^{\ell-1} \end{vmatrix} = \prod_{i,j=1,i<j}^{\ell} (x_j - x_i), \tag{4.4}$$

we have

$$\Psi_{g+2}((x_1,y_1),\dots,(x_g,y_g),(x_1',\pm y_1'),(x_2',\pm y_2')) = \begin{vmatrix} 1 & x_1 & \cdots & x_1^g & y_1 \\ 1 & x_2 & \cdots & x_2^g & y_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & x_g & \cdots & x_g^g & y_g \\ 1 & x'_1 & \cdots & {x'}_1^g & \pm y'_1 \\ 1 & x'_2 & \cdots & {x'}_2^g & \pm y'_2 \end{vmatrix} \tag{4.5}$$

$$= \sum_{i=1}^{g} \Delta(x_1,\dots,\check{x}_i,\dots,x_g,x_1',x_2')y_i \pm \sum_{i=1}^{2}(-1)^i \Delta(x_1,\dots,x_g,x_{3-i}')y_i'.$$

Thus for the case $m=g$ and $n=2$, the right-hand side of the formula in Theorem 4.6 is equal to (4.3). ☐

Baker [4, §11.217], [6, p. 138] proves the following:

Proposition 4.8. — *Let* $(x_i,y_i) \in X_g$ $(i=1,\dots,g)$ *and* $u \in \mathbb{C}^g$ *such that* $u = w((x_1,y_1),\dots,(x_g,y_g))$. *The following relation holds for generic* x_i' $(i=1,2)$,

$$\sum_{i=1}^{g}\sum_{j=1}^{g} \wp_{ij}(u){x_1'}^{i-1}{x_2'}^{j-1} = F(x_1')F(x_2')\left(\sum_{i=1}^{g}\frac{y_i}{(x_1'-x_i)(x_2'-x_i)F'(x_i)}\right)^2 - \frac{f(x_1')F(x_2')}{(x_1'-x_2')^2F(x_1')} - \frac{f(x_2')F(x_1')}{(x_1'-x_2')^2F(x_2')} + \frac{f(x_1',x_2')}{(x_1'-x_2')^2}, \tag{4.6}$$

where

$$f(x_1,x_2) = \sum_{i=0}^{g} x_1^i x_2^i(\lambda_{2i+1}(x_1+x_2)+2\lambda_{2i}).$$

Corollary 4.7 and Proposition 4.8 yield the key proposition in this article.

Proposition 4.9. — *For the variables in Corollary 4.7, the following relation holds:*

$$\frac{\sigma(u+v)\sigma(u-v)}{\sigma(u)^2\sigma_\flat(v)^2} = \frac{f(x'_1,x'_2)-2y'_1y'_2}{(x'_1-x'_2)^2} - \sum_{i=1}^{g}\sum_{j=1}^{g}\wp_{ij}(u){x'_1}^{i-1}{x'_2}^{j-1}. \tag{4.7}$$

This corresponds to Fay's formula [19, (39)], which is the basis of "Fay's trisecant identity".

Remark 4.10. — For the $g=1$ case, $\dfrac{f(x'_1,x'_2)-2y'_1y'_2}{(x'_1-x'_2)^2} = \wp(v_1+v_2)$. Thus if we let $v_3 = v_1+v_2$ and $x'_3 = \wp(v_3)$, (4.7) recovers (2.4).

Corollary 4.11. — *For the variables in Corollary 4.7, with $v^{[1]} = v^{[2]}$, we have*

$$\begin{aligned}\frac{\sigma(u+2v^{[1]})\sigma(u-2v^{[1]})}{\sigma(u)^2\sigma_\flat(2v^{[1]})^2} &= f_{1,2}(x'_1) - \sum_{i=1}^{g}\sum_{j=1}^{g}\wp_{ij}(u){x'_1}^{i+j-2}\\ &= -\lim_{x'_2\to x'_1}\Xi(u,v).\end{aligned} \tag{4.8}$$

where

$$f_{1,2}(x) := \frac{\partial_x^2 f(x)}{2f(x)} - f^I_{1,2}(x), \quad f^I_{1,2}(x) := \sum_{i=0}^{g}(i^2\lambda_{2i+1}x^{2i-1} + i(i-1)\lambda_{2i}x^{2i}).$$

Belokolos, Enolskii, and Salerno gave the following relation [7, Theorem 3.2]:

Corollary 4.12. — *For the variables in Corollary 4.7, with $v^{[2]} = 0$ or $(x'_2,y'_2)=\infty$, we have*

$$\begin{aligned}\frac{\sigma(u+v^{[1]})\sigma(u-v^{[1]})}{\sigma(u)^2\sigma_\flat(v^{[1]})^2} &= {x'_1}^{g} - \sum_{i=1}^{g}\wp_{gi}(u){x'_1}^{i-1}\\ &= F(x'_1) = (x'_1-x_1)(x'_1-x_2)\cdots(x'_1-x_g)\\ &\equiv \mu_g((x'_1,y'_1);(x_1,y_1),\ldots,(x_g,y_g)).\end{aligned} \tag{4.9}$$

Proof. — We divide both sides of (4.7) by ${x'_2}^{g-1}$. By taking the appropriate limit, we obtain the equality. □

Remark 4.13. — We conclude this Section by elaborating on the geometric properties of the σ-function which make the addition formulas particularly suited for integrating equations of dynamics. As stated in the Introduction, originally the sigma function was defined by Weierstrass [52] in order to express a symmetric function of the points of a hyperelliptic curve, which he called "al" function, in terms of rational functions. The al function is a root function, equal to $\sqrt{F(b_a)}$ for a zero b_a of $f(x)$, giving one of the branch points of X_g. The focus was on constructing a dictionary between Abelian and rational functions. In particular, a goal was the solution of the "Jacobi inversion", namely returning the symmetric functions of the divisor from a point on the universal cover of the Jacobian, as in the genus-2 case: $\wp_{22} = x_1 + x_2$, $\wp_{21} = x_1 x_2$, where the points of the divisor $P_1 + P_2$ are $P_i = (x_i, y_i)$, $i = 1, 2$. (see also Remark 5.7(2)(b)). The most natural application is then the explicit realization of the group structure of the Jacobian (the generalized Frobenius-Stickelberger relation in Proposition 4.2, which gives the addition structure, shows a simple connection between the affine coordinate ring R_g and the group law on the Jacobian $\mathcal{J}_g$), reflecting addition in the free Abelian divisor group, and this is achieved in Corollary 4.11, vis-à-vis Proposition 4.4.

5. The σ-function solution of the Toda lattice

In this Section, we give the σ-function solution of the Toda lattice equation.

First, we identify some algebraic identities that hold for vector fields on the Jacobian. Vector fields on the Jacobian are understood to be translation invariant; equivalently, they are elements of the tangent space at the origin. It is important that we use algebraic functions on the curve to write their coordinates in the Abelian variables (u_i), but in doing this, we make the convention that we are on a suitable coordinate patch on the Jacobian, where the Abel map from $\mathrm{S}^g X_g$ can be inverted; as explained in [43, §3], there are choices involved and one has to check that the vector field in question is well defined. Our formulas would hold on this suitable affine patch anyway, since outside it, the Toda orbits become of smaller dimension, as mentioned in the Introduction; that case will be addressed in our forthcoming work. Moreover, as usual, we view the vector fields as derivations on the universal cover, namely on the ring of functions in $\Gamma(\mathbb{C}^g, \mathcal{O}(\mathbb{C}^g))$.

In order to give our solution of the Toda lattice equation, we need only two vector fields on the Jacobian, each associated in the same way to a fixed point $(x_j', y_j') \in X_g$, $j = 1, 2$; however, in order to relate our solution to Baker's differential equations (Remark 5.7), we distinguish two further fixed points $(x_j', y_j') \in X_g$, $j = 3, 4$, for which we use Baker's expression of the derivatives in terms of algebraic functions on the curve as opposed to Abelian coordinates on the Jacobian; the dictionary between the two expressions is given in Lemma 5.2 (b).

Definition 5.1. — *For* $(x_i, y_i) \in X_g$ $(i = 1, \dots, g)$, $u = w((x_1, y_1), \dots, (x_g, y_g)) \in \mathbb{C}^g$, *and* $(x'_j, y'_j) \in X_g$ $(j = 1, \dots, 4)$, *we let*

$$D_j := \sum_{i=1}^{g} {x'_j}^{i-1} \frac{\partial}{\partial u_i}.$$

Lemma 5.2. — *Let* $(x_i, y_i) \in X_g$ $(i = 1, \dots, g)$, $(x'_j, y'_j) \in X_g$ $(j = 1, \dots, 4)$, $u \in \mathbb{C}^g$, *and* $v^{[j]} \in \kappa^{-1}(\mathcal{W}_1)$ $(j = 1, \dots, 4)$ *be points such that* $u = w((x_1, y_1), \dots, (x_g, y_g))$ *and* $v^{[j]} = w((x'_j, y'_j))$, $(j = 1, \dots, 4)$. *We have the following expressions:*

(a) *For each* $j = 1, 2, 3,$ *or* 4,

$$D_j = \frac{1}{\Delta(x_1, x_2, \dots, x_g)} \begin{vmatrix} 1 & x_1 & \cdots & x_1^{g-1} & 2y_1\partial_{x_1} \\ 1 & x_2 & \cdots & x_2^{g-1} & 2y_2\partial_{x_2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & x_g & \cdots & x_g^{g-1} & 2y_g\partial_{x_g} \\ 1 & x'_j & \cdots & {x'_j}^{g-1} & 0 \end{vmatrix}$$

$$= \sum_{i=1}^{g} \frac{y_i F(x'_j)}{F'(x_i)(x'_j - x_i)} \frac{\partial}{\partial x_i},$$

and for each $j, j' = 1, \dots, 4$,

$$[D_j, D_{j'}] = D_j D_{j'} - D_{j'} D_j = 0.$$

(b) *For* $v^{(j)} = w(x'_j, y'_j)$ *with* $j = 1, \dots, 4$,

$$\frac{\partial}{\partial x'_j} = \frac{1}{2y'_j} \sum_{i=1}^{g} {x'_1}^{i-1} \frac{\partial}{\partial v_i^{(j)}}.$$

(c) *For* $h \in \Gamma(\mathbb{C}^g, \mathcal{O}(\mathbb{C}^g))$ *and* $j = 1, \dots, 4$,

$$D_{j'} h(u + v^{(j)}) := 2y'_j \frac{\partial}{\partial x'_j} h(u + v^{(j)}) = D_j h(u + v^{(j)}).$$

Proof. — A direct calculation gives the results. □

Lemma 5.3. — *For the variables in Corollary 4.7, we have*

(a)

$$D_1 \log \sigma(u + v) = \frac{1}{2} \left(D_1 \log \Xi(u, v) + D_{1'} \log \Xi(u, v) \right) \\ + D_1 \log \sigma(u) + D_{1'} \log \sigma_\flat(v),$$

(b)

$$\begin{aligned}D_1 D_2 \log \sigma(u+v) \\ &= \frac{1}{2} D_1 D_2 \log \Xi(u,v) + \frac{1}{2} D_1 D_{2'} \log \Xi(u,v) + D_1 D_2 \log \sigma(u) \\ &= \frac{1}{2} D_1 D_2 \log \Xi(u,v) + \frac{1}{2} D_2 D_{1'} \log \Xi(u,v) + D_1 D_2 \log \sigma(u).\end{aligned}$$

Proof. — By taking the derivatives of the logarithm of (4.3), we obtain

$$D_1 \log \Xi(u,v) = D_1 \log \sigma(u+v) - 2D_1 \log \sigma(u) + D_1 \log \sigma(u-v),$$
$$D_{1'} \log \Xi(u,v) = D_{1'} \log \sigma(u+v) - 2D_{1'} \log \sigma_\flat(v) + D_{1'} \log \sigma(u-v),$$

and Lemma 5.2 (c) yields the claims. □

Now we can give the σ-function solution of the Toda lattice equation:

Theorem 5.4. — *Let $(x_i, y_i) \in X_g$ $(i = 1, \dots, g)$, $(x'_1, y'_1) \in X_g$ $u \in \mathbb{C}^g$, and $v^{[1]} \in \kappa^{-1}(\mathcal{W}_1)$ be points such that $u = w((x_1, y_1), \dots, (x_g, y_g))$ and $v^{[1]} = w((x'_1, y'_1))$. We define $c := 2v^{[1]}$, $\tilde{D}_1 = \sigma_\flat(c) D_1$,*

$$\mathcal{V}(u) := \mathcal{V}(u, v^{[1]}) := \sum_{i=1}^{g} \sum_{j=1}^{g} \wp_{ij}(u) {x'_1}^{i+j-2}, \quad \mathcal{V}_c(c) := f_{1,2}(x'_1),$$

and $t := (t_{11}, t_{12}, \dots, t_{1g}) \in \mathbb{C}^g$ with

$$t_{1j} := (x'_1)^{1-j} \sum_{i=1}^{g} \int_\infty^{(x_i,y_i)} {\nu^{\mathrm{I}}}_j, \qquad (j = 1, 2, \dots, g).$$

Then with the coordinate change $u = nc + t^\perp + t$ (which defines $t^\perp$), we have

(1)

(5.1)
$$\begin{aligned}-D_1^2 \log \big(\mathcal{V}(t+nc+t^\perp) - \mathcal{V}_c(c)\big) \\ &= \mathcal{V}(t+(n+1)c+t^\perp) - 2\mathcal{V}(t+nc+t^\perp) + \mathcal{V}(t+(n-1)c+t^\perp).\end{aligned}$$

(2) *The Hirota bilinear equation,*

$$\begin{aligned}\sigma(t+nc+t^\perp)\tilde{D}_1^2\sigma(t+nc+t^\perp) - \tilde{D}_1\sigma(t+nc+t^\perp)\tilde{D}_1\sigma(t+nc+t^\perp) \\ - \mathcal{V}_c(c)\sigma(t+nc+t^\perp)^2 - \sigma(t+(n+1)c+t^\perp)\sigma(t+(n-1)c+t^\perp) = 0.\end{aligned} \tag{5.2}$$

(3) *The Toda-lattice equation, by letting $\mathcal{V}_n(t+t^\perp) := -\mathcal{V}(t+nc+t^\perp)$ and $q_n(t) := -\log\big(\mathcal{V}_n(t+t^\perp) - \mathcal{V}_c(c)\big)$,*

$$-D_1^2 q_n(t) = \mathrm{e}^{-q_{n+1}} - 2\mathrm{e}^{-q_n} + \mathrm{e}^{-q_{n-1}}. \tag{5.3}$$

Proof. — The claims follow from Corollary 4.11, by direct verification, as in the genus-1 case, using the definition of the vector fields. ☐

We also have the σ-function solutions of the two-(time-)dimensional Toda lattice equation [25, 3.5.1, (3.107-108)], $\partial^2 q_n/\partial t_1 \partial t_2 = V_{n+1} - 2V_n + V_{n-1},\ q_n = \log(1+V_n)$, n any integer:

Theorem 5.5. — *Let $(x_i, y_i) \in X_g$ $(i = 1, \ldots, g)$, $(x'_j, y'_j) \in X_g$ $(j = 1, 2)$, $u \in \mathbb{C}^g$, $v := v^{[1]} + v^{[2]} \in \kappa^{-1}(\mathcal{W}_2)$, and $v^{[j]} \in \kappa^{-1}(\mathcal{W}_1)$ $(j = 1, 2)$ be points such that $u = w((x_1, y_1), \ldots, (x_g, y_g))$ and $v^{[j]} = w((x'_j, y'_j))$, $(j = 1, 2)$. We define $c := v^{[1]} + v^{[2]}$,*

$$\hat{\mathcal{V}}(u, v^{[1]}, v^{[2]}) := \sum_{i=1}^{g} \sum_{j=1}^{g} \wp_{ij}(u) {x'_1}^{i-1} {x'_2}^{j-1},$$

$$\hat{\mathcal{V}}_c(v^{[1]}, v^{[2]}) := \frac{2y'_1 y'_2 - f(x'_1, x'_2)}{(x'_1 - x'_2)^2},$$

and $t_j := (t_{j1}, t_{j2}, \ldots, t_{jg}) \in \mathbb{C}^g$ with

$$t_{jk} := (x'_j)^{1-k} \sum_{i=1}^{g} \int_{\infty}^{(x_i, y_i)} {\nu^{\mathrm{I}}}_k, \qquad (j = 1, 2, \text{ and } k = 1, 2, \ldots, g).$$

Then with $u = nc + t^{\perp} + t_1 + t_2$, we have

$$\begin{aligned} - D_1 D_2 \log(\tilde{\mathcal{V}}(nc + t_1 + t_2 + t^{\perp}) - \tilde{\mathcal{V}}_c(c)) &= \tilde{\mathcal{V}}(t_1 + t_2 + (n+1)c + t^{\perp}) \\ &\quad - 2\tilde{\mathcal{V}}(t_1 + t_2 + nc + t^{\perp}) + \tilde{\mathcal{V}}(t_1 + t_2 + (n-1)c + t^{\perp}). \end{aligned}$$

Proof. — The claims follow from Proposition 4.9 by direct verification. ☐

Remark 5.6. — As Hirota notes, a two-(space-)dimensional version of the Toda system ought to have two independent discrete variables, but at the time, he had only found solitons for the two-times case [25, §3.5.1, Remark]. Instead of Theorem 5.5, as a different type of generalization of Theorem 5.4 involving the σ-function on the Jacobian, implementing a second spatial (discrete) variable would entail choosing another $z^{[1]} \in \mathcal{W}_1$ say, and adding to the argument of the σ-function a vector $mc_1 + nc_2$, with $c_1 = 2v^{[1]}$ and $c_2 = 2z^{[1]}$, and "considering a two-dimensional version of the force term on the right-hand side" of (1) in Theorem 5.4 [*ibid.*]. We would need a different kind of addition formula, which we have not yet developed.

Remark 5.7. — We would like to stress that a single formula underlies the validity of the integrable hierarchies: the differential operators that Baker used to obtain what we call the KdV and KP equations are always of the type D_1 we defined, namely a linear combination of the Abelian coordinates given by the basis of holomorphic differentials which involve the equation of the plane curve, against coefficients that are powers of the x-coordinate of one point of the curve; generically, distinct points yield independent vector fields. Specific to our situation:

(1) Corollary 4.7, Baker's formula (4.6) in [4, p. 328] and [6, p. 138], and Fay's formula (4.7) in [19, (39) in p. 26] are essentially the same. We notice the following two facts:
  (a) Baker derived the KdV hierarchy and KP equation by using his formula (4.6), using the vector fields D_3 and D_4, cf. Definition 5.1 [33].
  (b) Fay derived his famous trisecant identity, which is equivalent to the KP hierarchy, by using formula (4.7), which is the hyperelliptic σ-function version.

(2) The following relationships hold among the formulae:
  (a) The two-dimensional Toda equation in Theorem 5.5 is the same as formula (4.7).
  (b) When $v^{[1]} = v^{[2]}$ in Theorem 5.5, we obtain the Toda lattice equation of Theorem 5.4, and when $v^{[2]} = 0$, the Toda equations are obtained by differentiating $F(x_1')$ (Corollary 4.12). The function $F(x_1')$ is a polynomial of degree g in the variable x_1'. By applying the vector field D_3, we obtain the "Mumford triple" (which is called (U, V, W) in [43], three polynomials that parametrize the Jacobian outside a theta divisor (excluded are certain g-tuples of points (x_i, y_i) in special position). We note that the coefficients of Mumford's (U, V, W) involve only the Abelian functions $\wp_{gi}$, $i \geqslant 0$ [43, §10]. The KdV hierarchy follows again using formula (4.7).

(3) We stress again the connection between algebraic and Abelian functions: the differential operator D_j has an expression that involves only (and acts upon) the x_i's (cf. Lemma 5.2). On the other hand, (4.3) is also given by the affine coordinates of points of X_g. Hence the Toda lattice equation is an identity defined over $\mathrm{S}^g X_g$. It can also be regarded as a relation among the functions over $\mathcal{J}_g$ and $\mathrm{S}^g X_g$.

Remark 5.8. — We conclude by comparing our solution to some of the methods that were used to obtain algebro-geometric Toda flows.

(1) van Moerbeke [41], following his work with Kac on the Toda lattice and Jacobi matrices, reported below in Section 6, gave a description of the isospectral flows in terms of linear flows on a Jacobian, and with Mumford [42], the algebraic coordinates for the flows and the solutions in terms of θ-functions. The flows are linear combinations of, essentially, the D_i defined above, for (x_i', y_i') a branchpoint of the hyperelliptic involution. Fay's trisecant identity and its derivatives along the D_i are used [42, §5, Proposition] to show that the flows are Hamiltonian vector fields that preserve the spectrum of the matrices.

(2) Algebro-geometric solutions to the Toda lattice can be found in [21]. These authors in their extensive work also used the spectral curve of the tridiagonal matrices whose deformations, equivalent to the Toda lattice, are given below in Section 6. To solve them, using the divisors given by auxiliary spectra and via eigenvectors expressed in terms of theta functions, the authors work out a "discrete Floquet theory" (as had done Kac and van Moerbeke) and solve the "Dubrovin equations" on the expansion of the Green's function by Abelian functions.

6. Periodicity of the Toda-lattice solution

In this section we turn to the problem of periodic solutions of the Toda lattice. Spatial periodicity is of physical interest, in the lattice case, so the requirement amounts to finding a point of finite order N on the hyperelliptic curve: $c = 2w((x_1', y_1'))$ such that for $u \in \mathbb{C}^g$,

$$u + Nc = u \quad \text{modulo} \quad \Lambda.$$

Hyperelliptic curves that admit such a point are special, and were called "Toda curves" by McKean and van Moerbeke [39], who proved that they are dense in moduli space. For the same reason, points of finite order in the Jacobian that come from the curve, or from the sum of a specific number of points on the curve, give periodic orbits in the billiard (completely integrable Hamiltonian) system in the ellipsoid [13], and are related to Poncelet's closure (cf. Appendix).

The finite-point condition was investigated by Cantor and Ônishi [45, 11] using the division polynomial ψ_{2N}, an element of R_g. Similarly, we investigate the periodic solutions of the Toda lattice.

6.1. Division polynomials ψ_{2N}

Traditionally, division polynomials $F_n(x,y)$ arise in expressing the coordinates of nP in terms of those of P, a point of an elliptic curve in Weierstrass form. In particular, given our present interest, we call "the division polynomial" an element of the ring of functions on the affine curve, whose solutions are points of finite order in the Jacobian (where the Abel map is, as usual, based at ∞). The division polynomial for the genus-one case was studied by Kiepert [28] and Brioschi [8]. We call Kiepert-type polynomial and Brioschi-type polynomial the genus-one version of the division polynomial ψ_n, a polynomial whose zeros P satisfy the condition $nP \equiv 0$, more precisely, $nw(P) \equiv 0$ modulo Λ.

Referring to [45, Definition 9.2], the hyperelliptic version of the ψ_n function for genus X_g over $w(X_g) = \kappa^{-1}\mathcal{W}_1$ is defined by

$$\psi_n(u) = \frac{\sigma_{\natural^n}(nu)}{\sigma_\sharp(u)^{n^2}}.$$

A zero $u(\neq 0)$ of ψ_n has the property that $nu \in \kappa^{-1}\mathcal{W}_{g-1}$. The transformation law under translation (3.3) allows one to check that ψ_n belongs to R_g. Thus it is a natural generalization of the classical Kiepert formula, or the division polynomial. By taking limits of Proposition 4.2 along the Abelian variables, we can give an expression for ψ_n in terms of ϕ_i's in R_g [45, Theorem 9.3]. In [38], we proved the following:

Theorem 6.1. — *Let $n \geqslant 1$ be a positive integer. For*

$$\psi_n(u) = \frac{\varepsilon_{n,g}}{1!2!\cdots(n-1)!}\begin{vmatrix} \frac{\partial\phi_1}{\partial u_1} & \frac{\partial\phi_2}{\partial u_1} & \cdots & \frac{\partial\phi_{n-1}}{\partial u_1} \\ \frac{\partial^2\phi_1}{\partial {u_1}^2} & \frac{\partial^2\phi_2}{\partial {u_1}^2} & \cdots & \frac{\partial^2\phi_{n-1}}{\partial {u_1}^2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^{n-1}\phi_1}{\partial {u_1}^{n-1}} & \frac{\partial^{n}\phi_2}{\partial {u_1}^{n-1}} & \cdots & \frac{\partial^{n-1}\phi_{n-1}}{\partial {u_1}^{n-1}} \end{vmatrix},$$

with $\psi_1 \equiv 1$, the vanishing of ψ_n at a point P of the hyperelliptic curve X_g is a necessary and sufficient condition for $\omega(n\cdot P)$ to belong to $\mathcal{W}_{g-1}$, where ω is the Abel map $\omega : X_g \to \mathcal{J}_g$. Here $\varepsilon_{n,g}$ is a plus or minus sign.

Further let $n(\geqslant g)$, $k(< g)$ and $\ell := g-k-1$ be non-negative integers. For a hyperelliptic curve X_g, the vanishing of $\psi_{n+\ell}$, …, ψ_{n+1}, ψ_n, ψ_{n-1}, …, $\psi_{n-\ell}$, at a point P of X_g is a necessary and sufficient condition for $\omega(n\cdot P)$ to belong to $\mathcal{W}_k$.

Cantor gave a Brioschi-type expression for the ψ_n-function [11, 35],

$$\psi_n(u) = \begin{cases} \varepsilon'_{n,g}(2y)^{n(n-1)/2} \cdot T^{(g+2)}_{(n-g-1)/2}(y, \frac{d}{dx}) & \text{for } n \not\equiv g \bmod 2 \\ \varepsilon'_{n,g}(2y)^{n(n-1)/2} \cdot T^{(g+1)}_{(n-g)/2}(y, \frac{d}{dx}) & \text{for } n \equiv g \bmod 2. \end{cases}$$

Here $\varepsilon'_{n,g}$ is a plus or minus sign and $T^{(m)}_n$ is a Toeplitz determinant [35],

$$T^{(m)}_n\left(g(s), \frac{d}{ds}\right) = \begin{vmatrix} g^{[m+n-1]} & g^{[m+n-2]} & \cdots & g^{[m+1]} & g^{[m]} \\ g^{[m+n]} & g^{[m+n-1]} & \cdots & g^{[m+2]} & g^{[m+1]} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ g^{[m+2n-3]} & g^{[m+2n-4]} & \cdots & g^{[m+n-1]} & g^{[m+n-2]} \\ g^{[m+2n-2]} & g^{[m+2n-3]} & \cdots & g^{[m+n]} & g^{[m+n-1]} \end{vmatrix},$$

and $T^{(m)}_\ell\left(g(s), \dfrac{d}{ds}\right) \equiv 1$ when m is a non-negative integer and ℓ is a negative integer, $g(s)$ is a function of an argument s and

$$g^{[n]}(s) := \frac{1}{n!}\frac{d^n}{ds^n}g(s).$$

We note for $y^2 = f(x)$ that $y^{2n-1}d^ny/dx^n$ is a polynomial in x and coprime (in the sense of not vanishing together on a point of the curve) to $f(x)$ in general. Hence the function $y^{n(2m+2n-3)}T^{(m)}_n(y, \frac{d}{dx})$, that is,

$$\begin{vmatrix} y^{2m+2n-3}y^{[m+n-1]} & y^{2m+2n-5}y^{[m+n-2]} & \cdots & y^{2m+1}y^{[m+1]} & y^{2m-1}y^{[m]} \\ y^{2m+2n-1}y^{[m+n]} & y^{2m+2n-3}y^{[m+n-1]} & \cdots & y^{2m+3}y^{[m+2]} & y^{2m+1}y^{[m+1]} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ y^{2m+4n-7}y^{[m+2n-3]} & y^{2m+4n-9}y^{[m+2n-4]} & \cdots & y^{2m+2n-3}y^{[m+n-1]} & y^{2m+2n-5}y^{[m+n-2]} \\ y^{2m+4n-3}y^{[m+2n-2]} & y^{2m+4n-7}y^{[m+2n-3]} & \cdots & y^{2m+2n+1}y^{[m+n+1]} & y^{2m+2n-3}y^{[m+n-1]} \end{vmatrix}$$

is an element of $\mathbb{C}[x]$ and coprime to $y^2 = f(x)$. Hence $\psi_n(u)$ can be expressed by

$$\psi_n = \begin{cases} (2y)^{g(g+1)/2}\alpha_n(x) & \text{for } n-g = \text{odd}, n > g+1 \\ (2y)^{g(g-1)/2}\alpha_n(x) & \text{for } n-g = \text{even}, n > g+1 \\ (2y)^{n(n-1)/2}\alpha_n(x) & \text{otherwise}, \end{cases}$$

where $\alpha_n(x)$ is a polynomial of x and coprime of y for $n > g+1$, and $\alpha_n = \varepsilon'_{n,g}1$ for $n \leqslant g+1$. As shown in [11], the degree of $\alpha_n(x)$, $(n \geqslant g+2)$ is

$$\deg(\alpha_n) = \begin{cases} \dfrac{g(n+g)(n-g) - g(2g+1)}{2} & \text{for } n-g = \text{odd} \\ \dfrac{g(n+g)(n-g)}{2} & \text{for } n-g = \text{even}. \end{cases}$$

Definition 6.2. — *We define*

$$\Phi_n := \{P \in X_g \mid P \text{ is a zero of } \alpha_n\},$$

and for $n > g$,

$$\Xi_n := \Phi_{n-g+1} \cap \ldots \cap \Phi_{n-1} \cap \Phi_n \cap \Phi_{n+1} \cap \ldots \cap \Phi_{n+g-1}.$$

One should note here that there is no guarantee that Ξ_n is not empty.

Theorem 6.3. — *Let* $2N \geqslant 2g+1$. *For a hyperelliptic curve of genus* g *which has a point* $(x_1', y_1') \in \Xi_{2N}$, $\mathcal{V}(u)$ *in Theorem 5.4 is the periodic solution of the Toda lattice equation such that* $\mathcal{V}(u) = \mathcal{V}(u+Nc)$ *with* $c = 2w(x_1', y_1')$.

Remark 6.4. —

(1) In the $g = 1$ case, since an elliptic curve is a divisible group, there exist a point (x_1', y_1') which is a zero of a ψ_N; as mentioned in Remark 4.10, for every point (x_1', y_1') we have a point $(x_2', y_2') = 2(x_1', y_1')$. Thus, $\mathcal{V}$ in Theorem 5.4, is a periodic solution of the Toda lattice equation, provided $\mathcal{V}(u) = \mathcal{V}(u+Nc)$ with $c = w(x_2', y_2')$.

(2) We do not address in the present work the important problem of finding real-valued solutions $\mathcal{V}(u)$.

Example 6.5. — Case $g = 1$, $N = 3$ and $N = 4$.

We consider the elliptic curve $y^2 = x^3 - x$ (which has an extra automorphism of order two – a property that is not related to points of finite order but is usually accompanied by a large number of exact solutions for the coordinates of points on the curve that satisfy algebraic conditions). The division polynomials are given by

$$\begin{aligned}
\psi_1 &= 1,\\
\psi_2 &= -2y,\\
\psi_3 &= 34y^2(x^4 - 6x^2 - 1),\\
\psi_4 &= -2y(x^2+1)(x^2+2x-1)(x^2-2x-1),\\
\psi_5 &= 32x^{14} - 187x^{12} - 64x^{11} + 2x^{10} + 320x^9 - 233x^8\\
&\quad + 320x^7 - 52x^6 - 64x^5 - 61x^4 + 50x^2 + 1.
\end{aligned}$$

For $x_3' = (1/3)\sqrt{9+6\sqrt{3}}$, we have $N = 3$ and for $x_3' = \sqrt{2}+1$ or a zero of ψ_4 we have $N = 4$.

Lemma 6.6. — *Let* $2N \geqslant 2g+1$. *If* $P := (x_1', y_1') \in \Xi_{2N}$ *has the property that* ℓP *are distinct for different* $\ell \in \{1, 2, \cdots, 2N\}$, *then*

$$(x - x_1')(x - x_2') \cdots (x - x_{2N}') | \psi_{2N+m}(x, y)$$

where $(x'_\ell, y'_\ell) := \ell(x'_1, y'_1)$ *and* $m = -g+1, \ldots, 0, \ldots, g-1$.

Proof. — By assumption, $\pm\ell P$ are exactly the points of the set Ξ_{2N}. □

6.2. A periodic Toda lattice

In this subsection, we consider the relation between Theorem 6.3 and an algebraically completely integrable system[(5)] originally studied by Kac and van Moerbeke [27]. Using the solution given in Section 5, we find the explicit form of Flaschka's coordinates for the Toda lattice. The Hamiltonian of the Toda lattice equation is

$$H = \frac{1}{2}\sum_{k=1}^{N} P_k^2 + \sum_{k=1}^{N} \exp(Q_k - Q_{k+1}),$$

where $P_k = P_{k+N}$ and $Q_k = Q_{k+N}$. For Flaschka's coordinates, $a_k = \exp(Q_k - Q_{k+1})$ and $b_k = -P_k$, the equations of motion under H become

$$\begin{cases} \dfrac{d}{dt}a_k = a_k(b_{k+1} - b_k), \\ \dfrac{d}{dt}b_k = a_k - a_{k-1}. \end{cases} \qquad (k = 1, 2, \ldots, N).$$

Remark 6.7. — The Toda Hamiltonian system admits the time inversion $t \mapsto -t$, and this corresponds to the hyperelliptic involution on X_g.

For brevity, we introduce the notation

$$\sigma^{(n)}(t; t^\perp) := \sigma(t + nc + t^\perp), \quad \sigma^{(c)} := \sigma_\flat(c),$$

$$\zeta^{(n)}(t; t^\perp) := \sum_{i=1}^{g} {x'_1}^{i-1}\zeta_i(t + nc + t^\perp), \quad \zeta^{(c)} := \frac{1}{2}D_{1'} \log \sigma_\flat(c),$$

$$\wp^{(n)}(t; t^\perp) := \sum_{i,j=1}^{g} {x'_1}^{i+j-2}\wp_{ij}(t + nc + t^\perp), \quad \wp^{(c)}(t^\perp) := f_{1,2}(x'_1).$$

Proposition 6.8. — *Using the σ-function solution of the Toda lattice equation in Theorem 5.4 (2), the Flaschka coordinates for the Toda lattice*

(5) Different systems have been variously referred to in the literature as "periodic (generalized) Toda systems"; for extensive information on definitions and solutions we refer to the monographs [3, 51].

are expressed as follows:

$$a_n = \wp^{(n)}(t;t^\perp) - \wp^{(c)}(t^\perp) = \frac{\sigma^{(n+1)}(t;t^\perp)\sigma^{(n-1)}(t;t^\perp)}{\sigma^{(n)}(t;t^\perp)^2 {\sigma^{(c)}}^2},$$

$$b_{n-1} = D_t \log \frac{\sigma^{(n)}(t;t^\perp)}{\sigma^{(n-1)}(t;t^\perp)} - \zeta_c = \zeta^{(n)}(t;t^\perp) - \zeta^{(n-1)}(t;t^\perp) - \zeta^{(c)}.$$

Proof. — By definition of $P_k = -b_k$,

$$P_{n-1} - P_n = \zeta^{(n+1)}(t;t^\perp) - 2\zeta^{(n)}(t;t^\perp) + \zeta^{(n-1)}(t;t^\perp),$$
$$\cdots\cdots$$
$$P_2 - P_3 = \zeta^{(4)}(t;t^\perp) - 2\zeta^{(3)}(t;t^\perp) + \zeta^{(2)}(t;t^\perp),$$
$$P_1 - P_2 = \zeta^{(3)}(t;t^\perp) - 2\zeta^{(2)}(t;t^\perp) + \zeta^{(1)}(t;t^\perp),$$
$$P_0 - P_1 = \zeta^{(2)}(t;t^\perp) - 2\zeta^{(1)}(t;t^\perp) + \zeta^{(0)}(t;t^\perp),$$

with

$$-P_n = \zeta^{(n+1)}(t;t^\perp) - \zeta^{(n)}(t;t^\perp) - (\zeta^{(1)}(t;t^\perp) - \zeta^{(0)}(t;t^\perp)) - P_0.$$

Since the total momentum should be invariant, we set

$$P_0 = -(\zeta^{(1)}(t;t^\perp) - \zeta^{(0)}(t;t^\perp)) + p_0.$$

where p_0 is a constant corresponding to $\zeta^{(c)}$. Then the equation $db_k/dt = a_k - a_{k-1}$ holds by Lemma 5.3 and Corollary 4.7, equation $da_k/dt = a_k(b_{k+1} - b_k)$ by Theorem 5.4. $\square$

Proposition 6.9. — *The a_n's and b_n's given in Proposition 6.8 are rational functions of x_i, y_i $(i = 1, \dots, g)$ and x'_1, y'_1.*

Proof. — The translation law (3.3) for the $\sigma_{\natural^n}$ functions shows that a_n and b_n are meromorphic functions over the Jacobian $\mathcal{J}_g$; Lemma 5.2 gives b_n as a meromorphic function over $S^g X_g \times X_g$, derived using algebraic vector fields whose coordinates are rational functions of x_i, y_i $(i = 1, \dots, g)$, x'_1, y'_1.

On the other hand, using Theorem 4.6, a_n is given by

$$\frac{\sigma(u + 2nv^{[1]} + 2v^{[1]})\sigma(u + 2nv^{[1]} - 2v^{[1]})}{\sigma(u + 2nv^{[1]})^2 \sigma_{\natural^n}(2v^{[1]})^2}$$
$$= \lim_{(x'_i,y'_i)\to(x'_1,y'_1)} \left[\frac{\prod_{i=0}^{1} \Psi_{g+2n}((x_1,y_1),\dots,(x'_{2n+2},y'_{2n+2}),(x'_1,(-1)^i y'_1),(x'_2,(-1)^i y'_2))}{(\Psi_{g+2n}((x_1,y_1)\dots,(x'_{2n+2},y'_{2n+2}))\Psi_2((x'_1,y'_1),(x'_2,y'_2)))^2} \right.$$
$$\left. \times \prod_{i=1}^{g}\prod_{j=1}^{2} \frac{1}{\Psi_2((x_i,y_i),(x'_j,y'_j))} \prod_{i=3}^{2n+2}\prod_{j=1}^{2} \frac{1}{\Psi_2((x'_i,y'_i),(x'_j,y'_j))} \right].$$

Here $((x_1, y_1), \ldots, (x'_{2n+2}, y'_{2n+2}))$ in both numerator and denominator of the equation is an abbreviation for

$$((x_1, y_1), \ldots, (x_g, y_g), (x'_3, y'_3), \ldots, (x'_{2n+2}, y'_{2n+2})).$$

This completes the proof of the statement. □

Remark 6.10. — To exemplify Proposition 6.9 we write b_n in the $g = 1$ case:

$$\zeta(u + v) - \zeta(u) - \zeta(v) = \frac{y(u) - y(v)}{x(u) - x(v)}.$$

Now we give an inverse of Theorem 6.3; together, they constitute our main result relating the periodic Toda lattice to the σ-function solution:

THEOREM 6.11. — *Let $2N \geqslant 2g + 1$. If a hyperelliptic curve of genus g has a point $(x'_1, y'_1) \in X_g \setminus \infty$ such that $\mathcal{V}(u)$ in Theorem 5.4 is the periodic solution of the Toda lattice equation, i.e., $\mathcal{V}(u) = \mathcal{V}(u + Nc)$ with $c = 2w(x'_1, y'_1)$, then (x'_1, y'_1) belongs to Ξ_{2N}.*

Proof. — The periodicity condition $\mathcal{V}(u) = \mathcal{V}(u+Nc)$ is equivalent to the periodicity of a_n and b_n due to Kac and van Moerbeke [27]. The expression for the Flaschka coordinates in Proposition 6.8 allows for a prefactor $\mathrm{e}^{{}^t\beta u}$ of $\sigma(u)$, β a constant vector in $\mathbb{C}^g$, in terms of which the periodicity of a_n and b_n, $(a_n = a_{n+N}, b_n = b_{n+N})$, is equivalent to the equality:

$$\mathrm{e}^{2N{}^t\beta v^{[1]}}\sigma(u + 2Nv^{[1]}) = \sigma(u) \tag{6.1}$$

for every $\pm u \in \mathcal{W}_k$ $(k = 0, \ldots, g)$, where $v^{[1]} = w(x'_1, y'_1) \neq 0$ (by assumption). Noting $2N - (g - 1) > g$, this implies in particular, setting $u = \pm kv^{[1]} \neq 0$,

$$\mathrm{e}^{2N{}^t\beta v^{[1]}}\sigma((2N \pm k)v^{[1]}) = \sigma(\pm kv^{[1]}).$$

The right-hand side vanishes for $(k = 0, \ldots, g-1)$ since $u = \pm kv^{[1]}$ belongs to the (translated) theta divisor, whereas $\sigma_\sharp(v^{[1]})$ does not vanish.

Moreover, we recall that a point (x, y) in X_g with $y = 0$ is such that $2w(x, y) \in W_{g-1}$, which still satisfies the conclusion since $N > 1$. Hence (6.1) implies that $(x'_1, y'_1) \in \Xi_{2N}$. □

Remark 6.12. — Given the connection between Theorem 6.3 and theory of Kac and van Moerbeke provided by Theorem 6.11, we note in addition: since it is known that for certain multi-indices $\gamma = (\gamma_1, ..., \gamma_g)$ and for $\pm u \in \kappa^{-1}\mathcal{W}_k$ $(k = 1, \ldots, g-1)$, letting $\partial^\gamma := \partial_{u_1}^{\gamma_1} \ldots \partial_{u_g}^{\gamma_g}$, the derivative $\partial^\gamma\sigma(u)$ vanishes [45, 37], by differentiating the equation, the σ function satisfies $(k = 1, \ldots, \ell)$ for a suitable ℓ,

$$\partial^\gamma\sigma((2N - \ell)v^{[1]}) = \partial^\gamma\sigma((2N - \ell - 1)v^{[1]}) = \ldots = \partial^\gamma\sigma((2N + \ell)v^{[1]}) = 0.$$

As we have assumed that $v^{[1]} \in \Xi_{2N}$ does not vanish, the vanishing of sigma and its derivatives on multiples of $v^{[1]}$ is a condition of flag-variety type (cf. [2, 1, 20, 12, 30]). Studying the topology of these Toda orbits of smaller (than generic) dimension was the original motivation for our work, which we plan to use for a sequel to this paper.

6.3. Hyperelliptic curve $\hat{X}_{g,N-1}$ for the periodic Toda lattice

The Lax matrix for the periodic Toda lattice is now given by

$$\mathcal{L} := \begin{pmatrix} b_1 & 1 & 0 & \cdots & a_N \hat{w}^{-1} \\ a_1 & b_2 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & a_{N-2} & b_{N-1} & 1 \\ \hat{w} & \cdots & \cdots & a_{N-1} & b_N \end{pmatrix}.$$

The characteristic equation for $\mathcal{L}$ defines the hyperelliptic spectral curve:

$$\det(\mathcal{L} - z) = -\left(\hat{w} + \frac{\prod_{i=1}^N a_i}{\hat{w}} - \mathcal{P}(z) \right) = 0,$$

which gives the affine curve $(\hat{w}, z)$ of genus $N-1$,

$$\hat{X}_{g,N-1} \ : \ \hat{w}^2 - \mathcal{P}(z)\hat{w} + \prod_{i=1}^N a_i = 0. \tag{6.2}$$

Here $\mathcal{P}$ is given by

$$\mathcal{P}(z) := \Delta^{\mathrm{per}}{}_{1,N}(z) - \Delta^{\mathrm{per}}{}_{2,N-1}(z),$$

where

$$\Delta^{\mathrm{per}}{}_{n,m} := \begin{vmatrix} b_m - z & 1 & 0 & \cdots & 0 \\ a_m & b_{m+1} - z & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & a_{n-2} & b_{n-1} - z & 1 \\ 0 & \cdots & \cdots & a_{n-1} & b_n - z \end{vmatrix}.$$

$$\mathcal{P}(z) := (-1)^N z^N + \sum_{k=1} N(-1)^{N+k} \mathcal{I}_k z^{N-k}.$$

The coefficients of the powers of z are given by

$$\begin{aligned} &\mathcal{I}_1 = \sum_{i=1}^{N} b_i, \quad \mathcal{I}_2 = \sum_{i>j} b_i b_j - \sum_{i=1}^{N} a_i, \quad \cdots\cdots \\ &\cdots\cdots \qquad \cdots\cdots \\ &\mathcal{I}_N = \Delta^{\mathrm{per}}{}_{1,N}(0) - a_N \Delta^{\mathrm{per}}{}_{2,N-1}(0), \quad \mathcal{I}_{N+1} = \prod_{i=1}^{N} a_i. \end{aligned} \tag{6.3}$$

We refer to $\hat{X}_{g,N-1}$ as the periodic Toda curve. Since the characteristic polynomial $\det(\mathcal{L} - z)$ is invariant under the Toda flow, these coefficients give the Hamiltonians, i.e.

$$\frac{\partial}{\partial t}\mathcal{I}_j = 0 \qquad (j = 1, 2, \ldots, N+1). \tag{6.4}$$

The set $\{\mathcal{I}_j : j = 1, \ldots, N+1\}$ gives an involutive, complete family of integrals of motion, which guarantees the complete integrability of the Toda lattice. In particular, $\mathcal{I}_1$ and $\mathcal{I}_{N+1}$ can be expressed as follows:

Example 6.13. — From the formulae of (a_n, b_n) in Proposition 6.8, the following expression for two Hamiltonians in terms of Abelian functions follows directly:

$$\mathcal{I}_1 = \sum_i b_i = N\zeta^{(c)}, \qquad \mathcal{I}_{N+1} = \prod_{i=1}^{N} a_i = (\sigma^{(c)})^{-2N}.$$

Lemma 6.14. — *$\mathcal{I}_j$ $(j = 1, \ldots, N+1)$ can be expressed as rational functions of (x_1', y_1') only.*

Proof. — Condition (6.4) shows that $\mathcal{I}_j$ $(j = 1, \ldots, N+1)$ does not depend upon (x_j, y_j), $j = 1, \ldots, g$. □

Proposition 6.15. — *The hyperelliptic curve $\hat{X}_{g,N-1}$ of (6.2) admits a morphism to the curve:*

$$w^2 = \prod_{i=1}^{2N}(z - z_i) = \mathcal{P}(z)^2 - 4\prod_{i=1}^{N} a_i,$$

with $w := 2\hat{w} - \mathcal{P}(z)$. The coordinates of the Weierstrass points of $\hat{X}_{g,N-1}$ are rational functions of x_1' and y_1'.

Proof. — The Weierstrass points have z-coordinates which correspond to $w = 0$. The coefficients of the corresponding polynomial in z are rational functions of x_1' and y_1' by Lemma 6.14, and the fundamental theorem of elementary symmetric functions gives the conclusion. □

Example 6.16. — For $g=1$, we consider again the curve $y^2 = x^3 - x$ or $\hat{y}^2 = 4(x^3 - x)$. Then,

$$
\begin{aligned}
&a_0(t) = x - x_3', \\
&a_1(t) = \frac{(3xx_3'^{\,2} - x - x_3'^{\,3} - 2\hat{y}\hat{y}_3' - x_3')}{(x - x_3')^2}, \\
&a_2(t) = 2(x - x_3')\Big(\frac{(\hat{y}_3'^{\,4} + 8\hat{y}\hat{y}_3'^{\,3} + (-36xx_3'^{\,2} + (2(12+9x^2))x_3' - 12x)\hat{y}_3'^{\,2}}{(-3x_3'^{\,3} + x_3' + 2\hat{y}_3'^{\,2} - 2\hat{y}\hat{y}_3' + 3xx_3'^{\,2} - x)^2} \\
&\qquad + \frac{x^2 + 9x_3'^{\,6} - 6x_3'^{\,4} + x_3'^{\,2} + 12xx_3'^{\,3} - 2xx_3' - 18xx_3'^{\,5} - 6x^2x_3'^{\,2} + 9x^2x_3'^{\,4}}{(-3x_3'^{\,3} + x_3' + 2\hat{y}_3'^{\,2} - 2\hat{y}\hat{y}_3' + 3xx_3'^{\,2} - x)^2}\Big),
\end{aligned} \tag{6.5}
$$

$$
b_0(t) = \frac{\hat{y} - \hat{y}_3'}{x - x_3'}, \quad b_1(t) = \frac{\eta_{\hat{y},1} - \hat{y}_3'}{\eta_{x,1} - x_3'}, \quad b_2(t) = \frac{\eta_{\hat{y},2} - \hat{y}_3'}{\eta_{x,2} - x_3'}, \tag{6.6}
$$

where $(\eta_{x,1}, \eta_{\hat{y},1})$ and $(\eta_{x,2}, \eta_{\hat{y},2})$ are the solutions of the equations $\mu_1((z,w);(x,\hat{y}),(x_3',\hat{y}_3')) = 0$ and $\mu_2((z,w);(x,\hat{y}),2(x_3',\hat{y}_3')) = 0$ in the variables (z,w), with:

$$
\mu_1((z,w);(x,\hat{y}),(x_3',\hat{y}_3')) := \frac{(x_3'w - \hat{y}_3'z + z\hat{y}_1 - xw + x\hat{y}_3' - \hat{y}x_3')}{(x_3' - x)},
$$

$$
\begin{aligned}
\mu_2((z,w);(x,\hat{y}),2(x_3',\hat{y}_3')) &:= \frac{-zx^2 + xz^2 + 2xx_3'w\hat{y}_3' - 2z\hat{y}x_3'\hat{y}_3' - zx_3'^{\,2} - 3xz^2x_3'^{\,2}}{x - 3xx_3'^{\,2} + x_3'^{\,3} + 2\hat{y}\hat{y}_3' + x_3'} \\
&+ \frac{3zx^2x_3'^{\,2} + xx_3'^{\,2} + xx_3'^{\,4} - zx_3'^{\,4} - 2wx^2\hat{y}_3' + 2\hat{y}z^2\hat{y}_3' + z^2x_3'^{\,3} - x^2x_3'^{\,3} + x_3'z^2 - x_3'x^2}{x - 3xx_3'^{\,2} + x_3'^{\,3} + 2\hat{y}\hat{y}_3' + x_3'}.
\end{aligned}
$$

6.4. Remark on a relation between $\hat{X}_{g,N-1}$ and X_g

Even though the Toda time flows have been linearized on both the Jacobians of the curves X_g and $\hat{X}_{g,N-1}$ (cf. Remark 5.8), and solutions have been expressed in terms of hyperelliptic abelian functions on each, the relationship between these curves is non-trivial. To name only one other example, the Kowalevski solution of the top has been compared by several authors with the Lax-pair solution, and the classical Arithmetic-geometric Mean has been shown to relate the (genus-2) curves [31], the Jacobian of the one is the quotient of the Jacobian of the other by a group of order

4. A different Lax pair provides a curve of genus 3, which covers an elliptic curve: the Prym variety of the cover is again isogenous to the genus-2 Jacobian [32].

In our case, we make some observations. The coefficients of $\hat{X}_{g,N-1}$ are given by hyperelliptic ζ- and σ- functions, which are transcendental functions of their arguments (points of the Jacobian). However, Propositions 6.8 and 6.9 show that they are rational functions of x_i's and y_i's; moreover, these coefficients are Hamiltonians of motion, so the t-dependence of the a_n and b_n fixes the curve $\hat{X}_{g,N-1}$, associated to a point on the Jacobian of X_g.

We note that the cyclic group C_N acts on the sets of a's and b's by: $a_n \mapsto a_{n+1}$ and $b_n \mapsto b_{n+1}$, via the addition on $2P$ (the image of the point P in the Jacobian has order $2N$): $2\ell P \mapsto (2\ell+2)P$ and $(2\ell+1)P \mapsto (2\ell+3)P$, and that the curve $\hat{X}_{g,N-1}$ is invariant under this action, although this does not guarantee that $\hat{X}_{g,N-1}$ admits a C_N action. There could be a relationship of Kowalevski type between the two Jacobians (which have different dimensions in general), such as quotienting by the group C_N. We plan to investigate this relationship, starting with small-genus examples.

For example, in genus 1, the relation between X_1 and $\hat{X}_{1,N-1}$ could give another solution to Poncelet's closure problem (for Cayley's solution, cf. [13, 22]): as we review in the Appendix, the periodic Toda flow corresponding to a point of order $2N$ on a given elliptic curve, which plays the role of our X_g, also corresponds to a closed Poncelet polygon with $2N$ sides; in this case, as we saw in subsection 6.3 the periodic Toda curve has genus $N-1$, and could be viewed as an algebraic solution to the porism.

We conclude the Appendix by a reference to another classical problem solved in terms of transcendental functions over an algebraic curve, which therefore could play the role of X_g, a construction similar to ours yielding algebraic solutions over $\hat{X}_{g,N-1}$.

A. Appendix: Toda lattice and Poncelet's closure problem

Poncelet's porism (cf. [13, 22]) can be stated as follows:

Theorem A.1. — (Poncelet) *Let C and D be two smooth conics in the real affine plane, such that C includes D. For an integer $N > 2$, if there exists a closed N-polygon inscribed in C and circumscribed about D, for every point P in C there exists a polygon whose vertices are in C and includes P, and segments are tangent to D.*

More generally, in the complex projective plane, the existence of such an N-sided Poncelet polygon corresponds to a point of order N on the elliptic curve determined by the two conics, and a point in the incidence correspondence of points of C and tangents to D. This interpretation in terms of a transcendental problem is one of the deeper ways to prove the theorem (cf. [23]), whereas Poncelet used elementary projective geometry, a subject that did not exist at the time.

As can be expected, a point of finite order gives rise to many applications in the theory of periodic motion, and recently in [10] it was applied in a novel way to give a condition of Fritz John type on a Dirichlet problem for a planar domain bounded by an ellipse; in the same paper, the authors give the following explicit parametrization of the conics, and show that the vertices of a Poncelet N-gon give a solution to the N-periodic Toda chain, which we reproduce below.

Without loss of generality, we assume the conic C to be given by the equation $x^2 = yz$ and parametrize it by $(x, x^2, 1)$. Let the vertices of the Poncelet polygon be the N-points $(x_i^{(0)}, x_i^{(0)^2}, 1)$ $(i = 1, \dots, N)$. Let D be defined by:

$$(x, y, z)A^t(x, y, z) = 0,$$

where

$$A = \begin{pmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{pmatrix}.$$

Assume $a_5 = 0$. The dual conic D^* of D is given by $(X, Y, Z)A^{-1t}(X, Y, Z) = 0$. A pair $(P, L) \in C \times D^*$ such that $P \in L$ satisfies

$$xX + yY + zZ = 0,$$

for $P = (x, y, z)$ and $L = (X, Y, Z)$. The relation is reduced to the elliptic curve E_1

$$w^2 = \frac{1}{a_2 + a_4}(x, x^2, 1)A^t(x, x^2, 1),$$

where

$$w = \frac{1}{\sqrt{\det A}}\left(h_1(x)\frac{Y}{X} - h_2(x)\right),$$

and the h_1, h_2 are polynomials in x. Poncelet's closure is equivalent to finding a matrix A as above and a point (x, w) belonging to E_1 such that it satisfies the equation of Kiepert and Brioschi,

$$\psi_N((x, w)) = 0,$$

a criterion attributed to Cayley and proved in [22]. We regard the equation of $\psi_N((x,w)) = 0$ as the moduli equation for a given x.

For such an A, Poncelet's theorem means that the vertex $P_n \equiv (x_n, x_n^2, 1) \in C$ $(n = 1, 2, \dots, N)$ satisfies the periodic Toda lattice, $x_n = \wp((n-1)u_0 + t)$ [10, §7.1],

$$\begin{aligned}(\mathrm{A.1})\qquad -\frac{d^2}{dt^2}\log[\wp(nu_0+t)-\wp(u_0)] &= [\wp((n+1)u_0+t)-\wp(u_0)] \\ &\quad - 2[\wp(nu_0+t)-\wp(u_0)] + [\wp((n-1)u_0+t)-\wp(u_0)],\end{aligned}$$

where

$$(\mathrm{A.2})\qquad u_0 = \int_\infty^{(x_1^{(0)}, w_1^{(0)})} \frac{dx}{2w}$$

(in the previous notation, $x_n^{(0)} = \wp((n-1)u_0)$).

Lastly, we cite the problem of finding the general solution of the fifth-degree algebraic equation with Galois group $\mathrm{PSL}(2, \mathbb{F}_5) = A_5$, alternating group on five elements. While the solution cannot be algebraic in the coefficients of the polynomial equation, classical authors such as Jacobi, Galois, and Klein [29], gave a solution in terms of the zeros of ψ_5 for an elliptic curve X_1[6]. In the classical, fifth-degree case, Humbert [26] expressed the period-5 condition in terms of a Poncelet pentagon, but in addition proved that it is equivalent to the curve $y^2 = (x - x_1)\cdots(x - x_5)$ having real multiplication by the quadratic order of discriminant 5. Hashimoto and Sakai [24] expressed the general condition for a hyperelliptic curve of genus 2, $H_{1,2}$, $y^2 = (x - x_1)\cdots(x - x_5)\cdot(x - x_6)$ to have real multiplication of discriminant 5 again translating it into a condition for the existence of closed Poncelet pentagons. Mestre [40] generalized the condition of real multiplication to genus g. Lemma 6.6 gives a criterion for the points of order $2N$ in X_g in terms of the (genus-g) division polynomial; by analogy $H_{1,2}$ might be replaced by a hyperelliptic curve $H_{g,N-1}$,

$$\tilde{y}^2 = (x - x_1')(x - x_2')\cdots(x - x_{2N}').$$

Proposition 6.15 gives an algebraic relation between $H_{g,N-1}$ and the moduli of $\hat{X}_{g,N-1}$.

(6) More recently, H. Umemura [43, Chapter III.c] gave a solution of an algebraic equation of any degree n in terms of theta constants of a hyperelliptic curve of genus n, in other words, in terms of a Siegel modular function.

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