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AN ANALOGUE OF THE VARIATIONAL PRINCIPLE FOR GROUP AND PSEUDOGROUP ACTIONS

by Andrzej BIŚ

Abstract. — We generalize to the case of finitely generated groups of homeomorphisms the notion of a local measure entropy introduced by Brin and Katok [7] for a single map. We apply the theory of dimensional type characteristics of a dynamical system elaborated by Pesin [25] to obtain a relationship between the topological entropy of a pseudogroup and a group of homeomorphisms of a metric space, defined by Ghys, Langevin and Walczak in [12], and its local measure entropies. We prove an analogue of the Variational Principle for group and pseudogroup actions which allows us to study local dynamics of foliations.

Résumé. — On généralise au cas des groupes d'homéomorphismes de type fini la notion d'entropie mesure locale introduite par Brin et Katok [7] pour une seule transformation. On applique la théorie des caractéristiques de type dimension d'un système dynamique élaborée par Pesin [25] pour obtenir une relation entre l'entropie topologique d'un pseudogroupe et d'un groupe d'homéomorphismes d'un espace métrique, définie par Ghys, Langevin et Walczak dans [12], et ses entropies mesure locale. On prouve un analogue du principe variationnel pour les actions de groupe et de pseudogroupe qui nous permet d'étudier les dynamiques locales des feuilletages.

1. Introduction

A classical discrete-time dynamical system consists of a non-empty set X endowed with a structure and a cyclic group or a cyclic semigroup $G =< f >$ generated by a map $f : X \to X$ which preserves the structure of X. Topological dynamical system consists of topological space X and continuous map $f : X \to X$. A measure-preserving dynamical system is a probability space X with a measure-preserving transformation on it.

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A fundamental invariant of a continuous map $f : X \to X$ is its topological entropy $h_{\text{top}}(f)$ which measures the complexity of the system in the sense of the rate at which the action of the transformation disperses points. When the entropy is positive, it reflects some chaotic behavior of the map f.

It is known that a continuous map $f : X \to X$ determines an f-invariant measure μ and one can define a measure-theoretic entropy $h_\mu(f)$ with respect to μ. A relationship between topological entropy and measure-theoretic entropy of a map $f : X \to X$ is established by the Variational Principle, which asserts that

$$h_{\text{top}}(f) = \sup\{h_\mu(f) : \mu \in M(X, f)\}$$

i.e., topological entropy is equal to the supremum $h_\mu(f)$, where μ ranges over the set $M(X, f)$ of all f-invariant Borel probability measures on X. If an f-invariant Borel probability measure μ_0 on X satisfies the equality $h_{\text{top}}(f) = h_{\mu_0}(f)$ then it is called a maximal entropy measure. Measures of maximal entropy reflect the complexity of the dynamical systems and the subset where the dynamics concentrates.

We obtain a generalized dynamical system by exchanging the cyclic group $G =< f >$, generated by a single homeomorphism $f : X \to X$ of the metric space X, for a finitely generated group of homeomorphisms or by a pseudogroup of local homeomorphisms of a topological space X. Ghys, Langevin and Walczak notice in [12] that a foliation of a compact manifold defines a dynamics determined by a finitely generated holonomy pseudogroup of the foliation. We apply the notion of topological entropy $h_{\text{top}}(G, G_1)$ of a finitely generated group or a pseudogroup G generated by a finite symmetric set G_1 of homeomorphisms (resp. local homeomorphisms) of a compact metric space (X, d), introduced in [12]. If $s(n, \epsilon)$ denotes the maximal cardinality of any $(n, \epsilon)-$separated subset of X then

$$h_{\text{top}}(G, G_1) := \lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{\log(s(n, \epsilon))}{n}.$$

Recall that a subset $A \subset X$ is $(n, \epsilon)-$separated if for any two distinct points $x, y \in A$ there exists a map $g \in G$ such that g is a composition of at most n generators from G_1 and $d(g(x), g(y)) \geqslant \epsilon$.

We prove in Theorem 2.5 that any finitely generated group or pseudogroup admits a point which the entropy concentrates on. As a result, we are able to show that any two holonomy pseudogroups of a foliation of a compact manifold have simultaneously either positive or vanishing topological entropy. The problem of defining good measure-theoretical entropy for foliated manifolds which would provide an analogue of the Variational

Principle for geometric entropy of foliations is still open. In general, there are many examples of foliations that do not admit any non-trivial invariant measure. Even in a case when an invariant measure exists, it is not clear how to define its measure-theoretic entropy.

We introduce by Definition 4.5 a concept of a G-homogeneous measure which is a natural generalization of f-homogeneous measures considered by Bowen [5], for the case of a finitely generated group or a pseudogroup of homeomorphisms of a compact metric space. We prove in Theorem 4.12 that if a group, or a pseudogroup, admits a G-homogeneous measure then the G-homogeneous measure is the measure of maximal entropy.

Brin and Katok [7] consider a compact metric space (X, d) with a continuous mapping $f : X \to X$ preserving a Borel probability non-atomic measure m. They define a local measure entropy $h_m(f, x)$ of f with respect to m at a point $x \in X$ by

$$h_m(f, x) := \lim_{\delta \to 0} \liminf_{n \to \infty} \frac{-\log(m(B_n^f(x, \delta)))}{n},$$

where $B_n^f(x, \delta)$ denotes the d_n^f−ball centered at x of radius δ, with respect to the metric $d_n^f(x, y) := \max\{d(f^i(x), f^i(y)) : 0 \leqslant i \leqslant n-1\}$. They prove (Theorem 1 in [7]) that for m−almost every $x \in X$ the local entropy $h_m(f, x)$ is f-invariant and $\int_X h_m(f, x) dm = h_m(f)$.

Brin and Katok show in [7] the interrelations between a measure-theoretic entropy and dimension-like characteristics of smooth dynamical systems. Ma and Wen [20] apply a dimensional type characteristic of the entropy $h(f, Y)$ of $f : X \to X$ restricted to $Y \subset X$, in the sense of Bowen ([6]), to obtain the following relation between local measure entropies of f and the dimensional type entropy $h(f, Y)$ of f:

THEOREM 1.1 (Theorem 1 in [20]). — *Let μ be a Borel probability measure on X, E be a Borel subset of X and $0 < s < \infty$.*

1. *If $h_\mu(f, x) \leqslant s$ for all $x \in E$, then $h(f, E) \leqslant s$.*
2. *If $h_\mu(f, x) \geqslant s$ for all $x \in E$ and $\mu(E) > 0$, then $h(f, E) \leqslant s$.*

We generalize in Definition 4.9 the notion of local measure entropy for the case of a group or a pseudogroup of homeomorphisms of a metric space and we introduce an upper local measure entropy $h_\mu^G(x)$ and a lower local measure entropy $h_{\mu,G}(x)$ of a group G with respect to the measure μ. We apply the theory of C-structures, elaborated by Pesin in [25], to construct a dimensional type entropy-like invariant and we prove that it coincides with the topological entropy of groups and of pseudogroups. This approach allows us to obtain an analogue of the variational principle for group and pseudogroup actions which is stated in Theorem 5.2 and Theorem 5.3.

Theorem 5.2 relates the topological entropy of a homeomorphism group of a closed manifold to the upper local measure entropies with respect to the natural volume measure.

THEOREM 5.2. – *Let* (G, G_1) *be a finitely generated group of homeomorphisms of a compact closed and oriented manifold* (M, d). *Let* E *be a Borel subset of* M, $s \in (0, \infty)$ *and* μ_v *the natural volume measure on* M. *If*

$$h^G_{\mu_v}(x) \leqslant s \quad \text{for all } x \in E \quad \text{then} \quad h_{top}((G, G_1), E) \leqslant s.$$

Theorem 5.3 relates the topological entropy of a pseudogroup of a compact metric space to the common upper bound of lower local measure entropies with respect to a Borel probability measure on the space.

THEOREM 5.3. – *Let* (G, G_1) *be a finitely generated pseudogroup on a compact metric space* (X, d). *Let* E *be a Borel subset of* X *and* $s \in (0, \infty)$. *Denote by* μ *a Borel probability measure on* X. *If*

$$h_{\mu,G}(x) \geqslant s \quad \text{for all } x \in E \text{ and } \mu(E) > 0 \quad \text{then} \quad h_{top}((G, G_1), E) \geqslant s.$$

Theorem 5.2 and Theorem 5.3 are a generalization of Theorem 1 of Ma and Wen [20].

The concept of entropy of a finitely generated group is closely related to entropy of a finitely generated semigroup which appears both in real and complex dynamics. However, a few different definitions of entropy of a semigroup are known ([11], [8], [26], [3], [4]) and most of them are unrelated. For example, both Bufetov in [8] and Sumi in [26] apply the idea of skew-product transformations. They assign a skew-product transformation

$$F : \sum_m \times X \to \sum_m \times X$$

with a fibre X to the action of a semigroup $G =< f_1, ..., f_m >$ where each $f_i : X \to X$ is a continuous map. The base space $\sum_m := \{1, ..., m\}^{\mathbb{N}}$ consists of one-sided sequences of m-symbols endowed with a product topology. The transformation F is defined by

$$F(\omega, x) = (\sigma(\omega), f_{\omega_1} x),$$

where the shift map $\sigma : \sum_m \to \sum_m$ assigns $(\omega_1, \omega_2, ...) \to (\omega_2, \omega_3, ...)$. This method allows for reduction of the dynamics of semigroups to the dynamics of single transformations.

2. Finitely generated pseudogroup and its topological entropy

Given a topological space X, denote by Homeo(X) the family of all homeomorphisms between open subsets of X. For $g \in$ Homeo(X) denote by D_g its domain and by $R_g = g(D_g)$ its range.

Definition 2.1. — *A pseudogroup* Γ *on* X *is a collection of homeomorphisms* $h : D_h \to R_h$ *between open subsets* D_h *and* R_h *of* X *such that:*

1. *If* $g, f \in \Gamma$, *then* $g \circ f : f^{-1}(R_f \cap D_g) \to g(R_f \cap D_g)$ *is in* Γ.
2. *If* $g \in \Gamma$, *then* $g^{-1} \in \Gamma$.
3. $id_X \in \Gamma$.
4. *If* $g \in \Gamma$ *and* $W \subset D_g$ *is an open subset, then* $g|_W \in \Gamma$.
5. *If* $g : D_g \to R_g$ *is a homeomorphism between open subsets of* X *and if, for each point* $p \in D_g$, *there exists a neighbourhood* N *of* p *in* D_g *such that* $g|_N \in \Gamma$, *then* $g \in \Gamma$.

For any set $G \subset$ Homeo(X) which satisfies the condition

$$\bigcup_{g \in G} \{D_g \cup R_g : g \in G\} = X$$

there exists a unique smallest (in the sense of inclusion) pseudogroup $\Gamma(G)$ which contains G. Notice that $g \in \Gamma(G)$ if and only if $g \in$ Homeo(X) and for any $x \in D_g$ there exist maps $g_1, ..., g_k \in G$, exponents $e_1, ..., e_k \in \{-1, 1\}$ and an open neighbourhood U of x in X such that

$$U \subset D_g \text{ and } g|_U = g_1^{e_1} \circ ... \circ g_k^{e_k}|_U.$$

The pseudogroup $\Gamma(G)$ is said to be **generated by** G. If the set G is finite, then we say that $\Gamma(G)$ is **finitely generated**.

A concept of a pseudogroup is essential in the study of geometry and dynamics of foliated manifolds. As we know, the notion was introduced to foliation theory by Haefliger ([14] and [15]). The formal definition of a C^r-foliation, where $r = 1, 2, ..., \infty$, reads as follows:

Definition 2.2. — *A p-dimensional* C^r-*foliation* F *of codimension* q *on an n-dimensional manifold* M *is a decomposition of* M *into connected submanifolds* $\{L_\alpha\}_{\alpha \in A}$, *called leaves of the foliation* F, *such that for any point* $x \in M$ *there exist a neighbourhood* U *of x and a* C^r-*differentiable chart* $\phi = (\phi_1, \phi_2) : U \to \mathbb{R}^n = \mathbb{R}^p \times \mathbb{R}^q$ *and for any leaf* L_α *of* F *the connected components of* $L_\alpha \cap U$ *are described by the equation* $\phi_2 = const$.

The connected components of $L_\alpha \cap U$ *are called plaques.*

We say that a foliated manifold (M, F) admits a **"nice" foliated atlas** $\mathcal{A}$ (equivalently, the covering of M by the domains D_g of the charts $g \in \mathcal{A}$ is **"nice"**) if

1. the covering $\{D_g : g \in \mathcal{A}\}$ is locally finite,
2. for any chart $g \in \mathcal{A}$ the range $g(D_g) \subset \mathbb{R}^n$ is an open cube,
3. for any $g, h \in \mathcal{A}$ satisfying the condition $D_g \cap D_h \neq \emptyset$ there exists a chart f distinguished by the foliation F such that: $f(D_f)$ is an open cube, D_f contains the closure of $D_g \cup D_h$, and $g = f|_{D_g}$.

It is well known that any foliation of a compact manifold admits a finite "nice covering" and any nice covering $\mathcal{U}$ determines a finitely generated holonomy pseudogroup (see Chapter 1 in [29]).

Now, consider a finitely generated pseudogroup (G, G_1) acting on a compact metric space (X, d). Let G_1 be a finite symmetric generating set of G and

$$G_n := \{g_{i_1} \circ ... \circ g_{i_n} : g_{i_j} \in G_1\}.$$

Usually, it is assumed that $\mathrm{id}_X \in G_1$ which implies the inclusion $G_m \subset G_n$ for any $m \leqslant n$. We emphasize the generating set G_1 of the pseudogroup G writing (G, G_1) instead of G. Following [12] we say that two points $x, y \in E \subset X$ are $(n, \epsilon, E)-$separated by (G, G_1) if there exists $g \in G_n$ such that $x, y \in D_g$ and $d(g(x), g(y)) \geqslant \epsilon$. Let $s(n, \epsilon, E)$ denote the maximal number of $(n, \epsilon, E)-$separated points of E. The quantity

$$h_{\mathrm{top}}((G, G_1), E) := \lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log s(n, \epsilon, E)$$

is called the **topological entropy of G** restricted to E, with respect to G_1. The topological entropy $h_{\mathrm{top}}((G, G_1), E)$ can be defined not only in terms of $(n, \epsilon, E)-$separated sets but also in terms of $(n, \epsilon, E)-$spanning sets. We say that a set $F \subset E$ is (n, ϵ, E)-spanning whenever for any $x \in E$ there exists a point $y_0 \in F$ such that the inequality $d(g(x), g(y_0)) < \epsilon$ holds for any $g \in G_n$ such that $x, y_0 \in D_g$. The minimal cardinality of (n, ϵ, E)-spanning subset of E is denoted by $r(n, \epsilon, E)$. It is known (see [12] or [29]) that

$$\lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log s(n, \epsilon, E) = \lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log r(n, \epsilon, E).$$

Thus these two approaches to the topological entropy of pseudogroups are equivalent.

It is known ([12]) that that the topological entropy of a finitely generated pseudogroup depends on the generating set. However, Lemma 2.3 ([12]). — *If G_1 and G_1' are two generating sets of the same pseudogroup G and the topological entropy $h_{top}(G, G_1) = 0$, then the topological entropy $h_{top}(G, G_1')$ is vanishing as well.*

Lemma 2.3 shows that one can distinguish pseudogroups with zero entropy from non-zero topological entropy. In a similar way, we intend to present a distinction of foliations with positive topological entropy from those with vanishing entropy. To this end, we take a nice atlas $\mathcal{A}_1$ of a p-dimensional foliation F of a compact manifold M. Compactness of M allows us to consider a finite and nice subatlas $\mathcal{A}$ of $\mathcal{A}_1$. Then, for any $g \in \mathcal{A}$ the range $g(D_g)$ in an open cube in $\mathbb{R}^n$, so it can be written in the form $U_1(g) \times U_2(g)$, where $U_1(g)$ (resp., $U_2(g)$) is an open cube in $\mathbb{R}^p$ (resp., in $\mathbb{R}^q$). Therefore, each $U_1(g) \times \{y_0\}$, where $y_0 \in U_2(g)$, is isomorphic to a plaque in D_g and the points of $U_2(g)$ parametrize the plaques in D_g. If we select a point $x_0 \in U_1(g)$, then $g^{-1}(\{x_0\} \times U_2(g))$ is a submanifold of D_g which intersects every plaque of D_g exactly once. The submanifold $g^{-1}(\{x_0\} \times U_2(g))$ is called a **local transversal** T_g of D_g.

The domains of the finite atlas $\mathcal{A}$ constitute a finite nice covering $\mathcal{U} = \{U_1, ..., U_k\}$ of M which determines a finitely generated holonomy pseudogroup $(H_{\mathcal{U}}, H_{\mathcal{U},1})$ described below.

Pick $U_i, U_j \in \mathcal{U}$ and choose local transversals $T_i \subset U_i$ and $T_j \subset U_j$. Assume that $U_i \cap U_j \neq \emptyset$, then for any $x \in U_i \cap U_j$ there exists a unique plaque $P_i(x) \in U_i$ and there exists a unique plaque $P_j(x) \in U_j$ such that $x \in P_i(x) \cap P_j(x)$. The map h_{U_i,U_j} transforming a plaque $P_i(x)$ onto $P_j(x)$ is called a **local holonomy transformation**. Since T_i intersects every plaque of U_i exactly once, we may identify a plaque $P_i(x)$ with $T_i \cap P_i(x)$ and view at h_{U_i,U_j} as a map defined on an open subset of T_i with range in T_j. The finite set

$$H_{\mathcal{U},1} = \{h_{U_i,U_j} : U_i, U_j \in \mathcal{U} \text{ and } U_i \cap U_j \neq \emptyset\}$$

generates a pseudogroup $H_{\mathcal{U}}$ called the **holonomy pseudogroup** (determined by the nice covering $\mathcal{U}$).

Definition 2.4. — *We say that a finitely generated pseudogroup (G, G_1) acting on a compact metric space (X, d) admits an* **entropy point** *x_0 if for any open neighbourhood U of x_0 the inequality $h_{top}((G, G_1), \overline{U}) = h_{top}((G, G_1), X)$ holds.*

Theorem 2.5. — *For any finitely generated pseudogroup $(G, G_1) \subset$ Homeo(X), where X is a compact metric space, there exists a point $x_0 \in X$*

and an arbitrary small open neighbourhood U of x_0 such that

$$h_{top}((G,G_1),X) = h_{top}((G,G_1),\overline{U}).$$

Proof. — Let (G,G_1) act on a compact metric space (X,d). The result is obvious if $h_{\mathrm{top}}((G,G_1),X)=0$. Assume that $h_{\mathrm{top}}(G,G_1)>0$ and denote by $B^k(x)$ a closed ball in X, centered at x of radius $r=1/k$. Let

$$X \subset B^k(x_1) \cup B^k(x_2) \cup \ldots \cup B^k(x_m)$$

for some points $x_1,x_2,\ldots,x_m \in X$. Fix $\epsilon>0$. By definition $s(n,\epsilon,X) \leqslant s(n,\epsilon,B^k(x_1)) + \ldots + s(n,\epsilon,B^k(x_m))$. Notice that for any positive integer n there exists $i(n,\epsilon)\in\mathbb{N}$ such that

$$s(n,\epsilon,B^k(x_{i(n,\epsilon)})) = \max\{s(n,\epsilon,B^k(x_j)) : j=1,2,\ldots,m\}.$$

Therefore, $s(n,\epsilon,X) \leqslant m\cdot s(n,\epsilon,B^k(x_{i(n,\epsilon)}))$.

Choose an increasing sequence of integers $\{n_j\}_{j\in\mathbb{N}}$ such that the sequence $\{\frac{1}{n_j}\log s(n_j,\epsilon,X)\}_{j\in\mathbb{N}}$ tends to $\limsup_{n\to\infty}\frac{1}{n}\log s(n,\epsilon,X)$ with $j\to\infty$. At least one element of the set $\{B^k(x_1),B^k(x_2),\ldots B^k(x_m)\}$ appears infinitely many times in the infinite sequence $\{B^k(x_{i(n_j,\epsilon)})\}_{j\in\mathbb{N}}$, say $B^k(x_{i^*})$. The ball $B^k(x_{i^*})$ certainly depends on ϵ, therefore we write $B^k(x_{i^*}) = B^k(x_{i^*}(\epsilon))$. Again choosing a subsequence of the sequence $\{n_j\}_{j\in\mathbb{N}}$, for simplicity denoting it again by $\{n_j\}_{j\in\mathbb{N}}$, we may assume that $B^k(x_{i(n_j,\epsilon)})) = B^k(x_{i^*}(\epsilon))$ for any $j\in\mathbb{N}$. It yields

$$\begin{aligned}(2.1)\qquad \lim_{j\to\infty}\frac{1}{n_j}\log s(n_j,\epsilon,X) &\leqslant \lim_{j\to\infty}\frac{1}{n_j}\log s(n_j,\epsilon,B^k(x_{i(n_j,\epsilon)})) \\ &= \lim_{j\to\infty}\frac{1}{n_j}\log s(n_j,\epsilon,B^k(x_{i^*}(\epsilon))).\end{aligned}$$

Now, take a sequence $\{\epsilon_p\}_{p\in\mathbb{N}}$ of positive real numbers, convergent to zero.

At least one ball of the set $\{B^k(x_1),B^k(x_2),\ldots B^k(x_m)\}$, say $B^k(x_*)$, appears infinitely many times in the infinite sequence $\{B^k(x_{i^*})(\epsilon_p)\}_{p\in\mathbb{N}}$, so taking a subsequence $\{\epsilon_{p_l}\}_{l\in\mathbb{N}}$ we get the equality $B^k(x_{i^*}(\epsilon_{p_l})) = B^k(x_*)$, which holds for any $l\in\mathbb{N}$. By (2.1) we conclude that

$$\begin{aligned}h_{\mathrm{top}}((G,G_1),X) &= \lim_{l\to\infty}\lim_{j\to\infty}\frac{1}{n_j}\log s(n_j,\epsilon_{p_l},X) \\ &\leqslant \lim_{l\to\infty}\lim_{j\to\infty}\frac{1}{n_j}\log s(n_j,\epsilon_{p_l},B^k(x_*)) = h_{\mathrm{top}}((G,G_1),B^k(x_*)).\end{aligned}$$

The inequality $h_{\mathrm{top}}((G,G_1),B^k(x_*)) \leqslant h_{\mathrm{top}}((G,G_1),X)$ is obvious. □

Corollary 2.6. — *A pseudogroup (G,G_1) admits an entropy point.*

Consider two pseudogroups (G, G_1) and (H, H_1) acting on topological spaces X and Y, respectively. Following Haefliger ([16]) we say that an **étale morphism** $\Phi : G \to H$ is a maximal collection Φ of homeomorphisms of open subsets of X to open subsets of Y such that:

(1) If $\phi \in \Phi$, $g \in G$ and $h \in H$, then $h \circ \phi \circ g \in \Phi$,
(2) domains D_ϕ of the elements of Φ form a covering of X, and
(3) if $\phi, \psi \in \Phi$, then $\phi \circ \psi^{-1} \in H$.

An étale morphism Φ is called an **equivalence** if the collection $\Phi^{-1} = \{\phi^{-1} : \phi \in \Phi\}$ is also an étale morphism of H into G. We say that an étale morphism $\Phi : G \to H$ is **generated** by a subset $\Phi_0 \subset \Phi$ if

$$\Phi = \{h \circ \phi \circ g : g \in G,\ h \in H,\ \phi \in \Phi_0\}.$$

Finally, the pseudogroups (G, G_1) and (H, H_1) are said to be **equivalent** if there exists an equivalence $\Phi : G \to H$. Moreover, G and H are **finitely equivalent** if the equivalence $\Phi : G \to H$ is generated by a finite collection Φ_0.

Holonomy pseudogroups, acting on different transversals, of a given foliation $\mathcal{F}$ are equivalent. Moreover, they are finitely equivalent when the foliated space under consideration is compact.

Proposition 2.7. — *Let* (G,G_1) *and* (H,H_1) *be holonomy pseudogroups which correspond to nice coverings* $\mathcal{U}$ *and* $\mathcal{W}$ *of a compact foliated manifold* (M, F). *Then the inequality* $h_{top}(G, G_1) > 0$ *implies that* $h_{top}(H, H_1) > 0$.

Proof. — Take an entropy point x_* of (G, G_1) and an equivalence $\Phi : G \to H$. Due to the compactness of (M, F) the equivalence is generated by a finite collection Φ_0. Choose $\phi_0 \in \Phi_0$ such that x_* belongs to the domain D_{ϕ_0} of ϕ_0. Take an open neighbourhood U of x_* which closure $\overline{U} \subset D_{\phi_0}$. Certainly, $h_{\text{top}}((G, G_1), \overline{U}) > 0$. Denote by S a symmetric set of generators of G that is closed under compositions. We may also assume (remark (ii) of Definition 8.4 in [1]) that S is closed under restrictions to open sets, thus each $g \in G_1$ is a composition of maps from S.

Using the same arguments as in the proof of Lemma 8.8 in [1], it is ascertained that the set

$$S' := \{\phi \circ g \circ \psi^{-1} : g \in S, \phi, \psi \in \Phi_0\}$$

is symmetric, generates H_1 and is closed under compositions. In particular, (G, G_1) restricted to U is conjugate by ϕ_0 to a pseudogroup P generated by

$$S'' := \{\phi_0 \circ g \circ \phi_0^{-1} : g \in S\}$$

which is a subpseudogroup of (H, H_1). Notice that $\phi_0(x_*)$ is an entropy point of (P, S''), thus $h_{\mathrm{top}}(H, H_1) \geqslant h_{\mathrm{top}}(P, S'') > 0$, which completes the proof. □

Therefore, we can distinguish foliations of compact foliated manifolds with vanishing entropy from those with non-vanishing entropy.

3. Topological entropy and Hausdorff dimension

The notion of topological entropy can be introduced similarly to the definition of Hausdorff dimension. We briefly recall this notion, one can find a detailed introduction to Hausdorff dimension and its properties in [10] or in [22].

A countable collection of subsets $U_i \subset \mathbb{R}^n$ is called a $\delta-$cover of a set $E \subset \mathbb{R}^n$ if for any i the diameter $\mathrm{diam}(U_i) \leqslant \delta$ and E is covered by the union of U_i. Let $\mathbb{I}$ denotes the family of all subsets of $\mathbb{N}$. For a subset $E \subset \mathbb{R}^n$, $s \geqslant 0$ and $\delta > 0$ we define

$$\mathcal{H}^s_\delta(E) = \inf \left\{ \sum_{i \in I} [\mathrm{diam}(U_i)]^s : (U_i)_{i \in I} \text{ is a } \delta - \text{cover of } E, I \in \mathbb{I} \right\}. \tag{3.1}$$

As δ decreases, the collection of $\delta-$covers of E is reduced, thus the infimum increases and approaches a limit with δ tending to 0.

Definition 3.1. — *The quantity*

$$\mathcal{H}^s(E) = \lim_{\delta \to 0} \mathcal{H}^s_\delta(E)$$

is called the **s-dimensional Hausdorff measure** *of E.*

Definition 3.2. — *The real number* $\dim_H(E)$, *called the* **Hausdorff dimension** *of E, is such that* $\mathcal{H}^s(E) = \infty$ *if* $s < \dim_H(E)$ *and* $\mathcal{H}^s(E) = 0$ *if* $s > \dim_H(E)$.

A direct conclusion is obtained from the above definition

$$\dim_H(E) = \inf\{s : \mathcal{H}^s(E) = 0\} = \sup\{s : \mathcal{H}^s(E) = \infty\}.$$

3.1. Carathéodory dimension structure

In this section, we present a general approach to a construction of $\alpha-$measures on a metric space, elaborated by Pesin [24], which is a generalization of the Hausdorff measure and the classical Carathéodory construction. Pesin introduced axiomatically a structure, called the Carathéodory structure (or C-structure), by describing its elements and relation between them.

Let X be a set and F a collection of subsets of X. Following Pesin [25] we assume that there exist two set functions $\eta, \psi : F \to \mathbb{R}_+$ satisfying the following conditions:

- **A1.** $\emptyset \in F$ and $\eta(\emptyset) = 0 = \psi(\emptyset)$; for any non-empty $U \in F$ we get $\eta(U) > 0$ and $\psi(U) > 0$.
- **A2.** For any $\delta > 0$ there exists $\epsilon > 0$ such that $\eta(U) \leqslant \delta$ for any $U \in F$ with $\psi(U) \leqslant \epsilon$.
- **A3.** For any $\epsilon > 0$ there exists a finite or countable subcolection $G \subset F$ which covers X and $\psi(G) := \sup\{\psi(U) : U \in G\} \leqslant \epsilon$.

Definition 3.3. — *Let $\xi : F \to \mathbb{R}_+$ be a set function. We say that the collection of subsets F and the set functions ξ, η, ψ satysfying conditions A1, A2 and A3, introduce a* **Carathéodory dimension structure** *or C-***structure** *τ on X and we write $\tau = (F, \xi, \eta, \psi)$.*

Now, consider a set X endowed with a C-structure $\tau = (F, \xi, \eta, \psi)$. For any subset $Z \subset X$, real number α and $\epsilon > 0$ we define

$$M_C(Z, \alpha, \epsilon) := \inf_G \left\{ \sum_{U \in G} \xi(U) \cdot \eta(U)^\alpha \right\},$$

where the infimum is taken over all finite or countable subcollections $G \subset F$ which cover Z and satisfy the condition $\psi(G) \leqslant \epsilon$. Therefore, the limit $m_C(Z, \alpha) = \lim_{\epsilon \to 0} M_C(Z, \alpha, \epsilon)$ exists.

The set function $m_C(\cdot, \alpha)$ becomes an outer measure on X, according to the general measure theory it induces a $\sigma-$additive measure called the α-**Carathéodory measure**. Moreover

Lemma 3.4 (Proposition 1.2 in [25]). — *There exists a critical value α_C, $-\infty \leqslant \alpha_C \leqslant \infty$ such that $m_C(Z, \alpha) = \infty$ for $\alpha \leqslant \alpha_C$ and $m_C(Z, \alpha) = 0$ for $\alpha > \alpha_C$.*

The **Carathéodory dimension of a set** $Z \subset X$ with respect to the C-structure τ, is defined as follows

$$\dim_{C,\tau} Z = \alpha_C = \inf\{\alpha : m_C(Z, \alpha) = 0\}.$$

3.2. Carathéodory capacity of sets

Assume that a C-structure $\tau = (F, \xi, \eta, \psi)$ satisfies Condition A3. It is useful to require a slightly stronger condition. Pesin (p. 16 in [25]) introduced another type of Carathéodory dimension characteristic of a set and defined A3' condition as follows:

A3'. There exists $\epsilon_0 > 0$ such that for any $\epsilon \in (0, \epsilon_0)$ one can find subcollection $G \subset F$ covering X such that $\psi(U) = \epsilon$ for any $U \in G$.

It is clear that Condition A3' is stronger than Condition A3. For any subset $Z \subset X$, real number α and $\epsilon > 0$ we define

$$R_C(Z, \alpha, \epsilon) := \inf_G \left\{ \sum_{U \in G} \xi(U) \cdot \eta(U)^\alpha \right\},$$

where the infimum is taken over all finite or countable subcollections $G \subset F$ which cover Z and satisfy the condition $\psi(U) = \epsilon$ for all $U \in G$. Due to A3' the quantity $R_C(Z, \alpha, \epsilon)$ is well defined, it yields the existence of the limits

$$\underline{r}_C(Z, \alpha) = \underline{\lim}_{\epsilon \to 0} R_C(Z, \alpha, \epsilon) \text{ and } \overline{r}_C(Z, \alpha) = \overline{\lim}_{\epsilon \to 0} R_C(Z, \alpha, \epsilon).$$

The behaviour of $\underline{r}_C(\cdot, \alpha)$ and $\overline{r}_C(\cdot, \alpha)$ is described by the following result.

Proposition 3.5 (Proposition 2.1 in [25]). — *For any $Z \subset X$, there exist $\underline{\alpha}_C, \overline{\alpha}_C \in \mathbb{R}$ such that*

1. *$\underline{r}_C(Z, \alpha) = \infty$ for $\alpha < \underline{\alpha}_C$ and $\underline{r}_C(Z, \alpha) = 0$ for $\alpha > \underline{\alpha}_C$;*
2. *$\overline{r}_C(Z, \alpha) = \infty$ for $\alpha < \overline{\alpha}_C$ and $\overline{r}_C(Z, \alpha) = 0$ for $\alpha > \overline{\alpha}_C$.*

Given $Z \subset X$, the **lower** and the **upper Carathéodory capacities** of a set Z are defined by

$$\underline{\mathrm{Cap}}_C Z = \underline{\alpha}_C = \inf\{\alpha : \underline{r}_C(Z, \alpha) = 0\} = \sup\{\alpha : \underline{r}_C(Z, \alpha) = \infty\};$$
$$\overline{\mathrm{Cap}}_C Z = \overline{\alpha}_C = \inf\{\alpha : \overline{r}_C(Z, \alpha) = 0\} = \sup\{\alpha : \overline{r}_C(Z, \alpha) = \infty\}.$$

The upper Carathéodory capacity of a set has the following property.

Lemma 3.6 (Theorem 2.1 in [25]). — *If $Z_1 \subset Z_2 \subset X$, then*

$$\overline{\mathrm{Cap}}_C Z_1 \leqslant \overline{\mathrm{Cap}}_C Z_2.$$

We will use the following properties of the lower and the upper Carathéodory capacities of a set. For $\epsilon > 0$ and any $Z \subset X$ we define

$$\Lambda(Z, \epsilon) := \inf_G \left\{ \sum_{U \in G} \xi(U) \right\},$$

where the infimum is taken over all finite or countable subcollection $G \subset F$ covering Z for which the condition $\psi(U) = \epsilon$ holds for all $U \in G$.

Let us assume that the set function η satisfies the following condition:

A4. $\eta(U_1) = \eta(U_2)$ for any $U_1, U_2 \in F$ for which $\psi(U_1) = \psi(U_2)$.

Then, the lower and upper Carathéodory capacities have the following properties.

Lemma 3.7 (Theorem 2.2 in [25]). — *If the set function η satisfies Condition A4, then for any $Z \subset X$*

$$\overline{\mathrm{Cap}}_C Z = \overline{\lim}_{\epsilon\to 0} \frac{\log \Lambda(Z,\epsilon)}{\log(\frac{1}{\eta(\epsilon)})} \quad \textit{and} \quad \underline{\mathrm{Cap}}_C Z = \underline{\lim}_{\epsilon\to 0} \frac{\log \Lambda(Z,\epsilon)}{\log(\frac{1}{\eta(\epsilon)})}.$$

Lemma 3.8 (Theorem 2.4 in [25]). — *Under Condition A4 the equality*

$$\overline{\mathrm{Cap}}_C(Z_1 \cup Z_2) = \max\{\overline{\mathrm{Cap}}_C(Z_1), \overline{\mathrm{Cap}}_C(Z_2)\}$$

holds for any subsets $Z_1, Z_2 \subset X$.

3.3. C-structures and topological entropy of a pseudogroup

An important application of C-structures is to illustrate the relationship between topological entropy and dimensional characteristic of a dynamical system. We apply Pesin's theory to a finitely generated pseudogroup (H, H_1) acting on a compact metric space (X, d) to describe its topological entropy. To this end, we construct a C-structure determined by (H, H_1) acting on X. First, recall that an n-ball of radius r, centered at $x \in X$, is defined by

$$B_n^H(x,r) := \{y \in X : d(h(x), h(y)) < r \text{ for any } h \in H_{n-1} \text{ such that } x, y \in D_h\}.$$

Fix $\delta > 0$. Define the collection F_δ of subsets of X by

$$F_\delta = \{B_n^H(x,\delta) : x \in X, n \in \mathbb{N}\}$$

and three set functions $\xi, \eta, \psi : F_\delta \to \mathbb{R}$ as follows

$$\xi(B_n^H(x,\delta)) \equiv 1,\ \eta(B_n^H(x,\delta)) = \exp(-n),\ \psi(B_n^H(x,\delta)) = \frac{1}{n}. \tag{3.2}$$

It is easy to verify that F_δ and three set functions ξ, η, ψ satisfy conditions A1, A2, A3 and A3', therefore they determine a C-structure $\Gamma_\delta = (F_\delta, \xi, \eta, \psi)$ on X.

The Carathéodory function $\overline{r}_C(Z,\alpha,\delta)$, where $Z \subset X$ and $\alpha \in \mathbb{R}$, depends on the covering F_δ and is given by

$$\begin{aligned}&\overline{r}_C(Z,\alpha,\delta)\\ &\quad = \limsup_{N\to\infty} \inf_G \left\{ \sum_{B_N^H(x,\delta)\in G} e^{-\eta(B_N^H(x,\delta))\cdot\alpha} : Z \subset \bigcup_{B_N^H(x,\delta)\in G} B_N^H(x,\delta) \right\}.\end{aligned}$$

The C-structure Γ_δ generates an upper Carathéodory capacity of Z, denoted here by $\overline{CP}_Z(\delta)$, specified by the covers F_δ and the pseudogroup (H,H_1). We have that

$$\overline{CP}_Z(\delta) = \inf\{\alpha : \overline{r}_C(Z,\alpha,\delta) = 0\} = \sup\{\alpha : \overline{r}_C(Z,\alpha,\delta) = \infty\}.$$

By Theorem 11.1 in [25] the limit $\overline{CP}_Z := \lim_{\delta\to 0} \overline{CP}_Z(\delta)$ exists. Notice that the functions η and ψ satisfy Condition A4, therefore by Lemma 3.6 and Theorem 11.1 in [25] we obtain.

Lemma 3.9. — *For any $Z \subset X$ there exists a limit*

$$\overline{CP}_Z := \lim_{\delta\to 0} \limsup_{N\to\infty} \frac{1}{N} \log \Lambda(Z,\delta,N),$$

where $\Lambda(Z,\delta,N) = \inf_G\{\mathrm{card}(G)\}$ and the infimum is taken over all finite or countable collections $G \subset F_\delta$ of N-balls such that $Z \subset \bigcup_{B_N^H(x,\delta)\in G} B_N^H(x,\delta)$.

Corollary 3.10. — *For any pseudogroup (H,H_1) acting on X and $Z \subset X$ we get*

$$\overline{CP}_Z = h_{top}((H,H_1),Z).$$

Proof. — One can directly verify that $\Lambda(Z,\delta,N)$ coincides with $s(N,\delta,Z)$, the maximal cardinality of $(N,\delta,Z)-$separated subset of Z, with respect to (H,H_1). The claim follows from the definition of the topological entropy of (H,H_1) restricted to Z. □

It is known (see [29]) that the topological entropy of any finitely generated group of homeomorphisms H (w.r. t. H_1), of a compact metric space X, coincides with the entropy of the pseudogroup generated by H_1. Therefore:

Corollary 3.11. — *For a finitely generated group (H,H_1) of homeomorphisms of a compact metric space (X,d) and a subset $Z \subset X$ we have*

$$\overline{CP}_Z = h_{top}((H,H_1),Z).$$

4. Local measure entropy of a pseudogroup

4.1. G-homogeneous measure

Let (G, G_1) be a finitely generated pseudogroup acting on a compact metric space (X, d) equipped with a Borel measure μ. Denote by $B_d(x, r)$ an open ball with center $x \in X$ and radius r. Our main goal here is to show the relationship between local μ-measure entropy of (G, G_1) and the topological entropy $h_{\text{top}}(G, G_1)$.

Definition 4.1. — *A Borel probability measure m defined on a compact metric space (X, d) satisfies the* **doubling property** *provided that there exists a constant $D > 0$ such that*

$$m(B_d(x, 2 \cdot r)) < D \cdot m(B_d(x, r)) \tag{4.1}$$

for all $x \in X$ and $r > 0$. We say that m is a **doubling measure**.

It is known (Theorem 5.2.2 in [2]) that the doubling property of the measure m implies the density lower bound, i. e. there are constants $C > 0$ and $s > 0$ such that the inequality

$$\frac{m(B_d(x, r))}{m(B_d(y, R))} \geqslant C \cdot \left(\frac{r}{R}\right)^s$$

holds for all $0 < r < R < \infty$ and all $x \in B_d(y, R)$.

Definition 4.2. — *A metric space (X, d) has the* **doubling property** *if any ball $B(x, 2r)$ in X may be covered by finitely many, say $N(x, r)$ balls of radius r, and there exists a finite upper bound N of the set $\{N(x, r) : x \in X \text{ and } r \in \mathbb{R}\}$ which is independent of x and r.*

Coifman and Weiss [9] observed that a space admitting a doubling measure has the doubling property, Vol'berg and Konyagin [27], [28] proved that any compact subset of $\mathbb{R}^n$ with induced metric admits a doubling measure. In 1998 Luukkainen and Saksman [19] showed that every complete doubling metric space carries a doubling measure.

An important class of doubling measures is formed by so called s-regular measures. For an s-regular measure there exist $C > 0$ and $s > 0$ such that the condition

$$\frac{1}{C} r^s \leqslant \mu(B(x, r)) \leqslant C \cdot r^s$$

holds for $x \in X$ and $0 < r < \operatorname{diam}(X)$. The s-regular measures are closely related to the Hausdorff measure H^s due to the following result.

Proposition 4.3 (Theorem 4.6 in [17]). — *If μ is an s-regular measure, then there is a constant $C \geqslant 1$ so that $C^{-1}\mu(E) \leqslant H^s(E) \leqslant C\mu(E)$ for every $E \subset X$. In particular H^s is s-regular too.*

Now, we consider a sequence of metrics d_n on a set X and assume that all of (X, d_n) carry a common doubling measure μ, *i.e.*, there exists a sequence $\{D_n\}_{n\in\mathbb{N}}$ of positive numbers such that for any $n \in \mathbb{N}$

$$\mu(B_{d_n}(x, 2 \cdot r)) < D_n \cdot \mu(B_{d_n}(x, r)) \tag{4.2}$$

for all $x \in X$ and $r > 0$. If there exists a finite upper bound D for the sequence $\{D_n\}_{n\in\mathbb{N}}$, the asymptotic behavior of the sequence of metric measure spaces (X, d_n, μ) seems to be interesting. A dynamical system $f : X \to X$ determines a sequence of metrics

$$d_n(x, y) = \{d(f^i(x), f^i(y)) : 0 \leqslant i \leqslant n - 1\},$$

then an open ball $B_{d_n}(x, r)$ has the following form

$$B_{d_n}(x, r) = \bigcap_{i=0}^{n-1} f^{-i} B_d(f^i(x), r).$$

For simplicity we write $B_n(x, r) := B_{d_n}(x, r)$.

Definition 4.4 (Definition 6 in [5]). — *We say that a Borel probability measure μ on X is a* **homogeneous measure** *with respect to a dynamical system $f : X \to X$ if:*

1. *there exists $E_0 \subset X$ with $\mu(E_0) > 0$ and*
2. *for any $\epsilon > 0$ there exist $\delta > 0$ and $c > 0$ such that the inequality*

$$\mu(B_n(y, \delta)) \leqslant c \cdot \mu(B_n(x, \epsilon))$$

holds for all $n \in \mathbb{N}$ and all $x, y \in X$.

Bowen [5] observed that Haar measures on some homogeneous spaces are invariant under affine transformations and have such properties. The reader may also find the general approach to homogeneous measures in analysis and geometry of metric measure spaces with no priori smooth structure in [18] or [2]). The notion of a homogeneous measure (or f-homogeneous measure) was very fruitful in dynamics of a single continuous map $f : X \to X$. It can be adopted to a finitely generated pseudogroup (G, G_1) of a metric space (X, d). Let

$$B_n^G(x, \epsilon) := \bigcap_{g\in G_n^x} g^{-1} B_d(g(x), \epsilon),$$

where $B_d(z, r) = \{y \in X : d(z, y) < r\}$ and $G_n^x := \{g \in G_n : x \in D_g\}$.

DEFINITION 4.5. — *We say that a Borel measure* μ *on a metric space* (X,d) *is* $G-$*homogeneous with respect to a finitely generated pseudogroup* (G,G_1) *if*

1. $\mu(K)<\infty$ *for any compact* $K\subset X$,
2. *there exists a compact* $K_0\subset X$ *such that* $\mu(K_0)>0$, *and*
3. *for any* $\epsilon>0$ *there exist* $\delta>0$ *and* $c>0$ *such that the inequality*
$$\mu(B_n^G(y,\delta))\leqslant c\cdot\mu(B_n^G(x,\epsilon))$$
holds for all $n\in\mathbb{N}$ *and all* $x,y\in X$.

4.2. Examples of G-homogeneous measures

We describe two examples of G-homogeneous measures.

1) The canonical volume form dV on a closed, compact and oriented Riemannian manifold M, determines a G-homogeneous measure μ with respect to a finitely generated group G of isometries of M. Indeed, for $\mu(A):=\int_A dV$ and finitely generated group G of isometries of M we get

$$B_n^G(x,\epsilon)=\bigcap_{g\in G_n}g^{-1}(B(g(x),\epsilon))=B(x,\epsilon).$$

Since M is compact, then for any $\delta<\epsilon$, arbitrary $n\in\mathbb{N}$ and $x,y\in M$, we have

$$\mu(B_n^G(y,\delta))\leqslant C\cdot\mu(B_n^G(x,\epsilon)),$$

where

$$C=\frac{\sup\{\mu(B(z,\epsilon)):z\in M\}}{\inf\{\mu(B(z,\epsilon)):z\in M\}}.$$

2) Let X be a locally compact topological group endowed with a right invariant Haar measure μ. It is known (see [23]) that X admits a right invariant metric d. Choose a homeomorphisms $A:X\to X$ which is an isomorphism of the group X onto itself. Fix $g_1,g_2,...,g_k\in X$ and denote by $T_i=R_{g_i}\circ A$, where $R_{g_i}(x)=x\cdot g_i$, for $x\in X$ and $i=1,2,...k$. We claim that:

PROPOSITION 4.6. — *The group* G *generated by the finite set of homeomorphisms* $G_1=\{id_X,T_1,T_1^{-1},T_2,T_2^{-1},...,T_k,T_k^{-1}\}$ *of the locally compact topological group* X, *admits* μ *as its G-homogeneous measure.*

To prove the claim we need two auxiliary lemmas. Let e stand for the identity element of X.

Lemma 4.7. — *For each* $T_i \in G_1, x \in X$ *and* $r > 0$ *the equality*

$$T_i^{-1}(B(T_i(x), r)) = A^{-1}[B(e, r)] \cdot x$$

holds.

Proof. — Choose any two points $x, y \in X$ and let $y' := A^{-1}(y)$, then

$$A^{-1}[y \cdot A(x)] = A^{-1}[A(y' \cdot x)] = A^{-1}(y) \cdot x. \tag{4.3}$$

Notice that due to the right invariant metric d we get the second equality

$$T_i^{-1}[B(T_i(x), r)] = A^{-1}\{R_{g_i}^{-1}[B(A(x) \cdot g_i, r)]\} = A^{-1}[B(A(x), r)].$$

Since A is the group homomorphism and d is the right invariant metric, we obtain

$$\begin{aligned} A^{-1}[B(A(x), r)] &= A^{-1}[B(A(x) \cdot A(e)), r)] = A^{-1}[B(A(e), r) \cdot A(x)] \\ &= A^{-1}[B(e, r)] \cdot x, \end{aligned}$$

where the last equality is due to (4.3). ☐

Lemma 4.8. — *For any* $T_i, T_j \in G_1$ *the equality*

$$(T_i \circ T_j)^{-1}(B((T_i \circ T_j)(x), r)) = A^{-2}[B(e, r)] \cdot x$$

holds for all $x \in X$ *and* $r > 0$.

Proof. — Applying Lemma 4.7 we arrive at

$$\begin{aligned} &(T_i \circ T_j)^{-1}\{B((T_i \circ T_j)(x), r)\} = T_j^{-1}\{T_i^{-1}[B((T_i(T_j(x)), r)]\} \\ &= T_j^{-1}\{A^{-1}[B(e, r)] \cdot T_j(x)\} = (A^{-1} \circ R_{g_j}^{-1})\{A^{-1}[B(e, r)] \cdot A(x) \cdot g_j\} \\ &= A^{-1}\{A^{-1}[B(e, r)] \cdot A(x)\} = A^{-2}[B(e, r)]x \end{aligned}$$

which proves our claim. ☐

Proof of Proposition 4.6. — For any $T \in G$ we write $\operatorname{ord}(T) = m$ if and only if $m = \min\{n : T \in G_n \setminus G_{n-1}\}$. In the view of Lemma 4.8 we get

$$B_n^G(x, r) = \bigcap_{T \in G_n} T^{-1}[B(T(x), r)] = \bigcap_{T \in G_n} A^{-\operatorname{ord}(T)}[B(e, r)] \cdot x$$

Thus the right invariance of the Haar measure μ yields

$$\mu[B_n^G(x, r)] = \mu\left\{\bigcap_{T \in G_n} A^{-\operatorname{ord}(T)}[B(e, r)]\right\},$$

so for any points $x, y \in X$ we have

$$\mu[B_n^G(x, r)] = \mu[B_n^G(y, r)].$$

The proof is complete. ☐

4.3. G-homogeneous measures and topological entropy

Brin and Katok [7] introduced a notion of the local measure entropy for a single continuous map $f : X \to X$. We adapt a notion of the local measure entropy to a finitely generated pseudogroup (G, G_1) acting on X in the following way:

Definition 4.9. — *For any $x \in X$ and a Borel probability measure μ on X the quantity*

$$h_\mu^G(x) = \lim_{\epsilon\to 0} \limsup_{n\to\infty} -\frac{1}{n} \log \mu(B_n^G(x,\epsilon))$$

is called a **local upper $\mu-$measure entropy** *at the point x, with respect to (G, G_1), while the quantity*

$$h_{\mu,G}(x) = \lim_{\epsilon\to 0} \liminf_{n\to\infty} -\frac{1}{n} \log \mu(B_n^G(x,\epsilon))$$

is called a **local lower $\mu-$measure entropy** *at the point x, with respect to (G, G_1).*

Lemma 4.10. — *If μ is a G-homogeneous measure on X, then the equalities $h_\mu^G(x) = h_\mu^G(y)$ and $h_{\mu,G}(x) = h_{\mu,G}(y)$ hold for any $x, y \in X$.*

Proof. — By definition of a G-homogeneous measure, for $\epsilon > 0$ there exist $0 < \delta(\epsilon) < \epsilon$ and $c > 0$ such that

$$\mu(B_n^G(y,\delta(\epsilon))) \leqslant c \cdot \mu(B_n^G(x,\epsilon)).$$

Thus

$$\frac{1}{n} \log \mu(B_n^G(y,\delta(\epsilon))) \leqslant \frac{\log(c)}{n} + \frac{1}{n} \log \mu(B_n^G(x,\epsilon)),$$

so

$$\limsup_{n\to\infty} -\frac{1}{n} \log \mu(B_n^G(y,\delta(\epsilon))) \geqslant \limsup_{n\to\infty} -\frac{1}{n} \log \mu(B_n^G(x,\epsilon))$$

and

$$\liminf_{n\to\infty} -\frac{1}{n} \log \mu(B_n^G(y,\delta(\epsilon))) \geqslant \liminf_{n\to\infty} -\frac{1}{n} \log \mu(B_n^G(x,\epsilon)).$$

Taking the limit as $\epsilon \to 0$ we arrive at $h_\mu^G(y) \geqslant h_\mu^G(x)$ and $h_{\mu,G}(y) \geqslant h_{\mu,G}(x)$. Similarly, for $\epsilon' > 0$ there exist $\delta'(\epsilon') > 0$ and $c' > 0$ such that

$$\mu(B_n^G(x,\delta'(\epsilon'))) \leqslant c' \cdot \mu(B_n^G(y,\epsilon')).$$

Applying the same arguments, we obtain the inequalities $h_\mu^G(x) \geqslant h_\mu^G(y)$ and $h_{\mu,G}(x) \geqslant h_{\mu,G}(y)$. □

Definition 4.11. — *If μ is a G-homogeneous measure on X, then the common value of local upper measure entropies is denoted by h_μ^G.*

THEOREM 4.12. — *For a finitely generated pseudogroup (G, G_1) acting on a compact metric space X, and admitting a $G -$ homogeneous measure μ on X, we have*

$$h_{top}(G, G_1) = h_\mu^G.$$

Proof. — Take an $(n, \epsilon)-$separated subset $E \subset X$, with respect to (G,G_1), with maximal cardinality equal to $s(n, \epsilon, (G, G_1))$. Then,

$$B_n^G(x, \epsilon/2) \cap B_n^G(y, \epsilon/2) = \emptyset,$$

for any distinct points $x, y \in E$. So

$$s(n, \epsilon, (G, G_1)) \cdot \mu(B_n^G(x, \epsilon/2)) \leqslant \mu(X).$$

The $G -$ *homogenity* of the measure μ allows us to choose $0 < \delta(\epsilon) < \epsilon$ and $c > 0$ so that

$$\mu(B_n^G(y, \delta(\epsilon))) \leqslant c \cdot \mu(B_n^G(x, \epsilon/2))$$

for all x and y. Thus

$$s(n, \epsilon, (G, G_1)) \cdot \mu(B_n^G(y, \delta(\epsilon))) \leqslant c \cdot \mu(X)$$

and

$$\limsup_{n\to\infty} \frac{1}{n} \log s(n, \epsilon, (G, G_1)) \leqslant \limsup_{n\to\infty} -\frac{1}{n} \log \mu(B_n^G(y, \delta(\epsilon))).$$

Taking the limit as $\epsilon \to 0$ we obtain

$$h_{\text{top}}(G, G_1) \leqslant h_\mu^G.$$

Now take an $(n, \delta)-$spanning subset $F \subset X$, with respect to (G, G_1), with minimal cardinality equal to $r(n, \delta, (G, G_1))$. Notice that $X \subset \bigcup_{x\in F} B_n^G(x, 2\delta)$. Given $\epsilon > 0$ choose $0 < \delta(\epsilon) < \epsilon$ and $c > 0$ so that

$$\mu(B_n^G(x, 2\delta(\epsilon))) \leqslant c \cdot \mu(B_n^G(y, \epsilon))$$

for all $x, y \in X$ and $n \in \mathbb{N}$. Then the inequality

$$c \cdot \mu(B_n^G(y, \epsilon)) \cdot r(n, \delta(\epsilon), (G, G_1)) \geqslant \mu(X) > 0$$

yields that

$$\limsup_{n\to\infty} \frac{1}{n} \log r(n, \delta(\epsilon), (G, G_1)) \geqslant \limsup_{n\to\infty} -\frac{1}{n} \log \mu(B_n^G(y, \epsilon)).$$

Finally, as $\epsilon \to 0$ we obtain

$$h_{\text{top}}(G, G_1) \geqslant h_\mu^G.$$

□

Corollary 4.13. — *For a finitely generated group* (G, G_1) *of homeomorphisms of a compact metric space* X, *which admits a* $G-$*homogeneous measure* μ *on* X, *we have*

$$h_{top}(G, G_1) = h_\mu^G.$$

5. Partial variational principle

Let B_j denote an open ball of radius r centered at x in a metric space (X, d). Then $m \cdot B_j$, where $m \in \mathbb{N}$, denotes the open ball of radius $m \cdot r$ centered at x. The diameter of the set $A \subset X$ is denoted by $\mathrm{diam}(A)$. A metric space X is called boundedly compact if all bounded closed subsets of X are compact. In particular $\mathbb{R}^n$ and Riemannian manifolds (see Gromov [13], p. 9) are boundedly compact.

Lemma 5.1 (Vitali Covering Lemma, Theorem 2.1 in [21]). — *Let X be a boundedly compact metric space and* $\mathcal{B}$ *a family of closed balls in X such that*

$$\sup\{\mathrm{diam}(B) : B \in \mathcal{B}\} < \infty.$$

Then there is a finite or countable sequence $B_i \in \mathcal{B}$ *of disjoint balls such that*

$$\bigcup_{B\in\mathcal{B}} B \subset \bigcup_i 5 \cdot B_i.$$

Any oriented Riemannian manifold M has a natural volume form dV which gives rise to a natural volume measure μ_v on the Borel sets defined as

$$\mu_v(A) = \int_A dV.$$

Theorem 5.2. — *Let* (G, G_1) *be a finitely generated group of homeomorphisms of a compact closed and oriented manifold* (M, d). *Let* E *be a Borel subset of* M, $s \in (0, \infty)$ *and* μ_v *the natural volume measure on* M. *If*

$$h_{\mu_v}^G(x) \leqslant s \quad \text{for all } x \in E \quad \text{then} \quad h_{top}((G, G_1), E) \leqslant s.$$

Proof. — Assume that $h_{\mu_v}^G(x) \leqslant s$, for any $x \in E$. Fix $\epsilon > 0$. For $k \in \mathbb{N}$ we define a set

$$E_k := \left\{ x \in E : \limsup_{n\to\infty} \frac{-\log \mu_v(B_n^G(x, r))}{n} < s + \epsilon \text{ for all } r \in (0, 1/k] \right\},$$

then clearly

$$E = \bigcup_{k\in\mathbb{N}} E_k.$$

For any fixed $r \in (0, \frac{1}{5\cdot k}]$ and arbitrary point $x \in E_k$ there exists $n(x) \in \mathbb{N}$ such that for any $N \geqslant n(x)$ the inequality

$$\mu_v(B_N^G(x,r)) \geqslant e^{-(s+\epsilon)\cdot N}$$

holds. Since M has bounded geometry each function $f_m : E_k \to \mathbb{R}$, defined by $f_m(x) := \mu_v(B_m^G(x,r))$, is continuous which implies that

$$N_0 := \sup\{n(x) : x \in E_k\} < \infty.$$

Due to Vitaly Covering Lemma for any $N \geqslant N_0$ we can choose from the cover $C_N := \{\overline{B_N^G(x,r)} : x \in E_k\}$ of $\overline{E_k}$ a family $D_N := \{\overline{B_N^G(x,r)} : x \in F_N\}$ of disjoint balls such that

$$E_k \subset \overline{E_k} \subset \bigcup_{x \in F_N} 5 \cdot \overline{B_N^G(x,r)} \subset \bigcup_{x \in F_N} 6 \cdot B_N^G(x,r)$$

and

$$\mu_v(B_N^G(x,r)) \geqslant e^{-(s+\epsilon)\cdot N}, \text{ for } x \in F_N.$$

Therefore

$$\operatorname{card}(F_N) \cdot e^{-(s+\epsilon)\cdot N} \leqslant \sum_{x \in F_N} e^{-(s+\epsilon)\cdot N} \leqslant \sum_{x \in F_N} \mu_v(B_N^G(x,r)) \leqslant 1,$$

which gives the upper bound for the Carathéodory function

$$\overline{r}_C(E_k, s+\epsilon, r) = \limsup_{N\to\infty} \inf_{F_N} \Big\{ \operatorname{card}(F_N) \cdot e^{-(s+\epsilon)\cdot N} \\ : E_k \subset \bigcup_{x \in F_N} 6 \cdot B_N^G(x,r) \Big\} \leqslant 1.$$

The above estimation implies that

$$\overline{\operatorname{Cap}}_{E_k} \leqslant s + \epsilon.$$

Since $\{E_k\}_{k\in\mathbb{N}}$ is an ascending sequence of sets, by Lemma 3.6 and Lemma 3.8, we obtain

$$\overline{\operatorname{Cap}}_E = \max\left\{\overline{\operatorname{Cap}}_{E_k}, \overline{\operatorname{Cap}}_{E\setminus E_k}\right\} \leqslant \sup_{k\in\mathbb{N}} \overline{\operatorname{Cap}}_{E_k}$$

and due to Corollary 3.11

$$h_{\text{top}}((G,G_1),E) = \overline{\operatorname{Cap}}_E \leqslant \sup_{k\in\mathbb{N}} \overline{\operatorname{Cap}}_{E_k} \leqslant s + \epsilon.$$

Finally, since ϵ is arbitrary small we get the inequality

$$h_{\text{top}}((G,G_1),E) \leqslant s.$$

□

Theorem 5.3. — *Let (G, G_1) be a finitely generated pseudogroup on a compact metric space (X, d). Let E be a Borel subset of X and $s \in (0, \infty)$. Denote by μ a Borel probability measure on X. If*

$$h_{\mu,G}(x) \geqslant s \quad \textit{for all } x \in E \textit{ and } \mu(E) > 0 \quad \textit{then} \quad h_{top}((G, G_1), E) \geqslant s.$$

Proof. — For any fixed $\epsilon > 0$ we have the equality $E = \bigcup_{k\in\mathbb{N}} E_k$, where

$$E_k := \left\{ x \in E : \liminf_{n\to\infty} \frac{-\log \mu(B_n^G(x, r))}{n} > s - \epsilon/2 \text{ for all } r \in (0, 1/k) \right\}.$$

The inequality

$$0 < \mu(E) \leqslant \sum_{k\in\mathbb{N}} \mu(E_k)$$

yields that $\mu(E_{k_0}) > 0$ for some $k_0 \in \mathbb{N}$. Notice that $E_{k_0} = \bigcup_{N\in\mathbb{N}} E_{k_0,N}$, where

$$E_{k_0,N} = \Big\{ x \in E_{k_0} : \frac{-\log \mu(B_n^G(x, r))}{n} > s - \epsilon, \text{ for all } n \geqslant N \text{ and } r \in (0, 1/k) \Big\}.$$

Again since $\mu(E_{k_0}) > 0$ and $E_{k_0} = \bigcup_{N\in\mathbb{N}} E_{k_0,N}$, we conclude that $\mu(E_{k_0,N_0}) > 0$, for some $N_0 \in \mathbb{N}$. This condition is equivalent to the following inequality

$$\mu(B_n^G(x, \delta)) \leqslant e^{-(s-\epsilon)\cdot n} \tag{5.1}$$

which holds for any point $x \in E_{k_0,N_0}$, radius $\delta \in (0, \frac{1}{k_0})$ and $n \geqslant N_0$.

For any positive integer $N \geqslant N_0$ we consider a cover $F_N = \{B_N^G(x, \delta) : x \in E_{k_0,N_0}\}$ of E_{k_0,N_0}. Applying (5.1) to a subcover C of F_N we obtain the following estimations

$$\inf_C \left\{ \sum_{B_N^G(x,\delta)\in C} e^{-N\cdot(s-\epsilon)} : E_{k_0,N_0} \subset \bigcup_{B_N^G(x,\delta)\in C} B_N^G(x, \delta) \right\}$$
$$\geqslant \inf_C \left\{ \sum_{B_N^G(x,\delta)\in C} \mu(B_N^G(x, \delta)) \right\} \geqslant \mu(E_{k_0,N_0}) > 0.$$

Thus for any $\delta \in (0, \frac{1}{k_0})$ the Carathéodory function $\overline{r}_C(E_{k_0,N_0}, s - \epsilon, \delta)$ is positive and therefore $\overline{CP}_{(E_{k_0,N_0})} \geqslant s - \epsilon$. Applying Lemma 3.6 and Corollary 3.10 we obtain

$$h_{\mathrm{top}}((G, G_1), E) = \overline{CP}_E \geqslant \overline{CP}_{E_{k_0,N_0}} \geqslant s$$

since ϵ is arbitrarily small. □

COROLLARY 5.4. — *Let* (G, G_1) *be a finitely generated group on a compact metric space* (X, d)*. Let* E *be a Borel subset of* X *and* $s \in (0, \infty)$*. Denote by* μ *a Borel probability measure on* X*. If*

$$h_{\mu,G}(x) \geqslant s \quad \textit{for all } x \in E \textit{ and } \mu(E) > 0 \quad \textit{then} \quad h_{top}((G, G_1), E) \geqslant s.$$

Remark 5.5. — The proof of Theorem 5.3 was inspired by Theorem 1 in [20] by Ma and Wen who related the lower measure entropy of a single continuous map $f : X \to X$ of a compact metric space (X, d) with a dimensional type characteristic of the dynamical system $f : X \to X$.

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BIBLIOGRAPHY

[1] J. A. Álvarez López & A. Candel, "Equicontinuous foliated spaces", *Math. Z.* **263** (2009), no. 4, p. 725-774.

[2] L. Ambrosio & P. Tilli, *Topics on analysis in metric spaces*, Oxford Lecture Series in Mathematics and its Applications, vol. 25, Oxford University Press, Oxford, 2004, viii+133 pages.

[3] A. Biś, "Entropies of a semigroup of maps", *Discrete Contin. Dyn. Syst.* **11** (2004), no. 2-3, p. 639-648.

[4] A. Biś & M. Urbański, "Some remarks on topological entropy of a semigroup of continuous maps", *Cubo* **8** (2006), no. 2, p. 63-71.

[5] R. Bowen, "Entropy for group endomorphisms and homogeneous spaces", *Trans. Amer. Math. Soc.* **153** (1971), p. 401-414.

[6] ———, "Topological entropy for noncompact sets", *Trans. Amer. Math. Soc.* **184** (1973), p. 125-136.

[7] M. Brin & A. Katok, "On local entropy", in *Geometric dynamics (Rio de Janeiro, 1981)*, Lecture Notes in Math., vol. 1007, Springer, Berlin, 1983, p. 30-38.

[8] A. Bufetov, "Topological entropy of free semigroup actions and skew-product transformations", *J. Dynam. Control Systems* **5** (1999), no. 1, p. 137-143.

[9] R. R. Coifman & G. Weiss, *Analyse harmonique non-commutative sur certains espaces homogènes*, Lecture Notes in Mathematics, Vol. 242, Springer-Verlag, Berlin, 1971, Étude de certaines intégrales singulières, v+160 pages.

[10] K. Falconer, *Techniques in fractal geometry*, John Wiley & Sons Ltd., Chichester, 1997, xviii+256 pages.

[11] S. Friedland, "Entropy of graphs, semigroups and groups", in *Ergodic theory of* $\mathbf{Z}^d$ *actions (Warwick, 1993–1994)*, London Math. Soc. Lecture Note Ser., vol. 228, Cambridge Univ. Press, Cambridge, 1996, p. 319-343.

[12] É. Ghys, R. Langevin & P. Walczak, "Entropie géométrique des feuilletages", *Acta Math.* **160** (1988), no. 1-2, p. 105-142.

[13] M. Gromov, *Metric structures for Riemannian and non-Riemannian spaces*, english ed., Modern Birkhäuser Classics, Birkhäuser Boston Inc., Boston, MA, 2007, Based on the 1981 French original, With appendices by M. Katz, P. Pansu and S. Semmes, Translated from the French by Sean Michael Bates, xx+585 pages.

[14] A. Haefliger, "Variétés feuilletées", *Ann. Scuola Norm. Sup. Pisa (3)* **16** (1962), p. 367-397.

[15] ———, "Groupoïdes d'holonomie et classifiants", *Astérisque* (1984), no. 116, p. 70-97, Transversal structure of foliations (Toulouse, 1982).

[16] ———, "Pseudogroups of local isometries", in *Differential geometry (Santiago de Compostela, 1984)*, Res. Notes in Math., vol. 131, Pitman, Boston, MA, 1985, p. 174-197.

[17] P. Hajłasz, "Sobolev spaces on metric-measure spaces", in *Heat kernels and analysis on manifolds, graphs, and metric spaces (Paris, 2002)*, Contemp. Math., vol. 338, Amer. Math. Soc., Providence, RI, 2003, p. 173-218.

[18] J. Heinonen, *Lectures on analysis on metric spaces*, Universitext, Springer-Verlag, New York, 2001, x+140 pages.

[19] J. Luukkainen & E. Saksman, "Every complete doubling metric space carries a doubling measure", *Proc. Amer. Math. Soc.* **126** (1998), no. 2, p. 531-534.

[20] J.-H. Ma & Z.-Y. Wen, "A Billingsley type theorem for Bowen entropy", *C. R. Math. Acad. Sci. Paris* **346** (2008), no. 9-10, p. 503-507.

[21] P. Mattila, *Geometry of sets and measures in Euclidean spaces*, Cambridge Studies in Advanced Mathematics, vol. 44, Cambridge University Press, Cambridge, 1995, Fractals and rectifiability, xii+343 pages.

[22] R. D. Mauldin & M. Urbański, *Graph directed Markov systems*, Cambridge Tracts in Mathematics, vol. 148, Cambridge University Press, Cambridge, 2003, Geometry and dynamics of limit sets, xii+281 pages.

[23] D. Montgomery & L. Zippin, *Topological transformation groups*, Interscience Publishers, New York-London, 1955, xi+282 pages.

[24] Y. B. Pesin, "Dimension Type Characteristics for Invariant Sets of Dynamical Systems", *Russian Math. Surveys* **43** (1988), p. 111-151.

[25] ———, *Dimension theory in dynamical systems*, Chicago Lectures in Mathematics, University of Chicago Press, Chicago, IL, 1997, Contemporary views and applications, xii+304 pages.

[26] H. Sumi, "Skew product maps related to finitely generated rational semigroups", *Nonlinearity* **13** (2000), no. 4, p. 995-1019.

[27] A. L. Vol'berg & S. V. Konyagin, "There is a homogeneous measure on any compact subset of $\mathbb{R}^n$", *Soviet Math. Dokl.* **30** (1984), p. 453-456.

[28] ———, "On measure with the doubling condition", *Math. USSR-Izv.* **30** (1988), p. 629-638.

[29] P. Walczak, *Dynamics of Foliations, Groups and Pseudogroups*, Monografie Matematyczne, vol. 64, Birkhäuser, Basel, 2004.

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