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A REMARK ON A CONJECTURE OF HAIN AND LOOIJENGA

by Carel FABER

ABSTRACT. — We show that the natural generalization of a conjecture of Hain and Looijenga to the case of pointed curves holds for all g and n if and only if the tautological rings of the moduli spaces of curves with rational tails and of stable curves are Gorenstein.

RÉSUMÉ. — Nous montrons que la généralisation naturelle d'une conjecture de Hain et Looijenga au cas des courbes époutées tient pour tout g et n si et seulement si les anneaux tautologiques des espaces des modules des courbes à queues rationnelles et des courbes stables sont des anneaux de Gorenstein.

Let $M_{g,n}$ (resp. $\overline{M}_{g,n}$) be the moduli space of smooth (resp. stable) n -pointed curves of genus g and let $M_{g,n}^{ct}$ be the moduli space of pointed curves of compact type, the complement of the boundary divisor Δ_{irr} of irreducible singular curves and their degenerations. Let $M_{g,n}^{rt}$ be the moduli space of pointed curves with rational tails; for $g \geq 2$, it is the inverse image of M_g under the natural morphism $\overline{M}_{g,n} \rightarrow \overline{M}_g$, while $M_{1,n}^{rt} = M_{1,n}^{ct}$ and $M_{0,n}^{rt} = \overline{M}_{0,n}$ by definition. Here, (g, n) is a pair of nonnegative integers such that $2g - 2 + n > 0$. There is a natural partial ordering of these pairs: $(h, m) \leq (g, n)$ if and only if $h \leq g$ and $2h - 2 + m \leq 2g - 2 + n$, or, in other words, if and only if there exists a stable n -pointed curve of genus g whose dual graph contains a vertex of genus h with valency m .

We recall the definition of the tautological algebras $R^\bullet(\overline{M}_{g,n})$ from [4]: the system $\{R^\bullet(\overline{M}_{g,n})\}_{(g,n)}$ is defined as the set of smallest \mathbb{Q} -subalgebras of the rational Chow rings $A^\bullet(\overline{M}_{g,n})$ that is closed under push-forward via all maps forgetting markings and all standard gluing maps. The well-known ψ -, κ -, and λ -classes are tautological. The system is also closed under pull-back via the forgetting and gluing maps. The successive quotients $R^\bullet(M_{g,n}^{ct})$, $R^\bullet(M_{g,n}^{rt})$, and $R^\bullet(M_{g,n})$ are defined as the restrictions

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to the respective open subsets. (Observe that it is in general not known whether the corresponding tautological localization sequences are exact in the middle.)

The following results are known:

- (a) $R^\bullet(M_{g,n}^{rt})$ vanishes in degrees $> g - 2 + n - \delta_{0g}$ and is 1-dimensional in degree $g - 2 + n - \delta_{0g}$.
- (b) $R^\bullet(M_{g,n}^{ct})$ vanishes in degrees $> 2g - 3 + n$ and is 1-dimensional in degree $2g - 3 + n$.
- (c) $R^\bullet(\overline{M}_{g,n})$ (vanishes in degrees $> 3g - 3 + n$ and) is 1-dimensional in degree $3g - 3 + n$.

Statement (a) was proved by Looijenga [10] and Faber [1], [3]. Statements (b) and (c) were proved by Graber and Vakil [6], [7] and Faber and Pandharipande [4].

Recall the following three conjectures:

- (A) $R^\bullet(M_{g,n}^{rt})$ is Gorenstein with socle in degree $g - 2 + n - \delta_{0g}$.
- (B) $R^\bullet(M_{g,n}^{ct})$ is Gorenstein with socle in degree $2g - 3 + n$.
- (C) $R^\bullet(\overline{M}_{g,n})$ is Gorenstein with socle in degree $3g - 3 + n$.

(For a graded \mathbb{Q} -algebra R^\bullet , to be Gorenstein with socle in degree m means that it vanishes in degrees $> m$, that R^m is isomorphic to \mathbb{Q} , and that the pairings $R^i \times R^{m-i} \rightarrow R^m$ are perfect.)

In the case $g = 0$, the three conjectures coincide and have been proved by Keel [9]. Conjecture (A) in the case $n = 0$ is due to the author [1] and is true for $g \leq 23$. Hain and Looijenga [8] raised (C) as a question and (A), (B), and (C) were formulated in [11] (see also [3], [2]).

Hain and Looijenga also introduce a compactly supported version of the tautological algebra: they define $R_c^\bullet(M_{g,n})$ as the set of elements in $R^\bullet(\overline{M}_{g,n})$ that restrict trivially to the Deligne-Mumford boundary (i.e., the pull-back via any standard map from a product of moduli spaces \overline{M}_{g_i, n_i} onto the closure of a boundary stratum vanishes). It is a graded ideal in $R^\bullet(\overline{M}_{g,n})$ and a module over $R^\bullet(M_{g,n})$. They then formulate the following conjecture in the case $n = 0$:

CONJECTURE 1 (Hain and Looijenga [8]). — *The intersection pairings*

$$R^k(M_g) \times R_c^{3g-3-k}(M_g) \rightarrow R_c^{3g-3}(M_g) \cong \mathbb{Q}, \quad k = 0, 1, 2, \dots$$

are perfect (Poincaré duality) and $R_c^\bullet(M_g)$ is a free $R^\bullet(M_g)$ -module of rank one.

Observe that $\lambda_1 \in R_c^1(M_{1,1})$ and $\lambda_g \lambda_{g-1} \in R_c^{2g-1}(M_g)$ for $g > 1$. (The author's proof of the nonvanishing of $R^{g-2+n}(M_{g,n}^{rt})$ for $g > 0$ uses

this fact.) So this class is supposed to be a generator of the $R^\bullet(M_g)$ -module $R_c^\bullet(M_g)$ (the unique generator of degree $2g - 1$ up to a scalar).

However, the pull-backs of these classes to $\overline{M}_{g,n}$ don't lie in $R_c^\bullet(M_{g,n})$ for $n \geq 2$, since they don't vanish on the boundary strata corresponding to curves with rational tails. Let us therefore define $R_c^\bullet(M_{g,n}^{rt})$ as the set of elements in $R^\bullet(\overline{M}_{g,n})$ that restrict trivially to $\overline{M}_{g,n} \setminus M_{g,n}^{rt}$. Consider the following conjectures:

(D) The intersection pairings

$$R^k(M_{g,n}^{rt}) \times R_c^{3g-3+n-k}(M_{g,n}^{rt}) \rightarrow R_c^{3g-3+n}(M_{g,n}^{rt}) \cong \mathbb{Q}$$

are perfect for $k \geq 0$.

(E) In addition to (D), $R_c^\bullet(M_{g,n}^{rt})$ is a free $R^\bullet(M_{g,n}^{rt})$ -module of rank one.

Conjecture (E) appears to be the natural generalization of Conjecture 1 to the case $n > 0$. For reasons that will become clear in a moment, we also include the weaker statement (D). Observe that (E) implies that $\lambda_g \lambda_{g-1}$ is a generator of $R_c^\bullet(M_{g,n}^{rt})$ for $g > 0$ (the unique one of degree $2g - 1$ up to a scalar), by (a) above.

THEOREM 1. — *Conjectures (A) and (C) are true for all (g, n) if and only if Conjecture (E) is true for all (g, n) . More precisely,*

$$A_{(g,n)} \text{ and } C_{(g,n)} \Rightarrow E_{(g,n)} \Rightarrow A_{(g,n)} \text{ and } D_{(g,n)}$$

and

$$\{D_{(g',n')}\}_{(g',n') \leq (g,n)} \Rightarrow \{C_{(g',n')}\}_{(g',n') \leq (g,n)}.$$

Proof. Suppose first that (C) is not true for all (g, n) and let a minimal counterexample be given by $0 \neq \alpha \in R^\bullet(\overline{M}_{g,n})$, i.e., $R^\bullet(\overline{M}_{g',n'})$ is Gorenstein for all $(g', n') < (g, n)$ and $\deg(\alpha\beta) = 0$ for all $\beta \in R^\bullet(\overline{M}_{g,n})$. (We write \deg for the degree homomorphism on $R_0(\overline{M}_{g,n})$ and its extension by zero to all of $R^\bullet(\overline{M}_{g,n})$). It follows that $g > 0$.

Let π denote the standard map $\overline{M}_{g-1,n+2} \rightarrow \overline{M}_{g,n}$ onto the boundary divisor Δ_{irr} . Let $\gamma \in R^\bullet(\overline{M}_{g-1,n+2})$ be arbitrary. Then

$$\deg((\pi^*\alpha)\gamma) = \deg(\pi_*((\pi^*\alpha)\gamma)) = \deg(\alpha\pi_*\gamma) = 0,$$

since $\pi_*\gamma$ is tautological. Since $R^\bullet(\overline{M}_{g-1,n+2})$ is Gorenstein, it follows that $\pi^*\alpha = 0$.

Next, let π denote one of the standard maps $\overline{M}_{g_1,n_1} \times \overline{M}_{g_2,n_2} \rightarrow \overline{M}_{g,n}$ onto a boundary component parametrizing reducible singular curves ($g_1 + g_2 = g$ and $n_1 + n_2 = n + 2$). We have the push-forward map

$$\pi_* : R^\bullet(\overline{M}_{g_1,n_1}) \otimes_{\mathbb{Q}} R^\bullet(\overline{M}_{g_2,n_2}) \rightarrow R^\bullet(\overline{M}_{g,n})$$

and the pull-back map in the other direction (cf. [5]). The tensor product is Gorenstein, with perfect pairing given by

$$\deg((\beta_1 \otimes \beta_2)(\gamma_1 \otimes \gamma_2)) = \deg(\beta_1 \gamma_1) \deg(\beta_2 \gamma_2).$$

Let γ_1 resp. γ_2 be arbitrary elements of $R^\bullet(\overline{M}_{g_1, n_1})$ resp. $R^\bullet(\overline{M}_{g_2, n_2})$. Then $\deg((\pi^* \alpha)(\gamma_1 \otimes \gamma_2)) = \deg(\pi_*((\pi^* \alpha)(\gamma_1 \otimes \gamma_2))) = \deg(\alpha \pi_*(\gamma_1 \otimes \gamma_2)) = 0$,

since $\pi_*(\gamma_1 \otimes \gamma_2)$ is tautological. Again, it follows that $\pi^* \alpha = 0$.

Therefore, $0 \neq \alpha \in R_c^\bullet(M_{g,n})$ and a fortiori $0 \neq \alpha \in R_c^\bullet(M_{g,n}^{rt})$. But it pairs to zero with all β and this contradicts $D_{(g,n)}$. The implication in the second display follows as an immediate consequence.

The next step is to prove the implication $E_{(g,n)} \Rightarrow A_{(g,n)}$. As mentioned above, if $g > 0$ and $E_{(g,n)}$ holds, then $\lambda_g \lambda_{g-1}$ generates $R_c^\bullet(M_{g,n}^{rt})$ freely. Suppose that $A_{(g,n)}$ fails: let $0 \neq \alpha \in R^\bullet(M_{g,n}^{rt})$ be such that it pairs to zero with all $\beta \in R^\bullet(M_{g,n}^{rt})$, i.e., $\deg(\alpha \beta \lambda_g \lambda_{g-1}) = 0$ for all β (note that $g > 0$). From $D_{(g,n)}$, it follows that $\alpha \lambda_g \lambda_{g-1} = 0$, but this contradicts $E_{(g,n)}$. This proves the second implication in the first display.

To prove the first implication, we first show that $A_{(g,n)}$ and $C_{(g,n)}$ imply $D_{(g,n)}$. Assume that $D_{(g,n)}$ fails; the perfect pairing may fail on either side. Suppose first that $0 \neq \alpha \in R_c^\bullet(M_{g,n}^{rt})$ pairs to zero with all of $R^\bullet(M_{g,n}^{rt})$. We know that $\pi^* \alpha = 0$, for every standard map π associated to a stratum in $\overline{M}_{g,n} \setminus M_{g,n}^{rt}$. This means that the product of α and a Chow class pushed forward via such a map is zero (hence the pairing is well-defined). Since α pairs to zero with all of $R^\bullet(M_{g,n}^{rt})$, it gives a counterexample to $C_{(g,n)}$. If instead $0 \neq \alpha \in R^\bullet(M_{g,n}^{rt})$ pairs to zero with all of $R_c^\bullet(M_{g,n}^{rt})$, then it pairs to zero with all classes of the form $\beta \lambda_g \lambda_{g-1}$, for $\beta \in R^\bullet(M_{g,n}^{rt})$ (note that $g > 0$). In this case, α gives a counterexample to $A_{(g,n)}$.

We conclude by showing that $A_{(g,n)}$ and $C_{(g,n)}$ imply $E_{(g,n)}$. We already have $D_{(g,n)}$. If $E_{(g,n)}$ doesn't hold, then $g > 0$ and certainly $\lambda_g \lambda_{g-1}$ fails to be a basis for $R_c^\bullet(M_{g,n}^{rt})$, i.e., multiplication by $\lambda_g \lambda_{g-1}$ fails to be surjective or injective. From $A_{(g,n)}$ and $D_{(g,n)}$, it follows that the surjectivity and injectivity of this map are equivalent (recall from [5], Cor. 1, that $R^\bullet(\overline{M}_{g,n})$ is finite-dimensional). But if $0 \neq \alpha \in R^\bullet(M_{g,n}^{rt})$ and $\alpha \lambda_g \lambda_{g-1} = 0$, then $A_{(g,n)}$ fails. □

There is an analogous result in the compact type case. Begin by defining $R_c^\bullet(M_{g,n}^{ct})$ as the set of elements in $R^\bullet(\overline{M}_{g,n})$ that pull back to zero via the standard map $\overline{M}_{g-1, n+2} \rightarrow \overline{M}_{g,n}$ onto Δ_{irr} . Conjectures (D) and (E)

have obvious analogues (D^{ct}) and (E^{ct}). We have that

$$B_{(g,n)} \text{ and } C_{(g,n)} \Rightarrow E_{(g,n)}^{ct} \Rightarrow B_{(g,n)} \text{ and } D_{(g,n)}^{ct}$$

and

$$\{D_{(g',n')}^{ct}\}_{(g',n') \leq (g,n)} \Rightarrow \{C_{(g',n')}\}_{(g',n') \leq (g,n)}.$$

The proof proceeds entirely analogously; the class λ_g now plays the role of $\lambda_g \lambda_{g-1}$ (it is no longer necessary to treat the case $g = 0$ separately).

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Note added in the second version (November 2010). Tavakol [13] has proved that the tautological ring of $M_{1,n}^{ct} = M_{1,n}^{rt}$ is Gorenstein with socle in degree $n - 1$ (Conjectures (A) and (B) for $g = 1$). From Theorem 1, the tautological rings $R^\bullet(\overline{M}_{1,n})$ are Gorenstein if and only if $E_{(1,n)}$ holds for all $n \geq 1$, in other words, if and only if $R_c^\bullet(M_{1,n}^{rt})$ is generated by λ_1 as an $R^\bullet(M_{1,n}^{rt})$ -module.

Note added in the third version (April 2012). Tavakol [14] has now also proved Conjecture (A) for $g = 2$: the tautological ring of $M_{2,n}^{rt}$ is Gorenstein with socle in degree n .

Note added in the fourth version (June 2012). Petersen [12] has proved that the tautological ring of $\overline{M}_{1,n}$ is Gorenstein with socle in degree n (Conjecture (C) for $g = 1$).

BIBLIOGRAPHY

- [1] C. FABER, “A conjectural description of the tautological ring of the moduli space of curves”, in *Moduli of curves and abelian varieties*, Aspects Math., E33, Vieweg, Braunschweig, 1999, p. 109-129.
- [2] ———, “Hodge integrals, tautological classes and Gromov-Witten theory”, Proceedings of the Workshop “Algebraic Geometry and Integrable Systems related to String Theory” (Kyoto, 2000), no. 1232, Sūrikaiseikikenkyūsho Kōkyūroku, 2001, p. 78-87.
- [3] C. FABER & R. PANDHARIPANDE, “Logarithmic series and Hodge integrals in the tautological ring”, with an appendix by Don Zagier. Dedicated to William Fulton on the occasion of his 60th birthday. *Michigan Math. J.* **48** (2000), p. 215-252.
- [4] ———, “Relative maps and tautological classes”, *J. Eur. Math. Soc. (JEMS)* **7** (2005), no. 1, p. 13-49.
- [5] T. GRABER & R. PANDHARIPANDE, “Constructions of nontautological classes on moduli spaces of curves”, *Michigan Math. J.* **51** (2003), no. 1, p. 93-109.

- [6] T. GRABER & R. VAKIL, “On the tautological ring of $\overline{M}_{g,n}$ ”, *Turkish J. Math.* **25** (2001), no. 1, p. 237-243.
- [7] ———, “Relative virtual localization and vanishing of tautological classes on moduli spaces of curves”, *Duke Math. J.* **130** (2005), no. 1, p. 1-37.
- [8] R. HAIN & E. LOOIJENGA, “Mapping class groups and moduli spaces of curves”, in *Algebraic geometry—Santa Cruz 1995*, Proc. Sympos. Pure Math., vol. 62, Part 2, Amer. Math. Soc., Providence, RI, 1997, p. 97-142.
- [9] S. KEEL, “Intersection theory of moduli space of stable n -pointed curves of genus zero”, *Trans. Amer. Math. Soc.* **330** (1992), no. 2, p. 545-574.
- [10] E. LOOIJENGA, “On the tautological ring of M_g ”, *Invent. Math.* **121** (1995), no. 2, p. 411-419.
- [11] R. PANDHARIPANDE, “Three questions in Gromov-Witten theory”, Proceedings of the International Congress of Mathematicians, Vol. II (Beijing, 2002), Higher Ed. Press, Beijing, 2002, p. 503-512.
- [12] D. PETERSEN, “The structure of the tautological ring in genus one”, Preprint, arXiv:1205.1586.
- [13] M. TAVAKOL, “The tautological ring of $M_{1,n}^{ct}$ ”, *Ann. Inst. Fourier (Grenoble)* **61.7** (2011).
- [14] ———, “The tautological ring of the moduli space $M_{2,n}^{rt}$ ”, Preprint, arXiv:1101.5242.

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