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The tautological ring of $M_{1,n}^{ct}$

THE TAUTOLOGICAL RING OF $M_{1,n}^{ct}$

by Mehdi TAVAKOL

ABSTRACT. — We describe the tautological ring of the moduli space of stable n-pointed curves of genus one of compact type. It is proven that it is a Gorenstein algebra.

RÉSUMÉ. — Nous décrivons l'anneau tautologique de l'espace des modules des courbes stables de genre un de type compact avec n points marqués. On prouve que c'est une algèbre de Gorenstein.

Introduction

Let $M_{g,n}^{ct}$ be the moduli space of stable n-pointed genus g curves of compact type and denote by $R^*(M_{g,n}^{ct})$ its tautological ring. Here, we study this ring in genus one. It is known that the tautological ring $R^*(M_{1,n}^{ct})$ is additively generated by boundary cycles, and it is the subalgebra of the Chow ring $A^*(M_{1,n}^{ct})$ (taken with $\mathbb{Q}$-coefficients throughout) of $M_{1,n}^{ct}$ generated by divisor classes D_I, for $I \subset \{1, \ldots, n\}$ with $|I| > 1$. For example, see the appendix of [11]. Recall that a boundary cycle of $M_{1,n}^{ct}$ parameterizes stable curves whose dual graphs are trees, and D_I is associated to those with one edge, for which I is the marking set on the genus zero component. We study this ring to understand the space of relations among the generators. In particular, we prove that the tautological ring is Gorenstein.

We begin this note by recalling the definitions and known facts about the tautological algebras as well as the conjectural structure of them.

In the second section we consider a fixed pointed elliptic curve $(C; O)$, and we describe the reduced fiber of the projection $M_{1,n}^{ct} \to M_{1,1}^{ct}$ over

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$[(C; O)] \in M_{1,1}^{ct}$, which is denoted by $\overline{U}_{n-1}$, as a sequence of blow-ups of C^{n-1}. As a result, we get a map

$$F : \overline{U}_{n-1} \to M_{1,n}^{ct}.$$

There is a description of the Chow ring $A^*(\overline{U}_{n-1})$ of $\overline{U}_{n-1}$ in the third section.

Then we define the tautological ring $R^*(C^n)$ of C^n as a subring of its Chow ring $A^*(C^n)$. We give a description of the pairing

$$R^d(C^n) \times R^{n-d}(C^n) \to \mathbb{Q}$$

for $0 \leqslant d \leqslant n$. In particular, we will see that this pairing is perfect.

The fifth section starts with the definition of the tautological ring $R^*(\overline{U}_{n-1})$ of $\overline{U}_{n-1}$. It is defined to be the subalgebra of its Chow ring generated by the tautological classes in $R^*(C^{n-1})$ and the classes of proper transforms of the exceptional divisors introduced in the construction of $\overline{U}_{n-1}$. The study of the pairing

$$R^d(\overline{U}_{n-1}) \times R^{n-1-d}(\overline{U}_{n-1}) \to \mathbb{Q},$$

for $0 \leqslant d \leqslant n - 1$, shows that it is perfect as well.

In the last section we study the fibers of the map $F : \overline{U}_{n-1} \to M_{1,n}^{ct}$, and we will see that the images of the tautological classes in $M_{1,n}^{ct}$ under the induced pull-back

$$F^* : A^*(M_{1,n}^{ct}) \to A^*(\overline{U}_{n-1})$$

are elements of the tautological ring $R^*(\overline{U}_{n-1})$ of $\overline{U}_{n-1}$ and hence, it induces a map

$$R^*(M_{1,n}^{ct}) \to R^*(\overline{U}_{n-1}),$$

which is denoted by the same letter F^*, by abuse of notation. Then, we will see that F^* induces an isomorphism between the tautological rings involved. This gives a description of the ring $R^*(M_{1,n}^{ct})$ in terms of the generators D_I's and the space of relations. This leads us to a proof of the following result:

THEOREM 0.1. — *The ring homomorphism* $F^* : R^*(M_{1,n}^{ct}) \to R^*(\overline{U}_{n-1})$ *is an isomorphism. In particular, for any* $0 \leqslant d \leqslant n - 1$, *the pairing*

$$R^d(M_{1,n}^{ct}) \times R^{n-1-d}(M_{1,n}^{ct}) \to \mathbb{Q}$$

is perfect. In other words, $R^*(M_{1,n}^{ct})$ *is a Gorenstein ring.*

was done during my graduate studies at KTH, the Royal Institute of Technology.

1. Review of known facts and conjectures about the tautological ring $R^*(M_{g,n}^{ct})$

Let $\overline{M}_{g,n}$ be the moduli space of stable curves of genus g with n marked points. In [8] the system of tautological rings is defined to be the set of smallest $\mathbb{Q}$-subalgebras of the Chow rings,

$$R^*(\overline{M}_{g,n}) \subset A^*(\overline{M}_{g,n}),$$

satisfying the following two properties:
- The system is closed under push-forward via all maps forgetting markings:
$$\pi_* : R^*(\overline{M}_{g,n}) \to R^*(\overline{M}_{g,n-1}).$$
- The system is closed under push-forward via all gluing maps:
$$\iota_* : R^*(\overline{M}_{g_1,n_1\cup\{*\}}) \otimes R^*(\overline{M}_{g_2,n_2\cup\{\bullet\}}) \to R^*(\overline{M}_{g_1+g_2,n_1+n_2}),$$
$$\iota_* : R^*(\overline{M}_{g,n\cup\{*,\bullet\}}) \to R^*(\overline{M}_{g+1,n}),$$
with attachments along the markings $*$ and $\bullet$.

The quotient $R^*(M_{g,n})$ of the tautological ring is defined as the restriction to the open subset $M_{g,n}$. In [3] it was conjectured that the tautological ring $R^*(M_g)$ is a Gorenstein algebra with socle in degree $g-2$. It was raised as a question in [14] whether the tautological ring of $\overline{M}_{g,n}$ satisfy Poincaré duality and has the Lefschetz property with respect to κ_1, which was known to be ample [2]. In [4] the following conjecture about the tautological ring $R^*(\overline{M}_{g,n})$ is stated:

CONJECTURE 1.1. — $R^*(\overline{M}_{g,n})$ is Gorenstein with socle in degree $3g - 3 + n$.

We now define the moduli space $M_{g,n}^{ct}$ and its tautological ring. To every stable n-pointed curve $(C; x_1, \ldots, x_n)$ there is an associated dual graph. Its vertices correspond to the irreducible components of C and edges correspond to intersection of components. Note that self intersection is allowed. The curve C is of compact type if its dual graph is a tree, or equivalently, the Jacobian of C is an abelian variety. The moduli space $M_{g,n}^{ct}$ parametrizes stable n-pointed curves of genus g of compact type. One can also define $M_{g,n}^{ct}$ as the complement of the boundary divisor Δ_{irr} of irreducible singular curves and their degenerations.

The tautological ring, $R^*(M_{g,n}^{ct}) \subset A^*(M_{g,n}^{ct})$, for the moduli space $M_{g,n}^{ct}$, is defined to be the image of $R^*(\overline{M}_{g,n})$ via the natural map,

$$R^*(\overline{M}_{g,n}) \subset A^*(\overline{M}_{g,n}) \to A^*(M_{g,n}^{ct}).$$

The quotient ring $R^*(M_{g,n}^{ct})$ admits a canonical non-trivial linear evaluation ϵ to $\mathbb{Q}$ obtained by integration involving the λ_g class, the Euler class of the Hodge bundle.

Recall that the Hodge bundle $\mathbb{E}$ on $\overline{M}_g$ for $g > 1$ (resp. $\overline{M}_{1,1}$ for $g = 1$), is the locally free Q-sheaf of rank g defined by $\mathbb{E} = \pi_*\omega$, where $\pi : \overline{M}_{g,1} \to \overline{M}_g$ (resp. $\pi : \overline{M}_{1,2} \to \overline{M}_{1,1}$) is the universal curve of genus g and ω denotes its relative dualizing sheaf. The Hodge bundle on $\overline{M}_{g,n}$ is defined as the pull-back of $\mathbb{E}$ via the natural projection $\pi : \overline{M}_{g,n} \to \overline{M}_g$ for $g > 1$ (resp. $\pi : \overline{M}_{1,n} \to \overline{M}_{1,1}$ for $g = 1$) and is denoted by the same letter. The fiber of $\mathbb{E}$ over a moduli point $[(C; x_1, \ldots, x_n)]$ is the g-dimensional vector space $H^0(C, \omega_C)$. The class λ_i on $\overline{M}_{g,n}$ is defined to be the ith Chern class $c_i(\mathbb{E})$ of the Hodge bundle.

The class ψ_i is the pull back $\sigma_i^*(K)$ of K along $\sigma_i : \overline{M}_{g,n} \to \overline{M}_{g,n+1}$, where $\sigma_1, \ldots, \sigma_n$ are the natural sections of the map $\pi : \overline{M}_{g,n+1} \to \overline{M}_{g,n}$, which forgets the last marking on the curve and stabilizes, and K is the class of the relative dualizing sheaf of the projection π. It is the first Chern class of the cotangent line bundle $\mathbb{L}_i$ on the moduli space whose fiber at the moduli point $[(C; x_1, \ldots, x_n)]$ is the cotangent space to C at the ith marking. The class κ_i on $\overline{M}_{g,n}$ is defined to be the push-forward $\pi_*(\psi_{n+1}^{i+1})$, where the projection $\pi : \overline{M}_{g,n+1} \to \overline{M}_{g,n}$ is defined as above. The ψ, κ and λ classes in $A^*(\overline{M}_{g,n})$ all lie in the tautological ring.

The class λ_g vanishes when restricted to the complement Δ_{irr}. This gives rise to an evaluation ϵ on $A^*(M_{g,n}^{ct})$:

$$\xi \mapsto \epsilon(\xi) = \int_{\overline{M}_{g,n}} \xi \cdot \lambda_g.$$

The non-triviality of the ϵ evaluation is proven by explicit integral computations. The following formula for λ_g integrals is proven in [7]:

$$\int_{\overline{M}_{g,n}} \psi_1^{\alpha_1} \cdots \psi_n^{\alpha_n} \lambda_g = \binom{2g-3+n}{\alpha_1, \ldots, \alpha_n} \int_{\overline{M}_{g,1}} \psi_1^{2g-2} \lambda_g.$$

The integrals on the right side are evaluated in terms of the Bernoulli numbers:

$$\int_{\overline{M}_{g,1}} \psi_1^{2g-2} \lambda_g = \frac{2^{2g-1} - 1}{2^{2g-1}} \frac{|B_{2g}|}{(2g)!}.$$

This proves the non-triviality of the evaluation since B_{2g} doesn't vanish.

It is proven in [12] that $R^*(M_{g,n}^{ct})$ vanishes in degrees $> 2g - 3 + n$ and is 1-dimensional in degree $2g - 3 + n$. It was speculated in [6] that $R^*(M_g^{ct})$ is a Gorenstein algebra with socle in codimension $2g - 3$. The following conjecture is stated in [4]:

CONJECTURE 1.2. — $R^*(M_{g,n}^{ct})$ is Gorenstein with socle in degree $2g - 3 + n$.

A compactly supported version of the tautological algebra is defined in [14]. The algebra $R_c^*(M_{g,n})$ is defined to be the set of elements in $R^*(\overline{M}_{g,n})$ that restrict trivially to the Deligne-Mumford boundary. This is a graded ideal in $R^*(\overline{M}_{g,n})$ and the intersection product defines a map

$$R^*(M_{g,n}) \times R_c^*(M_{g,n}) \to R_c^*(M_{g,n})$$

that makes $R_c^*(M_{g,n})$ a $R^*(M_{g,n})$-module. In [14] they formulated the following conjecture for the case $n = 0$:

CONJECTURE 1.3.

(A) The intersection pairings

$$R^k(M_g) \times R_c^{3g-3-k}(M_g) \to R_c^{3g-3}(M_g) \cong \mathbb{Q}$$

are perfect for $k \geqslant 0$.

(B) In addition to (A), $R_c^*(M_g)$ is a free $R^*(M_g)$-module of rank one.

In a similar fashion one defines $R_c^*(M_{g,n}^{ct})$ as the set of elements in $R^*(\overline{M}_{g,n})$ that pull back to zero via the standard map $\overline{M}_{g-1,n+2} \to \overline{M}_{g,n}$ onto Δ_{irr}. The analogue of the conjectures above for the spaces $M_{g,n}^{ct}$ instead of M_g and its relation with the conjecture 1.2 is discussed in [5]. First consider the analogue of the conjectures 1.3 as follows:

CONJECTURE 1.4.

(C) The intersection pairings

$$R^k(M_{g,n}^{ct}) \times R_c^{3g-3+n-k}(M_{g,n}^{ct}) \to R_c^{3g-3+n}(M_{g,n}^{ct}) \cong \mathbb{Q}$$

are perfect for $k \geqslant 0$.

(D) In addition to (C), $R_c^*(M_{g,n}^{ct})$ is a free $R^*(M_{g,n}^{ct})$-module of rank one.

In [5] it is proven that for a given (g, n), the statement (D) in 1.4 follows if the statements 1.1 and 1.2 hold. On the other hand, for such (g, n) the statements 1.2 and (C) in 1.4 follow from (D) in 1.4. It is also proven that a counterexample to the conjecture 1.1 leads to a disproof of the conjecture (C) in 1.4.

In this note we consider the case $g = 1$ and prove that the conjecture 1.2 is true in this case.

2.   The space $\overline{U}_{n-1}$

Let C be a fixed elliptic curve and choose a point $O \in C$ as its origin. For a given natural number $n \in \mathbb{N}$, the space U_{n-1} is defined to be the open subset

$$\left\{ (x_1, \ldots, x_{n-1}) \in C^{n-1} : x_i \neq O \text{ and } x_i \neq x_j \text{ for } i \neq j \right\}$$

of C^{n-1}. The projection $\pi : U_{n-1} \times C \to U_{n-1}$ admits n disjoint sections with smooth fibers and defines a map

$$F : U_{n-1} \to M_{1,n},$$

where $M_{1,n}$ denotes the moduli space of smooth n-pointed curves of genus one. The map F sends the point $P = (x_1, \ldots, x_{n-1})$ of U_{n-1} to the class of the pointed curve $(C; x_1, \ldots, x_{n-1}, O)$.

For a proper subset I of $\{1, \ldots, n\}$, let $X_I \subset C^{n-1}$ be the $|I|$-dimensional subvariety defined by

$$\begin{cases} x_i = x_j & \text{for } i, j \in \{1, \ldots, n\} - I & \text{if } n \in I \\ x_i = O & \text{for } i \in \{1, \ldots, n-1\} - I & \text{if } n \notin I. \end{cases}$$

The space $\overline{U}_{n-1}$ is constructed from C^{n-1} by the following sequence of blow-ups: At step zero blow-up C^{n-1} at the point X_0, and at the kth step, for $1 \leqslant k \leqslant n-3$, blow-up the space obtained in the previous step along the regularly embedded union of the proper transforms of the subvarieties X_I, where $|I| = k$.

The space $\overline{U}_{n-1}$ contains U_{n-1} as an open dense subset. There exists a family of stable curves of genus one of compact type over $\overline{U}_{n-1}$, whose total space is isomorphic to $\overline{U}_n$. The resulting family is denoted by $\pi : \overline{U}_n \to \overline{U}_{n-1}$ by abuse of notation. Since $\pi^{-1}(U_{n-1})$ is isomorphic to the product $U_{n-1} \times C$, on which π is projection onto the first factor, and this coincides with the former definition of π given above, there is no danger of confusion. The map π admits n disjoint sections in the smooth locus of the fibers, and defines a morphism

$$F : \overline{U}_{n-1} \to M_{1,n}^{ct}.$$

The morphism F sends a geometric point $P \in \overline{U}_{n-1}$ to the moduli point of the pointed curve $(\pi^{-1}(P); x_1, \ldots, x_n)$, where the x_i's are the n distinct

points on the fiber $\pi^{-1}(P)$ obtained by intersecting the fiber $\pi^{-1}(P)$ with the n disjoint sections of π.

3. The Chow ring $A^*(\overline{U}_{n-1})$

In this section we recall some facts about the intersection ring of the blow-up $\widetilde{Y}$ of a smooth variety Y along a smooth irreducible subvariety Z from [9]. When the restriction map from $A^*(Y)$ to $A^*(Z)$ is surjective, S. Keel has shown in [16] that the computations become simpler. Assuming the surjectivity of the restriction map, we denote the kernel of the restriction map by $J_{Z/Y}$ so that

$$A^*(Z) = \frac{A^*(Y)}{J_{Z/Y}}.$$

Define a Chern polynomial for $Z \subset Y$, denoted by $P_{Z/Y}(t)$, to be a polynomial

$$P_{Z/Y}(t) = t^d + a_1 t^{d-1} + \cdots + a_{d-1}t + a_d \in A^*(Y)[t],$$

where d is the codimension of Z in Y and $a_i \in A^i(Y)$ is a class whose restriction in $A^i(Z)$ is $c_i(N_{Z/Y})$, where $N_{Z/Y}$ is the normal bundle of Z in Y. We also require that $a_d = [Z]$, while the other classes a_i, for $0 < i < d$, are determined only modulo $J_{Z/Y}$.

Let us verify the surjectivity of the restriction map from $A^*(Y)$ to $A^*(Z)$ in our case, when $Y = C^{n-1}$ and $Z = X_I$, for a subset I of the set $\{1, \ldots, n\}$. First assume that n doesn't belong to the set I. Denote by $i_I : X_I \to C^{n-1}$ the inclusion map and by $\pi : C^{n-1} \to X_I$ the canonical projection. From the equality $\pi \circ i_I = id_{X_I}$ we conclude that the restriction map i_I^* is surjective. It also follows that the push-forward map $(i_I)_*$ is injective. This will be used in 5.1. The case $n \in I$ is treated in a similar manner. In this case there is not a canonical projection $\pi : C^{n-1} \to X_I$, and one has to make a choice.

The following lemma can be used to compute $P_{Z/Y}$ when the subvariety Z is a transversal intersection of divisor classes:

LEMMA 3.1.

(a) If $Z = D$ is a divisor, then $P_{D/Y}(t) = t + D$.

(b) If $V \subset Y$ and $W \subset Y$ are subvarieties meeting transversally in a variety Z, and V and W have Chern polynomials $P_{V/Y}(t)$ and $P_{W/Y}(t)$, then Z has a Chern polynomial

$$P_{Z/Y}(t) = P_{V/Y}(t) \cdot P_{W/Y}(t).$$

In addition the restriction from $A^*(Y)[t]$ to $A^*(V)[t]$ maps $P_{W/Y}(t)$ to a Chern polynomial $P_{Z/V}(t)$ for $Z \subset V$.

Proof. — This is Lemma 5.1 in [9].   □

We identify $A^*(Y)$ as a subring of $A^*(\widetilde{Y})$ by means of the map π^* : $A^*(Y) \to A^*(\widetilde{Y})$, where $\pi : \widetilde{Y} \to Y$ is the birational morphism. Let $E \subset \widetilde{Y}$ be the exceptional divisor. The formula of Keel is as follows:

LEMMA 3.2. — *With the above assumptions and notations, the Chow ring $A^*(\widetilde{Y})$ is given by*

$$A^*(\widetilde{Y}) = \frac{A^*(Y)[E]}{(J_{Z/Y} \cdot E, P_{Z/Y}(-E))}.$$

Proof. — This is Lemma 5.3 in [9].   □

The next lemma relates a Chern polynomial $P_{\widetilde{V}/\widetilde{Y}}(t)$ of the proper transform $\widetilde{V}$ of a subvariety $V \subset Y$ to $P_{V/Y}(t)$:

LEMMA 3.3. — *Let V be a subvariety of Y not contained in Z and let $\widetilde{V} \subset \widetilde{Y}$ be its proper transform. Suppose that $P_{V/Y}(t)$ is a Chern polynomial for V.*

  (1) *If V meets Z transversally, then $P_{V/Y}(t)$ is a Chern polynomial for $\widetilde{V}$ in $\widetilde{Y}$.*
  (2) *If V contains Z, then $P_{V/Y}(t-E)$ is a Chern polynomial for $\widetilde{V} \subset \widetilde{Y}$.*

Proof. — This is Lemma 5.2 in [9].   □

In the second section we saw that the space $\overline{U}_{n-1}$ is obtained as a result of a sequence of blow-ups of the variety C^{n-1}. As we have observed before in this section, the restriction map from the intersection ring of C^{n-1} to $A^*(Z)$ is surjective when $Z = X_I$ for a subset I of $\{1, \ldots, n\}$. The following lemmas show that the same surjectivity holds for all of the blow-up centers at every step during the process of the construction of $\overline{U}_{n-1}$. They also relate the ideal $J_{\widetilde{V}/\widetilde{Y}}$ to the ideal $J_{V/Y}$ for a subvariety V of Y when we blow-up the variety Y along a subvariety Z assuming the surjectivity of the restriction map $A^*(Y) \to A^*(Z)$:

LEMMA 3.4. — *Suppose that V is a nonsingular subvariety of Y that intersects Z transversally in an irreducible variety $V \cap Z$, and that the restriction $A^*(V) \to A^*(V \cap Z)$ is also surjective. Let $\widetilde{V} = Bl_Z V$. Then $A^*(\widetilde{Y}) \to A^*(\widetilde{V})$ is surjective, with kernel $J_{V/Y}$ if $V \cap Z$ is not empty, and kernel $(J_{V/Y}, E)$ if $V \cap Z$ is empty.*

Proof. — This is Lemma 5.4 in [9].   □

LEMMA 3.5. — *Suppose that Z is the transversal intersection of nonsingular subvarieties V and W of Y, and that the restrictions $A^*(Y) \to A^*(V)$ and $A^*(Y) \to A^*(W)$ are also surjective, let $\widetilde{V} = Bl_Z V$. Then*

(1) *$A^*(\widetilde{Y}) \to A^*(\widetilde{V})$ is surjective, with kernel $(J_{V/Y}, P_{W/Y}(-E))$;*
(2) *$A^*(\widetilde{Y}) \to A^*(E \cap \widetilde{V})$ is surjective, with kernel $(J_{Z/Y}, P_{W/Y}(-E))$.*

Proof. — This is Lemma 5.5 in [9]. $\qquad\square$

From now on we will always assume that all of the restriction maps to the blow-up centers are surjective. Using the general results mentioned above we are able to express certain monomials that belong to the Chow ring $A^*(\widetilde{Y})$ in terms of elements in $A^*(Y)$:

LEMMA 3.6. — *Suppose that Z is the transversal intersection $D_1 \cap \cdots \cap D_r$ of divisor classes $D_1, \ldots, D_r$ on Y and let $f \in A^*(Y)$ be an element of degree $d = \dim(Z)$. The following relation holds in $A^*(Bl_Z Y)$:*

$$f \cdot E^r = (-1)^{r-1} f \cdot Z.$$

Proof. — Multiply both sides of the equality

$$(D_1 - E) \cdots (D_r - E) = 0$$

by f. For any element $g \in A^*(Y)$ of positive degree, the pull back $i_Z^*(fg)$ of fg along the inclusion $i_Z : Z \to Y$ is zero, which means that the product $fg \cdot E$ is zero as well. This proves the claim. $\qquad\square$

We also state the non-vanishing criteria of the product $E_I \cdot E_J$ for a pair of exceptional divisors E_I and E_J :

PROPOSITION 3.7. — *Let $I, J \subset \{1, \ldots, n\}$ be subsets satisfying $|I|, |J| \leqslant n - 3$. The product $E_I \cdot E_J \in A^2(\overline{U}_{n-1})$ is zero unless $I \subseteq J$ or $J \subseteq I$ or $I \cup J = \{1, \ldots, n\}$.*

Proof. — If $I \cup J \neq \{1, \ldots, n\}$, then $X_{I \cap J}$ is equal to the intersection $X_I \cap X_J$, and it is a proper subset of X_I and X_J both if $I \not\subseteq J$ and $J \not\subseteq I$. Under the assumption $I \not\subseteq J, J \not\subseteq I$ and $I \cup J \neq \{1, \ldots, n\}$, the proper transforms of the subvarieties X_I and X_J become disjoint after blowing up along that of $X_{I \cap J}$. This means that the product $E_I \cdot E_J \in A^2(\overline{U}_{n-1})$ is zero. $\qquad\square$

4. The tautological ring $R^*(C^n)$

DEFINITION 4.1. — *Suppose $(C; O)$ is a fixed pointed elliptic curve, and let $n \in \mathbb{N}$ be a natural number. The tautological ring, $R^*(C^n) \subset A^*(C^n)$,*

is defined to be the $\mathbb{Q}$-subalgebra generated by the following classes:

$$a_i = \big\{(x_1, \ldots, x_n) \in C^n : x_i = O\big\},$$
$$d_{j,k} = \big\{(x_1, \ldots, x_n) \in C^n : x_j = x_k\big\},$$

where $1 \leqslant i \leqslant n$ and $1 \leqslant j < k \leqslant n$. If we define $b_{j,k} := d_{j,k} - a_j - a_k$, then another set of generators for $R^*(C^n)$ is $\{a_i, b_{j,k} : 1 \leqslant i \leqslant n \text{ and } 1 \leqslant j < k \leqslant n\}$.

PROPOSITION 4.2.

(A) *Over the polynomial ring with generators $a_i, b_{j,k}$ the ideal of relations is generated by the following ones:*

$$a_i^2 = 0, \quad a_i b_{i,j} = 0, \quad b_{i,j}^2 = -2a_i a_j, \quad b_{i,j} b_{i,k} = a_i b_{j,k},$$
$$b_{i,j} b_{k,l} + b_{i,k} b_{j,l} + b_{i,l} b_{j,k} = 0,$$

where in each relation the indices are distinct.

(B) *For any $0 \leqslant d \leqslant n$, the pairing $R^d(C^n) \times R^{n-d}(C^n) \to \mathbb{Q}$ is perfect.*

Proof. — We first verify that the relations above hold in $R^2(C^n)$. The relations $a_i^2 = a_i b_{i,j} = 0$ and $b_{i,j}^2 = -2a_i a_j$ obviously hold. E. Getzler proved in [10] that the following relation holds in $A^2(\overline{M}_{1,4})$:

$$(1) \qquad 12\delta_{2,2} - 4\delta_{2,3} - 2\delta_{2,4} + 6\delta_{3,4} + \delta_{0,3} + \delta_{0,4} - 2\delta_\beta = 0,$$

where the classes above are defined in [10].

In [17], R. Pandharipande gives a direct construction of Getzler's relation via a rational equivalence in the Chow group $A_2(\overline{M}_{1,4})$. If we restrict the relation (1) to the space $M_{1,4}^{ct}$, pull it back along the morphism $F : \overline{U}_3 \to M_{1,4}^{ct}$, and push it down to C^3 via the blow-down map, we get the relation

$$(2) \qquad\qquad\qquad 12(a_1 b_{2,3} - b_{1,2} b_{1,3}) = 0.$$

The pull back of the relation above to $R^2(\overline{U}_4)$ via the morphism $\pi : \overline{U}_4 \to \overline{U}_3$ which forgets the last coordinate gives the relation

$$(3) \qquad\qquad\qquad 12(b_{1,2} b_{3,4} + b_{1,3} b_{2,4} + b_{1,4} b_{2,3}) = 0.$$

For more details about the derivation of (2) and (3), please see the appendix.

Now, we study the pairing

$$R^d(C^n) \times R^{n-d}(C^n) \to \mathbb{Q}$$

for $0 \leqslant d \leqslant n$. From the relations above, we see that the tautological group $R^d(C^n)$ is spanned by monomials of the form $v = a(v)b(v)$, where $a(v)$ is a product $\prod a_i$ of a_i's for $i \in A_v$, and $b(v)$ is a product $\prod b_{j,k}$ of $b_{j,k}$'s, for $j, k \in B_v$, such that A_v and B_v are disjoint subsets of the set $\{1, \ldots, n\}$

satisfying $d = |A_v| + \frac{1}{2}|B_v|$. Under this circumstance, the monomial v is said to be standard. To any standard monomial v we associate a dual element $v^* \in R^{n-d}(C^n)$, which is defined to be the product of all a_i's, for $i \in \{1, \ldots, n\} - A_v \cup B_v$, with $b(v)$. The following lemma enables us to study the pairing:

LEMMA 4.3. — Let $v \in R^d(C^n)$ and $w \in R^{n-d}(C^n)$ be standard monomials. If the product $v \cdot w$ is nonzero, then $B_v = B_w$, and the disjoint union of the sets A_v, A_w and $B_v = B_w$ is equal to the set $\{1, \ldots, n\}$.

Proof. — By assumption, we obtain the following inequalities:

$$n = \left(|A_v| + \frac{1}{2}|B_v|\right) + \left(|A_w| + \frac{1}{2}|B_w|\right) \leqslant |A_v| + |A_w| + |B_v \cup B_w| \leqslant n.$$

This forces the inequalities to be equalities. The equality

$$\left(|B_v \cup B_w| - |B_v|\right) + \left(|B_v \cup B_w| - |B_w|\right) = 0$$

implies that $|B_v \cup B_w| = |B_v| = |B_w|$, which shows that $B_v = B_w$. The equality $|A_v| + |A_w| + |B_v| = n$ proves the second part of the claim. □

So, after a suitable enumeration of generators for $R^d(C^n)$, the resulting intersection matrix of the pairing between standard monomials and their dual consists of square blocks along the main diagonal and the off-diagonal blocks are all zero. To prove that the pairing is perfect we need to study the square blocks on the main diagonal. These matrices and their eigenvalues are studied in [15]. In particular, from their result it follows that the kernel of any such matrix is generated by relations of the form (3):

LEMMA 4.4. — Let $m \geqslant 2$ be a natural number and S be the set of all standard monomials v of the form $b_{i_1,j_1} \cdots b_{i_m,j_m}$ in $R^m(C^{2m})$. The kernel of the intersection matrix $(v \cdot w)$ for v, w in S is the ideal generated by expressions of the form

$$R_{\{i,j,k,l\}} := b_{i,j}b_{k,l} + b_{i,k}b_{j,l} + b_{i,l}b_{j,k},$$

where the indices are distinct elements varying over the set $\{1, \ldots, 2m\}$.

Proof. — The intersection matrix $(v \cdot w)$ for $v, w \in S$ in [15] is denoted by $T_r(x)$ for $r = m$ and $x = -2$. The S_{2m}-module generated by elements of S decomposes into the sum $\oplus_\lambda V_\lambda$, where λ varies over all partitions of $2m$ into even parts. For each such λ the space V_λ is an eigenspace of $T_r(x)$. The corresponding eigenvalue is zero when $\lambda \neq 2^m$ and it is $(-1)^m(m+1)!$ when $\lambda = 2^m$. We identify the space V_λ with a subspace of $R^m(C^{2m})$, defined below, which is generated by expressions of the form $R_{\{i,j,k,l\}}$, for $\lambda \neq 2^m$.

Recall that a tabloid is an equivalence class of numberings of a Young diagram, two being equivalent if corresponding rows contain the same entries. The tabloid determined by a numbering T is denoted $\{T\}$. The space V_λ is generated by elements of the form

$$v_T = \sum_{q \in C(T)} \mathrm{sgn}(q)\{q \cdot T\},$$

where $C(T)$ is the subgroup of S_{2m} consisting of permutations preserving the columns of T.

Note that the sum $R_{\{1,\dots,2m\}} := \sum_{v \in S} v$ is a symmetric expression, which is clearly a linear combination of terms of the form R_T, where $|T| = 4$. This proves the claim when $\lambda = 2m$ gives the trivial representation. For other partitions λ we use the proven result for the symmetric relations. Let $\lambda = (\lambda_1 \geqslant \cdots \geqslant \lambda_r > 0)$ be a partition of $2m$. For each numbering of a Young diagram T let T_i denote the subset of $\{1, \dots, 2m\}$ containing elements of the ith row of T, for $i = 1, \dots, r$. Consider the product $P_T := \prod_{i=1}^{r} R_{T_i}$, where $R_{\{i,j\}}$ is defined to be $b_{i,j}$, while the other R_{T_i}'s are defined above when $|T_i| \geqslant 4$. Note that P_T doesn't change as T varies in an equivalence class $\{T\}$ since R_{T_i}'s are symmetric. This means that the assignment

$$v_T \to \sum_{q \in C(T)} \mathrm{sgn}(q) P_{q \cdot T},$$

is a well-defined S_{2m}-module morphism. This map is non-zero, hence an isomorphism onto its image. The result follows.   □

Since the relations of the form (3) hold in the tautological ring $R^*(C^n)$, we conclude that the pairing is perfect. This also shows that the relations stated in the proposition generate all relations in the tautological ring.   □

Remark 4.5. — The tautological ring $R^*(C^n)$, for a smooth curve C of genus g, was defined by C. Faber and R. Pandharipande (unpublished) as the $\mathbb{Q}$-subalgebra of $A^*(C^n)$ generated by the standard classes K_i and $D_{i,j}$. They show that the image $RH^*(C^n)$ in cohomology is Gorenstein. In [13] M. Green and P. Griffiths have shown that $R^*(C^2)$ is not Gorenstein, for C a generic complex curve of genus $g \geqslant 4$.

5. The tautological ring $R^*(\overline{U}_{n-1})$

DEFINITION 5.1. — *The tautological ring, $R^*(\overline{U}_{n-1})$ of $\overline{U}_{n-1}$, is defined to be the subalgebra of the Chow ring $A^*(\overline{U}_{n-1})$ generated by the tautological classes in $R^*(C^{n-1})$ and the classes E_I for $|I| \leqslant n - 3$.*

5.1. Relations in $R^*(\overline{U}_{n-1})$

- The first class of relations in $R^*(\overline{U}_{n-1})$ are those between the generators of $R^*(C^{n-1})$ described in Section 4.

- For subsets $I, J \subset \{1, \ldots, n\}$, where $|I|, |J| \leqslant n - 3$, the product $E_I \cdot E_J \in R^2(\overline{U}_{n-1})$ is zero unless

$$(*) \quad I \subseteq J, \quad \text{or } J \subseteq I, \quad \text{or } I \cup J = \{1, \ldots, n\}.$$

- For any subset $I \subset \{1, \ldots, n\}$, where $|I| \leqslant n - 3$, consider the inclusion

$$i_I : X_I \to C^{n-1}.$$

The relations

$$\{x \cdot E_I = 0 : x \in \ker(i_I^* : R^*(C^{n-1}) \to R^*(X_I))\}$$

hold. Note that the kernel of i_I^* coincides with the kernel of the map

$$R^*(C^{n-1}) \to R^*(C^{n-1})$$

defined by $x \to x \cdot X_I$. This follows since $(i_I)_*(i_I)^*(x) = x \cdot X_I$ for $x \in R^*(C^{n-1})$, and $(i_I)_*$ is injective in our case.

- As we saw in the third section, in blowing-up the variety Y along a subvariety $Z \subset Y$, if the center Z can be written as the transversal intersection of the subvarieties V and W of Y, then the class $P_{W/Y}(-E_Z)$ is in the ideal $J_{\widetilde{V}/\widetilde{Y}}$. This means that the product $P_{W/Y}(-E_Z) \cdot E_{\widetilde{V}}$ is zero, where $E_{\widetilde{V}}$ is the class of the exceptional divisor of the blow-up along the subvariety $\widetilde{V}$. We get a class of relations of this type by writing the centers of blow-ups introduced in the construction of the space $\overline{U}_{n-1}$ as transversal intersections in different ways. If the subvariety V can be written as a transversal intersection $V_1 \cap \cdots \cap V_k$, we obtain the relation $P_{W/Y}(-E_Z) \cdot E_{V_1} \cdots E_{V_k} = 0$.

- For each subvariety $Z \subset Y$ with a Chern polynomial $P_{Z/Y}(t)$, there is a relation

$$P_{Z/Y}(-E_Z) = 0,$$

where E_Z is the class of the exceptional divisor of the blow-up of Y along Z. These give another class of relations in $R^*(\overline{U}_{n-1})$. Note that for each subset I of $\{1, \ldots, n\}$, a Chern polynomial $P_{X_I/C^{n-1}}$ of the subvariety X_I is in $R^*(C^{n-1})[t]$. This means that a Chern polynomial of its proper transform under later blow-ups belongs to $R^*(\overline{U}_{n-1})$. It follows from Lemma 3.3, which relates a Chern polynomial $P_{V/Y}(t)$ of a subvariety V to that of its proper transform $\widetilde{V}$.

Example 5.2. — Suppose $Y = C^5$.

- Let X_0 be the point $O^5 \in C^5$. Then $\ker(i^* : R^*(C^5) \to R^*(X_0))$ consists of all elements of positive degree. It follows that $a_i \cdot E_0 = b_{i,j} \cdot E_0 = 0$ for all i and j.
- Let $X_1 = a_2 a_3 a_4 a_5$. From $a_1 \cap X_1 = X_0$ we get the relation $(a_1 - E_0) \cdot E_1 = 0$. If $X_{1,2,3} = a_4 a_5$ and $X_{4,5,6} = d_{1,2} d_{1,3}$, then the relation $(a_1 - E_0) \cdot E_{1,2,3} \cdot E_{4,5,6} = 0$ follows from the equality $a_1 \cap X_{1,2,3} \cap X_{4,5,6} = X_0$.
- A Chern polynomial of the subvariety X_0 is $\prod_{i=1}^{5}(a_i + t)$, from which we get the following relation:

$$\prod_{i=1}^{5}(a_i - E_0) = a_1 a_2 a_3 a_4 a_5 - E_0^5 = 0.$$

There are several special cases of the relations above which will be useful in the definition of standard monomials and in defining the dual elements:

LEMMA 5.3. — *Let I be a subset of the set $\{1, \ldots, n\}$ with at most $n - 3$ elements, containing n. For any $i \in I$ and $j, k \in \{1, \ldots, n\} - I$, the following relations hold in $A^2(\overline{U}_{n-1})$:*

$$a_j \cdot E_I = a_k \cdot E_I, \quad b_{j,k} \cdot E_I = -2a_j \cdot E_I,$$

$$b_{i,k} \cdot E_I = \left(\sum_{J \subseteq I - \{i\}} E_J - a_i - a_j \right) \cdot E_I.$$

Proof. — Recall that E_I is the exceptional divisor of the blow-up along the proper transform of the subvariety

$$X_I = \cap_{j \neq r \in \{1, \ldots, n\} - I} d_{j,r} = \cap_{k \neq r \in \{1, \ldots, n\} - I} d_{k,r}.$$

The equality $a_j \cdot E_I = a_k \cdot E_I$ follows since

$$a_j - a_k \in \ker\left(i_I^* : R^*(C^{n-1}) \to R^*(X_I)\right),$$

where $i_I : X_I \to C^{n-1}$ denotes the inclusion. We give another proof as well: From the following equality

$$X_I \cap a_j = X_I \cap a_k = X_{I-\{n\}},$$

we obtain the relation

$$\left(a_j - \sum_{J \subseteq I - \{n\}} E_J \right) \cdot E_I = \left(a_k - \sum_{J \subseteq I - \{n\}} E_J \right) \cdot E_I = 0.$$

This gives the first relation after canceling out $\left(\sum_{J \subseteq I - \{n\}} E_J \right) \cdot E_I$ on both sides.

The second equality results from the definition $b_{j,k} = d_{j,k} - a_j - a_k$, the relation $d_{j,k} \cdot E_I = 0$, and from the previous case.

To prove the last statement, first note that $b_{i,k} = d_{i,k} - a_i - a_k$, by definition. From the equality $X_I \cap d_{i,k} = X_{I-\{i\}}$, we get the relation

$$\left(d_{i,k} - \sum_{J \subseteq I-\{i\}} E_J \right) \cdot E_I = 0.$$

We conclude that

$$b_{i,k} \cdot E_I = \left(\sum_{J \subseteq I-\{i\}} E_J - a_i - a_k \right) \cdot E_I,$$

which proves the last statement, using that $a_j \cdot E_I = a_k \cdot E_I$. $\square$

5.2. Standard monomials

The existence of the relations stated above makes it possible to obtain a smaller set of generators for the tautological ring $R^*(\overline{U}_{n-1})$. Any monomial $v \in R^d(\overline{U}_{n-1})$ can be written as a product $a(v)b(v) \cdot E(v)$, where $a(v)$ is a product of a_i's, $b(v)$ is a product of $b_{j,k}$'s, and $E(v)$ is a product of exceptional divisors. To simplify the enumeration of the generators for $R^*(\overline{U}_{n-1})$, we introduce the directed graph associated to a monomial. We first define an ordering on the *polynomial ring*

$$R := \mathbb{Q}\Big[a_i, b_{j,k}, \ E_I : 1 \leqslant i \leqslant n-1, \ 1 \leqslant j < k \leqslant n-1,$$

$$I \subset \{1, \ldots, n\}, \text{ where } |I| \leqslant n-3 \Big].$$

DEFINITION 5.4. — *Let $I, J \subset \{1, \ldots, n\}$, we say that $I < J$ if*

- $|I| < |J|$
- *or if $|I| = |J|$ and the smallest element in $I - I \cap J$ is less than the smallest element of $J - I \cap J$.*

Put an arbitrary total order on monomials in

$$\mathbb{Q}[a_i, b_{j,k} : 1 \leqslant i \leqslant n-1, \ 1 \leqslant j < k \leqslant n-1].$$

Suppose $v_1, v_2 \in R$ are monomials. We say that $v_1 < v_2$ if we can write them as

$$v_1 = a(v_1)b(v_1) \cdot \prod_{r=1}^{r_0} E_{I_r}^{i_r} \cdot E, \quad \text{and} \quad v_2 = a(v_2)b(v_2) \cdot \prod_{r=1}^{r_0} E_{I_r}^{j_r} \cdot E,$$

where $E = \prod_{r=r_0+1}^{m} E_{I_r}^{i_r}$, for $I_1 < \cdots < I_m$, and $i_{r_0} < j_{r_0}$; or if $r_0 = 0$ and $a(v_1)b(v_1) < a(v_2)b(v_2)$.

Furthermore, we say that $v_1 \ll v_2$, if for any factor E_I of v_2 we have that $v_1 < E_I$.

DEFINITION 5.5. — Let $v = a(v)b(v) \cdot E_{I_1}^{i_1} \cdots E_{I_m}^{i_m}$, where $i_r \neq 0$ for $r = 1, \ldots, m$ and $I_1 < \cdots < I_m$, be a monomial. The directed graph $\mathcal{G} = (V_{\mathcal{G}}, E_{\mathcal{G}})$ associated to v is defined by the following data:

- A set $V_{\mathcal{G}}$ and a one-to-one correspondence between members of $V_{\mathcal{G}}$ and members of the set $\{1, \ldots, m\}$. Elements of $V_{\mathcal{G}}$ are called the vertices of $\mathcal{G}$.
- A set $E_{\mathcal{G}} \subset V_{\mathcal{G}} \times V_{\mathcal{G}}$ consisting of all pairs (r, s), where I_s is a minimal element of the set

$$\{I_i : I_r \subset I_i\}$$

with respect to inclusion. Elements of $E_{\mathcal{G}}$ are called the edges of $\mathcal{G}$.

For a vertex $i \in V_{\mathcal{G}}$, the closure $\bar{i} \subset V_{\mathcal{G}}$ is defined to be the subset

$$\{r \in V_{\mathcal{G}} : I_i \subseteq I_r\}$$

of $V_{\mathcal{G}}$. The degree $\deg(i)$ of i is defined to be the number of the elements of the set

$$\{j \in V_{\mathcal{G}} : (i, j) \in E_{\mathcal{G}}\}.$$

A vertex $i \in V_{\mathcal{G}}$ is called a root of $\mathcal{G}$ if I_i is minimal with respect to inclusion of sets. Maximal vertices of $\mathcal{G}$ are called external and all the other vertices will be called internal.

In the following, we use the letters $I_1, \ldots, I_m$ to denote the vertices of $\mathcal{G}$.

Remark 5.6. — We can define a directed graph associated to any collection of subsets of the set $\{1, \ldots, n\}$ in a similar way. In general there may be a loop in the resulting graph after forgetting the directions. Loops don't occur in the graph associated to a non-zero monomial v. Hence, we refer to $\mathcal{G}$ as the associated forest of the monomial v, or of the collection $\{I_1, \ldots, I_m\}$.

The next lemma turns out to be useful in defining the dual element:

LEMMA 5.7. — Suppose that $I_1, \ldots, I_m$ are proper subsets of the set $\{1, \ldots, n\}$, containing at most $n-3$ elements, with the property that each pair I_r and I_s satisfy $(*)$. Let $\mathcal{G}$ be the associated forest. If $n \notin \cap_{r=1}^{m} I_r$, then there is a unique root of $\mathcal{G}$ not containing n.

Proof. — By assumption, there is a root I_r such that $n \notin I_r$. Uniqueness follows since for any two roots I_r, I_s of $\mathcal{G}$ the equality $I_r \cup I_s = \{1, \ldots, n\}$ holds. This means that their complements I_r^c, I_s^c are disjoint. Hence, n belongs to the complement of at most one root. $\square$

DEFINITION 5.8. — *Let v be as in Definition 5.5, $\mathcal{G}$ be the associated forest, and $J_1, \ldots, J_s$, for some $s \leqslant m$, be roots of $\mathcal{G}$. For each $1 \leqslant r \leqslant s$ such that $n \in J_r$, let $\alpha_r \in \{1, \ldots, n\} - J_r$ be the smallest element. The subset S of the set $\{1, \ldots, n-1\}$ is defined as follows:*

- *If $n \in \cap_{r=1}^{m} I_r$, put*

$$S := \{\alpha_1, \ldots, \alpha_s\} \cup (\cap_{r=1}^{m} I_r - \{n\}),$$

- *if $n \notin \cap_{r=1}^{m} I_r$, let J_1 be the unique root of $\mathcal{G}$ not containing n. In this case*

$$S := \{\alpha_2, \ldots, \alpha_s\} \cup (\cap_{r=1}^{m} I_r).$$

The monomial v is said to be standard if

- *The monomial $a(v)b(v) \in R^*(C^S)$ is in standard form according to the definition given in the fourth section.*
- *For each r we have that*

$$i_r \leqslant \min(n - 2 - |I_r|, |\cap_{I_r \subset I_s} I_s| - |I_r| + \deg(I_r) - 2).$$

We now prove that standard monomials form a basis of the tautological ring:

PROPOSITION 5.9. — *The tautological ring $R^*(\overline{U}_{n-1})$ of $\overline{U}_{n-1}$ is spanned by standard monomials.*

Proof. — Let v be a monomial given as in Definition 5.5. We have seen that any monomial in $R^d(C^{n-1})$ can be written in standard form for $0 \leqslant d \leqslant n-1$. By Lemma 5.3, we may assume that $a(v)b(v)$ is an element of the tautological ring $R^*(C^S)$ of C^S, where S is defined according to Definition 5.8. The statement is proven using induction and from the following observations:

- From the last class of relations in 5.1, for any subset I_r of the set $\{1, \ldots, n\}$, where $|I_r| \leqslant n - 3$, we can write $E_{I_r}^{n-1-|I_r|}$ as a sum of elements which are strictly less than it.
- Let $\{J_1, \ldots, J_s\}$ be the set of minimal elements of the set

$$\{I_i : I_r \subset I_i, \text{ where } 1 \leqslant i \leqslant m\}.$$

From the third class of relations in 5.1, the monomial $E_{I_r}^{j} \cdot \prod_{i=1}^{s} E_{J_i}$ can be written as a sum of terms which are strictly less than it, where $j = |\cap_{i=1}^{s} J_i| - |I_r| + s - 1$.

□

5.3. Definition of the dual element

Now suppose $v \in R^d(\overline{U}_{n-1})$ is an element of degree d written in standard form. Below, we define the element v^*, which is an element of $R^{n-1-d}(\overline{U}_{n-1})$. As we will see, the property $v^{**} = v$ holds. This shows that there is a one-to-one correspondence between standard monomials in degree d and $n-1-d$.

DEFINITION 5.10. — *Suppose $v = a(v)b(v) \cdot E(v)$ is a standard monomial, where $a(v)b(v)$ is in the tautological ring $R^*(C^{n-1})$ of C^{n-1}, and*

$$E(v) = \prod_{r=1}^{m} E_{I_r}^{i_r},$$

where $i_r \neq 0$ for $r = 1, \ldots, m$, and $I_1 < \cdots < I_m$. Let $\mathcal{G}$ be the associated forest, and $J_1, \ldots, J_s$, the set S and $\alpha_r \in J_r$ for $1 \leqslant r \leqslant s$ be as in Definition 5.8. The subset T of the set S is defined as follows:

- *If $n \in \cap_{r=1}^m I_r$ put*

$$T := S - (A_v \cup B_v \cup \{n\}),$$

- *if $n \notin \cap_{r=1}^m I_r$, let J_1 be the unique root of $\mathcal{G}$ not containing n. In this case*

$$T := S - (A_v \cup B_v).$$

For each $1 \leqslant r \leqslant m$, define j_r to be

$$\begin{cases} |\cap_{I_r \subset I_s} I_s| - |I_r| + \deg(I_r) - 1 - i_r & \text{if } I_r \text{ is an internal vertex of } \mathcal{G} \\ \\ n - 1 - |I_r| - i_r & \text{if } I_r \text{ is an external vertex of } \mathcal{G}. \end{cases}$$

We define $v^* = a(v^*)b(v^*) \cdot E(v^*)$, where

$$a(v^*) = \prod_{i \in T} a_i, \quad b(v^*) = b(v), \quad E(v^*) = \prod_{r=1}^{m} E_{I_r}^{j_r}.$$

Remark 5.11. — We verify that the dual element v^* is well-defined. We need to show that the integers j_r are non-negative. These integers are indeed positive numbers for $r = 1, \ldots, m$. In the definition of standard monomials we have seen that

$$i_r \leqslant |\cap_{I_r \subset I_s} I_s| - |I_r| + \deg(I_r) - 2.$$

This shows that $j_r \geqslant 1$ when $i_r > 0$ and I_r is an internal vertex. The case of external vertices is treated using the inequality $i_r \leqslant n - 2 - |I_r|$.

The following corollary follows from Definition 5.10.

COROLLARY 5.12. — *Suppose $v \in R^d(\overline{U}_{n-1})$ is a standard monomial and let $v^* \in R^{n-1-d}(\overline{U}_{n-1})$ be its dual. Then v^* is a standard monomial, and furthermore $v^{**} = v$.*

The next lemma will be useful in the proof of Proposition 5.16 and identity (5):

LEMMA 5.13. — *Let $v = a(v)b(v) \cdot E(v)$ be as in Definition 5.10, and $\mathcal{G}$ be the associated forest. For a vertex $i \in V_{\mathcal{G}}$ corresponding to the subset I_i of the set $\{1, \ldots, n\}$, the equality*

$$\sum_{\bar{i}} (i_r + j_r) = n - 1 - |I_i|$$

holds. Here $\bar{i}$ is the closure of i in $\mathcal{G}$ defined in Definition 5.5.

Proof. — It is immediate from the definition of the j_r's above. $\qquad\square$

5.4. The pairing $R^d(\overline{U}_{n-1}) \times R^{n-1-d}(\overline{U}_{n-1})$

In the previous part, we defined dual elements for standard monomials. Below, we will see that with respect to the ordering of the generators of the Chow groups given in 5.4 the resulting intersection matrix between the standard monomials and their duals consists of square blocks on the main diagonal, whose entries are, up to a sign, intersection numbers in $R^{|S|}(C^S)$, for certain sets S, and all blocks below the diagonal are zero. To prove the stated properties of the intersection matrix, we introduce a natural filtration [1] on the tautological ring.

DEFINITION 5.14. — *Let v be a standard monomial as given in Definition 5.8, and let $J_1, \ldots, J_s$ be roots of the associated forest. Define $p(v)$ to be the degree of the element*

$$a(v)b(v) \cap_{r=1}^s X_{J_r} \in A^*(C^{n-1}),$$

which is the same as the integer

$$\deg a(v)b(v) + n - |\cap_{r=1}^s J_r| - s.$$

---

[1] The definition of this filtration on the tautological ring was formulated after a question of E. Looijenga.

The subspace $F^p R^*(\overline{U}_{n-1})$ of the tautological ring is defined to be the $\mathbb{Q}$-vector space spanned by standard monomials v satisfying $p(v) \geqslant p$.

PROPOSITION 5.15.

(a) For any integer p, we have that $F^{p+1} R^*(\overline{U}_{n-1}) \subseteq F^p R^*(\overline{U}_{n-1})$.

(b) Let $v \in F^p R^*(\overline{U}_{n-1})$ and $w \in R^d(\overline{U}_{n-1})$ be such that $w \ll v$. If $p + d \geqslant n$, then $v \cdot w$ is zero. In particular, $F^n R^*(\overline{U}_{n-1})$ is zero.

Proof. — The first statement is immediate from Definition 5.14. Let us prove (b). Let v be given as in Definition 5.8. Denote by Y the blow-up of C^{n-1} corresponding to the collection

$$\{J \subset \{1, \ldots, n\} : J < J_r \text{ for } 1 \leqslant r \leqslant s\}$$

Note that the dimension $\dim \cap_{r=1}^s X_{J_r}$ of the transversal intersection $\cap_{r=1}^s X_{J_r}$ is equal to $|\cap_{r=1}^s J_r| + s - 1$. The product

$$a(v)b(v) \cdot \prod_{r=1}^s \widetilde{X}_{J_r} \cdot w \in A^*(Y)$$

is zero since its degree, which is

$$\deg(a(v)b(v)) + n - |\cap_{r=1}^s J_r| - s + d,$$

is at least n, by assumption. This means that the product

$$a(v)b(v) \cdot \prod_{r=1}^s E_{J_r} \cdot w \in R^*(\overline{U}_{n-1}),$$

which is a factor of $v \cdot w$, is zero as well.   □

Using the proven lemma we are able to prove the following vanishing result:

PROPOSITION 5.16. — Suppose $v_1, v_2 \in R^d(\overline{U}_{n-1})$ are standard monomials satisfying $E(v_1) < E(v_2)$. Then $v_1 \cdot v_2^* = 0$.

Proof. — It is enough to write $v_1 \cdot v_2^*$ as a product $v \cdot w$, for $v, w \in R^*(\overline{U}_{n-1})$ satisfying the properties given in Proposition 5.15. To find v and w, let v_1, v_2 be given as in Definition 5.4, and denote by $\{J_1, \ldots, J_s\}$ the set of roots of the graph associated to the monomial $E = \prod_{r=r_0+1}^m E_{I_r}^{i_r}$. By relabeling the roots we may assume that there is an $s_0 \geqslant 0$ such that $I_{r_0} \subset J_r$ for $1 \leqslant r \leqslant s_0$, and the equality $I_{r_0} \cup J_r = \{1, \ldots, n\}$ holds for $s_0 < r \leqslant s$. Let w be the product of all monomials in $v_1 \cdot v_2^*$ which are strictly less than $E_{I_{r_0}}$ and v be the product of the other factors, so that

$v_1 \cdot v_2^* = v \cdot w$. Notice that $w \ll v$, by the definition of v and w. The degree d of w is computed using Lemma 5.13:

$$d = n + j_{r_0} - i_{r_0} + s - s_0 - |I_{r_0}^c| - \sum_{r=s_0+1}^{s} |J_r^c|$$

$$= j_{r_0} - i_{r_0} + s - s_0 + |I_{r_0} \cap J_{s_0+1} \cap \cdots \cap J_s|$$

$$\geqslant s - s_0 + 1 + |I_{r_0} \cap J_{s_0+1} \cap \cdots \cap J_s| = n - p(v).$$

From $d + p(v) \geqslant n$ we see that the product $v \cdot w$ is zero. □

To study the blocks on the main diagonal we proceed as follows: We first prove an identity which reduces the number of exceptional divisors in the product for certain monomials. Let Y be a blow-up of C^{n-1} at some step in the construction of $\overline{U}_{n-1}$. Suppose that

$$V_1 \cap \cdots \cap V_k \cap W = Z$$

is a transversal intersection of tautological classes, where $W = D_1 \cap \cdots \cap D_r$ is a transversal intersection of divisors $D_1, \ldots, D_r \in R^1(Y)$, and let $f \in R^*(Y)$ be an element of degree $d = dim(Z)$. Denote by E_Z the exceptional divisor of the blow-up $Bl_Z Y$ of Y along Z and by $E_{V_1}, \ldots, E_{V_k}$ those of the blow-up $\widetilde{Y}$ of $Bl_Z Y$ along the proper transform of the subvarieties $V_1, \ldots, V_k$. It follows from Lemma 3.5 that $P_{W/Y}(-E_Z) \in J_{\widetilde{V}/\widetilde{Y}}$, for $V = V_1 \cap \cdots \cap V_k$. Using the same argument as in Lemma 3.6 we observe that the equality

$$f \cdot E_Z^r \cdot E_{V_1} \cdots E_{V_k} = (-1)^{r-1} f \cdot W \cdot E_{V_1} \cdots E_{V_k}$$

holds in $R^{r+d+k}(\widetilde{Y})$. If the codimension of the subvariety V_i is r_i and that of Z is r_0, then from the proven result in Lemma 3.6 one gets the following identity:

$$(4) \qquad f \cdot E_Z^r \cdot E_{V_1}^{r_1} \cdots E_{V_k}^{r_k} = (-1)^{r_0-k-1} f \cdot Z.$$

Notice that this identity reduces the computation of certain monomials containing the exceptional divisors to one which belongs to the Chow ring $A^*(Y)$. We now use this identity to compute the numbers occurring on the main diagonal of the intersection matrix.

Let $I_1, \ldots, I_m$ be subsets of the set $\{1, \ldots, n\}$ containing at most $n-3$ elements such that for every pair I_r and I_s the property $(*)$ holds. Let $\mathcal{G}$ be the associated forest of the collection $I_1, \ldots, I_m$ and $J_1, \ldots, J_s$, the set S and $\alpha_r \in J_r$ for $1 \leqslant r \leqslant s$ be as in Definition 5.8. If $n \notin \cap_{r=1}^m I_r$, let J_1

be the unique root of $\mathcal{G}$ not containing n. Define

$$E := \prod_{r=1}^{m} E_{I_r}^{i_r},$$

where

$$i_r = \begin{cases} |\cap_{I_r \subset I_s} I_s| - |I_r| + \deg(I_r) - 1 & \text{if } I_r \text{ is an internal vertex of } \mathcal{G} \\ \\ n - 1 - |I_r| & \text{if } I_r \text{ is an external vertex of } \mathcal{G}. \end{cases}$$

Consider an element $f \in \mathbb{Q}[a_i, b_{i,j} : i, j \in S]$, of degree $|\cap_{r=1}^{m} I_r| + s - 1$. Then from the identity (4) it follows that

$$(5) \qquad\qquad f \cdot E = (-1)^{\varepsilon} f \cdot \prod_{i \in \{1,\ldots,n-1\} - S} a_i.$$

where $\varepsilon = n + |\cap_{r=1}^{m} I_r| + \sum_{i \in V(G)} \deg(i)$, using Lemma 5.13.

Note that for any $v \in R^d(\overline{U}_{n-1})$, the product $E := E(v) \cdot E(v^*)$ is in the form given above according to Definition 5.10.

THEOREM 5.17. — For any $0 \leqslant d \leqslant n-1$, the pairing

$$R^d(\overline{U}_{n-1}) \times R^{n-1-d}(\overline{U}_{n-1}) \to \mathbb{Q}$$

is perfect.

Proof. — Let $A := \{v_1, \ldots, v_r\} \subset R^d(\overline{U}_{n-1})$, where $v_1 < \cdots < v_r$, be the set of standard monomials of degree d, and $\{v_1^*, \ldots, v_r^*\} \subset R^{n-1-d}(\overline{U}_{n-1})$ be the set of their duals defined above. For a monomial

$$E \in \mathbb{Q}[E_I : I \subset \{1, \ldots, n\} \text{ and } |I| \leqslant n-3],$$

define

$$A_E := \{v \in A : E(v) = E\}.$$

Let $\mathcal{G}$ be the graph associated to the monomial E, and define S as in Definition 5.8. For $v_i, v_j \in A_E$ the number

$$v_i \cdot v_j^* \in R^{n-1}(\overline{U}_{n-1}) = \mathbb{Q}$$

is the same as

$$(-1)^{\varepsilon} a(v_i) b(v_i) \cdot a(v_j^*) b(v_j^*) \in R^S(C^S) = \mathbb{Q},$$

by the identity (5), where $\varepsilon = n + |\cap_{r=1}^{m} I_r| + \sum_{i \in V(\mathcal{G})} \deg(i)$.

This means that the intersection matrices $(v_i \cdot v_j^*)$ and $(a(v_i) b(v_i) \cdot a(v_j^*) b(v_j^*))$, for v_i, v_j in the set A_E, are the same up to a sign after the identifications above. From the study of the tautological ring $R^*(C^S)$ of C^S, we know that the kernel of the matrix above is generated by relations

in $R^*(C^S)$. After choosing a basis for $R^{d-\deg(E)}(C^S)$, the resulting matrix is invertible. It means that the intersection matrix of the pairing

$$R^d(\overline{U}_{n-1}) \times R^{n-1-d}(\overline{U}_{n-1}) \to \mathbb{Q}$$

with this choice of basis elements for the tautological groups consists of invertible blocks on the main diagonal and zero blocks below the diagonal, hence is invertible. This proves the claim. $\qquad\square$

6. The tautological ring $R^*(M_{1,n}^{ct})$

In the first part we obtained the morphism $F : \overline{U}_{n-1} \to M_{1,n}^{ct}$, induced from the family of curves $\pi : \overline{U}_n \to \overline{U}_{n-1}$. The morphism F induces a ring homomorphism

$$F^* : A^*(M_{1,n}^{ct}) \to A^*(\overline{U}_{n-1}).$$

For a subset $J \subset \{1, \ldots, n\}$, the pull back $F^*(D_J)$ of the divisor class D_J is a subvariety of $\overline{U}_{n-1}$ for which the fiber $\pi^{-1}(P)$ is a nodal curve of type given by D_J, when P is its generic point. It follows that P is a point of the proper transform of the subvariety X_I, where $I := \{1, \ldots, n\} - J$, when $|J| \geqslant 3$, and belongs to the proper transform of the divisors $a_i, d_{j,k}$ if $J = \{i, n\}, \{j, k\}$, for $1 \leqslant i \leqslant n - 1$ and $1 \leqslant j < k \leqslant n - 1$. But the proper transform of X_I is equal to E_I, when $|I| \leqslant n - 3$, and those of the divisors $a_i, d_{j,k}$ are $a_i - \sum_{i \notin I \subset \{1, \ldots, n-1\}} E_I$ and $d_{j,k} - \sum_{I \subset \{1, \ldots, n\} - \{j,k\}} E_I$ respectively, for $1 \leqslant i \leqslant n - 1$ and $1 \leqslant j < k \leqslant n - 1$. This means that

$$F^*(D_{\{i,n\}}) = a_i - \sum_{i \notin I \subset \{1, \ldots, n-1\}} E_I \text{ for } 1 \leqslant i \leqslant n - 1,$$

$$F^*(D_{\{j,k\}}) = d_{j,k} - \sum_{I \subset \{1, \ldots, n\} - \{j,k\}} E_I \text{ for } 1 \leqslant j < k \leqslant n - 1,$$

$$F^*(D_J) = E_I, \quad \text{where } I = \{1, \ldots, n\} - J \text{ for } |J| \geqslant 3.$$

From this we see that the pull-back homomorphism F^* sends tautological classes to tautological classes, and defines a ring homomorphism

$$F^* : R^*(M_{1,n}^{ct}) \to R^*(\overline{U}_{n-1}).$$

If we rewrite the expressions above, we get that

$$a_i = F^*\Big(\sum_{i,n \in J} D_J\Big) \text{ for } 1 \leqslant i \leqslant n - 1,$$

$$d_{i,j} = F^*\Big(\sum_{j,k \in J} D_J\Big) \text{ for } 1 \leqslant j < k \leqslant n - 1,$$

$$E_I = F^*(D_J), \quad \text{where } J = \{1, \ldots, n\} - I \text{ for } |I| \leqslant n - 3.$$

This shows that F^* is a surjection. We prove that F^* is injective by extending the function

$$G : \{a_i, d_{j,k}, E_I : 1 \leqslant i \leqslant n-1, 1 \leqslant j < k \leqslant n-1, I \subset \{1,\dots,n\}$$
$$\text{and } |I| \leqslant n-3\} \to R^*(M_{1,n}^{ct}),$$

defined by the rule

$$G(a_i) = \sum_{i,n \in J} D_J \text{ for } 1 \leqslant i \leqslant n-1,$$

$$G(d_{j,k}) = \sum_{j,k \in J} D_J \text{ for } 1 \leqslant i \leqslant n-1 \text{ and } 1 \leqslant j < k \leqslant n-1,$$

$$G(E_I) = D_{I^c} \text{ for } |I| \leqslant n-3$$

to a ring homomorphism

$$G : R^*(\overline{U}_{n-1}) \to R^*(M_{1,n}^{ct}).$$

This is done by verifying that all relations between elements $a_i, d_{j,k}, E_I$'s on the left hand side hold between classes on the right hand side. To simplify the notation, we drop the letter G for tautological classes in $R^*(M_{1,n}^{ct})$. For instance, we write $a_i = \sum_{i,n \in J} D_J$ and $E_I = D_{I^c}$ for a subset $I \subset \{1,\dots,n\}$ with $|I| \leqslant n-3$.

Let us introduce the following notation: Suppose S is a subset of the set $\{1,\dots,n\}$. By $M_{1,S}^{ct}$, we mean the moduli space of stable curves of genus one of compact type whose marking set is S. Let

$$\pi_S : M_{1,n}^{ct} \to M_{1,S}^{ct}$$

be the projection which forgets the markings in $\{1,\dots,n\} - S$ and contracts unstable components.

- We first deal with relations among generators of $R^*(C^{n-1})$: Notice that

$$a_i = \pi^*_{\{i,n\}}(D_{\{i,n\}}), \quad d_{j,k} = \pi^*_{\{j,k\}}(D_{\{j,k\}}),$$
$$b_{j,k} = \pi^*_{\{j,k,n\}}(D_{\{j,k\}} - D_{\{j,n\}} - D_{\{k,n\}} - D_{\{j,k,n\}}).$$

From the relations $D_{\{i,n\}}^2 = D_{\{j,k\}}^2 = 0$ in $R^2(M_{1,\{i,n\}}^{ct})$ and $R^2(M_{1,\{j,k\}}^{ct})$, we obtain that the relations $a_i^2 = d_{j,k}^2 = 0$ hold in $R^2(M_{1,n}^{ct})$, for $1 \leqslant i \leqslant n-1$ and $1 \leqslant j < k \leqslant n-1$. On the other hand, the relation

$$(D_{\{i,j\}} - D_{\{i,n\}} - D_{\{j,n\}} - D_{\{i,j,n\}}) \cdot (D_{\{i,n\}} + D_{\{i,j,n\}})$$
$$= 0 \in R^2(M_{1,\{i,j,n\}}^{ct})$$

says that $a_i b_{i,j} = 0$. From the relation $d_{j,k}^2 = 0$ obtained above, we get that $b_{j,k}^2 = -2a_j a_k$. Now suppose that i, j, k, l are distinct elements of the set $\{1, \ldots, n-1\}$. As we saw in the fourth section, the relation (1) in $R^2(M_{1,\{i,j,k,n\}}^{ct})$ can be written as

$$12(a_i b_{j,k} - b_{i,j} b_{i,k}) = 0.$$

The relation

$$12(b_{i,j} b_{k,l} + b_{i,k} b_{j,l} + b_{i,l} b_{j,k}) = 0$$

is the pull-back of the relation above to $R^2(M_{1,\{i,j,k,l,n\}}^{ct})$ along the morphism

$$\pi_{\{i,j,k,n\}} : M_{1,\{i,j,k,l,n\}}^{ct} \to M_{1,\{i,j,k,n\}}^{ct}.$$

This shows that the classes $a_i, b_{j,k} \in R^*(M_{1,n}^{ct})$ satisfy all relations among $a_i, b_{j,k} \in R^*(\overline{U}_{n-1})$.

- Note that the following

$$D_I \cdot D_J \neq 0 \Rightarrow I \subseteq J \quad \text{or } J \subseteq I \quad \text{or } I \cap J = \emptyset$$

is true. But this can be written as

$$E_I \cdot E_J \neq 0 \Rightarrow I \subseteq J \quad \text{or } J \subseteq I \quad \text{or } I \cup J = \{1, \ldots, n\}.$$

This proves that the E_I's in $R^*(M_{1,n}^{ct})$ satisfy this class of relations between E_I's in $R^*(\overline{U}_{n-1})$ obtained above.

- For any $I \subset \{1, \ldots, n\}$ with $|I| \leqslant n - 3$, we found the relations $x.E_I = 0$ for $x \in \ker(i_I^*)$, where $i_I : X_I \to C^{n-1}$ denotes the inclusion. If $n \notin I$, then $\ker(i_I^*)$ is generated by divisor classes $a_i, b_{i,j}$, where $i \in J := \{1, \ldots, n-1\} - I$, and $j \in \{1, \ldots, n-1\}$ is different from i. Let us see that $a_i \cdot E_I = 0$ in this case:

$$a_i \cdot E_I = D_J^2 + \sum_{J_0 : i, n \in J_0 \subset J} D_{J_0} \cdot D_J + \sum_{J_0 : i, n \in J_0, J \subset J_0} D_{J_0} \cdot D_J.$$

But this expression is zero from the following known formula for ψ classes in genus zero and one:

PROPOSITION 6.1.
(a) The following relation holds in $A^1(\overline{M}_{0,n})$:

$$\psi_i = \sum_{j,k \notin J, i \in J, |J| \geqslant 2} D_J$$

for some fixed distinct $j, k \in \{1, \ldots, n\} - \{i\}$.

(b) *The following relation holds in $A^1(M_{1,n}^{ct})$:*

$$\psi_i = \sum_{i \in J, |J| \geqslant 2} D_J.$$

Proof.

(a) is Proposition 1.6 in [1].

To prove (b), it is enough to restrict the divisor classes given in Proposition 1.9 of [1], to the space $M_{1,n}^{ct}$. □

If $i \in \{1, \ldots, n-1\} - I$, and $j \in \{1, \ldots, n-1\}$ is distinct from i, we saw that $a_i \cdot E_I = 0$, and by the same argument as above we see that $a_j \cdot E_I = d_{i,j} \cdot E_J$, from which it follows that

$$b_{i,j} \cdot E_I = (d_{i,j} - a_i - a_j) \cdot E_I = 0.$$

The case $n \in I$ is proven by the same argument.

• We get a relation $P_{W/Y}(-E_Z) \cdot E_{V_1} \cdots E_{V_k} = 0$, when the subvariety Z is a transversal intersection $V_1 \cap \cdots \cap V_k \cap W$. After possibly relabeling the indices, we can assume that

$$Z = X_{I_0}, \quad V_i = X_{I_i}, \text{ for } 1 \leqslant i \leqslant k,$$

$$W = \prod_{i=r_0+1}^{r_1} a_i \cdot \prod_{j=1}^{k-1} a_{r_j+1},$$

where $r_0 \leqslant r_1 < \cdots < r_k < n$, and $I_0 = \{1, \ldots, r_0\}, I_i^c = \{r_i + 1, \ldots, r_{i+1}\}$, for $1 \leqslant i < k$, and $I_k^c = \{r_k + 1, \ldots, n\}$. Let us prove that

$$P_{W/C^{n-1}}\left(-\sum_{J \subseteq I_0} E_I\right) \cdot E_{I_1} \cdots E_{I_k} = 0 \in R^{r_1-r_0+2k-1}(M_{1,n}^{ct})$$

by showing that any monomial in the expansion of this expression is zero. Let

$$\prod_{i=r_0+1}^{r_1} E_{J_i} \cdot \prod_{j=1}^{k-1} E_{J_{r_j+1}} \cdot E_{I_1} \cdots E_{I_k},$$

where

$$i, n \in J_i^c \text{ for } r_0 + 1 \leqslant i \leqslant r_1, \text{ and } r_j + 1, n \in J_j^c \text{ for } 1 \leqslant j \leqslant k-1,$$

be any such monomial.

For $r_0 + 1 \leqslant i \leqslant r_1$, if the product $E_{J_i} \cdot E_{I_k}$ is non-zero, then $J_i \subseteq I_k - \{i\}$. For $1 \leqslant j \leqslant k-1$, if the product $E_{J_{r_j+1}} \cdot E_{I_j}$ is non-zero, then $J_{r_j+1} \subseteq I_j$. On the other hand, since $n \notin J_i, J_{r_j+1}$ for all i, j, the product $E_{J_{i_1}} \cdot E_{J_{i_2}}$ is non-zero only if $J_{i_1} \subseteq J_{i_2}$ or

$J_{i_2} \subseteq J_{i_1}$. It follows that the subsets J_i's are totally ordered by inclusion, which means that their intersection is one of them. We conclude that for some i the inclusion $J_i \subseteq I_0$ holds. But this term is excluded from expression above, whence the product is zero.

- For any subset $I \subset \{1, \ldots, n\}$, where $|I| \leqslant n - 3$, we prove that $P_{X_I/C^{n-1}}(-\sum_{J \subseteq I} E_J)$ is zero by the same argument as in the previous case, by showing that the monomials occurring in the expansion of the expression above are all zero.

The argument above shows that F^* is an isomorphism, and hence, our proof of Theorem 0.1 is complete.

Appendix: Derivation of the relations (2) and (3)

In this appendix we explain why the relations (2) and (3) follow from Getzler's relation (1). First note that the restriction of the relation (1) to the space $M_{1,4}^{ct}$ becomes

(6) $$12\delta_{2,2} - 4\delta_{2,3} - 2\delta_{2,4} + 6\delta_{3,4} = 0.$$

Then we compute the pull-back of the classes above to the space $\overline{U}_3$ along the morphism

$$F : \overline{U}_3 \to M_{1,4}^{ct}.$$

Recall that

$$\delta_{2,2} = D_{\{1,2\}} \cdot D_{\{3,4\}} + D_{\{1,3\}} \cdot D_{\{2,4\}} + D_{\{1,4\}} \cdot D_{\{2,3\}},$$

$$\delta_{2,3} = D_{\{1,2\}} \cdot D_{\{1,2,3\}} + D_{\{1,2\}} \cdot D_{\{1,2,4\}} + D_{\{1,3\}} \cdot D_{\{1,2,3\}}$$
$$+ D_{\{1,3\}} \cdot D_{\{1,3,4\}} + D_{\{1,4\}} \cdot D_{\{1,2,4\}} + D_{\{1,4\}} \cdot D_{\{1,3,4\}}$$
$$+ D_{\{2,3\}} \cdot D_{\{1,2,3\}} + D_{\{2,3\}} \cdot D_{\{2,3,4\}} + D_{\{2,4\}} \cdot D_{\{1,2,4\}}$$
$$+ D_{\{2,4\}} \cdot D_{\{2,3,4\}} + D_{\{3,4\}} \cdot D_{\{1,3,4\}} + D_{\{3,4\}} \cdot D_{\{2,3,4\}},$$

$$\delta_{2,4} = D_{\{1,2\}} \cdot D_{\{1,2,3,4\}} + D_{\{1,3\}} \cdot D_{\{1,2,3,4\}} + D_{\{1,4\}} \cdot D_{\{1,2,3,4\}}$$
$$+ D_{\{2,3\}} \cdot D_{\{1,2,3,4\}} + D_{\{2,4\}} \cdot D_{\{1,2,3,4\}} + D_{\{3,4\}} \cdot D_{\{1,2,3,4\}},$$

$$\delta_{3,4} = D_{\{1,2,3\}} \cdot D_{\{1,2,3,4\}} + D_{\{1,2,4\}} \cdot D_{\{1,2,3,4\}} + D_{\{1,3,4\}} \cdot D_{\{1,2,3,4\}}$$
$$+ D_{\{2,3,4\}} \cdot D_{\{1,2,3,4\}}.$$

From the argument given in Section 6 we see that

$$F^*(D_I) = E_{\{1,2,3,4\}-I} \text{ when } |I| = 3, 4,$$
$$F^*(D_{\{i,4\}}) = a_i - E_0 - E_j - E_k,$$
$$F^*(D_{\{j,k\}}) = d_{j,k} - E_0 - E_i - E_4,$$

for $1 \leqslant i \leqslant 3$ and $j \neq k \in \{1, 2, 3\} - \{i\}$, from which we conclude that

$$F^*(\delta_{2,2}) = a_1 d_{2,3} + a_2 d_{1,3} + a_3 d_{1,2} + 3E_0^2,$$
$$F^*(\delta_{2,3}) = 3(a_1 a_2 + a_1 a_3 + a_2 a_3 + d_{1,2} d_{1,3})$$
$$\qquad\qquad + 3(4E_0 + E_1 + E_2 + E_3 + E_4)E_0,$$
$$F^*(\delta_{2,4}) = -3(2E_0 + E_1 + E_2 + E_3 + E_4)E_0,$$
$$F^*(\delta_{3,4}) = (E_1 + E_2 + E_3 + E_4)E_0.$$

If we substitute the expressions above into the relation (6), we get that

$$12(a_1 d_{2,3} + a_2 d_{1,3} + a_3 d_{1,2} - a_1 a_2 - a_1 a_3 - a_2 a_3 - d_{1,2} d_{1,3})$$
$$= 12(a_1 b_{2,3} - b_{1,2} b_{1,3}) = 0.$$

The push-forward of the relation above via the blow-down map to C^3 gives the relation (2).

To get the relation (3), notice that the pull-back of the divisor class a_i, for $1 \leqslant i \leqslant 3$, via the morphism $\pi : \overline{U}_4 \to \overline{U}_3$, is $d_{i,4}$, and that of $b_{i,j}$, for $1 \leqslant i, j \leqslant 3$, is $d_{i,j} - d_{i,4} - d_{j,4}$. From this observation it is easy to see that the pull-back of the relation above to $R^2(\overline{U}_4)$ along the morphism π gives a relation whose push-forward to $R^2(C^4)$ via the blow-down map is the relation (3).

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