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QUANTUM SINGULARITY THEORY FOR $A_{(r-1)}$ AND r-SPIN THEORY

by Huijun FAN, Tyler JARVIS & Yongbin RUAN (*)

Abstract. — We give a review of our construction of a cohomological field theory for quasi-homogeneous singularities and the r-spin theory of Jarvis-Kimura-Vaintrob. We further prove that for a singularity W of type A our construction of the stack of W-curves is canonically isomorphic to the stack of r-spin curves described by Abramovich and Jarvis. We further prove that our theory satisfies all the Jarvis-Kimura-Vaintrob axioms for an r-spin virtual class. Therefore, the Faber-Shadrin-Zvonkine proof of the Witten Integrable Hierarchies Conjecture for r-spin curves applies to our theory for A-type singularities; that is, the total descendant potential function of our theory for A-type singularities satisfies the corresponding Gelfand-Dikii integrable hierarchy.

Résumé. — Nous passons en revue notre construction d'une théorie cohomologique des champs pour les singularités quasi-homogènes et la théorie des courbes r-spin de Jarvis-Kimura-Vaintrob. De plus, nous prouvons que pour une singularité W de type A notre construction du champ algébrique des W-courbes est canoniquement isomorphe au champ algébrique des courbes r-spin décrit par Abramovich et Jarvis. En outre, nous prouvons que notre théorie satisfait tous les axiomes de Jarvis-Kimura-Vaintrob pour une classe virtuelle r-spin. Par conséquent, la preuve de Faber-Shadrin-Zvonkine de la conjecture des hiérarchies intégrables de Witten pour les courbes r-spin s'applique à notre théorie des singularités de type A. C'est-à-dire, la fonction potentielle descendante totale de notre théorie des singularités de type A satisfait la hiérarchie intégrable de Gelfand-Dikii.

Keywords: FJRW, Mirror symmetry, r-spin curve, spin curve, Witten, Cohomological field theory, moduli, Gelfand-Dikii, integrable hierarchy.
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1. Introduction

In the paper [5] we introduce a family of moduli spaces, a virtual cycle, and a corresponding cohomological field theory associated to each non-degenerate, quasi-homogeneous hypersurface singularity $W \in \mathbb{C}[x_1, \ldots, x_N]$ and for each admissible subgroup G of the diagonal automorphism group $G_{max} := \{(\gamma_1, \ldots, \gamma_N) \in (\mathbb{C}^*)^N | W(\gamma_1 x_1, \ldots, \gamma_N x_N) = W(x_1, \ldots, x_N)\}$.

When the singularity is $A_{r-1} = x^r$, then the only admissible subgroup is the full automorphism group $G_{max} = \mu_r \subset \mathbb{C}^*$, which is the group of rth roots of unity.

In this paper we prove that for the singularity $A_{r-1} = x^r$ (with the group $G = \mu_r$), the stack of A_{r-1}-curves of [5] is canonically isomorphic to the stack of r-spin curves described in [1] and [7], and that the theory of [5] satisfies all the axioms of [8] for an r-spin virtual class. Therefore, the results of [8, 9, 4, 6] all apply to the A_{r-1}-theory of [5]. In particular, this shows that the Witten Integrable Hierarchies Conjecture is true for the A_{r-1}-theory of [5]; that is, the total descendant potential function of the A_{r-1}-theory satisfies the r-th Gelfand-Dikii hierarchy.

1.1. Conventions and Notation

Throughout this paper we will assume that $r > 1$ is an integer and we will set $W := x^r$. The group of automorphisms of W is $G = \mu_r = \{J^k | k \in \mathbb{Z}/r,\ J = \exp(2\pi i/r)\}$. Although some of the results described in this paper hold in a more general setting, we will always work over $\mathbb{C}$.

2. The stacks

In this section we review the definition and basic properties of the stack of r-spin curves used in [8, 7] and of the stack of A_{r-1}-curves of [5]. These two stacks are isomorphic, as proved in [1]. We will briefly review that isomorphism in this section as well.

The stack of A_{r-1} curves has a much simpler definition, and we will use that definition rather than the more complicated r-spin curve definition whenever possible.

2.1. The stack of A_{r-1}-curves

The definition in [5] of an n-pointed A_{r-1}-curve $(\mathscr{C}, \mathscr{L}, p_1, \ldots, p_n, \varphi)$ is an n-pointed stable orbicurve $(\mathscr{C}, p_1, \ldots, p_n)$, an orbifold line bundle $\mathscr{L}$, and an isomorphism $\varphi : \mathscr{L}^r \longrightarrow \omega_{\log}$, where $\omega_{\log}$ is the *log-canonical bundle*—specifically, the bundle of meromorphic 1-forms having at worst a single pole at each mark p_i for $i \in \{1, \ldots, n\}$. That is, the sheaf of sections of $\omega_{\log}$ is locally generated by the one-form dz/z, where z is a local coordinate on $\mathscr{C}$ near a marked point p_i. We describe these structures in more detail below.

Recall that an orbicurve $\mathscr{C}$ with marked points $p_1, \ldots, p_n$ is a (possibly nodal) Riemann surface C with orbifold structure at each p_i and each node. That is to say, for each marked point p_i there is a local group G_{p_i} and (since we are working over $\mathbb{C}$) a canonical isomorphism $G_{p_i} \cong \mathbb{Z}/m_i$ for some positive integer m_i. A neighborhood of p_i is uniformized by the branched covering map $z \longrightarrow z^{m_i}$. For each node p there is again a local group $G_p \cong \mathbb{Z}/n_j$ whose action is complementary on the two different branches. That is to say, a neighborhood of a nodal point (viewed as a neighborhood of the origin of $\{zw = 0\} \subset \mathbb{C}^2$) is uniformized by a branched covering map $(z, w) \longrightarrow (z^{n_j}, w^{n_j})$, with $n_j \geqslant 1$, and with group action $e^{2\pi i/n_j}(z, w) = (e^{2\pi i/n_j} z, e^{-2\pi i/n_j} w)$.

DEFINITION 2.1.1. — *We call the orbicurve $\mathscr{C}$* smooth *if the underlying curve C is smooth, and we will call the orbicurve* nodal *if the underlying curve C is nodal.*

We denote by $\varrho : \mathscr{C} \longrightarrow C$ the natural projection to the underlying (coarse, or non-orbifold) Riemann surface C. If $\mathscr{L}$ is a line bundle on C, it can be pulled back to an orbifold line bundle $\varrho^*\mathscr{L}$ over $\mathscr{C}$. When there is no danger of confusion, we use the same symbol $\mathscr{L}$ to denote its pullback.

DEFINITION 2.1.2. — *Let ω_C be the canonical bundle of C. We define the* log-canonical bundle of C *to be the line bundle*

$$\omega_{C,\log} := \omega_C \otimes \mathscr{O}(p_1) \otimes \cdots \otimes \mathscr{O}(p_n),$$

where $\mathscr{O}(p)$ is the holomorphic line bundle of degree one whose sections may have a simple pole at p. This bundle $\omega_{C,\log}$ can be thought of as the canonical bundle of the punctured Riemann surface $C - \{p_1, \ldots, p_n\}$.

The log-canonical bundle of $\mathscr{C}$ *is defined to be the pullback to $\mathscr{C}$ of the log-canonical bundle of C:*

$$\omega_{\mathscr{C},\log} := \varrho^*\omega_{C,\log}. \tag{2.1}$$

Near a marked point p of C with local coordinate x, the bundle $\omega_{C,\log}$ is locally generated by the meromorphic one-form dx/x. If the local coordinate near p on $\mathscr{C}$ is z, with $z^m = x$, then the lift $\omega_{\mathscr{C},\log} := \varrho^*(\omega_{C,\log})$ is still locally generated by $m\,dz/z = dx/x$. When there is no risk of confusion, we will denote both $\omega_{C,\log}$ and $\omega_{\mathscr{C},\log}$ by $\omega_{\log}$. Near a node with coordinates z and w satisfying $zw = 0$, both ω and $\omega_{\log}$ are locally generated by the one-form $dz/z = -dw/w$.

2.1.1. Pushforward to the underlying curve

If $\mathscr{L}$ is an orbifold line bundle on a smooth orbicurve $\mathscr{C}$, then the sheaf of locally invariant sections of $\mathscr{L}$ is locally free of rank one, and hence dual to a unique line bundle $|\mathscr{L}|$ on $\mathscr{C}$. We also denote $|\mathscr{L}|$ by $\varrho_*\mathscr{L}$, and it is called the "desingularization" of $\mathscr{L}$ in [2, Prop 4.1.2]. It can be constructed explicitly as follows.

We keep the local trivialization at non-orbifold points, and change it at each orbifold point p. If $\mathscr{L}$ has a local chart $\Delta \times \mathbb{C}$ with coordinates (z, s), and if the generator $1 \in \mathbb{Z}/m \cong G_p$ acts locally on $\mathscr{L}$ by $(z, s) \mapsto (\exp(2\pi i/m)z, \exp(2\pi i v/m)s)$, then we use the $\mathbb{Z}/m$-equivariant map $\Psi : (\Delta - \{0\}) \times \mathbb{C} \longrightarrow \Delta \times \mathbb{C}$ given by

$$(z, s) \longrightarrow (z^m, z^{-v}s), \tag{2.2}$$

where $\mathbb{Z}/m$ acts trivially on the second $\Delta \times \mathbb{C}$. Since $\mathbb{Z}/m$ acts trivially, this gives a line bundle over C, which is $|\mathscr{L}|$.

If the orbicurve $\mathscr{C}$ is nodal, then the pushforward $\varrho_*\mathscr{L}$ of a line bundle $\mathscr{L}$ may not be a line bundle on C. In fact, if the local group G_p at a node acts non-trivially on $\mathscr{L}$, then the invariant sections of $\mathscr{L}$ form a rank-one torsion-free sheaf on C (see [1]). However, we may take the normalizations $\widetilde{\mathscr{C}}$ and $\widetilde{C}$ to get (possibly disconnected) smooth curves, and the pushforward of $\mathscr{L}$ from $\widetilde{\mathscr{C}}$ will give a line bundle on $\widetilde{C}$. Thus $|\mathscr{L}|$ is a line bundle away from the nodes of C, but its fiber at a node can be two-dimensional; that is, there is (usually) no gluing condition on $|\mathscr{L}|$ at the nodal points. The situation is slightly more subtle than this (see [1]), but for our purposes, it will be enough to consider the pushforward $|\mathscr{L}|$ as a line bundle on the normalization $\widetilde{C}$ where the local group acts trivially on $\mathscr{L}$.

It is also important to understand more about the sections of the pushforward $\varrho_*\mathscr{L}$. Suppose that s is a section of $|\mathscr{L}|$ having local representative $g(u)$. Then $(z, z^v g(z^m))$ is a local section of $\mathscr{L}$. Therefore, we obtain a section $\varrho^*(s) \in \Omega^0(\mathscr{L})$ which equals s away from orbifold points under the identification given by Equation 2.2. It is clear that if s is holomorphic,

so is $\varrho^*(s)$. If we start from an analytic section of $\mathscr{L}$, we can reverse the above process to obtain a section of $|\mathscr{L}|$. In particular, $\mathscr{L}$ and $|\mathscr{L}|$ have isomorphic spaces of holomorphic sections:

$$\varrho^* : H^0(C, |\mathscr{L}|) \widetilde{\longrightarrow} H^0(\mathscr{C}, \mathscr{L}).$$

In the same way, there is a map $\varrho^* : \Omega^{0,1}(|\mathscr{L}|) \longrightarrow \Omega^{0,1}(\mathscr{L})$, where $\Omega^{0,1}(\mathscr{L})$ is the space of orbifold $(0,1)$-forms with values in $\mathscr{L}$. Suppose that $g(u)d\bar{u}$ is a local representative of a section of $t \in \Omega^{0,1}(|\mathscr{L}|)$. Then $\varrho^*(t)$ has a local representative $z^v g(z^m) m \bar{z}^{m-1} d\bar{z}$. Moreover, ϱ induces an isomorphism

$$\varrho^* : H^1(C, |\mathscr{L}|) \widetilde{\longrightarrow} H^1(\mathscr{C}, \mathscr{L}).$$

Example 2.1.3. — The pushforward $|\omega_{\mathscr{C},log}|$ of the log-canonical bundle of any orbicurve $\mathscr{C}$ is again the log-canonical bundle of C, because at a point p with local group $G_p \cong \mathbb{Z}/m$ the one-form $m\,dz/z = dx/x$ is invariant under the local group action.

Similarly, the pushforward $|\omega_{\mathscr{C}}|$ of the canonical bundle of $\mathscr{C}$ is just the canonical bundle of C:

$$|\omega_{\mathscr{C}}| = \varrho_* \omega_{\mathscr{C}} = \omega_C, \tag{2.3}$$

because the local group $\mathbb{Z}/m$ acts on the one-form dz by $\exp(2\pi i/m)dz$, and the invariant holomorphic one-forms are precisely those generated by $mz^{m-1}dz = dx$.

Definition 2.1.4. — *An A_{r-1}-structure on an orbicurve $\mathscr{C}$ is a choice of a line bundle $\mathscr{L}$ and an isomorphism of line bundles*

$$\varphi : \mathscr{L}^r \longrightarrow \omega_{\log},$$

with the additional condition that for each point $p \in \mathscr{C}$, the induced representation $\rho_p : G_p \longrightarrow \mathrm{Aut}(\mathscr{L}) \cong U(1)$ be faithful.

An *isomorphism of A_{r-1}-structures* $\Upsilon : (\mathscr{L}, \varphi) \longrightarrow (\mathscr{L}', \varphi')$ on $\mathscr{C}$ is defined to be an isomorphism $\xi : \mathscr{L} \longrightarrow \mathscr{L}'$ such that $\varphi = \varphi' \circ \xi$.

Different choices of maps φ give isomorphic A_{r-1}-structures.

Proposition 2.1.5. — *For a given orbicurve $\mathscr{C}$, any two A_{r-1}-structures $(\mathscr{L}, \varphi)$ and $(\mathscr{L}, \varphi')$ on $\mathscr{C}$ which have identical structure bundle $\mathscr{L}$ are isomorphic.*

Proof. — The composition $\varphi^{-1} \circ \varphi'$ is an automorphism of $\omega_{\log}$ and hence defined by an element $\exp(\alpha) \in \mathbb{C}^*$. Let $\beta = \alpha/r$. This induces an automorphism $\exp(\beta) : \mathscr{L} \longrightarrow \mathscr{L}$ which takes φ to $\exp(\alpha)\varphi = \varphi'$, and thus induces an isomorphism of A_{r-1}-structures $(\mathscr{L}, \varphi) \widetilde{\longrightarrow} (\mathscr{L}, \varphi')$. □

DEFINITION 2.1.6. — *For each orbifold marked point p_i we will denote the image $\rho_{p_i}(1)$ of the canonical generator $1 \in \mathbb{Z}/m \cong G_{p_i}$ in $U(1)$ by*

$$\gamma_i := \gamma_{p_i} := \rho_{p_i}(1) = \exp(2\pi i \Theta^{\gamma_i}),$$

where $\Theta^{\gamma_i} \in \mathbb{Q} \cap [0,1)$

The choices of orbifold structure for the line bundles in the A_{r-1}-structure is severely restricted by W; specifically, the faithful representation $\rho_{p_i} : G_{p_i} \longrightarrow U(1)$ factors through G so $\gamma_i \in \mu_r$.

DEFINITION 2.1.7. — *A marked point p of an A_{r-1}-curve is called* narrow *(called Neveu-Schwarz in* [5]*) if $\gamma_p = 1$. The point p is called* broad *(called Ramond in* [5]*) otherwise.*

2.2. Stack of stable A_{r-1}-orbicurves

DEFINITION 2.2.1. — *A triple $(\mathscr{C}, \mathscr{L}, \varphi)$ consisting of an orbicurve $\mathscr{C}$ with n marked points and with A_{r-1}-structure $(\mathscr{L}, \varphi)$ is called a* stable A_{r-1}-orbicurve *if the underlying (coarse) curve C is a stable curve.*

DEFINITION 2.2.2. — *A* genus-g, stable A_{r-1}-orbicurve with n marked points over a base T *is given by a flat family of genus-g, n-pointed orbicurves $\mathscr{C} \longrightarrow T$ with (gerbe) markings $\mathscr{S}_i \subset \mathscr{C}$ and sections $\sigma_i : T \longrightarrow \mathscr{S}_i$, and the data $(\mathscr{L}, \varphi)$. The sections σ_i are required to induce isomorphisms between T and the coarse moduli of $\mathscr{S}_i$ for $i \in \{1, \ldots, n\}$. The bundle $\mathscr{L}$ is an orbifold line bundle on $\mathscr{C}$. And $\varphi : \mathscr{L}^r \xrightarrow{\sim} \omega_{\mathscr{C}/T,log}$ is an isomorphism to the relative log-canonical bundle which, together with the $\mathscr{L}$, induces an A_{r-1}-structure on every fiber $\mathscr{C}_t$.*

DEFINITION 2.2.3. — *We denote the stack of stable A_{r-1}-orbicurves by $\mathscr{W}_{g,n}(A_{r-1})$ or simply $\mathscr{W}_{g,n}$.*

Forgetting the A_{r-1}-structure and the orbifold structure gives a morphism

$$st : \mathscr{W}_{g,n} \longrightarrow \overline{\mathscr{M}}_{g,n}.$$

The morphism st plays a role similar to that played by the stabilization morphism of stable maps.

THEOREM 2.2.4 ([1]). — *The stack $\mathscr{W}_{g,n}$ is a smooth, compact orbifold (Deligne-Mumford stack) with projective coarse moduli. In particular, the morphism $st : \mathscr{W}_{g,n} \longrightarrow \overline{\mathscr{M}}_{g,n}$ is flat, proper and quasi-finite (but not representable).*

2.2.1. Decomposition of $\mathscr{W}_{g,n}$ into components

The orbifold structure, and the image $\gamma_i = \rho_{p_i}(1)$ of the canonical generator $1 \in \mathbb{Z}/m_i \cong G_{p_i}$ at each marked point p_i is locally constant, and hence is constant for each component of $\mathscr{W}_{g,n}$. Therefore, we can use these decorations to decompose the moduli space into components.

DEFINITION 2.2.5. — *For any choice $\boldsymbol{\gamma} := (\gamma_1, \ldots, \gamma_n) \in G^n$ we define $\mathscr{W}_{g,n}(\boldsymbol{\gamma}) \subseteq \mathscr{W}_{g,n}$ to be the open and closed substack with orbifold decoration $\boldsymbol{\gamma}$. We call $\boldsymbol{\gamma}$ the* type *of any A_{r-1}-orbicurve in $\mathscr{W}_{g,n}(\boldsymbol{\gamma})$.*

We have the decomposition

$$\mathscr{W}_{g,n} = \sum_{\boldsymbol{\gamma}} \mathscr{W}_{g,n}(\boldsymbol{\gamma}).$$

The following proposition is proved in [5].

PROPOSITION 2.2.6. — *A necessary and sufficient condition for $\mathscr{W}_{g,n}(\boldsymbol{\gamma})$ to be non-empty is*

$$(2g-2+n)/r - \sum_{l=1}^{n} \Theta^{\gamma_l} \in \mathbb{Z}, \tag{2.4}$$

where Θ^{γ_l} is given in Definition 2.1.6.

Example 2.2.7. — For three-pointed, genus-zero A_{r-1}-curves, the choice of orbifold line bundle $\mathscr{L}$ providing the A_{r-1}-structure is unique, if it exists at all. Hence, if the selection rule is satisfied, $\mathscr{W}_{0,3}(\boldsymbol{\gamma})$ is isomorphic to the classifying stack $\mathscr{B}\mu_r := [pt/\mu_r]$.

2.3. The stack of r-spin curves

2.3.1. Smooth r-spin curves

Let g and n be integers such that $2g-2+n>0$. Let $\mathbf{m} = (m_1, \ldots, m_n)$ be an n-tuple of integers. A nonsingular n-pointed r-spin curve of genus g and type $\mathbf{m}$ over a base S, denoted $(C \longrightarrow S, s_i, \mathscr{L}, c)$, is the data of

(1) a smooth, n-pointed curve $(C \longrightarrow S, s_i : S \longrightarrow C)$ of genus g,
(2) an invertible sheaf $\mathscr{L}$ on C, and
(3) an isomorphism $c : \mathscr{L}^{\otimes r} \xrightarrow{\sim} \omega_{C/S}(-\sum_{i=1}^{n} m_i S_i)$, where S_i is the image of s_i.

The category $\mathscr{M}_{g,n}^{1/r,\mathbf{m}}$ of nonsingular, n-pointed r-spin curves of genus g and type $\mathbf{a}$, with morphisms given by fiber diagrams, is a Deligne-Mumford stack with quasi-projective coarse moduli space. See [1, 7] for a detailed proof. When $\mathbf{m}$ is congruent to $\mathbf{m}'$ mod r, the two stacks $\mathscr{M}_{g,n}^{1/r,\mathbf{m}}$ and $\mathscr{M}_{g,n}^{1/r,\mathbf{m}'}$ are canonically isomorphic. We denote by $\mathscr{M}_{g,n}^{1/r}$ the disjoint union

$$\mathscr{M}_{g,n}^{1/r} = \coprod_{\substack{\mathbf{m} \\ 0 \leqslant m_i < r}} \mathscr{M}_{g,n}^{1/r,\mathbf{m}}.$$

Note that if $(C \longrightarrow S, s_i, \mathscr{L}, c)$ is a nonsingular, n-pointed r-spin curve of genus g and type $\mathbf{m}$, then

$$\deg \mathscr{L} = (2g - 2 - \sum m_i/r) \in \mathbb{Z}. \tag{2.5}$$

Moreover, the stack $\mathscr{M}_{g,n}^{1/r,\mathbf{m}}$ is non-empty if and only if

$$\sum m_i \equiv 2g - 2 \bmod r.$$

2.3.2. The stack of stable r-spin curves

To compactify the stack $\mathscr{M}_{g,n}^{1/r,\mathbf{m}}$ one must replace rth-root line bundles by rank-one torsion-free sheaves on nodal curves as described in [1, 7]. Although the full definition of the stable r-spin curve includes additional data, for our purposes the most important structure is the choice of an n-pointed stable curve $(C, p_1, \ldots, p_n)$, a rank-one, torsion-free sheaf $\mathscr{E}$ on C, and a morphism $\varphi : \mathscr{E}^{\otimes r} \longrightarrow \omega(-\sum_{i=1}^n m_i p_i)$ such that away from the nodes of C the morphism φ is an isomorphism. Additionally, we require that at each node q of C one of the following two conditions holds.

(1) The morphism φ is an isomorphism (in which case $\mathscr{F}$ is locally free at the node). In this case we call the spin structure *Ramond* at the node q.

(2) The sheaf $\mathscr{F}$ is not locally free, but $\mathscr{F}$ and the morphism φ are induced from a line bundle $\widetilde{\mathscr{F}}$ and an isomorphism $\widetilde{\varphi} : \widetilde{\mathscr{F}}^{\otimes r} \longrightarrow \omega_{\widetilde{C}}(-m_+ q_+ - m_- q_-)$ on the normalization $\nu : \widetilde{C} \longrightarrow C$ of C at the node q. Here the normalization $\widetilde{C}$ has two smooth points q_+ and q_- lying over the node q, and the integers m_+ and m_- must satisfy $m_+ + m_- \equiv r - 2 \pmod r$. We call these integers the *type* of the r-spin structure at the node q. The inclusion $\omega_{\widetilde{C}} \hookrightarrow \omega_{\widetilde{C},log} = \nu^* \omega_C$ gives a morphism $\widetilde{\mathscr{F}}^{\otimes r} \longrightarrow \nu^* \omega_C$ which induces (by adjointness) the morphism $\varphi : \mathscr{F}^{\otimes r} \longrightarrow \omega_C$. In this case we say that the spin structure is *Neveu-Schwarz* at the node q.

Additional data and restrictions must also be placed on the torsion-free sheaves to ensure that the compactification is smooth. These restrict the way the spin structures may vary in families and it involves additional data in the form of intermediate roots of ω. For more details on these structures, see [1, 7].

The category $\overline{\mathscr{M}}_{g,n}^{1/r,\mathbf{a}}$ of n-pointed, stable r-spin curves of genus g and type $\mathbf{a}$, with morphisms given by fiber diagrams, is a smooth, proper Deligne-Mumford stack with projective coarse moduli space. The forgetful map $\overline{\mathscr{M}}_{g,n}^{1/r} \longrightarrow \overline{\mathscr{M}}_{g,n}$ is quasi-finite and proper, but not representable [7].

We must also consider decorated dual graphs.

DEFINITION 2.3.1. — *A* stable decorated graph *is a stable graph with a marking of each half-edge by an integer m with $-1 \leqslant m < r$, such that for each edge e the marks m^+ and m^- of the two half-edges of e satisfy*

$$m^+ + m^- \equiv r - 2 \pmod{r}.$$

Given a stable r-spin curve $\mathfrak{C}$ of type $\mathbf{m} = (m_1, \ldots, m_n)$, the decorated dual graph *of $\mathfrak{C}$ is the dual graph Γ of the underlying curve C, with the following additional markings. The i-th tail is marked by m_i, and each half-edge associated to a node of C is marked by the type (m^+ or m^-) of the r-spin structure along the branch of the node associated to that half-edge if the node is Neveu-Schwarz. We mark the half edges with $r-1$ if the corresponding node is Ramond.*

Given a decorated stable graph Γ, we denote by $\overline{\mathscr{M}}_{\Gamma}^{1/r}$ the locus of r-spin curves in $\overline{\mathscr{M}}_{g,n}^{1/r}$ with dual graph Γ.

2.3.3. Change of type and roots of other bundles

In the definitions given above, one could also have replaced the bundle ω with another line bundle defined on every stable curve. Specifically, let $\mathbf{1} := (1, 1, \ldots, 1) \in \mathbb{Z}^n$ be the n-tuple of all ones. It will be important in this paper to compare r-spin curves to the stack obtained by replacing ω by $\omega_{\log} := \omega(\sum S_i)$, or by $\omega(-r\ell\mathbf{1}) := \omega(-\sum r\ell S_i)$, or by $\omega_{\log}(-r\ell\mathbf{1}) := \omega(\sum(1-r\ell)S_i)$, for any choice of $\ell \in \mathbb{Z}$. We denote the corresponding stacks by $\overline{\mathscr{M}}_{g,n,log}^{1/r,\mathbf{m}}$, $\overline{\mathscr{M}}_{g,n}^{1/r,\mathbf{m}+r\ell\mathbf{1}}$, and $\overline{\mathscr{M}}_{g,n,log}^{1/r,\mathbf{m}+r\ell\mathbf{1}}$, respectively. More precisely, points of the smooth locus $\mathscr{M}_{g,n,log}^{1/r,\mathbf{m}}$ are tuples $(C, \mathscr{L}, c)$ such that C is an n-pointed,smooth, genus-g curve, $\mathscr{L}$ is a line bundle on C, and $c : \mathscr{L}^{\otimes r} \widetilde{\longrightarrow} \omega_{C,log}(-\sum m_i p_i) = \omega_C(\sum(1-m_i)p_i)$ is an isomorphism of line

bundles on C. Points of the smooth locus $\mathscr{M}_{g,n}^{1/r,\mathbf{m}+r\ell\mathbf{1}}$ are tuples $(C,\mathscr{L},c)$ such that C is an n-pointed,smooth, genus-g curve, $\mathscr{L}$ is a line bundle on C, and $c:\mathscr{L}^{\otimes r}\widetilde{\longrightarrow}\omega_C(-\sum(m_i+r\ell)p_i)$ is an isomorphism of line bundles on C. And points of the smooth locus $\mathscr{M}_{g,n,log}^{1/r,\mathbf{m}+r\ell\mathbf{1}}$ are tuples $(C,\mathscr{L},c)$ such that C is an n-pointed,smooth, genus-g curve, $\mathscr{L}$ is a line bundle on C, and $c:\mathscr{L}^{\otimes r}\widetilde{\longrightarrow}\omega_{C,log}(-\sum(m_i+r\ell)p_i)=\omega_C(\sum(1-m_i-r\ell)p_i)$ is an isomorphism of line bundles on C.

The stack $\overline{\mathscr{M}}_{g,n}^{1/r,\mathbf{m}}$ is canonically isomorphic to the stack $\overline{\mathscr{M}}_{g,n}^{1/r,\mathbf{m}+r\ell\mathbf{1}}$ for any $m\in\mathbb{Z}$. Similarly, the stacks $\overline{\mathscr{M}}_{g,n,log}^{1/r,\mathbf{m}}$ and $\overline{\mathscr{M}}_{g,n,log}^{1/r,\mathbf{m}+r\ell\mathbf{1}}$ are canonically isomorphic for any $\ell\in\mathbb{Z}$. In both cases, when C is smooth, the isomorphism sends $(C,\mathscr{L},\varphi)$ to $(C,\mathscr{L}\otimes\omega^m,\varphi\otimes I)$, where $I:(\omega^{\otimes\ell})^{\otimes r}\widetilde{\longrightarrow}\omega^{\otimes\ell r}$ is the obvious isomorphism.

Similarly, the stack $\overline{\mathscr{M}}_{g,n,log}^{1/r,\mathbf{m}}$ is isomorphic to the stack $\overline{\mathscr{M}}_{g,n}^{1/r,\mathbf{m}-\mathbf{1}}$. On the smooth locus this is immediate from the definitions.

2.4. The isomorphism between $\mathscr{W}_{g,n}$ and $\overline{\mathscr{M}}_{g,n}^{1/r}$

In [1, §4] it is shown that the stack $\mathscr{W}_{g,n}(A_{r-1})$ of stable A_{r-1}-curves (in [1] this stack is denoted $\mathcal{B}_{g,n}(\mathbb{G}_m,\omega_{\log}^{1/r})$) is isomorphic to the stack $\overline{\mathscr{M}}_{g,n}^{1/r}$ of r-spin curves. On the smooth locus the isomorphism is given simply by pushing forward the orbifold line bundle to the underlying (coarse) curve. Before describing the isomorphism, we will review some facts about pushforwards of A_{r-1} structures to the underlying curve.

2.4.1. Pushforward of A_{r-1}-structures

We now briefly recall some facts about the behavior of A_{r-1}-structures when forgetting the orbifold structure at marked points, that is, when they are pushed down to the underlying (coarse) curve.

An A_{r-1}-structure consists of a line bundle $\mathscr{L}$ with an isomorphism $\mathscr{L}^{\otimes r}\cong\omega_{\log}$ such that near an orbifold point p with local coordinate z the canonical generator $1\in\mathbb{Z}/m\cong G_p$ of the local group G_p acts on $\mathscr{L}$ by $(z,s)\mapsto(\exp(2\pi i/m)z,\exp(2\pi i(v/m))s)$ for some $v\in\{0,\dots,m-1\}$. Since $\omega_{\log}$ is invariant under the local action of G_p, we must have $rv=\ell m$ for some $\ell\in\{0,\dots,r-1\}$, and $\frac{v}{m}=\frac{\ell}{r}$. Denote the (invariant) local coordinate on the underlying curve C by $u=z^m$. Any section in $\sigma\in\Omega^0(|\mathscr{L}|)$ must locally be of the form $\sigma=g(u)z^vs$, in order to be $\mathbb{Z}/m$-invariant. So σ^r has local representative $z^{rv}g^r(u)\frac{dz}{z}=u^\ell g^r(u)\frac{du}{mu}$. Hence, $\sigma^r\in\Omega^0(\omega_{\log}\otimes\mathscr{O}((-\ell)p)$, and thus when $\ell\neq0$, we have $\sigma^r\in\Omega^0(K)$.

From this we get the following proposition.

PROPOSITION 2.4.1. — *If $\mathscr{C}$ is a smooth orbicurve (i.e., C is a smooth curve), let γ_i define the action of the local group G_{p_i} near p_i. The isomorphism $\varphi : \mathscr{L}^r \longrightarrow \omega_{\log}$ induces an isomorphism*

$$|\varphi| : |\mathscr{L}|^r \longrightarrow \omega_{\overline{\mathscr{C}},\log} \otimes \mathscr{O}\left(-\sum_{i=1}^{n} r\Theta^{\gamma_i} p_i\right). \tag{2.6}$$

2.4.2. The isomorphism between $\mathscr{W}_{g,n}$ and $\overline{\mathscr{M}}_{g,n}^{1/r}$

On the smooth locus, the isomorphism between $\mathscr{W}_{g,n}$ and $\overline{\mathscr{M}}_{g,n}^{1/r}$ is a straightforward consequence of Proposition 2.4.1. Specifically, for any smooth A_{r-1}-curve $(\mathscr{C}, \mathscr{L}, \varphi)$ in $\mathscr{W}_{g,n}(\boldsymbol{\gamma})$, the pushforward along $\varrho : \mathscr{C} \longrightarrow C$ to the underlying coarse curve induces a line bundle $|\mathscr{L}|$ on C and an isomorphism

$$|\varphi| : |\mathscr{L}|^r \longrightarrow \omega_{\log}\left(\sum_{i=1}^{n} -r\Theta^{\gamma_i} p_i\right) \cong \omega\left(\sum_{i=1}^{n} (1 - r\Theta^{\gamma_i}) p_i\right).$$

This describes the isomorphisms

$$\mathscr{W}_{g,n}(\boldsymbol{\gamma}) \longrightarrow \overline{\mathscr{M}}_{g,n,log}^{1/r,(r\Theta^{\gamma_1},\ldots,r\Theta^{\gamma_n})} \longrightarrow \overline{\mathscr{M}}_{g,n}^{1/r,(r\Theta^{\gamma_1},\ldots,r\Theta^{\gamma_n})-\mathbf{1}}$$

on the smooth locus.

The details of the isomorphism on nodal curves are more messy. These details are not essential for our purposes here, but the interested reader may find them in [1, §4].

3. The state spaces

3.1. The A_{r-1} state space of [5]

We now describe the state space $\mathscr{A} := \mathscr{A}_{A_{r-1},\langle J\rangle}$ of the theory of [5] for the singularity A_{r-1} with symmetry group $G = \mu_r = \langle J\rangle$, where $J = \exp(2\pi i/r)$.

DEFINITION 3.1.1. — *For each $\gamma \in G \subset (\mathbb{C}^*)^N$ we define* $\mathrm{Fix}(\gamma) \subseteq \mathbb{C}^N$ *to the be fixed locus of γ acting on $\mathbb{C}^N$. We also define N_γ to be the dimension of* $\mathrm{Fix}(\gamma)$ *(as a $\mathbb{C}$-vector space).*

As described in [5], the state space is a direct sum of the $\langle J\rangle$-invariant part of certain middle-dimensional relative cohomology groups:

$$\mathscr{A}_W = \bigoplus_{\gamma\in G} \mathscr{A}_\gamma := \bigoplus_{k=0}^{r-1} \mathscr{A}_{J^k} = \bigoplus_{k=0}^{r-1} \left(H^{N_\gamma}(\mathrm{Fix}(J^k), W_{J^k}^\infty, \mathbb{C})\right)^{\langle J\rangle}.$$

In the case that $k \neq 0$, the fixed locus of J^k is $\{0\} \subset \mathbb{C}$ so the middle cohomology for these elements is just

$$H^{N_\gamma}(\mathrm{Fix}(J^k), W_{J^k}^\infty, \mathbb{C}) = H^0(\{0\}, W_{J^k}^\infty, \mathbb{C}) \cong \mathbb{C},$$

and the J-action on $H^0(\{0\}, W_{J^k}^\infty, \mathbb{C})$ is trivial.

On the other hand, when $k = 0$ we have $H^{N_\gamma}(\mathrm{Fix}(J^0), W_{J^k}^\infty, \mathbb{C}) = H^1(\mathbb{C}, W_{J^0}^\infty, \mathbb{C})$, and a theorem of Wall [12] shows that this is isomorphic, as a graded G-module, to the following space of germs of one-forms on $\mathbb{C}$:

$$H^1(\mathrm{Fix}(J^0), W_{J^0}^\infty, \mathbb{C}) \cong \Omega^1/(x^{r-1}dx).$$

This space has the following simple basis:

$$\{dx, xdx, \ldots, x^{r-2}dx\}.$$

None of the elements in the J^0 sector are $\langle J\rangle$-invariant, and all the elements in the J^k sectors are invariant when $k \neq 0$. Thus we have

$$\mathscr{A}_{J^k} = \mathbb{C}\mathbf{a}_{J^k} \quad \text{and} \quad \mathscr{A} = \bigoplus_{k=1}^{r-1} \mathbb{C}\mathbf{a}_{J^k},$$

where $\mathbf{a}_{J^k} := 1 \in H^{N_{J^k}}(\mathbb{C}^N_{J^{-i}}, W^\infty_{J^{-i}}, \mathbb{C})$.

In the notation of the previous section, the group element J^k has $\Theta^{J^k} = k/r$. Therefore the J^k-sectors correspond to marked points p where the A_{r-1}-structure gives $|\mathscr{L}|^r \cong \omega_{\log}(-kp)$ near p.

When $k \neq 0$, we call these sectors *narrow*. These will correspond to the *Neveu-Schwarz* sectors of r-spin theory.

For any non-degenerate W and any admissible G, the state space $\mathscr{A}_{W,G}$ admits a grading and a non-degenerate pairing $\langle\,\rangle^W$. The pairing in the A_{r-1} case is given by

$$\langle \mathbf{a}_{J^k}, \mathbf{a}_{J^\ell}\rangle = \delta_{k+\ell\equiv 0 \,(\mathrm{mod}\ r)}.$$

The grading is more subtle, as we now describe.

Definition 3.1.2. — *The* central charge *of the singularity W is denoted $\hat{c}^{FJRW}$:*

$$\hat{c}^{FJRW} := \sum_{i=1}^{N} (1 - 2q_i). \tag{3.1}$$

Suppose that $\gamma = (e^{2\pi i\Theta_1^\gamma}, \dots, e^{2\pi i\Theta_N^\gamma})$ *for rational numbers* $0 \leqslant \Theta_i^\gamma < 1$. *We define the* degree shifting number

$$\iota_\gamma = \sum_i (\Theta_i^\gamma - q_i) \tag{3.2}$$

For a class $\alpha \in \mathscr{A}_\gamma$, *we define*

$$\deg_W(\alpha) = \frac{N_\gamma}{2} + \iota_\gamma. \tag{3.3}$$

Specifically, in the A_{r-1}-case, we have

$$\hat{c} = \frac{r-2}{r},$$

and

$$\deg_{A_{r-1}}(\mathbf{a}_{J^k}) = \frac{k-1}{r},$$

provided we choose $0 < k < r$.

3.2. The r-spin state space of [8]

The r-spin state space of [8] is a $\mathbb{C}$-vector space $\mathscr{S}$ of dimension $r-1$ with a basis $\{\boldsymbol{s}_0, \dots, \boldsymbol{s}_{r-2}\}$ and metric η^{JKV} given by

$$\eta^{JKV}(\boldsymbol{s}_\mu, \boldsymbol{s}_\nu) := \eta^{JKV}_{\mu\nu} := \delta_{\mu+\nu,r-2}. \tag{3.4}$$

Furthermore, in [8] the elements $\boldsymbol{s}_\mu$ are given a grading of

$$\deg_{JVK}(\boldsymbol{s}_\mu) = \mu/r,$$

and the central charge of the theory is

$$\hat{c}^{JKV} = \frac{r-2}{r}.$$

Remark 3.2.1. — In [8, 15] the span of each element $\boldsymbol{s}_\mu$ in $\mathscr{S}$ is called a *Neveu-Schwarz sector*. There is an additional one-dimensional *Ramond sector* that appears in the r-spin theory, but this sector completely decouples from the rest of the theory and so can be omitted (see [8, Rem 3.10] for more about this sector). This Ramond sector can be thought of as roughly corresponding to the case of $\gamma = J^0$ in the A_{r-1} theory.

3.3. The State Space Isomorphism

The two state spaces are isomorphic as graded vector spaces with metric. As suggested by the grading, the isomorphism matches the element $\mathbf{a}_{J^k} \in \mathscr{A}_{J^k}$ to the element $\boldsymbol{s}_{k-1} \in \mathscr{S}$. It is straightforward to see that this is indeed an isomorphism preserving the grading, that $\deg_{FJRW}(\mathbf{a}_{J^k}) = (k-1)/r = \deg_{JKV} \boldsymbol{s}_{k-1}$, and that $\hat{c}_{FJRW} = \hat{c}_{JKV} = (r-2)/r$. From now on we will drop the subscripts and just write deg to denote both $\deg_{A_{r-1}}$ and $\deg_{JKV}$, write η or $\langle\,,\rangle$ to denote the pairing, and write $\hat{c}$ to denote $\hat{c}_{FJRW} = \hat{c}_{JKV} = (r-2)/r$.

4. The virtual classes

In this section we show how to construct an r-spin virtual cohomology class using the virtual class of the A_{r-1} theory. We begin by reviewing the axioms of the r-spin virtual cohomology class and the properties of the A_{r-1} virtual cycle.

4.1. The [8] axioms of an r-spin virtual class

The r-spin theory depends on the existence of a *virtual cohomology class* $\boldsymbol{c}^{1/r}$ satisfying a list of axioms outlined in [8, §4.1]. We briefly review those axioms here.

Definition 4.1.1. — *An* r-spin virtual class *is an assignment of a cohomology class*

$$\boldsymbol{c}_\Gamma^{1/r} \in H^{2D}\left(\overline{\mathscr{M}}_\Gamma^{1/r}, \mathbb{Q}\right) \tag{4.1}$$

to every genus g*, stable, decorated graph* Γ *with* n*-tails. Here, if the tails of* Γ *are marked with the* n*-tuple* $\mathbf{m} = (m_1, \ldots, m_n)$*, then the dimension* D *is*

$$D = \hat{c}(g-\alpha) + \sum_{i=1}^n m_i/r = \hat{c}(g-\alpha) + \sum_{i=1}^n \deg(\boldsymbol{s}_{m_i}), \tag{4.2}$$

and α *is the number of connected components of* Γ*. In the special case where* Γ *has one vertex and no edges, we denote* $\boldsymbol{c}_\Gamma^{1/r}$ *by* $\boldsymbol{c}_{g,n}^{1/r}(\mathbf{m})$*. These classes must satisfy the axioms below.*

Axiom 1a (Connected Graphs): *Let Γ be a connected, genus g, stable, decorated graph with n tails. Let $E(\Gamma)$ denote the set of edges of Γ. For each edge e of Γ, let $l_e := \gcd(m_e^+ + 1, r)$, where m_e^+ is an integer decorating a half-edge of e. The classes $\boldsymbol{c}_\Gamma^{1/r}$ and $\boldsymbol{c}_{g,n}^{1/r}(\mathbf{m})$ are related by*

$$\boldsymbol{c}_\Gamma^{1/r} = \left(\prod_{e \in E(\Gamma)} \frac{r}{l_e} \right) \tilde{i}^* \, \boldsymbol{c}_{g,n}^{1/r}(\mathbf{m}) \in H^{2D}(\overline{\mathscr{M}}_\Gamma^{1/r}), \tag{4.3}$$

where $\tilde{i} : \mathscr{M}_\Gamma^{1/r} \hookrightarrow \mathscr{M}_{g,n}^{1/r,\mathbf{m}}$ is the canonical inclusion map.

Axiom 1b (Disconnected Graphs): *Let Γ be a stable, decorated graph which is the disjoint union of connected graphs $\Gamma^{(d)}$, then the classes $\boldsymbol{c}_\Gamma^{1/r}$ and $\boldsymbol{c}_{\Gamma^{(d)}}^{1/r}$ are related by*

$$\boldsymbol{c}_\Gamma^{1/r} = \bigotimes_d \boldsymbol{c}_{\Gamma^{(d)}}^{1/r} \in H^\bullet(\mathscr{M}_\Gamma^{1/r}).$$

Axiom 2 (Convexity): *If $m_i \neq r-1$ for all $i \in \{1, \ldots, n\}$, let $\mathscr{F}$ denote the universal rth root rank-one torsion-free sheaf on the universal r-spin curve $\pi : \mathcal{C}_{g,n}^{1/r,\mathbf{m}} \longrightarrow \overline{\mathscr{M}}_{g,n}^{1/r,\mathbf{m}}$. For each irreducible (and connected) component of $\mathscr{M}_{g,n}^{1/r,\mathbf{m}}$ (denoted here by $\mathscr{M}_{g,n}^{1/r,\mathbf{m},(d)}$ for some index d), if $\pi_*\mathscr{F} = 0$ on $\mathscr{M}_{g,n}^{1/r,\mathbf{m},(d)}$, then $\boldsymbol{c}_{g,n}^{1/r}(\mathbf{m})$ restricted to $\mathscr{M}_{g,n}^{1/r,\mathbf{m},(d)}$ is the top Chern class $(-1)^D c_D(R^1\pi_*\mathscr{F})$, of the dual of the first derived pushforward of $\mathscr{F}$.*

Axiom 3 (Cutting edges): *Given any genus-g, decorated stable graph Γ with n tails marked with $\mathbf{m}$, we have a diagram*

$$\begin{array}{ccccccc}
\mathscr{M}_{\tilde{\Gamma}} \times_{\mathscr{M}_\Gamma} \mathscr{M}_\Gamma^{1/r} & \xrightarrow{\tilde{\mu}} & \mathscr{M}_\Gamma^{1/r} & \xrightarrow{\tilde{i}} & \mathscr{M}_{g,n}^{1/r} \\
\downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p} \\
\mathscr{M}_{\tilde{\Gamma}}^{1/r} & & & & \\
\downarrow{\scriptstyle p_2} & & & & \\
\mathscr{M}_{\tilde{\Gamma}} & \xrightarrow{\mu} & \mathscr{M}_\Gamma & \xrightarrow{i} & \mathscr{M}_{g,n}.
\end{array} \tag{4.4}$$

where $\mathscr{M}_{\tilde{\Gamma}}$ is the stack of stable curves with graph $\tilde{\Gamma}$, the graph obtained by cutting all edges of Γ, and $\mathscr{M}_{\tilde{\Gamma}}^{1/r}$ is the stack of stable

r-spin curves with graph $\tilde{\Gamma}$ (still marked with $m^{\pm}$ on each half edge). p_1 is the following morphism: The fiber product consists of triples of an r-spin curve $(X/T, \mathscr{F}, \varphi)$, a stable curve $\tilde{X}/T$, and a morphism $\nu : \tilde{X} \longrightarrow X$, making $\tilde{X}$ into the normalization of X. Also, the dual graphs of X and $\tilde{X}$ are Γ and $\tilde{\Gamma}$, respectively. The associated r-spin curve in $\mathscr{M}_{\tilde{\Gamma}}^{1/r}$ is simply $(\tilde{X}/T, \nu^*\mathscr{F}, \nu^*\varphi)$. We require that

$$p_{1*}\tilde{\mu}^* \boldsymbol{c}_{\Gamma}^{1/r} = r^{|E(\Gamma)|} \boldsymbol{c}_{\tilde{\Gamma}}^{1/r},$$

where $E(\Gamma)$ is the set of edges of Γ that are cut in $\tilde{\Gamma}$.

Axiom 4 (Ramond Vanishing): *If Γ contains a tail marked with $m_i = r-1$ or $m_i = -1$, then $\boldsymbol{c}_{\Gamma}^{1/r} = 0$.*

Axiom 5 (Forgetting tails): *Let $\widehat{\Gamma}$ be a stable graph whose i-th tail is marked by $m_i = 0$, Γ be the stable graph obtained by removing the i-th tail, and*

$$\pi : \overline{\mathscr{M}}_{\widehat{\Gamma}}^{1/r} \longrightarrow \overline{\mathscr{M}}_{\Gamma}^{1/r}$$

be the forgetful morphism. The classes $\boldsymbol{c}_{\widehat{\Gamma}}^{1/r}$ and $\pi^*\boldsymbol{c}_{\Gamma}^{1/r}$ are related by

$$\boldsymbol{c}_{\widehat{\Gamma}}^{1/r} = \pi^* \boldsymbol{c}_{\Gamma}^{1/r}.$$

4.2. The properties satisfied by the [5] virtual cycle for A_{r-1}

Let Γ be a stable graph (not necessarily connected) with tails $T(\Gamma)$, and with each tail $\tau \in T(\Gamma)$ decorated by an element $\gamma_\tau \in G$. Denote by $n = |T(\Gamma)|$ the number of tails of Γ.

The theory of [5] provides a homology *virtual cycle*

$$\begin{aligned}[\mathscr{W}(\Gamma)]^{vir} \in & H_*(\mathscr{W}(\Gamma), \mathbb{Q}) \otimes \prod_{\tau \in T(\Gamma)} \left(H_{N_\gamma}(\mathrm{Fix}(J^k), W_{J^k}^{\infty}, \mathbb{Q})\right)^{\langle J \rangle} \\ = & H_*(\mathscr{W}(\Gamma), \mathbb{Q}) \otimes \prod_{\tau \in T(\Gamma)} \mathscr{A}_{\gamma_\tau}^*,\end{aligned}$$

where $\mathscr{A}_{\gamma_\tau}^*$ is the dual of $\mathscr{A}_{\gamma_\tau}$. And the virtual cycle satisfies several axioms similar to those of the virtual cohomology class of r-spin theory. Here we briefly review those properties of the cycle that are relevant to this paper.

When Γ has a single vertex of genus g, n tails, and no edges (i.e, Γ is a corolla), we denote the virtual cycle by $[\mathscr{W}(\boldsymbol{\gamma})]^{vir}$, where $\boldsymbol{\gamma} := (\gamma_1, \ldots, \gamma_n)$.

The following properties axioms hold for the virtual cycle $[\mathscr{W}(\Gamma)]^{vir}$:

(1) **Dimension:** As in the r-spin case, define

$$D = \hat{c}(\gamma - \alpha) + \sum_{\tau \in T(\Gamma)} \deg(\mathbf{a}_{\gamma_\tau}),$$

where α is the number of connected components of Γ. If D is not an integer, then $[\mathscr{W}(\Gamma)]^{vir} = 0$. Otherwise, the cycle $[\mathscr{W}(\Gamma)]^{vir}$ has degree

$$2\left((\hat{c}-3)(1-g) + n - \sum_{\tau \in T(\Gamma)} \iota_\tau\right). \tag{4.5}$$

So the cycle lies in $H_d(\mathscr{W}(\Gamma), \mathbb{Q}) \otimes \prod_{\tau \in T(\Gamma)} \mathscr{A}^*_{\gamma_\tau}$, where

$$d := 6g - 6 + 2n - 2D = 2\left((\hat{c}-3)(1-g) + n - \sum_{\tau \in T(\Gamma)} \deg(\mathbf{a}_{\gamma_\tau})\right).$$

(2) **Symmetric group invariance**: There is a natural S_n-action on $\mathscr{W}_{g,n}$ obtained by permuting the tails. This action induces an action on homology. That is, for any $\sigma \in S_n$ we have:

$$\sigma_* : H_*(\mathscr{W}_{g,n}, \mathbb{Q}) \otimes \prod_i \mathscr{A}^*_{\gamma_i} \longrightarrow H_*(\mathscr{W}_{g,k}, \mathbb{Q}) \otimes \prod_i \mathscr{A}^*_{\gamma_i}.$$

For any decorated graph Γ, let $\sigma\Gamma$ denote the graph obtained by applying σ to the tails of Γ.

We have

$$\sigma_* [\mathscr{W}(\Gamma)]^{vir} = [\mathscr{W}(\sigma\Gamma)]^{vir}. \tag{4.6}$$

(3) **Degenerating connected graphs:** Let Γ be a connected, genus-g, stable graph decorated with γ_i on the ith tail.

The cycles $[\mathscr{W}(\Gamma)]^{vir}$ and $[\mathscr{W}_{g,n}(\boldsymbol{\gamma})]^{vir}$ are related by

$$[\mathscr{W}(\Gamma)]^{vir} = \tilde{i}^* [\mathscr{W}_{g,n}(\boldsymbol{\gamma})]^{vir}, \tag{4.7}$$

where $\tilde{i} : \mathscr{W}(\Gamma) \longrightarrow \mathscr{W}_{g,n}(\boldsymbol{\gamma})$ is the canonical inclusion map.

(4) **Disconnected graphs:** Let $\Gamma = \coprod_i \Gamma_i$ be a stable, decorated W-graph which is the disjoint union of connected W-graphs Γ_i. The classes $[\mathscr{W}(\Gamma)]^{vir}$ and $[\mathscr{W}(\Gamma_i)]^{vir}$ are related by

$$[\mathscr{W}(\Gamma)]^{vir} = [\mathscr{W}(\Gamma_1)]^{vir} \times \cdots \times [\mathscr{W}(\Gamma_d)]^{vir}. \tag{4.8}$$

(5) **Concavity**: Suppose that all the decorations on tails of Γ are *narrow*, meaning that $\mathrm{Fix}(\gamma) = \mathbb{C}^{N_{\gamma_i}} = \{0\}$, and so we can omit $\mathscr{A}^*_{\gamma_i} = H_{N_{\gamma_i}}(\mathbb{C}^{N_{\gamma_i}}, W^\infty_{\gamma_i}, \mathbb{C}) \cong \mathbb{C}$ from our notation.

Consider the universal A_{r-1}-structure bundle $\mathscr{L}$ on the universal curve $\pi : \mathscr{C} \longrightarrow \mathscr{W}(\Gamma)$.

If $\pi_*(\mathscr{L}) = 0$, then the virtual cycle is given by capping the top Chern class of the dual $\left(R^1\pi_*(\mathscr{L})\right)^*$ of the pushforward with the usual fundamental cycle of the moduli space:

$$\begin{aligned}[\mathscr{W}(\Gamma)]^{vir} &= c_{top}\left((R^1\pi_*\mathscr{L})^*\right) \cap [\mathscr{W}(\Gamma)] \\ &= (-1)^D c_D\left(R^1\pi_*\mathscr{L}\right) \cap [\mathscr{W}(\Gamma)]\,.\end{aligned} \tag{4.9}$$

(6) **Composition law:** Given any genus-g decorated stable W-graph Γ with k tails, and given any edge e of Γ, let $\widehat{\Gamma}$ denote the graph obtained by "cutting" the edge e and replacing it with two unjoined tails τ_+ and τ_- decorated with γ_+ and γ_-, respectively.

The fiber product

$$F := \mathscr{W}(\widehat{\Gamma}) \times_{\mathscr{W}(\Gamma)} \mathscr{W}(\Gamma)$$

has morphisms

$$\mathscr{W}(\widehat{\Gamma}) \xleftarrow{q} F \xrightarrow{pr_2} \mathscr{W}(\Gamma).$$

We have

$$\left\langle \left[\mathscr{W}(\widehat{\Gamma})\right]^{vir} \right\rangle_{\pm} = \frac{1}{\deg(q)} q_* pr_2^* \left([\mathscr{W}(\Gamma)]^{vir}\right), \tag{4.10}$$

where $\langle\rangle_\pm$ is the map from

$$H_*(\mathscr{W}(\widehat{\Gamma})) \otimes \prod_{\tau \in T(\Gamma)} \mathscr{A}^*_{\gamma_\tau} \otimes \mathscr{A}^*_{\gamma_+} \otimes \mathscr{A}^*_{\gamma_-}$$

to

$$H_*(\mathscr{W}(\widehat{\Gamma})) \otimes \prod_{\tau \in T(\Gamma)} \mathscr{A}^*_{\gamma_\tau}$$

obtained by contracting the last two factors via the usual dual pairing

$$\langle\,,\rangle : \mathscr{A}^*_{\gamma_+} \otimes \mathscr{A}^*_{\gamma_-} \longrightarrow \mathbb{C}.$$

(7) **Forgetting tails:**

(a) Let Γ have its ith tail decorated with J, where J is the exponential grading element of G. Further let Γ' be the decorated W-graph obtained from Γ by forgetting the ith tail and its decoration. Assume that Γ' is stable, and denote the forgetting tails morphism by

$$\vartheta : \mathscr{W}(\Gamma) \longrightarrow \mathscr{W}(\Gamma').$$

We have

$$[\mathscr{W}(\Gamma)]^{vir} = \vartheta^* [\mathscr{W}(\Gamma')]^{vir}. \tag{4.11}$$

(b) In the case of $g = 0$ and $k = 3$, the space $\mathscr{W}(\gamma_1, \gamma_2, J)$ is empty if $\gamma_1\gamma_2 \neq 1$, and otherwise $\mathscr{W}_{0,3}(\gamma, \gamma^{-1}, J) = \mathscr{B}G$. We omit $\mathscr{A}_J^* = \mathbb{C}$ from the notation. In this case, the cycle

$$\left[\mathscr{W}_{0,3}(\gamma, \gamma^{-1}, J)\right]^{vir} \in H_*(\mathscr{B}G, \mathbb{Q}) \otimes \mathscr{A}_\gamma^* \otimes \mathscr{A}_{\gamma^{-1}}^*$$

is the fundamental cycle of $\mathscr{B}G$ times the Casimir element. Here the Casimir element is defined as follows. Choose a basis $\{\alpha_i\}$ of $\mathscr{A}_\gamma^*$, and a basis $\{\beta_j\}$ of $\mathscr{A}_{\gamma^{-1}}^*$. Let $\eta_{ij} = \langle \alpha_i, \beta_j \rangle$ and (η^{ij}) be the inverse matrix of (η_{ij}). The Casimir element is defined as $\sum_{ij} \alpha_i \eta^{ij} \otimes \beta_j$.

The virtual cycle satisfies several other properties as well, but those additional properties are not needed for this paper.

4.3. The A_{r-1} virtual cycle defines an r-spin class

As noted above, the stack of r-spin curves and the stack of A_{r-1}-curves are isomorphic as stacks, and the state spaces of the two theories are isomorphic. All that remains is to use the A_{r-1} virtual cycle to construct a cohomology class which satisfies the axioms of an r-spin virtual class.

4.3.1. r-spin class from the A_{r-1} virtual cycle

To complete the connection to the r-spin theory of [8], we define a cohomology class $c_{g,k}^{1/r}(\mathbf{m})$ as follows.

DEFINITION 4.3.1. — *Given a stable graph Γ with tails (indexed by $i \in \{1, \dots, n\}$) decorated by integers $m_i \in \{0, \dots, r-1\}$ and half-edges (indexed by e_+ and e_- for $e \in E(\Gamma)$) decorated by integers m_{e_+} and m_{e_-} in $\{0, \dots, r-1\}$, such that for any edge the two decorations m_{e_+} and m_{e_-} on the half edges e_+ and e_-, respectively, satisfy the relation $m_{e_+} + m_{e_-} \equiv r - 2 \pmod r$, we let $\widetilde{\Gamma}$ be the stable decorated W-graph whose tails are decorated with the group elements $\gamma_i := J^{m_i+1}$ and whose half-edges are decorated by the group elements $\gamma_{e_+} := J^{m_{e_+}+1}$ and $\gamma_{e_-} := J^{m_{e_-}+1}$, respectively.*

We define $\boldsymbol{c}_\Gamma^{1/r}$ *to be the cohomology class*

$$\boldsymbol{c}_\Gamma^{1/r} := \begin{cases} 0 \text{ if } m_i \equiv -1 \pmod r \text{ for any } i \in \{1,\dots,n\} \text{ or if } D \text{ is not in } \mathbb{Z}. \\ \prod_{e\in E(\Gamma)} |\langle \gamma_{e_+}\rangle| PD\left(\left[\overline{\mathscr{M}}_{A_{r-1}}(\widetilde{\Gamma})\right]^{vir} \cap \prod_{i=1}^k \mathbf{a}_{\gamma_i}\right) \text{ otherwise.} \end{cases} \tag{4.12}$$

Here PD *denotes the Poincaré dual, and* $\cap$ *denotes the obvious contraction*

$$\cap : \left(H_*(\mathscr{W}(\Gamma),\mathbb{Q}) \otimes \prod_{\tau\in T(\Gamma)} \mathscr{A}_{\gamma_\tau}^*\right) \otimes \left(\prod_{\tau\in T(\Gamma)} \mathscr{A}_{\gamma_\tau}\right) \longrightarrow H_*(\mathscr{W}(\Gamma),\mathbb{C}).$$

We will write $\boldsymbol{c}_{g,k}^{1/r}(\mathbf{m}) := \boldsymbol{c}_\Gamma^{1/r}$ *when* Γ *is a genus-*g *corolla with* k *tails labeled by* $\mathbf{m} = (m_1,\dots,m_n)$.

4.3.2. Verification of the r-spin axioms

We will continue to use the notation of Definition 4.3.1, and we will write

$$\overline{\mathscr{M}}_\Gamma^{1/r} := \overline{\mathscr{M}}_{A_{r-1}}(\widetilde{\Gamma}). \tag{4.13}$$

This is legitimate, since as discussed earlier, the stack $\overline{\mathscr{M}}_\Gamma^{1/r}$ of stable r-spin curves with graph Γ is (canonically) isomorphic to the stack $\overline{\mathscr{M}}_{A_{r-1}}(\widetilde{\Gamma})$.

PROPOSITION 4.3.2. — *The collection of classes* $\boldsymbol{c}_\Gamma^{1/r}$ *satisfies all the axioms of an* r*-spin virtual class outlined in* [8].

Proof. — It is clear from the definition that the class $\boldsymbol{c}^{1/r}$ lies in $H^{2D}(\overline{\mathscr{M}}_\Gamma^{1/r})$. And, indeed, the only axiom that does not immediately follow from the definition and the corresponding axioms for A_{r-1}-curves is the cutting edges axiom.

The axiom requires in Diagram (4.4) that

$$q_* p r_2^* \boldsymbol{c}_\Gamma^{1/r} = r^{|E(\Gamma)|} \boldsymbol{c}_{\widehat{\Gamma}}^{1/r},$$

where $E(\Gamma)$ is the set of edges of Γ that are cut in $\widehat{\Gamma}$. This follows by induction on the number of edges from the corresponding axiom for the virtual cycle. For a single edge labeled by γ, the degree of the map q in that axiom is $|G/\langle\gamma\rangle|$ and the class $\boldsymbol{c}_\Gamma^{1/r}$ is $|\langle\gamma\rangle|$ times the Poincaré dual of the virtual cycle, so the overall factor introduced is $|G|^{|E(\Gamma)|} = r^{|E(\Gamma)|}$, as desired.

The only complication is the fact that we have defined the class $\boldsymbol{c}^{1/r}$ to be zero if any tail is marked with $r-1$ or -1 (corresponding to $\gamma = J^0$). But in the A_{r-1} case the only invariant element in the broad/Ramond (J^0-) sector is 0, so any time a cut graph $\widehat{\Gamma}$ introduces a new tail decorated with $m = r-1$ (or -1), the corresponding class will vanish, as required. □

5. Conclusion

The cohomological field theory arising from the A_{r-1} theory is given by

$$\begin{aligned}\Lambda_{g,n}^{A_{r-1}}(\mathbf{a}_{J^{k_1}},\dots,\mathbf{a}_{J^{k_n}}) : &= \frac{|G|^g}{\deg(st)} PD(st_*([\mathscr{W}_{g,n}]^{vir} \cap \mathbf{a}_{J^{k_1}} \otimes \cdots \otimes \mathbf{a}_{J^{k_n}})) \\ &= \frac{r^g}{r^{2g-1}} PD(st_*([\mathscr{W}_{g,n}]^{vir} \cap \mathbf{a}_{J^{k_1}} \otimes \cdots \otimes \mathbf{a}_{J^{k_n}})) \\ &= \frac{1}{r^{g-1}} st_* \boldsymbol{c}^{1/r}(k_1-1,\dots,k_n-1)\end{aligned}$$

And this is precisely the r-spin cohomological field theory $\Lambda_{g,n}^{1/r}(\boldsymbol{s}_{k_1-1},\dots,\boldsymbol{s}_{k_n-1})$ as defined in [8], so the two theories are identical.

Faber, Shadrin, and Zvonkine [4] have proved that any r-spin cohomological field theory arising from an r-spin virtual class is completely determined by the $g=0$ theory, and in [8] it is proved that the $g=0$ r-spin theory is completely determined by the axioms. Therefore the proof in [4] that the r-spin theory satisfies the Witten Integrable Hierarchies Conjecture also applies to the A_{r-1} theory, as expected.

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