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COXETER ELEMENTS FOR VANISHING CYCLES OF TYPES $A_{\frac{1}{2}\infty}$ AND $D_{\frac{1}{2}\infty}$

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ABSTRACT. — We introduce two entire functions $f_{A_{\frac{1}{2}\infty}}$ and $f_{D_{\frac{1}{2}\infty}}$ in two variables. Both of them have only two critical values 0 and 1, and the associated maps $\mathbf{C}^2 \to \mathbf{C}$ define topologically locally trivial fibrations over $\mathbf{C} \setminus \{0, 1\}$. All critical points in the singular fibers over 0 and 1 are ordinary double points, and the associated vanishing cycles span the middle homology group of the general fiber, whose intersection diagram forms bi-partitely decomposed infinite quivers of type $A_{\frac{1}{2}\infty}$ and $D_{\frac{1}{2}\infty}$, respectively. Coxeter elements of type $A_{\frac{1}{2}\infty}$ and $D_{\frac{1}{2}\infty}$, acting on the middle homology group, are introduced as the product of the monodromies around 0 and 1. We describe the spectra of the Coxeter elements by embedding the middle homology group into a Hilbert space. The spectra turn out to be strongly continuous on the interval $\left(-\frac{1}{2}, \frac{1}{2}\right)$ except at 0 for type $D_{\frac{1}{2}\infty}$.

RÉSUMÉ. — Nous introduisons deux fonctions entières de deux variables $f_{A_{\frac{1}{2}\infty}}$ et $f_{D_{\frac{1}{2}\infty}}$. Leurs seules valeurs critiques sont 0 et 1, et les applications associées $\mathbf{C}^2 \to \mathbf{C}$ définissent des fibrations localement topologiquement triviales au-dessus de $\mathbf{C} \setminus \{0, 1\}$. Tous les points critiques dans les fibres singulières au-dessus de 0 et 1 sont des points doubles ordinaires, les cycles évanescents associés engendrent le groupe d'homologie en dimension moitié de la fibre générique, et leur diagramme d'intersection est un carquois infini bipartite de type $A_{\frac{1}{2}\infty}$ et $D_{\frac{1}{2}\infty}$, respective-ment. Les transformations de Coxeter de type $A_{\frac{1}{2}\infty}$ et $D_{\frac{1}{2}\infty}$, agissant sur le groupe d'homologie en dimension moitié, sont définies par le produit des monodromies en 0 et 1. On décrit les spectres de ces transformations de Coxeter en plongeant le groupe d'homologie en dimension moitié dans un espace de Hilbert. Ces spectres sont fortement continus sur l'intervalle $\left(-\frac{1}{2}, \frac{1}{2}\right)$, sauf en 0 en type $D_{\frac{1}{2}\infty}$.

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1. Introduction

We introduce two particular entire transcendental functions in two variables, which we will call the functions of types $A_{\frac{1}{2}\infty}$ and $D_{\frac{1}{2}\infty}$, respectively. They are introduced in the hope that period maps associated with them should contribute to the understanding of KP- and KdV-hierarchies. For this purpose, we need to develop a theory of primitive forms for these transcendental functions by analogy with the classical theory of primitive forms for polynomial local singularities [6],[1] where the spectrum of Milnor monodromies is basic. Therefore, as a first step to this theory, we study the spectrum of the large circle monodromy, which we call the Coxeter element, acting on the lattice of vanishing cycles of these functions. The goal is to show that the spectrum is contained in the interval $(-\frac{1}{2}, \frac{1}{2})$.

We are still in early stage in studing transcendental functions in such geometric contexts. Many of classical algebraic tools are not available due to the lack of compactness/finiteness nature of them. However, the transcendency of the functions which we study in the present paper as the test cases, is still not "wild", and we handle them by "hand". Even though each step of the calculation is elementary, we want to be cautious and will proceed with the calculations in down to earth fashion.

Let us explain the contents of the present paper in more details. We first explain a classical analogue ([7, §2.5,3]) of the present work. Then we make some comparisons between the classical case and the present case.

For a Dynkin graph Γ_W of type $W \in \{A_l \ (l \in \mathbf{Z}_{\geqslant 1}), D_l \ (l \in \mathbf{Z}_{\geqslant 4}), E_6, E_7, E_8\}$, there exists a real polynomial $f_W(x, y, z)$ satisfying the following i)-iii).[2]

i) *All critical points of f_W are Morse (i.e. Hessians at the critical points are non-degenerate), and f_W has only two critical values 0 and 1.*

ii) *The map $f_W : \mathbf{C}^3 \to \mathbf{C}$ is a locally trivial fibration over $\mathbf{C} \setminus \{0, 1\}$. Let us denote by X_t the fiber $f_W^{-1}(t)$ over a point $t \in \mathbf{C}$.*

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[1] In the classical theory, a primitive form is defined on the universal unfolding of a function having an isolated critical point. Whereas, in the present program, the "generating center of the unfoldings" are these functions of types $A_{\frac{1}{2}\infty}$ and $D_{\frac{1}{2}\infty}$, having infinitely many critical points, where there is not yet a general frame work available.

[2] This is a consequence of a result in [7, §2.5, 3]([4, §6.5 Remark 19 and §8.9 Remark 27]). Let us briefly recall the result. For each simply-laced Dynkin type W, there exists a two parameter family $f_W(x, y, z, R, S)$ of real polynomials of type W, having only two critical values and having properties ii) and iii) (choose f_W such that its deformation class $[f_W]$ belongs to the vertex orbit line O in the deformation parameter space T_W of simple polynomials of type W). Then, fix the parameter values of (R, S) to re-size the critical values to $\{0, 1\}$. In particular, if $W = A_l$, the polynomial is given by (Chebyshev polynomial in x) $+y^2 + z^2$.

Instead of the formulation in 3-variables as given here, we may formulate results in 2-variables by replacing an intersection diagram by a quiver diagram (see §3.3).

iii) *For $t \in (0,1)$, let $\{\gamma_0^{(i)}\}_{i \in C_0}$ (resp. $\{\gamma_1^{(i)}\}_{i \in C_1}$) be the set of cycles in the middle homology group $H_2(X_t\mathbf{Z})$ which vanish at a critical point in the fiber X_0 as $t \downarrow 0$ (resp. X_1 as $t \uparrow 1$). Then, a) the union $\{\gamma_0^{(i)}\}_{i \in C_0} \cup \{\gamma_1^{(i)}\}_{i \in C_1}$ forms a basis of $H_2(X_t, \mathbf{Z})$, b) the intersection diagram of the basis gives a bi-partite decomposition of the Dynkin graph Γ_W.*

In the first half of the present paper, we show that the functions of type $A_{\frac{1}{2}\infty}$ and $D_{\frac{1}{2}\infty}$ satisfy exactly the properties parallel to i), ii) and iii) by replacing Γ_W by the infinite quivers of type $A_{\frac{1}{2}\infty}$ and $D_{\frac{1}{2}\infty}$ introduced in §3.2. This fact explains the naming of the functions. Here we should note that the middle homology group is an infinite rank lattice.

In the classical polynomial f_W case, the product of the two monodromies of the fibration around 0 and around 1, acting on the lattice $H_2(X_t, \mathbf{Z})$, is called the *Coxeter element.* The eigenvalues of the Coxeter elements are given by the set $\exp(2\pi\sqrt{-1}\frac{m_i}{h})$ $(i=1,\cdots,l)$, where $h \in \mathbf{Z}_{>0}$ is the Coxeter number of type W and $0 < m_1 < m_2 \leqslant \cdots < m_l < h$ are called exponents (see [2, ch.V,§6,n°2]). The data of exponents, or equivalently, the spectrum $\frac{m_i}{h}$ $(i=1,\cdots,l)$ are quite important both for the Lie theory of type W [2] and for the primitive forms of type W [6]. For instance, the fact that the spectrum is contained in the interval $(\frac{d}{2}-1, \frac{d}{2})$ means that the primitive form is of "simple type" of dimension d (see Remark at the end of §3 and [5]). However, the eigenvalues of the Coxeter element alone are not sufficient to recover the spectrum due to an ambiguity modulo integers. The clue to recover the spectrum is the study of eigenvalues of the intersection form on the root lattice, as will be described in §4 in the present paper.

Returning to the transcendental functions of types $A_{\frac{1}{2}\infty}$ and $D_{\frac{1}{2}\infty}$, in analogy with the classical case, we introduce the Coxeter element as the product of the local monodromy around 0 and around 1. In order to be able to discuss about the "eigenvalues" of the Coxeter element, we embed the "lattice" into the Hilbert space (see §4.1) such that the simple root basis turns to the ortho-normal basis of the Hilbert space.[3]

The main result of the present paper is that the spectrum of the Coxeter element is given by the function $\theta - \frac{1}{2}$ on the interval $\theta \in [0,1]$ with a Stieltjes measure $\xi_{W,\theta}$ which is strongly continuous (§4 Theorem 7). Actually, $\xi_{W,0} = \xi_{W,1} = 0$ (i.e. there is no discrete spectrum at $\theta = 0,1$) so that the spectrum is contained in the open interval $(-\frac{1}{2}, \frac{1}{2})$.

This is what was expected from the analogy with classical theory.

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[3] This view point is already implicitly in [2, ch.V,§6,n°2]. The Hilbert space lies between the homology group $H_1(X_t, \mathbf{C})$ and the cohomology group $H^1(X_t, \mathbf{C})$ (i.e. a sort of "intersection cohomology group", See end of §4.2), which fit with our original intention to develop a period map theory for this cohomology groups.

2. Functions of types $A_{\frac{1}{2}\infty}$ and $D_{\frac{1}{2}\infty}$

We introduce functions of type $A_{\frac{1}{2}\infty}$ and $D_{\frac{1}{2}\infty}$ and associated fibrations.

2.1. Definition of $f_{A_{\frac{1}{2}\infty}}$ and $f_{D_{\frac{1}{2}\infty}}$

DEFINITION. — *The function f_W of type $W \in \{A_{\frac{1}{2}\infty}, D_{\frac{1}{2}\infty}\}$* [4] *is a real entire function* [5] *in two variables x and y given by*

$$f_{A_{\frac{1}{2}\infty}}(x,y) := xs^2(x) - y^2 = 1 - c^2(x) - y^2 \qquad (2.1.1)$$

$$f_{D_{\frac{1}{2}\infty}}(x,y) := xs^2(x) - xy^2 = 1 - c^2(x) - xy^2. \qquad (2.1.2)$$

Here $s(x)$ and $c(x)$ are real entire functions [6] *in a variable x given by*

$$s(x) := \frac{\sin\sqrt{x}}{\sqrt{x}} = \prod_{n=1}^{\infty}\left(1 - \frac{x}{n^2\pi^2}\right) \qquad (2.1.3)$$

$$c(x) := \cos\sqrt{x} = \prod_{n=1}^{\infty}\left(1 - \frac{4x}{(2n-1)^2\pi^2}\right). \qquad (2.1.4)$$

2.2. Real level sets $X_{A_{\frac{1}{2}\infty},0,\mathbf{R}}$ and $X_{D_{\frac{1}{2}\infty},0,\mathbf{R}}$

We introduce the real level-0 set of the function f_W of type W by

$$X_{W,0,\mathbf{R}} := \mathbf{R}^2 \cap f_W^{-1}(0) .$$

Conceptual figures of them are drawn in the following.

Fig. 1

$X_{A_{\frac{1}{2}\infty},0,\mathbb{R}}$

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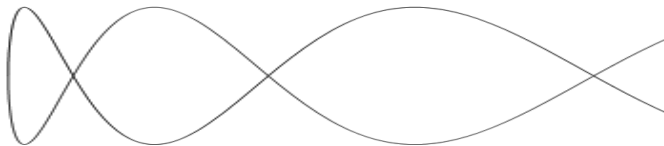
[4] In the present paper, the expression "*of type W*" automatically implies $W \in \{A_{\frac{1}{2}\infty}, D_{\frac{1}{2}\infty}\}$. Meaning for this name is given in §3.2 Quiver and its Remark.

[5] We mean by a *real entire function of n-variables* a holomorphic function on $\mathbf{C}^n$ which is real valued on the real form $\mathbf{R}^n$ of $\mathbf{C}^n$.

[6] In the sequel of the present paper, we shall freely use the following equalities:
$c(0) = s(0) = 1$, $xs^2(x) + c^2(x) = 1$, $s'(x) = \frac{1}{2x}(c(x) - s(x))$ and $c'(x) = -\frac{1}{2}s(x)$
without referring to them explicitly (here $f'(x) =$ the differentiation of $f(x)$).

Fig. 2

$X_{D_{\frac{1}{2}\infty},0,\mathbb{R}}$
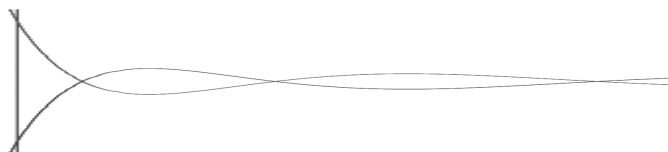
Terminology 1. By a *bounded connected component* (*bcc* for short) of type W, we mean a bounded connected component of $\mathbf{R}^2 \setminus X_{W,0,\mathbf{R}}$.

2. By a *node* of type W, we mean a point on the real curve $X_{W,0,\mathbf{R}}$ where two local smooth irreducible components are crossing normally.

3. We say that a node of type W is *adjacent* to a bcc of type W if the node belongs to the closure of the bcc.

We state some immediate observations on the level set $X_{W,0,\mathbf{R}}$, which can be easily verified by a use of absolutely convergent infinite products (2.1.3) and (2.1.4).

Observation 1. *For $n = 0, 1, 2, \cdots$, there exists exactly one bounded connected component of type W, containing the interval $(n^2\pi^2, (n+1)^2\pi^2)$ on the x-axis and contained in the domain $(n^2\pi^2, (n+1)^2\pi^2) \times$ y-axis.*

2. *For $n = 1, 2, 3, \cdots$, the point $c_{W,0}^{(n)} := (n^2\pi^2, 0)$ on the x-axis is a node of type W, which is adjacent to two bcc containing the interval $((n-1)^2\pi^2, n^2\pi^2)$ and the interval $(n^2\pi^2, (n+1)^2\pi^2)$.*

2.3. Fibrations over $\mathbf{C} \setminus \{0, 1\}$

For each type $W \in \{A_{\frac{1}{2}\infty}, D_{\frac{1}{2}\infty}\}$, let us consider the holomorphic map

$$f_W : \mathbf{X}_W \longrightarrow \mathbf{C}, \qquad (2.3.5)$$

where the domain $\mathbf{X}_W := \mathbf{C}^2$ of f_W is regarded as a contractible Stein manifold equipped with the real form $\mathbf{R}^2$. The fiber $X_{W,t} := f_W^{-1}(t)$ over $t \in \mathbf{C}$ is an *open* Riemann surface, closely embedded in $\mathbf{C}^2$.

Remark. — As we shall see in sequel, the fiber $X_{W,t}$ ($t \in \mathbf{C}$) has infinite genus. It is "wild" in the sense that the closure $\bar{X}_{W,t}$ in $\mathbf{P}_{\mathbf{C}}^2$ is equal to $X_{W,t} \cup \mathbf{P}_{\mathbf{C}}^1$ (i.e. the "ends" of $X_{W,t}$ is the $\mathbf{P}_{\mathbf{C}}^1$, this fact can be easily shown by the value distribution theory of one variable). By putting

$$\bar{\mathbf{X}}_W := \mathbf{X}_W \cup (\mathbf{P}_{\mathbf{C}}^1 \times \mathbf{C}) := \cup_{t \in \mathbf{C}} (\bar{X}_{W,t}, t) \subset \mathbf{P}_{\mathbf{C}}^2 \times \mathbf{C}, \qquad (2.3.6)$$

we obtain a proper map, i.e. a "compactification" of (2.3.5):

$$\bar{f}_W : \bar{\mathbf{X}}_W \longrightarrow \mathbf{C}. \qquad (2.3.7)$$

However, the spaces $\bar{X}_{W,t}$ and $\bar{\mathbf{X}}_W$ are not manifolds with boundary (note that their "boundaries" $\mathbf{P}_{\mathbf{C}}^1$ and $\mathbf{P}_{\mathbf{C}}^1 \times \mathbf{C}$, respectively, have the same dimension as the "interior" $X_{W,t}$ and $\mathbf{X}_W$).

By a lack of tools to handle such objects at present, we shall not use this compactification in the present paper. Nevertheless, in the following Theorem 3, we show that f_W induces a locally topologically trivial fibration over $\mathbf{C} \setminus \{0,1\}$. The proof is an elementary handwork, however it is not standard due to the transcendental nature of f_W mentioned. Therefore, we write the proof in down to earth fashion.

THEOREM. — *For each type* $W \in \{A_{\frac{1}{2}\infty}, D_{\frac{1}{2}\infty}\}$, *we have the following.*

1. *The function* f_W *has only two critical values 0 and 1. That is, the set of critical points* C_W *of* f_W *is contained in two fibers* $X_{W,0}$ *and* $X_{W,1}$.

2. **i)** *The critical set* C_W *lies in the real form* $\mathbf{R}^2$ *of* $\mathbf{X}_W$.

ii) *The Hessian form of* $f_W|_{\mathbf{R}^2}$ *at a critical point is non-degenerate. More precisely, the Hessian form is indefinite at a point in* $C_{W,0} := C_W \cap X_{W,0}$ *and is negative definite at a point in* $C_{W,1} := C_W \cap X_{W,1}$.

iii) *We have the natural bijections:*

$$C_{W,0} \simeq \{\text{nodes of type } P\} \quad (\text{identity map}), \tag{2.3.8}$$

$$C_{W,1} \simeq \{\text{bcc's of type } W\} \quad (c \mapsto B_c := \begin{smallmatrix}\text{the bcc}\\\text{containing}\end{smallmatrix} \, c) \tag{2.3.9}$$

3. *The restriction of the map* f_W *to the smooth fibers:*

$$f_W|_{\mathbf{X}_W \setminus (X_{W,0} \cup X_{W,1})} : \mathbf{X}_W \setminus (X_{W,0} \cup X_{W,1}) \to \mathbf{C} \setminus \{0,1\} \tag{2.3.10}$$

is a topologically locally trivial fibration.

Proof. — **1.** We proceed by direct calculations separately for each type. $A_{\frac{1}{2}\infty}$: The defining equations for $C_{A_{\frac{1}{2}\infty}}$ are $\partial_x f_{A_{\frac{1}{2}\infty}} = cs = 0, \partial_y f_{A_{\frac{1}{2}\infty}} = -2y = 0$. Hence, $C_{A_{\frac{1}{2}\infty}} = \{(x,0) \mid s(x)=0 \text{ or } c(x)=0\}$, where we have

$$f_{A_{\frac{1}{2}\infty}}(x,0) = \begin{cases} 0 & \text{if } s(x) = 0, \\ 1 & \text{if } c(x) = 0. \end{cases}$$

$D_{\frac{1}{2}\infty}$: The defining equations for $C_{D_{\frac{1}{2}\infty}}$ are $\partial_x f_{D_{\frac{1}{2}\infty}} = cs - y^2 = 0, \partial_y f_{D_{\frac{1}{2}\infty}} = -2xy = 0$. Hence, $C_{D_{\frac{1}{2}\infty}} = \{(0,\pm 1)\} \cup \{(x,0) \mid s(x)=0 \text{ or } c(x)=0\}$, where we have

$$f_{D_{\frac{1}{2}\infty}}(0,\pm 1)=0 \quad \text{and} \quad f_{D_{\frac{1}{2}\infty}}(x,0) = \begin{cases} 0 & \text{if } s(x) = 0, \\ 1 & \text{if } c(x) = 0. \end{cases}$$

2. **i)** Due to the descriptions of C_W in **1.**, we have only to show that the zero loci of $s(x)=0$ and $c(x)=0$ are real numbers. This follows from the

fact that the infinite product expressions (2.1.3) and (2.1.4) are absolutely convergent and the zero loci of $s(x)=0$ and $c(x)=0$ are given by the union of zero loci of factors of the expressions, respectively.

ii) Let us calculate the Hessian at a critical point.

The statement for the two critical points $(0, \pm 1)$ on $X_{D_{\frac{1}{2}\infty},0}$ can be verified directly. The other critical points are on the x-axis, i.e. one always has $y = 0$. Since $\partial_x \partial_y f_W |_{y=0} = 0$ for each type $W \in \{A_{\frac{1}{2}\infty}.D_{\frac{1}{2}\infty}\}$, the Hessian is a diagonal matrix of the form

$$[\partial_x(c(x)s(x)), -2]_{diag} \quad \text{for type } P = A_{\frac{1}{2}\infty},$$

$$[\partial_x(c(x)s(x)), -2x]_{diag} \quad \text{for type } P = D_{\frac{1}{2}\infty},$$

where the second diagonal component is always negative. We calculate the sign of the first diagonal component by

$$\partial_x(c(x)s(x)) |_{c=0} = -\frac{1}{2}s^2 = -\frac{1}{2x} < 0 \text{ and } \partial_x(c(x)s(x)) |_{s=0} = \frac{1}{2x} > 0,$$

implying the statement **ii)**.

iii) Combining the explicit descriptions of the set $C_{W,0}$, $C_{W,1}$ in Proof of **1.** with Observations 2. and 3. in §2.2, the correspondences are defined and are injective (see Figure 1 and 2.). So, we need only to show their surjectivity. But, this is again trivial since i) any node of a curve is a critical point of the defining equation of the curve, where Hessian is indefinite, and ii) inside of any bounded connected component of a complement of a real curve in $\mathbf{R}^2$, there exists at least a point where f_W takes local maximum, then the Hessian at the point should be negative definite since we saw in **2. ii)** that it is already non-degenerate.

3. Let us show that the fibration (2.3.10) is locally topologically trivial. Since our map is neither proper nor extendable to a suitably stratified proper map (recall Remark 1.3.), we cannot use standard techniques such as Thom-Ehresmann theorems. Instead, we use the elementary fact that $X_{W,t}$ is a ramified covering space: namely, in view of the equations (2.3.8) and (2.3.9), the projection map $(x, y) \in \mathbf{C}^2 \mapsto x \in \mathbf{C}$ to the x-plane induces a proper and ramified double covering map $\pi_{W,t}$:

$$X_{A_{\frac{1}{2}\infty},t} \to \mathbf{C} \ (t \in \mathbf{C}) \quad \text{and} \quad X_{D_{\frac{1}{2}\infty},t} \to \mathbf{C} \setminus \{0\} \ (t \in \mathbf{C} \setminus \{0\}), \qquad (2.3.11)$$

(for $X_{D_{\frac{1}{2}\infty},0}$, see [7]). Let us denote by $\mathbf{C}_W$ the base space of this covering, i.e. $\mathbf{C}_W := \mathbf{C}$ if $W = A_{\frac{1}{2}\infty}$ and $:= \mathbf{C} \setminus \{0\}$ if $W = D_{\frac{1}{2}\infty}$. In view of the

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[7] Since the fiber $X_{D_{\frac{1}{2}\infty},0}$ contains an irreducible component $L := \{x=0\}$, the map on $X_{D_{\frac{1}{2}\infty},0}$ is not a covering, but its restriction to $X_{D_{\frac{1}{2}\infty},0} \setminus L$ is a covering.

defining equation of $X_{W,t}$, the covering is ramifying at $X_{W,t} \cap \{y=0\}$, i.e. at solutions $x \in \mathbf{C}_w$ of the equation

$$xs^2(x) - t \;=\; 0, \tag{2.3.12}$$

which, apparently, has infinitely many solutions, depending on $t \in \mathbf{C}$.

We, now, state an elementary but crucial fact on the function xs^2.

Fact. *The correspondence* $\pi : \mathbf{C}_W \to \mathbf{C}$, $x \mapsto t := xs^2(x) = \sin^2(\sqrt{x})$ *is ramifying exactly and only at the inverse images of the points* 0 *and* 1, *and induces a (topological) covering map over* $\mathbf{C} \setminus \{0, 1\}$.

Proof of **Fact.** The critical points of the map $t = xs^2(x)$ are given by the equation $s(x)c(x) = 0$, and are exactly the points where $t = 0$ or 1 (recall Proof of **1.**). Thus, the restricted map $\pi' := \pi|_{\pi^{-1}(\mathbf{C} \setminus \{0,1\})}$ over $\mathbf{C} \setminus \{0, 1\}$ is a local homeomorphism. To see that π' is a covering (i.e. a proper map on each component of an inverse image of a simply connected open subset of $\mathbf{C} \setminus \{0, 1\}$), we need to show that the inverse map of $xs^2(x) = t$ as a multivalued function in t is analytically continuable everywhere on the set $\mathbf{C} \setminus \{0, 1\}$. Since the equation is equivalent to $\sqrt{x} = \pm \sin^{-1}(\sqrt{t})$, this fact follows from the fact that the multivalued function $\sin^{-1}(u)$ has singular points (i.e. points where the function cannot be analytically continued) only at $u = \pm 1$, easily seen from the integral expression $\sin^{-1}(u) = \int_0^u \frac{du}{\sqrt{1-u^2}}$. $\square$

Owing to **Fact**, we find a disc neighbourhood $\mathfrak{U}$ for any $t_0 \in \mathbf{C} \setminus \{0, 1\}$ so that $\pi^{-1}(\mathfrak{U})$ decomposes into components homeomorphic to U. For each $x_i \in \pi^{-1}(t_0)$ ($i \in I$ index set), let $s_i(t)$ be the function on $t \in \mathfrak{U}$, defining a section of π such that $s_i(t_0) = x_i$ (actually, $s_i(t) = \left(\sqrt{x_i} + \int_{\sqrt{t_0}}^{\sqrt{t}} \frac{du}{\sqrt{1-u^2}}\right)^2$ for choices of $\sqrt{t_0}$ and $\sqrt{x_i}$ such that $\sqrt{t_0} = \sin(\sqrt{x_i})$ and path of integral in the connected component of $\pm\sqrt{\mathfrak{U}}$ containing $\sqrt{t_0}$).

We can find a differentiable map $\varphi : \mathfrak{U} \times \mathbf{C}_W \to \mathbf{C}_W$ such that i) $\varphi(t_0, x) = x$, ii) for each $t \in U$, the $\varphi_t := \varphi(t, \cdot)$ is a diffeomorphism of $\mathbf{C}_W$, and iii) for each $i \in I$, $\varphi(t, s_i(t))$ is constant (equal to $s_i(t_0) = x_i$). The diffeomorphism φ_t can be uniquely lifted to a diffeomorphism $\hat{\varphi}_t : X_{W,t} \simeq X_{W,t_0}$ of the double covers such that $\varphi_t \circ \pi_{W,t} = \pi_{W,t_0} \circ \hat{\varphi}_t$. The $\hat{\varphi}_t$ gives the local trivialization of (2.3.10). $\square$

This completes a proof of Theorem 1., 2. and 3. $\square$

3. Vanishing cycles of types $A_{\frac{1}{2}\infty}$ and $D_{\frac{1}{2}\infty}$

We show that the middle homology group of a generic fiber of the map (2.3.5) has a basis consisting of vanishing cycles. The intersection form among them forms the *principal quiver*[8] of type $A_{\frac{1}{2}\infty}$ or $D_{\frac{1}{2}\infty}$.

3.1. Middle homology groups

In the present paragraph, we describe the middle homology group of the general fibers of (2.3.10) in terms of vanishing cycles of the function f_W of type $W \in \{A_{\frac{1}{2}\infty}, D_{\frac{1}{2}\infty}\}$.

Vanishing cycles: For a critical point $c \in C_W = C_{W,0} \sqcup C_{W,1}$, we define an oriented 1-cycle $\gamma_{W,c}$ in $X_{W,t}$ for $t \in (0,1)$ as follows.

Due to Theorem 2, we can choose holomorphic local coordinates (u,v) in a neighborhood $\mathfrak{U}$ of c in $\mathbf{X}_W$ such that i) u and v are real valued on $\mathfrak{U}_{\mathbf{R}} := \mathfrak{U} \cap \mathbf{R}^2$, ii) $\frac{\partial(u,v)}{\partial(x,y)}|_{\mathfrak{U}_R} > 0$ and iii) $f_W|_{\mathfrak{U}} = u^2 - v^2$ if $c \in C_{W,0}$ and $f_W|_{\mathfrak{U}} = 1 - u^2 - v^2$ if $c \in C_{W,1}$. Then, define cycles:

$$\gamma_{W,c} := \begin{cases} (\sqrt{t}\cos(\theta), \sqrt{-1}\sqrt{t}\sin(\theta)) \ (0 \leqslant \theta \leqslant 2\pi), & \text{if } c \in C_{W,0} \\ (\sqrt{1-t}\cos(\theta), \sqrt{1-t}\sin(\theta)) \ (0 \leqslant \theta \leqslant 2\pi), & \text{if } c \in C_{W,1}. \end{cases} \tag{3.1.1}$$

Fact. *The oriented cycle $\gamma_{W,c}$ in the surface $X_{W,t}$ is, up to free homotopy, unique and independent of a choice of coordinates (u,v).*

DEFINITION. — *We shall denote the homology class in $H_1(X_{W,t}, \mathbf{Z})$ of the cycle $\gamma_{W,c}$ by the same $\gamma_{W,c}$, and call it the vanishing cycle of the function f_W at the critical point $c \in C_W$ (vanishing along the path $t \downarrow 0$ or $t \uparrow 1$).*

Sign convention of intersection numbers of 1-cycles on $X_{W,t}$.

i) Let I be the skew symmetric intersection form between two oriented 1-cycles on a oriented surface. Then we define the convention of the sign of intersection number locally as follows:

Fig. 3 $I(\gamma_1, \gamma_2) = 1$ if   , $I(\gamma_1, \gamma_2) = -1$ if

---

[8] We mean by a *quiver* an oriented graph. It is called *principal*, if the set of vertices has a bipartite decomposition $\Gamma_0 \sqcup \Gamma_1$ such that the head (resp. tail) of any edge belongs to Γ_0 (resp. Γ_1) (e.g. Figure 3 and 4). See [Sa2,3].

ii) The orientation of the surface $X_{W,t}$ is $\sqrt{-1}dz \wedge d\bar{z} = 2dx \wedge dy$ for a local holomorphic coordinate $z = x + iy$ on $X_{W,t}$. Eg. Cycles γ_x and γ_y locally homotopic to x-axis and y-axis intersect with $I(\gamma_x, \gamma_y) = 1$ at $z = 0$.

THEOREM. — **4.** *The middle homology group of $X_{W,t}$, $t \in (0,1)$ is given by*

$$H_1(X_{W,t}, \mathbf{Z}) \simeq H_W := H_{W,0} \oplus H_{W,1}, \qquad (3.1.2)$$

where

$$H_{W,0} := \oplus_{c \in C_{W,0}} \mathbf{Z}\gamma_{W,c} \qquad (3.1.3)$$

$$H_{W,1} := \oplus_{c \in C_{W,1}} \mathbf{Z}\gamma_{W,c} \qquad (3.1.4)$$

are formally defined free abelian group spanned by vanishing cycles.

5. *Let $I_W : H_1(X_{W,t}, \mathbf{Z}) \times H_1(X_{W,t}, \mathbf{Z}) \to \mathbf{Z}$ be the intersection form on the middle homology group. Then we have*

$$I_W = J_W - {}^t J_W \qquad (3.1.5)$$

where J_W and ${}^t J_W$ are integral bilinear forms on H_W given by

$$J_W(\gamma_{W,c}, \gamma_{W,c'}) := \begin{cases} 1 & \text{if } c = c', \\ -1 & \text{if } c \in C_{W,0}, \ c' \in C_{W,1} \text{ and } c \in \overline{B}_{c'}, \\ 0 & else, \end{cases} \qquad (3.1.6)$$

and

$${}^t J_W(\gamma_{W,c}, \gamma_{W,c'}) := \begin{cases} 1 & \text{if } c = c', \\ -1 & \text{if } c \in C_{W,1}, \ c' \in C_{W,0} \text{ and } c' \in \overline{B}_c, \\ 0 & else. \end{cases} \qquad (3.1.7)$$

Remark. — The meaning to use the form J_W shall be clarified in §3.3.

Proof. — We first calculate intersection numbers between vanishing cycles $\gamma_{W,c}$ and $\gamma_{W,c'}$ as given in **5.**

Suppose both critical points c, c' belong to $C_{W,0}$ (resp. $C_{W,1}$). If $c \neq c'$ then, for t close enough to 0 (resp. 1), the supports of the vanishing cycles are close to c and c' so that they are disjoint, i.e. $\gamma_{W,c} \cap \gamma_{W,c'} = \emptyset$ and we get $I_W(\gamma_{W,c}, \gamma_{W,c'}) = 0$. Then, this equality holds for any $t \in (0,1)$. If $c = c'$, then $I_W(\gamma_{W,c}, \gamma_{W,c}) = 0$ due to skew-symmetry of I_W.

Next, we consider a cycle $\gamma_{W,c}$ for $c \in C_{W,0}$ and a cycle $\gamma_{W,c'}$ for $c' \in C_{w,1}$. From their expressions in (3.1.1), we observe the following two facts:

i) The cycle $\gamma_{W,c}$ intersects only with each connected component of $\mathbf{R}^2 \setminus X_{W,0,\mathbf{R}}$ adjacent to c at one point $(u,v) = (\varepsilon\sqrt{t}, 0)$ for $\varepsilon \in \{\pm 1\}$.

ii) The underlying set $|\gamma_{W,c'}|$ is presented by a circle of radius $1-t$ in the bcc $B_{c'}$ containing c', i.e. it is equal to $\{(u',v') \in B_{c'} \mid f_W(u',v') = t\}$.

This means that cycles $\gamma_{W,c}$ and $\gamma_{W,c'}$ for the same $t \in (0,1)$ intersect if and only if the critical point c is adjacent to the bounded component $B_{c'}$, and, then, they intersect transversely at one point, say p. Let (u', v') be the coordinates for the cycle $\gamma_{W,c'}$ in (3.1.1). Then, by an orientation preserving orthogonal linear transformation of the coordinates, the intersection point p may be given by $(u', v') = (\sqrt{1-t}, 0)$

We determine the sign of the intersection as follows: in a neighbourhood of p, we have an equality $f_W = u^2 - v^2 = 1 - u'^2 - v'^2$. Then the differentiation at p of the equation gives $df|_p = \varepsilon\sqrt{t}\,du|_p = -\sqrt{1-t}\,du'|_p$. Since $du \wedge dv|_p = c\,du' \wedge dv'|_p$ for some positive $c \in \mathbf{R}_{>0}$, we get

a) $$\frac{\partial v}{\partial v'}\Big|_p = \varepsilon c \frac{\sqrt{t}}{\sqrt{1-t}}.$$

On the other hand, since du and du' are co-normal vectors to $X_{W,t}$ at p (i.e. $df|_p \mathbin{/\!/} du|_p \mathbin{/\!/} du'|_p$), we use dv and dv' as for complex coordinates of the 1-dimensional complex tangent space $T(X_{W,t})_p$ at p, which are compatible with the sign convention ii) of the surface $X_{W,t}$.

Using these coordinates, the infinitesimal direction $\frac{\partial}{\partial\theta}|_p$ of $\gamma_{W,c}$ at p is evaluated by

b) $$\frac{\partial v}{\partial\theta}\Big|_p = \varepsilon\sqrt{-1}\sqrt{t}$$

and the infinitesimal direction $\frac{\partial}{\partial\theta'}|_p$ of $\gamma_{W,c'}$ at p is evaluated by

c) $$\frac{\partial v'}{\partial\theta'}\Big|_p = \sqrt{1-t}.$$

Combining a), b) and c), we obtain that the angle from the cycle $\gamma_{W,c'}$ to the cycle $\gamma_{W,c}$ at their intersection point p is given by the angle of the complex number

d) $$\left(\frac{\partial v}{\partial\theta}\Big|_p \Big/ \frac{\partial v'}{\partial\theta'}\Big|_p\right) \Big/ \frac{\partial v}{\partial v'}\Big|_p = \frac{\sqrt{-1}}{c},$$

i.e. the angle is $\frac{\pi}{2}$. Then due to our sign convention, we obtain

$$I_W(\gamma_{W,c}, \gamma_{W,c'}) = -1 \quad \text{and} \quad I_W(\gamma_{W,c'}, \gamma_{W,c}) = 1,$$

which is independent of the sign $\varepsilon \in \{\pm 1\}$. Thus, (3.1.5) is shown.

Finally in the following i)-v), we prove **4**.
We formally put (3.1.3) and (3.1.4).
i) Let us first show a natural isomorphism.

$$H_1(X_{W,0}, \mathbf{Z}) \simeq H_{W,1}. \tag{3.1.8}$$

Proof of (3.1.8). We first show that $X_{W,0,\mathbf{R}}$ is a deformation retract of $X_{W,0}$. For the proof of it, recall the double cover expression of $X_{W,0}$ over $\mathbf{C}_W$, used in the proof of THEOREM **3**. In case of type $W = A_{\frac{1}{2}\infty}$, the deformation retract of the plane $\mathbf{C}_W$ to the half real axis $\mathbf{R}_{\geqslant 0}$ induces the retract of the covering space $X_{W,0}$ to its real form $X_{W,0,\mathbf{R}}$. In case of type

$W = \mathrm{D}_{\frac{1}{2}\infty}$, we do the retraction irreducible-componentwisely to the real axis $\mathbf{R}$ (details are left to the reader). Thus, in view of Figures 1 and 2, we have a natural isomorphism:

$$\mathrm{H}_1(X_{W,0}, \mathbf{Z}) \;\simeq\; \mathrm{H}_1(X_{W,0,\mathbf{R}}, \mathbf{Z}) \;\simeq\; \mathrm{H}_{W,1}. \qquad \Box\,)$$

ii) Using the double cover expressions of fibers $X_{W,t}$ in the proof of THEOREM **3.**, we can show that $f_W^{-1}([0,t])$ $(t \in (0,1))$ retracts to its subset $X_{W,0}$. Then composing with the inclusion map $X_{W,t} \subset f_W^{-1}([0,t])$, we get an exact sequence

$$\mathrm{H}_{W,0} \;\to\; \mathrm{H}_1(X_{w,t}, \mathbf{Z}) \;\overset{r}{\to}\; \mathrm{H}_1(X_{W,0}, \mathbf{Z}) \;\to\; 0,$$

where the restriction of r to the submodule $H_{W,1}$ composed with the isomorphism (3.1.8) induces the identity on $H_{W,1}$. This implies that $\mathrm{H}_{W,1}$ is a factor of $\mathrm{H}_1(X_{W,t}, \mathbf{Z})$.

iii) What remains to show is that $\mathrm{H}_{W,0}$ is injectively embedded in $\mathrm{H}_1(X_{W,t}, \mathbf{Z})$. This can be partially shown by using the non-degeneracy of the intersection relations (3.1.6) as follows.

Let $\gamma \in H_{W,0}$ be a non-zero element, whose image in $\mathrm{H}_1(X_{W,t}, \mathbf{Z})$ is zero. Then solving the relation $I_W(\gamma, \gamma_{W,c}) = 0$ for $c = c_{W,1}^{(n)} \in C_{W,1}$ (see Notation in §3.2) from large enough $n \in \mathbf{Z}_{>0}$ backwards to 1, we see successive vanishings of the coefficients of γ, and finally see that γ, up to a constant factor, is equal to $\gamma_{D,0}^+ - \gamma_{D,0}^-$ (see §3.2 for Notation $\gamma_{D,0}^+$ and $\gamma_{D,0}^-$). In order to show that this is not possible, we prepare a fact.

iv) **Fact.** *The function f_W of type W is invariant by the involution σ :* $\mathbf{X}_W \to \mathbf{X}_W$, $(x,y) \mapsto (x,-y)$ *on its domain, i.e. $f_W \circ \sigma = f_W$. The induced involution on the surface $X_{W,t}$, denoted again by σ, is equivariant with the covering map $\pi_{W,t}$ (2.3.11), i.e. $\pi_{W,t} \circ \sigma = \pi_{W,t}$. Then, one has $\sigma_*(\gamma_{W,c}) = -\gamma_{W,c}$ for all $c \in C_W$, except for the following two cases*

$$\sigma_*(\gamma_{D,0}^+) = -\gamma_{D,0}^- \quad \text{and} \quad \sigma_*(\gamma_{D,0}^-) = -\gamma_{D,0}^+.$$

Proof of **Fact.** Except for the cases $\gamma_{D,0}^+$ and $\gamma_{D,0}^-$, we can choose the coordinate in (3.1.1) in such manner that $\sigma(u,v) = (u,-v)$. $\Box$

v) Assuming $\gamma_{D,0}^+ = \gamma_{D,0}^-$, let us show a contradiction. Consider the homomorphism $(\pi_D)_* : \mathrm{H}_1(X_{D,t}, \mathbf{Z}) \to \mathrm{H}_1(\mathbf{C}_D, \mathbf{Z}) \simeq \mathbf{Z}$. Above **Fact.** implies $(\pi_D)_*(\gamma_{D,0}^+) = (\pi_D \circ \sigma)_*(\gamma_{D,0}^+) = (\pi_D)_* \circ \sigma_*(\gamma_{D,0}^+) = -(\pi_D)_*(\gamma_{D,0}^-)$ which, by the assumption, is equal to $-(\pi_D)_*(\gamma_{D,0}^+)$. Thus, we get $(\pi_D)_*(\gamma_{D,0}^+) = 0$. This contradicts the fact that $(\pi_D)_*(\gamma_{D,0}^+)$ generates $\mathrm{H}_1(\mathbf{C}_D, \mathbf{Z}) \simeq \mathbf{Z}$ (observed easily from the fact that the equation $x = 0$ defines i) a branch of

$X_{D,0,\mathbf{R}}$ at the nodal point $c_{D,0}^+$ and also ii) the puncture in $\mathbf{C}_D$, and from the description of $\gamma_{D,0}^+$ in (3.1.1)).

This completes a proof of Theorem 4. and 5. □

Remark. — In the step v) in above proof, we may use a σ-invariant form $\omega := \mathrm{Res}\left[\frac{ydxdy}{f_D - t}\right]$. Since $\int_{\gamma_{D,0}^+} \omega = \int_{\gamma_{D,0}^+} \sigma^*(\omega) = \int_{\sigma_*(\gamma_{D,0}^+)} \omega = -\int_{\gamma_{D,0}^-} \omega$, the assumption $\gamma_{D,0}^+ = \gamma_{D,0}^-$ implies $\int_{\gamma_{D,0}^+} \omega = 0$. On the other hand, $\omega = \mathrm{Res}\left[\frac{ydxdy}{f_D - t}\right] = \frac{dx}{2x}|_{X_{D,t}}$, and hence $\int_{\gamma_{D,0}^+} \omega = \pm\sqrt{-1}\pi \neq 0$. A contradiction!

3.2. Quivers of type $A_{\frac{1}{2}\infty}$ and $D_{\frac{1}{2}\infty}$

We encode homological data of vanishing cycles of f_W in a quiver Γ_W.

DEFINITION. — *A quiver Γ_W of type $W \in \{A_{\frac{1}{2}\infty}, D_{\frac{1}{2}\infty}\}$ is defined by i) The set of vertices of Γ_W is in bijection with $\{\gamma_{W,c} \mid c \in C_{W,0} \cup C_{W,1}\}$. ii) We put an oriented edge from $\gamma_{W,c}$ to $\gamma_{W,c'}$ if and only if $c \in C_{W,0}$, $c' \in C_{W,1}$ and $c \in \overline{B}_{c'}$, that is, when $J_W(\gamma_{W,c}, \gamma_{W,c'}) = -1$.*

Let us fix a numbering of elements in $C_{W,0} \cup C_{W,1}$ as follows.

$$
\begin{aligned}
C_{A,0} &= \{c_{A,0}^{(n)} := (n^2\pi^2, 0)\}_{n\in\mathbf{Z}_{>0}} \\
C_{A,1} &= \{c_{A,1}^{(n)} := ((n - \tfrac{1}{2})^2\pi^2, 0)\}_{n\in\mathbf{Z}_{>0}} \\
C_{D,0} &= \{c_{D,0}^{(n)} := (n^2\pi^2, 0)\}_{n\in\mathbf{Z}_{>0}} \cup \{c_{D,0}^+ := (0,1), c_{D,0}^- := (0,-1)\} \\
C_{D,1} &= \{c_{D,1}^{(n)} := ((n - \tfrac{1}{2})^2\pi^2, 0)\}_{n\in\mathbf{Z}_{>0}}.
\end{aligned}
$$

According to them, the vertices of the quiver Γ_W are numbered as below.

$$\Gamma_{A_{\frac{1}{2}\infty}} : \quad \gamma_{A,1}^{(1)} \longrightarrow \gamma_{A,0}^{(1)} \longleftarrow \gamma_{A,1}^{(2)} \longrightarrow \gamma_{A,0}^{(2)} \longleftarrow \gamma_{A,1}^{(3)} \longrightarrow \gamma_{A,0}^{(3)} \longleftarrow \quad \cdots$$

$$\Gamma_{D_{\frac{1}{2}\infty}} : \quad
\begin{matrix}
\gamma_{D,0}^+ \\
\nwarrow \\
\\
\swarrow \\
\gamma_{D,0}^-
\end{matrix}
\quad \gamma_{D,1}^{(1)} \longrightarrow \gamma_{D,0}^{(1)} \longleftarrow \gamma_{D,1}^{(2)} \longrightarrow \gamma_{D,0}^{(2)} \longleftarrow \gamma_{D,1}^{(3)} \longrightarrow \quad \cdots$$

Note that the decomposition of the critical set C_W into $C_{W,0} \cup C_{W,1}$ gives rise to the bi-partite (or principal) decomposition of the quiver Γ_W.

3.3. Suspensions to higher dimensions

In this subsection, we briefly describe the suspensions of the results in previous subsections to higher dimensional cases.

For a type $W \in \{A_{\frac{1}{2}\infty}, D_{\frac{1}{2}\infty}\}$ and $d \in \mathbf{Z}_{\geqslant 1}$, let us introduce the d-th suspension $f_W^{(d)}$ of f_W (where $f_W^{(1)} = f_W$) as the entire functions in $d+1$ variables x, y and $\underline{z} = (z_2, \cdots, z_d)$ defined by

$$f_W^{(d)}(x, y, \underline{z}) := f_W(x, y) - z_2^2 - \cdots - z_d^2. \qquad (3.3.9)$$

Then, replacing the function f_W by $f_W^{(d)}$ and the domain $\mathbf{X}_W = \mathbf{C}^2$ by $\mathbf{X}_W^{(d)} = \mathbf{C}^2 \times \mathbf{C}^{d-1}$, we obtain a holomorphic map $(2.3.5)^{(d)}$ whose *fibers*, denoted by $X_{W,t}^{(d)}$ $(t \in \mathbf{C})$, are Stein varieties of complex dimension d.

Replacing, further, the real form $\mathbf{R}^2$ of $\mathbf{X}_W$ by the real form $\mathbf{R}^2 \times \mathbf{R}^{d-1}$ of $\mathbf{X}_W^{(d)}$, THEOREM **1., 2., 3.** in §1.3 hold completely parallely for $f_W^{(d)}$, where the set of critical points of $f_W^{(d)}$ is bijective to that of f_W by the natural embedding $\mathbf{X}_W \subset \mathbf{X}_W^{(d)}$ so that we identify them. Then the signature of Hessians of $f_W^{(d)}$ at points of $C_{W,0}$ is $(1, d)$ and that at points of $C_{W,1}$ is $(0, d+1)$. The suspended fibration shall be referred by $(2.3.10)^{(d)}$. The proof is reduced to the original case $d=1$.

Applying $d-1$-times suspension S on a homology class γ in $\mathrm{H}_1(X_{W,t}, \mathbf{Z})$, we obtain an element $S^{d-1}\gamma$ of the middle homology group $\mathrm{H}_d(X_{W,t}^{(d)}, \mathbf{Z})$ of the d-dimensional fiber $X_{W,t}^{(d)}$. In particular, the suspension $S^{d-1}\gamma_{W,c}$ of a vanishing cycle $\gamma_{W,c}$ of f_W at a critical point $c \in C_W$ is a vanishing cycle of $f_W^{(d)}$ at the same critical point, which, for simplicity, we shall denote again by $\gamma_{W,c}$. Then replacing $\mathrm{H}_1(X_{W,t}, \mathbf{Z})$ by the middle homology group $\mathrm{H}_d(X_{W,t}^{(d)}, \mathbf{Z})$, THEOREM **4.** in §2.1 holds completely parallely, where we keep notations (3.1.2) and (3.1.3).

The intersection form $I_W^{(d)}$ on the middle $(= d)$-dimensional homology group is well known to be symmetric or skew-symmetric according as cycles are even or odd dimensional (i.e. according as d is even or odd). It is also well known that $I_W^{(d)}(\gamma_{W,c}, \gamma_{W,c}) = (-1)^{\frac{d}{2}} 2$ for even d-dimensional vanishing cycles. Therefore, the formula (3.1.5) of the intersection form in THEOREM **5.** need to be slightly modified as in the following theorem, where we keep the notation J_W and $^t J_W$ together with the formulae (3.1.6) and (3.1.7).

THEOREM **5$^{(d)}$.** — *Let $I_W^{(d)} : \mathrm{H}_d(X_{W,t}^{(d)}, \mathbf{Z}) \times \mathrm{H}_d(X_{W,t}^{(d)}, \mathbf{Z}) \to \mathbf{Z}$ be the intersection form on middle-homology groups of the fibers of the fibration*

$(2.3.10)^{(d)}$. *Then we have the following 4-periodic expression.*

$$I_W^{(d)} = (-1)^{[\frac{d}{2}]} J_W - (-1)^{[\frac{d-1}{2}]} \, {}^t J_W. \qquad (3.3.10)$$

The proof of Theorem is standard, and is omitted. Actually, the form $I_W^{(d)}$ is symmetric for d even and is skew symmetric for d odd.

Remark. — We may regard that the form J_W is an infinite rank analogue of a *Seifert matrix* with respect to a "suitable compactification" of the threefold $f_W^{-1}(S^1)$, where S^1 is a circle in the base space $\mathbf{C}$ of $(2.3.5)$ which encloses the two points 0 and 1. However, we do not pursue any further this analogy (see §1.3 Remark and the next subsection §2.4).

3.4. Monodromy Transformations and Coxeter elements

The fundamental group $\pi_1(\mathbf{C} \setminus \{0,1\}, t_0)$ with $t_0 \in (0,1)$ of the base space of the fibration $(2.3.10)^{(d)}$ has two generators g_0 and g_1 which are presented by circular paths in $\mathbf{C} \setminus \{0,1\}$ starting at t_0 and turning once around the point 0 and 1 counterclockwise, respectively. Let $\sigma_{W,0}^{(d)}$ (resp. $\sigma_{W,1}^{(d)}$) be the monodromy action of g_0 (resp. g_1) on the middle homology group $(3.1.2)^{(d)}$ of the fiber of the family $(2.3.10)^{(d)}$, which preserves the intersection form $(3.3.10)$. Though the singular fibers $X_{W,0}^{(d)}$ and $X_{W,1}^{(d)}$ have infinitely many critical points, we can apply Picard-Lefschetz formula. That is, for $u \in H_W := H_{W,0} \oplus H_{W,1}$

$$
\begin{aligned}
\sigma_{W,0}^{(d)}(u) &= u + (-1)^{[\frac{d-1}{2}]} \sum_{c \in C_{W,0}} I_W^{(d)}(u, \gamma_{W,c}) \gamma_{W,c} \\
&= u + \sum_{c \in C_{W,0}} ((-1)^{d-1} J_w(u, \gamma_{w,c}) - J_W(\gamma_{W,c}, u)) \gamma_{W,c} \\
&= \begin{cases} (-1)^{d-1} u & \text{if } u \in H_{W,0} \\ u - \sum_{c \in C_{W,0}} J_W(\gamma_{W,c}, u) \gamma_{w,c} & \text{if } u \in H_{W,1} \end{cases}
\end{aligned}
$$
$$(3.4.11)$$

$$
\begin{aligned}
\sigma_{w,1}^{(d)}(u) &= u + (-1)^{[\frac{d-1}{2}]} \sum_{c \in C_{W,1}} I_W^{(d)}(u, \gamma_{W,c}) \gamma_{W,c} \\
&= u + \sum_{c \in C_{W,1}} ((-1)^{d-1} J_W(u, \gamma_{W,c}) - J_W(\gamma_{W,c}, u)) \gamma_{W,c}, \\
&= \begin{cases} u + (-1)^{d-1} \sum_{c \in C_{W,1}} J_W(u, \gamma_{W,c}) \gamma_{W,c} & \text{if } u \in H_{W,0} \\ (-1)^{d-1} u & \text{if } u \in H_{W,1}. \end{cases}
\end{aligned}
$$
$$(3.4.12)$$

Note that $\sigma_{W,0}^{(d)} = \sigma_{W,0}^{(d+2)}$ and $\sigma_{W,1}^{(d)} = \sigma_{W,1}^{(d+2)}$ for $d \in \mathbf{Z}_{\geqslant 1}$.

Note. Let d be even. Then the reflections $\sigma_{W,0}^{(d)}$, $\sigma_{W,1}^{(d)}$ are involutive:

$$(\sigma_{W,0}^{(d)})^2 = (\sigma_{W,1}^{(d)})^2 = \mathrm{id}_{H_W}. \tag{3.4.13}$$

In the next section, we shall see that the eigenvalues in a suitable sense of the product $\sigma_{W,0}^{(d)} \circ \sigma_{W,1}^{(d)}$ are "dense" in the unit circle S^1 in $\mathbf{C}^\times$, and hence $\sigma_{W,0}^{(d)} \circ \sigma_{W,1}^{(d)}$ is of infinite order. As a consequence, there are no more relations among $\sigma_{W,0}^{(d)}$ and $\sigma_{W,1}^{(d)}$, and the monodromy group is isomorphic to $\mathbf{Z}/2\mathbf{Z} * \mathbf{Z}/2\mathbf{Z}$.

DEFINITION. — *In analogy with the classical simple singularities, let us call the product of the two monodromy transformations $\sigma_{W,0}^{(d)}$ and $\sigma_{W,1}^{(d)}$ a Coxeter element. Two Coxeter elements depending on the order of the product are conjugate to each other. We fix one order as follows and call the product the Coxeter element.*

$Cox_W^{(d)}(u) \;:=\; \sigma_{W,0}^{(d)} \circ \sigma_{W,1}^{(d)} \, (u)$

$$= \begin{cases} (-1)^{d-1}\big(u + \sum_{c \in C_{W,1}} J_W(u, \gamma_{W,c})\gamma_{W,c} \\ \quad - \sum_{c \in C_{W,1}} \sum_{d \in C_{W,0}} J_W(u, \gamma_{W,c}) J_W(\gamma_{W,d}, \gamma_{W,c})\gamma_{W,d}\big) & \text{if } u \in H_{W,0} \\ (-1)^{d-1}\big(u - \sum_{c \in C_{W,0}} J_W(\gamma_{W,c}, u)\gamma_{W,c}\big) & \text{if } u \in H_{W,1}. \end{cases}$$
$$\tag{3.4.14}$$

Observation. *The Coxeter element is, up to the sign factor $(-1)^{d-1}$, independent of the suspensions for $d \in \mathbf{Z}_{\geqslant 1}$ (3.3.9).*

Remark. — It is well known that a classical Coxeter element for a root system of finite type W is semisimple of finite order, and $\frac{1}{2\pi\sqrt{-1}}$ log of its eigenvalues, referred as *spectra* and given by $\frac{m_i}{h}$ $(i = 1,\cdots,n)$, play an important role in Lie theory ([2]). They appear also as exponents of the primitive forms associated with simple polynomials of type W [6] and the fact that they lie in the interval $(0,1)$ for the case $d = 2$ characterizes that they are primitive forms associated with simple polynomials [5].

The Coxeter elements of types $A_{\frac{1}{2}\infty}$ and $D_{\frac{1}{2}\infty}$ are no longer of finite order. However, in the next section, we show that they are diagonalizable in suitable sense and the *spectra* for them are introduced. Then, the sign factor $(-1)^{d-1}$ in (3.4.14) of the Coxeter element $Cox_W^{(d)}$ is lifted to the shift of the spectra by $\frac{d-1}{2}$ so that the spectra of $Cox_W^{(d)}$ is contained in the interval $(\frac{d}{2} - 1, \frac{d}{2})$. The spectra should play a key role for primitive forms of type $A_{\frac{1}{2}\infty}$ and $D_{\frac{1}{2}\infty}$ in a forthcoming paper, where the shift of the spectra corresponds to the $\frac{d-1}{2}$-shift of the primitive forms in the semi-infinite Hodge filtration.

4. Spectra of Coxeter elements of types $A_{\frac{1}{2}\infty}$ and $D_{\frac{1}{2}\infty}$

We study spectra of the Coxeter element $Cox_W^{(d)}$ for $W \in \{A_{\frac{1}{2}\infty}, D_{\frac{1}{2}\infty}\}$. For the purpose, we extend the domain of the Coxeter element to the completion of $H_{W,\mathbf{C}} := H_W \otimes_{\mathbf{Z}} \mathbf{C}$ with respect to the l^2-norm with the ortho-normal basis $\{\gamma_{W,c}\}_{c \in \mathbf{C}_W}$. The Coxeter element action on this space is diagonalizable (in a suitable sense), and its "eigenvalues" take values in the unit circle $S^1 \subset \mathbf{C}^\times$. We want to determine $\frac{1}{2\pi\sqrt{-1}}$ log of the "eigenvalues", called the *spectra* of the Coxeter element. Actually, it is calculated by a help of the intersection form $\dot{I}_W$, since it, as a positive symmetric operator, has only positive real eigenvalues. It turns out that we get a continuous spectra on the interval $(\frac{d}{2} - 1, \frac{d}{2})$.

4.1. Hilbert space $\overline{H}_{W,\mathbf{C}}$

Consider $\mathbf{C}$-vector spaces obtained by the complexification of the $\mathbf{Z}$-lattices $H_{W,0}$, $H_{W,1}$ and H_W (recall (3.1.2),(3.1.3) and (3.1.4)):

$$H_{W,0,\mathbf{C}} := H_{W,0} \otimes_{\mathbf{Z}} \mathbf{C}, H_{W,1,\mathbf{C}} := H_{W,1} \otimes_{\mathbf{Z}} \mathbf{C} \quad \text{and} \quad H_{W,\mathbf{C}} := H_W \otimes_{\mathbf{Z}} \mathbf{C}. \quad (4.1.1)$$

We equip them with a hermitian inner product $\langle \cdot, \cdot \rangle$ defined by

$$\langle \sum_{c \in C_W} a_c \gamma_{W,c}, \sum_{c \in C_W} b_c \gamma_{W,c} \rangle := \sum_{c \in C_W} a_c \bar{b}_c, \quad (4.1.2)$$

where a_c, b_c $(c \in C_W)$ are complex numbers. Then, the l^2-completions of the spaces with respect to this inner product are separable Hilbert spaces, denoted by $\overline{H}_{W,0,\mathbf{C}}$, $\overline{H}_{W,1,\mathbf{C}}$ and $\overline{H}_{W,\mathbf{C}}$, respectively. We have the orthogonal direct sum decomposition:

$$\overline{H}_{W,\mathbf{C}} = \overline{H}_{W,0,\mathbf{C}} \oplus \overline{H}_{W,1,\mathbf{C}}. \quad (4.1.3)$$

Let us denote by π_0 and π_1 the orthogonal projections of the space $\overline{H}_{W,\mathbf{C}}$ to the subspaces $\overline{H}_{W,0,\mathbf{C}}$ and $\overline{H}_{W,1,\mathbf{C}}$, respectively, so that the sum

$$id_{\overline{H}_{W,\mathbf{C}}} = \pi_0 + \pi_1$$

is the identity map on $\overline{H}_{W,\mathbf{C}}$.

Remark that the lattice H_W is self-dual: $\text{Hom}_{\mathbf{Z}}(H_W, \mathbf{Z}) \cap \overline{H}_{W,\mathbf{C}} = H_W$.

Convention. In the sequel of the present paper, we freely identify a continuous bilinear form A on $\overline{H}_{W,\mathbf{C}}$ (resp. $H_{W,\mathbf{C}}$) and a continuous endomorphism $\dot{A}$ on $\overline{H}_{W,\mathbf{C}}$ (resp. $H_{W,\mathbf{C}}$) by the following relations:

$$A(\xi, \eta) = \langle \dot{A}(\xi), \eta \rangle \quad \text{and} \quad \sum_{c \in C_W} A(u, \gamma_{W,c}) \gamma_{W,c} = \dot{A}(u).$$

Transposes ${}^t A$ of A and ${}^t(\dot{A})$ of $\dot{A}$ are defined by the relations ${}^t A(\xi, \eta) = A(\eta, \xi)$ and $\langle \dot{A}(u), v \rangle = \langle u, {}^t(\dot{A})(v) \rangle$, respectively. Then, ${}^t(\dot{A}) = (\dot{{}^t A})$.

4.2. Extendability of $I_W^{(d)}$ and $Cox_W^{(d)}$ on $\overline{H}_W$

In order to calculate the eigenvalues of the intersection forms $I_W^{(d)}$ and the Coxeter elements $Cox_W^{(d)}$, we use the identification mentioned at the end of §3.1. Before we do this, we need to check that they are continuously extendable to the completion $\overline{H}_{W,\mathbf{C}}$. This is achieved by using the extendabilities of the endomorphisms $\dot{J}_W$, ${}^t\dot{J}_W$ associated with the bilinear forms (3.1.6) and (3.1.7). Put

$$
\begin{aligned}
\dot{J}_W(u) \;&:=\; \sum_{c \in C_W} J(u, \gamma_{W,c}) \gamma_{W,c} \\
&=\; \begin{cases} u + \sum_{c \in C_{W,1}} J_W(u, \gamma_{W,c}) \gamma_{W,c} & \text{if } u \in H_{W,0} \quad\quad (4.2.4) \\ u & \text{if } u \in H_{W,1} \end{cases}
\end{aligned}
$$

$$
\begin{aligned}
{}^t\dot{J}_W(u) \;&:=\; \sum_{c \in C_W} {}^t J(u, \gamma_{W,c}) \gamma_{W,c} \\
&=\; \begin{cases} u & \text{if } u \in H_{W,0} \quad\quad (4.2.5) \\ u + \sum_{c \in C_{W,0}} J_W(\gamma_{W,c}, u) \gamma_{W,c} & \text{if } u \in H_{W,1} \end{cases}
\end{aligned}
$$

which are endomorphisms on $H_{W,\mathbf{C}}$, since the quiver Γ_W in §2.2 is locally finite, i.e. any vertex is connected with only a finite number of other vertices. The inverse action of $\dot{J}_W$ (resp. ${}^t\dot{J}_W$) on $H_{W,\mathbf{C}}$ can be obtained by just replacing "+" by "−" in RHS of (4.2.4) (resp. (4.2.5)).

Assertion 1. *The endomorphisms $\dot{J}_W$, ${}^t\dot{J}_W$ and their inverses $\dot{J}_W^{-1}$, ${}^t\dot{J}_W^{-1}$ acting on $H_{W,\mathbf{C}}$ are extendable to bounded endomorphisms on $\overline{H}_{W,\mathbf{C}}$. The extensions are transpose to each other.*

Proof. — We show only the extendability of the domain of endomorphisms $\dot{J}_W$, ${}^t\dot{J}_W$ and their inverses $\dot{J}_W^{-1}$, ${}^t\dot{J}_W^{-1}$ from $H_{W,\mathbf{C}}$ to $\overline{H}_{W,\mathbf{C}}$, where the extensions are denoted by the same notation. Then the relations ${}^t(\dot{J}_W) = {}^t\dot{J}_W$, $\dot{J}_W \dot{J}_W^{-1} = \mathrm{id}_{H_W}$, ..., etc. are automatically preserved for the extensions.

The quivers $\Gamma_{A_{\frac{1}{2}\infty}}$ and $\Gamma_{D_{\frac{1}{2}\infty}}$ show that any critical point $c \in C_{W,0}$ is adjacent to at most two bdd components. In view of (4.2.4), this implies the inequality $\| \dot{J}_W(u) - u \| \leqslant 2 \| u \|$. Hence $\dot{J}_W$ is extendable to a bounded endomorphims on $\overline{H}_{W,\mathbf{C}}$, denoted by the same $\dot{J}_W$.

We observe also that, to any bdd component, at most 3 critical points in $C_{W,0}$ are adjacent (actually, 3 occurs only one bdd component for the

critical point $c_{D,1}^{(1)}$ of type $D_{\frac{1}{2}\infty}$). In view of (4.2.5), we get an inequality $\| \ ^t\dot{J}_W(u) - u \| \leqslant 3 \| u \|$, implying again the extendability of $^t\dot{J}_W$ to a bounded endomorphism on $\overline{H}_{W,\mathbf{C}}$, denoted by the same $^t\dot{J}_W$.

Similar arguments shows the extendability of the inverses. □

An immediate consequence of **Assertion 1** is that *the endomorphism*

$$(3.3.22)^{\bullet} \qquad\qquad \dot{I}_W^{(d)} := (-1)^{[\frac{d}{2}]} \dot{J}_W - (-1)^{[\frac{d-1}{2}]} \ ^t\dot{J}_W$$

defined on $H_{W,\mathbf{C}}$ *is extendable to a bounded endomorphism on* $\overline{H}_{W,\mathbf{C}}$.

Another important consequence of **Assertion 1** is the following.

COROLLARY. — *The Coxeter element* $Cox_W^{(d)}$ $(d \in \mathbf{Z}_{\geqslant 1})$ *defined on* $H_{W,\mathbf{C}}$ *is extendable to an invertible bounded automorphism on* $\overline{H}_{W,\mathbf{C}}$.

Proof. — Let us, first, show a formula:

$$Cox_W^{(d)} = (-1)^{d-1} \ (^t\dot{J}_W)^{-1} \dot{J}_W, \qquad\qquad (4.2.6)$$

on H_W by a direct calculation using formulae (3.4.14), (4.2.4) and

$$(4.2.30)^{-1} \qquad (^t\dot{J}_W)^{-1}(u) = \begin{cases} u & \text{if } u \in H_{W,0} \\ u - \sum_{c \in C_{W,0}} J_W(\gamma_{W,c}, u)\gamma_{W,c} & \text{if } u \in H_{W,1}. \end{cases}$$

Then, RHS of (4.2.6) is extendable to a bounded operator on $\overline{H}_{W,\mathbf{C}}$.

Invertibility of $Cox_W^{(d)}$ follows from that of $\dot{J}_W$ and $^t\dot{J}_W$. □

Remark. — Let $\check{H}_{W,\mathbf{C}} := \mathrm{Hom}_{\mathbf{C}}(H_{W,\mathbf{C}}, \mathbf{C})$ be the (formal) dual vector space of $H_{W,\mathbf{C}}$. The contragredient actions on $\check{H}_{W,\mathbf{C}}$ of the endomorphisms $\dot{J}_W, \ ^t\dot{J}_W, \ \dot{I}_W^{(d)}, \ ^t\dot{I}_W^{(d)}, \ Cox_W^{(d)}$ and $^tCox_W^{(d)}$ on $H_{W,\mathbf{C}}$ shall be denoted, as usual, by the super script " $^t(\text{-})$ " such that " $^{tt}(\text{-})=(\text{-})$ ".

On the other hand, by regarding $\{\gamma_{W,c}\}_{c \in C_W}$ as the self-dual basis, $\check{H}_{W,\mathbf{C}}$ is identified with the direct product $\prod_{c \in C_W} \mathbf{C}\gamma_{W,c}$ so that we have natural inclusions of $\mathbf{C}$-vector spaces:

$$H_{W,\mathbf{C}} \subset \overline{H}_{W,\mathbf{C}} \subset \check{H}_{W,\mathbf{C}}.$$

Then it is easy to verify that the extensions of $\dot{J}_W, \ ^t\dot{J}_W, \ \dot{I}_W^{(d)}, \ ^t\dot{I}_W^{(d)}, \ Cox_W^{(d)}$ and $^tCox_w^{(d)}$ to the spaces $\overline{H}_{w,\mathbf{C}}$ and $\check{H}_{W,\mathbf{C}}$ are naturally compatible with respect to the above inclusions. The relationships between these extensions and the transpositions are given as follows:

$$^t\dot{I}_W^{(d)} = (-1)^d \dot{I}_W^{(d)} \quad \text{and} \quad (^tCox_W^{(d)})^{-1} = \dot{J}_W \ Cox_W^{(d)} \ \dot{J}_W^{-1}.$$

However, the bilinear form $I_{W,\mathbf{C}}$ itself is no longer extandable to $\check{H}_{W,\mathbf{C}}$ and the endomorphism $\dot{I}_W$ on $\check{H}_{W,\mathbf{C}}$ has non-trivial kernel.

4.3. Spectral decomposition of $I_W^{(d)}$ for even d

Using the fact (3.3.10), the bilinear form $I_W^{(d)}$ is symmetric for even d. Let us consider the operator for the cases $d \in \mathbf{Z}_{\geqslant 1}$ with $d \equiv 0 \bmod 4$,[9]

$$\dot{I}_W := \dot{I}_W^{(d)} = \dot{J}_W + {}^t\dot{J}_W. \tag{4.3.7}$$

We, first, determine the point spectrum of the symmetric operator $\dot{I}_W$ on $\overline{H}_{W,\mathbf{C}}$. Let us consider the following two eigenspaces for $\lambda \in \mathbf{C}$:

$$\check{H}_{W,\lambda} := \{\xi \in \check{H}_{W,\mathbf{C}} \mid \dot{I}_W(\xi) = \lambda\xi\} \text{ and } \overline{H}_{W,\lambda} := \check{H}_{W,\lambda} \cap \overline{H}_{W,\mathbf{C}}. \tag{4.3.8}$$

Assertion 2. *For each type* $W \in \{A_{\frac{1}{2}\infty}, D_{\frac{1}{2}\infty}\}$ *and all* $\lambda \in \mathbf{C}$, *we have*

$$\dim_{\mathbf{C}} \check{H}_{W,\lambda} = 1 \quad and \quad \dim_{\mathbf{C}} \overline{H}_{W,\lambda} = 0, \tag{4.3.9}$$

except for the case $W = D_{\frac{1}{2}\infty}$ *and* $\lambda = 2$, *where we have*

$$\dim_{\mathbf{C}} \check{H}_{D_{\frac{1}{2}\infty},2} = 2 \quad and \quad \dim_{\mathbf{C}} \overline{H}_{D_{\frac{1}{2}\infty},2} = 1, \tag{4.3.10}$$

and $\overline{H}_{D_{\frac{1}{2}\infty},2}$ *is spanned by the vector* $\gamma_{D,0}^+ - \gamma_{D,0}^-$.

Proof. — This is shown by solving the equation $\dot{I}_W(\xi) = \lambda\xi$ for the coefficients of $\xi = \sum_{c \in C_W} a_c \gamma_{W,c} \in \check{H}_{W,\mathbf{C}}$ formally and inductively according to the following labeling and ordering of coefficients:

$$\Gamma_{A_{\frac{1}{2}\infty}} : \quad a_0 \longrightarrow a_1 \longleftarrow a_2 \longrightarrow a_3 \longleftarrow a_4 \longrightarrow a_5 \longleftarrow \quad \cdots$$

$$\Gamma_{D_{\frac{1}{2}\infty}} : \quad \begin{matrix} b_0^+ \\ \\ b_0^- \end{matrix} \searrow\hspace{-0.3em}\nearrow \; b_1 \longrightarrow b_2 \longleftarrow b_3 \longrightarrow b_4 \longleftarrow b_5 \longrightarrow \quad \cdots$$

Details of the calculation are omitted. Results are summarized as:

$A_{\frac{1}{2}\infty}$: The space $\check{H}_{A_{\frac{1}{2}\infty},\lambda}$ for any $\lambda \in \mathbf{C}$ is spanned by

$$\check{\xi}_{A_{\frac{1}{2}\infty},\lambda} : \quad a_n = \frac{\exp((n+1)\sqrt{-1}\pi\theta) - \exp(-(n+1)\sqrt{-1}\pi\theta)}{\exp(\sqrt{-1}\pi\theta) - \exp(-\sqrt{-1}\pi\theta)} \quad (n \geqslant 0)$$

where θ is any complex number satisfying $\lambda = 4\sin^2(\frac{\pi}{2}\theta)$. In case $\lambda = 0$ or 4 (i.e. when $\theta \in \mathbf{Z}$), we interpret this formula as $a_n = \pm(n+1)$.

$D_{\frac{1}{2}\infty}$: For all $\lambda \in \mathbf{C}$, let us introduce a vector

$$\check{\xi}_{D_{\frac{1}{2}\infty},\lambda} : \quad b_0^+ = 1, \; b_0^- = 1, \; b_n = \exp(n\sqrt{-1}\theta) + \exp(-n\sqrt{-1}\theta) \quad (n \geqslant 1)$$

where θ is any complex number satisfying the equation $\lambda = 4\sin^2(\frac{\pi}{2}\theta)$. Then, the space $\check{H}_{D_{\frac{1}{2}\infty},\lambda}$ for any $\lambda \neq 2$ is spanned by $\check{\xi}_{D_{\frac{1}{2}\infty},\lambda}$. The space $\check{H}_{D_{\frac{1}{2}\infty},2}$ is spanned by $\check{\xi}_{D_{\frac{1}{2}\infty},2}$ and

$$\gamma_{D,0}^+ - \gamma_{D,0}^- : \quad b_0^+ = 1, \; b_0^- = -1, \; b_n = 0 \; (n \geqslant 1).$$

---

[9] We choose $d \equiv 0 \bmod 4$ so that the form I_W is positive and symmetric, defining a "root lattice structure of infinite rank" on H_W (cf. Proof of Assertion 3.).

The norm $\langle \check{\xi}_{W,\lambda}, \check{\xi}_{W,\lambda} \rangle$ (4.1.2) for any $W \in \{A_{\frac{1}{2}\infty}, D_{\frac{1}{2}\infty}\}$ and any $\lambda \in \mathbf{C}$ is unbounded, whereas $\gamma_{D,0}^+ - \gamma_{D,0}^-$ has the bounded norm$=2$. $\qquad\square$

COROLLARY. — *The point spectrum of $\dot{I}_{A_{\frac{1}{2}\infty}}$ on $\overline{H}_{W,\mathbf{C}}$ is empty, and that of $\dot{I}_{D_{\frac{1}{2}\infty}}$ consists of a single eigenvalue $\lambda = 2$ with multiplicity 1. In particular, the operator $\dot{I}_W$ for any $W \in \{A_{\frac{1}{2}\infty}, D_{\frac{1}{2}\infty}\}$ is non-degenerate on $\overline{H}_{W,\mathbf{C}}$ in the sense that $\ker(\dot{I}_W) \cap \overline{H}_{W,\mathbf{C}} = \{0\}$.*

Remark. — By introducing the double cover of the λ-plane by $\mu := \exp(\pi\sqrt{-1}\theta) \in \mathbf{C} \setminus \{0\}$ with the relation $2 - \lambda = \mu + \mu^{-1}$, the base $\check{\xi}_{W,\lambda}$ in the proof of **Assertion 2** can be expressed in terms of Laurent polynomials in μ. The reader may be puzzled by the use of θ instead of μ in the above proof. We used the parameter θ since it shall parametrize the spectra of Coxeter elements in the next paragraph. We remark also that $\lambda \in [0, 4] \Leftrightarrow \theta \in \mathbf{R}$.

For a symmetric operator $\dot{I}_W$ on $\overline{H}_{W,\mathbf{C}}$, the *greatest lower bound* and the *least upper bound* are defined as the maximal real number m and the minimal real number M satisfying the following inequalities, respectively (see [3, §104]).

$$m\langle \xi, \xi \rangle \;\leqslant\; \langle \dot{I}_W(\xi), \xi \rangle = I_W(\xi, \xi) \;\leqslant\; M\langle \xi, \xi \rangle \quad \forall \xi \in \overline{H}_{W,\mathbf{C}} \qquad (4.3.11)$$

Assertion 3. *The greatest lower bound m and the least upper bound M of $\dot{I}_W$ for both $W \in \{A_{\frac{1}{2}\infty}, D_{\frac{1}{2}\infty}\}$ are given by $m = 0$ and $M = 4$.*

Proof. — For the definition of m and M, it is sufficient to run ξ only in H_W in the defining relation (4.3.11), since $H_{W,\mathbf{C}}$ is dense in $\overline{H}_{W,\mathbf{C}}$. Any $\xi \in H_W$ is contained in a sublattice L of H_W generated by the vertices of a finite (connected) subdiagram Γ of Γ_W (recall §2.2). Actually, Γ is a diagram of type either A_l or D_l for some $l \in \mathbf{Z}_{>0}$ and $I_W|_L$ gives a root lattice structure of that type on L. That is, $\{I_W(\gamma_{W,c}, \gamma_{w,d})\}_{c,d \in \Gamma \subset C_W}$ is the Cartan matrix of type Γ. In particular, the eigenvalues of $\dot{I}_W|_L$ are given by $4\sin^2\left(\frac{\pi}{2}\frac{m_i}{h}\right)$ $(i = 1, \cdots, l = \mathrm{rank}(L))$, where m_i are the exponents and h is the Coxeter number of the root system of type Γ ([2, ch.V,§6,n°2]). Since the smallest and the largest exponent of the (finite) root system are 1 and $h - 1$, respectively, the minimal and the maximal of the eigenvalues are $4\sin^2\left(\frac{\pi}{2}\frac{1}{h}\right)$ and $4\cos^2\left(\frac{\pi}{2}\frac{1}{h}\right)$, respectively. Since $h \to \infty$ according as Γ "exhaust" Γ_W, we obtain

$$m \;=\; \inf_{\Gamma \subset \Gamma_W} 4\sin^2\left(\frac{\pi}{2}\frac{1}{h}\right) = \lim_{h \to \infty} 4\sin^2\left(\frac{\pi}{2}\frac{1}{h}\right) = 0.$$

$$M \;=\; \sup_{\Gamma \subset \Gamma_W} 4\cos^2\left(\frac{\pi}{2}\frac{1}{h}\right) = \lim_{h \to \infty} 4\cos^2\left(\frac{\pi}{2}\frac{1}{h}\right) = 4. \qquad\square$$

We apply the spectral decomposition theory of bounded symmetric operators (see [3, §107 Theorem]) to the operator $\dot{I}_W$. Let us reformulate the result in [ibid] by adjusting the notation to our setting.

THEOREM 6. — *For each type* $W \in \{A_{\frac{1}{2}\infty}, D_{\frac{1}{2}\infty}\}$, *there exists a unique spectral family* $\{E_{W,\lambda}\}_{\lambda \in \mathbf{R}}$ *(i.e. a family of projection operators*[10] *on* $\overline{H}_{W,\mathbf{C}}$*) satisfying the following* a), b), c):

a) *For* $\lambda \leqslant \mu$, *one has* $E_{W,\lambda} \leqslant E_{W,\mu}$ $(\underset{def}{\Leftrightarrow} E_{W,\lambda}E_{W,\mu} = E_{W,\lambda})$.

b) *The family is strongly continuous with respect to* λ, *i.e.*
$$E_{W,\lambda+0}(:= \lim_{\mu \downarrow 0} E_{w,\lambda+\mu}) = E_{W,\lambda-0}(:= \lim_{\mu \uparrow 0} E_{w,\lambda-\mu}),$$
for all λ *except at* $\lambda = 2$ *for type* $W = D_{\frac{1}{2}\infty}$. *We have*
$$E_{D_{\frac{1}{2}\infty},2+0} - E_{D_{\frac{1}{2}\infty},2-0} = \eta_D \tag{4.3.12}$$
where η_D *is the projection:* $\overline{H}_{D_{\frac{1}{2}\infty},\mathbf{C}} \to \overline{H}_{D_{\frac{1}{2}\infty},2} = \mathbf{C}(\gamma_{D,0}^+ - \gamma_{D,0}^-)$.

c) *One has* $E_{W,\lambda} = 0$ *for* $\lambda \leqslant 0$ *and* $E_{w,\lambda} = \mathrm{Id}_{\overline{H}_{w,\mathbf{C}}}$ *for* $\lambda \geqslant 4$, *so that the following* (4.3.13) *holds.*
$$(\dot{I}_W)^r = \int_0^4 \lambda^r dE_{W,\lambda} \qquad (for \quad r = 0, 1, 2, \cdots), \tag{4.3.13}$$

where the integral is in the sense of Lebesgue-Stieltjes. [11]

4.4. Spectra of Coxeter elements

Recall that $\lambda \in [0,4]$ in §4.3 Theorem 6 is the parameter for the spectra of the intersection form $I_W := I_W^{(d)}$ for $d \equiv 0 \bmod 4$. What is wonderful, is the fact that this parameter gives a clue to parametrize the spectra θ of the Coxeter elements $Cox_W^{(d)}$ for all $d \in \mathbf{Z}_{\geqslant 1}$. In order to achieve this, we re-parametrize λ by a new parameter θ and by the relation (which we once used in a proof of **Assertion 2**.)
$$\lambda = 4 \sin^2\left(\theta \frac{\pi}{2}\right) \quad \text{for } 0 \leqslant \theta \leqslant 1. \tag{4.4.14}$$

---

[10] Here, we mean by a *projection operator* an orthogonal projection map from $\overline{H}_{W,\mathbf{C}}$ to its closed subspace such that the real form $\overline{H}_{W,\mathbf{R}}$ is mapped into itself. The fact that $E_{W,\lambda}$ is real, is not explicitly stated in the literature [3], but follows trivially from its construction and from the fact that $\dot{I}_W$ is real.

[11] Furthermore, [3, §107 Theorem], for any complex valued continuous function $u(\lambda)$ on the interval $[0,4]$, we have an equality $u(\dot{I}_W) = \int_0^4 u(\lambda)dE_{W,\lambda}$ between bounded operators, where LHS is defined by a (monotone decreasing) polynomial approximation of u and RHS is given by the norm-limit of the Stieltjes type summation. Then, for any $\xi, \eta \in \overline{H}_{W,\mathbf{C}}$, we have $\langle u(\dot{I}_W)\xi, \eta \rangle = \int_0^4 u(\lambda)d\langle E_{W,\lambda}\xi, \eta \rangle$.

Let us introduce a Stieltjes measure on the interval $\theta \in [0,1]$:

$$\xi_{W,\theta} := U_\theta \cdot dE_{W,\lambda} \cdot U_\theta^{-1} \tag{4.4.15}$$

for each $W \in \{A_{\frac{1}{2}\infty}, D_{\frac{1}{2}\infty}\}$, where
(i) $\{E_{W,\lambda}\}_{\lambda \in [0,4]}$ is the spectral family in §4.3 THEOREM **6**,
(ii) U_θ $(0 \leqslant \theta \leqslant 1)$ is a family of unitary operators on $\overline{H}_{W,\mathbf{C}}$ given by

$$U_\theta := \exp\left(-\frac{\pi}{2}\sqrt{-1}\theta\right)\pi_0 - \exp\left(\frac{\pi}{2}\sqrt{-1}\theta\right)\pi_1, \tag{4.4.16}$$

where π_i $(i=0,1)$ is the orthogonal projection to the subspace $\overline{H}_{W,\mathbf{C},i}$.

THEOREM **7**. — *We have the following* a) *and* b).

a) *The $\xi_{W,\theta}$ is a Stieltjes measure on $\theta \in [0,1]$, which is strongly continuous except at $\theta = \frac{1}{2}$ for the type $D_{\frac{1}{2}\infty}$. We have*

$$\xi_{D_{\frac{1}{2}\infty},\frac{1}{2}+0} - \xi_{D_{\frac{1}{2}\infty},\frac{1}{2}-0} = \eta_D, \tag{4.4.17}$$

where we recall (4.3.12) for η_D.

b) *The following two formulae hold:*

$$Cox_W^{(d)} \cdot \xi_{W,\theta} = \exp\left(2\pi\sqrt{-1}\left(\theta + \frac{d-2}{2}\right)\right)\xi_{W,\theta}, \tag{4.4.18}$$

and

$$\int_{\theta=0}^{\theta=1} \xi_{W,\theta} = \frac{1}{2}\,\dot{I}_W. \tag{4.4.19}$$

Proof. — a) The first half of the statement is obvious. Since $\overline{H}_{D_{\frac{1}{2}\infty},2}$ is a subspace of $\overline{H}_{D_{\frac{1}{2}\infty},0}$, we have $\pi_0\eta_D = \eta_D\pi_0 = \eta_D$ and $\pi_1\eta_D = \eta_D\pi_1 = 0$. Then, LHS of (4.4.17) is given by $U_{\frac{1}{2}}dE_{D_{\frac{1}{2}\infty},2+0}U_{\frac{1}{2}}^{-1} - U_{\frac{1}{2}}dE_{D_{\frac{1}{2}\infty},2-0}U_{\frac{1}{2}}^{-1}$
$= U_{\frac{1}{2}}(dE_{D_{\frac{1}{2}\infty},2+0} - dE_{D_{\frac{1}{2}\infty},2-0})U_{\frac{1}{2}}^{-1} = U_{\frac{1}{2}}\eta_D U_{\frac{1}{2}}^{-1} = \eta_D$ (cf. (4.3.12)).
b) 1. Proof of (4.4.18).
Consider the infinitesimal form of the formula (4.3.13) for $r=1$:

$$\dot{I}_W \cdot dE_{W,\lambda} = \lambda dE_{W,\lambda}. \tag{4.4.20}$$

Substitute the decomposition $dE_{W,\lambda} = \pi_0 dE_{W,\lambda} + \pi_1 dE_{W,\lambda}$ in this formula. Then, using (4.3.7), the LHS is equal to

$$\begin{aligned}
\dot{I}_W \cdot dE_{W,\lambda} &= (\dot{J}_W + {}^t\dot{J}_W)(\pi_0 \cdot dE_{W,\lambda} + \pi_1 \cdot dE_{W,\lambda})\\
&= 2\,\pi_0 \cdot dE_{W,\lambda} + 2\,\pi_1 \cdot dE_{W,\lambda}\\
&\quad + (\dot{J}_W - id)(\pi_0 \cdot dE_{W,\lambda}) + (\dot{J}_W - id)(\pi_1 \cdot dE_{W,\lambda})\\
&\quad + ({}^t\dot{J}_w - id)(\pi_0 \cdot dE_{w,\lambda}) + ({}^t\dot{J}_W - id)(\pi_1 \cdot dE_{w,\lambda}).
\end{aligned}$$

On the other hand, recalling (4.2.4) and (4.2.5), we know that

$$(\dot{J}_W - id)(\pi_1 \cdot dE_{W,\lambda}) = 0, \quad (\dot{J}_W - id)(\pi_0 \cdot dE_{W,\lambda}) \in \mathrm{Hom}(\overline{H}_{W,\mathbf{C}}, \overline{H}_{W,\mathbf{C},1}),$$
$$({}^t\dot{J}_w - id)(\pi_0 \cdot dE_{W,\lambda}) = 0, \quad ({}^t\dot{J}_W - id)(\pi_1 \cdot dE_{W,\lambda}) \in \mathrm{Hom}(\overline{H}_{W,\mathbf{C}}, \overline{H}_{W,\mathbf{C},0}).$$

Equating this with $\lambda dE_{W,\lambda} = \lambda\pi_0 \cdot dE_{w,\lambda} + \lambda\pi_1 \cdot dE_{w,\lambda}$ (4.4.19), we obtain

$$({}^t\dot{J}_W - id)(\pi_1 dE_{W,\lambda}) = (\lambda-2)\pi_0 dE_{W,\lambda}, \quad (\dot{J}_W - id)(\pi_0 dE_{W,\lambda}) = (\lambda-2)\pi_1 dE_{W,\lambda}.$$

Rewriting these together in matrix expressions, we obtain

$$\dot{J}_W \begin{pmatrix} \pi_0 \cdot dE_{W,\lambda} \\ \pi_1 \cdot dE_{W,\lambda} \end{pmatrix} = \begin{pmatrix} 1 & \lambda - 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \pi_0 \cdot dE_{W,\lambda} \\ \pi_1 \cdot dE_{W,\lambda} \end{pmatrix}. \tag{4.4.21}$$

$$^t\dot{J}_W \begin{pmatrix} \pi_0 \cdot dE_{W,\lambda} \\ \pi_1 \cdot dE_{W,\lambda} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \lambda - 2 & 1 \end{pmatrix} \begin{pmatrix} \pi_0 \cdot dE_{W,\lambda} \\ \pi_1 \cdot dE_{W,\lambda} \end{pmatrix}. \tag{4.4.22}$$

and, hence, also

$$({}^t\dot{J}_W)^{-1} \begin{pmatrix} \pi_0 \cdot dE_{W,\lambda} \\ \pi_1 \cdot dE_{W,\lambda} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 2 - \lambda & 1 \end{pmatrix} \begin{pmatrix} \pi_0 \cdot dE_{W,\lambda} \\ \pi_1 \cdot dE_{W,\lambda} \end{pmatrix}.$$

Thus, combining these with the expression (4.2.6), we obtain

$$Cox_W^{(d)}\begin{pmatrix} \pi_0 \cdot dE_{W,\lambda} \\ \pi_1 \cdot dE_{W,\lambda} \end{pmatrix} = (-1)^{d-1}\begin{pmatrix} 1 & \lambda - 2 \\ 2 - \lambda & 1 - (\lambda-2)^2 \end{pmatrix}\begin{pmatrix} \pi_0 \cdot dE_{W,\lambda} \\ \pi_1 \cdot dE_{W,\lambda} \end{pmatrix}. \tag{4.4.23}$$

Substitute λ in the RHS matrix by the expression (4.4.14) :

$$(-1)^{d-1}\begin{pmatrix} 1 & \lambda - 2 \\ 2 - \lambda & 1 - (\lambda - 2)^2 \end{pmatrix} = (-1)^{d-1}\begin{pmatrix} 1 & -2\cos(\pi\theta) \\ 2\cos(\pi\theta) & \sin^2(\pi\theta) - 3\cos^2(\pi\theta) \end{pmatrix}.$$

We see that the matrix is semi-simple for any θ. The eigenvalues are

$$\exp\Big(\pm 2\pi\sqrt{-1}\Big(\theta + \frac{d-2}{2}\Big)\Big),$$

and associated row eigenvectors (independent of n) are

$$\Big(\exp\Big(\mp\frac{\pi}{2}\sqrt{-1}\theta\Big), -\exp\Big(\pm\frac{\pi}{2}\sqrt{-1}\theta\Big)\Big).$$

Therefore, by introducing the unitary operators

$$U_{\pm\theta} := \exp\Big(\mp\frac{\pi}{2}\sqrt{-1}\theta\Big)\pi_0 - \exp\Big(\pm\frac{\pi}{2}\sqrt{-1}\theta\Big)\pi_1 \tag{4.4.24}$$

satisfying relations: $^tU_{\pm\theta} = U_{\pm\theta} = \overline{U_{\mp\theta}}$ and $U_{\pm\theta} \cdot U_{\mp\theta} = \mathrm{id}_{\overline{H}_{W,\mathbf{C}}}$, we introduce a Stieltjes measure on $[0,4] := \{\lambda \in \mathbf{R} \mid 0 \leqslant \lambda \leqslant 4\} \simeq [0,1] := \{\theta \in \mathbf{R} \mid 0 \leqslant \theta \leqslant 1\}$:

$$\xi_{W,\theta}^{\pm} := U_{\pm\theta} \cdot dE_{W,\lambda} \cdot U_{\mp\theta}. \tag{4.4.25}$$

Then, from (4.4.23), we obtain

$$Cox_W^{(d)} \cdot \xi_{W,\theta}^{\pm} = \exp\Big(\pm 2\pi\sqrt{-1}\Big(\theta + \frac{d-2}{2}\Big)\Big)\xi_{W,\theta}^{\pm}. \tag{4.4.26}$$

Putting $\xi_{W,\theta} := \xi_{W,\theta}^{+}$, we obtain (4.4.17).

b) 2. Proof of (4.4.19).

Using (4.4.15) and (4.4.16), we decompose $\xi_{W,\theta}$ into 4 pieces:

$$\pi_0 \cdot dE_{W,\theta} \cdot \pi_0 + \pi_1 \cdot dE_{W,\theta} \cdot \pi_1 - \exp(\pi\sqrt{-1}\theta)\pi_1 \cdot dE_{W,\theta} \cdot \pi_0 - \exp(-\pi\sqrt{-1}\theta)\pi_0 \cdot dE_{W,\theta} \cdot \pi_1.$$

The first two terms are integrated easily by

$$\int_{\theta=0}^{\theta=1} \pi_0 \cdot dE_{W,\theta} \cdot \pi_0 = \pi_0 \cdot \left(\int_{\theta=0}^{\theta=1} dE_{W,\theta}\right) \cdot \pi_0 = \pi_0 \cdot \mathrm{id}_{\overline{H}_{W,\mathbf{C}}} \cdot \pi_0 = \pi_0,$$

$$\int_{\theta=0}^{\theta=1} \pi_1 \cdot dE_{W,\theta} \cdot \pi_1 = \pi_1 \cdot \left(\int_{\theta=0}^{\theta=1} dE_{W,\theta}\right) \cdot \pi_1 = \pi_1 \cdot \mathrm{id}_{\overline{H}_{W,\mathbf{C}}} \cdot \pi_1 = \pi_1.$$

The third and fourth terms are integrated by the use of Footnote 8.

First, we introduce bounded nilpotent operators $\dot{K}_W : \overline{H}_{W,0,\mathbf{C}} \to \overline{H}_{W,1,\mathbf{C}}$ and ${}^t\dot{K}_W : \overline{H}_{W,1,\mathbf{C}} \to \overline{H}_{W,0,\mathbf{C}}$, by $\dot{K}_W := \mathrm{id}_{\overline{H}_{W,\mathbf{C}}} - \dot{J}_W$ and ${}^t\dot{K}_W := \mathrm{id}_{\overline{H}_{W,\mathbf{C}}} - {}^t\dot{J}_W$ so that we have $\dot{K}_W^2 = {}^t\dot{K}_W^2 = 0$ and $\dot{I}_W = 2\,\mathrm{id}_{\overline{H}_{W,\mathbf{C}}} - \dot{K}_W - {}^t\dot{K}_W$. Then,[12]

$$
\begin{aligned}
&\int_{\theta=0}^{\theta=1} \exp(\pi\sqrt{-1}\theta)\pi_1 \cdot dE_{W,\theta} \cdot \pi_0 \\
={}& \pi_1 \left[\int_{\theta=0}^{\theta=1}\left(1 - 2\sin^2(\tfrac{\pi}{2}\theta) + \sqrt{-1}\,2\sqrt{1 - \sin^2(\tfrac{\pi}{2}\theta)}\sin(\tfrac{\pi}{2}\theta)\right)dE_{W,\lambda}\right]\pi_0 \\
={}& \pi_1 \left[\int_{\theta=0}^{\theta=1}\left(1 - \tfrac{\lambda}{2} + \tfrac{\sqrt{-1}}{2}\sqrt{(4-\lambda)\lambda}\right)dE_{W,\lambda}\right]\pi_0 \\
={}& \pi_1 \left[\mathrm{id}_{\overline{H}_{w,\mathbf{C}}} - \tfrac{\dot{I}_W}{2} + \tfrac{\sqrt{-1}}{2}\sqrt{(4\,\mathrm{id}_{\overline{H}_{w,\mathbf{C}}} - \dot{I}_W)\dot{I}_W}\right]\pi_0
\end{aligned}
$$

After sandwiching by π_1 and π_0, the first and the second terms turn out to be $\pi_1 \cdot \mathrm{id}_{\overline{H}} \cdot \pi_0 = 0$ and $\pi_1 \cdot \frac{\dot{I}_W}{2} \cdot \pi_0 = -\frac{\dot{K}_W}{2}$, respectively. The third term turns out to be zero, since the operator

$$
\begin{aligned}
\sqrt{(4\,\mathrm{id}_{\overline{H}_{W,\mathbf{C}}} - \dot{I}_W)\dot{I}_W} &= \sqrt{(2\,\mathrm{id}_{\overline{H}_{W,\mathbf{C}}} + \dot{K}_W + {}^t\dot{K}_W)(2\,\mathrm{id}_{\overline{H}_{W,\mathbf{C}}} - \dot{K}_W - {}^t\dot{K}_W)} \\
&= \sqrt{4\,\mathrm{id}_{\overline{H}_{W,\mathbf{C}}} - \dot{K}_W \cdot {}^t\dot{K}_W - {}^t\dot{K}_W \cdot \dot{K}_W}
\end{aligned}
$$

preserves the decomposition (4.1.3) so that it does not have the "cross" term sandwiched by π_1 and π_0. Thus, we get

$$\int_{\theta=0}^{\theta=1} \exp(\pi\sqrt{-1}\theta)\pi_1 \cdot dE_{w,\lambda} \cdot \pi_0 = \frac{\dot{K}_W}{2}.$$

Similarly, we obtain also

$$\int_{\theta=0}^{\theta=1} \exp(-\pi\sqrt{-1}\theta)\pi_0 \cdot dE_{W,\lambda} \cdot \pi_1 = \frac{{}^t\dot{K}_W}{2}.$$

These altogether show the formula (4.4.19) □

COROLLARY. — Let $\varphi(\theta) = \sum_{m\in\mathbf{Z}} a_m \exp(2\pi\sqrt{-1}m(\theta + \frac{d-2}{2}))$ be an absolutely convergent Fourier expansion of a complex valued continuous function on the interval $\theta \in [0,1]$. Then, we have

$$2\int_{\theta=0}^{\theta=1} \varphi(\theta) \cdot \xi_{W,\theta} = \sum_{m\in\mathbf{Z}} a_m (Cox_W^{(d)})^m \cdot \dot{I}_W. \tag{4.4.27}$$

___

[12] In the present paper, $\sqrt{X}$ takes the positive branch for a positive object X.

Remark. — Due to the integral formula (4.4.19), we get a factor $\dot{I}_W$ at the end of the formula (4.4.27). Since $\dot{I}_W$ is not invertible by a bounded operator (recall the comment at the end of §4.2), the meaning of this factor is unclear. It would be desirable to ask:

Question: Can $\xi_{W,\theta}$ be divisible by $\dot{I}_W$ from the right (cf. §4.3 Corollary to **Assertion 2**)?

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