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COMPLEX STRUCTURES ON PRODUCT OF CIRCLE BUNDLES OVER COMPLEX MANIFOLDS

by Parameswaran SANKARAN & Ajay Singh THAKUR

Abstract. — Let $\bar{L}_i \longrightarrow X_i$ be a holomorphic line bundle over a compact complex manifold for $i = 1, 2$. Let S_i denote the associated principal circle-bundle with respect to some hermitian inner product on $\bar{L}_i$. We construct complex structures on $S = S_1 \times S_2$ which we refer to as *scalar, diagonal, and linear types*. While scalar type structures always exist, the more general diagonal but non-scalar type structures are constructed assuming that $\bar{L}_i$ are equivariant $(\mathbb{C}^*)^{n_i}$-bundles satisfying some additional conditions. The linear type complex structures are constructed assuming X_i are (generalized) flag varieties and $\bar{L}_i$ negative ample line bundles over X_i. When $H^1(X_1; \mathbb{R}) = 0$ and $c_1(\bar{L}_1) \in H^2(X_1; \mathbb{C})$ is non-zero, the compact manifold S does not admit any symplectic structure and hence it is non-Kähler with respect to *any* complex structure.

We obtain a vanishing theorem for $H^q(S; \mathcal{O}_S)$ when X_i are projective manifolds, $\bar{L}_i^\vee$ are very ample and the cone over X_i with respect to the projective imbedding defined by $\bar{L}_i^\vee$ are Cohen-Macaulay. We obtain applications to the Picard group of S. When $X_i = G_i/P_i$ where P_i are maximal parabolic subgroups and S is endowed with linear type complex structure with "vanishing unipotent part" we show that the field of meromorphic functions on S is purely transcendental over $\mathbb{C}$.

Keywords: circle bundles, complex manifolds, homogeneous spaces, Picard groups, meromorphic function fields.
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Résumé. — Soient $\bar{L}_i \longrightarrow X_i$ des fibrés en droites holomorphes sur des variétés complexes compactes, pour $i = 1, 2$. Soit S_i le fibré en cercles associé par rapport à un produit scalaire hermitienne sur $\bar{L}_i$. On construit des structures complexes sur $S = S_1 \times S_2$ dites de type scalaire, diagonal, ou linéaire. Bien que des structures de type scalaire existent toujours, on construit des structures plus générales de type diagonal mais non-scalaire dans le cas où les $\bar{L}_i$ sont des $(\mathbb{C}^*)^{n_i}$ fibrés équivariants qui vérifient certaines hypothèses supplémentaires. Les structures complexes de type linéaire sont des variétés des drapeaux (généralisées) et les L_i sont des fibrés en droites amples négatifs. Lorsque $H^1(X_1; \mathbb{R}) = 0$ et $c_1(\bar{L}_1)$ est non-nulle la variété compacte S n'admet pas de structure symplectique et donc elle est non-Kählerienne par rapport à toute structure complexe.

On montre que $H^q(S; \mathcal{O}_S)$ s'annule quand les X_i sont des variétés projectives, les $\bar{L}_i^\vee$ son très amples et le cône sur X_i par rapport au plongement projectif défini par $\bar{L}_i^\vee$ sont Cohen-Macaulay. On applique ces résultats au groupe de Picard de S. Quand $X_i = G_i/P_i$ où P_i sont les sousgroupes paraboliques maximaux et la variété S est munie d'une structure complexe du type linéaire avec « la partie unipotente nulle » on montre que le corps des fonctions méromorphes sur S est purement transcendental sur $\mathbb{C}$.

1. Introduction

H. Hopf [9] gave the first examples of compact complex manifolds which are non-Kähler by showing that $\mathbb{S}^1 \times \mathbb{S}^{2n-1}$ admits a complex structure for any positive integer n. Calabi and Eckmann [5] showed that product of any two odd dimensional spheres admit complex structures. Douady [7], Borcea [3] and Haefliger [8] studied deformations of the Hopf manifolds, and, Loeb and Nicolau [14], following Haefliger's ideas, studied the deformations of complex structures on Calabi-Eckmann manifolds. More recently, there have been many generalizations of Calabi-Eckmann manifolds leading to new classes of compact complex non-Kähler manifolds by López de Madrano and Verjovsky [12], Meersseman [15], Meersseman and Verjovsky [16], and Bosio [4]. See also [21] and [22].

In this paper we obtain another generalization of the classical Calabi-Eckmann manifolds. Our approach is greatly influenced by the work of Haefliger [8] and of Loeb-Nicolau [14] in that the compact complex manifolds we obtain arise as orbit spaces of holomorphic $\mathbb{C}$-actions on the product of two holomorphic principal $\mathbb{C}^*$-bundles over compact complex manifolds. As a differentiable manifold it is just the product of the associated circle bundles. In fact we obtain a family of complex analytic manifolds which may be thought of as a deformation of the total space of a holomorphic elliptic curve bundle over the product of the compact complex manifolds, in much the same way the construction of Haefliger (resp. Loeb and Nicolau) yields a deformation of the classical Hopf (resp. Calabi-Eckmann) manifolds.

The basic construction involves the notion of standard action by the torus $(\mathbb{C}^*)^{n_1}$ on a principal $\mathbb{C}^*$-bundle L_1 over a complex manifold X_1. See Definition 2.1. Let $L = L_1 \times L_2$ and $X = X_1 \times X_2$. When $L_i \longrightarrow X_i$ admit standard actions, any choice of a sequence of complex numbers $\lambda = (\lambda_1, \ldots, \lambda_N)$, $N = n_1 + n_2$, satisfying the *weak hyperbolicity condition of type* (n_1, n_2) (in the sense of Loeb-Nicolau [14, p. 788]) leads to a complex structure on the product $S(L) := S(L_1) \times S(L_2)$ where $S(L_i)$ denotes the circle bundle over X_i associated to L_i. This is the *diagonal* type complex structure on $S(L)$. The complex structure on $S(L)$ is obtained by identifying $S(L)$ as the orbit space of a certain $\mathbb{C}$-action determined by λ on L and by showing that the quotient map $L \longrightarrow L/\mathbb{C}$ is the projection of a holomorphic principal $\mathbb{C}$-bundle (Theorem 2.9). The scalar type structure arises as a special case of the diagonal type where $(\mathbb{C}^*)^{n_i} = \mathbb{C}^*$ is the structure group of L_i, $i = 1, 2$. In the case of scalar type complex structure the differentiable $\mathbb{S}^1 \times \mathbb{S}^1$-bundle with projection $S(L) \longrightarrow X$ is a holomorphic principal bundle with fibre and structure group an elliptic curve.

Under the hypotheses that $H^1(X_1; \mathbb{R}) = 0$ and $c_1(\bar{L}_1) \in H^2(X_1; \mathbb{R})$ is non-zero, we show that $S(L)$ is not symplectic and is non-Kähler with respect to any complex structure (Theorem 2.13).

The construction of linear type complex structure is carried out under the assumption that X_i is a generalized flag variety G_i/P_i, $i = 1, 2$, where G_i is a simply connected semi simple linear algebraic group over $\mathbb{C}$ and P_i a parabolic subgroup and the associated line bundle $\bar{L}_i$ over X_i is negative ample. In this case L_i is acted on by the reductive group $\widetilde{G}_i = G_i \times \mathbb{C}^*$ in such a manner that the action of a maximal torus $\widetilde{T}_i \subset \widetilde{G}_i$ on L_i is standard (Proposition 3.1). Fix a Borel subgroup $\widetilde{B}_i \supset \widetilde{T}_i$ and choose an element $\lambda \in Lie(\widetilde{B})$ where $\widetilde{B} = \widetilde{B}_1 \times \widetilde{B}_2 \subset \widetilde{G}_1 \times \widetilde{G}_2 =: \widetilde{G}$. Writing the Jordan decomposition $\lambda = \lambda_s + \lambda_u$ where λ_s belongs to the the Lie algebra of $\widetilde{T} := \widetilde{T}_1 \times \widetilde{T}_2$, we assume that λ_s satisfies the weak hyperbolicity condition. For each such λ we have a complex structure on $S(L)$ of *linear type.* (See Theorem 3.2.) We show that $H^q(S_\lambda(L); \mathcal{O})$ vanishes for most values of q. This result is valid in greater generality; see Theorem 4.5 for precise statements. We deduce that $Pic^0(S_\lambda(L)) \cong \mathbb{C}$, assuming that if $X_i = \mathbb{P}^1$, then $\bar{L}_i$ is the generator of $Pic(\mathbb{P}^1) \cong \mathbb{Z}$. (See Theorem 4.7.) When P_i are maximal parabolic subgroups and $\lambda_u = 0$, we show that the meromorphic function field of $S_\lambda(L)$ is a purely transcendental extension of $\mathbb{C}$. (Theorem 4.8).

Our proofs in §2 follow mainly the ideas of Loeb and Nicolau [14]. The construction of linear type complex structure is a generalization of the linear type complex structures on $\mathbb{S}^{2m-1} \times \mathbb{S}^{2n-1}$ given in [14] to the more general context where the base space is a product of generalized flag varieties. We use the Künneth formula due to A. Cassa [6], besides projective normality and arithmetic Cohen-Macaulayness of generalized flag varieties ([19], [20]), for obtaining our results on the cohomology groups $H^q(S(L);\mathcal{O})$. Construction of linear type complex structure, applications to Picard groups and the field of meromorphic functions on $S(L)$ when $X_i = G_i/P_i$ involve some elementary concepts from representation theory of algebraic groups.

Notations

The following notations will be used throughout.

X_1, X_2	compact complex manifolds
L_1, L_2	principal $\mathbb{C}^*$-bundles
L	$L_1 \times L_2$
$\bar{L}_i$	line bundle associated to L_i
$\bar{L}^\vee$	dual of $\bar{L}$
$\hat{L}_i$	cone over X_i with respect to $\bar{L}_i^\vee$ when $\bar{L}_i$ is negative ample
$T_i, T, \widetilde{T}_i, \widetilde{T}$	complex algebraic tori $T = T_1 \times T_2$, $\widetilde{T}_i = T_i \times \mathbb{C}^*$, $\widetilde{T} = \widetilde{T}_1 \times \widetilde{T}_2$
G, G_i	semi simple complex algebraic groups
$\widetilde{G}_i, \widetilde{G}$	$\widetilde{G}_i = G_i \times \mathbb{C}^*$, $\widetilde{G} = G \times \mathbb{C}^*$
P_i	a parabolic subgroup of G_i
ω_i, ω	dominant integral weights
$V(\omega_i)$	finite dimensional irreducible G_i-module of highest weight ω_i
$V(\omega_i)^\vee$	vector space dual to $V(\omega_i)$
$V(\omega_1, \omega_2)$	$V(\omega_1) \times V(\omega_2) \setminus (V(\omega_1) \times 0 \cup 0 \times V(\omega_2))$
$\Lambda(\omega_i), \Lambda(\omega_1, \omega_2)$	weights of $V(\omega_i)$, resp. $V(\omega_1) \times V(\omega_2)$
$R(G), R$	roots of G with respect to a maximal torus T
Φ	set of simple roots
R^+, R_{P_i}	positive roots, resp. set of positive roots of G_i which are not the roots of Levi part of P_i
$V \hat{\otimes} V'$	completed tensor product of Fréchet-nuclear spaces V, V'
$X_\beta, Y_\beta, H_\beta$	Chevalley basis elements of a reductive Lie algebra
$\mathcal{O}$	structure sheaf of an analytic space
$\mathcal{R}^+, \mathcal{R}^-$	$\{x + \sqrt{-1}y \in \mathbb{C} \mid x > 0\}$, resp. $\{x + \sqrt{-1}y \in \mathbb{C} \mid x < 0\}$
I	unit interval $[0, 1] \subset \mathbb{R}$
$\mathcal{T}_p M$	tangent space to a manifold M at a point p.

2. Basic construction

Let X_1, X_2 be any two compact complex manifolds and let $p_1 : L_1 \longrightarrow X_1$ and $p_2 : L_2 \longrightarrow X_2$ be holomorphic principal $\mathbb{C}^*$-bundles over X_1 and X_2 respectively. Denote by $p : L_1 \times L_2 =: L \longrightarrow X := X_1 \times X_2$ the product $\mathbb{C}^* \times \mathbb{C}^*$-bundle. We shall denote by $\bar{L}_i$ the line bundle associated to L_i and identify X_i with the zero cross-section in L_i so that $L_i = \bar{L}_i \setminus X_i$. We put a hermitian metric on $\bar{L}_i$ invariant under the action of $\mathbb{S}^1 \subset \mathbb{C}^*$ and denote by $S(L_i) \subset L_i$ the unit sphere bundle with fibre and structure group $\mathbb{S}^1$. We shall denote by $S(L)$ the $\mathbb{S}^1 \times \mathbb{S}^1$-bundle $S(L_1) \times S(L_2)$. Our aim is to study complex structures on $S(L)$ arising from holomorphic principal $\mathbb{C}$-bundle structures on L with base space $L/\mathbb{C}$. Such a bundle arises from the holomorphic foliation associated to certain holomorphic vector field whose integral curves are biholomorphic to $\mathbb{C}$. For the vector fields we consider, the space of leaves $L/\mathbb{C}$ can be identified with $S(L)$ as a differentiable manifold and the complex structure on $S(L)$ is induced from that on $L/\mathbb{C}$ via this identification.

In this section we consider holomorphic $\mathbb{C}$-actions on L which lead to complex structure on $S(L)$ of *scalar* and *diagonal* types. Whereas scalar type complex structures always exist, in order to obtain the more general diagonal type complex structure which are *not* of scalar type we need additional hypotheses.

Given any complex number τ such that $\mathrm{Im}(\tau) > 0$, one obtains a proper holomorphic imbedding $\mathbb{C} \longrightarrow \mathbb{C}^* \times \mathbb{C}^*$ defined as

$$z \mapsto (\exp(2\pi i z), \exp(2\pi i \tau z)).$$

We shall denote the image by $\mathbb{C}_\tau$. The action of the structure group $\mathbb{C}^* \times \mathbb{C}^*$ on L can be restricted to $\mathbb{C}$ via the above imbedding to obtain a holomorphic principal $\mathbb{C}$-bundle with total space L and base space the quotient space $S_\tau(L) := L/\mathbb{C}_\tau$. Clearly the projection $L \longrightarrow X$ factors through $S_\tau(L)$ to yield a principal bundle $S_\tau(L) \longrightarrow X$ with fibre and structure group $\mathbb{E} := (\mathbb{C}^* \times \mathbb{C}^*)/\mathbb{C}_\tau$. Since $\mathbb{E}$ is a Riemann surface with fundamental group isomorphic to $\mathbb{Z}^2$, it is an elliptic curve. It can be seen that $\mathbb{E} \cong \mathbb{C}/\Gamma$ where Γ is the lattice $\mathbb{Z} + \tau\mathbb{Z} \subset \mathbb{C}$. It is easily seen that $S_\tau(L)$ is diffeomorphic to $S(L) = S(L_1) \times S(L_2)$. The resulting complex structure on $S(L)$ is referred to as *scalar* type.

Now suppose that $\bar{L}_i, X_i$ are acted on holomorphically by the torus group $T_i \cong (\mathbb{C}^*)^{n_i}$ such that $\bar{L}_i$ is a T_i-equivariant bundle over X_i. We identify T_i with $(\mathbb{C}^*)^{n_i}$ by choosing an isomorphism $T_i \cong (\mathbb{C}^*)^{n_i}$. Set $N = n_1 + n_2$ and $T := T_1 \times T_2 = (\mathbb{C}^*)^N$. We shall denote by $\epsilon_j : \mathbb{C}^* \subset (\mathbb{C}^*)^N$ the inclusion of the jth factor and write $t\epsilon_j$ to denote $\epsilon_j(t)$ for $1 \leqslant j \leqslant N$. Thus any $t = (t_1, \cdots, t_N) \in T$ equals $\prod_{1\leqslant j\leqslant N} t_j\epsilon_j$, and, under the exponential map $\mathbb{C}^N \longrightarrow (\mathbb{C}^*)^N$, $\sum_{1\leqslant j\leqslant N} z_j e_j$ maps to $\prod \exp(z_j)\epsilon_j$. (Here e_j denotes the standard basis vector of $\mathbb{C}^N$.)

We put a hermitian metric on $\bar{L}_i$ which is invariant under action of the maximal compact subgroup $K_i = (\mathbb{S}^1)^{n_i} \subset T_i$. The following definition will be very crucial for our construction of complex structures on $S(L)$.

Definition 2.1. — *Let d be a positive integer. We say that the $T_1 = (\mathbb{C}^*)^{n_1}$-action on L_1 is d-*standard *(or more briefly* standard*) if the following conditions hold:*
(i) the restricted action of the diagonal subgroup $\Delta \subset T_1$ on L_1 is via the d-fold covering projection $\Delta \longrightarrow \mathbb{C}^*$ onto the structure group $\mathbb{C}^*$ of $L_1 \longrightarrow X_1$. (Thus if $d = 1$, the action of Δ coincides with that of the structure group of L_1.)
(ii) For any $0 \neq v \in L_1$ and $1 \leqslant j \leqslant n_1$ let $\nu_{v,j} : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ be defined as $t \mapsto ||t\epsilon_j.v||$. Then $\nu'_{v,j}(t) > 0$ for all t unless $\mathbb{R}_+\epsilon_j$ is contained in the isotropy at v.

Examples of standard actions are given in 2.5 below. Note that condition (i) in the above definition implies that the Δ-orbit of any $p \in L_1$ is just the fibre of the bundle $L_1 \longrightarrow X_1$ containing p. The exact value of d will not be of much significance for us. However, it will be too restrictive to assume $d = 1$. (See Example 2.5(iii) and also §3.)

Suppose that there exists a one parameter subgroup $S \cong \mathbb{C}^*$ of T_1 such that the restricted S action on L_1 is the same as that induced by a covering $S \longrightarrow \mathbb{C}^*$ of the structure group of L_1. Then one may parametrise T_1 so that the diagonal subgroup of $(\mathbb{C}^*)^{n_1}$ maps isomorphically onto S. But it may so happen that there exists *no* one-parameter subgroup S satisfying condition (i) with Δ replaced by S. Indeed, this happens for the action of the diagonal subgroup T_1 of $SL(2, \mathbb{C})$ on the tautological line bundle over $\mathbb{P}^1$. See also §3.

Condition (ii) above controls the dynamics of the T_1-action and will have important implications as we shall see. Roughly speaking condition (ii) says that, for each $v \in L_1$, the smooth curve $\sigma_{v,j} : \mathbb{R}_+ \longrightarrow L_1$ defined as $t \mapsto t\epsilon_j.v$ always "grows outwards" unless it is a degenerate curve. For a counterexample, consider again the action of the diagonal subgroup $T_1 = \{diag(t, t^{-1}) \mid t \in \mathbb{C}^*\}$ of $SL(2, \mathbb{C})$ on $\mathbb{C}^2 \setminus \{0\} \longrightarrow \mathbb{P}^1$. Then $t\epsilon_1.e_1 = te_1, t\epsilon_1.e_2 = t^{-1}e_2$. Therefore $\sigma_{e_1,1}$ 'grows outwards' whereas $\sigma_{e_2,1}$ 'grows inwards'. On the other hand if $v = v_1e_1 + v_2e_2$, where v_1, v_2 are both non-zero, then the function $\nu_v(t) = ||tv_1e_1 + t^{-1}v_2e_2|| = (t^2|v_1|^2 + t^{-2}|v_2|^2)^{1/2}$ attains its minimum at some $t_0 > 0$.

It is obvious that if $T_i = \mathbb{C}^*$, the structure group of L_i, then the action of T_i on $L_i = \bar{L}_i \setminus X_i$ is standard.

Let $\lambda \in Lie(T) = \mathbb{C}^N$. There exists a unique Lie group homomorphism $\alpha_\lambda : \mathbb{C} \longrightarrow T$ defined as $z \mapsto \exp(z\lambda)$. When λ is clear from the context, we write α to mean α_λ. We denote by $\alpha_{\lambda,i}$ (or more briefly α_i) the composition $\mathbb{C} \xrightarrow{\alpha_\lambda} T \xrightarrow{pr_i} T_i, i = 1, 2$.

We recall the definition of weak hyperbolicity [14]. Let $\lambda = (\lambda_1, \ldots, \lambda_N)$, $N = n_1 + n_2$. One says that λ satisfies the *weak hyperbolicity condition of type* (n_1, n_2) if

$$0 \leqslant \arg(\lambda_i) < \arg(\lambda_j) < \pi, \ 1 \leqslant i \leqslant n_1 < j \leqslant N \tag{1}$$

If $\lambda_j = 1 \ \forall j \leqslant n_1, \lambda_j = \tau \ \forall j > n_1$, with $\mathrm{Im}(\tau) > 0$, we say that λ is of *scalar type*.

We denote by C_i the cone $\{\sum r_j\lambda_j \in \mathbb{C} \mid r_j \geqslant 0, n_{i-1}+1 \leqslant j \leqslant n_{i-1}+n_i\}$ where $n_0 = 0$. We shall denote $C_i \setminus \{0\}$ by C_i° and refer to it as the *deleted cone*. Weak hyperbolicity is equivalent to the requirement that the cones C_1, C_2 meet only at the origin and are contained in the half-space $\{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\} \cup \mathbb{R}_{\geqslant 0}$.

Definition 2.2. — *Suppose that the $T_i = (\mathbb{C}^*)^{n_i}$-action on L_i is d_i-standard for some $d_i \geqslant 1$, $i = 1, 2$, and let $\lambda \in \mathbb{C}^N = Lie(T)$. The analytic homomorphism $\alpha_\lambda : \mathbb{C} \longrightarrow T = (\mathbb{C}^*)^N$ defined as $\alpha_\lambda(z) = \exp(z\lambda)$ is said to be* admissible *if λ satisfies the weak hyperbolicity condition (1) of type (n_1, n_2) above. We denote the image of α_λ by $\mathbb{C}_\lambda$. If α_λ is admissible, we say that the $\mathbb{C}_\lambda$-action on L is of* diagonal type*. If λ is of scalar type, we say that $\mathbb{C}_\lambda$-action is of* scalar type*.*

The weak hyperbolicity condition implies that $(\lambda_1, \ldots, \lambda_N) \in \mathbb{C}^N$ belongs to the Poincaré domain [1], (that is, 0 is not in the convex hull of $\lambda_1, \ldots, \lambda_N \in \mathbb{C}$,) and that α_λ is a proper holomorphic imbedding. Thus $\mathbb{C}_\lambda \cong \mathbb{C}$. When there is no risk of confusion, we merely write $\mathbb{C}$ to mean $\mathbb{C}_\lambda$.

Note that if λ is of scalar type, then the action of $\mathbb{C}_\lambda$ leads to a scalar type complex structure on the orbit space $L/\mathbb{C}_\lambda = S(L)$. Moreover, if $d_1 = d_2 = 1$, then $L/\mathbb{C}_\lambda = S_{\lambda_{n_1+1}}(L)$.

Lemma 2.3. — *Suppose that $L_1 \longrightarrow X_1$ is a T_1-equivariant principal $\mathbb{C}^*$-bundle such that the T_1-action is d-standard. Then:*
(i) One has $||z\epsilon_j.v|| \leqslant ||v||$ for $0 < |z| < 1$ where equality holds if and only if $\mathbb{R}\epsilon_j$ is contained in the isotropy at v.

(ii) For any $t = (t_1, \dots, t_{n_1}) \in T_1$*, one has*

$$|t_{k_0}|^d.||v|| \leqslant ||t.v|| \leqslant |t_{j_0}|^d.||v||, \forall v \in L_1, \tag{2}$$

where $j_0 \leqslant n_1$ *(resp.* $k_0 \leqslant n_1$*) is such that* $|t_{j_0}| \geqslant |t_j|$ *(resp.* $|t_{k_0}| \leqslant |t_j|$*) for all* $1 \leqslant j \leqslant n_1$*. Also* $||t.v|| = |t_{j_0}|^d.||v||$ *if and only if* $|t_j| = |t_{j_0}|$ *for all* j *such that* $(t_j/t_{j_0})\epsilon_j.v \neq v$ *and* $||t.v|| = |t_{k_0}|^d.||v||$ *if and only if* $|t_j| = |t_{k_0}|$ *for all* j *such that* $(t_j/t_{k_0})\epsilon_j.v \neq v$*.*

Proof. — (i) Suppose that $\mathbb{R}_+\epsilon_j$ is not contained in the isotropy at v. Since K_1 preserves the norm, we may assume that $z \in \mathbb{R}_+$. In view of 2.1(ii), $\nu_{v,j}$ is strictly increasing. Hence $||z\epsilon_j.v|| < ||v||$ for $0 < z < 1$.

(ii) Write $s = (s_1, \dots, s_{n_1})$ where $s_j = t_j/t_{j_0}$ $\forall j$. Denoting the diagonal imbedding $\mathbb{C}^* \longrightarrow T_1$ by δ, we have $t = \delta(t_{j_0})s$. Now $\delta(t_{j_0}).v = t_{j_0}^d v$ in view of 2.1 (i).

By repeated application of (i) above, we see that $||t.v|| = ||s(\delta(t_{j_0})v)|| = ||s.t_{j_0}^d v|| \leqslant |t_{j_0}|^d.||v||$ where the inequality is strict unless $|t_j| = |t_{j_0}|$ for all j such that $s_j\epsilon_j.v \neq v$. A similar proof establishes the inequality $||t.v|| \geqslant |t_{k_0}|^d.||v||$ as well as the condition for equality to hold. □

As an immediate corollary, we obtain

Proposition 2.4. — *Any admissible* $\mathbb{C}_\lambda$*-action of diagonal type on* $L_1 \times L_2$ *is free.*

Proof. — . Suppose that $z \in \mathbb{C}, z \neq 0, (p_1, p_2) \in L$. Let $\alpha_\lambda(z).(p_1, p_2) = (q_1, q_2)$. It is readily seen that one of the deleted cones zC_1°, zC_2° lies entirely in the left-half space $\mathcal{R}_- := \{z \in \mathbb{C} \mid \mathrm{Re}(z) < 0\}$ or the right-half space $\mathcal{R}_+ := \{z \in \mathbb{C} \mid \mathrm{Re}(z) > 0\}$. Consider the case $zC_1^\circ \subset \mathcal{R}_-$. Then $|\exp(z\lambda_j)| < 1$ for all $j \leqslant n_1$. We claim that there is some j such that $\exp(z\lambda_j)\epsilon_j.p_1 \neq p_1$, for, otherwise, the action of T_1-action, restricted to the orbit through p_1 factors through the compact group $T_1/\langle \exp(z\lambda_j)\epsilon_j, 1 \leqslant j \leqslant n_1\rangle \cong (\mathbb{S}^1)^{2n_1}$. This implies that the T_1-orbit of p_1 is compact, contradicting 2.1 (i). Now it follows from Lemma 2.3 that

$$||q_1|| = ||\left(\prod_{1 \leqslant j \leqslant n_1} \exp(z\lambda_j)\epsilon_j\right).p_1|| < ||p_1||.$$

Thus $q_1 \neq p_1$ in this case. Similarly, we see that $(p_1, p_2) \neq (q_1, q_2)$ in the other cases also, showing that the $\mathbb{C}$-action on L is free. □

Example 2.5. — *(i) Let* $T_i = \mathbb{C}^*$ *be the structure group of* $L_i \longrightarrow X_i$ *so that the* T_i*-action on* L_i *is standard,* $i = 1, 2$*. If* $\tau \in \mathbb{C}^*$ *is such that*

$0 < \arg(\tau) < \pi$, *then the imbedding* $\alpha(z) = (\exp(z), \exp(\tau z)) \in \mathbb{C}^* \times \mathbb{C}^*$ *is admissible.*

(ii) Suppose that T_1 *action on* L_1 *is d-standard and that* $X_1' \subset X_1$ *is a* T_1*-stable complex analytic submanifold. Then the* T_1*-action on* $L_1|X_1'$ *is again d-standard. More generally, suppose* X_1' *is any compact complex manifold with a holomorphic* T_1*-action and that* $\bar{L}_1' \longrightarrow X_1'$ *is the pull-back of* $\bar{L}_1 \longrightarrow X_1$ *via a* T_1*-equivariant holomorphic map* $f : X_1' \longrightarrow X_1$*. The hermitian metric on* $\bar{L}_1$ *induces a hermitian metric on* $\bar{L}_1'$*. Then the* T_1*-action on* L_1' *is standard.*

(iii) Let $A = (a_{ij})$ *be an* $n \times n_1$ *matrix—the matrix of exponents—where* $a_{ij} \in \mathbb{Z}$*. Then* $T_1 := (\mathbb{C}^*)^{n_1}$ *acts linearly on* $\mathbb{C}^n$ *where* $t\epsilon_j.(z_1, \ldots, z_n) = (t^{a_{1j}} z_1, \ldots, t^{a_{n,j}} z_n), t \in \mathbb{C}^*, 1 \leqslant j \leqslant n_1$*. This action makes the tautological line bundle* $\bar{L}_1 \longrightarrow \mathbb{P}^{n-1}$ *a* T_1*-equivariant bundle. The action is almost effective if* A *has rank equals* n_1*. Condition (i) of Definition 2.1 is satisfied if* A *has positive constant row sums, that is,* $d := \sum_j a_{ij}$ *is independent of* i *and is positive. Condition (ii) is satisfied if* $a_{i,j} \geqslant 0$*, for all* $1 \leqslant i \leqslant n, 1 \leqslant j \leqslant n_1$*. Thus we obtain a d-standard* T_1*-action on* L_1 *when the matrix* A *satisfied both these conditions where* $d := \sum_j a_{1,j}$*.*

(iv) Consider the linear representation of $T_1 \cong (\mathbb{C}^*)^{n_1}$ *on* $\mathbb{C}^n$ *obtained from a matrix of exponents* A *of rank* n_1*, having positive integral entries and constant row sums as in (iii) above. This induces a linear action of* T_1 *on* $\Lambda^k(\mathbb{C}^n) \cong \mathbb{C}^{\binom{n}{k}}$ *for* $k < n$*. Denote by* $G_k(\mathbb{C}^n)$ *the Grassmann variety of* k *dimensional vector subspaces of* $\mathbb{C}^n$*. The standard* T_1*-action on* $L_1 = \Lambda^k(\mathbb{C}^n) \setminus \{0\}$ *where* $\bar{L}_1$ *is the tautological line bundle over* $\mathbb{P}(\Lambda^k(\mathbb{C}^n))$ *restricts to a standard* T_1*-action on the* $L_1|G_k(\mathbb{C}^n)$ *via the Plücker imbedding* $G_k(\mathbb{C}^n) \hookrightarrow \mathbb{P}(\Lambda^k(\mathbb{C}^n))$*. Note that* $\bar{L}_1|G_k(\mathbb{C}^n)$ *is a negative ample line bundle over* $G_k(\mathbb{C}^n)$ *which generates* $Pic(G_k(\mathbb{C}^n)) \cong \mathbb{Z}$*.*

Lemma 2.6. — *The orbits of an admissible* $\mathbb{C}_\lambda$*-action on* L *are closed and properly imbedded in* L*.*

Proof. — Let $p = (p_1, p_2) \in L$. Let (z_n) be any sequence of complex numbers such that $|z_n| \to \infty$. We shall show that $\alpha_\lambda(z_n).p$ has no limit points in L. Without loss of generality, we may assume that the z_n are such that $z_n/|z_n|$ have a limit point $z_0 \in \mathbb{S}^1$. By the weak hyperbolicity condition (1), one of the deleted cones $z_0 C_i^\circ$ is contained in one of the sectors $\mathcal{S}_+(\theta) := \{w \in \mathbb{C} \mid -\theta < \arg(w) < \theta\} \subset \mathcal{R}_+$ or $\mathcal{S}_-(\theta) = -\mathcal{S}_+(\theta) \subset \mathcal{R}_-$ for some θ, $0 < \theta < \pi/2$. Say $z_0 C_i^0 \subset \mathcal{S}_-(\theta)$. Then $z_n C_i^\circ \subset \mathcal{S}_-(\theta)$ for all n sufficiently large. It follows that $|\exp(z_n \lambda_j)| \to 0$ as $n \to \infty$ for $n_{i-1} < j \leqslant n_{i-1} + n_i$ (where $n_0 = 0$). By Lemma 2.3 we conclude that the sequence $(\alpha_i(z_n)(p_i))$ does not have a limit in L_i. □

Definition 2.7. — *Given standard* $T_i = (\mathbb{C}^*)^{n_i}$*-actions on the* L_i, $i = 1, 2$, *we obtain holomorphic vector fields* $v_1, \ldots, v_N$ *on* $L = L_1 \times L_2$ *as follows. Let* $p = (p_1, p_2) \in L$. *Suppose that* $1 \leqslant j \leqslant n_1$. *The holomorphic map* $\mu_{p_1} : T_1 \longrightarrow L_1$, $s \mapsto s.p_1$, *induces* $d\mu_{p_1} : Lie(T_1) = \mathbb{C}^{n_1} \longrightarrow \mathcal{T}_{p_1} L_1$. *Set* $v_j(p) := (d\mu_p(e_j), 0) \in \mathcal{T}_{p_1} L_1 \times \mathcal{T}_{p_2} L_2 = \mathcal{T}_p L$. *The vector fields* $v_j, n_1 < j \leqslant N$, *are defined similarly. The vector fields* $v_j, 1 \leqslant j \leqslant N$, *are referred to as* fundamental vector fields *on* L.

Remark 2.8. — *Let* $1 \leqslant j \leqslant n_1$. *Consider the differential* $d\nu_1 : \mathcal{T}_p L \longrightarrow \mathbb{R}$ *of the norm map* $\nu_1 : L \longrightarrow \mathbb{R}_+$ *defined as* $q = (q_1, q_2) \mapsto ||q_1||$. *It is readily verified that, if* $\mathbb{R}_+ \epsilon_j$ *is not contained in the isotropy at* p_1, *then by standardness of the action,* $d\nu_1(v_j(p)) = v_j(p)(\nu_1) = \nu'_{j,p_1}(1) > 0$. *(Here* ν_{j,p_1} *is as in the definition 2.1(ii) of standard action.) On the other hand, since* $\nu_1(s.p) = \nu_1(p)$ *for all* $s \in (\mathbb{S}^1)^{n_1} = \exp(\sqrt{-1}\mathbb{R}^{n_1}) \subset T_1$ *we obtain that* $d\nu_1(\sqrt{-1} v_j(p)) = 0$. *Thus, for any* $z \in \mathbb{C}$, *we obtain that* $d\nu_1(z v_j(p)) = Re(z) \nu'_{j,p_1}(1)$. *An entirely analogous statement holds when* $n_1 < j \leqslant N$.

Assume that $\lambda \in \mathbb{C}^N$ yields an admissible imbedding $\alpha : \mathbb{C} \longrightarrow T$, $\alpha(z) = \exp(z\lambda)$. We obtain a holomorphic vector field v_λ on L where

$$v_\lambda(p) = \sum_{1 \leqslant j \leqslant N} \lambda_j v_j(p) \in \mathcal{T}_p L.$$

The flow of the vector field v_λ yields a holomorphic action of $\mathbb{C}$ which is just the restriction of the T-action to $\mathbb{C}_\lambda$. This $\mathbb{C}$-action on L is free and the $\mathbb{C}$-orbits are the same as the leaves of the holomorphic foliation defined by the integral curves of the vector field v_λ. By Lemma 2.6 each leaf is biholomorphic to $\mathbb{C}$. It turns out that the leaf space $L/\mathbb{C}$ is a Haudorff complex analytic manifold and the projection $L \longrightarrow L/\mathbb{C}_\lambda$ is the projection of a holomorphic principal bundle with fibre and structure group the additive group $\mathbb{C}$. The underlying differentiable manifold of the leaf space is diffeomorphic to $S(L) = S(L_1) \times S(L_2)$. These statements will be proved in Theorem 2.9 below. We shall denote the complex manifold $L/\mathbb{C}_\lambda$ by $S_\lambda(L)$. The complex structure so obtained on $S(L)$ is referred to as *diagonal type*.

We shall denote by $D(\bar{L}) \subset \bar{L} = \bar{L}_1 \times \bar{L}_2$ the product of the unit disk bundles $D(\bar{L}_i) = \{p \in \bar{L}_i \mid ||p|| \leqslant 1\} \subset \bar{L}_i, i = 1, 2$. Also we denote by $\Sigma(\bar{L}) \subset \bar{L}$ the boundary of $D(\bar{L})$. Thus $\Sigma(\bar{L}) = D(\bar{L}_1) \times S(L_2) \cup S(L_1) \times D(\bar{L}_2)$. Observe that $S(L) = D(\bar{L}_1) \times S(L_2) \cap S(L_1) \times D(\bar{L}_2) \subset \Sigma(\bar{L})$.

Theorem 2.9. — *With the above notations, suppose that* $\alpha_\lambda : \mathbb{C} \longrightarrow T$ *defines an admissible action of* $\mathbb{C}$ *of diagonal type on* L. *Then* $L/\mathbb{C}$ *is a (Hausdorff) complex analytic manifold and the quotient map* $L \longrightarrow L/\mathbb{C}$ *is the projection of a holomorphic principal* $\mathbb{C}$*-bundle. Furthermore,* *each $\mathbb{C}$-orbit meets $S(L)$ transversely at a unique point so that $L/\mathbb{C}$ is diffeomorphic to $S(L)$.*

Proof of the above theorem, which is along the same lines as the proof of [14, Theorem 1] with suitable modifications to take care the more general setting we are in, will be based on the following two lemmata.

LEMMA 2.10. — *Each $\mathbb{C}_\lambda$-orbit in L meets $S(L)$ at exactly one point.*

Proof. — *Step 1:* We first show that each orbit meets $S(L)$ at not more than one point. Let $p = (p_1, p_2) \in S(L)$. Suppose that $0 \neq z \in \mathbb{C}$ is such that $q := \alpha_\lambda(z).p = \alpha(z).p \in S(L)$. This means that, writing $q = (q_1, q_2)$, we have

$$q_i = \alpha_i(z)(p_i) = \left(\prod_{n_{i-1}<j\leqslant n_{i-1}+n_i} \exp(\lambda_j z)\epsilon_j\right) p_i, i = 1, 2,$$

(where $n_0 = 0$). Now $||q_i|| = ||p_i|| = 1, i = 1, 2$, and $p \neq q$. Since the hermitian metric on L_1 is invariant under $(\mathbb{S}^1)^{n_1}$, we see that $||p_1|| = ||q_1|| = ||(\prod_{1\leqslant j\leqslant n_1}(\exp(t_j)\epsilon_j))p_1||$ where $t_j = \mathrm{Re}(\lambda_j z)$. Standardness of the T_1-action implies that either $\mathrm{Re}(\lambda_i z) = 0$ for all $i \leqslant n_1$ or there exist indices $1 \leqslant i_1 < i_2 \leqslant n_1$ such that $\mathrm{Re}(z\lambda_{i_1}).\mathrm{Re}(z\lambda_{i_2}) < 0$. In the latter case there exist positive reals a_1, a_2 such that $a_1\mathrm{Re}(z\lambda_{i_1}) + a_2\mathrm{Re}(z\lambda_{i_2}) = 0$. Similarly, either $\mathrm{Re}(z\lambda_j) = 0$ for all $n_1 < j \leqslant N$ or there exist indices $n_1 < j_1 < j_2 \leqslant N$ and positive reals b_1, b_2 such that $b_1\mathrm{Re}(z\lambda_{j_1}) + b_2\mathrm{Re}(z\lambda_{j_2}) = 0$. Suppose $\mathrm{Re}(a_1\lambda_{i_1}z + a_2\lambda_{i_2}z) = 0 = \mathrm{Re}(b_1\lambda_{j_1}z + b_2\lambda_{j_2}z)$. This implies that $a_1\lambda_{i_1} + a_2\lambda_{i_2} = r(b_1\lambda_{j_1} + b_2\lambda_{j_2})$ for some positive number r. This contradicts the weak hyperbolicity condition (1). Similarly we obtain a contradiction in the remaining cases as well.

Step 2: Next we show that $\mathbb{C}p \cap \Sigma(\bar{L})$ is path-connected for any $p \in L$. We shall write D_- and D_+ to denote the bounded and unbounded components of $L \setminus \Sigma(\bar{L})$.

Without loss of generality, suppose that $p = (p_1, p_2) \in \Sigma(\bar{L})$ and let $q = (q_1, q_2) \in \Sigma(\bar{L}) \cap \mathbb{C}p$ be arbitrary. Say, $q = \alpha(z_1).p$ with $z_1 \neq 0$. Then $r \mapsto \alpha(rz_1).p$ defines a path $\sigma : I \longrightarrow \mathbb{C}p$ with end points in $\Sigma(\bar{L})$. We modify the path σ to obtain a new path which lies in $\Sigma(\bar{L})$. For this purpose choose $z_0 \in \mathbb{C}$, $\arg(z_0) > \frac{\pi}{2}$ such that $z_0C_1^\circ \cup z_0C_2^\circ$ is contained in the left-half space $\mathcal{R}_- = \{z \in \mathbb{C} \mid \mathrm{Re}(z) < 0\}$ and $(-z_0)C_1^\circ \cup (-z_0)C_2^\circ$ is contained in the right-half space $\mathcal{R}_+ = \{z \in \mathbb{C} \mid \mathrm{Re}(z) > 0\}$. In particular, $\lim_{r\to\infty} |\exp(rz_0\lambda_j)| = 0$ and $\lim_{r\to\infty} |\exp(-rz_0\lambda_j)| = \infty, \forall j \leqslant N$, where r varies in $\mathbb{R}_+$. By (2), we see that for $i = 1, 2$, and any $x_i \in L_i$, $||\alpha_i(rz_0).x_i|| \to 0$ and $||\alpha_i(-rz_0).x_i|| \to \infty$ as $r \to +\infty$ in $\mathbb{R}$.

For any $r \in I$, let $\gamma(r) \in \mathbb{R}$ be least (resp. largest) such that $\alpha(\gamma(r)z_0).\sigma(r) \in \Sigma(\bar{L})$ when $\sigma(r) \in D_+$ (resp. $\sigma(r) \in D_-$). Then γ is a well-defined continuous function of r. Now $r \mapsto \alpha(\gamma(r)z_0 + rz_1).p$ is a path in $\mathbb{C}p \cap \Sigma(\bar{L})$ joining p to q.

Step 3: To complete the proof, we shall show that, for any $p \in L$, there exist points $q' = (q'_1, q'_2), q'' = (q''_1, q''_2) \in \mathbb{C}p \cap \Sigma(\bar{L})$ such that $||q'_1|| \leqslant 1, ||q'_2|| = 1$ and $||q''_1|| = 1, ||q''_2|| \leqslant 1$. Then any path in $\mathbb{C}p \cap \Sigma(\bar{L})$ joining q' and q'' must contain a point of $S(L)$.

Choose $w_k \in \mathbb{C}^*, 1 \leqslant k \leqslant 4$, such that the deleted cones $w_1 C_i^\circ \subset \mathcal{R}_+, w_2 C_i^\circ \subset \mathcal{R}_-$, for $i = 1, 2$, and, $w_3 C_1^\circ, w_4 C_2^\circ \subset R_-$, $w_3 C_2^\circ, w_4 C_1^\circ \subset \mathcal{R}_+$. Then $|\exp(rw_k\lambda_j)| \to 0$ (resp. ∞) as $r \to +\infty$ ($r \in \mathbb{R}_+$) if $\lambda_j \in C_i^\circ$ and $w_k C_i^\circ \subset \mathcal{R}_-$ (resp. $\mathcal{R}_+$). Now $||\alpha_i(rw_1)p_i|| > 1, ||\alpha_i(rw_2)p_i|| < 1, i = 1, 2$ for $r \in \mathbb{R}_+$ sufficiently large. It follows that any path in $\mathbb{C}p$ joining $\alpha(rw_k)(p), k = 1, 2$, must meet $\Sigma(\bar{L})$ for some $r = r_0$. Thus we may as well assume that $p \in \Sigma(\bar{L})$. Suppose that $||p_1|| = 1, ||p_2|| < 1$. For $r > 0$ sufficiently large, $||\alpha_1(rw_3).p_1|| < 1$ and $||\alpha_2(rw_3).p_2|| > 1$. Therefore there must exist an r_1 such that setting $q'_i := \alpha_i(r_1 w_3).p_i$, we have $||q'_1|| \leqslant 1$ and $||q'_2|| = 1$. Then $q' = (q'_1, q'_2) \in \mathbb{C}p \cap \Sigma(\bar{L})$ and $q'' := p$ meet our requirements.

If $||p_1|| < 1, ||p_2|| = 1$, we set $q' := p$ and find a $q'' \in \mathbb{C}p \cap \Sigma(\bar{L})$ by the same argument using w_4 in the place of w_3. □

Lemma 2.11. — *Every $\mathbb{C}_\lambda$-orbit $\mathbb{C}p$, $p \in S(L)$, meets $S(L)$ transversally.*

Proof. — Denote by $\pi : L \longrightarrow S(L)$ the projection of the principal $(\mathbb{C}^*/\mathbb{S}^1)^2 \cong \mathbb{R}_+^2$-bundle. Evidently, the inclusion $j : S(L) \hookrightarrow L$ is a cross-section and so $L \cong S(L) \times \mathbb{R}_+^2$. The second projection $\nu : L \longrightarrow \mathbb{R}_+^2$ is just the map $L \ni p = (p_1, p_2) \mapsto (\nu_1(p), \nu_2(p))$ where $\nu_i(p) = ||p_i|| \in \mathbb{R}_+$. One has therefore an isomorphism $\mathcal{T}_p L|_{S(L)} \cong \mathcal{T}_p S(L) \oplus \mathbb{R}^2$, and the corresponding second projection map $\mathcal{T}_p L \longrightarrow \mathbb{R}^2$ is the differential of ν. Therefore $\mathbb{C}p$ is *not* transverse to $S(L)$ if and only if $av_\lambda(p) \in \mathcal{T}_p S(L)$ for some complex number $a \neq 0$; equivalently, if and only if $d\nu_i(av_\lambda(p)) = 0, i = 1, 2$, for some $a \neq 0$.

By Remark 2.8 we have

$$d\nu_i(av_\lambda(p)) = \sum_{1 \leqslant j \leqslant n_1} d\nu_i(a\lambda_j v_j(p)) = \sum_{1 \leqslant j \leqslant n_1} \mathrm{Re}(a\lambda_j)\nu'_{j,p_1}(1).$$

Similarly,

$$av_\lambda(p)(\nu_2) = \sum_{n_1 < j \leqslant N} \mathrm{Re}(a\lambda_j)\nu'_{j,p_2}(1).$$

Therefore, $\mathbb{C}p$ is not transverse to $S(L)$ if and only if $\sum_{1\leqslant j\leqslant n_1}\mathrm{Re}(a\lambda_j)r_j = 0 = \sum_{n_1<j\leqslant N}\mathrm{Re}(a\lambda_j)s_j$ for some complex number $a\neq 0$ and reals $r_j, s_k \geqslant 0$ (not all zero). This means that $\sqrt{-1}\mathbb{R}\subset aC_1^\circ\cap aC_2^\circ$ and hence $C_1^\circ\cap C_2^\circ\neq\emptyset$, contradicting the weak hyperbolicity condition. □

Proof of Theorem 2.9: We shall first show that $L/\mathbb{C}$ is Hausdorff by showing that $\pi_\lambda : L \longrightarrow S(L)$ which sends $p\in L$ to the unique point in $\mathbb{C}p\cap S(L)$ is continuous.

Let (p_n) be a sequence in L that converges to a point $p_0\in L$. Let $q_n := \pi_\lambda(p_n)\in S(L)$ and choose $z_n\in\mathbb{C}$ such that $\alpha(z_n).p_n = q_n$. Since $||p_n||, ||q_n||, n\geqslant 1$, are bounded, it follows by an argument similar to the proof of Lemma 2.6 that (z_n) is bounded, and, passing to a subsequence if necessary, we may assume that it converges to a $z_0\in\mathbb{C}$. By the continuity of $\mathbb{C}$-action, $\alpha(z_m).p_n\to\alpha(z_0).p_0$ as $m,n\to\infty$. Therefore $\alpha(z_n).p_n = q_n\to\alpha(z_0).p_0$ and $\pi_\lambda(p_0) = q_0$ and so π_λ is continuous and that the restriction of π_λ to $S(L)$ is a homeomorphism whose inverse is the composition $S(L)\hookrightarrow L\longrightarrow L/\mathbb{C}$.

By what has just been shown, $L/\mathbb{C}$ is in fact a Hausdorff manifold and that $\pi_\lambda|_{S(L)}$ is a diffeomorphism. The orbit space $L/\mathbb{C}$ has a natural structure of a complex analytic space with respect to which the projection $L\longrightarrow L/\mathbb{C}$ is analytic. Using Lemma 2.11 we see that $L\longrightarrow L/\mathbb{C}$ is a submersion. It follows that L is the total space of a complex analytic principal bundle with fibre and structure group $\mathbb{C}$. The last statement of the theorem follows from Lemmata 2.10 and 2.11. □

Remark 2.12. — *(i) When X_1 is a point, one has $X\cong X_2, L\cong\mathbb{C}^*\times L_2$. In this case, the orbit space $L/\mathbb{C}$ is readily identified with $L_2/\mathbb{Z}$ where the $\mathbb{Z}$ action is generated by $v\mapsto\prod_{2\leqslant j\leqslant N}\exp(2\pi\sqrt{-1}\lambda_j/\lambda_1)\epsilon_j.v$ where $v\in L_2$. The projection $L_2\longrightarrow S_\lambda(L)$ is a covering projection with deck transformation group $\mathbb{Z}$.*
(ii) When λ is of scalar type, the projection $L\longrightarrow X$ factors through $S_\lambda(L)$ and yields a complex analytic bundle $S_\lambda(L)\longrightarrow X$ with fibre and structure group the elliptic curve $(\mathbb{C}^*\times\mathbb{C}^*)/\mathbb{C}$. When endowed with diagonal type complex structure the projection $S_\lambda(L)\longrightarrow X$ of the principal $\mathbb{S}^1\times\mathbb{S}^1$-bundle, which is smooth, is not complex analytic in general. (Cf. Theorem 4.8.)
(iii) When the X_i do not admit any non-trivial T_i- action, we obtain only scalar type complex structures on $S(L)$. For example, this happens when the X_i are compact Riemann surfaces of genus at least 2.
(iv) Let $X_1 = G_k(\mathbb{C}^n)$, $X_2 = G_l(\mathbb{C}^m)$. We start with the example 2.5(iv)

of standard actions of T_i, corresponding to matrices of exponents A_i, constructed on L_i where $\bar{L}_i$ are the negative ample generators of $Pic(X_i)$. For any admissible $\mathbb{C}_\lambda \subset T_1 \times T_2$ we obtain a complex structure of diagonal type on $S(L)$.

(v) Let $r_i, i = 1, 2$ be positive integers. Suppose that $p_i : (\mathbb{C}^*)^{n_i} \cong \widetilde{T}_i \longrightarrow T_i \cong (\mathbb{C}^*)^{n_i}$ is the covering projection defined as $\prod t_j \epsilon_j \mapsto \prod t_j^{r_i} \epsilon_j$ where $r_i \geqslant 1$. Then a standard T_i-action on L_i induces a standard $\widetilde{T}_i$-action. If $\alpha_\lambda : \mathbb{C} \longrightarrow T = T_1 \times T_2$ determines an admissible diagonal type action on $L = L_1 \times L_2$, then the lift $\widetilde{\alpha}_\lambda : \mathbb{C} \longrightarrow \widetilde{T}$ also determines an admissible diagonal type action $\alpha_{\widetilde{\lambda}}$ where $\widetilde{\lambda}_j = (1/r_1)\lambda_j, 1 \leqslant j \leqslant n_1$, and $\widetilde{\lambda}_j = (1/r_2)\lambda_j, n_1 < j \leqslant N$. Indeed the resulting $\mathbb{C}$ action is the 'same' and so $S_\lambda(L) = S_{\widetilde{\lambda}}(L)$. In particular, if $p'_i : \widetilde{T}_i \longrightarrow T'_i, i = 1, 2,$ is another pair of such coverings and if $\alpha_\lambda : \mathbb{C} \longrightarrow T$ and $\alpha_{\lambda'} : \mathbb{C} \longrightarrow T'$ define admissible diagonal type actions on L such that $\alpha_{\widetilde{\lambda}} : \mathbb{C} \longrightarrow \widetilde{T}$ is a common lift of both α_λ and $\alpha_{\lambda'}$, then $S_\lambda = S_{\widetilde{\lambda}} = S_{\lambda'}$.

We conclude this section with the following observation.

Theorem 2.13. — *Suppose that $H^1(X_1; \mathbb{R}) = 0$ and that $c_1(\bar{L}_1) \in H^2(X_1; \mathbb{R})$ is non-zero. Then $S(L)$ is not symplectic and hence non-Kähler with respect to* any *complex structure.*

Proof. — In the Leray-Serre spectral sequence over $\mathbb{R}$ for the $\mathbb{S}^1$-bundle with projection $q : S(L_1) \longrightarrow X$ the differential $d : E_2^{0,1} \cong H^1(\mathbb{S}^1; \mathbb{R}) \cong \mathbb{R} \longrightarrow E_2^{2,0} = H^2(X_1; \mathbb{R})$ is non-zero. It follows that $E_3^{0,1} = E_\infty^{0,1} = 0$. Since $H^1(X_1; \mathbb{R}) = 0$, we see that $H^1(S(L_1); \mathbb{R}) = 0$. Hence, by the Künneth formula, $H^2(S(L); \mathbb{R}) = H^2(S(L_1); \mathbb{R}) \oplus H^2(S(L_2); \mathbb{R})$.

Let $u_i \in H^2(S(L_i); \mathbb{R}), i = 1, 2,$ be arbitrary. Since $\dim S(L_i)$ is odd for $i = 1, 2$, $u_1^r u_2^s = 0$ for any $r, s \geqslant 0$ such that $r + s = n$, where $2n := \dim_{\mathbb{R}} S(L)$. Hence $\omega^n = 0$ for any $\omega \in H^2(S(L); \mathbb{R})$. □

3. Complex structures of linear type

Let $X_i = G_i/P_i, i = 1, 2,$ where the G_i are simply-connected complex simple linear algebraic groups, P_i any maximal parabolic subgroup, and $\bar{L}_i$ the negative ample generator of the Picard group of X_i. We endow $\bar{L}_i$ with a hermitian metric invariant under a suitable maximal compact subgroup $H_i \subset G_i$. Let $L = L_1 \times L_2$ and let $S(L)$ be product $S(L_1) \times S(L_2)$ where $S(L_i) \subset L_i$ is the unit circle bundle over $X_i, i = 1, 2$. It can been seen that $S(L_i)$ is simply-connected. Indeed it is a homogeneous space H_i/Q_i

where Q_i is connected and is the semi simple part of the centralizer of a circle subgroup contained in H_i (see [21], [22]). By a classical result of H.-C. Wang [23], it follows that $S(L)$ admits complex structures invariant under the action of $H_1 \times H_2$. The complex structures considered by Wang are the same as those of scalar type considered in §2. The $H := H_1 \times H_2$-action does not preserve the complex analytic structure when $S(L)$ is endowed with the more general diagonal type complex structures.

In this section we shall construct a complex structure on $S(L)$ which will be referred to as *linear type*. Our construction will be more general in that *we assume only that G_i is any simply connected semi simple Lie group and $P_i \subset G_i$ any parabolic subgroup.*

The first step towards construction of linear type complex structure on $S(L)$ is to produce a standard action of a torus $T_i' \subset G_i$ on $L_i \longrightarrow G_i/P_i$. The following consideration shows that there can be no such action for any torus of the semi simple group G_i.

Suppose that $\bar{L} \longrightarrow G/P$ is a G-equivariant line bundle where G is a simply-connected semi simple complex linear algebraic group and P *any* parabolic subgroup of G. We assume that G action on G/P is almost effective, as otherwise $G/P = G'/P'$ where G' is proper factor of G and $P' = P \cap G'$. (Almost effective action means that the subgroup that fixes every element of G/P is finite.) Now the subgroup of G which fixes every element of G/P is readily seen to be equal to the centre $Z(G)$ of G. Let T' be any torus of G. We claim that the T'-action on L is *not* d-standard for any $d \geqslant 1$ (with respect to any isomorphism $T' \cong (\mathbb{C}^*)^k$, where $k \leqslant l = rank(G)$). If the T'-action were d-standard, then T' would contain a subgroup $\Delta \cong \mathbb{C}^*$ whose restricted action is as described in Definition 2.1(i). Since the G-action commutes with that of the structure group $\mathbb{C}^*$ of $\bar{L}$, it follows that $z.g(v) = g.z(v)$ for all $v \in \bar{L}, z \in \Delta, g \in G$. Since the G-action on L is almost effective, we see $g^{-1}zg = \zeta z$ where $\zeta \in Z(G)$, the centre of G, which is a finite group. This implies that $\Delta/(\Delta \cap Z(G))$ is contained in the centre of $G/Z(G)$ contradicting our hypothesis that G is semi simple.

We shall show in Proposition 3.1 that, when $\bar{L}$ is a line bundle associated to a negative dominant integral weight, it is possible to extend the G action on $\bar{L}$ and on G/P to a larger group $\widetilde{G}$ which is reductive such that the bundle $\bar{L} \to G/P$ is $\widetilde{G}$-equivariant and the action of a maximal torus $\widetilde{T}$ of $\widetilde{G}$ on L is d-standard for a suitable $d \geqslant 1$.

In order to construct linear type complex structure on $S(L)$, we need to assume that $\bar{L}_i, i = 1, 2$, is a negative ample line bundle over G_i/P_i.

This assumption allows us to view $\bar{L}_i$ as the restriction of the tautological bundle over a projective space $\mathbb{P}^{N_i}$ to G_i/P_i via an imbedding $G_i/P_i \hookrightarrow \mathbb{P}^{N_i}$ defined by the very ample line bundle $\bar{L}^\vee$. As this fact will be exploited in our construction of linear type complex structure, it fails when line bundle $\bar{L}_i$ is not negative ample.

We briefly recall some basic facts and notions about the representation theory of G, referring the reader to [10] for details.

Let G be a semi simple, simply-connected complex linear algebraic group. Let T be a maximal torus and let B be a Borel subgroup B containing T. Let $l = \dim T$ be the rank of G. Denote by $R(G)$—or more briefly R—the set of roots, by R^+ the positive roots, by Λ the weight lattice and by $Q \subset \Lambda$ the root lattice determined by $T \subset B \subset G$. We shall denote the set of coroots by $R^\vee$. Since G is assumed to be simply connected, $\Lambda = \chi(T)$, the group of characters $T \longrightarrow \mathbb{C}^*$ of T. Since $B = T.B_u$, B_u being the unipotent, every character of T extends uniquely to a (algebraic) character of B and we have $\chi(T) = \chi(B)$. Let $\Phi^+ \subset R^+$ denote the set of simple positive roots and let $\Lambda^+ \subset \Lambda$ denote the dominant (integral) weights. We shall denote by W the Weyl group of G with respect to T. It is generated by the set S of the fundamental reflections $s_\alpha, \alpha \in \Phi^+$. (W, S) is a finite Coxeter group whose longest element will be denoted w_0.

For $\omega \in \Lambda^+$, $V(\omega)$ denotes the finite dimensional irreducible highest weight G-module with highest weight ω. Also, for any $\omega \in \Lambda$, one has a G-equivariant line bundle $\bar{L}_\omega \longrightarrow G/B$ whose total space is $G \times_B \mathbb{C}_{-\omega}$ where $\mathbb{C}_{-\omega}$ is the 1-dimensional B-module with character $-\omega : B \longrightarrow \mathbb{C}^*$. If ω is dominant, then $H^0(G/B, L_\omega)^\vee = V(\omega)$ as G-module. If $u_\omega \in V(\omega)$ is a highest weight vector, then P_ω, the subgroup of G which stabilizes the 1-dimensional vector space $\mathbb{C}u_\omega$ is a parabolic subgroup that contains B and $\bar{L}_\omega$ is isomorphic to the pull-back of a line bundle, again denoted $\bar{L}_\omega$ over G/P where P is any parabolic subgroup such that $B \subset P \subset P_\omega$. Every parabolic subgroup that contains B arises as P_ω for some $\omega \in \Lambda^+$. Moreover, $\bar{L}_\omega \longrightarrow G/P_\omega$ is (very) ample. If ω is a positive multiple of a fundamental weight ϖ, then P_ω is a maximal parabolic which corresponds to 'omitting' ϖ.

Let $\omega \in \Lambda^+$ and let $\Lambda(\omega) \subset \Lambda$ denote the set of all weights of $V(\omega)$. If $\mu \in \Lambda(\omega)$, we denote the multiplicity of μ in $V(\omega)$ by m_μ; thus $m_\mu = \dim V_\mu(\omega)$, where $V_\mu(\omega)$ is the μ-weight space $\{v \in V(\omega) \mid t.v = \mu(t)v \ \forall t \in T\}$. The set $\Lambda(\omega)$ is stable under the action of W. We put a hermitian inner product on $V(\omega)$ with respect to which the decomposition $V(\omega) = \oplus_{\mu \in \Lambda(\omega)} V_\mu(\omega)$ is orthogonal. Such an hermitian product is invariant under the compact torus $K \subset T$. Indeed, without loss of generality we may assume that the inner product is invariant under a maximal compact subgroup of G that contains K.

Let $\varpi_1, \ldots, \varpi_l$ be the fundamental weights. Consider the homomorphism $\psi : T \longrightarrow (\mathbb{C}^*)^l$ of algebraic groups defined as $t \mapsto (\varpi_1(t), \ldots, \varpi_l(t))$. It is an isomorphism since $\varpi_1, \ldots, \varpi_l$ is a $\mathbb{Z}$-basis for $\chi(T)$. We shall identify T with $(\mathbb{C}^*)^l$ via ψ. Let $\omega \in \Lambda^+$. It is not difficult to see that the T-action on $V(\omega) \setminus 0 \longrightarrow P(V(\omega))$ is *not* standard since $w_0(\omega) \in \Lambda(\omega)$ is *negative* dominant, i.e., $-w_0(\omega) \in \Lambda^+$. Write $\mu = \sum_{1 \leqslant j \leqslant l} a_{\mu,j} \varpi_j$ for $\mu \in \Lambda(\omega)$ so that $\mu(t) = \prod_{1 \leqslant j \leqslant l} t_j^{a_{\mu,j}}$ where $t = (t_1, \ldots, t_l) \in T$. If $v \in V_\mu(\omega)$, then $t.v = \prod t_j^{a_{\mu,j}}.v$. Set $d' := 1 + \sum |a_{\mu,j}|$ where the sum is over $\mu \in \Lambda(\omega), 1 \leqslant j \leqslant l$. The group $T' := T \times \mathbb{C}^*$ acts on $V(\omega)$ where the last factor acts via the covering projection $\mathbb{C}^* \longrightarrow \mathbb{C}^*$, $z \mapsto z^{-d'}$, where the target $\mathbb{C}^*$ acts as scalar multiplication. Thus $(t, z).v = \mu(t) z^{-d'} v$ where $v \in V_\mu(\omega), (t, z) \in T'$. Now consider the $(l+1)$-fold covering projection $\widetilde{T} := (\mathbb{C}^*)^{l+1} \longrightarrow T'$, defined as $(t_1, \ldots, t_{l+1}) \mapsto (t_{l+1}^{-1} t_1, \ldots, t_{l+1}^{-1} t_l, \prod_{1 \leqslant j \leqslant l+1} t_j^{-1})$. The torus $\widetilde{T}$ acts on the principal $\mathbb{C}^*$-bundle $V(\omega) \setminus \{0\} \longrightarrow \mathbb{P}(V(\omega))$ via the above surjection.

Denote by $\widetilde{\epsilon}_j : \mathbb{C}^* \longrightarrow \widetilde{T}$ the jth coordinate imbedding. For any $\mu \in \Lambda(\omega)$, and any $v \in V_\mu(\omega)$, we have $z\widetilde{\epsilon}_{l+1}.v = z^{d'} \prod_{1 \leqslant j \leqslant l} z^{-a_{\mu,j}} v = z^{d' - \sum a_{\mu,j}} v$, and, when $j \leqslant l$, we have $z\widetilde{\epsilon}_j.v = z^{d' + a_{\mu,j}} v$. Also, if $z = (z_0, \ldots, z_0) \in \widetilde{T}$, then $z.v = z_0^{(l+1)d'} v$. Observe that the exponent of z that occurs in the above formula for $z\widetilde{\epsilon}_j.v$ is positive for $1 \leqslant j \leqslant l+1$ by our choice of d'. We shall denote this exponent by $d_{\mu,j}$, that is,

$$d_{\mu,j} = \begin{cases} d' + a_{\mu,j}, & 1 \leqslant j \leqslant l, \\ d' - \sum_{1 \leqslant i \leqslant l} a_{\mu,i}, & j = l+1, \end{cases} \tag{3}$$

where $\mu = \sum_{1 \leqslant j \leqslant l} a_{\mu,j} \varpi_j \in \Lambda(\omega)$.

Next note that the compact torus $\widetilde{K} := K \times \mathbb{S}^1 \subset T \times \mathbb{C}^*$ preserves the hermitian product on $V(\omega)$ and hence the (induced) hermitian metric on the tautological line bundle over $\mathbb{P}(V(\omega))$. From the explicit description of the action just given, it is clear that conditions (i) and (ii) of Definition 2.1 hold. Thus we have extended the T-action to an action of $\widetilde{T}$-action which is standard. We are ready to prove

Proposition 3.1. — *We keep the above notations. Let $\omega \in \Lambda^+$ be any dominant weight of G and let P be any parabolic subgroup such that $B \subset P \subset P_\omega$. Then the T-action can be extended to a d-standard action of $\widetilde{T} := T \times \mathbb{C}^*$ on $L_{-\omega} \longrightarrow G/P$ where $d = d'(l+1)$.*

Proof. — First assume that $P = P_\omega$. Since $\bar{L}_\omega$ is a very ample line bundle over G/P_ω, one has a G-equivariant embedding $G/P_\omega \longrightarrow \mathbb{P}(V(\omega))$ where $V(\omega) = H^0(G/P_\omega, \bar{L}_\omega)^\vee$. By our discussion above, the T-action on the tautological bundle over the projective space $\mathbb{P}(V(\omega))$ has been extended to a d-standard action of $\widetilde{T}$ for an appropriate $d > 1$. The tautological bundle over $\mathbb{P}(V(\omega))$ restricts to $L_{-\omega}$ on G/P_ω. Clearly the $L_{-\omega}$ is $\widetilde{T}$-invariant. Put any $\widetilde{K}$-invariant hermitian metric on $V(\omega)$ where $\widetilde{K}$ denotes the maximal compact subgroup of $\widetilde{T}$. As observed above, $z\widetilde{\epsilon}_j.v = z^{d_{\mu,j}}v$ where $d_{\mu,j} > 0$ for $v \in V_\mu(\omega)$, it follows that condition (ii) of Definition 2.1 holds. Therefore the $\widetilde{T}$-action on $L_{-\omega}$ is d-standard.

Now let P be any parabolic subgroup as in the proposition. One has a $\widetilde{T}$-equivariant morphism $G/P \longrightarrow G/P_\omega$ under which $\bar{L}_{-\omega} \longrightarrow G/P_\omega$ pulls back to $\bar{L}_{-\omega} \longrightarrow G/P$. In view of Remark 2.5(ii), it follows that the $\widetilde{T}$-action on $L_{-\omega} \longrightarrow G/P$ is d-standard. □

Let $\pi : \widetilde{G} = G \times \mathbb{C}^* \longrightarrow G \times \mathbb{C}^*$ be the $(l+1)$-fold covering obtained from the $(l+1)$-fold covering of the last factor and identity on the first. The maximal torus $\pi^{-1}(T \times \mathbb{C}^*)$ of $\widetilde{G}$ can be identified with $\widetilde{T}$. With respect to an appropriate choice of identification $\widetilde{T} \cong (\mathbb{C}^*)^{l+1}$, we see that the action of G on $L_{-\omega} \longrightarrow G/P_\omega$ extends to $\widetilde{G}$ in such a manner that the $\widetilde{T}$-action is d-standard where $d = (l+1)d'$ as above. Since $G/P_\omega = \widetilde{G}/\widetilde{P}_\omega$, where $\widetilde{P}_\omega = \pi^{-1}(P_\omega \times \mathbb{C}^*)$, the $\mathbb{C}^*$-bundle $L_{-\omega} \longrightarrow \widetilde{G}/\widetilde{P}_\omega$ is $\widetilde{G}$-equivariant. The parabolic subgroup $\widetilde{P}_\omega$ contains the Borel subgroup $\widetilde{B} := \pi^{-1}(B \times \mathbb{C}^*)$. We shall refer to $\bar{L}_{-\omega} \longrightarrow \widetilde{G}/\widetilde{P}_\omega$ as a *d-standard $\widetilde{G}$-homogeneous line bundle.*

Let $i = 1, 2$. We shall write L_i, P_i to abbreviate $L_{-\omega_i}, P_{\omega_i}$, etc. Note that $\bar{L}_i$ is a negative ample line over $X_i := G_i/P_i = \widetilde{G}_i/\widetilde{P}_i$ and is d_i-standard $\widetilde{G}_i$-homogeneous. Let $\widetilde{G} = \widetilde{G}_1 \times \widetilde{G}_2$, $(\mathbb{C}^*)^N \cong \widetilde{T} = \widetilde{T}_1 \times \widetilde{T}_2$ where $N = rank(\widetilde{G}) = n_1 + n_2$ with $n_i := l_i + 1$ and $\widetilde{B} = \widetilde{B}_1 \times \widetilde{B}_2$.

Let $\lambda \in Lie(\widetilde{B})$ and let $\lambda = \lambda_s + \lambda_u$ be its Jordan decomposition, where $\lambda_s = (\lambda_1, \ldots, \lambda_N) \in \mathbb{C}^N = Lie(\widetilde{T})$ satisfies the weak hyperbolicity condition (1) of type (n_1, n_2) and $\lambda_u \in Lie(\widetilde{B}_u)$, the Lie algebra of the unipotent radical $\widetilde{B}_u$ of $\widetilde{B}$. Thus $[\lambda_u, \lambda_s] = 0$ in $Lie(\widetilde{B})$. The analytic imbedding $\alpha_\lambda : \mathbb{C} \longrightarrow \widetilde{B}$ where $\alpha_\lambda(z) = \exp(z\lambda) = \exp(z\lambda_s).\exp(z\lambda_u)$ defines an action, again denoted α_λ, of $\mathbb{C}$ on $L := L_1 \times L_2$ and an action $\widetilde{\alpha}_\lambda$ on $V(\omega_1) \times V(\omega_2)$. Denote by $\mathbb{C}_\lambda$ the image $\alpha_\lambda(\mathbb{C}) \subset \widetilde{B}$. We shall now give an explicit description of these actions. Let $v_i \in V(\omega_i)$ and write $v_i = \sum_{\mu \in \Lambda(\omega_i)} v_\mu$ where $v_\mu \in V_\mu(\omega_i)$. Set

$$\lambda_\mu := \sum \lambda_j d_{\mu,j} \tag{4}$$

where the sum ranges over $n_{i-1} < j \leqslant n_{i-1} + n_i$ with $n_0 = 0$. Then $\widetilde{\alpha}_{\lambda_s}(z)(v_1, v_2) = (u_1, u_2)$ where

$$u_i = \sum_{\mu\in\Lambda(\omega_i)} \prod_j \exp(z\lambda_j)\widetilde{\epsilon}_j.v_\mu = \sum_\mu \prod_j (\exp(z\lambda_j d_{\mu,j})v_\mu = \sum_\mu \exp(z\lambda_\mu)v_\mu. \tag{5}$$

where the product is over j such that $n_{i-1} < j \leqslant n_{i-1} + n_i$.

The $\mathbb{C}$-action α_{λ_s} on L is just the restriction to $L \subset V(\omega_1) \times V(\omega_2)$ of the $\mathbb{C}$-action $\widetilde{\alpha}_{\lambda_s}$. Since the λ_μ are all *positive* linear combination of the λ_j, the action of $\mathbb{C}$ on $V(\omega_1, \omega_2) := (V(\omega_1) \setminus \{0\}) \times (V(\omega_2) \setminus \{0\})$, the total space of the product of tautological bundles, is admissible.

Fix a basis for $V(\omega_i)$ consisting of weight vectors so that $GL(V(\omega_i))$ is identified with invertible $r_i \times r_i$-matrices, where $r_i := \dim V(\omega_i)$. Note that the action of the diagonal subgroup of $GL(V(\omega_i))$ on $V(\omega_i) \setminus \{0\}$ is standard and that $\widetilde{T}$ is mapped into D, the diagonal subgroup of $GL(V(\omega_1)) \times GL(V(\omega_2))$. We put a hermitian metric on $V(\omega_1) \times V(\omega_2)$ which is invariant under the compact torus $(\mathbb{S}^1)^{r_1+r_2} \subset D$. Considered as a subgroup of $GL(V(\omega_1)) \times GL(V(\omega_2))$, the $\mathbb{C}$-action $\widetilde{\alpha}_{\lambda_s}$ on $V(\omega_1, \omega_2)$ is the *same* as that considered by Loeb-Nicolau corresponding to $\lambda_s(\omega_1, \omega_2) := (\lambda_\mu, \lambda_\nu)_{\mu\in\Lambda(\omega_1),\nu\in\Lambda(\omega_2)} \in Lie(D) = \mathbb{C}^{r_1} \times \mathbb{C}^{r_2}$, where it is understood that each λ_μ occurs as many times as $\dim V_\mu(\omega_1), \mu \in \Lambda(\omega_1)$, and similarly for $\lambda_\nu, \nu \in \Lambda(\omega_2)$.

Observation: The $\lambda_s(\omega_1, \omega_2)$ satisfy the weak hyperbolicity condition of type (r_1, r_2) since the λ_μ are *positive* integral linear combinations of the λ_j.

The differential of the Lie group homomorphism $\widetilde{G}_1 \times \widetilde{G}_2 \longrightarrow GL(V(\omega_1)) \times GL(V(\omega_2))$ maps λ_s to the diagonal matrix $diag(\lambda_s(\omega_1, \omega_2))$ and λ_u to a nilpotent matrix $\lambda_u(\omega_1, \omega_2)$ which commutes with $\lambda_s(\omega_1, \omega_2)$. Indeed $\lambda(\omega_1, \omega_2) := \lambda_s(\omega_1, \omega_2) + \lambda_u(\omega_1, \omega_2)$ has a block decomposition compatible with weight-decomposition of $V(\omega_1) \times V(\omega_2)$ where the μ-th block is $\lambda_\mu I_{m(\mu)} + A_\mu$, where A_μ is nilpotent and $I_{m(\mu)}$ is the identity matrix of size $m(\mu)$, the multiplicity of $\mu \in \Lambda(\omega_i), i = 1, 2$.

Recall that, for the $\mathbb{C}$-action $\widetilde{\alpha}_\lambda$ on $V(\omega_1, \omega_2)$, the orbit space $V(\omega_1, \omega_2)/\mathbb{C} =: S_\lambda(\omega_1, \omega_2)$ is a complex manifold diffeomorphic to the product of spheres $\mathbb{S}^{2r_1-1} \times \mathbb{S}^{2r_2-1}$ by [14, Theorem 1]. Indeed, the canonical projection $V(\omega_1, \omega_2) \longrightarrow S_\lambda(\omega_1, \omega_2)$ is the projection of a holomorphic principal bundle with fibre and structure group $\mathbb{C}$.

Theorem 3.2. — *We keep the above notations. Let $\bar{L}_i = \bar{L}_{-\omega_i}$ be a d_i-standard $\widetilde{G}_i$-homogeneous line bundle over $X_i = \widetilde{G}_i/\widetilde{P}_i, P_i = P_{\omega_i}$ and let $L = L_1 \times L_2$. Suppose that $\lambda = \lambda_s + \lambda_u \in Lie(\widetilde{B})$ where $\lambda_s \in Lie(\widetilde{T}) =$*

$\mathbb{C}^N$ satisfies the weak hyperbolicity condition of type (n_1, n_2). (See Equation (1), §2.)
(i) The orbit space, denoted $L/\mathbb{C}_\lambda$, of the $\mathbb{C}$-action on L defined by λ is a Hausdorff complex manifold and the canonical projection $L \longrightarrow L/\mathbb{C}_\lambda$ is the projection of a principal $\mathbb{C}$-bundle. Furthermore, $L/\mathbb{C}_\lambda$ is analytically isomorphic to $L/\mathbb{C}_{\lambda_\varepsilon}$ where $\lambda_\varepsilon := Ad(t_\varepsilon)(\lambda)$ and $t_\varepsilon \in \widetilde{T}$ is such that $\gamma(t_\varepsilon) = \varepsilon$ for all $\gamma \in \Phi^+$.
(ii) If $|\varepsilon|$ is sufficiently small, then each orbit of $\mathbb{C}_{\lambda_\varepsilon}$ on L meets $S(L)$ transversally at a unique point. In particular, the restriction of the projection $L \longrightarrow L/\mathbb{C}_{\lambda_\varepsilon}$ to $S(L) \subset L$ is a diffeomorphism.

Proof. — When $\lambda_u = 0$, the theorem is a special case of Theorem 2.9. So assume $\lambda_u \neq 0$.

Since the $\mathbb{C}$-action $\alpha_{\lambda_\varepsilon}$ is conjugate by the analytic automorphism $t_\varepsilon : L \longrightarrow L$ to α_λ, we see that $L/\mathbb{C}_\lambda \cong L/\mathbb{C}_{\lambda_\epsilon}$ as a complex analytic space. Thus, it is enough to prove the theorem for $|\varepsilon| > 0$ sufficiently small.

Consider the projective embedding $\phi'_i : X_i = G_i/P_i \hookrightarrow \mathbb{P}(V(\omega_i))$ defined by the ample line bundle $\bar{L}_i^\vee$. The circle-bundle $S(L_i) \longrightarrow X_i$ is just the restriction to X_i of the circle-bundle associated to the tautological bundle over $\mathbb{P}(V(\omega_i))$. Thus ϕ'_i yields an imbedding $\phi_i : S(L_i) \longrightarrow \mathbb{S}^{2r_i-1}$. Let $\phi : S(L) \longrightarrow S(V(\omega_1, \omega_2)) = \mathbb{S}^{2r_1-1} \times \mathbb{S}^{2r_2-1}$ be the product $\phi_1 \times \phi_2$.

Set $\lambda_{u,\varepsilon} = Ad(t_\varepsilon)\lambda_u$ so that $\lambda_\varepsilon = \lambda_s + \lambda_{u,\varepsilon}$. Note that, if $\beta = \sum_{\gamma \in \Phi^+} k_{\beta,\gamma}\gamma$ where $\beta \in R^+$, then $Ad(t_\varepsilon)X_\beta = \varepsilon^{|\beta|}X_\beta$ where $|\beta| = \sum k_{\beta,\gamma} \geqslant 1$. (Here $X_\beta \in Lie(B_u)$ denotes a weight vector of weight β.) This implies that $\lambda_\varepsilon \to \lambda_s$ as $\varepsilon \to 0$, and, furthermore, $\lambda_\varepsilon(\omega_1, \omega_2) \to \lambda_s(\omega_1, \omega_2)$ as $\varepsilon \to 0$. By [14, Theorem 1], for $|\varepsilon|$ sufficiently small, each $\mathbb{C}$-orbit for the $\widetilde{\alpha}_{\lambda_\varepsilon}$-action on $V(\omega_1, \omega_2)$ is closed and properly imbedded in $L(\omega_1, \omega_2)$ and intersects $\mathbb{S}^{2r_1-1} \times \mathbb{S}^{2r_2-1}$ at a unique point. In particular, each orbit of the $\mathbb{C}$-action corresponding to λ_ε meets $S(L) \subset \mathbb{S}^{2r_1-1} \times \mathbb{S}^{2r_2-1}$ at a *unique* point when $|\varepsilon| > 0$ is sufficiently small.

Consider the map $\pi_{\lambda_\varepsilon} : L \longrightarrow S(L)$ which maps each $\alpha_{\lambda_\varepsilon}$ orbit to the unique point where it meets $S(L)$. This is just the restriction of $V(\omega_1, \omega_2) \longrightarrow \mathbb{S}^{2r_1-1} \times \mathbb{S}^{2r_2-1}$ and hence continuous. It follows that the orbit space $L/\mathbb{C}_{\lambda_\varepsilon}$ is Hausdorff and that the map $\bar{\pi}_{\lambda_\varepsilon} : L/\mathbb{C} \longrightarrow S(L)$ induced by $\pi_{\lambda_\varepsilon}$ is a homeomorphism, whose inverse is just the composition $S(L) \hookrightarrow L \longrightarrow L/\mathbb{C}$. Since each $\mathbb{C}$-orbit for α_{λ_s}-action meets $S(L)$ transversely by Lemma 2.11, and since $S(L)$ is compact, the same is true for the $\alpha_{\lambda_\varepsilon}$-action provided $|\varepsilon|$ is sufficiently small. For such an ε, the $\pi_{\lambda_\varepsilon}$ is a submersion and $\bar{\pi}_{\lambda_\varepsilon}$ is a diffeomorphism. The orbit space $L/\mathbb{C}_{\lambda_\varepsilon}$ has a natural structure of a complex analytic space with respect to which $\pi_{\lambda_\varepsilon}$ is analytic. We have shown above that $L/\mathbb{C}_{\lambda_\varepsilon}$ is a Hausdorff manifold and that $\pi_{\lambda_\varepsilon}$ is a submersion. It follows that $\pi_{\lambda_\varepsilon}$ is the projection of a principal complex analytic bundle with fibre and structure group $\mathbb{C}$. □

Let $P' = P'_1 \times P'_2$ be any parabolic subgroup of $G = G_1 \times G_2$ such that $B \subset P' \subset P$, where $P = P_1 \times P_2$ is an Theorem 3.2. Let $L' = L'_1 \times L'_2$ where $\bar{L}'_i$ is the line bundle over $X'_i := G_i/P'_i$ associated to $-\omega_i$, where ω_i is a dominant integral weight of T_i. Then $L' \longrightarrow X'$ is a $\widetilde{B}$-equivariant line bundle and so one obtains an action of $\mathbb{C}_\lambda$ on L' via restriction for any $\lambda \in Lie(\widetilde{B})$. Moreover, L' is equivariantly isomorphic to the pull back of L via the natural projection $X' \longrightarrow X$ where $L, X = X_1 \times X_2$ are as in Theorem 3.2. In particular, assuming that λ_s satisfies the weak hyperbolicity condition, one has a $\mathbb{C}_\lambda$-equivariant projection $L' \longrightarrow L$. If $\lambda_u = 0$, then, by Proposition 3.1 and Theorem 2.9, $L'/\mathbb{C}_\lambda$ is compact Hausdorff complex manifold and we have the following commuting diagram:

$$\begin{array}{ccc} L' & \longrightarrow & L \\ \downarrow & & \downarrow \\ L'/\mathbb{C}_\lambda & \longrightarrow & L/\mathbb{C}_\lambda. \end{array}$$

However, it is not clear to us whether the orbit space $L'/\mathbb{C}_\lambda$ is a Hausdorff manifold when $\lambda_u \neq 0$.

Definition 3.3. — *With notations as in Theorem 3.2, if λ_s is weakly hyperbolic, we shall refer to the analytic homomorphism $\alpha_\lambda : \mathbb{C} \longrightarrow \widetilde{B}$ as* admissible *and the action α_λ of $\mathbb{C}$ on L as* linear type. *In this case, complex structure on the manifold $S(L)$ induced from $L/\mathbb{C}_{\lambda_\varepsilon} \cong L/\mathbb{C}_\lambda$ will be said to be of* linear type *and the resulting complex manifold will be denoted $S_\lambda(L)$.*

We conclude this section with the following remarks.

Remark 3.4. — *(i) Loeb and Nicolau* [14] *consider more general $\mathbb{C}$-actions on $\mathbb{C}^m \times \mathbb{C}^n$ in which the corresponding vector field is allowed to have higher order resonant terms. In our setup we have only to consider linear actions—the corresponding vector fields can at most have terms corresponding to resonant relations of the form "$\lambda_i = \lambda_j$".*
(ii) One has a commuting diagram

$$\begin{array}{ccc} L & \hookrightarrow & V(\omega_1, \omega_2) \\ \pi_\lambda \downarrow & & \downarrow \pi_{\lambda(\omega_1,\omega_2)} \\ S_\lambda(L) & \hookrightarrow & \mathbb{S}^{2r_1-1} \times \mathbb{S}^{2r_2-1} \end{array}$$

which the horizontal maps are holomorphic and the vertical maps, projections of holomorphic principal $\mathbb{C}$*-bundles.*
(iii) We do not know if any diagonal type action of $\mathbb{C}$ *on* L *(with* $L \longrightarrow X$ *as in Definition 3.3) in the sense of Definition 2.2 is conjugate by an element of* $\widetilde{G}$ *to a linear type action* α_λ *with unipotent part* λ_u *equal to zero.*

4. Cohomology of $S_\lambda(L)$

Let $p : L \longrightarrow X$ be a holomorphic principal $\mathbb{C}^*$-bundle over a compact connected complex manifold. Denote by $\bar{p} : \bar{L} \longrightarrow X$ the associated line bundle (i.e., vector bundle of rank 1). We identify X with the image of the zero cross section in $\bar{L}$ and L with $\bar{L} \setminus X$.

We denote the structure sheaf of a complex analytic space Y by $\mathcal{O}_Y$, or more briefly by $\mathcal{O}$ when Y is clear from the context. Recall that a compact subset $A \subset Y$ is called *Stein compact* if every neighbourhood of A contains a Stein open subset U such that $A \subset U$.

Lemma 4.1. — *Suppose that* $f : \bar{L} \longrightarrow Y$ *is a complex analytic map where* Y *is a Stein space and is Cohen-Macaulay. Suppose that* $f(X) =: A \subset Y$ *is Stein compact and that* $f|_L : L \longrightarrow Y \setminus A$ *is a biholomorphism. Then* $H^q(L; \mathcal{O}_L) = 0$ *if* $0 < q < \dim Y - \dim A - 1$*, or if* $q = \dim L$*. Furthermore,* $H^0(L; \mathcal{O}_L) \cong H^0(Y; \mathcal{O}_Y)$ *and the topological vector spaces* $H^q(L; \mathcal{O}_L)$ *are separated Fréchet-Schwartz spaces for all* q*.*

Proof. — First note that $H^q(Y; \mathcal{O}_Y) = 0$ unless $q = 0$. By [2, Ch. II, Theorem 3.6, Corollary 3.9], the restriction map $H^q(Y; \mathcal{O}) \longrightarrow H^q(Y \setminus A; \mathcal{O})$ is an isomorphism for $0 \leqslant q < \operatorname{depth}_A \mathcal{O}_Y = \dim X - \dim A$ the last equality in view of our hypothesis that Y is Cohen-Macaulay. In view of [2, Ch. I, Theorem 2.19], our hypothesis that $A \subset Y$ is Stein compact implies that $H^i(Y \setminus A; \mathcal{O}) \cong H^i(L, \mathcal{O}_L)$ is separated and Fréchet-Schwartz. As for any open connected complex manifold, $H^{\dim L}(L; \mathcal{O}) = 0$. □

Note that the hypothesis of the above lemma are satisfied in the case when X is a smooth projective variety, $\bar{L}$ a line bundle such that $\bar{L}^\vee$ is very ample, and Y is the cone $\hat{L}$ over X with respect to the projective imbedding determined by $\bar{L}^\vee$ is Cohen-Macaulay at its vertex $a \in \hat{L}$. In this case one says that X is *arithmetically Cohen-Macaulay.* For example, if X is the homogeneous variety G/P where G is a complex linear algebraic group over $\mathbb{C}$ and P a parabolic subgroup and L any negative ample line bundle, then the above properties hold. We will also need the fact that $\hat{L}$ is normal—that is, $X = G/P$ is *projectively normal* with respect to any ample line bundle. See [19] for projective normality and [20] for arithmetic Cohen-Macaulayness, where these results are established for the more general case of Schubert varieties over arbitrary algebraically closed fields. If we assume that $\bar{L}$ itself is very ample, then it is not possible to blow-down X. However, in this case, the following lemma allows one to compute the cohomology groups of L.

Lemma 4.2. — *Let L be any holomorphic principal $\mathbb{C}^*$-bundle over a complex manifold X. Then $L \cong L^\vee$ as complex manifolds. In particular, $H^{p,q}_{\bar\partial}(L) \cong H^{p,q}_{\bar\partial}(L^\vee)$.*

Proof. — Let $\psi : L \longrightarrow L^\vee$ be the map $v \mapsto v^\vee$ where $v^\vee(\lambda v) = \lambda \in \mathbb{C}$. Then ψ is a biholomorphism. □

Suppose that $L_i \longrightarrow X_i, i = 1, 2$, are projections of holomorphic principal $\mathbb{C}^*$-bundles where $\bar{L}_i$ are negatively ample over complex projective manifolds. We assume that X_i are arithmetically Cohen-Macaulay. We shall apply the Künneth formula [6] for cohomology with coefficients in coherent analytic sheaves to obtain some vanishing results for the cohomology groups $H^q(L; \mathcal{O}_L)$.

A coherent analytic sheaf $\mathcal{F}$ on a complex analytic space X is a *Fréchet* sheaf if $\mathcal{F}(U)$ is a Fréchet space for any open set $U \subset X$ and if the restriction homomorphism $\mathcal{F}(U) \longrightarrow \mathcal{F}(V)$ is continuous for any open set $V \subset U$. A Fréchet sheaf $\mathcal{F}$ is said to be *nuclear* if $\mathcal{F}(U)$ is a nuclear space for any open set U in X. A Fréchet sheaf $\mathcal{F}$ is called *normal* if there exists a basis for X which is a Leray cover for $\mathcal{F}$.

If X is a complex manifold, then any coherent analytic sheaf is Fréchet-nuclear and normal. See [6, p. 927].

Let V_1, V_2 are Fréchet spaces. We denote by $V_1 \otimes_\epsilon V_2$ (resp. $V_1 \otimes_\pi V_2$) the inductive (resp. projective) topological tensor product of V_1 and V_2. If V_1 is nuclear, then $V_1 \otimes_\pi V_2 = V_1 \otimes_\epsilon V_2$ The completed inductive topological tensor product tensor product will be denoted $V_1 \hat{\otimes} V_2$. For a detailed exposition on nuclear spaces see [18].

One has the notion of completed tensor product $\mathcal{F} \hat{\otimes} \mathcal{G}$ of coherent analytic Fréchet nuclear sheaves $\mathcal{F}, \mathcal{G}$. For example, the structure sheaf of an analytic space is Fréchet nuclear and $\mathcal{O}_{X \times Y} = pr_X^* \mathcal{O}_X \hat{\otimes} pr_Y^* \mathcal{O}_Y$, where pr_X denotes the projection $X \times Y \longrightarrow X$.

We apply the Künneth formula for Fréchet-nuclear normal sheaves established by A. Cassa [6, Teorema 3] to obtain the following

THEOREM 4.3. — *(i) Suppose that $L = L_1 \times L_2$ where $L_i \longrightarrow X_i$ are $\mathbb{C}^*$-bundles over connected complex compact manifolds X_i of dimension $\dim X_i \geqslant 1$. Suppose that, for $i = 1, 2$, $\mathcal{F}_i$ is a coherent analytic sheaf over L_i such that $H^q(L_i; \mathcal{F}_i), q \geqslant 0$, are Hausdorff. Then*

$$H^q(L; \mathcal{F}_1 \hat{\otimes} \mathcal{F}_2) \cong \sum_{k+l=q} H^k(L_1; \mathcal{F}_1) \hat{\otimes} H^l(L_2; \mathcal{F}_2) \tag{6}$$

for $q \geqslant 0$.
(ii) Assume that X_i are projective manifolds and $\bar{L}_i^\vee$ are very ample line bundles such that the $\hat{L}_i$ are Cohen-Macaulay.Then $H^q(L; \mathcal{O}_L) = 0$ except possibly when $q = 0, \dim X_1, \dim X_2$,
$\dim X_1 + \dim X_2$. Also, $H^0(L; \mathcal{O}_L) \cong H^0(L_1; \mathcal{O}_{L_1}) \hat{\otimes} H^0(L_2; \mathcal{O}_{L_2})$

Proof. — (i) The isomorphism (6) follows from the Künneth formula [6, Theorema 3].
(ii) Let a_i denote the vertex of the cone $\hat{L}_i$ over X_i with respect to the projective imbedding determined by $L_i^\vee$. Then $\hat{L}_i$ is an affine variety and hence Stein. By [2, Corollary 2.21, Chapter I] the cohomology groups $H^q(L_i; \mathcal{O}_{L_i}) = H^q(\hat{L}_i \backslash \{a_i\}; \mathcal{O}_{\hat{L}_i}), q \geqslant 0$, are separated and Fréchet-Schwartz. Since, by hypothesis $\hat{L}_i$ is Cohen-Macaulay at a_i, we have $H^q(L_i, \mathcal{O}_{L_i}) \cong H^q(\hat{L}_i; \mathcal{O}_{\hat{L}_i}) = 0$ if $0 < q < \dim X_i$ by [2, Theorem 3.6, Chapter II]. The rest of the theorem now follows readily from (6). ☐

Remark 4.4. — *(i) We remark that the vanishing of the cohomology groups $H^q(L; \mathcal{O}_L)$ for $0 < q < \min\{\dim X_1, \dim X_2\}$ in Theorem 4.3 (ii) follows from* [2, Ch. I, Theorem 3.6]. *To see this, set $\hat{L} := \hat{L}_1 \times \hat{L}_2 \setminus A$ where A is the closed analytic space $A = \hat{L}_1 \times \{a_2\} \cup \{a_1\} \times \hat{L}_2$. The ideal $\mathcal{I} \subset \mathcal{O}_{\hat{L}}$ of A equals $\mathcal{I}_1.\mathcal{I}_2$ where $\mathcal{I}_1, \mathcal{I}_2$ are the ideals of the components $A_1 := \hat{L}_1 \times \{a_2\}, A_2 := \{a_1\} \times \hat{L}_2$ of A. Then $depth_A \mathcal{O}_L = depth_{\mathcal{I}} \mathcal{O}_{\hat{L}} = \min_j\{depth_{\mathcal{I}_j} \mathcal{O}_{\hat{L}}\} = \min_j\{depth_{a_j} \mathcal{O}_{\hat{L}_j}\} = \min\{\dim X_1 + 1, \dim X_2 + 1\}$. Thus we see that $depth_A \mathcal{O}_{\hat{L}} = \min\{\dim X_1 + 1, \dim X_2 + 1\}$. Therefore $H^q(\hat{L}_1 \times \hat{L}_2; \mathcal{O}_{\hat{L}_1 \times \hat{L}_2}) \cong H^q(L; \mathcal{O}_L)$ if $q < \min\{\dim X_1, \dim X_2\}$ by* [2, Ch. I, Theorem 3.6] *where the isomorphism is induced by the inclusion. Since $\hat{L}_1 \times \hat{L}_2$ is Stein, the cohomology groups $H^q(L; \mathcal{O}_L)$ vanish for $1 \leqslant q < \min\{\dim X_1, \dim X_2\}$.*

(ii) When $\pi_1 : L_1 \longrightarrow X_1$ is an algebraic *$\mathbb{C}^*$-bundle over a smooth projective variety X_i, one has an isomorphism of quasi-coherent* algebraic *sheaves $\pi_{1,*}(\mathcal{O}^{alg}_{L_i}) \cong \oplus_{k \in \mathbb{Z}} \bar{L}_i^k$ of $\mathcal{O}_X$-modules. (By the GAGA principle,*

$H^*_{alg}(X_1;\bar{L}_1^k) \cong H^*(X_1;\bar{L}_1^k)$ for all $k \in \mathbb{Z}$.) Therefore the algebraic cohomology groups $H^q_{alg}(L_i;\mathcal{O}^{alg})$ can be calculated as

$$H^q_{alg}(L_1;\mathcal{O}^{alg}_{L_1}) \cong H^q(X_1;\pi_{1,*}\mathcal{O}_{L_1}) \cong \oplus_{k\in\mathbb{Z}} H^q(X_1;\bar{L}^k).$$

If X_1 is a flag variety and $\bar{L}_1$, negative ample, then it is known that $H^q(X_1;\bar{L}_1^k) = 0$ except when $k > 0$ (resp. $k \leqslant 0$) and $q = \dim X_1$ (resp. $q = 0$). Furthermore, $H^0(X_1;\bar{L}_1^k) \cong H^{\dim X_1}(X_1;\bar{L}_1^{-k})^\vee$ for $k < 0$. Hence $H^q_{alg}(L_1;\mathcal{O}^{alg}) = 0$ unless $q = 0$, or $q = \dim X_1$. Now $H^{\dim X_1}_{alg}(L_1;\mathcal{O}^{alg}) \cong \oplus_{k>0} H^{\dim X_1}(X_1;\bar{L}_1^k)$ and $H^0_{alg}(L_1;\mathcal{O}^{alg}) \cong \oplus_{k\leqslant 0}(X_1;\bar{L}_1^k)$. We do not know the relation between $H^{\dim X_1}(L_1;\mathcal{O})$ and $H^{\dim X_1}_{alg}(L_1;\mathcal{O}^{alg}_{L_1})$.

Suppose that α_λ is an admissible $\mathbb{C}$-action on $L \longrightarrow X$ of scalar type, or diagonal type, or linear type. It is understood that in the case of diagonal type, there is a standard T_i-action on $L_i \longrightarrow X_i, i = 1, 2$, and that, in the case of linear type action, $X_i = G_i/P_i$ and $\bar{L}_i$ is negative ample. Here, and in what follows, the groups $G_i, i = 1, 2$, are semi simple and $P_i \subset G_i$ any parabolic subgroups, unless otherwise explicitly stated.

Denote by γ_λ (or more briefly γ) the holomorphic vector field on L associated to the $\mathbb{C}$-action. Thus the $\mathbb{C}$-action is just the flow associated to γ. We shall denote by $\mathcal{O}^{tr}_\gamma$ the sheaf of germs of local holomorphic functions which are constant along the $\mathbb{C}$-orbits. Thus $\mathcal{O}^{tr}_\gamma$ is isomorphic to $\pi^*_\lambda(\mathcal{O}_{S_\lambda(L)})$. One has an exact sequence of sheaves

$$0 \to \mathcal{O}^{tr}_\gamma \to \mathcal{O}_L \xrightarrow{\gamma} \mathcal{O}_L \to 0. \tag{7}$$

Since the fibre of $\pi_\lambda : L \longrightarrow S_\lambda(L)$ is Stein, we see that $H^q(L;\mathcal{O}^{tr}_\gamma) \cong H^q(S_\lambda(L);\mathcal{O}_{S_\lambda(L)})$ for all q. Thus, the exact sequence (7) leads to the following long exact sequence:

$$\begin{aligned} 0 &\to H^0(S_\lambda(L);\mathcal{O}_{S_\lambda(L)}) \to H^0(L;\mathcal{O}_L) \to H^0(L;\mathcal{O}_L) \\ &\qquad\longrightarrow H^1(S_\lambda(L);\mathcal{O}_{S_\lambda(L)}) \to \cdots \\ &\to H^q(S_\lambda(L);\mathcal{O}_{S_\lambda(L)}) \to H^q(L;\mathcal{O}_L) \to H^q(L;\mathcal{O}_L) \\ &\qquad\longrightarrow H^{q+1}(S_\lambda(L);\mathcal{O}_{S_\lambda(L)}) \to \cdots . \end{aligned} \tag{8}$$

Theorem 4.5. — *With the above notations, suppose that the L_i satisfy the hypotheses of Theorem 4.3(ii) and that α_λ is an admissible $\mathbb{C}$-action on L of scalar or diagonal or linear type. Suppose that $1 \leqslant \dim X_1 \leqslant \dim X_2$. Then $H^q(S_\lambda(L);\mathcal{O}) = 0$ provided $q \notin \{0, 1, \dim X_i, \dim X_i + 1, \dim X_1 + \dim X_2, \dim X_1 + \dim X_2 + 1; i = 1, 2\}$. Moreover one has $\mathbb{C} \subset H^1(S_\lambda(L);\mathcal{O})$, given by the constant functions in $H^0(L;\mathcal{O})$.*

Proof. — The only assertion which remains to be explained is that the constant function 1 is not in the image of $\gamma_* : H^0(L;\mathcal{O}) \longrightarrow H^0(L;\mathcal{O})$. All other assertions follow trivially from the long exact sequence (8) above and Theorem 4.3.

Suppose that $f : L \longrightarrow \mathbb{C}$ is such that $\gamma(f) = 1$. This means that $\frac{d}{dz}|_{z=0}(f \circ \mu_p)(z) = 1$ for all $p \in L, z \in \mathbb{C}$, where $\mu_p : \mathbb{C} \longrightarrow L$ is the map $z \mapsto \alpha_\lambda(z).p = z.p$. Since $\mu_{w.p}(z) = z.(w.p) = (z+w).p = \mu_p(z+w)$, it follows that $\frac{d}{dz}|_{z=w}(f \circ \mu_p) = 1\ \forall w \in \mathbb{C}$. Hence $f \circ \mu_p(z) = z + f(p)$. This means that the complex hypersurface $Z(f) := f^{-1}(0) \subset L$ meets each fibre at exactly one point. It follows that the projection $L \longrightarrow S_\lambda(L)$ restricts to a bijection $Z(f) \longrightarrow S_\lambda(L)$.

In fact, since $\gamma(f) \neq 0$ we see that $Z(f)$ is smooth and since γ_p is tangent to the fibres of the projection $L \longrightarrow S_\lambda(L)$ for all $p \in Z(f)$, we see that the bijective morphism of complex analytic manifolds $Z(f) \longrightarrow S_\lambda(L)$ is an immersion. It follows that $Z(f) \longrightarrow S_\lambda(L)$ is a biholomorphism. Thus $Z(f)$ is a *compact* complex analytic sub manifold of $L \subset \hat{L}$. Since $\hat{L}$ is Stein, this is a contradiction. □

Our next result concerns the Picard group of $S_\lambda(L)$.

Proposition 4.6. — *Let $L_i \longrightarrow X_i$ be as in Theorem 4.3 (ii). Suppose that X_i is simply connected. Then $Pic^0(S_\lambda(L)) \cong \mathbb{C}^l$ for some $l \geqslant 1$.*

Proof. — Since $\bar{L}_i$ is negative ample, $c_1(\bar{L}_i) \in H^2(X_i;\mathbb{Z})$ is a non-torsion element. Clearly $H^1(S_\lambda(L);\mathbb{Z}) = 0$ by a straightforward argument involving the Serre spectral sequence associated to the principal $\mathbb{S}^1 \times \mathbb{S}^1$-bundle with projection $S(L) \longrightarrow X_1 \times X_2$. Using the exact sequence $0 \to \mathbb{Z} \to \mathcal{O} \to \mathcal{O}^* \to 1$ we see that $Pic^0(S_\lambda(L)) \cong H^1(S_\lambda(L);\mathcal{O}) \cong \mathbb{C}^l$. Now $l \geqslant 1$ by Theorem 4.5. □

The above proposition is applicable when $X_i = G_i/P_i$ and $\bar{L}_i$ are negative ample. However, in this case we have the following stronger result.

Theorem 4.7. — *Let $X_i = G_i/P_i$ where G_i is semi simple and P_i is any parabolic subgroup and let $\bar{L}_i \longrightarrow X_i$ be a negative line bundle, $i = 1, 2$. We assume that, when $X_i = \mathbb{P}^1$, the bundle $\bar{L}_i$ is a generator of $Pic(X_i)$. Then $Pic^0(S_\lambda(L)) \cong \mathbb{C}$. If the P_i are maximal parabolic subgroups and the L_i are generators of $Pic(X_i) \cong \mathbb{Z}$, then $Pic(S_\lambda(L)) \cong Pic^0(S_\lambda(L)) \cong \mathbb{C}$.*

Proof. — It is easy to see that $H^1(S(L);\mathbb{Z}) = 0$ and that, when P_i are maximal parabolic subgroups and L_i generators of $Pic(X_i) \cong \mathbb{Z}$, $S(L)$ is 2-connected. If $\dim X_i > 1$ for $i = 1, 2$, then $H^1(L;\mathcal{O}) = 0$ by Theorem 4.3 and so we need only show that $coker(H^0(L;\mathcal{O}) \xrightarrow{\gamma_*} H^0(L;\mathcal{O}))$ is isomorphic to $\mathbb{C}$. In case $\dim X_i = 1$—equivalently $X_i = \mathbb{P}^1$—$\bar{L}_i$ is the tautological bundle by our hypothesis. Thus $L_i = \mathbb{C}^2 \setminus \{0\}$. In this case we need to also show that $\ker(H^1(L;\mathcal{O}) \xrightarrow{\gamma_*} H^1(L;\mathcal{O}))$ is zero. Note that the theorem is known due to Loeb and Nicolau [14, Theorem 2] when both the X_i are projective spaces and the $\bar{L}_i$ are negative ample generators—in particular when both $X_i = \mathbb{P}^1$.

The validity of the theorem for the case when λ is of diagonal type implies its validity in the linear case as well. This is because one has a family $\{L/\mathbb{C}_{\lambda_\varepsilon}\}$ of complex manifolds parametrized by $\varepsilon \in \mathbb{C}$ defined by $\lambda_\varepsilon = \lambda_s + \lambda_{u,\varepsilon}$, where $S_{\lambda_\varepsilon}(L) = L/\mathbb{C}_{\lambda_\varepsilon} \cong L/\mathbb{C}_\lambda$ if $\varepsilon \neq 0$ and $\lambda_0 := \lambda_s$ is of diagonal type. (See §3.) The semi-continuity property ([11, Theorem 6, §4]) for $\dim H^1(S_{\lambda_\varepsilon}(L);\mathcal{O})$ implies that $\dim H^1(S_\lambda(L);\mathcal{O}) \leqslant \dim H^1(S_{\lambda_s}(L);\mathcal{O})$. But Theorem 4.5 says that $\dim H^1(S_\lambda(L);\mathcal{O}) \geqslant 1$ and so equality must hold, if $H^1(S_{\lambda_s}(L);\mathcal{O}) \cong \mathbb{C}$. Therefore we may (and do) assume that the complex structure is of diagonal type.

First we show that $coker(\gamma_* : H^0(L;\mathcal{O}) \longrightarrow H^0(L;\mathcal{O}))$ is 1-dimensional, generated by the constant functions. Consider the commuting diagram where $\widetilde{\gamma}$ is the holomorphic vector field defined by the action of $\mathbb{C}$ given by $\lambda(\omega_1,\omega_2)$ on $V(\omega_1,\omega_2)$. Note that $\widetilde{\gamma}_x = \gamma_x$ if $x \in L$.

$$\begin{array}{ccc} H^0(V(\omega_1,\omega_2);\mathcal{O}) & \xrightarrow{\widetilde{\gamma}_*} & H^0(V(\omega_1,\omega_2);\mathcal{O}) \\ \downarrow & & \downarrow \\ H^0(L;\mathcal{O}) & \xrightarrow{\gamma_*} & H^0(L;\mathcal{O}) \end{array}$$

By Hartog's theorem, $H^0(V(\omega_1,\omega_2);\mathcal{O}) \cong H^0(V(\omega_1) \times V(\omega_2);\mathcal{O})$. Also, since $\hat{L}_i$ is normal at its vertex [19], again by Hartog's theorem, $H^0(L;\mathcal{O}) \cong H^0(\hat{L}_1 \times \hat{L}_2;\mathcal{O})$. Since $\hat{L}_i \subset V(\omega_i)$ are closed sub varieties, it follows that the both vertical arrows, which are induced by the inclusion of L in $V(\omega_1,\omega_2)$, are surjective. From what has been shown in the proof of Theorem 4.5, we know that the constant functions are not in the cokernel of γ_*. So it suffices to show that $coker(\widetilde{\gamma}_*)$ is 1-dimensional. This was established in the course of proof of Theorem 2 of [14]. For the sake of completeness we sketch the proof. We identify $V(\omega_i)$ with $\mathbb{C}^{r_i}$ where $r_i := \dim V(\omega_i)$, by choosing a basis for $V(\omega_i)$ consisting of weight vectors. Let $r = r_1 + r_2$ so that $\mathbb{C}^r \cong V(\omega_1) \times V(\omega_2)$. The problem is reduced to the following: Given a holomorphic function $f : \mathbb{C}^r \longrightarrow \mathbb{C}$ with $f(0) = 0$, solve for a holomorphic function ϕ satisfying the equation

$$\sum_{i \leqslant j} b_j z_j \frac{\partial \phi}{\partial z_j} = f, \tag{9}$$

where we may (and do) assume that $\phi(0) = 0$. In view of the *Observation* made preceding the statement of Theorem 3.2, we need only to consider the case where $(b_j) \in \mathbb{C}^r$ satisfies the weak hyperbolicity condition of type (r_1, r_2). Denote by $z^{\mathbf{m}}$ the monomial $z_1^{m_1} \dots z_r^{m_r}$ where $\mathbf{m} = (m_1, \dots, m_r)$ and by $|\mathbf{m}|$ its degree $\sum_{1 \leqslant j \leqslant r} m_j$. Let $f(z) = \sum_{|\mathbf{m}|>0} a_{\mathbf{m}} z^{\mathbf{m}} \in H^0(\mathbb{C}^r; \mathcal{O})$. Then $\phi(z) = \sum a_{\mathbf{m}}/(b.\mathbf{m}) z^{\mathbf{m}}$ where $b.\mathbf{m} = \sum b_j m_j$ is the unique solution of Equation (9). Note that weak hyperbolicity and the fact that $|\mathbf{m}| > 0$ imply that $b.\mathbf{m} \neq 0$, and, $b.\mathbf{m} \to \infty$ as $\mathbf{m} \to \infty$. Therefore ϕ is a convergent power series and so $\phi \in H^0(\mathbb{C}^r; \mathcal{O})$.

It remains to show that, when $X_1 = \mathbb{P}^1$, $L_1 = \mathbb{C}^2 \setminus \{0\}$, and $\dim X_2 > 1$, the homomorphism $\gamma_* : H^1(L; \mathcal{O}) \longrightarrow H^1(L; \mathcal{O})$ is injective. Let $z_j, 1 \leqslant j \leqslant r$, denote the coordinates of $\mathbb{C}^2 \times V(\omega_2)$ with respect to a basis consisting of $\widetilde{T}$-weight vectors. Since $\dim X_2 > 1$, we have $H^1(L_2, \mathcal{O}) = 0$. Also $H^1(L_1; \mathcal{O}) = H^1(\mathbb{C}^2 \setminus \{0\}; \mathcal{O})$ is the space $\mathcal{A}$ of convergent power series $\sum_{m_1, m_2 < 0} a_{m_1, m_2} z_1^{m_1} z_2^{m_2}$ in z_1^{-1}, z_2^{-1} without constant terms.

By Theorem 4.3, $H^1(L; \mathcal{O}) = \mathcal{A} \hat{\otimes} H^0(L_2; \mathcal{O}) \cong \mathcal{A} \hat{\otimes} H^0(\hat{L}_2; \mathcal{O})$. Let $\mathcal{I} \subset H^0(V(\omega_2); \mathcal{O})$ denote the ideal of functions vanishing on $\hat{L}_2$ so that $H^0(\hat{L}_2; \mathcal{O}) = H^0(V(\omega_2); \mathcal{O})/\mathcal{I}$. One has the commuting diagram

$$\begin{array}{ccccccc} 0 & \to & \mathcal{A}\hat{\otimes}\mathcal{I} & \to & \mathcal{A}\hat{\otimes}H^0(V(\omega_2);\mathcal{O}) & \to & \mathcal{A}\hat{\otimes}H^0(L_2;\mathcal{O}) & \to 0 \\ & & \widetilde{\gamma}_* \downarrow & & \widetilde{\gamma}_* \downarrow & & \gamma_* \downarrow & \\ 0 & \to & \mathcal{A}\hat{\otimes}\mathcal{I} & \to & \mathcal{A}\hat{\otimes}H^0(V(\omega_2);\mathcal{O}) & \to & \mathcal{A}\hat{\otimes}H^0(L_2;\mathcal{O}) & \to 0 \end{array}$$

where the rows are exact. Theorem 2 of [14] implies that $\widetilde{\gamma}_* : \mathcal{A}\hat{\otimes} H^0(V(\omega_2); \mathcal{O}) \longrightarrow \mathcal{A}\hat{\otimes}H^0(V(\omega_2); \mathcal{O})$ is an isomorphism. As before, this is equivalent to show that Equation (9) has a (unique) solution ϕ without constant term when $f = \sum_{\mathbf{m}} c_{\mathbf{m}} z^{\mathbf{m}} \in \mathcal{A}\hat{\otimes}H^0(V(\omega_2); \mathcal{O})$, is any convergent power series in $z_1^{-1}, z_2^{-1}, z_j, 3 \leqslant z_j \leqslant r$, where the sum ranges over $\mathbf{m} = (m_1, m_2, \dots, m_r) \in \mathbb{Z}^r, m_1, m_2 < 0, m_j \geqslant 0, \forall j \geqslant 3$. It is clear that $\phi(z) = \sum c_{\mathbf{m}}/(b.\mathbf{m}) z^{\mathbf{m}}$ is the unique formal solution. Note that weak hyperbolicity condition implies that $b.\mathbf{m} \neq 0$ and $b.\mathbf{m} \to \infty$ as $\sum_{j \geqslant 1} |m_j| \to \infty$. So $\phi(z)$ is a well-defined convergent power series in the variables $z_1^{-1}, z_2^{-1}, z_j, j \geqslant 3$ and is divisible by $z_1^{-1} z_2^{-1}$. Hence $\phi \in \mathcal{A}\hat{\otimes}H^0(V(\omega_2); \mathcal{O})$ and so $\widetilde{\gamma}_* : \mathcal{A}\hat{\otimes}H^0(V(\omega_2); \mathcal{O}) \longrightarrow \mathcal{A}\hat{\otimes}H^0(V(\omega_2); \mathcal{O})$ is an isomorphism. The ideal $\mathcal{I}$ is stable under the action of $\widetilde{T}_2$, and so is generated as an ideal by (finitely many) polynomials in $z_3, \dots, z_r$ which are $\widetilde{T}_2$-weight vectors. In particular, the generators are certain homogeneous polynomials $h(z_3, \dots, z_r)$ such that $\widetilde{\gamma}_*(z_1^{m_1} z_2^{m_2} h) = b.\mathbf{m} z_1^{m_1} z_2^{m_2} h \ \forall m_1, m_2 \in \mathbb{Z}$ where $z^{\mathbf{m}}$ is any monomial that occurs in $z_1^{m_1} z_2^{m_2} h$. It follows easily that $\widetilde{\gamma}_*$

maps $\mathcal{A}\hat{\otimes}\mathcal{I}$ isomorphically onto itself. A straightforward argument involving diagram chase now shows that $\gamma_* : \mathcal{A}\hat{\otimes}H^0(L_2;\mathcal{O}) \longrightarrow \mathcal{A}\hat{\otimes}H^0(L_2;\mathcal{O})$ is an isomorphism. This completes the proof. □

Assume that $P_i \subset G_i$ are maximal parabolic subgroups so that $Pic(G_i/P_i) \cong \mathbb{Z}$. Suppose that the $\bar{L}_i$ are the negative ample generators of the $Pic(G_i/P_i)$. We have the following description of the principal $\mathbb{C}$-bundles $L_z, z \in \mathbb{C}$, over $S_\lambda(L)$. When $z = 0$, L_z is the trivial bundle. So let $z \neq 0$. Let $\{g_{ij}\}$ be a 1-cocyle defining the principal $\mathbb{C}$-bundle $L \longrightarrow S_\lambda(L)$. Then the $\mathbb{C}$-bundle L_z representing the element $z[L] \in H^1(S_\lambda(L);\mathcal{O})$ is defined by the cocylce $\{zg_{i,j}\}$ for any $z \in \mathbb{C}$. We denote the corresponding $\mathbb{C}$-bundle by L_z. Note that the total space and the projection are the same as that of L. The $\mathbb{C}$-action on L_z is related to that on L where $w.v \in L_z$ equals $(w/z).v = \alpha_\lambda(w/z)(v) \in L$ for $w \in \mathbb{C}, v \in L$. The vector field corresponding to the $\mathbb{C}$-action on L_z is given by $(1/z)\gamma_\lambda$. Of course, when $z = 0$, L_z is just the product bundle.

We shall denote the line bundle (i.e. rank 1 vector bundle) corresponding to L_z by E_z. Observe that $E_z = L_z \times_{\mathbb{C}} \mathbb{C}$, where

$$(w.v, t) \sim (v, \exp(2\pi\sqrt{-1}w)t),\ w, t \in \mathbb{C}, v \in L_z,$$

when $z \neq 0$. If $z \neq 0$, any cross-section $\sigma : S_\lambda(L) \longrightarrow E_z = L_z \times_{\mathbb{C}} \mathbb{C}$ corresponds to a holomorphic function $h_\sigma : L \longrightarrow \mathbb{C}$ which satisfies the following:

$$h_\sigma(w.v) = \exp(-2\pi\sqrt{-1}w)h_\sigma(v) \tag{10}$$

for all $v \in L_z, w \in \mathbb{C}$. Equivalently, this means that

$$h_\sigma(\alpha_\lambda(w)v) = \exp(-2\pi\sqrt{-1}wz)h_\sigma(v)$$

for $w \in \mathbb{C}$ and $v \in L$. This implies that

$$\gamma_\lambda(h_\sigma) = -2\pi\sqrt{-1}zh_\sigma. \tag{11}$$

Conversely, if h satisfies (11), then it determines a unique cross-section of E_z over $S_\lambda(L)$.

We have the following result concerning the field of meromorphic functions on $S_\lambda(L)$ with $\lambda_u = 0$. The proof will be given after some preliminary observations.

Theorem 4.8. — *Let L_i be the negative ample generator of $Pic(G_i/P_i) \cong \mathbb{Z}$ where P_i is a maximal parabolic subgroup of G_i, i=1,2. Assume that $\lambda_u = 0$. Then the field $\kappa(S_\lambda(L))$ of meromorphic functions of $S_\lambda(L)$ is purely transcendental over $\mathbb{C}$. The transcendence degree of $\kappa(S_\lambda(L))$ is less than* $\dim S_\lambda(L)$.

Let U_i denote the *opposite big cell*, namely the B_i^--orbit of $X_i = G_i/P_i$ the identity coset where B_i^- is the Borel subgroup of G_i opposed to B_i. One knows that U_i is a Zariski dense open subset of X_i and is isomorphic to $\mathbb{C}^{r_i}$ where r_i is the number of positive roots in the unipotent part $P_{i,u}$ of P_i. The bundle $\pi_i : L_i \longrightarrow X_i$ is trivial over U_i and so $\widetilde{U}_i := \pi_i^{-1}(U_i)$ is isomorphic to $\mathbb{C}^{r_i} \times \mathbb{C}^*$. We shall now describe a specific isomorphism which will be used in the proof of the above theorem.

Consider the projective imbedding $X_i \subset \mathbb{P}(V(\omega_i))$. Let $v_0 \in V(\omega_i)$ be a highest weight vector so that P_i stabilizes $\mathbb{C}v_0$; equivalently, $\pi_i(v_0)$ is the identity coset in X_i. Let $Q_i \subset P_i$ be the isotropy at $v_0 \in V(\omega_i)$ for the G_i so that $G_i/Q_i = L_i$. The Levi part of P_i is equal to the centralizer of a one-dimensional torus $\mathcal{Z}$ contained in T and projects onto $P_i/Q_i \cong \mathbb{C}^*$, the structure group of $L_i \longrightarrow X_i$.

Let $F_i \in H^0(X_i; L_i^\vee) = V(\omega_i)^\vee$ be the lowest weight vector such that $F_i(v_0) = 1$. Then $U_i \subset X_i$ is precisely the locus $F_i \neq 0$ and $F_i|_{\pi_i^{-1}([v])} : \mathbb{C}v \longrightarrow \mathbb{C}$ is an isomorphism of vector spaces for $v \in \widetilde{U}_i$. We denote also by F_i the restriction of F_i to $\widetilde{U}_i$.

Let Y_β be the Chevalley basis element of $Lie(G_i)$ of weight $-\beta, \beta \in R^+(G_i)$. We shall denote by $X_\beta \in Lie(G_i)$ the Chevalley basis element of weight $\beta \in R^+(G_i)$. Recall that $H_\beta := [X_\beta, Y_\beta] \in Lie(T)$ is non-zero whereas $[X_\beta, Y_{\beta'}] = 0$ if $\beta \neq \beta'$.

Let $R_{P_i} \subset R^+(G_i)$ denote the set of positive roots of G_i complementary to positive roots of Levi part of P_i and fix an ordering on it. (Thus $\beta \in R_{P_i}$ if and only if $-\beta$ is a not a root of P_i.) Let $r_i = |R_{P_i}| = \dim X_i$. Then $Lie(P_{i,u}^-) \cong \mathbb{C}^{r_i}$ where $P_{i,u}^-$ denotes the unipotent radical of the parabolic subgroup opposed to P_i. Observe that $P_i \cap P_{i,u}^- = \{1\}$. The exponential map defines an *algebraic* isomorphism $\theta : \mathbb{C}^{r_i} \cong Lie(P_{i,u})^- \longrightarrow U_i$ where $\theta((y_\beta)_{\beta \in R_{P_i}}) = (\prod_{\beta \in R_{P_i}} \exp(y_\beta Y_\beta)).P_i \in G_i/P_i$. It is understood that, here and in the sequel, the product is carried out according to the ordering on R_{P_i}.

If $v \in \mathbb{C}v_0$, then θ factors through the map $\theta_v : \mathbb{C}^{r_i} \cong Lie(P_{i,u}^-) \longrightarrow \widetilde{U}_i$ defined by $(y_\beta)_{\beta \in R_{P_i}} \mapsto \prod \exp(y_\beta Y_\beta).v$. Moreover, F_i is constant —equal to $F_i(v)$—on the image of θ_v.

We define $\widetilde{\theta} : \mathbb{C}^{r_i} \times \mathbb{C}^* \cong Lie(P_{i,u}^-) \times \mathbb{C}^* \cong P_{i,u}^- \times \mathbb{C}^* \longrightarrow \widetilde{U}_i$ to be $\widetilde{\theta}((y_\beta), z) = (\prod \exp(y_\beta Y_\beta)).zv_0 = \theta_{z.v_0}((y_\beta))$. This is an isomorphism. We obtain coordinate functions $z, y_\beta, \beta \in R_{P_i}$ by composing $\widetilde{\theta}^{-1}$ with projections $\mathbb{C}^{r_i} \times \mathbb{C}^* \longrightarrow \mathbb{C}$. Note that $F_i(\widetilde{\theta}((y_\beta), z))) = z$. Thus the coordinate function z is identified with F_i.

Since F_i is the lowest weight vector (of weight $-\omega_i$), $Y_\beta F_i = 0$ for all $\beta \in R^+(G_i)$. Define $F_{i,\beta} := X_\beta(F_i), \beta \in R_{P_i}$. Then $Y_\beta(F_{i,\beta}) = -[X_\beta, Y_\beta]F_i = -H_\beta(F_i) = \omega_i(H_\beta)F_i$ for all $\beta \in R_{P_i}$. Note that $\omega_i(H_\beta) \neq 0$ as $H_\beta \in R_{P_i}$. If $\beta', \beta \in R_{P_i}$ are unequal, then $Y_{\beta'}F_{i,\beta} = 0$. It follows that $Y^m_{\beta'}(F_{i,\beta}) = 0$ unless $\beta' = \beta$ and $m = 1$.

The following result is well-known to experts in standard monomial theory. (See [13].)

Lemma 4.9. — *With the above notations, the map $\widetilde{U}_i \longrightarrow \mathbb{C}^{r_i} \times \mathbb{C}^*$ defined as $v \mapsto ((F_{i,\beta}(v))_{\beta \in R^+_{P_i}}; F_i(v)),\ v \in \widetilde{U}_i$, is an algebraic isomorphism for $i = 1, 2$.*

Proof. — It is easily verified that $\partial f/\partial y_\beta|_{v_0} = Y_\beta(f)(v_0)$ for any local holomorphic function defined in a neighbourhood of v_0. (Cf. [13].)

Let $y = \widetilde{\theta}((y_\gamma), z) = \prod_{\gamma \in R_{P_i}}(\exp(y_\gamma Y_\gamma) \in P_i^-$. Denote by $l_y : \widetilde{U}_i \longrightarrow \widetilde{U}_i$ the left multiplication by y. If $v = y.v_0 \in \widetilde{U}_i$, then $(\partial/\partial y_\beta|_v)(f)$ equals $(\partial/\partial y_\beta)|_{v_0}(f \circ l_y)$. Taking $f = F_{i,\beta}, \beta \in R_{P_i}$ a straightforward computation using the observation made preceding the lemma, we see that

$$(\partial/\partial y_\beta|_v)(F_{i,\gamma}) = Y_\beta|_{v_0}(F_{i,\gamma} \circ l_y) = F_i(v)\omega_i(H_\beta)\delta_{\beta,\gamma} \text{ (Kronecker } \delta).$$

We also have $(\partial/\partial y_\beta|_v)(F_i) = 0$ for all $v \in \widetilde{U}_i$. Hence $(\partial/\partial y_\beta)|_v(F_{i,\gamma}/F_i) = \omega_i(H_\beta)\delta_{\beta,\gamma}$. Thus the Jacobian matrix relating the $F_{i,\beta}/F_i$ and the $y_\beta, \beta \in R_{P_i}$, is a diagonal matrix of *constant functions*. The diagonal entries are non-zero as $\omega_i(H_\beta) \neq 0$ for $\beta \in R_{P_i}$ and the lemma follows. □

We shall use the coordinate functions $F_i, F_{i,\beta}, \beta \in R_{P_i}$, to write Taylor expansion for analytic functions on $\widetilde{U}_i$. In particular, the coordinate ring of the affine variety $\widetilde{U}_i$ is just the algebra $\mathbb{C}[F_{i,\beta}, \beta \in R_{P_i}][F_i, F_i^{-1}]$. The projective normality [19] of G_i/P_i implies that $\mathbb{C}[\hat{L}_i] = \oplus_{r \geqslant 0} H^0(X_i; L_i^{-r}) = \oplus_{r \geqslant 0} V(r\omega_i)^\vee$. Since $\widetilde{U}_i$ is defined by the non-vanishing of F_i, we see that $\mathbb{C}[\widetilde{U}_i] = \mathbb{C}[\hat{L}_i][1/F_i]$.

Now let $X = X_1 \times X_2$ and $\widetilde{T} = \widetilde{T}_1 \times \widetilde{T}_2 \cong (\mathbb{C}^*)^N, N = n_1 + n_2$, where the isomorphism is as chosen in §3. Let $d_i > 0, i = 1, 2$, be chosen as in Proposition 3.1 so that the $\widetilde{T}_i$-action on $L_i \longrightarrow G_i/P_i$ is d_i-standard. Let $\lambda = \lambda_s \in Lie(\widetilde{T})$. Suppose that λ satisfies the weak hyperbolicity condition of type (n_1, n_2).

Recall from (4) and (5) that for any weight $\mu_i \in \Lambda(\omega_i)$, there exist elements $\lambda_{\mu_1}, \lambda_{\mu_2} \in \mathbb{C}$ such that for any $v = (v_1, v_2) \in V_{\mu_1}(\omega_1) \times V_{\mu_2}(\omega_2)$, the α_λ-action of $\mathbb{C}$ is given by $\alpha_\lambda(z)v = (\exp(z\lambda_{\mu_1})v_1, \exp(z\lambda_{\mu_2})v_2)$. In fact $\lambda_{\mu_i} = \sum_{n_{i-1} < j \leqslant n_{i-1}+n_i} d_{\mu_i,j}\lambda_j$ where $d_{\mu_i,j}$ are certain *non-negative integers*. It follows that, as observed in the discussion preceding the statement of Theorem 3.2, the complex numbers $\lambda_{\mu_i} \in \mathbb{C},\ \mu_i \in \Lambda(\omega_i), i = 1, 2$ satisfy weak hyperbolicity condition:

$$0 \leqslant \arg(\lambda_{\mu_1}) < \arg(\lambda_{\mu_2}) < \pi,\ \forall \mu_i \in \Lambda(\omega_i), i = 1, 2. \tag{12}$$

We observe that if $\mu = \mu_1 + \cdots + \mu_r = \nu_1 + \cdots + \nu_r$, where $\mu_j, \nu_j \in \Lambda(\omega_i)$, then $\lambda_{\mu,r} := \sum \lambda_{\mu_j} = \sum \lambda_{\nu_j}$. (This is a straightforward verification using (3) and (4).) Therefore, if $v \in V(\omega_i)^{\otimes r}$ is any weight vector of weight μ, we get, for the diagonal action of $\mathbb{C}$, $z.v = \exp(\lambda_{\mu,r} z)v$.

Any finite dimensional $\widetilde{G}_i$-representation space V is naturally $\widetilde{G}_1 \times \widetilde{G}_2$-representation space and is a direct sum of its $\widetilde{T}$-weight spaces V_μ. If V arises from a representation of G_i via $\widetilde{G}_i \longrightarrow G_i$, then the $\widetilde{T}$-weights of V are the same as T-weights.

Definition 4.10. — *Let $Z_i(\lambda) \subset \mathbb{C}, i = 1, 2,$ be the abelian subgroup generated by $\lambda_\mu, \mu \in \Lambda(\omega_i)$ and let $Z(\lambda) := Z_1(\lambda) + Z_2(\lambda) \subset \mathbb{C}$.*

The λ-weight *of an element $0 \neq f \in Hom(V_\mu(\omega_i); \mathbb{C})$ is defined to be $wt_\lambda(f) := \lambda_\mu$. If $h \in Hom(V(\omega_i)^{\otimes r}, \mathbb{C})$ is a weight vector of weight $-\mu$, (so that $h \in Hom(V(\omega_i)^{\otimes r}_\mu; \mathbb{C})$) we define the* λ-weight of *of h to be $\lambda_{\mu,r}$.*

If $f \in Hom(V_\mu(r\omega_i), \mathbb{C})$ is a weight vector (of weight $-\mu$), then it is the image of a unique weight vector $\widetilde{f} \in Hom(V(\omega_i)^{\otimes r}, \mathbb{C})$ under the surjection induced by the $\widetilde{G}_i$-inclusion $V(r\omega_i) \hookrightarrow V(\omega_i)^{\otimes r} = V(r\omega_i) \oplus V'$ where $\widetilde{f}|V' = 0$. We define the λ-weight of *f to be $wt_\lambda(f) := wt_\lambda(\widetilde{f})$.*

If $h_i \in V(r_i\omega_i)^\vee \subset \mathbb{C}[\hat{L}_i]$, $i = 1, 2$, are weight vectors, then h_1h_2 is a weight vector of $V(r_1\omega_1)^\vee \otimes V(r_2\omega_2)^\vee \subset \mathbb{C}[\hat{L}_1 \times \hat{L}_2]$ and we have $wt_\lambda(h_1h_2) = wt_\lambda(h_1) + wt_\lambda(h_2) \in Z(\lambda)$. Note that

$$wt_\lambda(f_1 \dots f_k) = \sum_{1 \leqslant j \leqslant k} wt_\lambda(f_j) \in Z(\lambda)$$

where $f_j \in \mathbb{C}[\hat{L}_1 \times \hat{L}_2] = \oplus_{r_1, r_2 \geqslant 0} V(r_1\omega_1)^\vee \otimes V(r_2\omega_2)^\vee$ are weight vectors.

Also $wt_\lambda(f) \in Z(\lambda)$ is a *non-negative* linear combination of $\lambda_j, 1 \leqslant j \leqslant N$, for any $\widetilde{T}$-weight vector $f \in \mathbb{C}[\hat{L}_1 \times \hat{L}_2]$.

If $f \in V(\omega_i)^\vee$, it defines a holomorphic function on $V(\omega_1) \times V(\omega_2)$ and hence on L, and denoted by the same symbol f; explicitly $f(u_1, u_2) = f(u_i),\ \forall (u_1, u_2) \in L$.

Lemma 4.11. — *We keep the above notations. Assume that $\lambda = \lambda_s \in Lie(\widetilde{T}) = \mathbb{C}^N$.*

Fix $\mathbb{C}$-bases $\mathcal{B}_i$ for $V(\omega_i)^\vee$, consisting of $\widetilde{T}$-weight vectors. Let $z_0 \in Z(\lambda)$. There are only finitely many monomials $f := f_1 \dots f_k$, $f_j \in \mathcal{B}_1 \cup \mathcal{B}_2$ having λ-weight z_0. Furthermore, $v_\lambda(f) = wt_\lambda(f)f$.

Proof. — The first statement is a consequence of weak hyperbolicity (see (12)). Indeed, since $0 \leqslant \arg(\lambda_\mu) < \pi$ for all $\mu \in \Lambda(\omega_i), i = 1, 2$, given any complex number z_0, there are only finitely many non-negative integers c_j such that $\sum c_j \lambda_{\mu_j} = z_0$.

As for the second statement, we need only verify this for $f \in \mathcal{B}_i, i = 1, 2$. Suppose that $f \in \mathcal{B}_1$ and that f is of weight $-\mu$, $\mu \in \Lambda(\omega_1)$, say. Then, for any $(u_1, u_2) \in L$, writing $u_1 = \sum_{\nu \in \Lambda(\omega_1)} u_\nu$, using linearity and the fact that $f(u_1, u_2') = f(u_\mu)$ we have

$$\begin{aligned} v_\lambda(f)(u_1, u_2) &= \lim_{w \to 0} (f(\alpha_\lambda(w)(u_1, u_2)) - f(u_1, u_2))/w \\ &= \lim_{w \to 0} (f(\exp(\lambda_\mu w)u_\mu) - f(u_1))/w \\ &= \lim_{w \to 0} (\frac{\exp(\lambda_\mu w) - 1}{w}) f(u_1) \\ &= \lambda_\mu f(u_1) \\ &= \lambda_\mu f(u_1, u_2). \end{aligned}$$

This completes the proof. □

We assume that $F_i, F_{i,\beta}, \beta \in R_{P_i}$, are in $\mathcal{B}_i$, $i = 1, 2$.

Let $\mathcal{M} \subset \mathbb{C}(\widetilde{U}_1 \times \widetilde{U}_2)$ denote the multiplicative group of all Laurent monomials in $F_i, F_{i,\beta}, \beta \in R_{P_i}, i = 1, 2$. One has a homomorphism $wt_\lambda : \mathcal{M} \longrightarrow Z(\lambda)$. Denote by $\mathcal{K}$ the kernel of wt_λ. Evidently, $\mathcal{M}$ is a free abelian group of rank $\dim L$.

Lemma 4.12. — *With the above notations, $wt_\lambda : \mathcal{M} \longrightarrow Z(\lambda)$ is surjective. Any $\mathbb{Z}$-basis $h_1, \dots, h_k$ of $\mathcal{K}$ is algebraically independent over $\mathbb{C}$.*

Proof. — Suppose that $\nu \in Z_i(\lambda)$. Write $\nu = \sum a_\mu \lambda_\mu$ and choose $b_\mu \in \mathcal{B}_i$ to be of weight μ. Then $wt_\lambda(\prod_\mu b_\mu^{a_\mu}) = \nu$. On the other hand, $wt_\lambda(b_\mu)$ equals the λ-weight of any monomial in the $F_i^{-1}, F_i, F_{i,\beta}, \beta \in R_{P_i}$ that occurs in $b_\mu|\widetilde{U}_i$. The first assertion follows from this.

Let, if possible, $P(z_1, \dots, z_k) = 0$ be a polynomial equation satisfied by $h_1, \dots, h_k$. Note that the h_j are certain Laurent monomials in a transcendence basis of the field $\mathbb{C}(\widetilde{U}_1 \times \widetilde{U}_2)$ of *rational functions* on the affine variety $\widetilde{U}_1 \times \widetilde{U}_2$. Therefore there must exist monomials $z^{\mathbf{m}}$ and $z^{\mathbf{m}'}$, $\mathbf{m} \neq \mathbf{m}'$, occurring in $P(z_1, \dots, z_k)$ with non-zero coefficients such that $h^{\mathbf{m}} = h^{\mathbf{m}'} \in \mathbb{C}(\widetilde{U}_1 \times \widetilde{U}_2)$. Hence $h^{\mathbf{m}-\mathbf{m}'} = 1$. This contradicts the hypothesis that the h_j are linearly independent in the multiplicative group $\mathcal{K}$. □

We now turn to the proof of Theorem 4.8.

Proof of Theorem 4.8. — By definition, any meromorphic function on $S_\lambda(L)$ is a quotient f/g where f and g are holomorphic sections of a holomorphic line bundle E_z. Any holomorphic section $f : S(L) \longrightarrow E_z$ defines a holomorphic function on L, denoted by f, which satisfies Equation (11). By the normality of $\hat{L}_1 \times \hat{L}_2$, the function f then extends uniquely to a function on $\hat{L}_1 \times \hat{L}_2$ which is again denoted f. Thus we may write $f = \sum_{r,s\geqslant 0} f_{r,s}$ where $f_{r,s} \in V(r\omega_1)^\vee \otimes V(s\omega_2)^\vee$. Now $v_\lambda f = af$ and $v_\lambda f_{r,s} \in V(r\omega_1)^\vee \otimes V(s\omega_2)^\vee$ implies that $v_\lambda(f_{r,s}) = af_{r,s}$ for all $r, s \geqslant 0$ where $a = -2\pi\sqrt{-1}z$. This implies that $wt_\lambda(f_{r,s}) = a$ for all $r, s \geqslant 0$. This implies, by Lemma 4.11, that $f_{r,s} = 0$ for sufficiently large r, s and so f is *algebraic.*

Now writing f and g restricted to $\widetilde{U}_1 \times \widetilde{U}_2$ as a polynomial in the the coordinate functions $F_i^\pm, F_{i,\beta}, i = 1, 2$, introduced above, it follows easily that f/g belongs to the field $\mathbb{C}(\mathcal{K})$ generated by $\mathcal{K}$. Evidently $\mathcal{K}$—and hence the field $\mathbb{C}(\mathcal{K})$—is contained in $\kappa(S_\lambda(L))$. Therefore $\kappa(S_\lambda(L))$ equals $\mathbb{C}(\mathcal{K})$. By Lemma 4.12 the field $\mathbb{C}(\mathcal{K})$ is purely transcendental over $\mathbb{C}$.

Finally, since $Z(\lambda)$ is of rank at least 2 and since $wt_\lambda : \mathcal{M} \longrightarrow Z(\lambda)$ is surjective,

$$tr.deg(\kappa(S_\lambda(L)) = rank(\mathcal{K}) \leqslant rank(\mathcal{K}) - 2 = \dim(L) - 2 = \dim(S_\lambda(L)) - 1.$$

□

Remark 4.13. — *(i) We have actually shown that the transcendence degree of $\kappa(S_\lambda(L))$ equals the rank of $\mathcal{K}$. In the case when X_i are projective spaces, this was observed by* [14]*. When λ is of scalar type,*

$$tr.deg(\kappa(S_\lambda(L)) = \dim(S_\lambda(L)) - 1.$$

(ii) Theorem 4.8 implies that any algebraic reduction of $S_\lambda(L)$ is a rational variety. In the case of scalar type, one has an elliptic curve bundle $S_\lambda(L) \longrightarrow X_1 \times X_2$. (Cf. [22]*.) Therefore this bundle projection yields an algebraic reduction. In the general case however, it is an interesting problem to construct explicit algebraic reductions of these compact complex manifolds. (We refer the reader to* [17] *and references therein to basic facts about algebraic reductions.)*
(iii) We conjecture that $\kappa(S_\lambda(L))$ is purely transcendental for $X_i = G_i/P_i$ where P_i is any parabolic subgroup and $\bar{L}_i$ is any negative ample line bundle over X_i, where $S_\lambda(L)$ has any linear type complex structure.

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