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COVERS IN p-ADIC ANALYTIC GEOMETRY AND LOG COVERS I: COSPECIALIZATION OF THE (p')-TEMPERED FUNDAMENTAL GROUP FOR A FAMILY OF CURVES

by Emmanuel LEPAGE

Abstract. — The tempered fundamental group of a p-adic analytic space classifies covers that are dominated by a topological cover (for the Berkovich topology) of a finite étale cover of the space. Here we construct cospecialization homomorphisms between (p') versions of the tempered fundamental groups of the fibers of a smooth family of curves with semistable reduction. To do so, we will translate our problem in terms of cospecialization morphisms of fundamental groups of the log fibers of the log reduction and we will prove the invariance of the geometric log fundamental group of log smooth log schemes over a log point by change of log point.

Résumé. — Le groupe fondamental tempéré d'un espace analytique p-adique classifie les revêtements qui sont dominés par un revêtement topologique (pour la topologie de Berkovich) d'un revêtement étale fini de cet espace. Nous construisons ici des morphismes de cospécialisation entre les versions (p') du groupe fondamental tempéré des fibres d'une famille lisse avec réduction semistable. Pour ce faire, nous traduisons notre problème en termes de morphismes de cospécialisation de groupes fondamentaux des fibres logarithmiques de la réduction modulo p et prouvons l'invariance du groupe fondamental logarithmique géométrique d'un log-schéma log-lisse au-dessus d'un point logarithmique par changement de base.

Introduction

In general topology, the fundamental group of a connected locally contractible pointed space classifies its (unramified) covers. A. Grothendieck developed an avatar in abstract algebraic geometry: he attached to any algebraic variety a profinite fundamental group, which classifies its finite étale covers. For a complex algebraic variety, Grothendieck's fundamental

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group is canonically isomorphic to the profinite completion of the topologic fundamental group of the corresponding topological space.

Here we are interested in an analog in p-adic geometry. More precisely we will study the *tempered fundamental group* of p-adic varieties defined by Y. André. The profinite completion of the tempered fundamental group of any smooth p-adic algebraic variety coincides with Grothendieck's algebraic fundamental group. It also accounts for the usual (infinite) "uniformizations" in p-adic analytic geometry such as the uniformization of Tate elliptic curves, which are historically at the very basis of p-adic rigid geometry. Such uniformizations give infinite discrete quotients of the tempered fundamental group.

The framework of this paper for non-archimedean analytic geometry will be Berkovich spaces. The underlying space of the analytification of an affine algebraic variety $\operatorname{Spec} A$ in the sense of Berkovich is the set of multiplicative seminorms on A with value in $\mathbf{R}_{\geqslant 0}$ extending the norm of the base field, endowed with the coarsest topology that makes the evaluation of the norm of any element $f \in A$ continuous. The analytification of a smooth algebraic variety is locally contractible, which ensures the existence of universal topological covers. Since the analytification (in the sense of V. Berkovich or of rigid geometry) of a finite étale cover of a p-adic algebraic variety is not necessarily a topological cover, André had to consider a category of covers slightly bigger than just the category of topological covers. He defined tempered covers, which are (possibly infinite) étale covers in the sense of A.J. de Jong (that is to say, which are, Berkovich-locally on the base, direct sums of finite covers) such that, after pulling back by some finite étale cover, they become topological covers (for the Berkovich topology). The *tempered fundamental group* is the prodiscrete group that classifies those tempered covers. To give a more handful description, if one has a sequence of pointed finite Galois connected covers $((S_i, s_i))_{i\in\mathbf{N}}$ such that the corresponding pointed pro-cover of (X, x) is the universal pro-cover of (X, x), and if (S_i^∞, s_i^∞) is a universal topological cover of S_i, the tempered fundamental group of X can be seen as $\pi_1^{\mathrm{temp}}(X, x) = \varprojlim_i \operatorname{Gal}(S_i^\infty/X)$. Therefore, to understand the tempered fundamental group of a variety, one has to understand the topological behavior of its finite étale covers.

In the case of a curve, the question becomes more concrete since there is a natural embedding of the geometric realization of the graph of its stable model into the Berkovich space of the curve which is a homotopy equivalence.

Among applications of tempered fundamental groups, let us cite in passing the theory of p-adic orbifolds and p-adic triangle groups [2] and a p-adic version of Grothendieck-Teichmüller theory [1].

In this article we will be interested in the variation of the tempered fundamental group of the fibers of a family of curves. This article will be followed by another one [14], in which we will consider higher dimensional families.

If $\bar{y}_1, \bar{y}_2$ are to geometric points of a scheme Y, a specialization $\bar{y}_2 \to \bar{y}_1$ is a Y morphism from $\bar{y}_2$ to the strict localization $Y(\bar{y}_1)$ of Y at $\bar{y}_1$. Equivalently, a specialization $\bar{y}_2 \to \bar{y}_1$ is a morphism of functors $(\ )_{\bar{y}_1} \to (\ )_{\bar{y}_2}$, where $(\ )_{\bar{y}}$ is the functor from the étale topos of Y to the category of sets that maps an étale sheaf $\mathcal{F}$ to its stalk $\mathcal{F}_{\bar{y}}$. For a proper morphism of schemes $f : X \to Y$ with geometrically connected fibers and a specialization $\bar{y}_2 \to \bar{y}_1$ of geometric points of Y, A. Grothendieck has defined algebraic fundamental groups $\pi_1^{\mathrm{alg}}(X_{\bar{y}_i})$ and a specialization homomorphism $\pi_1^{\mathrm{alg}}(X_{\bar{y}_1}) \to \pi_1^{\mathrm{alg}}(X_{\bar{y}_2})$. Grothendieck's specialization theorem tells that this homomorphism is surjective if f is separable and induces an isomorphism between the prime-to-p quotients if f is smooth (here, p denotes the characteristic of $\bar{y}_2$), *cf.* [6, cor. X.2.4, cor. X.3.9].

In complex analytic geometry, a smooth and proper morphism is locally a trivial fibration of real differential manifolds, so that, in particular, all the fibers are homeomorphic, and thus have isomorphic (topological) fundamental groups.

The aim of this paper is to find some analog of the specialization theorem of Grothendieck in the case of the tempered fundamental group.

In this paper, we will concentrate on the case of curves.

One problem which appears at once in looking for some non archimedean analog of Grothendieck's specialization theorems is that there are in general no non trivial specializations between distinct points of a non archimedean analytic (Berkovich or rigid) space: for example a separated Berkovich space has a Hausdorff underlying topological space, so that if there is a cospecialization (for the Berkovich topology, the étale topology…) between two geometric points of a Berkovich space, the two geometric points must have the same underlying point. Thus we will assume we have a model over the ring of integers of our non-archimedean field (with good enough properties) and we will look at the specializations in the special fiber.

We want to understand how the tempered fundamental group of the geometric fibers of a smooth family varies. Let us for instance consider a family of elliptic curves. The tempered fundamental group of an elliptic curve over a complete algebraically closed non archimedean closed field is $\widehat{\mathbf{Z}}^2$ if it has good reduction, and $\widehat{\mathbf{Z}} \times \mathbf{Z}$ if it is a Tate curve. In particular, by looking at a moduli space of stable pointed elliptic curves with level structure[(1)], the tempered fundamental group (or any reasonable (p')-version) cannot be constant.

Moreover, if one looks at the moduli space over $\mathbf{Z}_p$, and considers a curve E_1 with bad reduction and a curve E_0 with generic reduction (hence good reduction), there cannot be a morphism $\pi_1^{\mathrm{temp}}(E_0) \to \pi_1^{\mathrm{temp}}(E_1)$ which induces Grothendieck's specialization on the profinite completion, although the reduction point corresponding to E_1 specializes to the reduction point corresponding to E_0. Therefore there cannot be any reasonable specialization theory.

On the contrary, if one has two geometric points η_1 and η_2 of the moduli space such that the reduction of η_1 specializes to the reduction of η_2, then E_{η_1} has necessarily better reduction than E_{η_2} and there is some morphism $\pi_1^{\mathrm{temp}}(E_{\eta_2}) \to \pi_1^{\mathrm{temp}}(E_{\eta_1})$ that induces an isomorphism between the profinite completions. Thus we want to look for a *cospecialization* of the tempered fundamental group.

The topological behavior of general finite étale covers is too complicated to hope to have a simple cospecialization theory without adding a (p') condition on the covers: for example two Mumford curves over some finite extension of $\mathbf{Q}_p$ with isomorphic geometric tempered fundamental group have the same metrized graph of stable reduction [15]. Thus even if two Mumford curves have isomorphic stable reduction (and thus the point corresponding to their stable reduction is the same), they may not have isomorphic tempered fundamental group in general. Thus we will only study here finite covers that are dominated by a finite Galois cover whose order is prime to p, where p is the residual characteristic (which can be 0; such a cover will be called a (p')-finite cover). Then, it becomes natural to introduce a (p')-tempered fundamental group which classifies tempered covers that become topological covers after pullback along some (p')-finite cover. It should be remarked that this (p')-tempered fundamental group cannot be in general recovered from the tempered fundamental group.

The (p')-tempered fundamental group of a curve was already studied by S. Mochizuki in [16]. It can be described in terms of a graph of profinite groups. From this description, one easily sees that the isomorphism class of the (p')-tempered fundamental group of a p-adic curve depends only of the

[(1)] to avoid stacks. However, the cospecialization homomorphisms we will construct will be local for the étale topology of the special fiber of the base. Thus, the fact that the base is a Deligne-Mumford stack is not really a problem.

stratum of the Knudsen stratification of the moduli space of stable curves in which the stable reduction lies. Moreover if one has two strata x_1 and x_2 in the moduli space of stable curves such that x_1 is in the closure of x_2, one can easily construct morphisms from the graph of groups corresponding to x_1 to the graph of groups corresponding to x_2 (inducing morphisms of tempered fundamental groups which induce isomorphisms of the pro-(p') completions).

We shall study the following situation. Let O_K be a complete discrete valuation ring, K be its fraction field, k be its residue field and p be its characteristic (which can be 0). A proper semistable pointed curve (X, D) over a scheme S is given by a flat and proper morphism $X \to S$ with semistable geometric fibers, and D is a closed subscheme of X which lies inside the smooth locus of $X \to S$ and such that $D \to S$ is étale. Let (X, D) be a proper semistable pointed curve over O_K smooth over K and let $U = X_\eta \backslash D_\eta$. Let us describe the tempered fundamental group of $U_{\bar\eta}^{\mathrm{an}}$ in terms of $X_{\bar s}$ [16].

Let us make sure at first that we can get such a description for the pro-(p') completion, *i.e.,* the algebraic fundamental group. One cannot apply directly Grothendieck's specialization theorems (even if $U = X_\eta$) since the special fiber is not smooth but only semistable. Indeed, a pro-(p') geometric cover of the generic fiber will generally only induce a Kummer cover on the special fiber. These are naturally described in terms of *log geometry*, more precisely in terms of *Kummer-étale* (két) covers of a log scheme. One can endow X (and thus X_s too by restriction) with a natural log structure such that the pro-(p') fundamental group of U is isomorphic to a pro-(p') log fundamental group (as defined in [8]) of $X_{\bar s}$. One then gets a description of $\pi_1^{\mathrm{alg}}(U_{\bar\eta})$ by taking the projective limit under tame covers of K, or equivalently under két covers of s endowed with its natural log structure: there is an equivalence between finite étale covers of $U_{\bar\eta}$ and "geometric két covers" of $X_{\bar s}$.

A két cover of X will still be a semistable model of its generic fiber if one replaces K by some tame extension. Thus, one can describe the topology of the corresponding cover of $U_{\bar\eta}$ in terms of the graph of the corresponding geometric két cover of $X_{\bar s}$.

Let us now come back to the problem of cospecialization. Let $X \to Y$ be a semistable curve over O_K with $X \to Y$ endowed with compatible log structure (see Definition 2.1).

Let $\bar\eta_1$ (resp. $\bar\eta_2$) be a (Berkovich) geometric point of $Y_0 := Y_{\mathrm{tr}}^{\mathrm{an}} \cap \mathfrak{Y}_\eta \subset Y_K^{\mathrm{an}}$, where Y_{tr} is the locus of Y where the log structure is trivial and $\mathfrak{Y}_\eta$

is the generic fiber of the formal completion of Y along its closed fiber (if Y is proper, then $Y_0 = Y_{\mathrm{tr}}^{\mathrm{an}}$). Let $\bar{s}_1$ (resp. $\bar{s}_2$) be its log reduction in Y_s .

To use the previous description of the tempered fundamental group of $U_{\bar{\eta}_1}$ and $U_{\bar{\eta}_2}$ in terms of $X_{\bar{s}_1}$ and $X_{\bar{s}_2}$, we have to assume that $\bar{\eta}_1$ and $\bar{\eta}_2$ lie over Berkovich points with discrete valuation.

The main result of this paper is the following:

Theorem 0.1 (Th. 3.6). — *Let K be a complete discretely valued field. Let $\mathbb{L}$ be a set of primes that does not contain the residual characteristic of K. Let $Y \to \operatorname{Spec} O_K$ be a morphism of log schemes of finite type. Let $Y_0 = Y_{\mathrm{tr}}^{\mathrm{an}} \cap \mathfrak{Y}_\eta \subset Y^{\mathrm{an}}$ where $\mathfrak{Y}$ is the completion of Y along its closed fiber. Let $X \to Y$ be a proper semistable curve with compatible log structure. Let $U = X_{\mathrm{tr}}$. Let η_1 and η_2 be two Berkovich points of Y_0 whose residue fields have discrete valuation, and let $\bar{\eta}_1, \bar{\eta}_2$ be geometric points above them. Let $\bar{s}_2 \to \bar{s}_1$ be a log specialization of their log reductions such that there exists a compatible specialization $\bar{\eta}_2 \to \bar{\eta}_1$, then there is a cospecialization homomorphism $\pi_1^{\mathbb{L}\text{-temp}}(U_{\bar{\eta}_1}) \to \pi_1^{\mathbb{L}\text{-temp}}(U_{\bar{\eta}_2})$. Moreover, it is an isomorphism if $\overline{M}_{Y,\bar{s}_1} \to \overline{M}_{Y,\bar{s}_2}$ is an isomorphism.*

Let us come back to our example of the moduli space of pointed stable elliptic curves with high enough level structure M over O_K, and let C be the canonical stable elliptic curve on M. Let $\mathbb{L}$ be a set of primes that does not contain the residual characteristic of K. If η_1 and η_2 are two Berkovich points of M_η, they are in M_η^{tr} if and only if C_{η_1} and C_{η_2} are smooth. $C \to M$, endowed with their natural log-structures over (O_K, O_K^*), is a semistable morphism of log schemes. One thus get a cospecialization outer morphism $\pi_1^{\mathbb{L}\text{-temp}}(C_{\overline{\eta}_1}) \to \pi_1^{\mathbb{L}\text{-temp}}(C_{\overline{\eta}_2})$ for every specialization $\overline{s}_2 \to \overline{s}_1$, which is an isomorphism if $\overline{s}_1$ and $\overline{s}_2$ are in the same stratum of M_s. Since the moduli stack of pointed stable elliptic curves over $\operatorname{Spec} k$ has only two strata, one corresponding to smooth elliptic curves M_0 and one to singular curves M_1, one gets that $\pi_1^{\mathbb{L}\text{-temp}}(E_1) \simeq \pi_1^{\mathbb{L}\text{-temp}}(E_2)$ if E_1 and E_2 are two curves with good reduction or two Tate curves (the isomorphism depends on choices of cospecializations). Since M_1 is in the closure of M_0 one gets a morphism from the tempered fundamental group of a Tate curve to the tempered fundamental group of an elliptic curve with good reduction.

The first thing we need in order to construct the cospecialization homomorphism for tempered fundamental groups is a specialization morphism between the (p')-log geometric fundamental groups of $X_{\bar{s}_1}$ and $X_{\bar{s}_2}$. Such a specialization morphism will be constructed by proving that one can extend any (p')-log geometric cover of X_{s_1} to a két cover of X_U where U is some két neighborhood of s_1. If one has such a specialization morphism, by comparing it to the fundamental groups of $X_{\bar{\eta}_1}$ and $X_{\bar{\eta}_2}$ and using Grothendieck's specialization theorem, we will easily get that it must be an isomorphism. This specialization morphism is easily deduced from [19] if s_1 is a strict point of Y (*i.e.,* the log structure of s_1 is simply the one induced by Y), *i.e.,* the log structure of s_1 is just the pull back of the log structure of Y, but is not straightforward when the log structure is really modified. Thus we will study the invariance of the log geometric fundamental group by change of fs base point. The main result we will prove (in any dimension) is the following :

Theorem 0.2 (Th. 1.15). — *Let $s' \to s$ be a morphism of fs log points with isomorphic algebraically closed underlying fields. Let $X \to s$ be a saturated morphism of log schemes with X noetherian and let $X' \to s'$ be the pull back to s'. Then the map $\pi_1^{\text{log-geom}}(X'/s', \bar{x}') \to \pi_1^{\text{log-geom}}(X/s, \bar{x})$ is an isomorphism.*

It is interesting to notice that, in this situation, this is an isomorphism for the full fundamental group, and not only of the pro-(p') part. This mainly comes from the fact that the morphism of underlying schemes $\mathring{X}' \to \mathring{X}$ is an isomorphism (so that the problem only comes from the logarithmic structure and not the schematic structure). This result is proved by a local study on X for the strict étale topology.

Then we have to construct cospecialization topological morphisms for a semistable curve, more precisely cospecialization morphisms of the graphs of the geometric fibers. This will be done étale locally. These morphisms are not morphisms of graphs in the usual sense, since an edge can be contracted over a vertex, but still give a map between their geometric realizations, whence a map of homotopy types $U_{\bar{\eta}_1} \to U_{\bar{\eta}_2}$. This can also be done for any két cover of $X_{\bar{s}_1}$: we thus get such a map of homotopy types for every (p')-cover of $U_{\bar{\eta}_1}$. Those maps are compatible, and thus glue together to give the wanted cospecialization of tempered fundamental groups.

The paper is organized as follows.

In the first section, we will study specialization of fundamental groups.

In the second section, we will construct cospecialization maps of graphs of the geometric fibers of a semistable curve.

In the last section, we will prove Theorem 0.1.

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1. Specialisation of log fundamental groups

The main result of this part will be the invariance of the log geometric fundamental group announced in Theorem 0.2. We will deduce from it morphisms of specialization for the pro-(p') log geometric fundamental group of the fibers of a proper log smooth saturated morphism.

1.1. Log fundamental groups

For a curve with bad reduction, one cannot apply Grothendieck's specialization theorem to describe the geometric fundamental group of the curve in terms of the fundamental group of its stable reduction since the family is not smooth. However, such a comparison result exists in the realm of log geometry. More precisely, if one considers a smooth and proper variety with semistable reduction, the semistable model can naturally be endowed with a log structure, and the pro-(p') fundamental group of the variety is canonically isomorphic to the pro-(p') log fundamental group of the semistable reduction. Here we recall the basic definitions and results about log fundamental groups.

First, recall some usual notations about monoids and log schemes. All the monoids we consider are commutative with unit. If P is a monoid, then P^* is the group of invertible elements of P and P^{gp} is the universal group together with a morphism of monoids $P \to P^{\mathrm{gp}}$. A monoid is integral if $P \to P^{\mathrm{gp}}$ is injective. A monoid P is *sharp* if P^* is trivial. The sharpification P/P^* is denoted by $\overline{P}$. If X is a log scheme, the sheaf of monoids defining its log structure will usually be denoted by M_X, the sharpification $M_X/O^*_{\mathring{X}}$ of M_X will be denoted by $\overline{M}_X$, the underlying scheme will be denoted by $\mathring{X}$ and the open subset of $\mathring{X}$ where the log structure is trivial will be denoted X_{tr}.

A morphism $X \to Y$ of log schemes is *strict* if the log structure on X is the pullback log structure of the log structure on Y. If M_X and M_Y are integral, then $f : X \to Y$ is strict if and only if, for every geometric point $\bar{x}$ of X, $\overline{M}_{Y,f(\bar{x})} \to \overline{M}_{X,\bar{x}}$ is an isomorphism.

If P is a monoid, one denotes by $\operatorname{Spec} P$ the set of primes of P. There is a natural map $\operatorname{Spec} \mathbf{Z}[P] \to \operatorname{Spec} P$.

A monoid P *fine and saturated* (or *fs* for short) if it is finitely generated, integral and, for every $a \in P^{\mathrm{gp}}$ such that there exists a positive integer n such that $a^n \in P$, then $a \in P$. A log scheme X is *fs* if, locally for the étale topology of X, there is an fs monoid P and a morphism $P \to M_X$ such that $P^a \to M_X$, where P^a is the log structure associated to $P \to O_X$, is an isomorphism ($P \to M_X$ is then called a *fs chart* modeled on P). Giving a chart $P \to M_X$ is equivalent to giving a strict morphism of log schemes $X \to \operatorname{Spec} \mathbf{Z}[P]$.

If K is a complete discretely valued field, $S = \operatorname{Spec} O_K$ will be endowed in this paper with the log structure associated to $O_K \backslash \{0\} \to O_K$. If π is a uniformizer of O_K, the map $\alpha : \mathbf{N} \to O_K$ defined by $\alpha(n) = \pi^n$ is an fs chart.

If $\mathbb{L}$ is a set of prime numbers, a $\mathbb{L}$-integer is a product of elements of $\mathbb{L}$.

Definition 1.1. — *A morphism $h : Q \to P$ of fs monoids is* Kummer *(resp. $\mathbb{L}$-Kummer) if h is injective and for every $a \in P$, there exists a positive integer (an $\mathbb{L}$-integer) n such that $a^n \in h(Q)$ (note that if $Q \to P$ is Kummer,* $\operatorname{Spec} P \to \operatorname{Spec} Q$ *is an homeomorphism).*

A morphism $f : X \to Y$ of fs log schemes is said to be Kummer *(resp.* exact*) if for every geometric point $\bar{x}$ of X, $\overline{M}_{Y,f(\bar{x})} \to \overline{M}_{X,\bar{x}}$ is Kummer (resp. exact).*

A morphism of fs log scheme is Kummer étale *(or két for short) if it is Kummer and log étale.*

A morphism f is két if and only if étale locally it is deduced by strict base change and étale localization from a map $\operatorname{Spec} \mathbf{Z}[P] \to \operatorname{Spec} \mathbf{Z}[Q]$ induced by a Kummer map $Q \to P$ such that $nP \subset Q$ for some n invertible on X.

In fact if $f : Y \to X$ is két, $\bar{y}$ is a geometric point of Y, and $P \to M_X$ is an exact chart of X at $f(\bar{y})$, there is an étale neighborhood U of $\bar{x}$ and a Zariski open neighborhood $V \subset f^{-1}(U)$ of $\bar{y}$ such that $V \to U$ is isomorphic to $U \times_{\operatorname{Spec} \mathbf{Z}[P]} \operatorname{Spec} \mathbf{Z}[Q]$ with $P \to Q$ a $\mathbb{L}$-Kummer morphism where $\mathbb{L}$ is the set of primes invertible on U ([22, prop. 3.1.4]).

Két morphisms are open and quasi-finite.

The category of két fs log schemes over X (any X-morphism between two such fs log schemes is then két) where the covering families $(T_i \to T)$ of T are the families that are set-theoretical covering families (being a set-theoretical covering két family is stable under fs base change) is a site. We will denote by $X_{\text{két}}$ the corresponding topos. Any locally constant finite object of $X_{\text{két}}$ is representable.

Definition 1.2. — *A két fs log scheme over X which represents such a locally constant finite sheaf is called a* két cover *of X. The category of két covers of $X_{\text{két}}$ is denoted by* $\mathrm{KCov}(X)$.

A log geometric point is a log scheme s such that $\mathring{s}$ is the spectrum of a separably closed field k and M_s is saturated and multiplication by n on $\overline{M}_s$ is an isomorphism for every n prime to the characteristic of k.

A *log geometric point* of X is a morphism $x : s \to X$ of log schemes where s is a log geometric point. A *pointed log scheme* (X, x) is a log scheme X endowed with a log geometric point x. A két neighborhood U of $x : s \to X$ in X is a morphism $s \to U$ of X-log schemes where $U \to X$ is két. Then if x is a log geometric point of X, the functor F_x from $X_{\text{két}}$ to Set defined by $\mathcal{F} \mapsto \varinjlim_U \mathcal{F}(U)$ where U runs through the directed category of két neighborhoods of x is a point of the topos $X_{\text{két}}$ and any point of this topos is isomorphic to F_x for some log geometric point and the family of points (F_x) where x runs through log geometric points of X is a conservative system of points.

Definition 1.3. — *The inverse limit in the category of saturated log schemes of the két neighborhoods of x is called the* log strict specialization, *and is denoted by $X(x)$.*

If x and y are log geometric points of x, a specialization *of log geometric points $x \to y$ is a morphism $X(x) \to X(y)$ over X.*

A specialization $x \to y$ induces a canonical morphism $F_y \to F_x$ of functors.

If there is a specialization $x \to y$ of the underlying topological points, then there is some specialization $x \to y$ of log geometric points.

If X is connected, for any log geometric point x of X, F_x induces a fundamental functor $\mathrm{KCov}(X) \to \mathrm{fSet}$ of the Galois category $\mathrm{KCov}(X)$.

Definition 1.4. — *The* két fundamental group $\pi_1^{\log}(X, x)$ *is the profinite group of automorphisms of the fundamental functor* $\mathrm{KCov}(X) \to \mathrm{fSet}$.

Strict étale surjective morphisms satisfy effective descent for két covers ([22, prop. 3.2.19]).

If $f : S' \to S$ is an exact morphism of fs log schemes such that $\mathring{f}$ is proper, surjective and of finite presentation, then f satisfies effective descent for két covers ([22, th. 3.2.25]).

Proposition 1.5. — *Let $X \to S$ be a morphism of fs log schemes such that $\mathring{S}$ is locally noetherian and $\mathring{X} \to \mathring{S}$ is of finite type. Let $\tilde{s}$ be a geometric point of $\mathring{S}$ and let $S(\tilde{s})$ be the strict localization of S at $\tilde{s}$ endowed* *with the pullback log structure. Then the functor* $F : \varinjlim_U \mathrm{KCov}(X_U) \to \mathrm{KCov}(X_{S(\tilde{s})})$*, where* U *goes through étale neighbrhoods of* $\tilde{s}$*, is an equivalence of categories.*

Proof. — Let Y_U be a két cover of X_U such that, cofinally on V, Y_V is connected. Then $Y_U \times_U S(\tilde{s})$ is connected according to [7, prop. 8.4.4]. This proves that F is fully faithful, or equivalently the outer morphism of fundamental groups of Galois categories is surjective ([6, prop. V.6.10]).

Let $Y \to X_{S(\tilde{s})}$ be a két cover. Since one knows that F is fully faithful for any X and surjective étale morphisms satisfy effective descent for két covers, one only has to prove the essential surjectivity két locally on X, so that one may assume that X has a fs chart $X \to \mathrm{Spec}\,\mathbf{Z}[P]$. Let P be the characteristic of $\tilde{s}$. Then there is a (p')-Kummer morphism of monoids $P \to Q$ such that $Y_Q := Y \times_{\mathrm{Spec}\,\mathbf{Z}[P]} \mathrm{Spec}\,\mathbf{Z}[Q]$ is strict étale over $X_{S(\tilde{s}),Q} := X_{S(\tilde{s})} \times_{\mathrm{Spec}\,\mathbf{Z}[P]} \mathrm{Spec}\,\mathbf{Z}[Q]$. There exists a neighborhood U and an étale cover $Y_{U,Q}$ of $X_{U,Q}$ such that $Y_Q = Y_{U,Q} \times_{X_U} X_{S(\tilde{s})}$. Thus Y_Q is in the essential image of F. This proves that the outer morphism of fundamental groups corresponding to F is injective ([6, cor. V.6.8]), and therefore F is an equivalence. □

Let us now state the main results to compare log fundamental groups of different log schemes (in particular specialization comparisons). According to [8, th. 7.6], if X is a log regular fs log scheme, $\mathrm{KCov}(X)$ is equivalent to the category of tamely ramified covers of X_{tr}. If $\mathbb{L}$ is a set of primes invertible on X, by taking the pro-$\mathbb{L}$ completion, one gets:

Theorem 1.6. — *If* X *is a log regular fs log scheme and all the primes of* $\mathbb{L}$ *are invertible on* X*, then* $\mathrm{KCov}(X)^{\mathbb{L}} \to \mathrm{Cov}^{\mathrm{alg}}(X_{\mathrm{tr}})^{\mathbb{L}}$ *is an equivalence of categories.*

For example, if X is a regular scheme and D is a normal crossing divisor and $j : U := X \backslash D \to X$ is the open immersion, then $M_X = O_X \cap j_* O^*_{X\backslash D}$ is a log structure on X for which X is log regular and $X_{\mathrm{tr}} = U := X \backslash D$ (for example, if $X = \mathrm{Spec}\, O_K$ where O_K is a complete discretely valued ring and D is the special point of X, then $M_X = O_X \cap j_* O^*_{X\backslash D}$ is the usual log structure of $\mathrm{Spec}\, O_K$). Thus there is an equivalence of categories $\mathrm{KCov}(X)^{\mathbb{L}} \to \mathrm{Cov}^{\mathrm{alg}}(U)^{\mathbb{L}}$.

Proposition 1.7 ([19, cor. 2.3]). — *Let* S *be a noetherian strictly local scheme with closed point* s *and let* X *be a connected fs log scheme such that* $\mathring{X}$ *is proper over* S*. Then*

$$\mathrm{KCov}(X) \to \mathrm{KCov}(X_s)$$

is an equivalence of categories.

Recall that a strictly local scheme is a henselian scheme such that the residue field at the closed point is separably closed. One can extend Proposition 1.7 to henselian schemes:

Theorem 1.8. — *Let S be a noetherian henselian scheme with closed point s, and let X be a connected fs log scheme such that $\mathring{X}$ is proper over S. Then*

$$\mathrm{KCov}(X) \to \mathrm{KCov}(X_s)$$

is an equivalence of categories.

Proof. — First assume X_s to be geometrically connected. Let x be a log geometric point of X_s. Then X is also connected and we have to prove that $\pi_1^{\log}(X_s, x) \to \pi_1^{\log}(X, x)$ is an isomorphism. Let $\overline{s}$ be a strict localization of s and let $\overline{S}$ be the strict localization of S at $\overline{s}$. Let $\bar{x}$ be a log geometric point of $X_{\overline{s}}$ above x. Let S_i be a pointed Galois cover of S, let G_i be its Galois group and let $s_i = s \times_S S_i$. Then we have a diagram with exact lines:

$$\begin{tikzcd}
1 \arrow[r] & \pi_1^{\log}(X_{s_i}, \bar{x}) \arrow[r] \arrow[d] & \pi_1^{\log}(X_s, x) \arrow[r] \arrow[d] & G_i \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & \pi_1^{\log}(X_{S_i}, \bar{x}) \arrow[r] & \pi_1^{\log}(X, x) \arrow[r] & G_i \arrow[r] & 1
\end{tikzcd}$$

By taking the projective limit when S_i runs through the category of pointed Galois cover of S, one gets a diagram with exact lines

$$\begin{tikzcd}
1 \arrow[r] & \varprojlim_{S_i} \pi_1^{\log}(X_{s_i}, \bar{x}) \arrow[r] \arrow[d] & \pi_1^{\log}(X_s, x) \arrow[r] \arrow[d] & \pi_1^{\mathrm{alg}}(S, \overline{s}) \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & \varprojlim_{S_i} \pi_1^{\log}(X_{S_i}, \bar{x}) \arrow[r] & \pi_1^{\log}(X, x) \arrow[r] & \pi_1^{\mathrm{alg}}(S, \overline{s}) \arrow[r] & 1
\end{tikzcd}$$

But, according to Proposition 1.5, $\pi_1^{\log}(X_{\overline{S}}, \bar{x}) \to \varprojlim_{S_i} \pi_1^{\log}(X_{S_i}, \bar{x})$ is an isomorphism. Similarly $\pi_1^{\log}(X_{\overline{s}}, \overline{x}) \to \varprojlim_{S_i} \pi_1^{\log}(X_{s_i}, \overline{x})$ is an isomorphism. Thanks to Proposition 1.7, $\pi_1^{\log}(X_{\overline{s}}, \overline{x}) \to \pi_1^{\log}(X_{\overline{S}}, \overline{x})$ is an isomorphism. Thus $\pi_1^{\log}(X_s, x) \to \pi_1^{\log}(X, x)$ is also an isomorphism.

In the general case, let $X \to S'$ be the Stein factorization of $X \to S$. For every connected component S'_j of S', let $X_j = X \times_{S'} S'_j$. Since S'_j is henselian and $X_j \to S'_j$ has geometrically connected fibers, one gets that $\mathrm{KCov}(X_{j,s}) \to \mathrm{KCov}(X_j)$ is an equivalence of category. Since $\mathrm{KCov}(X) =$

$\prod_j \mathrm{KCov}(X_j)$ and $\mathrm{KCov}(X_s) = \prod \mathrm{KCov}(X_{j,s})$, one gets that $\mathrm{KCov}(X) \to \mathrm{KCov}(X_s)$ is an equivalence of categories. □

Corollary 1.9. — *Let O_K be a complete discretely valued ring endowed with its natural log structure and let $\mathbb{L}$ a set of prime numbers invertible in O_K. Let $X \to \operatorname{Spec} O_K$ be a proper and log smooth morphism and let $U := X_{\mathrm{tr}} \subset X_\eta$. There is a natural equivalence of categories*

$$\mathrm{KCov}(X_s)^{\mathbb{L}} \simeq \mathrm{Cov}^{\mathrm{alg}}(U)^{\mathbb{L}}.$$

In particular, if $\mathring{X} \to \operatorname{Spec} O_K$ is a semistable model of X_η, and the log structure on X is given by $M_X = O_X \cap j_* O^*_{X_\eta}$ where $j : X_\eta \to X$, then $X \to \operatorname{Spec} O_K$ is log smooth and $X_{\mathrm{tr}} = X_\eta$. We get an equivalence of categories $\mathrm{KCov}(X_s)^{\mathbb{L}} \simeq \mathrm{Cov}^{\mathrm{alg}}(X_\eta)^{\mathbb{L}}$, and thus an isomorphism

$$\pi_1^{\mathrm{alg}}(X_\eta)^{\mathbb{L}} \to \pi_1^{\log}(X_s)^{\mathbb{L}}.$$

Here we recall basic results about saturated morphisms of fs log schemes. The main reference on the subject is [23], which is unfortunately unpublished.

Definition 1.10. — *A morphism of fs monoids $P \to Q$ is* integral *if, for any morphism of integral monoids $P \to Q'$, the amalgamated sum $Q \oplus_P Q'$ is still integral.*

An integral morphism of fs monoids $P \to Q$ is saturated *if, for any morphism of fs monoids $P \to Q'$, the amalgamated sum $Q \oplus_P Q'$ is still fs.*

A morphism $f : Y \to X$ of fs log schemes is saturated *if for any geometric point $\bar{y}$ of Y, $\bar{M}_{X,f(\bar{y})} \to \bar{M}_{Y,\bar{y}}$ is saturated.*

If $Y \to X$ is saturated and $Z \to X$ is a morphism of fs log schemes, then the underlying scheme of $Z \times_X Y$ is $\mathring{Z} \times_{\mathring{X}} \mathring{Y}$.

If $P \to Q$ is a local and integral (resp. saturated) morphism of fs monoids and P is sharp, the morphism $\operatorname{Spec} \mathbf{Z}[Q] \to \operatorname{Spec} \mathbf{Z}[P]$ is flat (resp. separable, *i.e.*, flat with geometrically reduced fibers, *cf.* [17, cor. I.4.3.16] and [9, rem. 6.3.3]).

Let $f : X \to Y$ be log smooth, let $\bar{x}$ be a geometric point of X and let $\bar{y}$ be its image in Y. Étale locally on Y, there is a chart $Y \to \operatorname{Spec} P$ such that $P \to \overline{M}_{Y,\bar{y}}$ is an isomorphism. Then, according to [11, th. 3.5], there is étale locally at x a fs chart $\phi : P \to Q$ of $X \to Y$ such that $Y \to \operatorname{Spec} \mathbf{Z}[Q] \times_{\mathbf{Z}[P]} X$ is étale such that ϕ is injective and the torsion part of $\mathrm{Coker}(\phi^{\mathrm{gp}})$ has order invertible on X. Up to localizing Q by the face corresponding to $\bar{x}$, one can assume that $Q \to M_{X,\bar{x}}$ is local (and thus exact according to [17, def. II.2.2.8]). Thus if f is integral (resp. saturated),

$P \to Q$ is a local and integral (resp. saturated) morphism of fs monoids and P is sharp. Thus f is flat (resp. separable).

If $P \to Q$ is an integral morphism of fs monoids, there exists an integer n such that the pullback $P_n \to Q'$ of $P \to Q$ along $P \xrightarrow{n} P = P_n$ is saturated (theorem [9, A.4.2]).

Moreover if $P \to Q$ factors through Q_0 such that $P \to Q_0$ is saturated and $Q_0 \to Q$ is $\mathbb{L}$-Kummer, n can be chosen to be an $\mathbb{L}$-integer.

A morphism $X \to S$ of fs log schemes is said to be *log geometrically saturated* if there exists a két covering $U \to S$ such that $X \times_S U$ is saturated.

For example, if $Y \to S$ is a morphism of fs log schemes, with $\mathring{S}$ locally noetherian and $\mathring{Y} \to \mathring{S}$ of finite type, which factors through X such that $X \to S$ is saturated and $Y \to X$ is két, then $Y \to S$ is log geometrically saturated.

1.2. Log geometric fundamental groups

Let $X \to S$ be a morphism of fs log schemes. Let $\bar{x}$ be a log geometric point of X and let $\bar{s}$ be its image in S. The morphism $X \to S$ is said to be *log geometrically connected* at $\bar{s}$ if there exists a cofinal family of két neighborhoods U of $\bar{s}$ in S such that X_U is connected.

The *log geometric fundamental group* of X at $\bar{x}$ to be

$$\pi_1^{\text{log-geom}}(X/(S,\bar{s}),\bar{x}) := \varprojlim_U \pi_1^{\log}(X_U,\bar{x}),$$

where U runs through két neighborhoods of $\bar{s}$ in S. If $X \to S$ is log geometrically connected, the category $\pi_1^{\text{log-geom}}(X/(S,\bar{s}),\bar{x})$-fSet of finite sets endowed with a continuous action of $\pi_1^{\text{log-geom}}(X/s,\bar{x})$ is equivalent to the category

$$\mathrm{KCov}_{\text{geom}}(X/(S,\bar{s})) := \varinjlim_U \mathrm{KCov}(X_U).$$

In particular, $\pi_1^{\text{log-geom}}(X/(S,\bar{s}),\bar{x})$ does not depend on $\bar{x}$ up to outer isomorphism. Therefore, when we work in the category of groups with outer morphisms, the log geometric fundamental group will simply be denoted by $\pi_1^{\text{log-geom}}(X/(S,\bar{s}))$.

If $\bar{s}' \to \bar{s}$ is a specialization of log geometric points of S, there is a natural morphism of pro-log schemes "$\varprojlim_{\bar{s}'\in U}$" $U \to$ "$\varprojlim_{\bar{s}\in V}$" V, where U goes through két neighborhoods of $\bar{s}'$ and V goes through két neighborhoods of $\bar{s}$. This induces a functor, 2-functorially in $\bar{s}' \to \bar{s}$,

$$\mathrm{KCov}_{\text{geom}}(X/(S,\bar{s})) \to \mathrm{KCov}_{\text{geom}}(X/(S,\bar{s}')), \tag{1.1}$$

hence an outer morphism, functorially in $\bar{s}' \to \bar{s}$,

$$\pi_1^{\text{log-geom}}(X/(S,\bar{s}')) \to \pi_1^{\text{log-geom}}(X/(S,\bar{s})).$$

Let $(S',\bar{s}') \to (S,\bar{s})$ be a morphism of pointed fs log schemes. There is a natural morphism of pro-log schemes "$\varprojlim_{\bar{s}'\in U}$" $U \to$ "$\varprojlim_{\bar{s}\in V}$" V, where U goes through két neighborhoods of $\bar{s}'$ in S' and V goes through két neighborhoods of $\bar{s}$ in S. This induces a functor

$$\mathrm{KCov}_{\mathrm{geom}}(X/(S,\bar{s})) \to \mathrm{KCov}_{\mathrm{geom}}(X'/(S',\bar{s}'))$$

where $X' := X \times SS'$, hence an outer morphism

$$\pi_1^{\text{log-geom}}(X'/(S',\bar{s}')) \to \pi_1^{\text{log-geom}}(X/(S,\bar{s})).$$

Proposition 1.11. — *Let $X \to S$ be a morphism of fs log schemes such that $\mathring{S}$ is locally noetherian and $\mathring{X} \to \mathring{S}$ is of finite type. Let $\tilde{s}$ be the geometric point of $\mathring{S}$ defined by $\bar{s}$ and let $S(\tilde{s})$ be the strict localization of S at $\tilde{s}$ endowed with the pullback log structure. The morphism*

$$\pi_1^{\text{log-geom}}(X_{S(\tilde{s})}/(S(\tilde{s}),\bar{s}),\bar{x}) \to \pi_1^{\text{log-geom}}(X/(S,\bar{s}),\bar{x})$$

is an isomorphism.

Proof. — Let $\mathbb{L}$ be the set of primes invertible at $\bar{s}$. By replacing S by an étale neighborhood of $\tilde{s}$, one can assume that S has a chart $S \to \operatorname{Spec}\mathbf{Z}[P]$ such that the induced map $P \to \overline{M}_{S,\tilde{s}}$ is an isomorphism. Extend the map $P^{\mathrm{gp}} \to M^{\mathrm{gp}}_{\bar{s},\tilde{s}}$ into a map $P^{\mathrm{gp}} \otimes \mathbf{Z}[\frac{1}{\mathbb{L}}] \to M^{\mathrm{gp}}_{\bar{s},\tilde{s}}$: this defines for every $\mathbb{L}$-két morphism $P \to Q$ of sharp fs monoids a morphism $\bar{s}$-log point of $S(\tilde{s})_Q := S(\tilde{s}) \times_{\operatorname{Spec}\mathbf{Z}[P]} \operatorname{Spec}\mathbf{Z}[Q]$. When $P \to Q$ goes through $\mathbb{L}$-két morphism $P \to Q$ of sharp monoids, the family $(S(\tilde{s})_Q)$ goes through neighborhoods of $\bar{s}$ in $S(\tilde{s})$. One has also $\pi_1^{\text{log-geom}}(X_{S(\tilde{s})}/(S(\tilde{s}),\bar{s}),\bar{x}) = \varprojlim_{P\to Q}\varprojlim_U \pi_1^{\log}(X_U,\bar{x})$ where $P \to Q$ goes through $\mathbb{L}$-két morphisms of sharp monoids and U goes through étale neighborhoods of $\bar{s}$ in $X_Q := X \times_{\operatorname{Spec}\mathbf{Z}[P]} \operatorname{Spec}\mathbf{Z}[Q]$. Then $\pi_1^{\log}(X_{S(\tilde{s})_Q},\bar{x}) \to \varprojlim_U \pi_1^{\log}(X_U,\bar{x})$, where U goes through étale neighborhoods of $\bar{x}$ in X_Q, is an isomorphism according to Proposition 1.5. Since $\pi_1^{\text{log-geom}}(X/(S,\bar{s}),\bar{x}) = \varprojlim_Q \pi_1^{\log}(X_{S(\tilde{s})_Q},\bar{x})$ one gets the result. □

Assume $\mathring{S}$ to be a henselian local scheme. Let $(T,\bar{t})$ be a pointed Galois két cover of $(S,\bar{s})$. Then one has an exact sequence:

$$1 \to \pi_1^{\log}(X_t,\bar{x}_t) \to \pi_1^{\log}(X,\bar{x}) \to \mathrm{Gal}(t/s),$$

and the right map is onto if X_t is connected. By taking the projective limit of the previous exact sequence when $(t,\bar{t})$ runs through the directed category of pointed Galois connected covers of $(s,\bar{s})$, one gets an exact sequence

$$1 \to \pi_1^{\text{log-geom}}(X/(S,\bar{s}),\bar{x}) \to \pi_1^{\log}(X,\bar{x}) \to \pi_1^{\log}(S,\bar{s}),$$

and the right map is onto if $X \to S$ is log geometrically connected.

Let O_K be a complete discretely valued ring endowed with its natural log structure and let $\mathbb{L}$ a set of prime numbers invertible in O_K. Let $X \to \operatorname{Spec} O_K$ be a proper and log smooth morphism and let $U := X_{\mathrm{tr}} \subset X_\eta$. There is a geometric analog to the specialization isomorphism $\pi_1^{\mathrm{alg}}(X_\eta)^{\mathbb{L}} \to \pi_1^{\log}(X_s)^{\mathbb{L}}$ of Corollary 1.9:

Theorem 1.12 ([12, th. 1.4]). — *There is a natural equivalence of categories*

$$\mathrm{KCov}_{\mathrm{geom}}(X/s)^{\mathbb{L}} \simeq \mathrm{Cov}^{\mathrm{alg}}(U_{\bar{\eta}})^{\mathbb{L}}.$$

It can be deduced from Corollary 1.9 thanks to the fact that any algebraic cover of $U_{\bar{\eta}}$ is already defined over a tamely ramified extension of K ([12, prop. 1.15]).

1.3. Specialization of log fundamental groups

Let us study specialization of log geometric fundamental groups (that is the projective limit of the log fundamental groups after taking két extensions of the base log point).

The only result we will need later on is the following:

Proposition 1.13 (cor. 1.17). — *Let $X \to S$ be a proper and saturated morphism of log schemes such that $\mathring{S}$ is locally noetherian, and let $Y \to X$ be a két cover. Let $(s,\bar{s})$ and $(s',\bar{s}')$ be two pointed fs log points of S and let $\bar{s}' \to \bar{s}$ be a specialization of log geometric points. Let $\mathbb{L}$ be a set of primes that does not contain the characteristic of s. One has a specialization outer morphism*

$$\pi_1^{\text{log-geom}}(Y_{s'}/(s',\bar{s}'))^{\mathbb{L}} \to \pi_1^{\text{log-geom}}(Y_s/(s,\bar{s}))^{\mathbb{L}}.$$

Moreover this morphism factors through $\pi_1^{\text{log-geom}}(Y/(S,\bar{s}))^{\mathbb{L}}$.

To prove this, our main result will be the invariance of the log geometric fundamental group of an fs log scheme X saturated and of finite type over an fs log point S with separably closed field by fs base change that is an isomorphism on the underlying scheme. The assumptions implies that our base change induces an isomorphism of the underlying schemes. Working étale locally on this scheme, we are reduced to the case where this scheme is strictly local, where the log geometric fundamental group can be explicitly described in terms of the morphism of monoids $\overline{M}_X \to \overline{M}_S$.

Combining this base change invariance result with strict base change invariance of the $\mathbb{L}$-log geometric fundamental group and strict specialization of the $\mathbb{L}$-log geometric fundamental group ([19]), we will get that if $X \to S$ is a proper log smooth saturated morphism, and s_2, s_1 are fs points of S and $\bar{s}_2 \to \bar{s}_1$ is a specialization of log geometric points of S over s_2 and s_1, then there is a specialization morphism $\pi_1^{\text{log-geom}}(X_{s_2})^{\mathbb{L}} \to \pi_1^{\text{log-geom}}(X_{s_1})^{\mathbb{L}}$.

Lemma 1.14. — *Let $s' \to s$ be a strict morphism of fs log points such that $\mathring{s}'$ and $\mathring{s}$ are geometric points. Let $\mathbb{L}$ be a set of primes that does not contain the characteristic of s. Let $X \to s$ be a morphism of fs log schemes such that $\mathring{X} \to \mathring{s}$ is of finite type.*

Then $F : \mathrm{KCov}(X)^{\mathbb{L}} \to \mathrm{KCov}(X_{s'})^{\mathbb{L}}$ is an equivalence of categories.

Proof. — If T is a connected két cover of X, $T \overset{\circ}{\times}_s s' \to \mathring{T} \times_{\mathring{s}} \mathring{s}'$ is an isomorphism since $s' \to s$ is strict. The scheme $\mathring{T} \times_{\mathring{s}} \mathring{s}'$ is connected too, so we get that the functor F is fully faithful.

As one already knows that F is fully faithful for any X, and as strict étale surjective morphisms satisfy effective descent for két covers, one may prove the essential surjectivity étale locally, and thus assume that X has a global chart $X \to \operatorname{Spec} \mathbf{Z}[P]$.

Let Y' be a $\mathbb{L}$-két cover of $X_{s'}$. Then there exists a $\mathbb{L}$-Kummer morphism of monoids $P \to Q$ such that

$$Y'_Q := Y' \times_{\operatorname{Spec} \mathbf{Z}[P]} \operatorname{Spec} \mathbf{Z}[Q] \to X_{s',Q} := X_{s'} \times_{\operatorname{Spec} \mathbf{Z}[P]} \operatorname{Spec} \mathbf{Z}[Q]$$

is strict étale (and surjective).

But, since $\mathring{X}_{s',Q} \to \mathring{X}_Q \times_{\mathring{s}} \mathring{s}'$ is an isomorphism of schemes, $\mathrm{Cov}^{\mathrm{alg}}(\mathring{X}_{s',Q})^{\mathbb{L}} \to \mathrm{Cov}^{\mathrm{alg}}(\mathring{X}_Q)^{\mathbb{L}}$ is an equivalence of categories ([20, cor 4.5]). Thus, there is a strict étale cover Y_Q of X_Q (and thus $Y_Q \to X$ is a két cover) such that Y'_Q is $X_{s',Q}$-isomorphic to $Y_Q \times_s s'$.

Thus F is an equivalence of categories. □

Let now $s' \to s$ be a morphism of fs log points, such that the underlying morphism of schemes $\mathring{s}' \to \mathring{s}$ is an isomorphism of geometric points, and let $X \to s$ be a saturated morphism of fs log schemes with $\mathring{X}$ noetherian and $\mathring{X} \to \mathring{s}$ connected. Since $X \to s$ is saturated, it is log geometrically connected.

Let $\bar{x}'$ be a log geometric point of $X' = X \times_s s'$ and let $\bar{x}$, $\bar{s}'$ and $\bar{s}$ be its image in X, s' and s respectively.

Theorem 1.15. — *Let $s' \to s$ be a morphism of fs log points, such that the underlying morphism of schemes is an isomorphism of geometric points, and let $X \to s$ be a saturated morphism of connected noetherian fs log schemes. The map $\pi_1^{\text{log-geom}}(X'/(s',\bar{s}'),\bar{x}') \to \pi_1^{\text{log-geom}}(X/(s,\bar{s}),\bar{x})$ is an isomorphism.*

Proof. — Let $(s_i,\bar{s}_i)_{i\in I}$ be a cofinal system of pointed Galois connected két covers of $(s,\bar{s})$. Let $\tilde{s}_i$ be the reduced subscheme of s_i endowed with the inverse image log structure. Let us write $(X_i,\bar{x}_i)=(X\times_s \tilde{s}_i, \bar{x}\times_{\bar{s}} \bar{s}_i)$.

Let $(s'_j,\bar{s}'_j)_{j\in J}$ be a cofinal system of pointed Galois connected két covers of $(s',\bar{s}')$. Let $\tilde{s}'_j$ be the reduced subscheme of s'_j endowed with the inverse image log structure. Let us write $(X'_j,\bar{x}'_j)=(X\times_{s'} s'_j, \bar{x}'\times_{\bar{s}'} \bar{s}'_j)$.

One has to prove that

$$\varprojlim_j \pi_1^{\log}(X'_j,\bar{x}'_j) \to \varprojlim_i \pi_1^{\log}(X_i,\bar{x}_i)$$

is an isomorphism, or equivalently that

$$\underset{\substack{\longrightarrow\\ i}}{\mathrm{Lim}}\, \mathrm{KCov}(X_i) \to \underset{\substack{\longrightarrow\\ j}}{\mathrm{Lim}}\, \mathrm{KCov}(X'_j)$$

is an equivalence of categories.

Since strict étale surjective morphisms satisfy effective descent for két covers, the injective limits are filtering and X is quasicompact, it is enough to prove that

$$\underset{\substack{\longrightarrow\\ i}}{\mathrm{Lim}}\, \mathrm{KCov}(X_i) \to \underset{\substack{\longrightarrow\\ j}}{\mathrm{Lim}}\, \mathrm{KCov}(X'_j)$$

is an isomorphism locally on the étale topology of X.

According to Proposition 1.5, if $\bar{x}$ is a geometric point of $\mathring{X}$, then

$$\underset{\substack{\longrightarrow\\ \bar{x}\in U}}{\mathrm{Lim}}\, \mathrm{KCov}(U) \to \mathrm{KCov}(X(\bar{x})),$$

where U goes through étale neighborhoods of $\bar{x}$, is an equivalence of categories. Since X_i, X'_j and X have equivalent étale topoi, $\bar{x}$ also defines a point of the étale topoi of X_i and X'_j. According to [5, cor. III.2.1.5.8], one only has to prove that

$$\underset{\substack{\longrightarrow\\ i}}{\mathrm{Lim}}\, \mathrm{KCov}(X_i(\bar{x})) \to \underset{\longrightarrow}{\mathrm{Lim}}\, \mathrm{KCov}(X'_j(\bar{x}))$$

for every geometric point $\bar{x}$ of X.

We are thus reduced to the case where $\mathring{X}$ is a strictly local and noetherian scheme. But then ([22, prop. 3.1.11]), for $\mathring{X}$ a strictly local and noetherian scheme,

$$\begin{aligned}\varprojlim_i \pi_1^{\log}(X_{i,\text{két}},\bar{x}_i) &= \varprojlim_i \operatorname{Hom}(\overline{M}^{\text{gp}}_{X_i,x_i},\widehat{\mathbf{Z}}^{(p')}) \\ &= \operatorname{Hom}(\varinjlim_i \overline{M}^{\text{gp}}_{X_i,x_i},\widehat{\mathbf{Z}}^{(p')}) \\ &= \operatorname{Hom}(\operatorname{Coker}(\overline{M}^{\text{gp}}_s \to \overline{M}^{\text{gp}}_{X,x}),\widehat{\mathbf{Z}}^{(p')}),\end{aligned}$$

and one has a similar result for X'.

Since $\overline{M}_{X',x'} = \overline{M}_{X,x} \oplus_{\overline{M}_s} \overline{M}_{s'}$, one has $\overline{M}^{\text{gp}}_{X',x'} = \overline{M}^{\text{gp}}_{X,x} \oplus_{\overline{M}^{\text{gp}}_s} \overline{M}^{\text{gp}}_{s'}$. Thus, $\operatorname{Coker}(\overline{M}^{\text{gp}}_s \to \overline{M}^{\text{gp}}_{X,x}) \to \operatorname{Coker}(\overline{M}^{\text{gp}}_{s'} \to \overline{M}^{\text{gp}}_{X',x'})$ is an isomorphism.

One thus gets the wanted result. □

Assume now that $(s',\bar{s}') \to (s,\bar{s})$ is a morphism of pointed fs log points, and that $X \to s$ is log geometrically saturated. Recall that this assumption is satisfied if $X \to s$ goes through X_0 such that $X \to X_0$ is két and $X_0 \to s$ is saturated.

Let $\mathbb{L}$ be a set of prime that does not contain the characteristic of s.

Corollary 1.16. — *The map of profinite groups*

$$\pi_1^{\text{log-geom}}(X/(s,\bar{s}),\bar{x})^{\mathbb{L}} \to \pi_1^{\text{log-geom}}(X'/(s',\bar{s}'),\bar{x}')^{\mathbb{L}}$$

is an isomorphism.

Proof. — By replacing s (resp. s') by the closed reduced subscheme of a connected két cover of s (resp. s'), one can assume that $X \to s$ is saturated ($\mathring{X} \to \mathring{s}$ will still be of finite type).

If $(t,\bar{t}) \to (s,\bar{s})$ is a strict étale cover, then

$$\pi_1^{\text{log-geom}}(X_t/t,\bar{x}_t) \to \pi_1^{\text{log-geom}}(X/s,\bar{x})$$

is an isomorphism. Thus, by writing s_0 for the separable closure of s and by taking the projective limit over pointed strict étale covers (since $\pi_1^{\log}(X_{s_0}) = \varprojlim \pi_1^{\log}(X_t)$, where t runs through pointed strict étale covers of s), one gets that $\pi_1^{\text{log-geom}}(X_{s_0}/s_0,\bar{x}_0) \to \pi_1^{\text{log-geom}}(X/s,\bar{x})$ is an isomorphism. One thus may assume that $\mathring{s}$ and $\mathring{s}'$ are geometric points.

Let us consider the fs log scheme s'' whose underlying scheme is $\mathring{s}'$ and whose log structure is the inverse image of the log structure of s. Thus, one has morphisms $s' \to s'' \to s$, where $s' \to s''$ is an isomorphism on the underlying schemes and $s'' \to s$ is strict. But according to Lemma 1.14, $\pi_1^{\log}(X_{s''})^{\mathbb{L}} \to \pi_1^{\log}(X)^{\mathbb{L}}$ and $\pi_1^{\log}(s'')^{\mathbb{L}} \to \pi_1^{\log}(s)^{\mathbb{L}}$ are isomorphisms. Thus,

$$\pi_1^{\text{log-geom}}(X_{s''}/s'')^{\mathbb{L}} \to \pi_1^{\text{log-geom}}(X/s)^{\mathbb{L}}$$

is an isomorphism. By 1.15, $\pi_1^{\text{log-geom}}(X_{s'}/s')^{\mathbb{L}} \to \pi_1^{\text{log-geom}}(X_{s''}/s'')^{\mathbb{L}}$ is also an isomorphism. □

Corollary 1.17. — *Let $X \to S$ be a proper log geometrically saturated morphism of fs log schemes such that $\mathring{S}$ is locally noetherian. Let $(s, \bar{s})$ and $(s', \bar{s}')$ be two pointed fs log points and let $\bar{s}' \to \bar{s}$ be a specialization of log geometric points. Let $\mathbb{L}$ be a set of primes that does not contain the characteristic of s. The functor*

$$\phi_s : \mathrm{KCov}_{\mathrm{geom}}(X/(S, \bar{s}))^{\mathbb{L}} \to \mathrm{KCov}_{\mathrm{geom}}(X_s/(s, \bar{s}))$$

is an equivalence. Therefore, there is a pair $(\psi_{s/s'}, \alpha)$, where $\psi_{s/s'}$ is an exact functor

$$\psi_{s/s'} : \mathrm{KCov}_{\mathrm{geom}}(X_s/(s, \bar{s}))^{\mathbb{L}} \to \mathrm{KCov}_{\mathrm{geom}}(X_{s'}/(s', \bar{s}'))^{\mathbb{L}}$$

and a natural 2-isomorphism α

$$\begin{tikzcd}
\mathrm{KCov}_{\mathrm{geom}}(X/(S, \bar{s}))^{\mathbb{L}} \arrow[r, "\phi_{s/s'}"] \arrow[d, "\phi_s"] & \mathrm{KCov}_{\mathrm{geom}}(X/(S, \bar{s}'))^{\mathbb{L}} \arrow[d, "\phi_{s'}"] \\
\mathrm{KCov}_{\mathrm{geom}}(X_s/(s, \bar{s}))^{\mathbb{L}} \arrow[r, "\psi_{s/s'}"] \arrow[ur, phantom, "\Nearrow \alpha"] & \mathrm{KCov}_{\mathrm{geom}}(X_{s'}/(s', \bar{s}'))^{\mathbb{L}},
\end{tikzcd}$$

unique in the sense that if $(\psi'_{s/s'}, \alpha')$ satisfies the same conditions, there is a unique 2-isomorphism $\beta : \psi_{s/s'} \to \psi'_{s/s'}$ such that $\alpha' \cdot (\phi_s \circ \beta) = \alpha$. Moreover, if $(s'', \bar{s}'')$ is a pointed fs log point and $\bar{s}'' \to \bar{s}'$ is a specialization, then there is a unique isomorphism of functors $\psi_{s/s''} \simeq \psi_{s'/s''}\psi_{s/s'}$ such that the following diagram is 2-commutative:

$$\begin{tikzcd}
\mathrm{KCov}_{\mathrm{geom}}(X/(S, \bar{s}))^{\mathbb{L}} \arrow[r] \arrow[drr] \arrow[dd] & \mathrm{KCov}_{\mathrm{geom}}(X/(S, \bar{s}'))^{\mathbb{L}} \arrow[dr] \arrow[dd] & \\
 & & \mathrm{KCov}_{\mathrm{geom}}(X/(S, \bar{s}''))^{\mathbb{L}} \arrow[dd] \\
\mathrm{KCov}_{\mathrm{geom}}(X_s/(s, \bar{s}))^{\mathbb{L}} \arrow[r] \arrow[drr] & \mathrm{KCov}_{\mathrm{geom}}(X_{s'}/(s', \bar{s}'))^{\mathbb{L}} \arrow[dr] & \\
 & & \mathrm{KCov}_{\mathrm{geom}}(X_{s''}/(s'', \bar{s}''))^{\mathbb{L}}
\end{tikzcd}$$

Proof. — Let Z be the strictly local scheme of S at s endowed with the inverse image log structure, and let z be its closed point, endowed with the inverse image log structure. The three morphisms

$$\pi_1^{\text{log-geom}}(X_s/(s, \bar{s}))^{\mathbb{L}} \to \pi_1^{\text{log-geom}}(X_z/(z, \bar{s}))^{\mathbb{L}},$$
$$\pi_1^{\text{log-geom}}(X_z/(z, \bar{s}))^{\mathbb{L}} \to \pi_1^{\text{log-geom}}(X_Z/(Z, \bar{s}))^{\mathbb{L}},$$
$$\pi_1^{\text{log-geom}}(X_Z/(Z, \bar{s}))^{\mathbb{L}} \to \pi_1^{\text{log-geom}}(X/(S, \bar{s}))^{\mathbb{L}}$$

are isomorphisms according to Corollary 1.16, Theorem 1.7 and Corollary 1.11. Therefore ϕ_s is an equivalence. The functor $\psi_{s/s'}$ is then the composition of $\phi_{s'}\phi_{s/s'}$ with a quasi-inverse of ϕ_s and the uniqueness is obvious. The exactness of $\psi_{s/s'}$ comes from the exactness of $\phi_{s/s'}$. The compatibility with composition is a direct consequence of the uniqueness of $\psi_{s/s''}$. □

Let $\mathrm{Pt}(S)$ be the category whose objects are pointed fs log points $(s,\bar{s})$ of S, and whose morphisms from $(s,\bar{s})$ to $(s',\bar{s}')$ are specialization of log geometric points $\bar{s}\to\bar{s}'$. Corollary 1.17 tells us that there is a 2-functor $\mathrm{KCov}_{\mathrm{geom}}(X_{(\ )})$ from $\mathrm{Pt}(S)^{\mathrm{op}}$ to the 2-category of Galois categories where 1-morphisms are exact functors which maps $(s,\bar{s})$ to $\mathrm{KCov}_{\mathrm{geom}}(X_s/(s,\bar{s}))$.

This 2-functor induces a functor $\pi_1^{\text{log-geom}}(X_{(\ )})$ from $\mathrm{Pt}(S)$ to the category of groups with outer morphisms which maps $(s,\bar{s})$ to $\pi_1^{\text{log-geom}}(X_s/(s,\bar{s}))$.

2. Cospecialization of graphs of semistable curves

2.1. Graphs

A *graph* $\mathbb{G}$ is given by a set of edges $\mathcal{E}$ a set of vertices $\mathcal{V}$ and for any $e\in\mathcal{E}$ a set of branches $\mathcal{B}_e$ of cardinality 2 and a map $\psi_e:\mathcal{B}_e\to\mathcal{V}$. A branch b of e can be thought of as an orientation of e (or a half-edge), and $\psi_e(b)$ is to be thought of as the ending of e when e is oriented according to b.

One can also equivalently replace the data of edges and branches of each edge by the datum of the set of all branches $\mathcal{B}=\coprod_e\mathcal{B}_e$, with an involution ι without fixed points (which corresponds heuristically to the reversing of the orientation given by the branch), and a map $\psi:\mathcal{B}\to\mathcal{V}$. The set $\mathcal{E}$ is then the set of orbits of branches for ι.

A *genuine morphism of graphs* $\phi:\mathbb{G}\to\mathbb{G}'$ is given by a map $\phi_{\mathcal{E}}:\mathcal{E}\to\mathcal{E}'$, a map $\phi_{\mathcal{V}}:\mathcal{V}\to\mathcal{V}'$ and for every $e\in\mathcal{E}$ a bijection $\phi_e:\mathcal{B}_e\to\mathcal{B}'_{\phi_{\mathcal{E}}(e)}$ such that the following diagram commutes:

$$\begin{tikzcd} \mathcal{B}_e \arrow[r] \arrow[d] & \mathcal{B}'_{\phi_{\mathcal{E}}(e)} \arrow[d] \\ \mathcal{V} \arrow[r] & \mathcal{V}' \end{tikzcd}$$

Remark that $\phi_{\mathcal{E}}$ and $\phi_{\mathcal{V}}$ are not enough to define ϕ if $\mathbb{G}$ has a loop (*i.e.*, an edge whose to branches abut to the same vertex): one can define an automorphism of $\mathbb{G}$ just by inverting the two branches of the loop. Thus, to know how the branches are mapped is important as soon as $\mathbb{G}$ or $\mathbb{G}'$ has loops.

The topological cospecialization for semistable curves will be given by maps of graphs which are not genuine morphisms. A *generalized morphism of graphs* $\phi : \mathbb{G} \to \mathbb{G}'$ will be given by:

- a map $\phi_{\mathcal{V}} : \mathcal{V} \to \mathcal{V}'$;
- a map $\phi_{\mathcal{E}} : \mathcal{E} \to \mathcal{E}' \coprod \mathcal{V}'$ such that, for any $e \in \mathcal{E}$ such that $\phi_{\mathcal{E}}(e) \in \mathcal{V}'$ and for any $b \in \mathcal{B}_e$, $\phi_{\mathcal{V}}\psi(b) = \phi_{\mathcal{E}}(e)$;
- for any $e \in \mathcal{E}$ such that $\phi_{\mathcal{E}}(e) \in \mathcal{E}'$, a bijection $\phi_e : \mathcal{B}_e \to \mathcal{B}'_{\phi_{\mathcal{E}}(e)}$ such that the obvious diagram commutes (it is the same diagram as in the case of genuine morphisms).

One can replace the last two data by the data of $\phi_{\mathcal{B}} : \mathcal{B} \to \mathcal{B}' \coprod \mathcal{V}'$ such that, if $\phi_{\mathcal{B}}(b) \in \mathcal{B}'$, then $\phi_{\mathcal{B}}(\iota(b)) = \iota'(\phi_{\mathcal{B}}(b))$ and $\phi_{\mathcal{V}}\psi(b) = \psi'\phi_{\mathcal{B}}(b)$, and, if $\phi_{\mathcal{B}}(b) \in \mathcal{V}'$, then $\phi_{\mathcal{B}}(\iota(b)) = \phi_{\mathcal{B}}(b) = \phi_{\mathcal{V}}\psi(b)$.

In particular, a genuine morphism is a generalized morphism. Genuine morphisms and generalized morphisms can be composed in an obvious way.

One thus gets a category Graph of graphs with genuine morphisms and a category GenGraph of graphs with generalized morphisms.

There is a geometric realization functor $|\ | : \mathrm{GenGraph} \to \mathrm{Top}$ which maps a graph $\mathbb{G}$ to

$$|\mathbb{G}| := \mathrm{Coker}\Big(\coprod_{b\in\mathcal{B}} \mathrm{pt}_{1,b} \amalg \mathrm{pt}_{2,b} \rightrightarrows \coprod_{v\in\mathcal{V}} \mathrm{pt}_v \amalg \coprod_{b\in\mathcal{B}} [1/2,1]_b\Big),$$

where

- the upper map sends:
  - $\mathrm{pt}_{1,b}$ to $1/2$ in $[1/2,1]_b$
  - $\mathrm{pt}_{2,b}$ to 1 in $[1/2,1]_b$,
- the lower map sends
  - $\mathrm{pt}_{1,b}$ to $1/2$ in $[1/2,1]_{\iota(b)}$
  - $\mathrm{pt}_{2,b}$ to $\mathrm{pt}_{\psi(b)}$.

If $\phi : \mathbb{G} \to \mathbb{G}'$ is a generalized morphism, $|\phi|$ is obtained by mapping

- pt_v to $\mathrm{pt}_{\phi_{\mathcal{V}}(v)}$,
- $[1/2,1]_b$ to $[1/2,1]_{\phi_{\mathcal{B}}(b)}$ if $\phi_{\mathcal{B}}(b) \in \mathcal{B}'$ (by the identity of $[1/2,1]$),
- $[1/2,1]_b$ to $\mathrm{pt}_{\phi_{\mathcal{B}}(b)}$ if $\phi_{\mathcal{B}}(b) \in \mathcal{V}'$.

Remark that, if $\mathbb{G}$ is just a loop, then the geometric realization of the morphism induced by inverting the two branches is not homotopic to the identity: thus $\phi_{\mathcal{E}}$ and $\phi_{\mathcal{V}}$ are not enough in general to characterize the topological behavior of ϕ.

2.2. Semistable log curves

Definition 2.1. — *A morphism $X \to S$ of fs log schemes is a* semistable log curve *if étale locally on S there is a chart $S \to \operatorname{Spec} P$ such that one of the following is satisfied:*

- *$X \to S$ is a strict smooth curve,*
- *$X \to S$ factors through a strict étale morphism $X \to S \times_{\operatorname{Spec} \mathbf{Z}[P]} \operatorname{Spec} \mathbf{Z}[Q]$ with $Q = (P \oplus \langle u, v\rangle)/(u+v=p)$ and $p \in P$,*
- *$X \to S$ factors through a strict étale morphism $X \to S \times_{\operatorname{Spec} \mathbf{Z}[P]} \operatorname{Spec} \mathbf{Z}[P \oplus \mathbf{N}]$.*

A semistable log curve is *strictly semistable* if étale locally on S, there are such maps locally for the Zariski topology of X.

Proposition 2.2. — *A morphism $X \to S$ is a semistable log curve if and only if it is a log smooth and saturated morphism purely of relative dimension 1.*

Proof. — The direct sense is obvious. Let $X \to S$ be a saturated log smooth scheme of pure dimension 1. As the definition of a semistable log curve is local for the étale topology of X and S, one can assume that S has a chart $S \to \operatorname{Spec} \mathbf{Z}[P]$ and $X = S \times_{\operatorname{Spec} \mathbf{Z}[P]} \operatorname{Spec} \mathbf{Z}[Q]$ where $P \to Q$ is an injective local and saturated morphism of monoids, P is sharp and $Q^{\mathrm{gp}}/P^{\mathrm{gp}}$ is invertible on S. In particular $T^{\mathrm{gp}} := \overline{Q}^{\mathrm{gp}}/\overline{P}^{\mathrm{gp}}$ is torsionfree. Since $P \to Q$ is saturated, $\operatorname{Spec} \mathbf{Z}[P] \to \operatorname{Spec} \mathbf{Z}[Q]$ is flat and $1 = \dim \operatorname{Spec} \mathbf{Z}[P] - \dim \operatorname{Spec} \mathbf{Z}[Q] = \operatorname{rk} P^{\mathrm{gp}} - \operatorname{rk} Q^{\mathrm{gp}} = \operatorname{rk} Q^{\mathrm{gp}}/P^{\mathrm{gp}} \geqslant \operatorname{rk} T^{\mathrm{gp}}$. Thus T^{gp} is $\{0\}$ or $\mathbf{Z}$. For every $x \in T$, there exists a unique $\psi(x) \in \overline{Q}$ such that $f^{-1}(x) \cup Q = \psi(x) + \overline{P}$ where $f : \overline{Q}^{\mathrm{gp}} \to T^{\mathrm{gp}}$ ([17, prop. I.4.3.14]). In particular, if $T^{\mathrm{gp}} = \{0\}$, then $\overline{P} \to \overline{Q}$ is bijective, thus $X \to S$ is strict and thus $X \to S$ is smooth.

Assume $T^{\mathrm{gp}} = \mathbf{Z}$. Then $\operatorname{rk} Q^{\mathrm{gp}}/P^{\mathrm{gp}} = \operatorname{rk} T^{\mathrm{gp}}$ and thus $\operatorname{rk} Q^{\mathrm{gp}} = \operatorname{rk} \overline{Q}^{\mathrm{gp}}$. Since $\overline{Q}^{\mathrm{gp}}$ is a free abelian group, one can choose a splitting $Q = \overline{Q} \oplus Q^*$. Since $Q^* \hookrightarrow Q^{\mathrm{gp}}/P^{\mathrm{gp}}$ is finite of order invertible on S, $X \to S \times_{\operatorname{Spec} \mathbf{Z}[P]} \operatorname{Spec} \mathbf{Z}[\overline{Q}]$ is étale. Thus one can assume that Q is sharp. Let T be the image of Q in T^{gp}. Then $T = \mathbf{N}$ or $T = \mathbf{Z}$.

First assume $T = \mathbf{N}$. Then $n\psi(1) = \psi(n) + p$ with $p \in P$. Since $P \to Q$ is saturated and $p \leqslant n\psi(1)$, there exists $p' \in P$ such that $p \leqslant np'$ and $p' \leqslant \psi(1)$. Thus, by definition of $\psi(1)$, $p' = 0$, thus $p = 0$ and $\psi(n) = n\psi(1)$. Thus $Q = P \oplus \mathbf{N}\psi(1)$.

If $T = \mathbf{Z}$, let $u = \psi(1)$ and $v = \psi(-1)$. Since $\psi(u+v) = 0$, $p := u+v \in P$. As in the previous case, if $n \geqslant 0$, then $\psi(n) = n\psi(1)$ and $\psi(-n) = n\psi(-1)$. Thus $Q = P \oplus \langle u, v\rangle/(u+v=p)$. $\square$

The underlying morphism of schemes $\mathring{X} \to \mathring{S}$ is a semistable curve. In particular, if $\mathring{S}$ is a geometric point, one can associate to X a graph $\mathbb{G}(X)$ in the following way: the vertices are the irreducible components of X, the edges are the nodes. If x is a node, then the henselization $X(x)$ of X at x has two irreducible components: these components are the branches of the edge corresponding to x. If z is an irreducible component of $X(x)$ and z' is the irreducible component of X containing the image of z in X, the branch corresponding to z abuts to the vertex corresponding to z' (this graph does not depend of the log structure).

If $X \to S$ is a proper semistable log curve and $X' \to X$ is a két cover, then for any log geometric point $\bar{s}$ of S, there is a két neighborhood U of $\bar{s}$ such that $X'_U \to U$ is saturated. Then $X'_U \to U$ is also a semistable curve.

The morphism $\mathring{X}'_{\bar{s}} \to \mathring{X}_{\bar{s}}$ induces a genuine morphism $\mathbb{G}(X'_{\bar{s}}) \to \mathbb{G}(X_{\bar{s}})$ of graphs.

2.3. Topological cospecialization of semistable curves

Let $f : X \to S$ be a semistable curve such that S is locally noetherian, and let $\bar{s}_2 \to \bar{s}_1$ be a specialization of geometric points of S. In this section we will define a cospecialization map of graphs $\mathbb{G}(X_{\bar{s}_1}) \to \mathbb{G}(X_{\bar{s}_2})$.

Proposition 2.3. — *Let $f : X \to S$ be a strictly semistable curve such that S is strictly local and noetherian. Let s_1 be the closed point of S, and let s_2 be a point of S. Let x be a node or a generic point of X_{s_1}. Let $X(x)$ be the localization of X at x. Then $X(x)_{s_2}$ is either contained in the smooth locus of a geometrically irreducible component, denoted by $F(x)$, of $X(x)_{s_2}$ or contains a single node, denoted by $F(x)$, of $X(x)_{s_2}$, which is rational.*

Proof. — Let A be the noetherian strictly local ring such that $S = \operatorname{Spec} A$. By replacing S by the closure of s_2 endowed with the reduced scheme structure, one can assume that s_2 is the generic point of S and S is integral. Indeed, nonempty closed subschemes of henselian schemes are henselian ([21, cor. to § 3. prop 2]) and keep the same residue field at the special point, therefore the closure of s_2 is a strictly local scheme.

(i) If x is in the smooth locus of X_{s_1}, $X \to S$ is smooth at x, and $X(x)_{s_2}$ is geometrically connected by local 0-acyclicity of smooth morphisms.

(ii) If x is a node, one can assume that f factors through an étale morphism $X \to \operatorname{Spec} B$ with $B = A[u,v]/(uv - a)$ and $a(s_1) = 0$.

If $a = 0$ let $Z = X \times_{\operatorname{Spec} B} \operatorname{Spec} A$ where $g : B \to A$ is defined by $g(u) = g(v) = 0$ (this is the closed subscheme of X defined by the node; in particular Z_{s_2} is the union of all the nodes of X_{s_2}). The morphism $Z \to S$ is étale and thus $Z(x) \to S$ is an isomorphism. Thus $Z(x)_{s_2}$ is just a rational point $F(x)$.

If $a \neq 0$, then $a(s_2) \neq 0$ and thus X_{s_2} is smooth. Since $X \to S$ is a semistable curve, it is separable (*i.e.*, flat with separable geometric fibers). Let $B := O_{X,x}$ be the noetherian local ring such that $X(x) = \operatorname{Spec} B$. By applying [7, cor. 18.9.8] to $\operatorname{Spec} B = X(x) \to \operatorname{Spec} A = S$, one gets that $X(x)_{s_2}$ is geometrically connected.

□

Let $f : X \to S$ be a semistable curve, and let $\bar{s}_2 \to \bar{s}_1$ be a specialization of geometric points of S. One can apply Lemma 2.3 to $X_{S(\bar{s}_1)} \to S(\bar{s}_1)$ and to the Zariski point s_2 corresponding to $\bar{s}_2$. Let x be a node or a generic point of $X_{\bar{s}_1}$. If $F(x)$ is a rational node of X_{s_2}, then it defines an edge $F_0(x)$ of $\mathbb{G}_{\bar{s}_2}$. If $F(x)$ is a geometrically irreducible component of X_{s_2}, then it defines a vertex $F_0(x)$ of $\mathbb{G}_{\bar{s}_2}$.

Lemma 2.4. — *Let $S' = \operatorname{Spec} A' \to S = \operatorname{Spec} A$ be a local morphism of noetherian strictly local schemes. Let s'_1 be the closed point of S' and let s'_2 be a point of S' above s_2. Let $X' = X \times_S S'$. Let $x' \in X'_{s'_1}$ be above x. Let F' be defined analogously to F but with the curve $X' \to S'$. Then $F(x)$ is the image of $F'(x')$ by the map $X' \to X$.*

Proof. — Let z be the image of $F'(x')$ by the map $X' \to X$. Since x' is in the closure of $F'(x')$, x is in the closure of z, and therefore $F(x)$ is in the closure of z. One only has to prove that if $F(x)$ is a node, then z is also a node. One can assume that $X \to S$ factors through an étale morphism $X \to \operatorname{Spec} B$ with $B = A[u,v]/(uv - a)$ and $a(s_2) = 0$. Then $X' \to S'$ factors through the étale map $X \to \operatorname{Spec} B'$ with $B' = A'[u,v]/(uv - a')$ with $a'(s'_2) = 0$, and thus $F'(x')$ and z are also nodes. □

Lemma 2.5. — *If $\phi : X' \to X$ is a quasifinite open morphism of strictly semistable curves over S which maps nodes to nodes on every fiber, then $\phi F'_0 = F_0 \phi$.*

For example, the assumption is satisfied if $X' \to X$ is étale or if $X' \to X$ is a két morphism of strictly semistable log curves.

Proof. — One can assume that S is strictly local with closed point $\bar{s}_1$, so that one has to prove that $\phi F = F\phi$. Since $\phi(x)$ is in the closure of $\phi F'(x)$, $F\phi(x)$ is in the closure of $\phi F'(x)$. One only has to prove that if $F\phi(x)$ is a node, then $\phi F'(x)$ is also a node. Let us assume that $F\phi(x)$ is a node of X_{s_2}. Let z_1 and z_2 be the two generic points of the irreducible components of X_{s_2} whose closures contain $F\phi(x)$ (and thus also $\phi(x)$). Since ϕ is open, there exists z_1' and z_2' in $X(x)_{s_2}$ such that $\phi(z_1') = z_1$ and $\phi(z_2') = z_2$. Thus $X(x)_{s_2}$ cannot be in a single irreducible component of X_{s_2}, and thus $F'(x)$ is a node of X'_{s_2}. By assumption, $\phi F'(x)$ is a node of X_{s_2}. □

Proposition 2.6. — *Let S be a locally noetherian scheme and let $\bar{s}_2 \to \bar{s}_1$ be a specialization of geometric points. There is a unique way to associate to every semistable curve $X \to S$ a generalized morphism of graphs*

$$\psi : \mathbb{G}(X_{\bar{s}_1}) \to \mathbb{G}(X_{\bar{s}_2})$$

- *which is functorial for étale morphisms $X' \to X$,*
- *such that if $f : X \to S$ is strictly semistable, $\psi(x) = F_0(x)$ for any node or generic point x of $\mathbb{G}(X_{\bar{s}_1})$.*

Proof. — After replacing S by its strict localization at $\bar{s}_1$, one can assume that S is strictly local and $\bar{s}_1$ is the closed point.

First, let us prove the uniqueness. Let $X \to S$ be a semistable curve. Let x be a node or a vertex of $\mathbb{G}(X_{\bar{s}_1})$. Let $X' \to X$ be a surjective étale morphism such that $X' \to S$ is strictly semistable. Let x' be a preimage of x in $\mathbb{G}(X'_{\bar{s}_1})$. Then, by functoriality, $\psi(x)$ must be the image of $F_0(x')$ by the map $\mathbb{G}_{X'_{\bar{s}_2}} \to \mathbb{G}_{X_{\bar{s}_2}}$. Moreover if b is a branch of $\mathbb{G}(X_{\bar{s}_1})$, let b' be a preimage in $\mathbb{G}(X'_{\bar{s}_1})$. Since $\mathbb{G}_{X_{\bar{s}_2}}$ has no loop, $\psi(b')$ is uniquely defined by the vertex it is ending at, $\psi(b)$ is the image of $\psi(b')$ by the map $\mathbb{G}_{X'_{\bar{s}_2}} \to \mathbb{G}_{X_{\bar{s}_2}}$. This proves the uniqueness.

Let us now construct ψ.

Let $f : X \to S$ be a strictly semistable curve, and let $\bar{s}_2 \to \bar{s}_1$ be a specialization of geometric points of S.

If e is a vertex of $\mathbb{G}(X_{\bar{s}_1})$, then $\psi(e) := F_0(x)$ where x is the node of $X_{\bar{s}_1}$ corresponding to e. If v is a vertex of $\mathbb{G}(X_{\bar{s}_1})$, then $\psi(v) := F_0(x)$ where x is the generic point of the irreducible component of $X_{\bar{s}_1}$ corresponding to e. Let b be a branch of an edge e in $\mathbb{G}(X_{s_1})$ that abuts to a vertex v. Then $F(x) \subset F(s)$, where x is the node corresponding to e and s is the generic point of the irreducible component corresponding to v. If $F(x) = F(s)$, then $\psi(b) := F_0(x) = F_0(s)$. Otherwise, $\psi(e)$ is an edge and $\psi(v)$ is a vertex, and there is a branch b' of $\psi(e)$ abutting to $\psi(v)$. Since $X_{\bar{s}_2}$ is strictly semistable, this branch is unique. Let $\psi(b) = b'$.

The compatibility with étale morphisms is a direct consequence of Lemma 2.5.

If $X \to S$ is now a general semistable curve, one choses a surjective étale morphism $X' \to X$, and let $X'' = X' \times_X X'$.

One has a commutative diagram with genuine lines

$$\begin{tikzcd} \mathbb{G}(X''_{\bar{s}_1}) \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "\psi''"] & \mathbb{G}(X'_{\bar{s}_1}) \arrow[r] \arrow[d, "\psi'"] & \mathbb{G}(X_{\bar{s}_1}) \\ \mathbb{G}(X''_{\bar{s}_2}) \arrow[r, shift left] \arrow[r, shift right] & \mathbb{G}(X'_{\bar{s}_2}) \arrow[r] & \mathbb{G}(X_{\bar{s}_2}) \end{tikzcd} \tag{2.1}$$

such that

$$\mathcal{V}_{\mathbb{G}(X_{\bar{s}_2})} = \mathrm{Coker}\left(\mathcal{V}_{\mathbb{G}(X''_{\bar{s}_2})} \rightrightarrows \mathcal{V}_{\mathbb{G}(X'_{\bar{s}_2})}\right)$$

and

$$\mathcal{E}_{\mathbb{G}(X_{\bar{s}_2})} = \mathrm{Coker}\left(\mathcal{E}_{\mathbb{G}(X''_{\bar{s}_2})} \rightrightarrows \mathcal{E}_{\mathbb{G}(X'_{\bar{s}_2})}\right).$$

By taking the cokernel one gets maps $\psi_{\mathcal{V}} : \mathcal{V}_{\mathbb{G}(X_{\bar{s}_1})} \to \mathcal{V}_{\mathbb{G}(X_{\bar{s}_2})}$ and $\psi_{\mathcal{E}} : \mathcal{E}_{\mathbb{G}(X_{\bar{s}_1})} \to \mathcal{E}_{\mathbb{G}(X_{\bar{s}_2})} \coprod \mathcal{V}_{\mathbb{G}(X_{\bar{s}_2})}$.

Let e be an edge of $\mathbb{G}(X_{\bar{s}_1})$ such that $\psi_{\mathcal{E}}(e) \in \mathcal{E}_{\mathbb{G}(X_{\bar{s}_2})}$. Let e' be an edge of $\mathbb{G}(X_{\bar{s}_1})$ mapping to e. One has bijections $\mathcal{B}_e \leftarrow \mathcal{B}_{e'} \to \mathcal{B}_{\psi'_{\mathcal{E}}(e')} \to \mathcal{B}_{\psi_{\mathcal{E}}(e)}$, hence a bijection $\psi_e : \mathcal{B}_e \to \mathcal{B}_{\psi_{\mathcal{E}}(e)}$. Let e'_1 and e'_2 be edges of $\mathbb{G}(X_{\bar{s}_1})$ mapping to e. There exists an edge $e'' \in \mathcal{E}_{\mathbb{G}(X_{\bar{s}_2})}$ mapping to e'_1 and e'_2 by the two maps $\mathbb{G}(X''_{\bar{s}_1}) \to \mathbb{G}(X'_{\bar{s}_1})$. One gets a commutative diagram of bijections:

$$\begin{tikzcd} & \mathcal{B}_{e'_1} \arrow[r] \arrow[dl] & \mathcal{B}_{\psi'_{\mathcal{E}}(e'_1)} \arrow[dr] & \\ \mathcal{B}_e & \mathcal{B}_{e''} \arrow[l] \arrow[r] \arrow[u] \arrow[d] & \mathcal{B}_{\psi''_{\mathcal{E}}(e'')} \arrow[r] \arrow[u] \arrow[d] & \mathcal{B}_{\psi_{\mathcal{E}}(e)}, \\ & \mathcal{B}_{e'_2} \arrow[r] \arrow[ul] & \mathcal{B}_{\psi'_{\mathcal{E}}(e'_2)} \arrow[ur] & \end{tikzcd}$$

which proves that the bijection ψ_e does not depend on the choice of e'. The wanted compatibilities between $\psi_{\mathcal{E}}$, $\psi_{\mathcal{V}}$ and ψ_e come directly from the corresponding compatibilities between $\psi'_{\mathcal{E}}$, $\psi'_{\mathcal{V}}$ and $\psi'_{e'}$. Therefore, there is a unique generalized morphism of graphs $\psi : \mathbb{G}(X_{\bar{s}_1}) \to \mathbb{G}(X_{\bar{s}_2})$ making the diagram (2.1) commutative.

This morphism ψ does not depend of the choice of X'. Indeed let $X'_1 \to X$ and $X'_2 \to X$ be two surjective étale morphisms such that X'_1 and X'_2 are strictly semistable. By considering $X'_1 \times_X X'_2 \to X$, one can assume that there is a X-morphism $X'_2 \to X'_1$. Then one has a diagram

$$\begin{tikzcd}
\mathbb{G}(X'_{2,\bar{s}_1}) \arrow[rr] \arrow[dr] \arrow[dd] & & \mathbb{G}(X'_{1,\bar{s}_1}) \arrow[dr] \arrow[dd] & \\
& \mathbb{G}(X'_{2,\bar{s}_2}) \arrow[rr] \arrow[dd] & & \mathbb{G}(X'_{1,\bar{s}_2}) \arrow[dd] \\
\mathbb{G}(X_{\bar{s}_1}) \arrow[rr] \arrow[dr] & & \mathbb{G}(X_{\bar{s}_1}) \arrow[dr] & \\
& \mathbb{G}(X_{\bar{s}_2}) \arrow[rr] & & \mathbb{G}(X_{\bar{s}_2})
\end{tikzcd}$$

where the horizontal maps of the lower square are identities and the frontward maps of the lower square are the two versions of ψ defined in terms of X'_1 and X'_2. Since the upper face and the vertical faces are commutative and the vertical maps are surjective, the lower square is also commutative. Therefore ψ does not depend on X'.

Let us show the functoriality of ψ with respect to étale morphisms. Let $X_2 \to X_1$ be an étale morphism. Let $X'_1 \to X_1$ be a surjective étale morphism such that $X'_2 \to S$ is strictly semistable. Let $X'_2 := X'_1 \times_{X_1} X$. Consider the diagram

$$\begin{tikzcd}
\mathbb{G}(X'_{2,\bar{s}_1}) \arrow[rr] \arrow[dr] \arrow[dd] & & \mathbb{G}(X'_{1,\bar{s}_1}) \arrow[dr] \arrow[dd] & \\
& \mathbb{G}(X'_{2,\bar{s}_2}) \arrow[rr] \arrow[dd] & & \mathbb{G}(X'_{1,\bar{s}_2}) \arrow[dd] \\
\mathbb{G}(X_{2,\bar{s}_1}) \arrow[rr] \arrow[dr] & & \mathbb{G}(X_{1,\bar{s}_1}) \arrow[dr] & \\
& \mathbb{G}(X_{2,\bar{s}_2}) \arrow[rr] & & \mathbb{G}(X_{1,\bar{s}_2}).
\end{tikzcd}$$

Since the upper face and the vertical faces are commutative and the vertical maps are surjective, the lower square is commutative. $\square$

Proposition 2.7. — *Let $f : S' \to S$ be a morphism of locally noetherian schemes. Let $\bar{s}'_2 \to \bar{s}'_1$ be a specialization of geometric points of S', and let $\bar{s}_2 \to \bar{s}_1$ be the image in S. Let $X \to S$ be a semistable curve and let*

$X' = X \times_{S'} S$. Then the diagram

$$\begin{tikzcd} \mathbb{G}_{X'_{\bar{s}'_1}} \arrow[r, "\psi'"] \arrow[d, equal] & \mathbb{G}_{X'_{\bar{s}'_2}} \arrow[d, equal] \\ \mathbb{G}_{X_{\bar{s}_1}} \arrow[r, "\psi"] & \mathbb{G}_{X_{\bar{s}_2}}, \end{tikzcd} \tag{2.2}$$

where ψ and ψ' are the cospecialization maps, is commutative.

Proof. — Up to replacing S' by its strict localization at $\bar{s}'_1$ and S by its strict localization at $\bar{s}_1$, one can assume that $S' \to S$ is a local morphism of strictly local schemes and that $\bar{s}'_1$ and $\bar{s}_1$ are the closed points of S' and S. Let ψ_0 be the composition $\mathbb{G}_{X_{\bar{s}_1}} = \mathbb{G}_{X'_{\bar{s}'_1}} \xrightarrow{\psi'} \mathbb{G}_{X'_{\bar{s}'_2}} = \mathbb{G}_{X_{\bar{s}_2}}$. Since ψ' is compatible with étale morphisms, ψ_0 is also compatible with étale morphisms. Let $f : X \to S$ be a strictly semistable morphism and let x be a node or a vertex of $\mathbb{G}_{X_{\bar{s}_1}}$. Let x' be the corresponding node or vertex of $\mathbb{G}_{X'_{\bar{s}'_1}}$. Then $fF_0(x') = F_0(x)$ according to Lemma 2.4. Therefore $\psi_0(x) = f\psi'(x') = fF_0(x') = F_0(x)$. Therefore, by uniqueness in Proposition 2.6, one has $\psi_0 = \psi$. □

Proposition 2.8. — *Let $X \to S$ be a semistable curve. Let $\bar{s}_3 \to \bar{s}_2$ and $\bar{s}_2 \to \bar{s}_1$ be specializations of geometric points of X. Then the diagram*

$$\begin{tikzcd} \mathbb{G}_{X_{\bar{s}_1}} \arrow[r, "\psi_{12}"] \arrow[dr, "\psi_{13}"] & \mathbb{G}_{X_{\bar{s}_2}} \arrow[d, "\psi_{23}"] \\ & \mathbb{G}_{X_{\bar{s}_3}}, \end{tikzcd}$$

where ψ_{12}, ψ_{13} and ψ_{23} are cospecialization maps, is commutative.

Proof. — The morphism $\psi_{23}\psi_{12} : \mathbb{G}_{X_{\bar{s}_1}} \to \mathbb{G}_{X_{\bar{s}_3}}$ is functorial with respect to étale morphisms $X' \to X$. By uniqueness in Proposition 2.6, it is enough to prove that $\psi_{23}\psi_{12}(x) = \psi_{13}(x)$ for every node or edge x of $\mathbb{G}_{X_{\bar{s}_1}}$ assuming that X is strictly semistable. Since $\psi_{23}\psi_{12}(x)$ specializes to x, $\psi_{13}(x)$ is in the closure of $\psi_{23}\psi_{12}(x)$. Therefore, one only has to prove that if $\psi_{13}(x)$ is a node, $\psi_{23}\psi_{12}(x)$ is also a node. Then up to étale localization, one can assume $S = \operatorname{Spec} A$ and $X \to \operatorname{Spec} A$ goes through an étale morphism $X \to \operatorname{Spec} B$ where $B = A[u,v]/(uv - a)$ with $a(\bar{s}_3) = 0$, in which case it is obvious. □

Proposition 2.9. — *If $\phi : X' \to X$ is a quasifinite open morphism of semistable curves over S which maps nodes to nodes on every fiber,* *then* $\phi\psi' = \psi\phi$, *where* $\psi : \mathbb{G}_{X_{\bar{s}_1}} \to \mathbb{G}_{X_{\bar{s}_2}}$ *and* $\psi' : \mathbb{G}_{X'_{\bar{s}_1}} \to \mathbb{G}_{X'_{\bar{s}_2}}$ *are cospecialization maps.*

Proof. — Since the lemma is true if $X' \to X$ is étale, one only has to prove it locally on X' and X for the étale topology. Therefore one can assume that X and X' are strictly semistable curves over S. According to Lemma 2.5, for any node or edge x of $\mathbb{G}_{X'_{\bar{s}_1}}$, $\phi\psi'(x) = \phi F_0'(x) = F_0\phi(x) = \psi\phi(x)$. Since $\mathbb{G}_{X'_{\bar{s}_1}}$ has no loop, this implies that $\phi\psi' = \psi\phi$. □

We want to know when this generalized morphism of graphs is an isomorphism.

Proposition 2.10. — *Keeping the notations of Proposition 2.6, if* $\psi : \mathbb{G}(X_{\bar{s}_1}) \to \mathbb{G}(X_{\bar{s}_2})$ *is a genuine morphism of graphs and* f *is proper, then* ψ *is an isomorphism.*

Proof. — One may assume $S = \operatorname{Spec} A$ to be strictly local and integral with special point s_1 and generic point s_2. The assumption means that étale locally on the special fiber (and thus on X by properness), X is isomorphic to $\operatorname{Spec} A[u,v]/uv$ or is smooth.

Let $Z \subset X$ be the non smooth locus of $X \to S$, endowed with the reduced subscheme structure. $Z \to S$ is étale (as can be seen étale locally over X), and proper. One thus gets that F induces a bijection between nodes of $X_{\bar{s}_1}$ and $X_{\bar{s}_2}$.

Let $\widetilde{X}$ be the blowup of X along Z. When $X = \operatorname{Spec} A[u,v]/(uv)$, Z is defined by the ideal generated by u and v, and $\widetilde{X} = \operatorname{Spec} A[u] \coprod \operatorname{Spec} A[v]$. Thus by looking étale locally over X, one sees that $\widetilde{X}$ is smooth over S, and that $\widetilde{X}_s$ is simply the normalization of X_s. Since we assumed $X \to S$ to be proper, $\widetilde{X} \to S$ is smooth and proper, thus its Stein factorization induces a bijection between the connected components of $\widetilde{X}_{\bar{s}_1}$ and $\widetilde{X}_{\bar{s}_2}$, and thus the map between the irreducible components of $X_{\bar{s}_1}$ and $X_{\bar{s}_2}$ is a bijection too. □

Proposition 2.11. — *Let* $f : X \to S$ *be a log semistable curve and let* $\bar{s}_2 \to \bar{s}_1$ *be a specialization of log geometric point.*

Assume $\overline{M}_{S,\bar{s}_1} \to \overline{M}_{S,\bar{s}_2}$ *is an isomorphism. Then* $\psi : \mathbb{G}(X_{\bar{s}_1}) \to \mathbb{G}(X_{\bar{s}_2})$ *is a genuine morphism of graphs.*

Proof. — One can assume S to be strictly local, integral with generic point s_2: $S = \operatorname{Spec} A$, with a chart $P \to A$.

To show that it is a genuine morphism, one only has to prove that $\psi(e)$ is an edge if e is an edge of $\mathbb{G}(X_{s_1})$. This is not changed by an étale morphism, so that one can simply assume $X = \operatorname{Spec} A \otimes_{\mathbf{Z}[P]} \mathbf{Z}[Q]$ with

$Q = (P \oplus \langle u, v\rangle)/(u + v = p)$ and $p \in P$, such that the image of p in $M_{\bar{s}_1}$ is not invertible. Thus the image of p in $M_{\bar{s}_2}$ is not invertible and thus $X = \operatorname{Spec} A[u, v]/(uv = 0)$, which gives the wanted result. $\square$

2.4. Topological cospecialization and két morphisms

Proposition 2.12. — *Let S be a fs log scheme such that $\mathring{S}$ is locally noetherian. Let $(s_2, \bar{s}_2)$ and $(s_1, \bar{s}_1)$ be two pointed fs log points of S and let $\bar{s}_2 \to \bar{s}_1$ be a specialization of log geometric points. Let $X \to S$ be a proper semistable log curve and let $Y_{\bar{s}_2}$ be a log geometric két cover of $X_{s_1}/(s_1, \bar{s}_1)$. There is a unique morphism of graphs*

$$\phi : \mathbb{G}(Y_{\bar{s}_1}) \to \mathbb{G}(Y_{\bar{s}_2}),$$

where $Y_{\bar{s}_2}$ is the image of $Y_{\bar{s}_1}$ by the functor $\mathrm{KCov}(X_{\bar{s}_1}) \to \mathrm{KCov}(X_{\bar{s}_2})$ given by Corollary 1.17, such that, if U is a két neighborhood of $\bar{s}_1$ in S and $Z \to X_U := X \times_S U$ is an extension of $Y_{\bar{s}_1}$ such that $Z \to U$ is saturated (and therefore a semistable log curve), then the diagram

$$\begin{array}{ccc} \mathbb{G}_{Z_{\bar{s}_1}} & \xrightarrow{\psi} & \mathbb{G}_{Z_{\bar{s}_2}} \\ \| & & \| \\ \mathbb{G}_{Y_{\bar{s}_1}} & \xrightarrow{\phi} & \mathbb{G}_{Y_{\bar{s}_2}}, \end{array}$$

where ψ is the cospecialization morphism defined by Proposition 2.6, is commutative.

Moreover ϕ is functorial with respect to morphisms $Y'_{\bar{s}_1} \to Y_{\bar{s}_1}$ of log geometric két covers of $X_{s_1}/(s_1, \bar{s}_1)$ and with respect to composition of specializations of log geometric points.

If $\overline{M}_{S,s_1} \to \overline{M}_{S,s_2}$ is an isomorphism, then ϕ is an isomorphism.

Proof. — According to Corollary 1.17, there exists a két neighborhood U of $\bar{s}_1$ and a két cover $Z \to X_U$ which extends $Y_{\bar{s}_2}$. Up to replacing U by a smaller két neighborhood, one can assume that $Z \to U$ is saturated. This proves the uniqueness. One only has to prove that the morphism ϕ one gets does not depend on the choice of U and $Z \to X_U$. Let U and U' be two két neighborhoods of $\bar{s}_1$ and let $Z \to X_U$ and $Z' \to X_{U'}$ be két covers that extend $Y_{\bar{s}_2}$. Since $\mathrm{KCov}_{\mathrm{geom}}(X_{s_1}/(s_1, \bar{s}_1)) \to \mathrm{KCov}_{\mathrm{geom}}(X/(Z, \bar{s}_1))$ is an equivalnce there exists a két neighborhood U'' of $\bar{s}_1$ in $U \times_S U'$ and an isomorphism $Z'_{U''} \simeq Z_{U''}$. Therefore one can assume that there is a morphism $U' \to U$ over S and that $Z' = Z_{U'}$. Since the specializations maps of Proposition 2.6 are compatible with a base change $U' \to U$, U and U' define the same morphism ϕ. This proves the existence of ϕ.

Let $Y'_{\bar{s}_1} \to Y_{\bar{s}_1}$ be a morphism of log geometric két covers of $X_{s_1}/(s_1, \bar{s}_1)$. There exists a két neighborhood U of $\bar{s}_1$ and extensions $Z \to X_U$ and $Z' \to Z$ of $Y_{\bar{s}_1}$ and of $Y'_{\bar{s}_1} \to Y_{\bar{s}_1}$ such that $Z' \to U$ is saturated. The compatibility of ϕ with $Y'_{\bar{s}_1} \to Y_{\bar{s}_1}$ is equivalent to the compatibility of ψ with $Z' \to Z$, which is given by Proposition 2.9.

Let $(s_3, \bar{s}_3)$ be a pointed fs log point and let $\bar{s}_3 \to \bar{s}_2$ be a specialization. Let U be a két neighborhood of $\bar{s}_1$ and $Z \to X_U$ be an extension of $Y_{\bar{s}_1}$ such that $Z \to U$ is saturated. The compatibility of ϕ with the composition of specializations for $Y_{\bar{s}_1}$ is equivalent to the compatibility of ψ with the composition of specialization for Z, which is given by Lemma 2.8.

If $\overline{M}_{S,s_1} \to \overline{M}_{S,s_2}$ is an isomorphism, $\overline{M}_{U,s_1} \to \overline{M}_{U,s_2}$ is still an isomorphism, so that one can still apply Proposition 2.11 to an extension $Z \to U$ of $Y_{\bar{s}_1}$: the morphism ϕ is a genuine morphism of graphs. According to Proposition 2.10, ϕ is an isomorphism. □

If $Y'_{\bar{s}_1} \to Y_{\bar{s}_2}$ is a morphism of log geometric két covers, then the following diagram is commutative:

$$\begin{array}{ccc} \mathbb{G}(Y'_{\bar{s}_1}) & \to & \mathbb{G}(Y'_{\bar{s}_2}) \\ \downarrow & & \downarrow \\ \mathbb{G}(Y_{\bar{s}_1}) & \to & \mathbb{G}(Y_{\bar{s}_2}) \end{array}$$

If $\overline{M}_{S,s_1} \to \overline{M}_{S,s_2}$ is an isomorphism, $\overline{M}_{U,s_1} \to \overline{M}_{U,s_2}$ is still an isomorphism, so that one can still apply Proposition 2.11 to Y: the morphism $\mathbb{G}(Y_{\bar{s}_1}) \to \mathbb{G}(Y_{\bar{s}_2})$ is a genuine morphism of graphs.

3. Cospecialization of tempered fundamental groups

3.1. Tempered fundamental groups

Let K be a complete nonarchimedean field.

Let $\mathbb{L}$ be a set of prime numbers (for example, we will denote by (p') the set of all primes except the residual characteristic p of K). An $\mathbb{L}$-integer will be an integer which is a product of elements of $\mathbb{L}$.

If X is a K-algebraic variety, X^{an} will be the K-analytic space in the sense of Berkovich associated to X.

A morphism $f : S' \to S$ of analytic spaces is said to be an *étale cover* if S is covered by open subsets U such that $f^{-1}(U) = \coprod V_j$ and $V_j \to U$ is étale finite ([10]).

For example, étale $\mathbb{L}$-finite covers (*i.e.,* finite étale covers that are dominated by a Galois cover S'' of S such that $\#\operatorname{Gal}(S''/S)$ is an $\mathbb{L}$-integer), also called $\mathbb{L}$-*algebraic covers*, and covers in the usual topological sense for the Berkovich topology, also called *topological covers*, are étale covers.

Then, André defines tempered covers in [2, def. 2.1.1]. We generalize this definition to $\mathbb{L}$-tempered covers as follows:

Definition 3.1. — *An étale cover $S' \to S$ is* $\mathbb{L}$-tempered *if it is a quotient of the composition of a topological cover $T' \to T$ and of a $\mathbb{L}$-finite étale cover $T \to S$.*

This is equivalent to say that it becomes a topological cover after pullback by some $\mathbb{L}$-finite étale cover.

Let X be a K-analytic space. We denote by $\operatorname{Cov}^{\mathbb{L}\text{-temp}}(X)$ (resp. $\operatorname{Cov}^{\mathrm{alg}}(X)^{\mathbb{L}}$, $\operatorname{Cov}^{\mathrm{top}}(X)$) the category of $\mathbb{L}$-tempered covers (resp. $\mathbb{L}$-algebraic covers, topological covers) of X (with the obvious morphisms).

A geometric point of a K-analytic space X is a morphism of Berkovich spaces $\mathcal{M}(\Omega) \to X$ where Ω is an algebraically closed complete isometric extension of K.

Let $\bar{x}$ be a geometric point of X. Then one has a functor

$$F_{\bar{x}}^{\mathbb{L}} : \operatorname{Cov}^{\mathbb{L}\text{-temp}}(X) \to \mathrm{Set}$$

which maps a $\mathbb{L}$-tempered cover $S \to X$ to the set $S_{\bar{x}}$.

The $\mathbb{L}$-tempered fundamental group of X pointed at $\bar{x}$ is

$$\pi_1^{\mathbb{L}\text{-temp}}(X, \bar{x}) = \operatorname{Aut} F_{\bar{x}}^{\mathbb{L}}.$$

When X is a smooth algebraic K-variety, $\operatorname{Cov}^{\mathbb{L}\text{-temp}}(X^{\mathrm{an}})$ and $\pi_1^{\mathbb{L}\text{-temp}}(X^{\mathrm{an}}, \bar{x})$ will also be denoted simply by $\operatorname{Cov}^{\mathbb{L}\text{-temp}}(X)$ and $\pi_1^{\mathbb{L}\text{-temp}}(X, \bar{x})$.

By considering the stabilizers $(\operatorname{Stab}_{F_{\bar{x}}^{\mathbb{L}}(S)}(s))_{S \in \operatorname{Cov}^{\mathbb{L}\text{-temp}}(X), s \in F_{\bar{x}}^{\mathbb{L}}(S)}$ as a basis of open subgroups of $\pi_1^{\mathbb{L}\text{-temp}}(X, \bar{x})$, $\pi_1^{\mathbb{L}\text{-temp}}(X, \bar{x})$ becomes a topological group. It is a prodiscrete topological group.

When X is algebraic, K of characteristic zero and has only countably many finite extensions in a fixed algebraic closure $\overline{K}$, $\pi_1^{\mathbb{L}\text{-temp}}(X, \bar{x})$ has a countable fundamental system of neighborhood of 1 and all its discrete quotient groups are finitely generated ([2, prop. III.2.1.7]). When $\mathbb{L}$ is the set of all primes, we often forget it in the notations.

It should be remarked that in general, for a given $\mathbb{L}$, one cannot recover $\pi_1^{\mathbb{L}\text{-temp}}(X, \bar{x})$ from $\pi_1^{\mathrm{temp}}(X, \bar{x})$. For example, let us consider an Enriques surface X over a nonarchimedean field of residual characteristic zero. One has $\pi_1^{\mathrm{alg}}(X) = \mathbf{Z}/2\mathbf{Z}$ and X has a unique nontrivial connected finite cover;

it is given by a $K3$ surface Y. The surfaces X and Y have a semistable reduction, and according to [4], X^{an} and Y^{an} are homotopy equivalent to the dual simplicial sets of their semistable reduction. The possible simplicial sets are given by [13]. For the $K3$ surface Y, this dual simplicial set is always simply connected and therefore $\pi_1^{\mathrm{temp}}(X) = \pi_1^{\mathrm{alg}}(X) = \mathbf{Z}/2\mathbf{Z}$. If X is an Enriques surface with good reduction, $\pi_1^{\emptyset\text{-temp}}(X,\bar{x}) = \pi_1^{\mathrm{top}}(X,x) = \{1\}$. If the reduction of X is totally degenerate, *i.e.*, all the irreducible components of a semistable reduction are projective planes, the dual simplicial set is homotopy equivalent to a real projective plane and $\pi_1^{\emptyset\text{-temp}}(X,\bar{x}) = \pi_1^{\mathrm{top}}(X,x) = \mathbf{Z}/2\mathbf{Z}$. Therefore, two Enriques surface have isomorphic tempered fundamental groups but can have different $\emptyset$-tempered fundamental groups.

If $\bar{x}$ and $\bar{x}'$ are two geometric points, then $F_{\bar{x}}^{\mathbb{L}}$ and $F_{\bar{x}'}^{\mathbb{L}}$ are (non canonically) isomorphic ([10, th. 2.9]). Thus, as usual, the tempered fundamental group depends on the basepoint only up to inner automorphism (this topological group, considered up to conjugation, will sometimes be denoted simply $\pi_1^{\mathbb{L}\text{-temp}}(X)$).

The full subcategory of tempered covers S for which $F_{\bar{x}}^{\mathbb{L}}(S)$ is $\mathbb{L}$-finite is equivalent to $\mathrm{Cov}^{\mathrm{alg}}(S)^{\mathbb{L}}$, hence

$$\pi_1^{\mathbb{L}\text{-temp}}(X,\bar{x})^{\mathbb{L}} = \pi_1^{\mathrm{alg}}(X,\bar{x})^{\mathbb{L}}$$

(where $(\ )^{\mathbb{L}}$ denotes the pro-$\mathbb{L}$ completion).

For any morphism $X \to Y$, the pullback defines a functor $\mathrm{Cov}^{\mathbb{L}\text{-temp}}(Y) \to \mathrm{Cov}^{\mathbb{L}\text{-temp}}(X)$. If $\bar{x}$ is a geometric point of X with image $\bar{y}$ in Y, this gives rise to a continuous homomorphism

$$\pi_1^{\mathbb{L}\text{-temp}}(X,\bar{x}) \to \pi_1^{\mathbb{L}\text{-temp}}(Y,\bar{y})$$

(hence an outer morphism $\pi_1^{\mathbb{L}\text{-temp}}(X) \to \pi_1^{\mathbb{L}\text{-temp}}(Y)$).

One has the analog of the usual Galois correspondence:

Theorem 3.2 ([2, th. III.1.4.5]). — *$F_{\bar{x}}^{\mathbb{L}}$ induces an equivalence of categories between the category of direct sums of $\mathbb{L}$-tempered covers of X and the category $\pi_1^{\mathbb{L}\text{-temp}}(X,\bar{x})$* -Set *of discrete sets endowed with a continuous left action of $\pi_1^{\mathbb{L}\text{-temp}}(X,\bar{x})$.*

If S is a $\mathbb{L}$-finite Galois cover of X, its universal topological cover S^{∞} is still Galois and every connected $\mathbb{L}$-tempered cover is dominated by such a Galois $\mathbb{L}$-tempered cover.

If $((S_i,\bar{s}_i))_{i\in\mathbf{N}}$ is a cofinal projective system (with morphisms $f_{ij}: S_i \to S_j$ which maps s_i to s_j for $i \geqslant j$) of geometrically pointed Galois $\mathbb{L}$-finite étale covers of $(X,\bar{x})$, let $((S_i^{\infty},\bar{s}_i^{\infty}))_{i\in\mathbf{N}}$ be the projective system of its pointed universal topological covers (the transition maps will be denoted by f_{ij}^{∞}). It induces a projective system $(\mathrm{Gal}(S_i^{\infty}/X))_{i\in\mathbf{N}}$ of discrete groups. For every i, $\mathrm{Gal}(S_i^{\infty}/X)$ can be identified with $F_{\bar{x}}^{\mathbb{L}}(S_i^{\infty})$: this gives us compatible morphisms $\pi_1^{\mathbb{L}\text{-temp}}(X,\bar{x}) \to \mathrm{Gal}(S_i^{\infty}/X)$. Then, thanks to [2, lem. III.2.1.5],

Proposition 3.3.

$$\pi_1^{\mathbb{L}\text{-temp}}(X,\bar{x}) \to \varprojlim \mathrm{Gal}(S_i^{\infty}/X)$$

is an isomorphism.

In a more categorical way, we have a fibered category $\mathcal{D}_{\mathrm{top}}(X) \to \mathrm{Cov}^{\mathrm{alg}}(X)$, where the fiber $\mathcal{D}_{\mathrm{top}}(X)_S$ in an algebraic cover S of X is $\mathrm{Cov}^{\mathrm{top}}(X)$.

Since algebraic covers are of effective descent for tempered covers, the full subcategory of tempered covers T of X such that $T_S \to S$ is a topological cover is naturally equivalent to the category $\mathrm{DD}_{\mathrm{temp}_S}$ of descent data in the fibered category $\mathcal{D}_{\mathrm{top}}(X)$ with respect to $S \to X$.

If "$\varprojlim$" S_i is a universal procover of (X,x), one gets a natural equivalence

$$\mathrm{Cov}^{\mathrm{temp}}(X) = \varprojlim_i \mathrm{DD}_{\mathrm{temp}_{S_i}}$$

In particular one can recover the tempered fundamental group from the fibered category $\mathcal{D}_{\mathrm{top}}(X) \to \mathrm{Cov}^{\mathrm{alg}}(X)$.

If $S \to S$ is an isomorphism, the induced functor $\mathrm{DD}_{\mathrm{temp}_S} \to \mathrm{DD}_{\mathrm{temp}_S}$ is naturally isomorphic to the identity. Thus if α : "$\varprojlim_i$" $S_i \to$ "$\varprojlim_i$" S_i is an automorphism of the universal pro-cover, the induced functor $\varprojlim_i \mathrm{DD}_{\mathrm{temp}_{S_i}} \to \varprojlim_i \mathrm{DD}_{\mathrm{temp}_{S_i}}$ is naturally isomorphic to the identity. Thus the construction does not depend of the choice of the universal procover.

To give a more stacky and functorial description, let us consider $\mathrm{Cov}^{\mathrm{alg}}(X)$ with its canonical topology.

Let $\mathcal{D}_{\mathrm{temp}}(X) \to \mathrm{Cov}^{\mathrm{alg}}(X)$ be the fibered category whose fiber over U is the category $\mathrm{Cov}^{\mathrm{temp}}(U)$ of tempered covers of U. Then $\mathcal{D}_{\mathrm{temp}}(X)$ is a stack. The fully faithful cartesian functor of fibered categories $\mathcal{D}_{\mathrm{top}}(X) \to \mathcal{D}_{\mathrm{temp}}(X)$ induces a fully faithful cartesian functor of stacks $\mathcal{D}_{\mathrm{top}}(X)^a \to \mathcal{D}_{\mathrm{temp}}(X)$ where $\mathcal{D}_{\mathrm{top}}(X)^a$ is the stack associated to $\mathcal{D}_{\mathrm{top}}(X)$. Since a tempered cover is a topological cover locally on $\mathrm{Cov}^{\mathrm{alg}}(X)$, this functor is in fact an equivalence ([5, th. II.2.1.3]).

In a similar way:

Proposition 3.4. — *The stack* $(\mathcal{D}_{\mathrm{top}}(X)|_{\mathrm{Cov}^{\mathrm{alg}}(X)^{\mathbb{L}}})^a$ *is the stack* $\mathcal{D}_{\mathbb{L}\text{-temp}}(X)$ *of* $\mathbb{L}$*-tempered covers on* $\mathrm{Cov}^{\mathrm{alg}}(X)^{\mathbb{L}}$.

3.2. Homotopy types of analytic curves

Let K be a complete nonarchimedean field with separably closed residue field k and let O_K be its ring of integers. Let $X \to O_K$ be a proper semistable curve with smooth generic fiber. There is a canonical embedding $|\mathbb{G}(X_k)| \to X_\eta^{\mathrm{an}}$ which is a homotopy equivalence ([3, th. 4.3.2]). If K' is a complete isometric extenstion of K with separably closed field k', then the following diagram is commutative:

$$\begin{array}{ccc} |\mathbb{G}(X_{k'})| & \longrightarrow & X_{K'}^{\mathrm{an}} \\ \downarrow & & \downarrow \\ |\mathbb{G}(X_k)| & \longrightarrow & X_K^{\mathrm{an}} \end{array}$$

Moreover, if U is any dense Zariski open subset of X_η, $|\mathbb{G}(X_k)|$ is mapped into U^{an} and $|\mathbb{G}(X_k)| \to U^{\mathrm{an}}$ is still a homotopy equivalence.

If $X \to O_K$ is a semistable log curve and $X' \to X$ is a két morphism such that X' is still a semistable curve, the following diagram is commutative:

$$\begin{array}{ccc} |\mathbb{G}(X'_k)| & \to & X'^{\mathrm{an}}_\eta \\ \downarrow & & \downarrow \\ |\mathbb{G}(X_k)| & \to & X_\eta^{\mathrm{an}} \end{array}$$

3.3. Cospecialization of tempered fundamental groups

Let K be a complete discretely valued field. Let O_K be the ring of integers of K. Let $S \to O_K$ be a morphism of fs log schemes of finite type. Let S_{tr} be the open locus of S where the log structure is trivial ($S_{\mathrm{tr}} \subset S_\eta$). Let $\mathfrak{S}$ be the completion of S along its closed fiber. Then $\mathfrak{S}_\eta$ is an analytic domain of S^{an}. Let $S_0 = \mathfrak{S}_\eta \cap S_{\mathrm{tr}}^{\mathrm{an}} \subset S^{\mathrm{an}}$.

Let $\tilde{\eta}$ be a K'-point of S_0 where K' is a complete extension of K. One has a canonical morphism of log schemes $\mathrm{Spec}\, O_{K'} \to S$ where $\mathrm{Spec}\, O_{K'}$ is endowed with the log structure given by $O_{K'}\backslash\{0\} \to O_{K'}$. The *log reduction* $\tilde{s}$ of $\tilde{\eta}$ is the log point of S corresponding to the special point of $\mathrm{Spec}\, O_{K'}$ with the inverse image of the log structure of $\mathrm{Spec}\, O_{K'}$. If K' has discrete valuation, then $\tilde{s}$ is a fs log point. If K' is algebraically closed, $\tilde{s}$ is a geometric log point.

Definition 3.5. — *The category* $\widetilde{\mathrm{Pt}}^{\mathrm{an}}(S)$ *is the category whose objects are the geometric points* $\bar\eta$ *of* Y_0 *such that* $\mathcal{H}(\eta)$ *is discretely valued (where* η *is the underlying point of* $\bar\eta$*) and* $\mathrm{Hom}_{\mathrm{Pt}^{\mathrm{an}}(S)}(\bar\eta,\bar\eta')$ *is the set of két specializations in* S_k *from the log reduction* $\bar s$ *of* $\bar\eta$ *to the log reduction* $\bar s'$ *of* $\bar\eta'$ *such that there exists some specialization* $\bar\eta\to\bar\eta'$ *of geometric points in the sense of algebraic étale topology for which the following diagram commutes:*

$$\begin{tikzcd} \bar\eta \arrow[r] \arrow[d] & \bar s \arrow[d] \\ \bar\eta' \arrow[r] & \bar s' \end{tikzcd}$$

The category $\mathrm{Pt}_0^{\mathrm{an}}(S)$ *is the category obtained from* $\widetilde{\mathrm{Pt}}^{\mathrm{an}}(S)$ *by inverting the class of morphisms* $\bar\eta\to\bar\eta'$ *such that* $\overline{M}_{S,\bar s'}\to\overline{M}_{S,\bar s}$ *is an isomorphism.*

Let $\mathrm{OutGp}_{\mathrm{top}}$ be the category of topological groups with outer morphisms.

Theorem 3.6. — *Let* O_K *be a complete discretely valued ring of residue characteristic* $p\geqslant 0$*, let* $\mathbb{L}$ *be a set of integers such that* $p\notin\mathbb{L}$*. Let* $S\to\operatorname{Spec} O_K$ *be a morphism of fs log schemes of finite type and let* $X\to S$ *be a proper log semistable curve. Let* U *be the open locus of* X *where the log structure is trivial. Then there is a functor* $\pi_1^{\mathbb{L}\text{-temp}}(U_{(\cdot)}):\mathrm{Pt}_0^{\mathrm{an}}(S)^{\mathrm{op}}\to\mathrm{OutGp}_{\mathrm{top}}$ *sending* $\bar\eta$ *to* $\pi_1^{\mathbb{L}\text{-temp}}(U_{\bar\eta})$*.*

Proof. — Let $\bar\eta_2\to\bar\eta_1$ be a morphism of $\widetilde{\mathrm{Pt}}^{\mathrm{an}}(Y)$. Let us construct a cospecialization morphism $\pi_1^{\mathbb{L}\text{-temp}}(U_{\bar\eta_1})\to\pi_1^{\mathbb{L}\text{-temp}}(U_{\bar\eta_2})$, which is an isomorphism if $\overline{M}_{S,\bar s_1}\to\overline{M}_{S,\bar s_2}$ is an isomorphism.

One has a cospecialization functor

$$F:\mathrm{KCov}_{\mathrm{geom}}(X_{s_1}/s_1)^{\mathbb{L}}\to\mathrm{KCov}_{\mathrm{geom}}(X_{s_2}/s_2)^{\mathbb{L}}$$

which factors through $\mathrm{KCov}_{\mathrm{geom}}(X_T/T)^{\mathbb{L}}$ where T is the strict localization at s_1.

The cospecialization functor $\mathrm{KCov}_{\mathrm{geom}}(X_{s_i}/s_i)^{\mathbb{L}}\to\mathrm{Cov}^{\mathrm{alg}}(U_{\bar\eta_i})$ is an equivalence since $\eta_i\in\eta_{\mathrm{tr}}$ (1.12). If one choses a specialization $\bar\eta_2\to\bar\eta_1$ above $\bar s_2\to\bar s_1$, then one can apply [6, cor. XIII.2.9] to $U_K\subset X_K\to S_K$: one gets that the functor $\mathrm{Cov}^{\mathrm{alg}}(U_{\bar y_i})^{\mathbb{L}}\to\mathrm{Cov}^{\mathrm{alg}}(U_{\bar y_2})^{\mathbb{L}}$ is also an equivalence. Thus F is an equivalence.

Let $Y_{\bar s_1}$ be a log geometric két cover of $X_{s_1/(s_1,\bar s_1)}$ and let $Y_{\bar s_2}$ (resp. Y_1, Y_2) be the corresponding log geometric két cover of $X_{(s_2,\bar s_2)}$ (resp. $U_{\bar\eta_1}$, $U_{\bar\eta_2}$).

There are maps (functorially in Y):

$$|Y^{\mathrm{an}}_{\bar{s}_1}| \leftarrow |\mathbb{G}(Y_{\bar{s}_1})| \to |\mathbb{G}(Y_{\bar{s}_2})| \to |Y^{\mathrm{an}}_{\bar{s}_2}|$$

where the first and third map are the embedding of the skeleton of an anlytic curve. The first and third map are therefore homotopy equivalences.

One thus gets a morphism of homotopy types $|Y^{\mathrm{an}}_{\bar{s}_1}| \to |Y^{\mathrm{an}}_{\bar{s}_2}|$ functorially in Y. According to Proposition 2.12, if $\overline{M}_{S,\bar{s}_1} \to \overline{M}_{S,\bar{s}_2}$ is an isomorphism, $|Y^{\mathrm{an}}_{\bar{s}_1}| \to |Y^{\mathrm{an}}_{\bar{s}_2}|$ is an isomorphism of homotopy types.

With the notations of Proposition 3.4, one thus gets a functor of fibered categories:

$$\begin{array}{ccc} \mathcal{D}_{\mathrm{top}}(U_{\bar{\eta}_2}) & \to & \mathcal{D}_{\mathrm{top}}(U_{\bar{\eta}_1}) \\ \downarrow & & \downarrow \\ \mathrm{Cov}^{\mathrm{alg}}(U_{\bar{\eta}_2})^{\mathbb{L}} & \simeq & \mathrm{Cov}^{\mathrm{alg}}(U_{\bar{\eta}_1})^{\mathbb{L}} \end{array}$$

Using Proposition 3.4, this induces a functor of associated stacks:

$$\begin{array}{ccc} \mathcal{D}_{\mathbb{L}\text{-temp}}(U_{\bar{\eta}_2}) & \to & \mathcal{D}_{\mathbb{L}\text{-temp}}(U_{\bar{\eta}_1}) \\ \downarrow & & \downarrow \\ \mathrm{Cov}^{\mathrm{alg}}(U_{\bar{\eta}_2})^{\mathbb{L}} & \simeq & \mathrm{Cov}^{\mathrm{alg}}(U_{\bar{\eta}_1})^{\mathbb{L}} \end{array}$$

By taking the global sections one gets a functor:

$$\mathrm{Cov}^{\mathbb{L}\text{-temp}}(U_{\bar{\eta}_2}) \to \mathrm{Cov}^{\mathbb{L}\text{-temp}}(U_{\bar{\eta}_1}),$$

which is an equivalence if $\overline{M}_{S,\bar{s}_1} \to \overline{M}_{S,\bar{s}_2}$ is an isomorphism. It induces a cospecialization outer morphism of tempered fundamental groups

$$\pi_1^{\mathbb{L}\text{-temp}}(U_{\bar{\eta}_1}) \to \pi_1^{\mathbb{L}\text{-temp}}(U_{\bar{\eta}_2}),$$

which is an isomorphism if $\overline{M}_{S,\bar{s}_1} \to \overline{M}_{S,\bar{s}_2}$ is an isomorphism.

Let $\bar{\eta}_3 \to \bar{\eta}_2$ be a morphism of $\widetilde{\mathrm{Pt}}^{\mathrm{an}}(Y)$. According to Corollary 1.17, the diagram

$$\begin{tikzcd} \mathrm{KCov}_{\mathrm{geom}}(X_{s_1}/(s_1,\bar{s}_1))^{\mathbb{L}} \arrow[r, "F_{12}"] \arrow[dr, "F_{13}"] & \mathrm{KCov}_{\mathrm{geom}}(X_{s_2}/(s_2,\bar{s}_2))^{\mathbb{L}} \arrow[d, "F_{23}"] \\ & \mathrm{KCov}_{\mathrm{geom}}(X_{s_3}/(s_3,\bar{s}_3))^{\mathbb{L}} \end{tikzcd} \tag{3.1}$$

is 2-commutative. Let $Y_{\bar{s}_3}$ be the log geometric két cover of $X_{s_3}/(s_3/\bar{s}_3)$ corresponding to $Y_{\bar{s}_2}$ and let $Y_{\bar{\eta}_3}$ be the corresponding cover of $X_{\bar{\eta}_3}$. The diagram

$$\begin{tikzcd} |\mathbb{G}(Y_{\bar{s}_1})| \arrow[r] \arrow[dr] & |\mathbb{G}(Y_{\bar{s}_2})| \arrow[d] \\ & |\mathbb{G}(Y_{\bar{s}_3})| \end{tikzcd}$$

is commutative according to 2.12, and therefore the diagram of homotopy types

$$\begin{tikzcd} {|Y^{\mathrm{an}}_{\bar{\eta}_1}|} \arrow[r] \arrow[dr] & {|Y^{\mathrm{an}}_{\bar{s}_2}|} \arrow[d] \\ & {|Y^{\mathrm{an}}_{\bar{s}_3}|} \end{tikzcd}$$

is also commutative. One thus gets a 2-commutative diagram

$$\begin{tikzcd} \mathcal{D}_{\mathrm{top}}(U_{\bar{\eta}_3}) \arrow[r] \arrow[dr] & \mathcal{D}_{\mathrm{top}}(U_{\bar{\eta}_2}) \arrow[d] \\ & \mathcal{D}_{\mathrm{top}}(U_{\bar{\eta}_1}) \end{tikzcd}$$

of fibered categories above the inverse of (3.1). By taking the global sections of the associated functor, one gets that the diagram

$$\begin{tikzcd} \mathrm{Cov}^{\mathbb{L}\text{-temp}}(U_{\bar{\eta}_3}) \arrow[r] \arrow[dr] & \mathrm{Cov}^{\mathbb{L}\text{-temp}}(U_{\bar{\eta}_2}) \arrow[d] \\ & \mathrm{Cov}^{\mathbb{L}\text{-temp}}(U_{\bar{\eta}_1}) \end{tikzcd}$$

is 2-commutative, which proves the functoriality of $\pi_1^{\mathbb{L}\text{-temp}}(U_{(\cdot)})$. □

Remark 3.7. — Such a functor cannot exist if $p \neq 0$ and $\mathbb{L}$ is the set of all primes. Consider a moduli space S of stable curves over $\operatorname{Spec}\mathbf{Z}$, endowed with its canonical log structure. By [18, Th. 5.1.7], S classifies vertical stable log curves. Let $C \to S$ be the universal log curve. If $\bar{s}$ is a geometric point of S, $\overline{M}_{S,\bar{s}}$ can be identified with $\mathbf{N}^I$, where I is the set of double points of $C_{\bar{s}}$, in such a way that for any compatible local chart $U = \operatorname{Spec} A \xrightarrow{\phi} \operatorname{Spec}\mathbf{Z}[\mathbf{N}^I]$ of S at $\bar{s}$ modeled on $\mathbf{N}^I$, locally on the étale topology around the double point x of $C_{\bar{s}}$, the universal curve $C \to S$ has a chart

$$\begin{tikzcd} \mathbf{N}^I \oplus \mathbf{N}u \oplus \mathbf{N}v/(u+v=e_x) \arrow[r] & A[uv]/(uv-\phi^*(e_x)) \\ \mathbf{N}^I \arrow[u] \arrow[r, "\phi^*"] & A \arrow[u] \end{tikzcd}$$

where $e_x \in \mathbf{N}^I$ is defined by $(e_x)_x = 1$ and $(e_x)_{x'} = 0$ if $x' \neq x \in I$. Let $\bar{s}$ be a geometric point in S_k such that the corresponding stable curve is totally degenerate. Let I be the set of double points of $C_{\bar{s}}$. The set I is non empty and therefore $\overline{M}_{S,\bar{s}}$ is nontrivial. Let us choose a local chart $\phi : U =$

$\operatorname{Spec} A \to \operatorname{Spec} \mathbf{Z}[\mathbf{N}^I]$ of S at $\bar{s}$ compatible with the identification of $\overline{M}_{S,\bar{s}}$ with $\mathbf{N}^I$ mentioned before. Choose two different morphisms $a_1, a_2 : \mathbf{N}^I \to \mathbf{N}$ such that the preimage of 0 by a_1 and a_2 is 0. Put on $\bar{s}$ the log structure $k(\bar{s})^* \oplus \mathbf{N}$: one gets a log point s_0. Given the chart ϕ, the morphisms a_1, a_2 define two morphisms of fs log schemes $s_0 \to S$: we denote by s_1 and s_2 the corresponding fs log points of S. By choosing a uniformizer π of O_K, the morphism of schemes $\bar{s} \to \operatorname{Spec} O_K$ can be enriched in a morphism of fs log schemes $s_0 \to \operatorname{Spec} O_K$ by sending π to $(0,1) \in k(\bar{s})^* \oplus \mathbf{N}$. Thus s_1 and s_2 are lifted as fs log points of $S \times_{\operatorname{Spec} \mathbf{Z}} \operatorname{Spec} O_K$. Let η_1 and η_2 be discretely valued points of S_K^{an} whose log reductions are s_1 and s_2. Then, étale locally at x, $C_{O_{\mathcal{H}(\eta_i)}}$ is isomorphic to $\operatorname{Spec} O_K[u,v]/(uv - \pi^{a_i(e_x)})$. Since $a_i(e_x) > 0$ for every x, C_{η_i} is a smooth curve; since C_s is totally degenerate, C_{η_i} is a Mumford curve. The length of the corresponding edge of the graph of C_{η_i} is $-\log_p |\pi|^{a_i(e_x)}$. Since $a_1 \neq a_2$ the two Mumford curves C_{η_1} and C_{η_2} have different metric on the graph of their stable model, and thus have non isomorphic tempered fundamental groups ([15]). But the two geometric log points s_1 and s_2 are isomorphic with respect to specialization for két topology since they lie above the same Zariski point.

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