ANNALES

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Structure of leaves and the complex Kupka-Smale property

STRUCTURE OF LEAVES AND THE COMPLEX KUPKA-SMALE PROPERTY

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ABSTRACT. — We study topology of leaves of 1-dimensional singular holomorphic foliations of Stein manifolds. We prove that for a generic foliation all leaves, except for at most countably many, are contractible, the rest are topological cylinders. We show that a generic foliation is complex Kupka-Smale.

RÉSUMÉ. — Nous étudions la topologie des feuilles d'un feuilletage holomorphe singulier de dimension 1 sur des variétés de Stein. Nous prouvons que pour un feuilletage générique, toutes les feuilles, sauf au plus un nombre dénombrable, sont contractiles, les autres étant topologiquement des cylindres. Nous montrons aussi qu'un feuilletage générique est Kupka-Smale complexe.

1. Introduction

Consider a vector field $(f_1, \ldots, f_n)$ in $\mathbb{C}^n$, where $f_1, \ldots, f_n \in \mathcal{O}(\mathbb{C}^n)$. The phase space $\mathbb{C}^n$, outside the singular locus, is foliated by Riemann surfaces. The natural question is: what is the topological type of these leaves? For polynomial foliations of fixed degree this question was asked by Anosov and still remains unsolved. In general, it can be quite complicated. Consider, for example, a Hamiltonian foliation of $\mathbb{C}^2$: $H_n = \text{const}$, where H_n is a generic polynomial of degree n. All non-singular leaves are Riemann surfaces with $\frac{(n-1)(n-2)}{2}$ handles and n punctures. There are examples of foliations with dense leaves, having infinitely generated fundamental groups [17].

So one can restrict the question: what is the topological type of leaves for a generic foliation?

The genericity here is understood as follows: the space of holomorphic foliations can be naturally equipped with the (Baire) topology of uniform convergence on nonsingular compact sets. We recall the definition of the topology in Appendix 5.3. We call a foliation generic if it belongs to a residual set – an intersection of countably many open everywhere dense sets.

In our paper we describe the topological type of leaves for generic foliations on $\mathbb{C}^n$, and more generally, on arbitrary Stein manifolds. We prove the following theorem:

THEOREM 1.1. — *For a generic 1-dimensional singular holomorphic foliation on a Stein manifold X all leaves, except for at most countably many, are contractible, the rest are topological cylinders.*

We consider foliations with singular locus of codimension 2, i.e. foliations locally determined by holomorphic vector fields [14, Theorem 2.22].

Our technique is applicable in a more general setting. In particular, we establish the analog of the Kupka-Smale theorem for generic foliations on Stein manifolds:

THEOREM 1.2. — *A generic 1-dimensional singular holomorphic foliation on X is complex Kupka-Smale.*

DEFINITION 1.3. — *A foliation of a complex manifold is called complex Kupka-Smale if*
  (1) *all its singular points are complex hyperbolic;*
  (2) *all complex cycles are hyperbolic;*
  (3) *strongly invariant manifolds of different singular points intersect transversally;*
  (4) *invariant manifolds of complex cycles intersect transversally with each other and with strongly invariant manifolds of singular points.*

Let cycle γ be a phase curve of a real vector field, then γ is a loop on the phase curve of the complexified vector field. A complex cycle by definition is a free homotopy class of loops on a leaf of a foliation. Recall that by definition, a real Kupka-Smale vector field has hyperbolic cycles only. Condition (2) is a generalization of this property.

We review notions of complex hyperbolicity and invariant manifolds in the Appendix 5.1.

The above definition was suggested by Marc Chaperon in [5]. In this preprint he studies holomorphic 1-dimensional singular foliations on Stein manifolds. He shows that the property (1) holds for generic foliations. He

also gives the proof of the property (3) for generic foliations on $\mathbb{C}^n$ and states the result for generic foliations on Stein manifolds. Our technique also allows us to prove transversality results for strongly invariant manifolds of the same singular point:

THEOREM 1.4. — *For a generic 1-dimensional singular holomorphic foliation:*

(1) *all singular points are complex hyperbolic.*
(2) *Let a_1 be a complex hyperbolic singular point of the foliation. Let M_1 and M_2 be strongly invariant manifolds of the point a_1, such that $M_1^{loc} \cap M_2^{loc} = a_1$. Then M_1 and M_2 intersect transversally everywhere.*

Theorems 1.1, 1.2 for foliations of $\mathbb{C}^2$ are proved in [7]. Golenishcheva-Kutuzova [10] showed that for a generic foliation countable many cylinders do exist. We expect that for a generic singular holomorphic 1-dimensional foliation of a Stein manifold there are countably many cylinders.

It is known that any leaf of a generic polynomial foliation of degree n is hyperbolic [3], [8], [16]. We expect that the same answer is true for generic foliations of Stein manifolds and that technique from [3], [16], [8] can be adjusted to attack the problem. See the paper [13] for a vast discussion of open problems.

Greg Buzzard studied similar genericity questions for analytic automorphisms of $\mathbb{C}^n$. He proved that a generic analytic automorphism of $\mathbb{C}^n$ is Kupka-Smale [2].

1.1. Outline of the article

We establish generic properties of foliations by constructing perturbations that eliminate degeneracies. There are at most countably many isolated cycles. (This lemma is proved in [15] for foliations of $\mathbb{C}^2$. We included the proof for arbitrary Stein manifolds in Section 4.1 to explain our strategy of simultaneous elimination of degeneracies.) Therefore, once all nonisolated cycles are removed, all leaves, except for countably many, are contractible.

To prove that the rest have fundamental group $\mathbb{Z}$, one needs to eliminate all degeneracies from the following list:

(1) two cycles that belong to the same leaf of the foliation and are not multiples of the same cycle in the homology group of the leaf;

(2) saddle connections;

(3) cycles on a separatrix that are not multiples of the cycle around the critical point.

Recall that a separatrix is a leaf that can be holomorphically extended into a singular point and a saddle connection is a common separatrix of two singular points.

In the smooth category one can remove a degeneracy of a foliation locally. Say, one can destroy a homoclinic loop by changing the foliation only in a flow-box around a point on the loop.

In the holomorphic category, a priori, one cannot perturb a foliation in a flow-box without changing the foliation globally. Our strategy to remove degeneracies in the holomorphic category is the following:

In Section 2 we construct a family of foliations, that removes degeneracy, in a neighborhood of a degenerate object, rather than in a flow-box around a point. We say that a degenerate object is removed in a family of foliations if, roughly speaking, there are no degenerate objects of the same kind for perturbed foliations. See Definitions 2.6, 2.8, 2.9. A non-isolated cycle, a non-trivial pair of cycles are examples of degenerate objects. We give a complete list of degenerate objects in Section 2. All degenerate objects we consider are curves or collections of curves. Our technique allows us to construct an appropriate family only if a degenerate object is holomorphically convex. We expect though that it should be possible to carry out for any degenerate object.

In [7] our approach to construct a family of local foliations in a neighborhood of a degenerate object was to control the derivative of the holonomy map along the leaf with respect to a perturbation. This approach can not be adapted to remove a non-transversal intersection of strongly invariant manifolds. There are no leaf-wise paths, that connect singular points with a point of non-transversal intersection. Therefore, one cannot control the intersection of invariant manifolds this way.

In this paper we use a different approach, a more geometric one. First, we reglue the neighborhood (Subsection 2.3). Then we project the obtained manifold, together with a new foliation, to the original one. We use theorem [18], that states that a Stein manifold has a Stein neighborhood, to construct the projection.

We give a review of results on the holomorphic hulls of collections of curves in Section 3. We apply them to give geometric conditions for a degenerate object to be holomorphically convex. We also review the relevant results from the Approximation Theory on Stein manifolds and apply them

to pass from a local family of foliations in a neighborhood of a degenerate object to a global one.

When we remove a degenerate object, e.g. a complex cycle, we do not control the foliation outside a neighborhood of the degenerate object. Therefore, it might happen that eliminating one degenerate object we create many other in different places. We solve this problem as follows: We find a countable number of places where degenerate objects can be located. For each such location we prove that the complement to the set of foliations, which have the degenerate object at this particular location, is open and everywhere dense. Then we intersect these sets and get a residual set of foliations without holomorphically convex degenerate objects. We show that if a foliation has a degenerate object, then it has a holomorphically convex degenerate object. Therefore, the residual set constructed does not have degenerate objects. We describe this strategy in detail in Section 4. This strategy was previously used in [7] and [9].

We give background on the complex foliations in the Appendix 5.1.

1.2. Acknowledgements

2. Local removal of degenerate objects

2.1. List of degenerate objects

As we pointed out in the introduction one can not eliminate a homoclinic saddle connection by changing the foliation only locally in a flow-box. Rather than that one needs to perturb the foliation in the neighborhood of the separatrix loop. This leads us to considering degenerate objects.

Below we list degenerate objects. One can check that if a foliation does not have degenerate objects of type $1-5$, then it satisfies Theorem 1.1. If all singular points of a foliation are complex hyperbolic and it does not have degenerate objects of types $1-6$ and $8-9$, then it is complex Kupka-Smale. If all singular points of a foliation are complex hyperbolic and a foliation does not have degenerate objects of type 7, then it satisfies Theorem 1.4.

In, fact, we can require that there are no degenerate objects that satisfy additional geometric conditions. We prove these stronger statements in Theorems 4.11, 4.12.

DEFINITION 2.1. — *We say that γ is a degenerate object of a foliation $\mathcal{F}$ if γ is*

(1) *A non-trivial loop on a leaf L of $\mathcal{F}$, which is a representative of a non-hyperbolic cycle.*

(2) *A union of loops γ_1, γ_2 that belong to the same leaf L of $\mathcal{F}$. We assume γ_1 and γ_2 are not multiples of the same cycle. Moreover, γ_1, γ_2 are hyperbolic. (See Fig. 2.1.)*
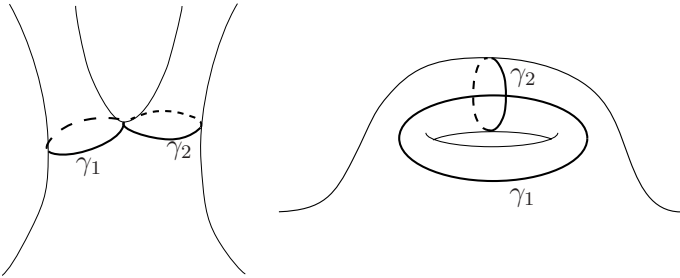
Figure 2.1. A pair of cycles

(3) *A path on a saddle connection, that connects two different hyperbolic singular points a_1 and a_2. (See Fig. 2.2).*

(4) *A loop on a homoclinic saddle connection S (See Fig.2.2):*
   - *a is a hyperbolic singular point;*
   - *S_1, S_2 are local separatrices of the singular point a; $S_1 \neq S_2$; $S_1, S_2 \subset S$;*
   - *$\gamma \subset S$ passes through the singular point a, starts at S_1, ends along S_2.*

Figure 2.2. A path on a saddle connection and a loop on a homoclinic saddle connection

(5) *A non-trivial loop γ on a separatrix that passes through a singular point a.*

(6) *A union of paths γ_1 and γ_2 (See Fig. 2.3):*
  - *a_1, a_2 are hyperbolic singular points of the foliation $\mathcal{F}$, $a_1 \neq a_2$;*
  - *M_1 and M_2 are strongly invariant manifolds of a_1 and a_2 correspondingly;*
  - *p is a point of A non-transversal intersection of M_1 and M_2;*
  - *$\gamma_1 \subset M_1$ and $\gamma_2 \subset M_2$ are paths that connect a_1 and a_2 with the point p;*
  - *$(\gamma_1 \cup \gamma_2) \backslash (M_1^{loc} \cup M_2^{loc}) \subset L$, where L is a leaf of the foliation $\mathcal{F}$.*
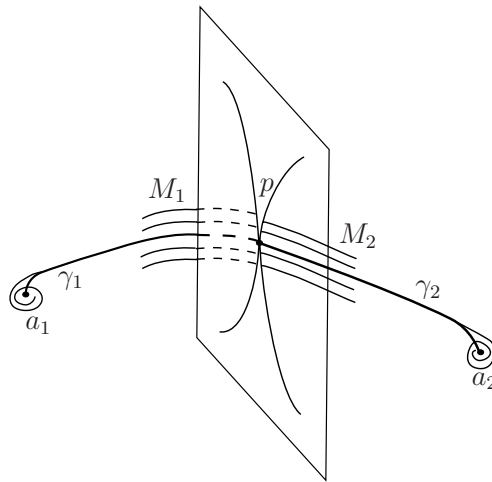
Figure 2.3. *A non-transversal intersection of strongly invariant manifolds. The leaf L on the picture is not a separatrix. Hence, it spirals around singular points a_1 and a_2.*
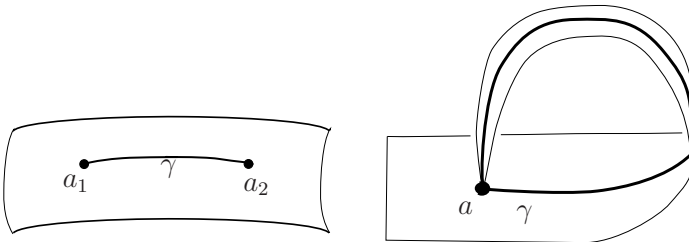
(7) *A loop $\gamma_1 \cup \gamma_2$:*
  - *a is a hyperbolic singular point of the foliation $\mathcal{F}$;*
  - *M_1 and M_2 are strongly invariant manifolds of the point a;*
  - *$M_1^{loc} \cap M_2^{loc} = a$;*
  - *paths $\gamma_1 \subset M_1$, $\gamma_2 \subset M_2$ connect a with p;*
  - *$(\gamma_1 \cup \gamma_2) \backslash (M_1^{loc} \cup M_2^{loc}) \subset L$, where L is a leaf of the foliation $\mathcal{F}$.*

(8) *A union $\gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \gamma_4$:*
  - *γ_1, γ_2 are hyperbolic loops on leaves of $\mathcal{F}$;*
  - *M_1, M_2 are invariant manifolds of γ_1, γ_2 correspondingly;*

- $\gamma_3 \subset M_1$, $\gamma_4 \subset M_2$ are paths that connect points on γ_1, γ_2 with a point of a non-transversal intersection of M_1, M_2.
- $(\gamma_3 \cup \gamma_4) \backslash \left(M_1^{loc} \cup M_2^{loc} \right) \subset L$, where L is a leaf of $\mathcal{F}$.

(9) *A union $\gamma = \gamma_1 \cup \gamma_2 \cup \gamma_3$:*
  - γ_1 is a hyperbolic loop on a leaf;
  - M_1 is an invariant manifold of γ_1;
  - a is a hyperbolic singular point;
  - M_2 is a strongly invariant manifold of a;
  - $\gamma_2 \subset M_1$, $\gamma_3 \subset M_2$ are paths on invariant manifolds that connect a point on γ_1 and the point a correspondingly with the point of a non-transversal intersection of M_1 and M_2
  - $(\gamma_2 \cup \gamma_3) \backslash \left(M_1^{loc} \cup M_2^{loc} \right) \subset L$, where L is a leaf of the foliation $\mathcal{F}$.

2.2. Local Removal Lemma

In this section we find a neighborhood of a degenerate object and a family of holomorphic foliations in this neighborhood that removes the degenerate object in the neighborhood.

Our technique allows us to do that only if the degenerate object is holomorphically convex. We expect, though, that it is possible to omit this assumption.

Let U be a neighborhood of the degenerate object. First, we allow not only the foliation, but the neighborhood itself to change with the parameter λ. We get a family of foliations $\mathcal{F}_\lambda$ on manifolds U_λ. Then we find the way to "project" U_λ to some neighborhood of the degenerate object. Thus, we produce a family of foliations in the neighborhood of the degenerate object that removes it.

The following lemma summarizes the results of the following two subsections.

Let γ be a union of curves on a Stein manifold X, endowed with a foliation $\mathcal{F}_0$. Assume that γ is holomorphically convex. Fix a point $p \in \gamma$, assume that $p \notin \Sigma(\mathcal{F})$. Let $\alpha \subset \gamma$ be a small arc, a neighborhood of p on γ. We assume that $\alpha \subset L$, where L is a leaf of the foliation $\mathcal{F}_0$. We can fix coordinates $(z_1, \ldots, z_{n-1}, t)$ in a neighborhood of the point p, so that z-coordinates do not change along the foliation. We can assume that t is a coordinate in a neighborhood $U(\alpha)$ of α on the leaf L. Consider the flow-box

$$V := \{|z| < 1\} \times U(\alpha).$$

We rescale z-coordinates if necessary so that V is compactly contained in the coordinate neighborhood of the point p. Take a pair of points $q_1, q_2 \in \gamma \backslash \alpha$, that lie on different sides of α and $q_1, q_2 \in U(\alpha)$. Let T_1, T_2 be transversal sections to $\mathcal{F}_0$ that pass through q_1, q_2. Functions $(z_1, \ldots, z_{n-1})$ work as coordinates on T_1, T_2.
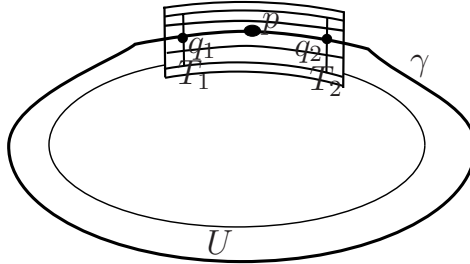

Figure 2.4. γ together with its neighborhood. The flow-box is foliated by leaves of the foliation $\mathcal{F}_0$. The break points of γ on the picture indicate that $\gamma \backslash \alpha$ does not necessarily belong to L.

Let Φ_λ be a family of germs of biholomorphisms, that depends on λ holomorphically.

$$\Phi_\lambda : \left(\mathbb{C}^{n-1}, 0\right) \to \left(\mathbb{C}^{n-1}, 0\right), \quad \Phi_0 = Id.$$

LEMMA 2.2. — There exist a neighborhood $\widetilde{U}$ of γ, that retracts to γ, and a family of foliations $\mathcal{F}_\lambda$ on $\widetilde{U}$ that depends holomorphically on λ satisfying the following conditions:

(1) in $\widetilde{U} \backslash V$, $\mathcal{F}_\lambda$ is biholomorphic to $\mathcal{F}_0$. More precisely, there exists a holomorphic on λ family of maps $\pi_\lambda : \left(\widetilde{U} \backslash V\right) \to X$, which are biholomorphisms to their images, such that π_λ maps the leaves of $\mathcal{F}_0$ to the leaves of $\mathcal{F}_\lambda$, $\pi_0 = Id$;

(2) The holonomy map inside the flow-box along the foliation $\mathcal{F}_\lambda$ between T_1 and T_2 is biholomorphically conjugate to Φ_λ, more precisely, in coordinates $(z_1, \ldots, z_{n-1})$ on T_1, T_2 it is $(\pi_\lambda^z)^{-1} \circ \Phi_\lambda \circ \pi_\lambda^z$, where π_λ^z and $(\pi_\lambda^z)^{-1}$ are first $(n-1)$ coordinates of π_λ and π_λ^{-1} correspondingly.

This lemma mimics the smooth case, where one can perturb the foliation only in the flow-box. In the holomorphic case this is not possible. Therefore, we need to adjust everything by the map π_λ. In the following two sections we describe the regluing and projection techniques. We use these techniques to prove Lemma 2.2 at the end of Section 2.4.

2.3. Regluing

We weaken the restriction on γ for this subsection. We do not assume it is holomorphically convex.

We start by constructing manifolds U_λ. They are obtained by regluing U in the flow-box around the point p. First, we describe the procedure informally and point out the technical difficulties that arise. Then we repeat the description paying attention to the technical difficulties.

We take a neighborhood U that can be retracted to γ. Let $\widehat{U}$ be the complex manifold obtained form U by doubling the preimage under retraction of a small arc α_1, $p \in \alpha_1$. One can assume that the preimage of α_1 is a flow-box. So $\widehat{U}$ comes with the natural projection $\widehat{U} \to U$, which is one-to-one everywhere except for the two flow-boxes around the preimages of p, which are glued together by the identity map. U_λ is obtained from $\widehat{U}$ by gluing the points in the flow-boxes by using the map $(\Phi_\lambda, \mathrm{Id})$. The problem is that $(\Phi_\lambda, \mathrm{Id})$ is not an isomorphism from the flow-box to itself. Thus, extra caution is needed to make U_λ Hausdorff. In the rest of the section we describe these precautions.

First, we choose a bigger neighborhood W that can be retracted to γ. Let ρ denote the retraction. Let $\widehat{W}$ be the connected complex manifold that projects one-to-one to $W \backslash \rho^{-1}(\alpha)$ and two-to-one to $\rho^{-1}(\alpha)$. Let π_1^{-1}, π_2^{-1} be the the two inverses of the projection $\widehat{W} \to W$, restricted to the preimage of $\rho^{-1}(\alpha) \subset W$.

We assume that the flow-box $V \subset \rho^{-1}(\alpha)$. We can assume that V is small enough so that $(\Phi_\lambda, \mathrm{Id})$ is a well-defined map on V and is a biholomorphism to its image. Let $V_1 = \pi_1^{-1}(V)$, $V_2 = \pi_2^{-1}(V)$.

Let $T_c \subset W$ be the tube of points that are at distance less than or equal to c from $\gamma \backslash \alpha$. Let $\widehat{T}_c \subset \widehat{W}$ be the tube of points that are at distance less than or equal to c from the preimage of $\gamma \backslash \alpha$. Take c small enough.

Take $U = T_c \cup V$, $\widehat{U} = V_1 \cup V_2 \cup \widehat{T}_c$. Note that U is obtained from $\widehat{U}$ by gluing the points from V_1 and V_2 that project to the same point in W.

Let $V_2^\lambda = \pi_2^{-1}\left((\Phi_\lambda, Id)(V)\right)$

Let $\widehat{U}_\lambda = V_1 \cup \widehat{T}_c \cup V_2^\lambda$. U_λ is a space obtained from $\widehat{U}_\lambda$ by gluing V_1 and V_2^λ by the map $(\Phi_\lambda, \mathrm{Id})$. The space U_λ inherits complex structure. If one takes c and λ small enough, then it is also Hausdorff.

We also consider the total space of reglued manifolds:

$$\widehat{\mathcal{U}} = \{(u, \lambda) \in \widehat{W} \times \Lambda \mid u \in V_1 \cup T_c \cup V_2^\lambda, \lambda \in \Lambda\}$$

$$\mathcal{U} = \widehat{\mathcal{U}}/ \sim, (u, \lambda) \sim \left((\Phi_\lambda, \mathrm{Id})(u), \lambda\right), \text{ where } u \in V_1, \lambda \in \Lambda$$

Note that both $\mathcal{U}, \widehat{\mathcal{U}}$ are complex manifolds. U is embedded into $\mathcal{U}$ as a submanifold, given by $\{\lambda = 0\}$.

2.4. Projection. Siu's Theorem

In this subsection we prove that for small enough λ, one can take a small neighborhood of γ in U_λ and project it biholomorphically to a neighborhood of γ in U.

Assume γ is holomorphically convex. By [11, Theorem 5.16] there is a neiborhood U_1 of γ, $U_1 \subset U$, such that U_1 is a Stein manifold.

By the theorem, formulated below there is a Stein neighborhood $\widetilde{\mathcal{U}}$ of U_1 in $\mathcal{U}$.

THEOREM 2.3. — [18] *Suppose X is a complex space and A is a subvariety of X. If A is Stein, then there exists an open neighborhood Ω of A in X such that Ω is Stein.*

Fix an embedding of $\widetilde{\mathcal{U}}$ into $\mathbb{C}^N$. We need the following lemma:

LEMMA 2.4. — *There exists a linear $(N-n)$-subspace $\alpha \subset \mathbb{C}^N$ such that the affine subspaces $\alpha_x \subset \mathbb{C}^N$ parallel to α passing through points $x \in \gamma$ are:*

   a) *transverse to U;*
   b) *pass through only one point on γ.*

Proof. — The set of all $(N-n)$-subspaces of $\mathbb{C}^N$ is $n(N-n)$-dimensional complex manifold $Gr(N-n, N)$.

Elements of $Gr(N-n, N)$ that are not transverse to a given subspace of complementary dimension form a complex (may be singular) subvariety of codimension 1. Path γ is a real manifold of dimension 1. Therefore, subspaces that do not satisfy (a) form a subvariety of $Gr(N-n, N)$ of real codimension 1.

A couple of points on γ form a real 2-dimensional manifold. Consider a line that passes through two given points in $\mathbb{C}^N$. Linear subspaces of $\mathbb{C}^N$ that contain this direction form $n(N-n-1)$-dimensional manifold. Therefore, subspaces that do not satisfy (b) form a submanifold of $Gr(N-n, N)$ of real codimension $2(n-1)$.

Since $n \geqslant 2$, a $(N-n)$-subspace α, that satisfies conditions (a) and (b), exists. □

Proof of Lemma 2.2: Take a hyperplane α that satisfies Lemma 2.4. Let $\tilde{\pi}_\lambda$ be the projection along α from a neighborhood U_1 of γ in U to

U_λ, given by Lemma 2.4. One can take U_1 to be small enough, so that $\tilde{\pi}_\lambda : U_1 \to U_\lambda$ is a biholomorphism to its image for all small $\lambda \in \Lambda_1 \subset \Lambda$. Recall that U_λ is obtained from U by regluing in the flow-box V. Therefore, the natural "inclusion" map $i_\lambda : U \backslash V \to U_\lambda$ is well-defined. We can take a small enough neighborhood $\tilde{U}$ of γ so that $i_\lambda(\tilde{U} \backslash V) \subset \pi_\lambda(U_1)$ for all $\lambda \in \Lambda_1$.

Since $U \subset X$, the inclusion map $i' : U \to X$ is well-defined.

Let $\pi_\lambda = i' \circ \tilde{\pi}_\lambda^{-1} \circ i_\lambda$. By construction, $\pi_\lambda : \tilde{U} \backslash V \to X$ is a biholomorphism to its image. The foliation $\mathcal{F}_\lambda$ is the image of $\mathcal{F}_0$ under the map π_λ.

$\pi_\lambda(T_1)$ and $\pi_\lambda(T_2)$ are cross-sections to $\mathcal{F}_\lambda$. Consider the holonomy map $h_\lambda : \pi_\lambda(T_1) \to \pi_\lambda(T_2)$ along the foliation $\mathcal{F}_\lambda$. In coordinates $\pi_\lambda(z)$ on $\pi_\lambda(T_1)$, and $\pi_\lambda(T_2)$, $h_\lambda = \Phi_\lambda$. Switching back to the original coordinate space $(z_1, \ldots, z_{n-1})$, one gets $h_\lambda = (\pi_\lambda^z)^{-1} \circ \Phi_\lambda \circ \pi_\lambda^z$.   □

2.5. Removal of a holomorphically convex degenerate object

As we pointed out in the introduction, a degenerate object is removed by a small perturbation if, roughly speaking, in some neighborhood of the object, there are no degenerate objects of the same kind for perturbed foliations.

Let γ be a degenerate object of a foliation $\mathcal{F}_0$ on a manifold X.

We say that $\mathcal{F}_\lambda$ is a local holomorphic family for γ if there exists a neighborhood U of γ, such that $\mathcal{F}_\lambda$ are well-defined in U for all $\lambda \in \Lambda$, where Λ is a neighborhood of the origin; and $\mathcal{F}_\lambda$ depend holomorphically on λ.

THEOREM 2.5. — *Let γ be a holomorphically convex degenerate object of a foliation $\mathcal{F}_0$. Then there exists a local holomorphic family of foliations $\mathcal{F}_\lambda$ that removes γ.*

In the following subsections we rigorously define what it means that a degenerate object is removed in a local holomorphic family of foliations. We also prove Theorem 2.5 for different types of degenerate objects.

2.6. Removal of a non-hyperbolic cycle

DEFINITION 2.6. — *Let γ be a non-hyperbolic cycle of a foliation $\mathcal{F}_0$. We say that it is removed in a local holomorphic family of foliations $\mathcal{F}_\lambda$ if*

(1) there is a transversal section T at a point $p \in \gamma$ to the foliation $\mathcal{F}_0$ such that holonomy maps along γ for the foliations $\mathcal{F}_\lambda$, $\Delta_\gamma^\lambda : D_r \to T$ are well-defined for $\lambda \in \Lambda$, where $D_r \subset T$ is the disk of radius r with the center in the point p;

(2) for all $\lambda \in \Lambda \backslash R$, Δ_γ^λ has a unique fixed point on D_r, where R is a one (or zero)-dimensional real-analytic set. Moreover, this fixed point is hyperbolic.

Proof of Theorem 2.5 for type 1:

Take a point $p \in \gamma$ and a transversal section T to $\mathcal{F}$, $p \in T$. Let $\Delta_\gamma : (T,p) \to (T,p)$ be the corresponding holonomy map. The cycle γ is hyperbolic by the definition if and only if all the eigenvalues of Δ_γ lie not on the unit circle.

First, we provide a specific perturbation of Δ_γ that has hyperbolic fixed points only.

The following lemma is the standard fact:

LEMMA 2.7. — *There exists a diagonal $n \times n$ matrix D and $a \in \mathbb{C}^n$ such that the map $\Delta_\gamma(z) + \lambda(Dz + a)$ is well-defined and has hyperbolic fixed points only for all $\lambda \in V \backslash R$, where V is a neighborhood of 0, R is a 1 (or 0)-dimensional real-analytic set, $0 \in R$.*

Take a, D such that Lemma 2.7 is satisfied.

Apply Lemma 2.2 to the cycle γ, the point p and the family of biholomorphisms $\Phi_\lambda = Id + \lambda(Dz + a)$. The map $\Delta_\gamma^\lambda = \pi_\lambda^{-1} \circ (\Delta_\gamma + \lambda(\Delta z + a)) \circ \pi_\lambda$ is the holonomy map along γ for the foliation $\mathcal{F}_\lambda$. For all λ outside a (possibly empty) one-dimensional real-analytic set R the map Δ_γ^λ has hyperbolic fixed points only on T. □

2.7. Splitting cycles to different leaves

Let $\gamma = \gamma_1 \cup \gamma_2$ be a degenerate object of type 2.

DEFINITION 2.8. — *We say that γ is removed in a holomorphic family of foliations $\mathcal{F}_\lambda$, $\lambda \in \Lambda$, if*

(1) *there is a transversal section T at a point $p \in \gamma_1 \cap \gamma_2$ to the foliation $\mathcal{F}_0$ such that holonomy maps $\Delta_{\gamma_1}^\lambda, \Delta_{\gamma_2}^\lambda : D_r \to T$ are well-defined for all $\lambda \in \Lambda$, where $D_r \subset T$ is a disk of radius r;*

(2) *$\Delta_{\gamma_1}^\lambda$ and $\Delta_{\gamma_2}^\lambda$ do not have a common fixed point on D_r for $\lambda \neq 0$.*

Thus, the degenerate object is removed if γ_1 and γ_2 split to leaves, that are different, at least in U.

Proof of Theorem 2.5 for type 2:

Let $q \in \gamma_1 \backslash \gamma_2$. Assume, it is not a point of self-intersection of γ_1. Apply Lemma 2.2 to the curve γ, the point q, and the family of biholomorphisms $\Phi_\lambda = z + \lambda$. Then $\pi_\lambda^{-1} \circ \Delta_{\gamma_2} \circ \pi_\lambda$ is a holonomy map along γ_2. Let T_1 be a transversal section to the foliation $\mathcal{F}_0$ in the point q. The holonomy map along γ_1 for the foliation $\mathcal{F}_0$ can be written as a composition $\Delta_{\gamma_1} = \Delta_2 \circ \Delta_1$, where Δ_1 is a holonomy map from transversal section T to T_1, Δ_2 is a holonomy map from T_1 to T. Then the holonomy map along γ_1 for the foliation $\mathcal{F}_\lambda$ is $\pi_\lambda^{-1} \circ \Delta_2 \circ \Phi_\lambda \circ \Delta_1 \circ \pi_\lambda$.

$\pi_\lambda^{-1}(p)$ is an isolated fixed point for the holonomy map along γ_2 and is not a fixed point for the holonomy map along γ_1. Thus, cycles split to leaves, that are different at least in the neihgborhood U. □

2.8. Removal of non-transversal intersections of invariant manifolds and saddle connections

Let a be a hyperbolic singular point. Local strongly invariant manifolds and separatrices of a exist and depend holomorphically on the perturbation. See Appendix 5.1.

Let γ be a complex hyperbolic cycle. Local stable and unstable manifolds of γ exist and depend holomorphically on the perturbation. See Appendix 5.1.

We refer to separatrices, strongly invarint manifolds and stable/unstable manifolds as invariant manifolds in the sequel.

For each degenerate object γ of a foliation $\mathcal{F}_0$ of types $3 - 9$, there are two invariant manifolds that meet nontransversally. Saddle connections are examples of a non-transversal intersection. We denote the corresponding local invariant manifolds by M_1^{loc} and M_2^{loc}. Let $\mathcal{F}_\lambda$ be a holomorphic family of foliations in a neighborhood of γ. Let $M_1^{loc}(\lambda)$, M_2^{loc} denote the perturbations of M_1^{loc} and M_2^{loc}.

Note that for the degeneracy of type 5, $M_1^{loc} = M_2^{loc}$.

For degenerate objects of types $3 - 5$, we need to fix a point p. We can take it to be any point in $\gamma \backslash \left(M_1^{loc} \cup M_2^{loc} \right)$.

Notice that for all degenerate objects of type $3 - 9$, $\gamma \backslash \left(M_1^{loc} \cup M_2^{loc} \right) \subset L$, where L is a leaf of foliation $\mathcal{F}_0$. Therefore, holomorphic extensions $M_1(\lambda)$ and $M_2(\lambda)$ of $M_1^{loc}(\lambda)$ and $M_2^{loc}(\lambda)$ along γ are well-defined. We omit parameter λ for invariant manifolds of $\mathcal{F}_0$.

DEFINITION 2.9. — *We say that γ can be eliminated in a holomorphic family of foliations $\mathcal{F}_\lambda$ if there exists a transversal section T to the foliation $\mathcal{F}_0$, $p \in T$, so that $M_1(\lambda)$ and $M_2(\lambda)$ intersect transversally on T.*

NOTE 2.10. — *Note that if M_1^{loc} and M_2^{loc} are separatrices, then the holomorphic family eliminates the saddle connection.*

Proof of Theorem 2.5 for types 3-9: Let U be a neighborhood of the point p. We can assume that $M_1 \cap U$ and $M_2 \cap U$, are biholomorphically equivalent to $m_1 \times D$, $m_2 \times D$, where $m_1 = M_1 \cap D_1$, $m_2 = M_2 \cap D_1$, D is a neighborhood of p on the leaf L; D_1 is a neighborhood of p on the transversal section T. Fix coordinates $(z_1, \ldots, z_{n-1})$ on T. Apply Lemma 2.2 to the curve γ, the point p and $\Phi_\lambda = z + \lambda a$. Assume that points $q_1 \in \gamma_1$, $q_2 \in \gamma_2$.

Outside of the flow-box $M_1(\lambda) = \pi_\lambda(M_1)$, $M_2(\lambda) = \pi_\lambda(M_2)$.

In a neighborhood of the point p:

$$T \cap M_2(\lambda) = \pi_\lambda^z(m_2)$$

$$T \cap M_1(\lambda) = \Phi_\lambda \circ \pi_\lambda^z(m_1).$$

Therefore, by Sard's Theorem, for almost all a they intersect transversally. □

3. Construction of a global removal family

In this section we give the geometric conditions for degenerate objects to be holomorphically convex and show how to pass from a local removal foliation to a global one.

3.1. Approximation Theory

Working in the category of smooth vector fields one can eliminate a non-transversality by perturbing the vector field only in a neighborhood of the non-transversality. In the holomorphic category there are no local perturbations allowed. However, approximation theory gives a way to work locally. In some cases one can perturb the local picture and then approximate your perturbation by a global one. In particular, for a holomorphic vector bundle on a Stein manifold holomorphic sections over a neighborhood of a holomorphically convex set can be approximated by global holomorphic sections. This follows from two theorems formulated below.

THEOREM 3.1. — [11, 5.6.2] *Let X be a Stein manifold and φ a strictly plurisubharmonic function in X such that $K_c = \{z \colon z \in X, \varphi(z) \leqslant c\} \Subset X$ for every real number c. Let B be an analytic vector bundle over X. Every analytic section of B over a neighborhood of K_c can then be uniformly approximated on K_c by global analytic sections of B.*

THEOREM 3.2. — [11, 5.1.6] *Let X be a Stein manifold, K a compact subset of X and U is an open neighborhood of holomorphic hull of K. Then there exists a function $\varphi \in C^\infty(X)$ such that*

(1) *φ is strictly plurisubharmonic,*
(2) *$\varphi < 0$ in K but $\varphi > 0$ in $X \backslash U$,*
(3) *$\{z \colon z \in X, \varphi(z) < c\} \Subset X$ for every $c \in \mathbb{R}$.*

THEOREM 3.3. — *Let γ be a holomorphically convex degenerate object of a foliation $\mathcal{F}_0$. Then there exists a holomorphic family $\mathcal{F}_\lambda$ of foliations on X, that removes γ.*

Proof. — By Theorem 2.5, there is a family of local holomorphic foliations $\mathcal{F}_\lambda$ that removes γ. By Lemma 5.8, foliations $\mathcal{F}_\lambda$ are determined by local sections s_λ of the analytic bundle $TX \otimes B_{\mathcal{F}}$. Let $\lambda_0 \in \Lambda$ be a parameter that does not belong to the exceptional real analytic set. By Theorems 3.1 and 3.2, there exists a global section S_{λ_0} that is ε-close to s_{λ_0} on U', where $\gamma \Subset U' \Subset U$. Therefore, the family of foliations determined by $S_\lambda = S_0 + \lambda(S_{\lambda_0} - S_0)$ removes the degenerate object. □

3.2. Holomorphic convexity of a curve

DEFINITION 3.4. — *Let K be a compact subset of a complex manifold X, the $\mathcal{O}(X)$-hull of K is the set*

$$h_X(K) = \{u \colon |f(u)| \leqslant \max\{f(x)|x \in K\} \text{ for all } f \in \mathcal{O}(X)\},$$

where $\mathcal{O}(X)$ are holomorphic functions on X.

Consider a collection of C^1 - smooth real curves $\gamma_1, \ldots, \gamma_m$ in $\mathbb{C}^N$. Their holomorphic hull is described by Stolzenberg's Theorem [19]:

THEOREM 3.5. — *Let $\gamma = \gamma_1 \cup \cdots \cup \gamma_m$. Then $h(\gamma) \backslash \gamma$ is a (possibly empty) one-dimensional analytic subset of $\mathbb{C}^N \backslash \gamma$.*

COROLLARY 3.6. — *The statement of the theorem is true if one replaces $\mathbb{C}^n$ by a Stein manifold X.*

Proof. — The corollary is proved by using a proper embedding of the Stein manifold X to $\mathbb{C}^N$ for some large enough N [11, Theorem 5.3.9]. □

3.3. Holomorphic convexity of a degenerate object

In this subsection we give the geometric conditions for the degenerate objects to be holomorphically convex. In the sequel we need the following corollary from the Stolzenberg's Theorem.

COROLLARY 3.7. — Let $\gamma_1, \ldots, \gamma_n$ be piecewise smooth curves, such that each intersection $\gamma_i \cap \gamma_j$ consists of finite number of points. Suppose that $h(\gamma) \not\subset \gamma$. Then there exists an arc $\alpha \subset \gamma_i$, such that $\alpha \subset \partial(h(\gamma) \backslash \gamma)$, where $\gamma = \cup \gamma_i$.

THEOREM 3.8. — [6, Section 18.5] Let M be a connected $(2p-1)$-dimensional C^1-submanifold of a complex manifold Ω. Let A_1, A_2 be irreducible p-dimensional analytic subsets of $\Omega \backslash M$ such that the closure of each of them contains M. Then either $A_1 = A_2$ or $A_1 \cup M \cup A_2$ is an analytic subset of Ω.

LEMMA 3.9. — Let $\alpha \subset \gamma$ be a real-analytic arc. Assume $\alpha \subset \partial(h(\gamma) \backslash \gamma)$. Let C be a holomorphic curve, $\alpha \subset C$. Then there exists a loop $\tilde{\gamma} \subset \gamma$, so that $\alpha \subset \tilde{\gamma} \subset \gamma \cap C$ and $\tilde{\gamma}$ is null homologous on C.

Proof. — One can take a neighborhood $U \subset X$ of the arc α, such that
(1) $U \cap \gamma = \alpha$;
(2) the connected component of $C \cap U$, that contains α, is a submanifold in U;
(3) the arc α separates this connected component into two pieces. Let Ω_1, Ω_2 be these pieces.

Let h_1 denote the connected component of $h(\gamma) \backslash \gamma$.

Apply Theorem 3.8 to the analytic sets h_1 and Ω_1, and the arc α. The closure of h_1 in U contains α. The closure of Ω_1 also contains α. Therefore, either $h_1 = \Omega_1$ or $h_1 \cup \alpha \cup \Omega_1$ is an analytic subset of U. In the second case $h_1 = \Omega_2$. Thus, $h_1 = \Omega_1$ or $h_1 = \Omega_2$. If two analytic sets coincide locally, then they coincide globally. Therefore, $h_1 \subset C$.

By Maximum Modulus Principle, $\partial h(\gamma) \subset \gamma$. Denote $\tilde{\gamma} = \partial h_1$. Then γ is a loop and is null-homologous on C. □

THEOREM 3.10. — Let γ be a degenerate object of a foliation $\mathcal{F}$ from the Definition 2.1. Then γ is a union of loops and curves. We assume that each loop and curve is simple and piecewise real-analytic. Moreover, for types $1, 2, 5, 8, 9$ we assume that γ satisfies the following geometric conditions:

type 1 : γ is not null-homologous on the leaf L.

type 2: (a) γ_1 and γ_2 have only one common point;
        (b) γ_1 and γ_2 are not null-homologous and are not multiples of the same cycle in the homology group of L.

type 5: γ is not null-homologous on S.

type 8: $\gamma_1 \subset L$ is not null-homologous on L; γ_1 and γ_2 have only one common point.

type 9: $\gamma_1 \subset L_1$, $\gamma_2 \subset L_2$ are not null-homologous on L_1, L_2 correspondingly; $L_1 \neq L_2$. Curves γ_1 and γ_3; γ_2 and γ_4 have only one common point.

Then γ is holomorphically convex.

NOTE 3.11. — *If γ satisfies the listed above geometric conditions, then we say that γ is a geometric degenerate object.*

Proof. — Suppose γ is not holomorphically convex.

type 1: Since γ is a simple cycle, by Lemma 3.9, γ is null-homologous on L, which contradicts the hypothesis of the theorem.

type 2: Since γ_1 and γ_2 are simple cycles and have only one point of intersection, there are three possibilities for the boundary of holomorphic hull of $\gamma : \partial h(\gamma) = \gamma_1$, $\partial h(\gamma) = \gamma_2$, $\partial h(\gamma) = \gamma_1 \cup \gamma_2$. Since γ_1 is not null-homologous on L, $\partial h(\gamma) \neq \gamma_1$. The same way, $\partial h(\gamma) \neq \gamma_2$. Since γ_1 and γ_2 are not multiples of the same cycle in the homology group of the leaf L, $h(\gamma) \neq \gamma_1 \cup \gamma_2$. Contradiction.

type 3: γ is simply connected, therefore, by Lemma 3.9, it bounds a region on S. This contradicts the hypothesis of the theorem.

type 4: Let $\tilde{S}$ be a surface obtained from $S \cup \{a\}$ by splitting the local components of S at the point a. Let $\pi : \tilde{S} \to S$ be the corresponding projection. Then $\pi^{-1}(\gamma)$ is a simple path on $\tilde{S}$. It does not bound a region on $\tilde{S}$. Therefore, its image does not bound a region on S. That contradicts Lemma 3.9.

type 5: The same as for type 1.

type 6: Let α be an arc, given by Corollary 3.7. Let us assume without loss of generality, $\alpha \subset \gamma_1$. Let C be a curve, given by Lemma 3.9. Then C is either a saddle connection or $C \subset M_1$. Saddle connections have been treated in (3) and (4). Thus, we may assume, $C \subset M_1$. Therefore, by Lemma 3.9, γ_1 bounds a region on C. This contradicts the hypothesis of the theorem.

type 7: The proof is the same as for type 6.

type 8: Let α be an arc, given by Corollary 3.7. Then $\alpha \subset \gamma_1$, or $\alpha \subset \gamma_2$, or $\alpha \subset \gamma_3$, or $\alpha \subset \gamma_4$. If $\alpha \subset \gamma_3$, or $\alpha \subset \gamma_4$, then we proceed by

the same reasoning as for type 6. We may assume $\alpha \subset \gamma_1$, then by Lemma 3.9, γ_1 is null-homologous on L_1. This contradicts the hypothesis of the theorem.

type 9: The proof is the same as for type 8. □

4. Simultaneous removal of degeneracies

Let X be a Stein manifold. Let Φ denote the space of 1-dimensional singular holomorphic foliations on X.

4.1. Landis-Petrovskii's Lemma

The idea is to encode degeneracies by countably many objects. To give a feeling of the method used, we first prove a version of the Landis-Petrovskii's Lemma [15] that we need in the sequel.

LEMMA 4.1. — *There are at most countably many isolated complex cycles on leaves of the foliation $\mathcal{F} \in \Phi$.*

Proof. — Since the manifold X is Stein, it can be embedded into $\mathbb{C}^N$. Take a cycle γ on a leaf L of the foliation $\mathcal{F}$. Fix coordinates $(z_1, \ldots, z_N)$ in $\mathbb{C}^N$. Let $C_1, \ldots, C_N$ be the coordinate lines,

$$C_i = \{z_1, \ldots, = \hat{z}_i = \cdots = z_N = 0\}.$$

Suppose that L does not belong to the hypersurface $\{z_i = c\}$ for any $c \in \mathbb{C}$. By perturbing γ on the leaf L one can assume that there exists a small neighborhood $U \supset \gamma$ so that $\pi_i|_U$ is a biholomorphism to the image (here $\pi_i : \mathbb{C}^N \to C_i$, $\pi_i(z) = z_i$ is the projection). Then one can perturb γ inside U so that $\pi_i(\gamma)$ becomes a piece-wise linear curve with rational vertices.

DEFINITION 4.2. — *We will say that the cycle γ' lies over the piece-wise linear curve g' if there exist a representative of γ' and its neighborhood U', such that U' is projected biholomorphically to its image and the representative is projected to g'. Note, that any cycle lies over countably many piece-wise linear curves.*

Take one of the vertices of $\pi_i(\gamma)$, say with coordinate $z_i = c$. The hypersurface $\{z_i = c\}$ intersects X by $(k-1)$-dimensional variety, such that for

any cycle γ', lying over $\pi_i(\gamma)$, it is transversal to the foliation in a neigh-
borhood of $\gamma' \cap \{z_i = c\}$. The holonomy map along γ is well-define in some
neighborhood of the intersection $\{z_i = c\} \cap \gamma$. The holonomy map does
not have any other fixed points in some smaller neighborhood. Thus, each
cycle that projects to the same piece-wise linear curve gives a neighbor-
hood on the hyperplane $\{z_i = c\} \subset \mathbb{C}^N$, so that two neighborhoods for two
different cycles do not intersect each other. Therefore, there are at most
countably many limit cycles that project to the same curve. Since there
are only countably many curves, there are at most countably many limit
cycles.   □

Landis-Petrovskii's Lemma implies that once all non-isolated cycles are
removed, all leaves except for countably many are homeomorphic to disks.

4.2. Simultaneous removal of non-isolated cycles

If there are non-isolated cycles on the leaves of a foliation $\mathcal{F}$, then the
number of the cycles is obviously uncountable. However, the strategy de-
scribed above can be applied. Our idea is to catch the degenerations by a
countable number of holonomy maps.

THEOREM 4.3. — *There exists a residual set in Φ with no geometric
degenerate objects of type 1.*

Proof. — Since X is Stein, it can be embedded into $\mathbb{C}^N$. We can restrict
ourselves to the foliations without leaves that belong to the hypersurfaces
$\{z_N = c\}$, $c \in \mathbb{C}$. The set of such foliations is open and dense. We describe
the holonomy maps that catch all the cycles for all foliations.

We introduce the following notations:

- $\mathcal{A}$ is a countable, everywhere dense subset in Φ;
- $\mathcal{G}$ is the set of all closed piecewise-linear curves with rational vertices
  on
  $$\{z_1 = \cdots = z_{N-1} = 0\},$$
  with one marked vertex.
- Let $\tau_q = \{z_n = q\} \cap X$, where $q \in \mathbb{Q} + i\mathbb{Q}$.
  Let $\mathcal{Q}_q$ be a countable everywhere dense set on τ_q.
  $\mathcal{Q} = \bigsqcup \mathcal{Q}_q$.

Let $\mathbf{z} = (z_1, \ldots, z_{N-1})$, $u = z_N$.

Consider a 4-tuple $\alpha = (\mathcal{F}, g, \mathbf{z}, r) \in (\mathcal{A}, \mathcal{G}, \mathcal{Q}_q, \mathbb{Q}_+)$. Let q denote the
marked point of g. Take a point z' in a neighborhood of $\mathbf{z}$ on τ_q. Lift g

to the leaf, starting from z'. Assume that the end point of the lift of g belongs to the same neighborhood. This defines a holonomy map for the foliation $\mathcal{F}$. If the listed above conditions are satisfied, we call a 4-tuple admissible. One can consider the germ of the holonomy map along the lifting of g, starting at $\mathbf{z}$, for foliations close to $\mathcal{F}$. Therefore, we think of Δ_α as of function of two variables: a foliation close to $\mathcal{F}$, and a point on the transversal section τ_q.

Below we fix a specific representative of Δ_α. We use the same notation for the specific representative as for the germ.

Let V_α be the connected component, containing $\mathcal{F}$, of the set of foliations, for which the holonomy map along g in the point $\mathbf{z}$ is well-defined and has radius of convergence greater than r. The domain of definition of Δ_α is

$$\{(\mathcal{F}', \mathbf{z}')|\ \mathcal{F}' \in V_\alpha,\ |\mathbf{z}' - \mathbf{z}| < r\}.$$

Note, that V_α is open.

From this point on, we work with fixed representatives, rather than germs.

LEMMA 4.4. — *Let γ be a complex cycle of a foliation $\mathcal{F}$. Then there exists an admissible α, such that γ corresponds to a fixed point of $\Delta_\alpha(\mathcal{F}, \cdot)$.*

Proof. — Let γ be a complex cycle on a leaf L of a foliation $\mathcal{F}$. One can perturb γ on L so that it projects to some $g \in \mathcal{G}$. Let $u(g)$ be one of the vertices of the projection, and let $\mathbf{z} \in \gamma$ be the preimage of $u(g)$. Consider the holonomy map along γ in a neighborhood of $\mathbf{z}$ in the transversal section $C = \{u = u(g)\}$. Take a point $\mathbf{z}_1 \in \mathcal{Q}$ such that $|\mathbf{z} - \mathbf{z}_1| < r_{\mathbf{z}}(\mathcal{F})/4$ where $r_{\mathbf{z}}(\mathcal{F})$ is the radius of convergence of the holonomy map in the point $\mathbf{z}$ along γ for the foliation $\mathcal{F}$. Note, that $r_{\mathbf{z}_1}(\mathcal{F}) > r_{\mathbf{z}}(\mathcal{F})/2$. One can take $\mathcal{F}_1$ close to $\mathcal{F}$ so that $r_{\mathbf{z}_1}(\mathcal{F}_1) > r_{\mathbf{z}}(\mathcal{F})/2$. Denote by $\alpha = (\mathcal{F}_1, g, \mathbf{z}_1, r)$, where $r \in \mathbb{Q}, r_{\mathbf{z}}(\mathcal{F})/4 < r < r_{\mathbf{z}}(\mathcal{F})/2$. Then $r < r_{\mathbf{z}_1}(\mathcal{F}_1)$. Also, $\mathcal{F} \in V_\alpha$, because $r_{\mathbf{z}_1}(\mathcal{F}) > r$. Since $r > r_{\mathbf{z}}(\mathcal{F})/4$, the point z belongs to the domain of definition of $\Delta_\alpha(\mathcal{F}_1, \cdot)$. $\square$

LEMMA 4.5. — *Fix Δ_α. The set $D_\alpha \subset V_\alpha$ of foliations $\mathcal{F}$ such that $\Delta_\alpha(\mathcal{F}, \cdot)$ has a non-hyperbolic fixed point, so that the corresponding cycle γ is a geometric degenerate object of type 1, is closed and nowhere dense in V_α.*

Proof. — We prove that by a finite number of steps, we can perturb the foliation $\mathcal{F}$ so that $\Delta_\alpha(\widetilde{\mathcal{F}}, \cdot)$ has isolated fixed points only, that correspond to geometric degenerate cycles, in the domain of definition discussed above. Assume that A is the set of fixed points of $\Delta_\alpha(\mathcal{F}, \cdot)$. Let A be k-dimensional.

As we show in the appendix, one can associate multiplicity $m(A)$ to the analytic set A. Take a point z that is a generic point of a k-dimensional stratum A_i. Assume z corresponds to a geometric degenerate object. By Theorem 3.3, there exists a neighborhood of z and a foliation $\widetilde{\mathcal{F}}$, arbitrary close to $\mathcal{F}$, such that the holonomy map of $\widetilde{\mathcal{F}}$ along γ has isolated fixed points only in this neighborhood.

This perturbation destroys the component A_i. Therefore, by Lemma 5.14 it either decreases the dimension of A, or it decreases the multiplicity $m(A)$. Therefore, after a finite number of steps, only isolated geometric cycles are left. By the Theorem 3.3, they can be turned into hyperbolic by a finite number of steps as well.   □

COROLLARY 4.6. — *The complement of D_α in Φ contains an open every where dense set.*

The residual set is obtained by intersecting open everywhere dense sets from the Corollary above.   □

4.3. Simultaneous splitting of cycles to different leaves

THEOREM 4.7. — *There exists a residual set in Φ with no geometric degenerate objects of type 2.*

Proof. — The construction is similar to Section 4.2. The difference is that one needs to consider pairs of holonomy maps. The analytic condition is that they do not have a common fixed point.   □

4.4. Simultaneous removal of separatrices and non-transversal intersections of invariant manifolds

THEOREM 4.8. — *There exists a residual set in Φ with no geometric degenerate objects of types $3 - 9$.*

Proof. — We outline the proof for strongly invariant manifolds of different singular points. For other types of degenerate objects the proof goes along the same lines.

Since X is a Stein manifold, it can be embedded into $\mathbb{C}^N$.

We fix the countable set of data $\alpha = (\mathcal{F}, a_1, M_1, a_2, M_2, g, z_1, r)$.

- $\mathcal{F} \in \mathcal{A}$, where $\mathcal{A}$ is a countable every-where dense set in Φ;

Foliations with complex hyperbolic singular points only form a residual set [5]. Therefore, we can assume that all singular points for all the foliations $\mathcal{F} \in \mathcal{A}$ are complex hyperbolic.

- a_1, a_2 are complex hyperbolic singular points of $\mathcal{F}$;
- M_1, M_2 are strongly invariant manifolds of a_1 and a_2 correspondingly;

We associate the maximal radius r_i to the singular point a_i.

DEFINITION 4.9. — *The radius r_i is the maximal radius, such that M_i is transversal to $\partial U_r(a_i)$ for all $r < r_i$.*

Not that maximal radius is a lower semicontinuous function on the space of foliations.

Let $\pi : X \to C$ be the projection to $C = \{z_1 = \cdots = z_{N-1} = 0\}$, $\pi(x_1, \ldots, x_N) = x_N$.

- $g \subset C$ is a piecewise linear curve with rational vertices. Let u_1, u_2 be the starting and the ending points of g correspondingly. We require that $u_1 \in \pi(U_{r_1}(a_1))$, $u_2 \in \pi(U_{r_2}(a_2))$;
- $z_1 \in \mathcal{Q}_q$, where $\mathcal{Q}_q$ is an everywhere dense set on the transversal section
  $$\tau_1 = \{z_n = u_1 = q\} \cap X \text{ in } U_{r_1}(a_1);$$

We require that there is a well-defined lift of g to the leaf L of the foliation $\mathcal{F}$, that starts from a point z_1. The lift is denoted by γ. Let z_2 be the lift of u_2. We require that $z_2 \in U_{r_2}(a_2)$

Let $\tau_2 = \{z_N = u_2\} \cap X$.

There is a well-defined germ $\Delta : \tau_1 \to \tau_2$ of the holonomy map along γ in the point z_1.

As before, we think of Δ as a function of two variables: a foliation $\mathcal{G}$, close to $\mathcal{F}$, and a point on the transversal section τ_1.

- $r \in \mathbb{Q}_+$. We require that
  (1) r is less than radius of convergence of Δ.
  (2) The disk $D_r(z_1)$ on the transversal section τ_1 of the radius r_1 with the center z_1 is compactly contained in $U_{r_1}(a_1)$.
  (3) $\Delta(D_r(z_1))$ is compactly contained in $U_{r_2}(a_2)$.

We fix a representative Δ_α of Δ. Below we describe the neighborhood U_α of $\mathcal{F}$. $\mathcal{G}$ belongs to U_α if

(1) there is a holomorphic family of foliations $\mathcal{F}_\lambda$, so that $\mathcal{F}_0 = \mathcal{F}$, $\mathcal{F}_1 = \mathcal{G}$; for all $\lambda \in D_1$ there are unique hyperbolic singular points $a_1^\lambda \in U_{r_1/2}(a_1)$ and $a_2^\lambda \in U_{r_2/2}(a_2)$ of the foliation $\mathcal{F}_\lambda$;

Let a_1', a_2' be singular points of $\mathcal{G}$, obtained via holomorphic continuation. Let M_1', M_2' be the corresponding strongly invariant manifolds. Let r_1', r_2' be the maximal radii for (a_1', M_1'), (a_2', M_2').

(2) $z_1 \in U_{r_1'}(a_1')$, $z_2' \in U_{r_2'}(a_2')$, where z_2' is the lift of u_2 along g for $\mathcal{G}$.

(3) $D_r(z_1)$ is compactly contained in $U_{r_1'}(a_1')$.

(4) $\Delta(\mathcal{G}, D_r(z_1))$ is compactly contained in $U_{r_2'}(a_2')$.

The domain of definition of Δ_α is $U_\alpha \times D_r(z_1)$.

LEMMA 4.10. — *For any α, the set $D_\alpha \subset U_\alpha$ of foliations $\mathcal{G} \subset U_\alpha$, for which there exists a leaf L such that*

(1) *the lift of u_1 to L is in $U_{r_1}(a_1)$, the lift of u_2 to L is in $U_{r_2}(a_2)$;*

(2) *the lift of g belongs to the strongly invariant manifold M_1' of the singular point a_1' of $\mathcal{G}$ (a_1' is a holomorphic continuation of a_1);*

(3) *the lift of u_2 belongs to the strongly invariant manifold M_2' of a singular point a_2' (a_2' is a holomorphic continuation of a_2);*

(4) *the lift of u_2 is a point of a non-transversal intersection of M_1' and M_2'.*

is a closed and nowhere dense set.

Proof. — The proof follows from the local Theorem 3.3 in the same way as in Lemma 4.5. $\qquad\square$

The desired residual set is obtained by intersecting the complements to closed nowhere dense sets from the previous lemma. $\qquad\square$

4.5. Proofs of the main theorems

THEOREM 4.11. — *A foliation $\mathcal{F}$ that does not have geometric degenerate objects of types $1 - 5$ satisfies Theorem 1.1.*

Proof. — Assume that L is a non-contractible leaf of the foliation $\mathcal{F}$. Then there exists a simple loop $\gamma \subset L$, non-homologous to zero on L. Consider two cases:

(1) The leaf L has a finitely generated fundamental group. Notice that L is non compact, since otherwise it would be a compact submanifold of the Stein manifold X. L is homeomorphic to a compact surface with finitely many punctures. Take γ to be a simple loop around a puncture. One can choose γ to be piecewise real-analytic.

(2) The leaf L has an infinitely generated fundamental group. Take a simple cycle α on L. If α is non-homologous to zero, take $\gamma := \alpha$.

If α is null-homologous, then it bounds a compact region $K \subset L$. Take any simple cycle $\gamma \subset K$, which is non-homologous to zero on K. Then γ is non-homologous to zero on L. One can choose γ to be piecewise real-analytic.

Suppose that γ is not isolated. Then it is a geometric degenerate object of type 1, which is impossible by the hypothesis of the theorem.

By Lemma 4.1, there are at most countably many isolated cycles. Thus, there are at most countably many noncontractible leaves.

Let L be a leaf of the foliation $\mathcal{F}$ that is not a separatrix. Moreover, assume that $H^1(L, \mathbb{Z}) \neq 0, \mathbb{Z}$. By [7, Lemma 6.9], there exists a pair of cycles $\gamma_1, \gamma_2 \subset L$, that form a geometric degenerate object of type 2. Since the foliation $\mathcal{F}$ does not have geometric degenerate objects of type 2, all non-separatrix leaves L are either contractible or $H^1(L, \mathbb{Z}) = \mathbb{Z}$.

Let L be a separatrix of a singular point a. Since the foliation $\mathcal{F}$ does not have geometric degenerate objects of types 3 and 4, L is not a saddle connection or a homoclinic saddle connection. Since, there are no geometric degenerate objects of type 2, by [7, Lemma 6.9], the union $L \cup a$ is either contractible or $H^1(L \cup a, \mathbb{Z}) = \mathbb{Z}$. If $H^1(L \cup a) = \mathbb{Z}$, then there is a simple piecewise real-analytic generator of $H^1(L \cup a)$ that passes through the point a. Let us denote it by γ. Then γ is a geometric degenarate object of type 5. That contradicts the hypothesis of the theorem. Therefore, $L \cup a$ is contractible, and L is homeomorphic to a cylinder.   □

Proof of Theorem 1.1: By Theorems 4.3, 4.7, 4.8, there are residual sets $R_1, R_2, R_3 \subset \Phi$, with no geometric degenerate objects of types $1, 2, 3 - 5$ correspondingly. By Theorem 4.11, any foliation $\mathcal{F} \subset R$, where $R = R_1 \cap R_2 \cap R_3$ satisfies Theorem 1.1.   □

THEOREM 4.12. — *If all singular points of a foliation $\mathcal{F}$ are complex hyperbolic and it does not have geometric degenerate objects of types $1 - 6$, $8 - 9$, then it is complex Kupka-Smale.*

Proof. — By Theorem 4.11, all leaves of the foliation $\mathcal{F}$ are either contractible or cylinders. Since the foliation does not have geometric non-hyperbolic cycles, all cycles are hyperbolic.

Suppose there is a non-transversal intersection of invariant manifolds M_1 and M_2. We assume that M_1 and M_2 are strongly invariant manifolds of singular points a_1 and a_2 correspondingly. The other cases are treated the same way. Let p be a point of nontransversal intersection of M_1 and M_2. Assume $p \in L$, where L is a leaf of the foliation $\mathcal{F}$. We can assume that L is not a saddle connection. Since $L \subset M_1$, there is a path $\gamma_1' \subset L$ that

connects p with a point $q \in M_1^{loc}$. Let $\gamma_1'' \subset M_1^{loc}$ be a path that connects q and a_1. Let $\gamma_1 = \gamma_1' \cup \gamma_1''$. We can assume that γ_1 is simple and piece-wise real analytic. The same way, we construct $\gamma_2 \subset M_2$. Thus, we constructed a geometric degenerate object of type 6, which contradicts the hypothesis of the theorem. $\qquad\square$

Proof of Theorem 1.2: Consider the set of foliations with complex hyperbolic singular points only. This set contains a residual subset R_4 [5]. By Theorem 4.8 there is a residual set R_5 with no geometric degenerate objects of types 6, 7, 9. Recall that R, defined in the proof of Theorem 1.1, is a subset with no geometric degenerate objects of types 1-5. By Theorem 4.12, any foliation $\mathcal{F} \subset R \cap R_4 \cap R_5$ is complex Kupka-Smale. $\qquad\square$

Proof of Theorem 1.4 : There is a residual set R_4 of foliations with complex hyperbolic singular points only [5].

Assume that a foliation has a non-transversal intersection of invariant manifolds of the same singular point. Then, by the same argument as in the proof of Theorem 4.12, we construct a geometric degenerate object of type 7. By Theorem 1.4, there is a residual set R_6 with no geometric degenerate objects of type 7. The set $R_4 \cap R_6$ is a desired residual set. $\quad\square$

5. Appendix

5.1. Complex foliations

Definitions 5.1-5.5 are from [14]. They are scatted through out the text, so we provide them here for the convenience of the reader. Definition 5.6, 5.7 can be found in [20],[4] correspondingly.

DEFINITION 5.1. — *Let $\mathcal{F}$ be a foliation on a complex manifold X. Let $\gamma : [0,1] \to X$ be a path on X. Let T_0 and T_1 be two transversal sections to $\mathcal{F}$, passing through $\gamma(0)$ and $\gamma(1)$ respectively. Then for any initial point $x \in T_0$, close to $\gamma(0)$, leaf-wise curves, starting from x, and staying close to γ, and arriving to T_1, arrive at a well defined point $\Delta_\gamma(x)$. Thus, we obtain a map $\Delta_\gamma(x)$, which we call the holonomy map. If $\gamma : [0,s] \to X$ is a closed curve, and T is a transversal section to $\mathcal{F}$, passing through $\gamma(0)$. The map $\Delta_\gamma : T \to T$ is called the holonomy map as well.*

DEFINITION 5.2. — *A complex cycle is a nontrivial free homotopy class of loops on a leaf of a foliation. It is called isolated if it corresponds to an isolated fixed point of its holonomy map. It is hyperbolic if its holonomy map is hyperbolic, i.e. its linearization is non-degenerate, and the eigenvalues of the linearization do not belong to the unit circle.*

DEFINITION 5.3. — *Let γ be a hyperbolic cycle. By Hadamard-Perron Theorem, the holonomy map Δ_γ has local stable and unstable manifolds m_1^{loc}, m_2^{loc} [14, Theorem 7.1] The union of leaves that pass through m_1^{loc}, m_2^{loc} are called the stable, unstable manifold of γ correspondingly.*

DEFINITION 5.4. — *A singular point is called complex hyperbolic if it is non-degenerate and the ratio of any two eigenvalues is not real.*

In this article we work only with complex hyperbolic singular points. So we reserve the word "hyperbolic" to complex hyperbolicity.

DEFINITION 5.5. — *A local complex separatrix of a singular holomorphic foliation $\mathcal{F}$ at a singular point $a \in \Sigma(\mathcal{F})$ is a local leaf $L \subset (U, a)\backslash\Sigma$, whose closure $L \cup a$ is a germ of an analytic curve. The leaf L is called separatrix.*

DEFINITION 5.6. — *A saddle connection is a common separatrix of two singular points. See Fig.2.2*

DEFINITION 5.7. — *Suppose a is a hyperbolic singular point of the foliation $\mathcal{F}$. Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of a. Let l be a line passing through the origin in $\mathbb{C}$. Let $\lambda = (\lambda_{i_1}, \ldots, \lambda_{i_k})$ be the eigenvalues of a that lie on one side of the line l. Let α_λ be a subspace spanned by the eigenspaces of all elements of λ. The local strongly invariant manifold M_λ^{loc} is a manifold tangent to α_λ. The global strongly invariant manifold M_λ is obtained by taking the union of leaves that belong to the local strongly invariant manifold.*

Strongly invariant manifolds exist [14, Theorem 7.4]. The proof can be easily modified to show that they depend holomorphically on a foliation.

Suppose that v is a vector field that determines a foliation locally. Strongly invariant manifolds are stable and unstable manifolds of the time-one map Φ_{cv}^1 of the vector field cv, where $c \in \mathbb{C}^*$ is taken so that l becomes the imaginary axis. If one considers the real flow of the vector field cv, then locally strongly invariant manifolds coincide with stable and unstable manifolds [4].

5.2. Holomorphic vector bundle associated to a foliation

Take a 1-dimensional singular holomorphic foliation $\mathcal{F}$ of a Stein manifold M. One can naturally associate a linear bundle $B_\mathcal{F}$ to $\mathcal{F}$.

Notice that a 1-dimensional holomorphic foliation with singular locus of codimension 2 is locally determined by a holomorphic vector field [14].

Consider a covering of a Stein manifold by open contractible sets U_i. On each set U_i the foliation is determined by a holomorphic vector field v_i. For a pair of intersecting sets U_i and U_j define a function $g_{ij} = v_i/v_j$. This function is well-defined on $(U_i \cap U_j)\setminus\{v_j = 0\}$. The set $\{v_j = 0\}$ has codimension 2. Therefore, g_{ij} can be extended to $U_i \cap U_j$.

The same way $g_{ji} = v_j/v_i$ can be extended to a well-defined function on $U_i \cap U_j$.

$$g_{ij}g_{ji} = 1 \quad \Rightarrow \quad g_{ij}\big|_{U_i \cap U_j} \neq 0$$

The set of functions $\{g_{ij}\}$ form a 2-cocycle, therefore, they define a linear bundle.

LEMMA 5.8. — *1-dimensional singular holomorphic foliation $\mathcal{F}$ of a Stein manifold X is determined by a global section of the vector bundle $TX \otimes B_{\mathcal{F}}$.*

Proof. — Lemma follows from the construction of $B_{\mathcal{F}}$. □

If $H^2(X, \mathbb{Z}) = 0$, then each foliation on X is determined by a global vector field. In particular, this holds for foliations on $\mathbb{C}^n$.

5.3. Topology of uniform convergence on compact non-singular sets

The description of topology on the space of foliations in $\mathbb{C}^n$ is given for example in [9]. Let X be a Stein manifold. We fix its compact exhaustion:

$$K_1 \Subset \cdots \Subset K_n \cdots \Subset X,$$

where $K_1, \ldots, K_n$ are compact subsets of X, closures of open connected subsets of X;

$$\cup_n K_n = X.$$

Let d_1 be a metric on X and d_2 be a metric on the projectivization of its tangent bundle PTX. A basis of neighborhoods of the foliation $\mathcal{F}$ is formed by

$$U_{n,\varepsilon,\delta} = \left\{ \begin{array}{l} \mathcal{G}|\ \mathcal{G} \text{ is nonsingular in } K_{\varepsilon,n} = K_n\setminus U_\varepsilon(\Sigma(\mathcal{F})) \text{ and the tangent} \\ \text{directions to the foliations } \mathcal{F} \text{ and } \mathcal{G} \text{ are } \varepsilon\text{-close on } K_{\varepsilon,n} \end{array} \right\}.$$

Note that the obtained topology does not depend on the choice of compact exhaustion and the choice of metrics d_1 and d_2. The set of foliations of X has countably many connected components, parametrized by Chern classes of the linear bundles, associated to the foliations.

The set of sections of $TX \otimes B_{\mathcal{F}}$ is equipped with the topology of uniform convergence on compact sets. The map from the space of sections to the space of foliations is continuous.

5.4. Multiplicity

We consider analytic subsets A of a polydisk $\bar{D}^n$, i.e. we assume that A is an analytic subset of some neighborhood of D^n. Suppose that A is given by a system of n equations:

$$f_1 = \cdots = f_n = 0.$$

Assume that A is k-dimensional. We define the multiplicity of A that does not increase under perturbations.

LEMMA 5.9. — *There are only finitely many strata of A of maximal dimension.*

Proof. — The number of strata is locally finite [6, Section 2.1]. Since A is an analytic subset of $\bar{D}^n$, it is globally finite. $\square$

Let $A_1, \ldots, A_m$ be the strata of maximal dimension.

Take a smooth point $z \in A_i$. Consider a transversal section T to A_i at the point z. Let $\tilde{f}_1, \ldots, \tilde{f}_n$ be the restriction of $f_1, \ldots, f_n$ to T. The point z is an isolated solution of the system:

$$\tilde{f}_1 = \cdots = \tilde{f}_n = 0.$$

DEFINITION 5.10. — *Let z be an isolated point of a system of equations:*

$$\tilde{f}_1 = \cdots = \tilde{f}_n = 0,$$

defined in $(n-k)$-dimensional polydisk D^{n-k}. The multiplicity $m(z)$ of a point z is

$$\dim \mathcal{O}_{D^{n-k},z} / < \tilde{f}_1, \ldots, \tilde{f}_n >,$$

where $\mathcal{O}_{D^{n-k},z}$ is the local ring of $z \in D^{n-k}$, i.e. functions, regular in a neighborhood of $z \in D^{n-k}$; $< \tilde{f}_1, \ldots, \tilde{f}_n >$ is the ideal in $\mathcal{O}_{D^{n-k},z}$ generated by $\tilde{f}_1, \ldots, \tilde{f}_n$.

LEMMA 5.11. — *The multiplicity does not increase under perturbations, i.e. if $z'_1, \ldots, z'_m$ are isolated solutions of a perturbed system in a neighborhood of a point z, then*

$$\sum_{i=1}^{m} m(z'_i) \leqslant m(z).$$

Proof. — In [1, Chapter 2.5.7] it is proved for $k = 0$. In general case the proof goes the same way. □

DEFINITION 5.12. — *The multiplicity of $z \in A_i$ is the multiplicity of the point z as an isolated solution of $\tilde{f}_1 = \cdots = \tilde{f}_n = 0$.*

The multiplicity does not depend on the choice of a generic point and a transversal section T.

DEFINITION 5.13. — *The multiplicity of a stratum A_i is the multiplicity of a generic point. The multiplicity of A is the sum of multiplicities of A_i.*

LEMMA 5.14. — *The multiplicity of A does not increase under perturbations, i.e. let $A'_1, \ldots A'_{m'}$ be strata of a perturbed system, then*

$$\sum_{i=1}^{m'} m(A'_i) \leqslant m(A).$$

Proof. — Let $T_1, \ldots, T_m$ be transversal sections to A_i's at generic points. Every A'_i intersect at least one of the sections $T_1, \ldots, T_m$. One can also assume that T_i's meet A_i's transversally. On each transversal section the result follows from the Lemma 5.11.

□

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