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# IRREDUCIBILITY OF AUTOMORPHIC GALOIS REPRESENTATIONS OF $GL(n)$, $n$ AT MOST 5

**by Frank CALEGARI & Toby GEE (*)**

Abstract. — Let $\pi$ be a regular, algebraic, essentially self-dual cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$, where $F$ is a totally real field and $n$ is at most 5. We show that for all primes $l$, the $l$-adic Galois representations associated to $\pi$ are irreducible, and for all but finitely many primes $l$, the mod $l$ Galois representations associated to $\pi$ are also irreducible. We also show that the Lie algebras of the Zariski closures of the $l$-adic representations are independent of $l$.

Résumé. — Nous prouvons l'irréductibilité pour $n$ inférieur ou égal à 5 des représentations galoisiennes $l$-adiques associées aux représentations automorphes cuspidales algébriques et régulières de $\mathrm{GL}_n$ sur un corps totalement réel qui sont autoduales à torsion près. Nous prouvons également l'irréductibilité des représentations galoisiennes modulo $l$ pour presque tout $l$, et nous montrons l'indépendance en $l$ de l'algèbre de Lie de la clôture Zariskienne de la représentation $l$-adique.

## 1. Introduction

### 1.1.

It is a folklore conjecture that the Galois representations (conjecturally) associated to algebraic cuspidal automorphic representations of $\mathrm{GL}_n(\mathbb{A}_F)$ over a number field $F$ are all irreducible. In general, rather little is known in this direction. Ribet ([24]) proved this result for classical modular forms, and his proof extends to the case of Hilbert modular forms ([25]). The result was proved for essentially self-dual representations of $\mathrm{GL}_3(\mathbb{A}_F)$, $F$

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totally real, in [7]. In [13], Dieulefait and Vila proved big image results for a compatible family arising from a rank four pure motive $M$ over $\mathbb{Q}$ with Hodge-Tate weights $(0,1,2,3)$, coefficients in a quadratic field $K$, and certain other supplementary hypotheses (see also [12]). In a 2009 preprint ([23]), Ramakrishnan proves irreducibility of the associated $l$-adic representations for essentially self-dual representations of $\mathrm{GL}_4(\mathbb{A}_{\mathbb{Q}})$ for sufficiently large $l$; his argument also applies without the assumption of self-duality *assuming* the existence of the corresponding Galois representations.

Until recently, very little was known in the general case. It is sometimes the case that the Galois representations can be proved to be irreducible for purely local reasons; if the automorphic representation is square-integrable at some finite place, then it is a consequence of the expected local-global compatibility that the corresponding local Galois representation is indecomposable, which implies that the global Galois representation, being semisimple, is irreducible. In [27], this observation was used to prove the irreducibility of the Galois representations considered in [17] whenever the square-integrability hypothesis holds.

One reason to suppose that the Galois representations should be irreducible is that if the Fontaine–Mazur–Langlands conjecture holds, then (by a standard $L$-function argument) the reducibility of an $l$-adic Galois representation associated to an automorphic representation would show that the automorphic representation could not be cuspidal. In fact, it is enough to know that any geometric Galois representation is potentially automorphic, as this $L$-function argument is compatible with the usual arguments involving Brauer's theorem. This observation was exploited in [4] to prove that if $K$ is an imaginary CM field and $\pi$ is a regular, algebraic, essentially self-dual cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_K)$ which has extremely regular weight (a notion defined in [4]), then for a density one set of primes $l$, the $l$-adic Galois representations associated to $\pi$ are irreducible. While this theorem is useful in practice, the condition that the weight of $\pi$ be extremely regular is sometimes too restrictive. For example, it is never satisfied by the base change of an automorphic representation over a totally real field if $n > 3$. In the present paper, we begin by extending the result of [4] to the case of totally real fields if $n \leqslant 5$, with no assumption on the weight of $\pi$. Just as in [4], we use potential automorphy theorems and the $L$-function argument mentioned above. The key difficulty with applying the potential automorphy theorems available to us is to show that any hypothetical summand of the Galois representation to be proved irreducible is both essentially self-dual and odd. It is here that the arguments of [4] make use of the condition of extreme regularity, which is not available to us. Instead, we observe that if $n \leqslant 5$ then any constituent of dimension at least 3 must be essentially self-dual for dimension reasons, and is then odd by the main result of [6]. One dimensional summands are trivial to deal with, which leaves us with the problem of dealing with 2-dimensional summands. However, any two-dimensional representation is essentially self-dual, so we need only show that we cannot have even two-dimensional constituents, at least outside of a set of places of density zero. To do this, we use a variant of the arguments of [8], together with an argument using class field theory to show that there cannot be too many residually dihedral representations.

The arguments outlined so far suffice to prove the result for a set of primes of density 1. In order to extend our result to all primes, we make use of a group-theoretic argument (in combination with our density 1 result) to show that the characteristic polynomials of the images of the Frobenius elements can only be divisible by the characteristic polynomials of a global character in certain special cases, which rules out the possibility of any of the Galois representations having a one-dimensional summand. We use the same argument to show that it is not possible for any of the representations to have a dihedral summand. We make use of the self-duality of the Galois representations we consider to reduce to these possibilities and so obtain the result.

In order to extend this argument to the characteristic $l$ representations, we show using class field theory that if infinitely many of the characteristic $l$ representations have a one-dimensional summand, then the characteristic polynomials of the images of the Frobenius elements are divisible by the characteristic polynomials of a global character, which reduces us to the cases above. We prove a similar result for dihedral representations. This quickly reduces us to one special case, that of an irreducible 4-dimensional subrepresentation which when reduced mod $l$ splits up as a sum of two irreducible 2-dimensional representations. In this case, we are able to exploit the connection between $\mathrm{GO}_4$ and $\mathrm{GL}_2 \times \mathrm{GL}_2$ to reach a contradiction.

Using similar arguments, we are also able to show that the Lie algebras of the Zariski closures of the images of the $l$-adic representations are independent of $l$. Following [23], we additionally extend our analysis to the Galois representations associated to regular algebraic cuspidal automorphic representations of $\mathrm{GL}_3$ or $\mathrm{GL}_4$ over a totally real field which are not assumed to be essentially self-dual, under the hypothesis that the Galois representations exist.

One may naturally ask whether these methods can be generalized to $n \geqslant 6$; we explain why this might be difficult. Suppose that $\pi$ is a regular algebraic cuspidal automorphic representation of $\mathrm{GL}_3(\mathbb{A}_{\mathbb{Q}})$ which is not essentially self-dual (they exist!). Then, conjecturally, there should exist a compatible system of three dimensional Galois representations $\mathcal{R} = \{r_\lambda(\pi)\}$ of $G_{\mathbb{Q}}$. For a sufficiently large integer $n$, the compatible system $\mathcal{R} \oplus (\epsilon^n \otimes \mathcal{R}^\vee)$ is a six dimensional compatible system of essentially self-dual regular Galois representations. Our method for ruling out that this (completely reducible) compatible system is associated to a regular, algebraic essentially self-dual cuspidal automorphic representation $\Pi$ of $\mathrm{GL}_6(\mathbb{A}_{\mathbb{Q}})$ would consist of recognizing it as an isobaric sum $\pi \boxplus (\pi^\vee \otimes |\cdot|^n)$ by proving the (potential) automorphy of a non-essentially self-dual representation $r_\lambda(\pi) : G_{\mathbb{Q}} \to \mathrm{GL}_3(\overline{\mathbb{Q}}_l)$ for some prime $l$. However, such automorphy results are out of reach at present.

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## 2. Preliminaries

### 2.1.

We recall some notions from [4]. Let $F$ be a totally real field. By a *RAESDC* (regular, algebraic, essentially self-dual, cuspidal) automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$, we mean a pair $(\pi, \chi)$ where:

- $\pi$ is a cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$ such that $\pi_\infty$ has the same infinitesimal character as some irreducible algebraic representation of the restriction of scalars from $F$ to $\mathbb{Q}$ of $\mathrm{GL}_n$,
- $\chi : \mathbb{A}_F^\times / F^\times \to \mathbb{C}^\times$ is a continuous character such that $\chi_v(-1)$ is independent of $v|\infty$,

- and $\pi \cong \pi^\vee \otimes (\chi \circ \det)$.

If $\Omega$ is an algebraically closed field of characteristic 0, we write $(\mathbb{Z}^n)^{\mathrm{Hom}(F,\Omega),+}$ for the set of $a = (a_{\tau,i}) \in (\mathbb{Z}^n)^{\mathrm{Hom}(F,\Omega)}$ satisfying

$$a_{\tau,1} \geqslant \cdots \geqslant a_{\tau,n}.$$

If $a \in (\mathbb{Z}^n)^{\mathrm{Hom}(F,\mathbb{C}),+}$, let $\Xi_a$ denote the irreducible algebraic representation of $\mathrm{GL}_n^{\mathrm{Hom}(F,\mathbb{C})}$ which is the tensor product over $\tau$ of the irreducible representations of $\mathrm{GL}_n$ with highest weights $a_\tau = (a_{\tau,i})_{1 \leqslant i \leqslant n}$. We say that a RAESDC automorphic representation $(\pi, \chi)$ of $\mathrm{GL}_n(\mathbb{A}_F)$ has *weight $a$* if $\pi_\infty$ has the same infinitesimal character as $\Xi_a^\vee$ (this is necessarily the case for some unique $a$). There is necessarily an integer $w$ such that

$$a_{\tau,i} + a_{\tau,n+1-i} = w$$

for all $\tau$, $i$ (*cf.* section 2.1 of [4]).

We refer the reader to section 5.1 of [4] for the definition of a compatible system of Galois representations, and for various attendant definitions. If $(\pi, \chi)$ is a RAESDC automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$, then there is a number field $M$ containing the images of all embeddings $F \hookrightarrow \overline{M}$ and weakly compatible systems of Galois representations

$$r_\lambda(\pi) : G_F \to \mathrm{GL}_n(\overline{M}_\lambda)$$

and

$$r_\lambda(\chi) : G_F \to \overline{M}_\lambda^\times$$

as $\lambda$ ranges over the finite places of $M$ (*cf.* the last paragraph of section 5.1 of [4]). Suppose that $\pi$ has weight $a \in (\mathbb{Z}^n)^{\mathrm{Hom}(F,\mathbb{C}),+}$, and regard each element of $\mathrm{Hom}(F, \mathbb{C})$ as an element of $\mathrm{Hom}(F, \overline{M})$. Then:

- if $S$ is the finite set of finite places $v$ of $F$ at which $\pi_v$ is ramified, then $r_\lambda(\pi)$ and $r_\lambda(\chi)$ are unramified unless $v \in S$ or $v|l$;
- $r_\lambda(\pi) \cong r_\lambda(\pi)^\vee \otimes \epsilon^{1-n} r_\lambda(\chi)$;
- if $v|l$ then $r_\lambda(\pi)|_{G_{F_v}}$ and $r_\lambda(\chi)|_{G_{F_v}}$ are de Rham. If furthermore $v \notin S$ then $r_\lambda(\pi)|_{G_{F_v}}$ and $r_\lambda(\chi)|_{G_{F_v}}$ are crystalline;
- for each $\tau : F \hookrightarrow \overline{M}$ and any $\overline{M} \hookrightarrow \overline{M}_\lambda$ over $M$, the set $\mathrm{HT}_\tau(r_\lambda(\pi))$ of $\tau$-Hodge-Tate weights of $r_\lambda(\pi)$ is equal to
$$\{a_{\tau,1} + (n-1), a_{\tau,2} + (n-2), \ldots, a_{\tau,n}\}.$$

In arguments it will occasionally be useful to replace $M$ with a finite extension, in order to compare two different compatible systems; we will do this without comment.

While we will not make explicit use of this fact, to orient the reader we remark that if $v \notin S, v \nmid l$ is a finite place of $F$, and $\mathrm{Frob}_v$ is a geometric

Frobenius element at $v$, then the characteristic polynomial of $r_\lambda(\pi)(\mathrm{Frob}_v)$ is

$$X^n - t_v^{(1)} X^{n-1} + \cdots + (-1)^j t_v^{(j)} (\mathbf{N}v)^{j(j-1)/2} X^{n-j} + \cdots + (-1)^j t_v^{(n)} (\mathbf{N}v)^{n(n-1)/2} X^n,$$

where the $t_v^{(j)}$ are the eigenvalues of the usual Hecke operators on $\pi_v^{\mathrm{GL}_2(\mathcal{O}_{F_v})}$.

If $\rho : G \to \mathrm{GL}(V)$ is any semi-simple two dimensional irreducible representation which is induced from an index two subgroup $G'$ of $G$, then, by abuse of notation, we call $r$ *dihedral*. The image of a dihedral representation is a generalized dihedral group; equivalently, the *projective* image of $\rho$ in $\mathrm{PGL}(V)$ is a dihedral group.

## 2.2. Oddness

We now recall from section 2.1 of [4] the notion of oddness for essentially self-dual representations of $G_F$. Let $l > 2$ be a prime number, and let $K = \overline{\mathbb{Q}}_l$ or $\overline{\mathbb{F}}_l$. If $r : G_F \to \mathrm{GL}_n(K)$ and $\mu : G_F \to K^\times$ are continuous homomorphisms, then we say that the pair $(r, \mu)$ is essentially self-dual if for some (so any) infinite place $v$ of $F$ there is an $\epsilon_v \in \{\pm 1\}$ and a non-degenerate pairing $\langle , \rangle$ on $K^n$ such that

$$\langle x, y \rangle = \epsilon_v \langle y, x \rangle$$

and

$$\langle r(\sigma)x, r(c_v \sigma c_v) y \rangle = \mu(\sigma) \langle x, y \rangle$$

for all $x$, $y \in K^n$ and all $\sigma \in G_F$. Equivalently, $(r, \mu)$ is essentially self-dual if and only if either $\mu(c_v) = -\epsilon_v$ and $r$ factors through $\mathrm{GSp}_n(K)$ with multiplier $\mu$, or $\mu(c_v) = \epsilon_v$ and $r$ factors through $\mathrm{GO}_n(K)$ with multiplier $\mu$.

We say that the pair $(r, \mu)$ is *odd* and essentially self-dual if it is essentially self-dual, and $\epsilon_v = 1$ for all $v | \infty$.

Lemma 2.1. — *If $(r, \mu)$ is essentially self-dual and $n$ is odd, then $(r, \mu)$ is odd.*

*Proof.* — Since $n$ is odd, $r$ factors through $\mathrm{GO}_n(K)$ with multiplier $\mu$. Taking determinants, we see that for each $v | \infty$, $\mu(c_v)^n = 1$, so that $\mu(c_v) = 1$, as required. □

We also have the following trivial lemma.

Lemma 2.2. — *If $\chi : G_F \to K^\times$ is a character, then $(\chi, \chi^2)$ is odd and essentially self-dual.*

We have the following important result of [6].

Theorem 2.3. — *Let $(\pi, \chi)$ be a RAESDC automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$, and denote the corresponding compatible systems of Galois representations by $(r_\lambda(\pi), r_\lambda(\chi))$. If for some $\lambda$ we have an irreducible subrepresentation $r$ of $r_\lambda(\pi)$ with $r \cong r^\vee \otimes \epsilon^{1-n} r_\lambda(\chi)$, then $(r, \epsilon^{1-n} r_\lambda(\chi))$ is essentially self-dual and odd.*

*Proof.* — This is Corollary 1.3 of [6]. □

Since $r_\lambda(\pi) \cong r_\lambda(\pi)^\vee \otimes \epsilon^{1-n} r_\lambda(\chi)$, if $r$ is an irreducible subrepresentation of $r_\lambda(\pi)$ then there must be an irreducible subrepresentation $r'$ of $r_\lambda(\pi)$ (possibly equal to $r$) with $r' \cong r^\vee \otimes \epsilon^{1-n} r_\lambda(\chi)$. In particular, we have:

Corollary 2.4. — *Let $(\pi, \chi)$ be a RAESDC automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$, and denote the corresponding compatible systems of Galois representations by $(r_\lambda(\pi), r_\lambda(\chi))$. If for some $\lambda$ we have an irreducible subrepresentation $r$ of $r_\lambda(\pi)$ with $\dim r > n/2$, then $(r, \epsilon^{1-n} r_\lambda(\chi))$ is essentially self-dual and odd.*

*Proof.* — There is an irreducible subrepresentation $r'$ of $r_\lambda(\pi)$ with $r' \cong r^\vee \otimes \epsilon^{1-n} r_\lambda(\chi)$; but $\dim r + \dim r' > \dim r_\lambda(\pi)$, so we must have $r' = r$. The result then follows from Theorem 2.3. □

Suppose now that $r : G_F \to \mathrm{GL}_2(\overline{\mathbb{Q}}_l)$. Then $r$ factors through $\mathrm{GSp}_2(\overline{\mathbb{Q}}_l)$ with multiplier $\det r$, so the pair $(r, \det r)$ is essentially self-dual and odd if $\det r(c_v) = -1$ for all $v|\infty$. We have the following variant on Theorem 1.2 of [8].

Proposition 2.5. — *Suppose that $l > 7$ is prime, and that $r : G_F \to \mathrm{GL}_2(\overline{\mathbb{Q}}_l)$ is a continuous representation. Assume that*

- *$r$ is unramified outside of finitely many primes.*
- *$\mathrm{Sym}^2 \bar{r}|_{G_{F(\zeta_l)}}$ is irreducible.*
- *$l$ is unramified in $F$.*
- *For each place $v|l$ of $F$ and each $\tau : F_v \hookrightarrow \overline{\mathbb{Q}}_l$, $\mathrm{HT}_\tau(r|_{G_{F_v}})$ is a set of 2 distinct integers whose difference is less than $(l-2)/2$, and $r|_{G_{F_v}}$ is crystalline.*

*Then the pair $(r, \det r)$ is essentially self-dual and odd.*

*Proof.* — Consider the representation $s = \mathrm{Sym}^2 r$. Since the pair $(r, \det r)$ is essentially self-dual, so is the pair $(s, (\det r)^2)$. By Lemma 2.1, $(s, (\det r)^2)$ is odd. By Corollary 4.5.2 and Lemma 1.4.3(2) of [4], there is a Galois totally real extension $F'/F$ such that $(s|_{G_{F'}}, (\det r)^2|_{G_{F'}})$ is automorphic. By Proposition A of [26], for any place $v|\infty$ of $F'$ we have

$$\mathrm{tr}\, s|_{G_{F'}}(c_v) = \pm 1,$$

so that

$$\det r|_{G_{F'}}(c_v) = -1,$$

and $(r, \det r)$ is odd, as required. □

### 2.3. Residually dihedral compatible systems

We now show that residually dihedral compatible systems are themselves dihedral up to a set of places of density zero.

Lemma 2.6. — *Suppose that $l > 2$ is unramified in $F$, and that $s : G_F \to \mathrm{GL}_2(\overline{\mathbb{Q}}_l)$ is a continuous irreducible representation, such that if $v|l$ is a place of $F$, then*

- *$s|_{G_{F_v}}$ is crystalline, and*
- *for all embeddings $F \hookrightarrow \overline{\mathbb{Q}}_l$, $\mathrm{HT}_\tau(s|_{G_{F_v}})$ consists of two distinct integers with difference less than $(l-2)/2$.*

*Assume that $\bar{s}$ is dihedral, so that $\bar{s}$ is induced from a character of a quadratic extension $K/F$. Then $l$ is unramified in $K$.*

*Proof.* — Let $v|l$ be a place of $F$. Assume for the sake of a contradiction that $v$ is ramified in $K$. If $\bar{s}|_{G_{F_v}}$ is irreducible, then $\bar{s}|_{G_{F_v}}$ is induced from a character $\chi$ of $G_L$, where $L$ is the quadratic unramified extension of $F_v$. But then

$$\bar{s}|_{G_{K_v}} \cong (\mathrm{Ind}_{G_L}^{G_{F_v}} \chi)|_{G_{K_v}} \cong \mathrm{Ind}_{G_{LK_v}}^{G_{K_v}} \chi|_{G_{LK_v}}$$

is irreducible, a contradiction.

If on the other hand $\bar{s}|_{G_{F_v}}$ is reducible, then since it is isomorphic to the induction from $K_v$ of some character, it must be of the form $\psi_1 \oplus \psi_2$ with $\psi_1\psi_2^{-1}$ quadratic. Let $k$ be the residue field of $F_v$, and for each $\sigma : k \hookrightarrow \overline{\mathbb{F}}_l$ let $\omega_\sigma$ be the corresponding fundamental character of $G_{F_v}$ of niveau 1. By Fontaine-Laffaille theory,

$$\psi_1\psi_2^{-1}|_{I_{F_v}} = \prod_{\sigma : k \hookrightarrow \overline{\mathbb{F}}_l} \omega_\sigma^{a_\sigma}$$

where $a_\sigma$ is the (positive or negative) difference between the elements of $\mathrm{HT}_\tau(s|_{G_{F_v}})$, where $\tau : F_v \hookrightarrow \overline{\mathbb{Q}}_l$ is the unique lift of $\sigma$. By assumption, we have $2a_\sigma \in [2-l, l-2]$, so $(\psi_1\psi_2^{-1})^2|_{I_{F_v}} \neq 1$, a contradiction. So $l$ is unramified in $K$, as claimed. □

Proposition 2.7. — *Suppose that $\mathcal{R}$ is a regular, weakly compatible system of $l$-adic representations of $G_F$. Then there is a set of rational primes $\mathcal{L}$ of density one such that if $\lambda$ lies over a place of $\mathcal{L}$ and $s$ is a* *two-dimensional irreducible summand of* $r_\lambda$ *such that* $\bar{s}$ *is dihedral, then* $s$ *is also dihedral, and the pair* $(s, \det s)$ *is essentially self-dual and odd.*

*Proof.* — Note firstly that if $s$ is dihedral, then it is induced from an algebraic character of a quadratic extension of $F$. If this quadratic extension is not totally imaginary, then this character would be a finite order character times a power of the cyclotomic character, contradicting the regularity of $s$. So the extension must be totally imaginary, in which case $\det s(c_v) = -1$ for all places $v|\infty$ of $F$, and the pair $(s, \det s)$ is essentially self-dual and odd.

Let $S$ be the finite set of primes at which $\mathcal{R}$ ramifies, and let $F'$ be the maximal abelian extension of $F$ of exponent 2 which is unramified outside $S$ (the extension $F'/F$ is finite). By Lemma 2.6, for all but finitely many $\lambda$, if $s$ is as in the statement of the proposition then $\bar{s}|_{G_{F'}}$ is reducible. Applying Proposition 5.2.2 of [4] to the regular weakly compatible system $\mathcal{R}|_{G_{F'}}$, we see that there is a set $\mathcal{L}$ of rational primes of density one such that if $\lambda$ lies over an element of $\mathcal{L}$ and $s$ is as in the statement of the proposition, then $s|_{G_{F'}}$ is reducible, so that $s$ is dihedral, as required. □

## 3. Irreducibility for a density one set of primes

### 3.1.

In this section, we establish the irreducibility of $r_\lambda(\pi)$ for a density one set of primes $\lambda$.

PROPOSITION 3.1. — *Let* $F$ *be a totally real field. Suppose that* $\pi$ *is a RAESDC automorphic representation of* $\mathrm{GL}_n(\mathbb{A}_F)$, *and that for some* $\lambda$ *we have a decomposition*

$$r_\lambda(\pi) = r_\lambda(\pi)_1 \oplus \cdots \oplus r_\lambda(\pi)_j,$$

*where each* $r_\lambda(\pi)_i$ *is irreducible. Suppose also that there is a totally real Galois extension* $F'/F$ *such that each* $r_\lambda(\pi)_i|_{G_{F'}}$ *is irreducible and automorphic. Then* $j = 1$, *so* $r_\lambda(\pi)$ *is irreducible.*

*Proof.* — This may be proved by an identical argument to the proof of Theorem 5.4.2 of [4]. □

THEOREM 3.2. — *Let* $F$ *be a totally real field. Suppose that* $(\pi, \chi)$ *is a RAESDC automorphic representation of* $\mathrm{GL}_n(\mathbb{A}_F)$ *with* $n \leqslant 5$. *Then there is a density one set of rational primes* $\mathcal{L}$ *such that if* $\lambda$ *lies over a prime in* $\mathcal{L}$, *then* $r_\lambda(\pi)$ *is irreducible.*

*Proof.* — Write

$$r_\lambda(\pi) = r_\lambda(\pi)_1 \oplus \cdots \oplus r_\lambda(\pi)_{j_\lambda},$$

with each $r_\lambda(\pi)_i$ irreducible. By Proposition 5.2.2 of [4] there is a density one set of rational primes $\mathcal{L}$ such that if $\lambda$ lies over a prime of $\mathcal{L}$, then each $\bar{r}_\lambda(\pi)_i|_{G_{F(\zeta_l)}}$ is irreducible. We may assume that every prime in $\mathcal{L}$ is at least 13.

If $\dim r_\lambda(\pi)_i \geqslant 3$, then by Corollary 2.4 and the hypothesis that $n \leqslant 5$ we see that the pair $(r_\lambda(\pi)_i, \epsilon^{1-n} r_\lambda(\chi))$ is essentially self-dual and odd. If $\dim r_\lambda(\pi)_i = 1$, then by Lemma 2.2, the pair $(r_\lambda(\pi)_i, r_\lambda(\pi)_i^2)$ is essentially self-dual and odd.

Suppose now that $\dim r_\lambda(\pi)_i = 2$. By removing finitely many primes from $\mathcal{L}$, we see from Proposition 2.5 that we may assume that if $\lambda$ lies over an element of $\mathcal{L}$, and $\operatorname{Sym}^2 \bar{r}_\lambda(\pi)_i|_{G_{F(\zeta_l)}}$ is irreducible, then the pair $(r_\lambda(\pi)_i, \det r_\lambda(\pi)_i)$ is essentially self-dual and odd. If $\lambda$ lies over an element of $\mathcal{L}$ and $\operatorname{Sym}^2 \bar{r}_\lambda(\pi)_i|_{G_{F(\zeta_l)}}$ is reducible, then since $\bar{r}_\lambda(\pi)_i|_{G_{F(\zeta_l)}}$ is irreducible, it follows from Lemmas 4.2.1 and 4.3.1 of [3] that $\bar{r}_\lambda(\pi)_i$ has dihedral image. By Proposition 2.7, after possibly replacing $\mathcal{L}$ with a subset of density one, the pair $(r_\lambda(\pi)_i, \det r_\lambda(\pi)_i)$ is essentially self-dual and odd.

Thus if $\lambda$ divides a prime in $\mathcal{L}$, for each $i$ there is a character $\chi_{\lambda,i}$ such that the pair $(r_\lambda(\pi)_i, \chi_{\lambda,i})$ is essentially self-dual and odd. After possibly removing a finite set of primes from $\mathcal{L}$, we may assume that every element of $\mathcal{L}$ is unramified in $F$, and that each $r_\lambda(\pi)_i$ is crystalline with Hodge-Tate weights in the Fontaine-Laffaille range. Fix some $l \in \mathcal{L}$ and some $\lambda | l$. Let $K$ be an imaginary quadratic extension of $F$ in which each place of $F$ above $l$ splits completely. By Theorem 4.5.1 of [4], there is a finite Galois CM extension $K'$ of $K$ such that each $r_\lambda(\pi)_i|_{G_{K'}}$ is irreducible and automorphic.

Let $F'$ be the maximal totally real subfield of $K'$. By Lemma 1.5 of [5] each $r_\lambda(\pi)_i|_{G_{F'}}$ is irreducible and automorphic. The result now follows from Proposition 3.1. □

## 4. Irreducibility for all primes

### 4.1.

In this section we prove that the representations $r_\lambda(\pi)$ are irreducible for all $\lambda$.

Theorem 4.1. — *Let $F$ be a totally real field. Suppose that $(\pi, \chi)$ is a RAESDC automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$ with $n \leqslant 5$. Then all of the representations $r_\lambda(\pi)$ are irreducible.*

By Theorem 3.2, $r_\lambda(\pi)$ is irreducible for a set of $\lambda$ of density one. By Proposition 5.2.2 of [4], this implies that $\bar{r}_\lambda(\pi)$ is irreducible for a set of $\lambda$ of density one.

Definition 4.2. — *Let $F$ be a number field. We say that a representation $r : G_F \to \mathrm{GL}_n(\overline{\mathbb{Q}}_l)$ is strongly irreducible if for all finite extensions $E/F$, $r|_{G_E}$ is irreducible.*

We would like to understand when the Galois representations $r_\lambda(\pi)$ which are irreducible can fail to be strongly irreducible. We begin with an easy group theory lemma.

Lemma 4.3. — *Suppose that $G$ acts irreducibly on a finite dimensional vector space $V$ of dimension $n$. Let $G'$ be a normal finite index subgroup of $G$, and suppose that $V|_{G'} \simeq \bigoplus W_k$ decomposes non-trivially as a $G'$ representation into $m$ distinct irreducible representations. Then $m|n$ and there exists a proper subgroup $H \supseteq G'$ of $G$ of index $m$ and an irreducible representation $W$ of $H$ such that $V \simeq \mathrm{Ind}_H^G W$.*

*Proof.* — Since the representations $W_k$ are distinct, and since $G'$ is normal, the group $G$ acts transitively on the set of representations $W_k$. In particular, all the $W_k$ have the same dimension. Let $W$ be one of these representations, and let $H$ denote the stabilizer of $W$. By the orbit–stabilizer theorem, the index of $H$ in $G$ is $m$. The representation $W$ extends to a representation of $H$. Since $\mathrm{Hom}_H(V, W)$ is non-trivial, by Frobenius reciprocity $\mathrm{Hom}_G(V, \mathrm{Ind}_H^G W)$ is also non-trivial. Yet $\mathrm{Ind}_H^G(W)$ has dimension $[G : H] \dim(W) = \dim(V)$ and $V$ is irreducible, and thus the homomorphism $V \to \mathrm{Ind}_H^G(W)$ must be both an injection and a surjection, and hence an isomorphism. □

Using this lemma, we shall see that the density one set of irreducible Galois representations $r_\lambda(\pi)$ remain irreducible upon restriction to any fixed finite extension, *except* in situations in which we can prove Theorem 4.1 directly.

Corollary 4.4. — *Let $F$ be a totally real field. Let $(\pi,\chi)$ be a RAESDC automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$ with $n \leqslant 5$. Let $\lambda$ be a prime such that $r_\lambda(\pi)$ is irreducible. Then either:*

(1) *$r_\lambda(\pi)$ is strongly irreducible, or*

(2) *$(\pi,\chi)$ is an automorphic induction, $r_\lambda(\pi)$ is irreducible for all $\lambda$, and $\bar{r}_\lambda(\pi)$ is irreducible for all but finitely many $\lambda$.*

*Proof.* — We claim that for any finite extension $E/F$, either $r_\lambda(\pi)|_{G_E}$ is irreducible or it decomposes into a sum of *distinct* irreducible representations. This follows immediately from the fact that $r_\lambda(\pi)|_{G_E}$ has distinct Hodge–Tate weights at any prime $w|l$. (Note that $r_\lambda(\pi)|_{G_E}$ is necessarily semisimple; if $V$ denotes any irreducible subrepresentation, then the various conjugates of $V$ are stable under the conjugates of $G_E$, and we see that $r_\lambda(\pi)|_{G_E}$ becomes completely decomposable under restriction to a finite index subgroup, so must already have been semisimple.) Suppose that $r_\lambda(\pi)|_{G_E}$ is reducible for some finite extension $E/F$. Replacing $E$ by its normal closure over $F$, we may assume that the extension $E/F$ is Galois, and hence by Lemma 4.3, we see that $r_\lambda(\pi)$ is the induction of an irreducible representation from some finite extension of degree dividing $n$. If $n=2$, 3 or 5, the only possibility is that $r_\lambda(\pi)$ is the induction of a character from some degree $n$ extension $H$ of $F$. This character is de Rham, and is thus the Galois representation associated to an algebraic Hecke character. If $n=3$ or 5, then $[H:F]$ is odd, and thus $H$ does not contain a CM field. It follows that the corresponding Galois representation is a finite order character times some power of the cyclotomic character. This contradicts the regularity of $r_\lambda(\pi)$. If $n=2$, then $H$ must be a CM field, and $r_\lambda(\pi)$ is the induction of an algebraic Hecke character. The claims regarding the irreducibility of $\bar{r}_\lambda(\pi)$ are elementary to verify in this case.

If $n=4$, then either $r_\lambda(\pi)$ is the induction of a character from some degree 4 extension $L/F$, or the induction of a two dimensional representation of some quadratic extension $K/F$. In the first case, since $r_\lambda(\pi)$ is regular, $L$ contains a CM field. It follows that $L$ must contain a subfield $K$ of index two, and thus in both cases $r_\lambda(\pi)$ is the induction of some two dimensional representation of some quadratic extension $K/F$. It follows that there is an isomorphism $r_\lambda(\pi)\simeq r_\lambda(\pi)\otimes\eta$, where $\eta$ is the quadratic character of $K/F$. By multiplicity one for $\mathrm{GL}_4(\mathbb{A}_F)$ ([18]) and by Theorem 4.2 (p.202) of [1], we deduce that $(\pi,\chi)$ is an automorphic induction from some quadratic field $K/F$.

It suffices to prove that when $n=4$ and $(\pi,\chi)$ is an induction of some cuspidal automorphic representation $\varpi$ of $\mathrm{GL}_2(\mathbb{A}_K)$ from some quadratic field $K/F$, then $r_\lambda(\pi)$ is irreducible for *all* $\lambda$, and $\bar{r}_\lambda(\pi)$ is irreducible for all but finitely many $\lambda$. If $K/F$ is totally real, then $\varpi$ corresponds to a Hilbert modular form with corresponding Galois representations $s_\lambda(\varpi)$, and there are isomorphisms $r_\lambda(\pi) \simeq \mathrm{Ind}_{G_K}^{G_F} s_\lambda(\varpi)$. The representation $s_\lambda(\varpi)$ is irreducible for all $\lambda$, and $\bar{s}_\lambda(\varpi)$ is irreducible for all but finitely many $\lambda$ ([14, Proposition 0.1]). If $r_\lambda(\pi)$ is reducible, then $s_\lambda(\varpi) \simeq s^c_\lambda(\varpi)$ where $c$ is the non-trivial element of $\mathrm{Gal}(K/F)$. Similarly, if $\bar{r}_\lambda(\pi)|_{G_F}$ is reducible for infinitely many primes $\lambda$, then $\bar{s}_\lambda(\varpi) \simeq \bar{s}^c_\lambda(\varpi)$ for infinitely many $\lambda$. In either case, by multiplicity one, we deduce that $\varpi \simeq \varpi^c$, and by Theorem 4.2 (p.202) of [1], we deduce that $\varpi$ itself arises from base change from $\mathrm{GL}_2(\mathbb{A}_F)$. If this is so, however, then $\pi$ is not cuspidal, contrary to assumption. Suppose instead that $K/F$ is not totally real. By (3.6.1) of [16]), the infinitesimal character of $\varpi$ at any pair of complex conjugate infinite places must be equal, contradicting the regularity of $\pi$. □

In the sequel, it will be useful to collate some information about irreducible representations of semi-simple Lie algebras of small dimension.

Proposition 4.5. — *Let $k$ be an algebraically closed field of characteristic zero. Let $G$ be the $k$-points of a reductive algebraic group acting faithfully and irreducibly on a vector space over $k$ of dimension $n$. Let $G^0$ denote the connected component of $G$, let $\mathfrak{g}$ be the Lie algebra of $G^0$, and write $\mathfrak{g} = \mathfrak{z} \oplus \mathfrak{h}$, with $\mathfrak{z}$ is abelian and $\mathfrak{h}$ semisimple. Suppose that $G^0$ is not abelian. Then, for $n \leqslant 6$, $\mathfrak{h}$ is one of the following algebras, where the columns of the table correspond to whether $G$ preserves a generalized orthogonal pairing, a generalized symplectic pairing, or does not preserve any such pairing respectively.*

| | GO | GSp | GL |
|---|---|---|---|
| 2 | * | $\mathfrak{sl}_2$ | * |
| 3 | $\mathfrak{sl}_2$ | * | $\mathfrak{sl}_3$ |
| 4 | $\mathfrak{so}_4 = \mathfrak{sl}_2 \times \mathfrak{sl}_2$ | $\mathfrak{sl}_2$, $\mathfrak{sl}_2 \times \mathfrak{sl}_2$, $\mathfrak{sp}_4$ | $\mathfrak{sl}_4$ |
| 5 | $\mathfrak{sl}_2$, $\mathfrak{so}_5 = \mathfrak{sp}_4$ | * | $\mathfrak{sl}_5$ |
| 6 | $\mathfrak{so}_6$ | $\mathfrak{sl}_2$, $\mathfrak{sl}_2 \times \mathfrak{sl}_2$, $\mathfrak{sp}_6$ | $\mathfrak{sl}_2 \times \mathfrak{sl}_3$, $\mathfrak{sl}_2 \times \mathfrak{sl}_2 \times \mathfrak{sl}_2$, $\mathfrak{sl}_3 \times \mathfrak{sl}_3$, $\mathfrak{sl}_4$, $\mathfrak{sl}_6$ |

*Proof.* — This can be checked directly by hand. In dimension $n$, the representation of $\mathfrak{sl}_2$ is the $(n-1)$st symmetric power of the tautological representation, which is orthogonal if $n$ is odd and symplectic if $n$ is even. The four dimensional symplectic representation of $\mathfrak{sl}_2 \times \mathfrak{sl}_2$ is reducible, but the image of $\mathrm{GL}(2) \times \mathrm{GL}(2)$ in $\mathrm{GL}(4)$ is normalized by a group $G$ which contains it with index two and does act irreducibly. The same is true of $\mathfrak{sl}_3 \times \mathfrak{sl}_3$ (index two) and $\mathfrak{sl}_2 \times \mathfrak{sl}_2 \times \mathfrak{sl}_2$ (index six) in dimension six. The algebra $\mathfrak{sl}_2 \times \mathfrak{sl}_2$ has a six dimensional representation which is the tensor product of the standard representation of the first factor and the symmetric square of the second. Finally, $\mathfrak{sl}_4$ has a six dimensional representation which is $\wedge^2$ of the tautological representation. □

The key idea of our argument is that, with certain caveats, we can detect reducibility on the level of compatible systems. Suppose that $\mathcal{R}$ and $\mathcal{S}$ are two weakly compatible systems of Galois representations.

Definition 4.6. — *Say that $\mathcal{S}$ weakly divides $\mathcal{R}$ if the characteristic polynomials of $\mathcal{S}$ divide the characteristic polynomials of $\mathcal{R}$.*

It is not true that if $\mathcal{S}$ weakly divides $\mathcal{R}$ then there is a corresponding splitting of Galois representations. A good example to keep in mind is as follows. Suppose that $\pi$ is a RAESDC automorphic representation for $\mathrm{GL}_2(\mathbb{A}_F)$ with central character $\psi$ which does not have CM. Then if $\mathcal{R}$ and $\mathcal{S}$ are the compatible systems associated to $\mathrm{Sym}^{n-2}(\pi) \otimes \psi$ and $\mathrm{Sym}^n(\pi)$ respectively then $\mathcal{S}$ weakly divides $\mathcal{R}$, but both compatible systems are irreducible. Nevertheless, we will be able to detect non-trivial information from weak divisibility. The key result is the following.

Theorem 4.7. — *Let $V$ be a vector space of dimension $n \leqslant 5$ over an algebraically closed field of characteristic zero. Let $H \subseteq \mathrm{GL}(V)$ be a Zariski closed subgroup, and assume that the connected component of the identity $H^0$ acts irreducibly on $V$. Suppose that every $h \in H$ has a fixed vector in $V$. Then $H = \mathrm{PSL}(2)$, $n$ is odd, and $V \simeq \mathrm{Sym}^{n-1}(W)$ where $W$ is the standard representation of $\mathrm{SL}(2)$. For a generic semi-simple element $h \in H$, one has $\dim(V|h = 1) = 1$.*

*Proof.* — Tautologically, $H$ admits a faithful irreducible representation into $\mathrm{GL}(V)$, and thus $H$ is reductive. Let $Z$ be the center of $H$. By Schur's lemma, $Z$ acts on $V$ via scalars. Yet (by assumption) any $z \in Z \subset H$ has a fixed vector, and thus $Z$ is trivial. In particular, $H$ is semi-simple. It suffices to assume that every $t \in T \subset H$ has a fixed vector for every torus $T$ on $H$. Since $H^0$ is connected, we may check this condition for $H^0$ on the level of Lie algebras. The only (semi-simple) Lie algebras $\mathfrak{h}$ which admit faithful irreducible representations of dimension at most five are $\mathfrak{sl}_2$, $\mathfrak{sl}_3$, $\mathfrak{so}_4 = \mathfrak{sl}_2 \times \mathfrak{sl}_2$, $\mathfrak{sp}_4 = \mathfrak{so}_5$, $\mathfrak{sl}_4$, and $\mathfrak{sl}_5$. It is easy to check that the only possibility is that $\mathfrak{h} = \mathfrak{sl}_2$. Since $H^0$ is connected with trivial center it must be the adjoint form of $\mathrm{SL}(2)$, which is $\mathrm{PSL}(2)$. The irreducible representations of $\mathrm{PSL}(2)$ are given by the even symmetric powers of the standard representation of $\mathrm{SL}(2)$. The group $H^0 = \mathrm{PSL}(2)$ has a trivial outer automorphism group. Hence the action of conjugation by $h \in H$ on $H^0$ is given by conjugation by a element $\gamma$ of $\mathrm{PSL}(2)$. It follows that $\gamma^{-1}h$ acts trivially on $\mathrm{PSL}(2)$, and thus, by Schur's lemma, $\gamma^{-1}h$ is a scalar, and so lies in the center $Z$ of $H$. Yet we have seen that $Z$ is trivial, and so it follows that $H = H^0$. A generic semi-simple element in $\mathrm{PSL}(2)$ is conjugate to an element of the form $h = \begin{pmatrix} z & \\ & z^{-1} \end{pmatrix}$, for which $\dim(V|h = 1) = 1$ unless $z$ is a root of unity of sufficiently small order. $\square$

Using this, we prove the following.

Lemma 4.8. — *Let $F$ be a totally real field. Let $(\pi, \chi)$ be a RAESDC automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$ with $n \leqslant 5$, and let $\mathcal{R}$ be the corresponding weakly compatible system. Suppose that either:*

(1) *$\mathcal{R}$ is weakly divisible by a compatible system of algebraic Hecke characters.*
(2) *For some finite extension $E/F$, $\mathcal{R}|_{G_E}$ is weakly divisibly by a direct sum of compatible systems of two algebraic Hecke characters over $E$.*

*Then either $(\pi, \chi)$ is an automorphic induction, or $n$ is odd, and there exists a finite Galois extension $F'/F$ and a compatible system of two dimensional irreducible Galois representations $\mathcal{S}$ of $G_{F'}$ such that $\mathcal{R}|_{G_{F'}}$ is a twist of $\mathrm{Sym}^{n-1}(\mathcal{S})$. In either case, $r_\lambda(\pi)$ is irreducible for all $\lambda$, and $\bar{r}_\lambda(\pi)$ is irreducible for all but finitely many $\lambda$.*

*Proof.* — We may assume that $(\pi, \chi)$ is not an automorphic induction. By Corollary 4.4, we may find a sufficiently large $\lambda$ such that $r_\lambda(\pi)$ is strongly irreducible. We may also assume that $\bar{r}_\lambda(\pi)$ is irreducible. Let us assume that we are in case (1). After twisting, we may assuming that the compatible system $\mathcal{R}$ is divisible by the trivial character. Let $H$ denote the Zariski closure of the image of $r_\lambda(\pi)$, and let $H^0$ denote the connected component of $H$. Since Frobenius elements are Zariski dense, we deduce that every $h \in H$ has a fixed vector. We deduce from Theorem 4.7 that $n$ is odd, and that the image of $r_\lambda(\pi)$ lands in the image of $\mathrm{PSL}_2(\overline{\mathbb{Q}}_l)$ in $\mathrm{GO}_n(\overline{\mathbb{Q}}_l)$ under the $(n-1)$-st symmetric power map. In particular, the image of $r_\lambda(\pi)$ lands in $\mathrm{SO}_n(\overline{\mathbb{Q}}_l)$ and $\det(r_\lambda(\pi)) = 1$. The obstruction to lifting a projective homomorphism from $\mathrm{PSL}_2(\overline{\mathbb{Q}}_l)$ to $\mathrm{GL}_2(\overline{\mathbb{Q}}_l)$ lies in $H^2(G_F, \overline{\mathbb{Q}}_l^\times)$, which vanishes by a result of Tate (for example, see Theorem 5.4 of [9]). Hence there exists a Galois representation $s_\lambda : G_F \to \mathrm{GL}_2(\overline{\mathbb{Q}}_l)$ and an isomorphism

$$r_\lambda(\pi) = \mathrm{Sym}^{n-1} s_\lambda \otimes \det(s_\lambda)^{\frac{1-n}{2}}.$$

We will show that $s_\lambda$ is potentially modular, and use known irreducibility results for the Galois representations associated to Hilbert modular forms to conclude.

We first show that $\mathrm{ad}^0(s_\lambda) = \mathrm{Sym}^2(s_\lambda)\det(s_\lambda)^{-1}$ is de Rham. We see this from the following plethysm for the standard representation $W$ of $\mathrm{GL}(2)$:

$$(\mathrm{Sym}^{n-1} W)^{\otimes 2} \otimes \det(W)^{-(n-1)} = \bigoplus_{i=0}^{n-1} \mathrm{Sym}^{2i} W \otimes \det(W)^{-i}.$$

In particular, $\mathrm{ad}^0(s_\lambda)$ is a constituent of $(r_\lambda(\pi))^{\otimes 2}$, and hence is crystalline with Hodge–Tate weights in the Fontaine–Laffaille range for sufficiently large $\lambda$. We claim that in fact $\mathrm{ad}^0(s_\lambda)$ is regular; in the cases $n = 1$, $3$ this is immediate from the regularity of $r_\lambda(\pi)$, and in the case $n = 5$ it follows from the regularity of $r_\lambda(\pi)$ and the relation

$$\mathrm{Sym}^2(\mathrm{ad}^0(s_\lambda)) = r_\lambda(\pi) \oplus 1.$$

Since $r_\lambda(\pi)$ is odd, it follows immediately that $s_\lambda$ and hence $\mathrm{ad}^0(s_\lambda)$ is also odd. Since it is 2-dimensional, $s_\lambda$ is automatically essentially self-dual, so that $\mathrm{ad}^0(s_\lambda)$ is also essentially self-dual. If $\bar{s}_\lambda$ was dihedral or reducible, then $\bar{r}_\lambda(\pi)$ would be reducible, contrary to assumption. Hence if $\lambda$ is sufficiently large, $\mathrm{ad}^0(\bar{s}_\lambda)|_{G_{F(\zeta_l)}}$ is irreducible, and so we deduce from Corollary 4.5.2 and Lemma 1.4.3(2) of [4] that there is a finite Galois extension of totally real fields $F'/F$ such that $\mathrm{ad}^0(s_\lambda)|_{G_{F'}}$ is automorphic. It follows that up to twist $s_\lambda|_{G_{F'}}$ itself is automorphic (using the characterization of the image of the symmetric square in Theorem A and Corollary B of [22]), say $s_\lambda|_{G_{F'}} \cong r_\lambda(\pi') \otimes \psi$ for some $\psi$. Since $r_\lambda(\pi)$ is strongly irreducible, $r_\lambda(\pi)|_{G_{F'}} \cong \mathrm{Sym}^{n-1} r_\lambda(\pi') \otimes \det(r_\lambda(\pi'))^{\frac{1-n}{2}}$ is irreducible, so $\pi'$ cannot be of CM type. Thus, for all $\lambda'$, we have $r_{\lambda'}|_{G_{F'}} \cong \mathrm{Sym}^{n-1} r_{\lambda'}(\pi') \otimes \det(r_\lambda(\pi'))^{\frac{1-n}{2}}$ is irreducible, and $\bar{r}_{\lambda'}|_{G_{F'}}$ is irreducible for all but finitely many $\lambda'$ (since for all but finitely many $\lambda'$, the image of $\bar{r}_{\lambda'}|_{G_{F'}}$ will contain $\mathrm{SL}_2(\mathbb{F})$ for some finite field $\mathbb{F}$ by [14, Proposition 0.1]).

Suppose instead that we are in case (2). After twisting, we may assume that the compatible system $\mathcal{R}|_{G_E}$ is divisible by the trivial character. If the second character is also trivial (after this twist), we obtain a contradiction with Theorem 4.7, since the generic multiplicity of the $h = 1$ eigenspace is 1. Hence the characters are different. It follows as in the first paragraph of this proof that both the representations $r_\lambda(\pi)$ and $r_\lambda(\pi) \otimes \chi$ have trivial determinant for some $\chi \neq 1$, which implies that $\chi$ has finite order. Replacing $E$ with the fixed field of the kernel of $\chi$, we reduce to the case that both characters are trivial, which is a contradiction. ☐

Corollary 4.9. — *Let $F$ be a totally real field. Let $(\pi,\chi)$ be a RAESDC automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$ with $n \leqslant 5$. Suppose that some*

*$r_\lambda(\pi)$ is reducible, say $r_\lambda(\pi) = s_\lambda \oplus t_\lambda$. Then* $\min(\dim s_\lambda, \dim t_\lambda) \geqslant 2$, *and neither of $s_\lambda$, $t_\lambda$ can be dihedral.*

*Proof.* — Suppose that without loss of generality $\dim s_\lambda = 1$. Since $r_\lambda(\pi)$ is de Rham, so is $s_\lambda$, so by *e.g.* Lemma 4.1.3 of [10] there is an algebraic character $\chi$ of $\mathbb{A}_F^\times/F^\times$ such that $s_\lambda = r_\lambda(\chi)$. Then the weakly compatible system $\{r_\lambda(\chi)\}$ weakly divides $\{r_\lambda(\pi)\}$, so by Lemma 4.8, we see that $r_\lambda(\pi)$ is irreducible for all $\lambda$, a contradiction.

Suppose now that $s_\lambda$ is dihedral. Then there is a quadratic extension $E/F$ such that $s_\lambda|_{G_E}$ is reducible, so is a sum of two de Rham characters. Arguing in the same way, we again obtain a contradiction from Lemma 4.8. □

We now prove Theorem 4.1. We proceed by contradiction, assuming that for some $\lambda$ we have $r_\lambda(\pi) = s_\lambda \oplus t_\lambda$. By Corollary 4.9, it suffices to consider the cases that both $s_\lambda$ and $t_\lambda$ are irreducible, that $n = 4$ or 5, and that $\dim s_\lambda = 2$.

## 4.2. The case $n = 5$

Since 5 is odd, we see from Lemma 2.1 that $r_\lambda(\pi)$ factors through $\mathrm{GO}_5$ with even multiplier. Since $2 \neq 3$, we see that $s_\lambda$ factors through $\mathrm{GO}_2$ with even multiplier, so $s_\lambda$ is dihedral. This contradicts Corollary 4.9.

## 4.3. The $n = 4$ symplectic case

Suppose that $r_\lambda(\pi)$ is symplectic with odd multiplier. If $V$ is a (generalized) symplectic representation of a group $G$ with multiplier $\chi$, then there is a surjection $\wedge^2 V \to \chi$. Hence the virtual representation $\wedge^2(V) - \chi$ is an actual representation of $G$. In particular, if $\mathcal{R}$ is the compatible system of Galois representations associated to $(\pi, \chi)$, then $\mathcal{A} := \wedge^2(\mathcal{R}) - \chi$ is a compatible system of Galois representations such that $a_\lambda : G_F \to \mathrm{GL}_5(\overline{\mathbb{Q}}_l)$ has image in $\mathrm{GO}_5(\overline{\mathbb{Q}}_l)$. Moreover, this compatible system is odd (automatic since 5 is odd) and regular.

Since $r_\lambda(\pi) = s_\lambda \oplus t_\lambda$, we have

$$a_\lambda \oplus \chi_\lambda = s_\lambda \otimes t_\lambda \oplus \det(s_\lambda) \oplus \det(t_\lambda).$$

In particular, as there are two characters on the right hand side, the representation $a_\lambda$ contains a character, and we deduce that the compatible system $\mathcal{A}$ is weakly divisible by the compatible system of a character.

For a density one set of primes $\lambda'$, the representation $r_{\lambda'}(\pi)$ is irreducible. For such a prime $\lambda'$, let $G$ denote the Zariski closure of the image, and $G^0$ the connected component of the identity. Let $Z$ denote the center of $H$. If $G^0$ acts reducibly, then $r_{\lambda'}$ is potentially reducible and hence $(\pi, \chi)$ is an automorphic induction by Corollary 4.4, and we would be done. Hence $G^0$ acts irreducibly. Let $\mathfrak{g}$ be the Lie algebra of $G^0$, and let $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{z}$ for $\mathfrak{h}$ semi-simple and $\mathfrak{z}$ central. We deduce that $\mathfrak{h}$ acts irreducibly, and hence $\mathfrak{h} = \mathfrak{sp}_4$ or $\mathfrak{sl}_2 \subset \mathfrak{sp}_4$ acting through the third symmetric power representation. In either case, the corresponding representation in dimension 5 of $\mathfrak{so}_5 = \mathfrak{sp}_4$ is irreducible, and thus $a_{\lambda'}$ is also irreducible, even after restricting to any finite extension $E/F$. Arguing as in the proof of Lemma 4.8, we deduce that there is a finite Galois extension $F'/F$ of totally real fields and a RAESDC automorphic representation $\pi'$ of $\mathrm{GL}_2(\mathbb{A}_{F'})$ such that, for all $\lambda''$, we have $a_{\lambda''}|_{G_{F'}} \cong \operatorname{Sym}^4 r_{\lambda''}(\pi') \otimes \det(r_{\lambda''}(\pi'))^{-2}$. Since $a_{\lambda'}$ is irreducible, $\pi'$ cannot be of CM type, so that in fact $a_{\lambda''}$ is irreducible for all $\lambda''$. This contradicts the reducibility of $a_\lambda$.

### 4.4. The $n = 4$ orthogonal case

Suppose finally that $r_\lambda(\pi)$ is orthogonal with even multiplier, and that $r_\lambda(\pi) = s_\lambda \oplus t_\lambda$. By Corollary 4.9 both $s_\lambda$ and $t_\lambda$ are non-dihedral two-dimensional representations.

Since $r_\lambda(\pi)$ factors through $\mathrm{GO}(4)$, it must either be the case that each of $s_\lambda$ and $t_\lambda$ factors through $GO(2)$, or that the orthogonal pairing identifies $s_\lambda$ with $t_\lambda^\vee \otimes \epsilon^{-3} r_\lambda(\chi)$. In the former case both $s_\lambda$ and $t_\lambda$ would be dihedral, a contradiction, so we must be in the latter case. Since we also have $t_\lambda^\vee \cong t_\lambda \otimes \det(t_\lambda)^{-1}$, we may write

$$r_\lambda(\pi) \cong t_\lambda \oplus t_\lambda \otimes \psi,$$

where $\psi = \epsilon^{-3} r_\lambda(\chi) \det(t_\lambda)^{-1}$. Since $r_\lambda(\pi)$ is de Rham, we see that $t_\lambda$ and $\psi$ are de Rham. Thus $\psi$ is pure of some weight. However, the representation $r_\lambda(\pi)$ is pure, so if $v \nmid l$ is a finite place of $F$ with $\pi_v$ unramified, then all of the eigenvalues of $r_\lambda(\pi)(\mathrm{Frob}_v)$ are Weil numbers of the same weight. This implies that $\psi$ must be pure of weight 0; but this contradicts the regularity of $r_\lambda(\pi)$.

## 5. Representations with small image

### 5.1.

With an eye to proving that $\bar{r}_\lambda(\pi)$ is irreducible for all but finitely many $\lambda$, in this section, we prove a variety of results on residually reducible or dihedral 2-dimensional representations, using class field theory and the main conjecture of Iwasawa theory.

Fix number fields $F$, $M$ such that $|\operatorname{Hom}_{\mathbb{Q}}(F,M)| = [F:\mathbb{Q}]$. Fix a positive integer $n$ and an element $\sigma \in (\mathbb{Z}^n)^{\operatorname{Hom}(F,M)}$. Let $\lambda$ be a place of $F$ with residue characteristic $l$ and residue field $k_\lambda$, and let $\rho_\lambda : G_F \to \operatorname{GL}_n(\overline{M}_\lambda)$ be a continuous de Rham representation. Then we say that $\rho$ has Hodge-Tate weights $\sigma$ if for each embedding $\tau \in \operatorname{Hom}(F,M)$ inducing a place $v$ of $F$ via $F \hookrightarrow M \hookrightarrow M_\lambda$, the $\tau$-Hodge-Tate weights of $\rho_\lambda|_{G_{F_v}}$ are $\sigma_\tau$. Similarly, we say that a continuous representation $\overline{\rho}_\lambda : G_F \to \operatorname{GL}_n(\overline{k}_\lambda)$ is Fontaine-Laffaille of weight $\sigma$ if $l$ is unramified in $F$, and for each $v|l$, $\overline{\rho}_\lambda|_{G_{F_v}}$ admits a crystalline lift with Hodge-Tate weights $\sigma$ in the Fontaine–Laffaille range (equivalently, it has Fontaine-Laffaille weights determined by $\sigma$ in the usual way).

Lemma 5.1. — *Let $F/\mathbb{Q}$ be a number field, and let $\sigma \in \mathbb{Z}^{\operatorname{Hom}(F,M)}$ denote a fixed weight. Suppose that there exist infinitely many primes $\lambda$ such that $G_F$ admits a character: $\chi_\lambda : G_F \to \overline{k}_\lambda^\times$ with the following properties:*

(1) *$\chi_\lambda$ is Fontaine-Laffaille of weight $\sigma$.*
(2) *The conductor of $\chi_\lambda$ away from $l$ is bounded independently of $l$.*

*Then, for infinitely many $\lambda$, there exists a finite order character $\phi$ of $G_F$ such that $\chi_\lambda \phi^{-1} = \overline{\psi_\lambda}$ for a fixed algebraic Hecke character $\psi$ of weight $\sigma$. If $F$ does not contain a CM field, then $\psi_\lambda$ is a power of the cyclotomic character times a finite order character.*

*Proof.* — The last sentence follows from the rest of the result by Weil's classification of algebraic Hecke characters. Note that since there are only finitely many finite order characters of $G_F$ with fixed ramification, it suffices to show that there is some algebraic Hecke character of weight $\sigma$. Without loss of generality we may assume that $M/\mathbb{Q}$ is Galois, and then by (the proof of) Weil's classification, it is enough to check that for any $g \in \operatorname{Gal}(M/\mathbb{Q})$, $g\sigma \otimes 1$ annihilates $\mathcal{O}_F^\times \otimes \mathbb{R}$. Replacing $\sigma$ by $g\sigma$ and each $\psi_\lambda$ by $g\psi_\lambda$, we see that it is enough to check that $\sigma \otimes 1$ annihilates $\mathcal{O}_F^\times \otimes \mathbb{R}$.

Regard each $\psi_\lambda$ as a character of $\mathbb{A}_F^\times/F^\times$ by class field theory. Since the ramification of the $\psi_\lambda$ outside of $l$ is bounded independently of $l$, we see that there is a finite index subgroup $U$ of $\mathcal{O}_F^\times$ such that each $\psi_\lambda|_U$ is just the composition of $\sigma$ and reduction mod $\lambda$. Thus, for any $u \in U$, we see that $(\sigma(u) - 1)$ is divisible by $\lambda$; since this holds for infinitely many $\lambda$, we see that $\sigma(u) = 1$. Since $U$ has finite index in $\mathcal{O}_F^\times$, the result follows. □

Corollary 5.2. — *Let $F$ be a totally real field. Suppose that there are infinitely many primes $l$ and 2-dimensional dihedral representations $\bar{s}_\lambda : G_F \to \mathrm{GL}_2(\overline{k}_\lambda)$ of fixed distinct Fontaine-Laffaille weights and fixed tame level. Then:*

(1) *For all but finitely many $l$, $\bar{s}_\lambda$ is induced from a quadratic CM extension $F'/F$ unramified at $l$. In particular, $\bar{s}_\lambda$ is (totally) odd. (The field $F'$ may depend on $l$.)*
(2) *For infinitely many $l$, $\bar{s}_\lambda$ is the reduction of the Galois representation associated to a fixed RAESDC automorphic representation $\pi_s$ of $\mathrm{GL}_2(\mathbb{A}_F)$ which arises from the induction of an algebraic Hecke character for $\mathbb{A}_{F'}^\times$, for some CM extension $F'/F$.*

*Proof.* — Each $\bar{s}_\lambda$ is induced from some quadratic extension $F_\lambda/F$ unramified outside of $l$ and a fixed set of places. By Lemma 2.6, for all but finitely many $\lambda$, $F_\lambda/F$ is unramified at places dividing $l$. Thus there are only finitely many possible extensions $F_\lambda/F$, so it suffices to show that any extension $F'/F$ which occurs infinitely often is CM. However, if $F'$ is not CM, then it does not contain a CM subfield, and Lemma 5.1 (applied to the characters of $G_{F'}$ from which the $s_\lambda$ are induced) contradicts the assumption that $s_\lambda$ has distinct Fontaine-Laffaille weights.

The second part now follows from Lemma 5.1 in the same way. □

Lemma 5.3. — *Let $F$ be a number field. Fix a dimension $n$ and distinct Fontaine–Laffaille weights, and fix a bound on the order of the projective images of the Galois representations under consideration (for example, suppose that the projective images are all $A_4$, $S_4$ or $A_5$). Then there are only finitely many $\lambda$ such that there exists an irreducible representation*

$$\overline{\rho}_\lambda : G_F \to \mathrm{GL}_n(\overline{k}_\lambda)$$

*such that $\overline{\rho}_\lambda$ has these fixed distinct Fontaine-Laffaille weights, and has projective image of bounded order.*

*Proof.* — Suppose to the contrary that there are infinitely many such representations. By Fontaine-Laffaille theory, we see that the order of the projective image of $\overline{\rho}_\lambda$ grows at least linearly with $l$; but it is also bounded, a contradiction. □

Lemma 5.4. — *Let $F$ be a totally real field. Then there are only finitely many $\lambda$ such that there exists an irreducible representation*

$$\rho_\lambda : G_F \to \mathrm{GL}_2(\overline{M}_\lambda)$$

*such that* $\det(\rho_\lambda(c_v))$ *is independent of any infinite place $v$ of $F$,* $\mathrm{ad}^0(\rho_\lambda)$ *is crystalline with fixed distinct Hodge-Tate weights and fixed tame level, and $\overline{\rho}_\lambda$ is reducible.*

*Proof.* — Assume not, so that infinitely many such representations exist. We may assume that $l$ is odd, unramified in $F$ and is sufficiently large that $\mathrm{ad}^0(\rho_\lambda)$ has Hodge-Tate weights in the Fontaine–Laffaille range. By Lemma 5.1, we may deduce that, for infinitely many $\lambda$, there is an isomorphism $(\overline{\rho}_\lambda)^{\mathrm{ss}} \simeq \psi\omega^k \oplus 1 \oplus \psi^{-1}\omega^{-k}$ for a fixed integer $k \neq 0$ and a fixed finite order character $\psi$. By Fontaine–Laffaille theory, we deduce that $\mathrm{ad}^0(\rho_\lambda)$ is ordinary at all primes $v|l$.

Suppose that $\overline{\rho}_\lambda \cong \psi\omega^k\phi \oplus \phi$. Applying Ribet's lemma, we obtain integral lattices in $\rho_\lambda$ which give nonzero classes in the groups $\mathrm{Ext}^1_{G_F}(\psi\omega^k\phi, \phi)$ and $\mathrm{Ext}^1_{G_F}(\phi, \psi\omega^k\phi)$. Consider the corresponding lattices in $\mathrm{ad}^0\, r_\lambda$. Looking at the "top extension" in $\mathrm{ad}^0(\overline{\rho}_\lambda)$, we obtain classes in the groups

$$H^1(F, \omega^k\psi), \qquad H^1(F, \omega^{-k}\psi^{-1}).$$

These classes are nonzero since $l \neq 2$, and are unramified outside $N$, $l$, and $\infty$ by construction. Moreover, in the second case, by the ordinarity of $\mathrm{ad}^0(\rho_\lambda)$ the class is also unramified and consequently trivial at all $v|l$. If $M$ is a $G_F$-module, we may define a set of Selmer conditions as follows. Let $\mathcal{L} = \{\mathcal{L}_v\}$ where $\mathcal{L}_v \subset H^1(G_v, M)$ is defined to be:

(1) $\mathcal{L}_v = H^1(G_v/I_v, M^{I_v})$ if $v \nmid l$.
(2) $\mathcal{L}_v = 0$ if $v|l$.

We note the following:

Proposition 5.5. — *Fix a place $v$, an integer $m \notin \{0, 1\}$, and a finite order character $\chi$. Let $\omega$ denote the mod-$l$ cyclotomic character. Then $H^1(F_v, \omega^m \cdot \chi) = 0$ for sufficiently large $l$.*

*Proof.* — We may assume that $v \nmid l$. Let $q = N(v)$ and let $p$ be the residue characteristic of $v$. There is an equality

$$|H^1(F_v, M)| = |H^0(F_v, M)||H^2(F_v, M)| = |H^0(F_v, M)||H^0(F_v, M^*)|.$$

Hence, if $H^1(F_v, \omega^m \cdot \chi)$ is non-trivial, then $\chi$ is unramified at $v$, and

$$\omega^m(\mathrm{Frob}_v)\chi(\mathrm{Frob}_v) \in \{1, q\}.$$

Since $\omega(\mathrm{Frob}_v) = q$, it follows that $\chi(\mathrm{Frob}_v) \in \{q^{-m}, q^{1-m}\}$. If $\chi$ has order $d$, it follows that

$$(q^{dm} - 1)(q^{d(m-1)} - 1) \equiv 0 \pmod{l}.$$

Since $m \notin \{0, 1\}$, this equality can only occur for finitely many $l$. □

It follows that for sufficiently large $l$ the classes constructed above lie in the Selmer groups $H^1_{\mathcal{L}^*}(F, \omega^k\psi)$ and $H^1_{\mathcal{L}}(F, \omega^{-k}\psi^{-1})$ respectively, where $\mathcal{L}^*$ is the dual Selmer condition (with no restriction on the class at $v|l$), with the possible exception of the class in $H^1(F, \omega\psi)$ when $k = 1$. We now consider three cases.

(1) Suppose that $\omega^k\psi$ is (totally) odd. Then the main conjecture for totally real fields as proven by Wiles [28] shows that (for $l$ odd) $l$ divides $|H^1_{\mathcal{L}}(F, \omega^{-k}\psi^{-1})|$ if and only if $l$ divides

$$L(0, \omega^{-k}\psi^{-1}) \equiv L(-k, \psi^{-1}) \neq 0.$$

(2) Suppose that $\omega^k\psi$ is even and $k > 1$. Then, by Theorem 2.19 of [11] (the global duality formula for Selmer groups, which is a reflection formula in this case), we deduce that

$$|H^1_{\mathcal{L}^*}(F, \omega^k\psi)| = |H^1_{\mathcal{L}}(F, \omega^{1-k}\psi^{-1})|.$$

Once more by Wiles this group is non-trivial if and only if $l$ divides

$$L(0, \omega^{1-k}\psi^{-1}) \equiv L(1-k, \psi^{-1}) \neq 0.$$

(3) Suppose that $\omega^k\psi$ is even and $k = 1$. Then we still have a class in $H^1_{\mathcal{L}}(F, \omega^{-1}\psi^{-1})$. Let $E = F(\psi)$. Then, by restriction, we obtain a class in

$$H^1_{\mathcal{L}}(F, \omega^{-1}\psi^{-1}) \hookrightarrow H^1_{\mathcal{L}}(E, \omega^{-1}) \hookrightarrow H^1_{\mathcal{L}}(E(\zeta_l), \mathbb{F}_l)^{\omega^{-1}}.$$

The latter group is isomorphic to the $\omega^{-1}$-part of the $l$-torsion of the class group of $E(\zeta_l)$. Yet, by Theorem 5.4 of [19], the $\omega^{-1}$ part of this group injects into $K_2(\mathcal{O}_E)/l$. Since $K_2(\mathcal{O}_E)$ is finite, this group is trivial for $l$ sufficiently large.

In each case, we deduce that $l$ divides a fixed non-zero rational number which is independent of $l$, and hence $l$ is bounded. □

## 6. Residual irreducibility for all but finitely many primes

### 6.1.

We now bootstrap our previous arguments to prove the following result.

Theorem 6.1. — *Let $F$ be a totally real field. Suppose that $(\pi, \chi)$ is a RAESDC automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$ with $n \leqslant 5$. Then all but finitely many of the residual representations $\bar{r}_\lambda(\pi)$ are irreducible.*

We firstly establish the following corollary of Lemma 4.8 and Corollary 4.9.

Lemma 6.2. — *Let $F$ be a totally real field. Let $(\pi, \chi)$ be a RAESDC automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$, $n \leqslant 5$, and let $\mathcal{R} = \{r_\lambda(\pi)\}$ be the associated weakly compatible system. Suppose that, for infinitely many primes $\lambda$, at least one of the following holds:*

(1) *$\bar{r}_\lambda(\pi)^{\mathrm{ss}}$ contains a character.*
(2) *$\bar{r}_\lambda(\pi)^{\mathrm{ss}}$ contains a two dimensional dihedral representation.*

*Then, respectively, at least one of the following also holds:*

(1) *$\mathcal{R}$ is weakly divisible by a compatible system of algebraic Hecke characters.*
(2) *For some finite extension $E/F$, $\mathcal{R}|_{G_E}$ is weakly divisibly by a direct sum of two compatible systems of algebraic Hecke characters.*

*In particular, $\bar{r}_\lambda(\pi)$ is irreducible for all but finitely many $\lambda$.*

*Proof.* — Denote the corresponding sub-representations of $\bar{r}_\lambda(\pi)^{\mathrm{ss}}$ by $\bar{s}_\lambda$. If the Hodge-Tate weights of $r_\lambda(\pi)$ are in the Fontaine–Laffaille range, then there are a fixed number of possible Fontaine-Laffaille weights of $\bar{s}_\lambda$, which are independent of $\lambda$. Similarly, there are finitely many possible Serre levels determined by the auxiliary ramification structure of $\mathcal{R}$. The result follows by Lemma 5.1 and Corollary 5.2 respectively (with the last sentence following from Lemma 4.8). □

We now prove Theorem 6.1. Assume for the sake of contradiction that there are infinitely many $\lambda$ with $\bar{r}_\lambda(\pi)$ reducible. By Lemma 6.2, there can only be finitely many $\lambda$ such that $\bar{r}_\lambda(\pi)^{\mathrm{ss}}$ contains a character. This is already a contradiction when $n \leqslant 3$, and when $n = 4$ or $n = 5$ it implies that there are infinitely many $\lambda$ for which $\bar{r}_\lambda(\pi)^{\mathrm{ss}} \cong \bar{s}_\lambda \oplus \bar{t}_\lambda$ with $\bar{s}_\lambda$ and $\bar{t}_\lambda$ both irreducible, $\dim \bar{s}_\lambda = 2$, and neither of $\bar{s}_\lambda$ and $\bar{t}_\lambda$ are dihedral.

### 6.2. The case $n = 5$

Since 5 is odd, we see from Lemma 2.1 that $r_\lambda(\pi)$ and thus $\bar{r}_\lambda(\pi)$ factors through $\mathrm{GO}_5$ with even multiplier. Since $2 \neq 3$, we see that $\bar{s}_\lambda$ factors through $\mathrm{GO}_2$ with even multiplier, so $\bar{s}_\lambda$ is dihedral. This can only happen for finitely many $\lambda$ by Lemma 6.2.

### 6.3. The $n = 4$ symplectic case

We argue as in section 4.3. Let $\mathcal{R}$ be the compatible system of Galois representations associated to $(\pi, \chi)$, and define $\mathcal{A} := \wedge^2(\mathcal{R}) - \chi$, a compatible system of Galois representations such that $a_\lambda : G_F \to \mathrm{GL}_5(\overline{\mathbb{Q}}_l)$ has image in $\mathrm{GO}_5(\overline{\mathbb{Q}}_l)$. Again, this compatible system is odd and regular.

Since $\bar{r}_\lambda(\pi) = \bar{s}_\lambda \oplus \bar{t}_\lambda$, we have

$$\overline{a}_\lambda \oplus \bar{\chi}_\lambda = \bar{s}_\lambda \otimes \bar{t}_\lambda \oplus \det(\bar{s}_\lambda) \oplus \det(\bar{t}_\lambda).$$

In particular, as there are two characters on the right hand side, the representation $\overline{a}_\lambda$ contains a character for infinitely many $\lambda$, and as in the proof of Lemma 6.2, we deduce that the compatible system $\mathcal{A}$ is weakly divisible by the compatible system of a character.

Arguing as in section 4.3, we deduce that there is a finite Galois extension $F'/F$ of totally real fields and a RAESDC automorphic representation $\pi'$ of $\mathrm{GL}_2(\mathbb{A}_{F'})$, which is not of CM type, such that, for all $\lambda$, we have $a_\lambda|_{G_{F'}} \cong \mathrm{Sym}^4 r_\lambda(\pi') \otimes \det(r_\lambda(\pi'))^{-2}$. Then for all but finitely many $\lambda$ the projective image of $\bar{r}_\lambda(\pi')$ contains $\mathrm{PSL}_2(\mathbb{F}_l)$ and $\overline{a}_\lambda|_{G_{F'}}$ is irreducible, a contradiction.

### 6.4. The $n = 4$ orthogonal case

It will be useful in the sequel to exploit the exceptional isomorphism of Lie groups $\mathfrak{so}_4 \simeq \mathfrak{sl}_2 \times \mathfrak{sl}_2$. More precisely:

Lemma 6.3. — *Let $F$ be a number field. Suppose that $r : G_F \to \mathrm{GO}_4(\overline{\mathbb{Q}}_l)$ is a continuous representation. Then either:*

(1) *there are continuous representations $a$, $b : G_F \to \mathrm{GL}_2(\overline{\mathbb{Q}}_l)$ with $r \cong a \otimes b$, or*

(2) *there is a quadratic extension $K/F$ and a continuous representation $a : G_K \to \mathrm{GL}_2(\overline{\mathbb{Q}}_l)$ with $r|_{G_K} \cong a \otimes a^c$, where $\mathrm{Gal}(K/F) = \{1, c\}$.*

*Proof.* — We have an exact sequence

$$0 \to \overline{\mathbb{Q}}_l^\times \to \mathrm{GL}_2(\overline{\mathbb{Q}}_l) \times \mathrm{GL}_2(\overline{\mathbb{Q}}_l) \to \mathrm{GO}_4(\overline{\mathbb{Q}}_l) \to \{\pm 1\} \to 0$$

(*cf.* section 1 of [21]). Suppose firstly that the composite $r : G_F \to \mathrm{GO}_4(\overline{\mathbb{Q}}_l) \to \{\pm 1\}$ is not surjective. Then the obstruction to lifting $r$ to a homomorphism $G_F \to \mathrm{GL}_2(\overline{\mathbb{Q}}_l) \times \mathrm{GL}_2(\overline{\mathbb{Q}}_l)$ lies in $H^2(G_F, \overline{\mathbb{Q}}_l^\times)$, which vanishes by the proof of Theorem 5.4 of [9]. If the composite $r : G_F \to \mathrm{GO}_4(\overline{\mathbb{Q}}_l) \to \{\pm 1\}$ is surjective, then we let $G_K$ be the kernel of this composite, and the result follows as in the previous case. $\square$

By Lemma 6.3, we may assume either that for infinitely many $\lambda$ we have $r_\lambda \cong a_\lambda \otimes b_\lambda$ for some $a_\lambda$, $b_\lambda : G_F \to \mathrm{GL}_2(\overline{\mathbb{Q}}_l)$, or that for infinitely many $\lambda$ there is a quadratic extension $K = K_\lambda/F$ with $r|_{G_K} \cong a_\lambda \otimes a_\lambda^c$ for some $a_\lambda : G_K \to \mathrm{GL}_2(\overline{\mathbb{Q}}_l)$.

Suppose that we are in the first case. If $\overline{a}_\lambda$ and $\overline{b}_\lambda$ are both reducible, then the semi-simplification of $\overline{a}_\lambda \otimes \overline{b}_\lambda$ consists of four characters, contrary to assumption. Hence, without loss of generality, we may assume that $\overline{b}_\lambda$ is irreducible infinitely often. There is an isomorphism

$$\wedge^2 r_\lambda(\pi) \otimes (\det r_\lambda(\pi))^{-1} \simeq \mathrm{ad}^0\, a_\lambda \oplus \mathrm{ad}^0\, b_\lambda,$$

from which we see that $\mathrm{ad}^0\, a_\lambda$ and $\mathrm{ad}^0\, b_\lambda$ are both de Rham, and are crystalline for all but finitely many $\lambda$. They are both regular (this can be seen from a consideration of the Hodge-Sen-Tate weights of $a_\lambda$ and $b_\lambda$).

After passing to a subset, we may assume that the Hodge-Tate weights of $\mathrm{ad}^0(a_\lambda)$ and $\mathrm{ad}^0(b_\lambda)$ are independent of $\lambda$. Assume firstly that $\overline{a}_\lambda$ and $\overline{b}_\lambda$ are irreducible for infinitely many $\lambda$.

By Lemma 5.3 and the classification of finite subgroups of $\mathrm{PGL}_2(\overline{\mathbb{F}}_l)$ (*cf.* Theorem 2.47 of [11]), we may assume that $\overline{a}_\lambda$ and $\overline{b}_\lambda$ are either dihedral or have images containing $\mathrm{SL}_2(\mathbb{F}_l)$.

If both $\overline{a}_\lambda$ and $\overline{b}_\lambda$ have image containing $\mathrm{SL}_2(\mathbb{F}_l)$, then $\overline{a}_\lambda \otimes \overline{b}_\lambda$ is either irreducible or breaks up as a sum of a character and a 3-dimensional representation, a contradiction. If without loss of generality $\overline{a}_\lambda$ is dihedral and $\overline{b}_\lambda$ has image containing $\mathrm{SL}_2(\mathbb{F}_l)$, let $K/F$ be the quadratic extension from which $\overline{a}_\lambda$ is induced. Then $\overline{b}_\lambda|_{G_K}$ is irreducible, so $\overline{a}_\lambda \otimes \overline{b}_\lambda$ is irreducible, a contradiction. The remaining case is that $\overline{a}_\lambda$ and $\overline{b}_\lambda$ are both dihedral. Then $\bar{r}_\lambda(\pi)$ is completely decomposable over some quartic extension, which implies that $\bar{s}_\lambda$ and $\bar{t}_\lambda$ are both dihedral, a contradiction.

We may thus assume that for infinitely many $\lambda$, $\overline{a}_\lambda$ is reducible and $\overline{b}_\lambda$ is irreducible. If $\overline{b}_\lambda$ is dihedral then one of $\bar{s}_\lambda$ and $\bar{t}_\lambda$ is dihedral, which can only occur finitely often, so by Lemma 5.3 (applied to $\mathrm{ad}^0\, b_\lambda$) we may assume that the image of $\overline{b}_\lambda$ contains $\mathrm{SL}_2(\mathbb{F}_l)$. Then for $\lambda$ sufficiently large $\mathrm{ad}^0\, \overline{b}_\lambda$ is irreducible, so as in the proof of Proposition 2.5 we see that $\mathrm{ad}^0\, b_\lambda$ is potentially automorphic and $b_\lambda$ is odd. Since the multiplier character of $r_\lambda(\pi) = a_\lambda \otimes b_\lambda$ is even, we see that $\det a_\lambda(c_v)$ is independent of $v|\infty$. Then Lemma 5.4 implies that there are only finitely many $\lambda$ for which such an $a_\lambda$ can exist.

This contradiction means that we may assume that we are in the second case, so that for infinitely many $\lambda$, there is a quadratic extension $K/F$ (which might depend on $\lambda$) and a continuous representation $a_\lambda : G_K \to \mathrm{GL}_2(\overline{\mathbb{Q}}_l)$ such that $r_\lambda(\pi)|_{G_K} \cong a_\lambda \otimes a_\lambda^c$.

We claim that $\overline{a}_\lambda$ is necessarily dihedral for all but finitely many of the $\lambda$ under consideration. We prove this by eliminating the other possibilities. Firstly, $\overline{a}_\lambda$ cannot be reducible, because if $\overline{a}_\lambda \cong \bar{\phi} \oplus \bar{\chi}$, then

$$\overline{a}_\lambda \otimes \overline{a}_\lambda^c \cong \bar{\phi} \otimes \bar{\phi}^c \oplus \bar{\chi} \otimes \bar{\chi}^c \oplus \bar{\phi} \otimes \bar{\chi}^c \oplus \bar{\chi} \otimes \bar{\phi}^c,$$

and the first two characters descend to $\mathbb{Q}$, so that $\bar{r}_\lambda(\pi)$ would have one-dimensional subrepresentations.

If $\overline{a}_\lambda$ has projective image $A_4$, $S_4$ or $A_5$, then the projective image of $\bar{r}_\lambda(\pi)$ is bounded independently of $\lambda$. By Lemma 5.3, this can only happen for finitely many $\lambda$.

The image of $\overline{a}_\lambda$ cannot contain $\mathrm{SL}_2(\mathbb{F}_\ell)$. If it did, then $\overline{a}_\lambda \otimes \overline{a}_\lambda^c$ would either be irreducible or a sum of an irreducible three-dimensional representation and a character, depending on whether the projective representation associated to $\overline{a}_\lambda$ extends to $F$ or not. This again contradicts the assumption that $\bar{r}_\lambda(\pi)$ is a sum of two irreducible 2-dimensional representations.

Having eliminated the other possibilities, we see that for infinitely many $\lambda$, $\overline{a}_\lambda$ is dihedral. Then for infinitely many $\lambda$, $\bar{s}_\lambda$ and $\bar{t}_\lambda$ each become reducible over quartic extensions of $F$, and are thus dihedral. This contradiction completes the proof.

## 7. Lie algebras

In this section we prove that the Lie algebras of the images of the $r_\lambda(\pi)$ are independent of $\lambda$. More specifically, we prove the following theorem.

Theorem 7.1. — *Let $F$ be a totally real field. Suppose that $(\pi, \chi)$ is a RAESDC automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$ with $n \leqslant 5$. Let $G_\lambda$ denote the Zariski closure of the image $r_\lambda(\pi)(G_F)$, and let $\mathfrak{g}_\lambda = \mathfrak{z}_\lambda \oplus \mathfrak{h}_\lambda$ denote the Lie algebra of $G_\lambda$, where $\mathfrak{z}_\lambda$ is abelian and $\mathfrak{h}_\lambda$ is semisimple. Then $\mathfrak{g}_\lambda$ is independent of $\lambda$, and $\mathfrak{h}_\lambda$ is either $\mathfrak{sl}_2$, $\mathfrak{so}_4$ or $\mathfrak{sp}_4$ (if $n = 4$), or $\mathfrak{so}_5 = \mathfrak{sp}_4$ (if $n = 5$).*

*Proof.* — The result is trivial if $n = 1$ and standard if $n = 2$, so we may assume that $n \geqslant 3$. If $(\pi, \chi)$ is the automorphic induction of a character, then certainly $\mathfrak{g}_\lambda$ is abelian and independent of $\lambda$. If $n = 4$ and $(\pi, \chi)$ is the automorphic induction of a RAESDC automorphic representation of $\mathrm{GL}_2(\mathbb{A}_{F'})$ for $F'/F$ a quadratic totally real extension, then either this representation is of CM type, and $(\pi, \chi)$ is the automorphic induction of a character, or $\mathfrak{g}_\lambda$ is independent of $\lambda$ and $\mathfrak{h}_\lambda$ is equal to $\mathfrak{sl}_2$ (acting reducibly).

Excluding these cases, by Corollary 4.4 (and its proof) we may assume that $(\pi, \chi)$ is not an automorphic induction, and that $r_\lambda(\pi)$ is strongly irreducible. In general the independence of $\mathfrak{z}_\lambda$ of $\lambda$ is an easy consequence of Schur's lemma. Therefore we need only determine $\mathfrak{h}_\lambda$. Suppose firstly that the compatible system $\{r_\lambda(\pi)\}$ is weakly divisible by a compatible system of algebraic Hecke characters. Then by Lemma 4.8 we see that $n = 3$ or 5, and that $\mathfrak{h}_\lambda = \mathfrak{sl}_2$, acting through the $(n-1)$st symmetric power representation, independently of $\lambda$.

Conversely, if $n = 3$ or 5 and for some $\lambda$ we have $\mathfrak{h}_\lambda = \mathfrak{sl}_2$ acting through the $(n-1)$st symmetric power representation, then we claim that the compatible system $\{r_\lambda(\pi)\}$ is weakly divisible by a compatible system of algebraic Hecke characters. To see this, write $G$ for the Zariski closure of $r_\lambda(\pi)(G_F)$, and $G^0$ for the connected component of the identity. Then the derived subgroup of $G^0$ must be $\mathrm{PSL}_2$, and since $\mathrm{PSL}_2$ has no outer automorphisms, Schur's lemma shows that $G$ is necessarily of the form $Z(G) \times \mathrm{PSL}_2$. Since $Z(G)$ acts via a character (again by Schur's lemma), we see that the compatible system $\{r_\lambda(\pi)\}$ is weakly divisible by a compatible system of algebraic Hecke characters, as required.

Examining the table in Proposition 4.5, we see that we are done unless $n = 4$. In this case if $\chi$ is odd then each $r_\lambda(\pi)$ has even multiplier and is thus orthogonal, and we see from the same table that $\mathfrak{h}_\lambda = \mathfrak{so}_4$ for all $\lambda$. If $\chi$ is even then for each $\lambda$ either $\mathfrak{h}_\lambda = \mathfrak{sl}_2$ (acting via $\mathrm{Sym}^3$) or $\mathfrak{h}_\lambda = \mathfrak{sp}_4$. We distinguish between these two possibilities by arguing as in Section 4.3. Consider the compatible system $\mathcal{A} := \wedge^2(\mathcal{R}) - \chi$. This is a compatible system of odd, regular Galois representations such that $a_\lambda : G_F \to \mathrm{GL}_5(\overline{\mathbb{Q}}_l)$ has image in $\mathrm{GO}_5(\overline{\mathbb{Q}}_l)$. If for some $\lambda$ we have $\mathfrak{h}_\lambda = \mathfrak{sl}_2$ then as above the compatible system $\mathcal{A}$ is weakly divisible by a compatible system of algebraic characters, and the argument of the proof of Lemma 4.8 shows that $\mathfrak{h}_\lambda = \mathfrak{sl}_2$ for all $\lambda$, as required. □

## 8. Non self-dual representations of $\mathrm{GL}_3$ and $\mathrm{GL}_4$

In this section, we follow [23] and sketch a proof that our earlier irreducibility results extend to the case of regular algebraic cuspidal automorphic representations $\pi$ of $\mathrm{GL}_3(\mathbb{A}_F)$ or $\mathrm{GL}_4(\mathbb{A}_F)$, $F$ a totally real field, without assuming that $\pi$ is essentially self-dual, but with the assumption that the Galois representations $r_\lambda(\pi)$ exist.

Assume throughout this section that $F$ is totally real, that $n = 3$ or 4 and that there is a weakly compatible system $\{r_\lambda(\pi)\}$ of Galois representations associated to $\pi$. We will demonstrate the required irreducibility results by reducing to the essentially self-dual case.

Note firstly that the proof of Corollary 4.4 made no use of the essential self-duality of $\pi$, so we have the following.

LEMMA 8.1. — *Let $\lambda$ be a prime such that $r_\lambda(\pi)$ is irreducible. Then either:*

(1) *$r_\lambda(\pi)$ is strongly irreducible, or*
(2) *$\pi$ is an automorphic induction, $r_\lambda(\pi)$ is irreducible for all $\lambda$, and $\bar{r}_\lambda(\pi)$ is irreducible for all but finitely many $\lambda$.*

The following Lemma and Corollary will be our main tool to reduce to the essentially self-dual case.

LEMMA 8.2. — *Suppose that $r : G_F \to \mathrm{GL}_4(\overline{\mathbb{Q}}_l)$ is strongly irreducible, and suppose that $\wedge^2 r : G_F \to \mathrm{GL}_6(\overline{\mathbb{Q}}_l)$ is not strongly irreducible. Then $r \simeq r^\vee \chi$ for some character $\chi$.*

*Proof.* — This is a standard argument (*cf.* Theorem 6.5 of [2]). Consider the Zariski closure $G$ of the image of $r$. Let $G^0$ denote the connected component of $G$, let $\mathfrak{g}$ be the Lie algebra of $G^0$, and write $\mathfrak{g} = \mathfrak{z} \oplus \mathfrak{h}$, with $\mathfrak{z}$ is abelian and $\mathfrak{h}$ semisimple. By assumption, $G^0$ acts irreducibly in dimension 4. If $\wedge^2 r$ is not strongly irreducible, then $\mathfrak{h} = \mathfrak{sl}_2$, $\mathfrak{so}_4$, or $\mathfrak{sp}_4$. It follows that $G^0$ preserves a symplectic or orthogonal form, from which it is easy to deduce (using, for example, the facts that the normalizers of $\mathrm{Sp}_4$ and $\mathrm{SO}_4$ in $\mathrm{GL}_4$ are respectively $\mathrm{GSp}_4$ and $\mathrm{GO}_4$) that the image of $r$ is symplectic or orthogonal, as required. □

COROLLARY 8.3. — *Suppose that $n = 4$, that $\pi$ is not essentially self-dual, and that for some $\lambda$, $r_\lambda(\pi)$ is strongly irreducible. Then $\wedge^2 r_\lambda(\pi)$ is strongly irreducible.*

*Proof.* — Suppose that $\wedge^2 r_\lambda(\pi)$ is not strongly irreducible. By Lemma 8.2, we see that $r_\lambda(\pi) \simeq r_\lambda(\pi)^\vee \chi_\lambda$ for some character $\chi_\lambda$; but then strong multiplicity one for $\mathrm{GL}_4$ implies that $\pi$ is essentially self-dual, a contradiction. □

LEMMA 8.4. — *If $n = 4$, then it is impossible for $r_\lambda(\pi)$ to have a 1-dimensional summand.*

*Proof.* — This may be proved in exactly the same way as Proposition 7.8 of [23]. Suppose that $r_\lambda(\pi) = \chi_\lambda \oplus s_\lambda$ with $\chi_\lambda$ a character. Then $\chi_\lambda$ and $\det s_\lambda$ are both algebraic characters of $G_F$, so arise from automorphic representations $\chi$ and $\nu$ of $\mathrm{GL}_1(\mathbb{A}_F)$. One easily obtains an equality of incomplete

$L$-functions

$$L^S(s,\pi\otimes\chi^{-1})L^S(s,\pi^\vee\otimes\nu\chi^{-1})=L^S(s,\wedge^2(\pi)\otimes\chi^{-1})\zeta^S(s)L^S(s,\nu\chi^{-3}).$$

Since $\wedge^2(\pi)$ is an isobaric automorphic representation of $\mathrm{GL}_6(\mathbb{A}_F)$ by [20], we see that the left hand side is holomorphic at $s=1$, but the right hand side has at least a simple pole at $s=1$, a contradiction. □

Lemma 8.5. — *For a density one set of primes $\lambda$, $r_\lambda(\pi)$ is irreducible.*

*Proof.* — Suppose not. By Lemma 8.4, there is a set of primes $\lambda$ of positive density such that $r_\lambda(\pi)$ decomposes as a sum of irreducible representations of dimension at most 2. The result now follows by the same proof as Theorem 3.2. □

Theorem 8.6. — *$r_\lambda(\pi)$ is irreducible for all $\lambda$.*

*Proof.* — Let $\mathcal{R}$ denote the compatible system $\{r_\lambda(\pi)\}$. By Theorem 4.1, we may assume that $\pi$ is not essentially self-dual. By Lemmas 8.1 and 8.5, and Proposition 5.2.2 of [4], we may assume that $r_\lambda(\pi)$ is strongly irreducible for some $\lambda$. Then the proof of Lemma 4.8 goes through verbatim, as does that of Corollary 4.9, and we see that it is impossible for $r_{\lambda'}(\pi)$ to have a one-dimensional summand for any $\lambda'$.

We are done if $n=3$. If $n=4$, the only possibility is that for some $\lambda'$, $r_{\lambda'}(\pi)\cong s_{\lambda'}\oplus t_{\lambda'}$, with $s_\lambda$ and $t_\lambda$ both 2-dimensional. Since we have assumed that $\pi$ is not essentially self-dual, we see from Corollary 8.3 that (using the same $\lambda$ as in the first paragraph) $\wedge^2 r_\lambda(\pi)$ is strongly irreducible. On the other hand,

$$\wedge^2 r_{\lambda'}(\pi)\cong s_{\lambda'}\otimes t_{\lambda'}\oplus\det(s_{\lambda'})\oplus\det(t_{\lambda'}).$$

It follows that the compatible system $\wedge^2\mathcal{R}$ is weakly divisible by the compatible system of a character (in fact, two characters). After twisting, we may suppose that this character is trivial. We then obtain a contradiction as in the proof of Theorem 4.7 (note that the semisimple part of the Lie algebra of the Zariski closure of $\wedge^2 r_\lambda(\pi)(G_F)$ is $\mathfrak{sl}_4$ acting via $\wedge^2$, and it is not the case that every element of $\mathfrak{sl}_4$ fixes some vector under this action). □

Theorem 8.7. — *For all but finitely many $\lambda$, $\bar{r}_\lambda(\pi)$ is irreducible.*

*Proof.* — By Theorem 6.1, it is enough to assume that $\pi$ is not essentially self-dual. Again, the proof of Lemma 6.2 goes over without change to the present setting, completing the proof if $n=3$. If $n=4$, it suffices to show that there cannot be infinitely many $\lambda$ for which $\bar{r}_\lambda(\pi)=\bar{s}_\lambda\oplus\bar{t}_\lambda$ with $s_\lambda$ and $t_\lambda$ 2-dimensional. However, we again note that in this case we have

$$\wedge^2 r_\lambda(\pi)=\bar{s}_\lambda\otimes\bar{t}_\lambda\oplus\det(\bar{s}_\lambda)\oplus\det(\bar{t}_\lambda),$$

and we deduce that the compatible system $\{\wedge^2 r_\lambda(\pi)\}$ is weakly divisible by the compatible system of a character. This gives a contradiction as in the proof of Theorem 8.6. □

Theorem 8.8. — *Let $\mathfrak{g}_\lambda$ be the Lie algebra of the Zariski closure of $r_\lambda(G_F)$. Then $\mathfrak{g}_\lambda$ is independent of $\lambda$.*

*Proof.* — As in the proof of theorem Theorem 7.1, if we write $\mathfrak{g}_\lambda = \mathfrak{h}_\lambda \oplus \mathfrak{z}_\lambda$ with $\mathfrak{h}_\lambda$ semisimple and $\mathfrak{z}_\lambda$ abelian, it suffices to show that $\mathfrak{h}_\lambda$ is independent of $\lambda$.

If $n = 3$, then the proof of Theorem 7.1 goes through unchanged, so we may assume that $n = 4$. By Theorem 7.1, we may assume that $\pi$ is not essentially self-dual, and by Lemma 8.1 and Theorem 8.6, we may assume that $r_\lambda(\pi)$ is strongly irreducible for all $\lambda$. It then follows from the proofs of Lemma 8.2 and Corollary 8.3 that $\mathfrak{h}_\lambda = \mathfrak{sl}_4$ for all $\lambda$, as required. □

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