ANNALES

DE

L'INSTITUT FOURIER

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Tome 63, n° 6 (2013), p. 2199-2221.

<http://aif.cedram.org/item?id=AIF_2013___63_6_2199_0>

Ann. Inst. Fourier, Grenoble
63, 6 (2013) 2199-2221

ASYMPTOTIC COHOMOLOGY VANISHING AND A CONVERSE TO THE ANDREOTTI-GRAUERT THEOREM ON SURFACES

by Shin-ichi MATSUMURA

ABSTRACT. — In this paper, we study relations between positivity of the curvature and the asymptotic behavior of the higher cohomology group for tensor powers of a holomorphic line bundle. The Andreotti-Grauert vanishing theorem asserts that partial positivity of the curvature implies asymptotic vanishing of certain higher cohomology groups. We investigate the converse implication of this theorem under various situations. For example, we consider the case where a line bundle is semi-ample or big. Moreover, we show the converse implication holds on a projective surface without any assumptions on a line bundle.

RÉSUMÉ. — Dans cet article, nous étudions les relations entre la positivité de la courbure et le comportement asymptotique de la cohomologie de degré supérieur des puissances tensorielles d'un fibré en droites holomorphe. Le théorème d'annulation d'Andreotti-Grauert affirme que la positivité partielle de la courbure implique l'annulation asymptotique de la cohomologie de certains degrés supérieurs. Nous étudions la réciproque de ce théorème dans plusieurs situations. Par exemple, nous considérons le cas d'un fibré en droite semi-ample ou gros. De plus, nous montrons que la réciproque du théorème d'Andreotti-Grauert est vraie sur les surfaces projectives sans aucune hypothèse sur le fibré en droites.

1. Introduction

In complex geometry, the positivity concept plays an important role. In particular, a positive line bundle is fundamental and important in the theory of several complex variables and algebraic geometry. For this reason, a positive line bundle has been characterized in various ways. For example, some positive multiple gives an embedding to the projective space (geometric characterization), all higher cohomology groups of some positive multiple are zero (cohomological characterization), and the intersection number

Keywords: Asymptotic cohomology groups, partial cohomology vanishing, q-positivity, hermitian metrics, Chern curvatures.
Math. classification: 14C20, 14F17, 32L15.

with any subvariety is positive (numerical characterization). The purpose of this paper is to generalize these characterizations to a *q-positive* line bundle.

Throughout this paper, let X be a compact complex manifold of dimension n, L a (holomorphic) line bundle on X and q an integer with $0 \leqslant q \leqslant n-1$. Sometimes we may assume that X is Kähler or projective.

In this paper, we study relations between the q-positivity and the cohomological q-amplitude of a line bundle. The fundamental relations are discussed in [9]. Küronya and Totaro investigated the cohomological q-amplitude of a line bundle in terms of algebraic geometry (see [13], [21]). We consider a q-ample line bundle in terms of complex geometry. Let us recall the definition of a q-positive (cohomologically q-ample) line bundle.

Definition 1.1. — *(1) A holomorphic line bundle L on X is called q-positive, if there exists a (smooth) hermitian metric h whose Chern curvature $\sqrt{-1}\Theta_h(L)$ has at least $(n-q)$ positive eigenvalues at any point on X as a $(1,1)$-form.*

(2) A holomorphic line bundle L on X is called cohomologically q-ample, if for any coherent sheaf $\mathcal{F}$ on X there exists a positive integer $m_0 = m_0(\mathcal{F}) > 0$ such that

$$H^i\big(X, \mathcal{F} \otimes \mathcal{O}_X(L^{\otimes m})\big) = 0 \quad \text{for } i > q,\ m \geqslant m_0.$$

A 0-positive line bundle is a positive line bundle in the usual sense. Further, it follows from the Serre vanishing theorem that a cohomologically 0-ample line bundle is an ample line bundle in the usual sense of algebraic geometry. Thanks to the Kodaira embedding theorem, we know that a positive line bundle coincides with an ample line bundle. We attend to generalize this relation to a q-positive line bundle.

Andreotti and Grauert proved that a q-positive line bundle is always q-ample. (see [1, Théorème 14], [9, Proposition 2.1]). It is of interest to know whether the converse implication of the Andreotti-Grauert theorem holds. In this paper, we mainly discuss the following problem.

Problem 1.2. — ([9, Problem 2.2]). *Does the converse implication of the Andreotti-Grauert theorem hold ? That is to say, is a q-ample line bundle always q-positive ?*

This problem was first posed by Demailly, Peternell and Schneider in [9]. Precisely speaking, they consider a uniformly q-ample line bundle. However, Totaro showed that the uniform q-amplitude is the same concept as the cohomological q-amplitude (see [21, Theorem 6.2]). When q is zero, this

problem is affirmative. Therefore it is a natural question. However it has been an open problem for a long time, except the case of $q = 0$.

In Section 2, we study this problem when X is a smooth projective surface. The main result of this section is an affirmative answer for Problem 1.2 on a surface (Theorem 1.3).

THEOREM 1.3. — *On a smooth projective surface X, the converse of the Andreotti-Grauert theorem holds. That is, the following conditions are equivalent.*
(A) *L is cohomologically q-ample.*
(B) *L is q-positive.*

For the proof of Theorem 1.3, we establish Theorem 2.1. Theorem 2.1 also leads to Corollary 2.6, which can be seen as a generalization of [12, Theorem 1]. The idea of the proof of Theorem 2.1 is to use a solution of the global equation (the Monge-Ampère equation).

In his paper [8], Demailly proved the converse of the holomorphic Morse inequality under various situations. These results can be seen as a "partial" converse of the Andreotti-Grauert theorem. The original part of this paper is to give an "exact" converse (see [8] and Section 5 of this paper for the precise statement). By combining Theorem 1.3 and the result of [8], the asymptotic behavior of the higher cohomology on a surface can be interpreted in terms of the curvature.

In Section 3, various characterizations of the q-positivity of a semi-ample line bundle are given on an arbitrary compact complex manifold. A line bundle L is called *semi-ample*, if its holomorphic global sections of some positive multiple of L have no common zero set. A semi-ample line bundle induces a holomorphic map to the projective space. (See [14] for more details on a semi-ample line bundle.)

Theorem 3.1 gives a relation between the fibre dimension of a holomorphic map and the q-positivity. When the map is the holomorphic map associated to a sufficiently large multiple of a semi-ample line bundle L, condition (B) in Theorem 3.1 is equivalent to the cohomological q-amplitude of L (see [20, Proposition 1.7]). It leads to the following theorem:

THEOREM 1.4. — *Let L be a semi-ample line bundle on a compact complex manifold X. Then the following conditions (A), (B) and (C) are equivalent.*
(A) *L is q-positive.*
(B) *The semi-ample fibration of L has fibre dimensions at most q.*
(C) *L is cohomologically q-ample.*

Further if X is projective, the conditions above are equivalent to condition (D).
(D) *For every subvariety Z with $\dim Z > q$, there exists a curve C on Z such that the degree of L on C is positive.*

Condition (B) (resp. (C), (D)) gives a geometric (resp. cohomological, numerical) characterization of a q-positive line bundle. In particular, the converse of the Andreotti-Grauert theorem holds for a semi-ample line bundle on an arbitrary compact complex manifold.

In section 4, we consider the Zariski-Fujita type theorem (Theorem 4.1) in order to investigate the q-positivity of a big line bundle. In particular, we know that the converse of the Andreotti-Grauert theorem for a big line bundle is reduced to the case of varieties of smaller dimension (the non-ample locus).

Acknowledgment

The author wishes to express his deep gratitude to his supervisor Professor Shigeharu Takayama for various comments, encouragement and carefully reading a preliminary version of this paper. He would like to thank Professor Takeo Ohsawa who taught him the paper [2] which is related with Theorem 3.1. Further he is deeply grateful to Professor Hajime Tsuji who pointed out some mistakes of a preliminary version of this paper. He also would like to express his thanks to the referee for many nice advices and carefully reading this paper. He is supported by the Grant-in-Aid for Scientific Research (KAKENHI No. 23-7228) and the Grant-in-Aid for JSPS fellows.

2. The Monge-Ampère equation and $(n-1)$-positivity

This section is devoted to prove Theorem 2.1 and its corollaries. Throughout this section, let L be a line bundle on a compact Kähler manifold X and ω a Kähler form on X. First we give the proof of Theorem 2.1.

THEOREM 2.1. — *Let L be a line bundle on a compact Kähler manifold X and ω a Kähler form on X. Assume that the intersection number $\left(L \cdot \{\omega\}^{n-1}\right)$ is positive. Here $\{\omega\}$ denotes the cohomology class in $H^{1,1}(X, \mathbb{R})$ which is defined by ω.*

Then L is $(n-1)$-positive. That is, there exists a smooth hermitian metric h whose Chern curvature $\sqrt{-1}\Theta_h(L)$ has at least 1 positive eigenvalue at every point on X.

Proof. — The main idea of the proof is to use a solution of the Monge-Ampère equation. In order to solve the Monge-Ampère equation, we make use of the following Calabi-Yau type theorem. It is a deep result which was proved as a special case in [22]. Roughly speaking, it says that the product of the eigenvalues of a Kähler form (which represents a given Kähler class) can be controlled.

THEOREM 2.2. — ([22]). *Let M be a compact Kähler manifold of dimension n and $\widetilde{\omega}$ a Kähler form on M. For a positive smooth (n,n)-form $F>0$ with $\int_M F = \int_M \widetilde{\omega}^n$, there exists a function $\varphi \in C^\infty(M,\mathbb{R})$ with the following properties* :

(1) $(\widetilde{\omega} + dd^c\varphi)^n = F$ *at every point on* M

(2) $(\widetilde{\omega} + dd^c\varphi)$ *is a Kähler form on* M.

Fix a smooth hermitian metric h of L. Then the Chern curvature $\sqrt{-1}\Theta_h(L)$ represents the first Chern class of L. We want to construct a real-valued smooth function φ on X such that $\sqrt{-1}\Theta_h(L) + dd^c\varphi$ is $(n-1)$-positive (that is, the $(1,1)$-form has at least 1 positive eigenvalue everywhere). If we obtain a function φ with the condition above, we can easily see that L is $(n-1)$-positive. In fact, the Chern curvature associated to the metric defined by $he^{-2\varphi}$ is equal to $\sqrt{-1}\Theta_h(L) + dd^c\varphi$. Therefore it is sufficient for the proof to construct a function φ with the condition above. To construct such function, we use Theorem 2.2.

Since ω is a positive form, the $(1,1)$-form $\sqrt{-1}\Theta_h(L) + k\omega$ is a Kähler form on X for a sufficiently large constant $k>0$. Now we consider the following Monge-Ampère equation:

$$\big(\sqrt{-1}\Theta_h(L) + k\omega + dd^c\varphi\big)^n = D_k\big(k\omega\big)^n,$$
$$\big(\sqrt{-1}\Theta_h(L) + k\omega + dd^c\varphi\big) > 0.$$

Here D_k is a positive constant which depends on k. In order to solve this equation, we need to define D_k by

$$D_k := \frac{\int_X \big(\sqrt{-1}\Theta_h(L) + k\omega\big)^n}{\int_X \big(k\omega\big)^n}.$$

When D_k is defined as above, we can take a solution of the equation, thanks to Theorem 2.2. In fact, by applying Theorem 2.2 to a Kähler form

$$\widetilde{\omega} := \big(\sqrt{-1}\Theta_h(L) + k\omega\big)$$

and a smooth (n,n)-form $F := D_k\big(k\omega\big)^n$, we can obtain a solution. Note that the equality $\int_X \widetilde{\omega}^n = \int_X F$ holds by the definition of D_k.

Now we shall show that the constant D_k is greater than 1 for a sufficiently large $k > 0$. We use the assumption of the theorem only for this argument. By the definition of D_k we have

$$\begin{aligned} D_k &= \frac{\int_X \big(\sqrt{-1}\Theta_h(L) + k\omega\big)^n}{\int_X \big(k\omega\big)^n} \\ &= \frac{\big(L + k\{\omega\}\big)^n}{k^n(\{\omega\}^n)} \\ &= \frac{\big(L^n\big) + kn\big(L^{n-1}\cdot\{\omega\}\big) + \cdots + k^{n-1}n\big(L\cdot\{\omega\}^{n-1}\big)}{k^n(\{\omega\}^n)} + 1. \end{aligned}$$

The molecule in the right hand is a polynomial of degree $(n-1)$ with respect to k. Further, the coefficient of the highest degree term is equal to $n(L\cdot\{\omega\}^{n-1})$. It is positive by the assumption. Therefore the first term in the right hand is greater than 0 for a sufficiently large $k > 0$. Hence D_k is greater than 1.

Finally we show that $\sqrt{-1}\Theta_h(L)+dd^c\varphi$ has at least 1 positive eigenvalue at every point on X. Here $\lambda_1(x) \geqslant \cdots \geqslant \lambda_n(x)$ denote the eigenvalues of

$$\big(\sqrt{-1}\Theta_h(L) + k\omega + dd^c\varphi\big)$$

with respect to $k\omega$ at $x \in X$. Then the function λ_i for $i = 1, 2, \ldots, n$ is well-defined as a function on X. Further λ_i for $i = 1, 2, \ldots, n$ is continuous, since the j-th symmetric function in $\lambda_1, \ldots, \lambda_n$ is smooth. Since φ is a solution of the Monge-Ampère equation, the functions λ_i satisfy the following equality everywhere:

$$\lambda_1(x)\lambda_2(x)\cdots\lambda_n(x) = D_k > 1, \quad \text{at everypoint } x \in X.$$

In addition, $\lambda_n(x)$ is positive for any point $x \in X$ since

$$\big(\sqrt{-1}\Theta_h(L) + k\omega + dd^c\varphi\big)$$

is a Kähler $(1,1)$-from. Thus we know $\lambda_1(x) > 1$ at every point on X since D_k is greater than 1.

Now the eigenvalues of

$$\sqrt{-1}\Theta_h(L) + dd^c\varphi = (\sqrt{-1}\Theta_h(L) + k\omega + dd^c\varphi) - k\omega$$

are $(\lambda_1 - 1), \ldots, (\lambda_n - 1)$ since all eigenvalues of $k\omega$ are 1. Thus $\big(\sqrt{-1}\Theta_h(L) + dd^c\varphi\big)$ has 1 positive eigenvalue $(\lambda_1 - 1)$ everywhere. It completes the proof. □

On the rest of this section, we give the proofs of Theorem 2.3 and Corollary 2.6.

THEOREM 2.3. — (=Theorem 1.3). *On a smooth projective surface X, the converse of the Andreotti-Grauert theorem holds. That is, the following conditions are equivalent.*
(A) *L is cohomologically q-ample.*
(B) *L is q-positive.*

In the statement of Theorem 2.3, it follows that condition (B) leads to condition (A) from the Andreotti-Grauert theorem. The converse is an open problem and a main subject in this paper. Theorem 2.3 claims Problem 1.2 is affirmatively solved on a smooth projective surface.

For the proof of Theorem 2.3, we shall prepare Lemma 2.4 and Lemma 2.5. These lemmata may be known facts. However we give proofs for readers' convenience. Lemma 2.4 can be proved even if X has singularities. However for our purpose, we need only the case where X is smooth.

LEMMA 2.4. — *Let L be a line bundle on a smooth projective variety X. Then the following conditions are equivalent.*
(1) *The dual line bundle $L^{\otimes -1}$ is not pseudo-effective.*
(2) *L is cohomologically $(n-1)$-ample.*

Proof. — This theorem follows from the Serre duality theorem.

First we confirm that condition (2) implies (1). For a contradiction, we assume that $L^{\otimes -1}$ is psuedo-effective. Then we can take an ample line bundle A on X such that $A \otimes L^{\otimes -m}$ has a non-zero section for every positive integer $m > 0$. Note that A does not depend on m. For the coherent sheaf $\mathcal{O}_X(K_X \otimes A^{\otimes -1})$, we can take a positive integer m such that

$$h^n\big(X, \mathcal{O}_X(K_X \otimes A^{\otimes -1} \otimes L^{\otimes m})\big) = 0$$

from condition (2). Here K_X denotes the canonical bundle on X. It follows from the Serre duality theorem that $h^0\big(X, \mathcal{O}_X(A \otimes L^{\otimes -m})\big) = 0$. This is a contradiction to the choice of A.

Conversely we show that condition (1) implies (2). Fix an ample line bundle B on X. For a given coherent sheaf $\mathcal{F}$ on X, we can take the following resolution of $\mathcal{F}$ by taking a large integer $k > 0$:

$$0 \longrightarrow \mathcal{G} \longrightarrow \oplus_{i=1}^{N} \mathcal{O}_X(B^{\otimes -k}) \longrightarrow \mathcal{F} \longrightarrow 0.$$

In fact, $\mathcal{F} \otimes \mathcal{O}_X(B^{\otimes k})$ is globally generated for a sufficient large k since B is ample. Therefore we obtain a surjective map $\oplus_{i=1}^{N} \mathcal{O}_X(B^{\otimes -k}) \longrightarrow \mathcal{F}$ as a sheaf morphism. We define $\mathcal{G}$ to be the kernel of its map.

Thus it is sufficient to show that there is a positive integer m_0 such that

$$h^n\big(X, \mathcal{O}_X(B^{\otimes -k} \otimes L^{\otimes m})\big) = 0 \quad \text{for } m \geqslant m_0$$

In fact, the long exact sequence yields $h^n\big(X, \mathcal{F} \otimes \mathcal{O}_X(L^{\otimes m})\big) = 0$ for $m \geqslant m_0$. It means that L is cohomologically $(n-1)$-ample.

Since $L^{\otimes -1}$ is not psuedo-effective, we can take a sufficiently large integer m_0 such that $K_X^{\otimes -1} \otimes B^{\otimes k} \otimes L^{\otimes -m}$ is not psuedo-effective for $m \geqslant m_0$. Since this line bundle is not psuedo-effective, we have

$$h^0\big(X, \mathcal{O}_X(K_X^{\otimes -1} \otimes B^{\otimes k} \otimes L^{\otimes -m})\big) = 0.$$

Again by using the Serre duality theorem, we have

$$h^n(X, \mathcal{O}_X(B^{\otimes -k} \otimes L^{\otimes m})) = 0$$

for $m \geqslant m_0$. □

Lemma 2.5. — *Let L be a line bundle on a smooth projective variety X of dimension n. Then the following conditions are equivalent.*
(2) *L is cohomologically $(n-1)$-ample.*
(3) *There exists a strongly movable curve C on X such that the degree of L on C is positive.*

Here a curve C is called a *strongly movable* curve if

$$C = \mu_*\big(A_1 \cap \cdots \cap A_{n-1}\big)$$

for suitable very ample divisors A_i on $\widetilde{X}$, where $\mu : \widetilde{X} \to X$ is a birational morphism. See [4, Definition 1.3] for more details.

Proof. — The deep result proved in [4] yields Lemma 2.5. It follows from [4, Theorem2.2] that the cone of pseudo-effective line bundles is the dual cone of strongly movable curves. That is, a line bundle is pseudo-effective if and only if the degree of the line bundle on every strongly movable curve is semi-positive. From Lemma 2.4, L is cohomological $(n-1)$-ample if and only if $L^{\otimes -1}$ is not psuedo-effective. Therefore then there exists a strongly movable curve C such that

$$(L^{\otimes -1} \cdot C) < 0.$$

It completes the proof. □

By applying Lemma 2.5 and Theorem 2.1, we shall complete the proof of Theorem 2.3.

Proof of Theorem 2.3. When X is a projective surface, the closure of the cone of strongly movable curves coincides with the closure of the cone of

ample line bundles (that is, the nef cone). Indeed, the dual cone of pseudo-effective line bundles equals to the closure of the cone of ample line bundles. Therefore, when L is cohomologically 1-ample, there exists an ample line bundle H with $(L \cdot H) > 0$ by Lemma 2.5.

Since H is ample, the first Chern class of H contains a Kähler form ω. Since the intersection number $(L \cdot H)$ equals to $(L \cdot \{\omega\})$, the line bundle L satisfies the assumption in Theorem 2.1. Therefore if follows that L is 1-positive from Theorem 2.1. □

At the end of this section, we prove Corollary 2.6, which can be seen as a generalization of [12, Theorem 1] to a psuedo-effective line bundle. In [12], in order to show that an effective line bundle is $(n-1)$-positive, Fuse and Ohsawa apply n-convexity of a non-compact complex manifold. We make use of the Monge-Ampère equation instead of n-convexity of a non-compact complex manifold.

COROLLARY 2.6. — *Let L be a pseudo-effective line bundle on a compact Kähler manifold X. Assume that the first Chern class of L is not trivial.*

Then L is $(n-1)$-positive.

A line bundle is called *pseudo-effective* if there exists a singular hermitian metric h whose curvature current $\sqrt{-1}\Theta_h(L)$ is positive on X as a $(1,1)$-current. A pseudo-effective line bundle (which is not numerically trivial) is cohomologically $(n-1)$-ample (see Lemma 2.4). Therefore a pseudo-effective line bundle should be $(n-1)$-positive if the converse of the Andreotti-Grauert theorem holds. Corollary 2.6 claims that it is affirmative on a compact Kähler manifold.

Proof. — Under the assumption of Corollary 2.6, we show that the line bundle L satisfies the assumption in Theorem 2.1.

Fix a Kähler form ω on X and take an arbitrary smooth $(n-1,n-1)$-form γ on X. The $(n-1,n-1)$-form ω^{n-1} is strictly positive. Therefore there exists a large constant $C > 0$ such that

$$-C\omega^{n-1} \leqslant \gamma \leqslant C\omega^{n-1}.$$

Here we used the compactness of X. Since L is psuedo-effective, there exists a singular hermitian metric such that $\sqrt{-1}\Theta_h(L)$ is a positive current. Since

$\sqrt{-1}\Theta_h(L)$ is a positive current, the following inequalities hold:

$$-C\int_X \sqrt{-1}\Theta_h(L)\wedge\omega^{n-1} \leqslant \int_X \sqrt{-1}\Theta_h(L)\wedge\gamma \leqslant C\int_X \sqrt{-1}\Theta_h(L)\wedge\omega^{n-1}.$$

The positive current $\sqrt{-1}\Theta_h(L)$ represents the first Chern class of L. Thus the integral $\int_X \sqrt{-1}\Theta_h(L)\wedge\omega^{n-1}$ agrees with the intersection number $\left(L\cdot\{\omega\}^{n-1}\right)$. If the intersection number is zero, it follows from the inequality above that $\int_X \sqrt{-1}\Theta_h(L)\wedge\gamma$ is zero for any smooth $(n-1,n-1)$-form γ. It means that $\sqrt{-1}\Theta_h(L)$ is a zero current. This is a contradiction to the assumption that the first Chern class of L is not trivial. Hence the intersection number $\left(L\cdot\{\omega\}^{n-1}\right)$ must be positive. It follows that L is $(n-1)$-positive from Theorem 2.1. □

3. The fiber dimension and q-positivity

The main purpose of this section is to give the proof of Theorem 1.4. For this purpose, we first consider Theorem 3.1.

THEOREM 3.1. — *Let $\Phi : X \longrightarrow Y$ be a holomorphic map (possibly not surjective) from X to a compact complex manifold Y. Then the following conditions are equivalent.*

(A) *Fix a Hermitian form ω (that is, a positive definite $(1,1)$-form) on Y. Then there exists a function $\varphi \in C^\infty(X,\mathbb{R})$ such that the $(1,1)$-form $\Phi^*\omega + dd^c\varphi$ is q-positive (that is, the form has at least $(n-q)$ positive eigenvalues at any point on X as a $(1,1)$-form).*

(B) *The map Φ has fibre dimensions at most q.*

Throughout this section, let $\Phi : X \longrightarrow Y$ be a holomorphic map from X to a compact complex manifold Y. Fix a hermitian form ω on Y. Set $\widetilde{\omega} := \Phi^*\omega$, which is a semi-positive $(1,1)$-form on X.

First we show that condition (A) implies (B). For a contradiction, we assume that there is a fibre F of the map Φ with $\dim F > q$. Then by condition (A), X allows a smooth function φ such that $\widetilde{\omega} + dd^c\varphi$ is q-positive. Since F is a fibre, the restriction to F of $\widetilde{\omega} = \Phi^*\omega$ is equal to zero. It implies that the restriction to F of $dd^c\varphi$ is q-positive. Even if F has singularities, we can define the q-positivity (see Definition 4.2). Since the dimension of F is strictly larger than q, the Levi-form of $\varphi|_F$ has at least 1 positive eigenvalue.

Since F is compact, the function $\varphi|_F$ must have the maximum value on F. Suppose that $p \in F$ attains the maximum value of $\varphi|_F$. Then the Levi-form of $\varphi|_F$ at p has no positive eigenvalues. It is a contradiction. Hence condition (A) leads to (B).

On the rest of this section, we shall show that condition (B) implies (A). From now on, we assume that the dimension of every fibre of the map Φ is at most q. Then we want to construct a function φ with condition (A). For this purpose, we define the degenerated locus of a hermitian form by the pull-back of the map Φ as follows:

DEFINITION 3.2. — *The degenerated locus by the pull-back of Φ is defined to be*

$$B_q := \{\ p \in X \ \big|\ \Phi^*\omega \text{ has at least } (q+1) \text{ zero eigenvalues at } p\}.$$

Since $\widetilde{\omega} = \Phi^*\omega$ is a semi-positive form, $\widetilde{\omega}$ is q-positive outside B_q. Therefore if B_q is empty, condition (A) is satisfied by taking $\varphi := 0$. Thus it is sufficient to consider the case where B_q is not empty. The following lemma asserts that the degenerated locus is locally the zero set of finite holomorphic functions.

PROPOSITION 3.3. — *The degenerated locus B_q is a closed analytic set on X.*

Proof. — First we show that B_q is a closed set in X. Fix a hermitian form $\overline{\omega}$ on X. We denote by $\lambda_1 \geqslant \cdots \geqslant \lambda_n \geqslant 0$, the eigenvalues of $\widetilde{\omega}$ with respect to $\overline{\omega}$. The j-th symmetric function in $\lambda_1, \ldots, \lambda_n$ are smooth since $\widetilde{\omega}$ and $\overline{\omega}$ is smooth forms. Therefore λ_i for $0 \leqslant i \leqslant n$ is a (well-defined) continuous function on X. Now p is contained in B_q if and only if $\lambda_{n-q}(p) = 0$. Thus the degenerated locus is closed since λ_{n-q} is a continuous function.

It remains to show that B_q is the zero set of finite holomorphic functions. Now we take a local coordinate $(w_1, \ldots, w_m)$ on Y. Here m denotes the dimension of Y. Then the degenerate locus of ω coincides with the degenerate locus of $\sum_{i=1}^m dd^c|w_i|^2$. In fact, it follows since we have

$$(1/C)\sum_{i=1}^m dd^c|w_i|^2 \leqslant \omega \leqslant C\sum_{i=1}^m dd^c|w_i|^2$$

for a sufficiently large constant $C > 0$.

The holomorphic map Φ can be locally written as

$$(z_1, \ldots, z_n) \longmapsto (f_1(z), \ldots, f_m(z))$$

for some holomorphic functions f_i. Here $(z_1, \ldots, z_n)$ denotes a local coordinate on X. Then B_q is equal to the locus where the hermitian form

$dd^c \sum_{i=1}^m dd^c |f_i(z)|^2$ has at least $(q+1)$ zero eigenvalues. In general, a semi-positive hermitian form has at least $(q+1)$ zero eigenvalues if and only if j-th symmetric function in the eigenvalues is zero for $n-q \leqslant j \leqslant n$. By a simple computation, j-th symmetric function σ_j in $\lambda_1, \ldots, \lambda_n$ is described as follows:

$$\begin{aligned}\sigma_j &= \sum_{0\leqslant i_1<\cdots<i_j\leqslant n} \det\left(\left\langle \mathbb{G}_{i_a}, \mathbb{G}_{i_b}\right\rangle\right)_{a,b=1,\ldots,n} \\ &= \Big| \sum_{0\leqslant i_1<\cdots<i_j\leqslant n} \sum_{0\leqslant \alpha_1<\cdots<\alpha_j\leqslant m} \det\left(g_{i_b}^{\alpha_a}\right)_{a,b=1,\ldots,j}\Big|^2.\end{aligned}$$

Here g_i^α denotes the differential $\partial f_\alpha/\partial z_i$, $\mathbb{G}_i$ a vector $(g_i^1, \ldots, g_i^m)$, and $\langle \cdot, \cdot \rangle$ a standard hermitian metric on $\mathbb{C}^m$. Therefore the set defined by $\sigma_j = 0$ $(n-q \leqslant j \leqslant n)$ is the zero set of finite holomorphic functions. □

Thanks to Proposition 3.3, we can consider the dimension of B_q. When the dimension of B_q is less than or equal to q, we can easily see condition (A) in Theorem 3.1 from Lemma 3.8. When the dimension is greater than q, we factor B_q to subvarieties of smaller dimension. For this purpose we need Lemma 3.5. The assumption on fibre dimensions is used in the proof of this lemma. Later we need to treat an analytic set which may not be closed. For that reason, Lemma 3.5 is formulated for an analytic set (possibly not closed).

Definition 3.4. — *A subset V in X is called an analytic set, if for every point p in V there exist a small neighborhood of p and finite holomorphic functions on the neighborhood such that V is the zero set of these functions.*

Note that an analytic set does "not" mean a closed analytic set in this paper. For example, the set $\left\{1/n \in \mathbb{C} \;\middle|\; n \in \mathbb{N}\right\}$ is an analytic set, but not a closed analytic set.

Lemma 3.5. — *Let W be an irreducible analytic set (possibly not closed, singular) on X. Assume that the dimension of W is greater than q.*

Then the degenerate locus D defined by

$$D := \left\{p \in W_{\mathrm{reg}} \;\middle|\; \text{The restriction } \widetilde{\omega}|_{W_{\mathrm{reg}}} \text{ has at least } (q+1) \text{ zero eigenvalues at } p\right\}$$

is a closed analytic set on W_{reg} and properly contained in W_{reg}. Here W_{reg} denotes the regular locus of W.

Proof. — We have already proved that D is a closed analytic set on W_{reg} in the proof of Proposition 3.3. It remains to show that D is a properly contained subset in W_{reg}. For a contradiction, we assume $D = W_{\mathrm{reg}}$.

We take a point p in W_{reg} such that $\Phi|_W : W \longrightarrow Y$ is a smooth morphism at p, and a fibre F of Φ containing p. Further we take an open ball U in W with a local coordinate $(z_1, \ldots, z_r)$ centered at p. We may assume that the first coordinate $(z_1, \ldots, .z_s)$ also becomes a local coordinate on F_{reg}. Here r (resp. s) denotes the dimension of W (resp. F).

Now we consider the restriction f of Φ defined by

$$f := \Phi|_{F^\perp} : F^\perp \longrightarrow Y, \text{ where } F^\perp = \{\ (0, \ldots, 0, z^{s+1}, \ldots, z^r) \in U\ \}.$$

Then the fibre of $\Phi(p)$ by f is a zero dimensional analytic set. It implies the Jacobian of f is not identically zero on some neighborhood of p. Hence the holomorphic map f gives a local biholomorphic at some point. This means the restriction of $\widetilde{\omega}$ to $F^\perp$ has $(n-s)$ positive eigenvalues. Note that s is less than or equal to q. This is a contradiction to $W_{\mathrm{reg}} = D$. □

Lemma 3.5 leads to Proposition 3.6. Later we shall apply this proposition to B_q. The set B_q is a closed analytic set. However we formulate this proposition for an analytic set to prove Proposition 3.6 with induction on the dimension. Remark that "dimension " in Proposition 3.6 does not necessarily mean the pure dimension .

PROPOSITION 3.6. — *Let V be an analytic set of dimension k (possibly, not closed, not irreducible, singular). Then there exist sets D_ℓ $(0 \leqslant \ell \leqslant k-1)$ with following properties:*

(1) *D_ℓ is an analytic set on X.*

(2) *$D_k := V \supseteq D_{k-1} \supseteq \cdots \supseteq D_1 \supseteq D_0$.*

(3) *$\dim D_\ell = \ell$ for $\ell = 0, 1, 2, \ldots, k$.*

(4) *$D_\ell \setminus D_{\ell-1}$ for $\ell = 1, 2, \ldots, k$ is a disjoint union of non-singular analytic sets.*

(5) *For an irreducible component W of $D_\ell \setminus D_{\ell-1}$ with $\dim W > q$, the $(1,1)$-form $\widetilde{\omega}|_W$ has $(\dim W - q)$ positive eigenvalues.*

Proof. — We prove this proposition by induction on the dimension $k = \dim V$. When k is zero, we set $D_0 = V$. Then the properties in Proposition 3.6 hold. From now on, we assume that k is greater than zero.

We consider the decomposition $V = V_{\mathrm{reg}} \cup V_{\mathrm{sing}}$. Here V_{reg} (resp. V_{sing}) denotes the regular locus (resp. the singular locus) of V. Note that this decomposition is a disjoint union. Since the dimension of V_{sing} is smaller than k, we obtain $\widetilde{D}_\ell$ $(0 \leqslant \ell \leqslant \dim V_{\mathrm{sing}})$ with the properties in Proposition 3.6 by the induction hypothesis.

Let $V_{\mathrm{reg}} = \bigcup_{i \in I} V_i$ be the irreducible decomposition of V_{reg}. For an irreducible component V_i, we set $D^i_{\dim V_i} := V_i$ if the dimension of V_i is less

than or equal to q. Otherwise, we investigate the degenerated locus D^i of V_i defined by

$$D^i := \{p \in (V_i)_{\mathrm{reg}} \mid \widetilde{\omega}|_{(V_i)_{\mathrm{reg}}} \text{ has at least } (q+1) \text{ zero eigenvalues at } p\ \}.$$

It follows that D^i is an analytic set and properly contained in V_i from Lemma 3.5. In particular, the dimension of D^i is smaller than k. Therefore by applying the induction hypothesis to D^i, we obtain D^i_ℓ $(0 \leqslant \ell \leqslant \dim V_i)$ with the properties in Proposition 3.6.

For each ℓ, we define the set D_ℓ to be the union of $\widetilde{D}_\ell$ and D^i_ℓ $(i \in I)$. Then we can easily see that D_ℓ has the properties in Proposition 3.6. In fact, it follows since $\widetilde{D}_\ell$ and D^i_ℓ satisfy the properties in Proposition 3.6 and D_ℓ is a disjoint union of them. □

For the proof of Theorem 3.1, we apply Proposition 3.6 to B_q. Then we obtain D_ℓ with the properties in Proposition 3.6. By using these properties, we shall construct a function φ whose Levi-form has $(n - \dim W)$ positive eigenvalues in the normal direction of an irreducible component W of $D_\ell \setminus D_{\ell-1}$. On the other hand, the restriction $\widetilde{\omega}|_W$ has $(\dim W - q)$ positive eigenvalues from property (5) if the dimension of W is greater than q. Thus if there is such function φ, the $(1,1)$-form $\widetilde{\omega} + dd^c\varphi$ has at least $(n-q)$ positive eigenvalues everywhere. To construct such function, we prepare Proposition 3.7, 3.9. The proofs of these Propositions are similar to the proof of [6, Theorem 4] (cf. [19]).

PROPOSITION 3.7. — *For $\ell = 0, 1, \ldots, k$, there exists a function $\varphi_\ell \in C^2(X, \mathbb{R})$ with the following properties* :

(1) *Let W be an irreducible component of $D_\ell \setminus D_{\ell-1}$. Then the Levi form $dd^c\varphi_\ell$ has $(n - \dim W)$ positive eigenvalues in the normal direction of W.*

(2) *The Levi form $dd^c\varphi_\ell$ is semi-positive at every point in $\overline{D_\ell}$.*

Here $\overline{D_\ell}$ denotes the closure of D_ℓ in X. Note $\overline{D_\ell}$ may not be a closed analytic set. For example, the closure of $\{(x,y) \in \mathbb{C}^2 \mid x = e^y\}$ in the 2-dimensional projective space is not a closed analytic set. In order to show Proposition 3.7, we prepare the following lemma. If D_ℓ for $\ell = 0, 1, \ldots, k$ is a closed analytic set, the statement of this lemma is same as that of Proposition 3.7.

LEMMA 3.8. — *Let ℓ be an integer with $0 \leqslant \ell \leqslant k$. For every open neighborhood U of $\overline{D_\ell} \setminus D_\ell$, there exists a function $\varphi_U \in C^\infty(X, \mathbb{R})$ with following properties* :

(1) *Let W be an irreducible component of $D_\ell \setminus D_{\ell-1}$. Then the Levi form $dd^c\varphi_U$ has $(n - \dim W)$ positive eigenvalues in the normal direction at every point in $W \setminus U$.*

(2) *The Levi form $dd^c\varphi_U$ is semi-positive at every point in $\overline{D_\ell}$.*

Proof. — For a given U, we take an open covering of X by open balls U_j $(j = 1, 2, \ldots, N)$. Further, we can assume the following properties for this covering.

(a) The members U_j $(j = 1, 2, \ldots, s)$ cover $D_\ell \setminus D_{\ell-1}$.

(b) The members U_j $(j = 1, 2, \ldots, s)$ are contained in U.

(c) Every U_k which intersects with U_j $(j = 1, 2, \ldots, s)$ is also contained in U.

We denote by $\mathcal{I}_{B_q}$, the ideal sheaf associated to a closed analytic set B_q. For every $j = 1, 2, \ldots, s$, we take holomorphic functions $\{f_{j,i}\}_{i=1}^{N_j}$ on U_j such that these functions generate the ideal sheaf $\mathcal{I}_{B_q}$. Further, for every $j = s+1, \ldots, N$, we take holomorphic functions $\{f_{j,i}\}_{i=1}^{N_j}$ on U_j such that these functions generate the ideal sheaf $\mathcal{I}_{D_\ell}$. Note D_ℓ is a closed analytic set on U_j $(j = s+1, \ldots, N)$. Therefore we can define the ideal sheaf $\mathcal{I}_{D_\ell}$ and take its generators.

Let $\{\rho_j\}_{j=1}^{N}$ be a partition of unity associated to the covering of X. Now we define φ_U to be

$$\varphi_U := \sum_{j=1}^{N} \rho_j \Big(\sum_{i=1}^{N_j} |f_{j,i}|^2 \Big).$$

Then φ_U satisfies properties (1), (2). In fact, an easy computation yields

$$\sqrt{-1}\partial\overline{\partial}\varphi_U = \sum_{j=1}^{N}\sum_{i=1}^{N_j} \Big\{ |f_{j,i}|^2 \sqrt{-1}\partial\overline{\partial}\rho_j + \sqrt{-1}\overline{f_{j,i}}\partial f_{j,i} \wedge \overline{\partial}\rho_j + \sqrt{-1} f_{j,i}\partial\rho_j \wedge \overline{\partial} f_{j,i} + \sqrt{-1}\rho_j \partial f_{j,i} \wedge \overline{\partial f_{j,i}} \Big\}.$$

By the definition, $f_{j,i}$ is identically zero on $\overline{D_\ell}$. (Notice that $\overline{D_\ell}$ is contained in B_q.) Therefore the first three terms are zero on $\overline{D_\ell}$. Further the last term is clearly semi-positive. Therefore property (2) is satisfied. For every point p in $W \setminus U$, we can take j_0 such that U_{j_0} does not intersect with $\overline{D_\ell} \setminus D_{\ell-1}$ and $\rho_{j_0}(p) \neq 0$ by the choice of the covering. Hence the last term has property (2) since $\partial f_{j_0,i} \wedge \overline{\partial f_{j_0,i}}$ has $(n - \dim W)$ positive eigenvalues at p in the normal direction of W. □

Before the proof of Proposition 3.7, we recall the definition of a C^2-norm $\| \cdot \|_{C^2}$ on $C^2(X, \mathbb{R})$. We take an open covering of X by open balls U_j

$(j=,1,2,\ldots,N)$ with a differential coordinate $(x_1^j,\ldots,x_{2n}^j)$. Let V_j be a relatively compact set in U_j such that $\{V_j\}_{j=1}^N$ is also an open covering of X. Then the C^2-norm $|\cdot|_{C^2}$ with respect to the open covering is defined to be

$$|f|_{C^2} := \sum_{j=1}^{N}\sum_{\alpha,\beta=1}^{2n,2n}\sup_{p\in\overline{V_j}}\left|\frac{\partial^2 f}{\partial x_\alpha^j\partial x_\beta^j}(p)\right| + \sum_{j=1}^{N}\sum_{\alpha=1}^{k}\sup_{p\in\overline{V_j}}\left|\frac{\partial f}{\partial x_\alpha^j}(p)\right| + \sup_{p\in X}\left|f(p)\right|$$

for every $f\in C^2(X,\mathbb{R})$. Certainly the norm depends on the choice of an open covering. However the topology induced by these norms is unique. For our purpose, we fix the norm. Let us begin the proof of Proposition 3.7.

Proof of Proposition 3.7. Choose a family of open neighborhoods $\{U_i\}_{i=1}^\infty$ of $\overline{D_\ell}\setminus D_{\ell-1}$ such that $\bigcap_{i=1}^\infty U_i = \overline{D_\ell}\setminus D_{\ell-1}$. For each positive integer i, we can take a function $\varphi_{U_i}\in C^\infty(X,\mathbb{R})$ with the properties in Lemma 3.8. We set

$$A_i = \| \varphi_{U_i} \|_{C^2} .$$

Here $\|\cdot\|_{C^2}$ denotes the fixed C^2-norm.

Now we define a function φ_ℓ on X to be

$$\varphi_\ell := \sum_{i=1}^{\infty}\frac{1}{2^i A_i}\varphi_{U_i}.$$

By the definition of A_i, the sum in the definition uniformly converges with respect to the C^2-norm. Hence we obtain

$$dd^c\varphi_\ell = \sum_{i=1}^{\infty}\frac{1}{2^i A_i}dd^c\varphi_{U_i}.$$

Properties (1), (2) in Lemma 3.8 and the choice of U_i lead to the properties in Proposition 3.7. □

Proposition 3.9. — *For every integer $\ell=0,1,\ldots,k$, there exists a function $\widetilde{\varphi_\ell}\in C^2(X,\mathbb{R})$ with property $(*)$.*

$(*)$ *Let m be an integer with $0\leqslant m\leqslant \ell$ and W an irreducible component of $D_m\setminus D_{m-1}$. Then the Levi-form $dd^c\widetilde{\varphi_\ell}$ has $(n-\dim W)$ positive eigenvalues in the normal direction of W.*

Before the proof of Proposition 3.9, we confirm that Proposition 3.9 and Proposition 3.6 complete the proof of Theorem 3.1. That is, there is a smooth function φ on X such that the $(1,1)$-form $\widetilde{\omega}+dd^c\varphi$ is q-positive.

First we obtain $\{D_\ell\}_{\ell=0}^k$ with properties in Proposition 3.6 by applying Proposition 3.6 to B_q. Now we take $\widetilde{\varphi_k}$ with property $(*)$ in Proposition

3.9. Recall k is the dimension of B_q. Then we shall show that $\widetilde{\omega} + \varepsilon dd^c\widetilde{\varphi}_k$ is q-positive for a sufficiently small $\varepsilon > 0$.

Now $\widetilde{\omega}$ is q-positive at $x \notin B_q$. Hence when x is not contained in B_q, the form $\widetilde{\omega} + \varepsilon dd^c\widetilde{\varphi}_k$ is q-positive for a small $\varepsilon > 0$. When x is contained in B_q, there is a positive integer ℓ such that $x \in D_\ell \setminus D_{\ell-1}$. (Otherwise x is contained in D_0. Then the Levi-form of $\widetilde{\varphi}_k$ has n positive eigenvalues at x.) For an irreducible component W of $D_\ell \setminus D_{\ell-1}$ containing x, the $(1,1)$-form $\widetilde{\omega}|_W$ has $(\dim W - q)$ positive eigenvalues along W. On the other hand, the Levi-form of $\widetilde{\varphi}_k$ has $(n - \dim W)$ positive eigenvalues in the normal direction of W (property $(*)$ in Proposition 3.9). Thus $\widetilde{\omega} + \varepsilon dd^c\widetilde{\varphi}_k$ has $(n-q)$ positive eigenvalues. The function $\widetilde{\varphi}_k$ may not be smooth. However we can approximate it with smooth functions without the loss of the q-positivity since it is C^2-function (for instance, see [18]). It completes the proof of Theorem 3.1.

Proof of Proposition 3.9. We prove Proposition 3.9 by induction on ℓ. When ℓ is zero, the claim is obvious. Thus we assume ℓ is greater than 0. By the induction hypothesis, we obtain a smooth function $\widetilde{\varphi}_{\ell-1}$ with property $(*)$. Further we take φ_ℓ with the properties in Proposition 3.7. We define a function $\widetilde{\varphi}_\ell$ to be $\varphi_\ell + \varepsilon\widetilde{\varphi}_{\ell-1}$. Then the function satisfies property $(*)$ for a sufficiently small $\varepsilon > 0$. □

Finally we see that it follows Theorem 1.4 from Theorem 3.1.

THEOREM 3.10. — (=Theorem 1.4). *Let L be a semi-ample line bundle on a compact complex manifold X. Then the following conditions (A), (B) and (C) are equivalent.*

(A) L is q-positive.

(B) The semi-ample fibration of L has fibre dimensions at most q.

(C) L is cohomologically q-ample.

Further if X is projective, the conditions above are equivalent to condition (D).

(D) For every subvariety Z with $\dim Z > q$, there exists a curve C on Z such that the degree of L on C is positive.

Proof. — We can easily confirm the equivalence between condition (B) and (C) from the standard argument of the spectral sequence (see [20, Proposition 1.7]).

The equivalence between condition (A) and (B) is directly followed from Theorem 3.1. Indeed, we apply Theorem 3.1 to the semi-ample fibration $\Phi := \Phi_{|L^{\otimes m}|} : X \longrightarrow \mathbb{P}^{N_m}$ and $\omega := \omega_{FS}$, where ω_{FS} is the Fubini-Study form. Then $\Phi^*\omega_{FS} + dd^c\varphi$ is q-positive for some function φ by Theorem 3.1.

The form represents the Chern class of $L^{\otimes m}$. Therefore condition (B) implies (A). Conversely, if L is q-positive, it is cohomologically q-ample by the Andreotti-Grauert theorem. Therefore the fibre dimension of the semi-ample fibration must be at most q.

It remains to show the equivalence between condition (B) and (D). In this step, we use the assumption that X is projective. Assume that the fibre dimension of the semi-ample fibration is at most q. Then for any subvariety Z with $\dim Z > q$, we can take a curve on Z which is not contracted by Φ. Then the degree of L on the curve is positive by the projection formula. Conversely, if there exists a fibre F with $\dim F > q$, the degree on every curve in F is zero. □

4. Zariski-Fujita type theorem for big line bundles

In this section, we prove Theorem 4.1. Theorem 4.1 says that, a big line bundle is q-positive if and only if the restriction to the non-ample locus of the line bundle is q-positive. See [10] or [3, Section 3.5] for the definition and properties of the non-ample locus. (Sometimes, the non-ample locus is called the augmented base locus or the non-Kähler locus.)

THEOREM 4.1. — *Let L be a big line bundle on a smooth projective variety X. Assume that the restriction of L to the non-ample locus $\mathbb{B}_+(L)$ is q-positive. Then L is q-positive on X.*

Recall that 0-positive is positive in the usual sense (that is, ample). Hence Theorem 4.1 claims that L is ample on X if the restriction to the non-ample locus of L is ample. It can be seen as the parallel of the the Zariski-Fujita theorem (see [23] and [11] for the Zariski-Fujita theorem).

Throughout this section, we denote by X a compact Kähler manifold and by L a line bundle on X. Moreover fix a smooth hermitian metric h of L. The Chern curvature $\sqrt{-1}\Theta_h(L)$ associated to h represents the first Chern class $c_1(L)$.

In this section, we treat a closed analytic set which may have the singularities on X. For this purpose, we extend the q-positivity concept from a manifold to an analytic space. Note the following definition does not depend on the choice of a hermitian metric h of L.

DEFINITION 4.2. — *Let V be a closed analytic set on X. The restriction $L|_V$ to V of L is q-positive if there exists a real-valued continuous function φ on V with the following condition:*

For every point p on V, there exist a neighborhood U of p on X and a C^2-function $\widetilde{\varphi}$ on U such that $\widetilde{\varphi}|_{V\cap U} = \varphi$ and the $(1,1)$-form $\sqrt{-1}\Theta_h(L)+dd^c\widetilde{\varphi}$ has at least $(n-q)$-positive eigenvalues on U.

For the proof of Theorem 4.1, we prepare the following lemma.

LEMMA 4.3. — *Let V be a closed analytic set (possibly not irreducible) on X. If the restriction $L|_V$ to V of L is q-positive, then X allows a function $\varphi_V \in C^\infty(X,\mathbb{R})$ on X such that $\sqrt{-1}\Theta_h(L) + dd^c\varphi_V$ has at least $(n-q)$ positive eigenvalues on some neighborhood of V.*

Proof. — We take a smooth extension $\widetilde{\varphi}$ to X of the potential function in Definition 4.2. Let $V = \bigcup_{i\in I} V_i$ be the irreducible decomposition. From the construction of $\widetilde{\varphi}$, the restriction to $(V_i)_{\mathrm{reg}}$ of $\sqrt{-1}\Theta_h(L) + dd^c\widetilde{\varphi}$ has at least $(\dim V_i - q)$ positive eigenvalues. Then we can revise the positivity in the normal direction of V with the same argument as Proposition 3.9. It leads to Lemma 4.3. □

Proof of Theorem 4.1. By the property of the non-ample locus (see [3, Theorem 3.17]), there exists a d-closed current T on X with following properties:

(1) T represents the first Chern class of L.
(2) T has analytic singularities along the non-ample locus of L.
(3) For some hermitian form ω on X, the inequality $T \geqslant \omega$ holds as a $(1,1)$-current.

From property (1), we obtain an L^1-function φ_s on X with

$$T = \sqrt{-1}\Theta_h(L) + dd^c\varphi_s.$$

On the other hand, by applying Lemma 4.3 to the non-ample locus, we can obtain a function $\varphi_{\mathbb{B}_+} \in C^\infty(X,\mathbb{R})$ such that $\sqrt{-1}\Theta_h(L) + dd^c\varphi_{\mathbb{B}_+}$ is q-positive on some neighborhood U on the non-ample locus.

Then we shall see that φ_s and $\varphi_{\mathbb{B}}$ can be glued. For a real number $C > 0$, we define the function ψ_C to be $\psi_C := \max\{\varphi_{\mathbb{B}_+} - C, \varphi_s\}$. For a large $C > 0$, the function $\varphi_{\mathbb{B}} - C$ is smaller than φ_s outside some neighborhood U_C of the non-ample locus. By taking a sufficiently large $C > 0$, we may assume that U_C is relatively compact in U.

On the other hand, the function φ_s has a polar set along the non-ample locus. That is, $\varphi_s(x) = -\infty$ for any point x on the non-ample locus. Hence there exists a neighborhood V_C of the non-ample locus such that φ_s is smaller than $\varphi_{\mathbb{B}_+} - C$ even if C is large. We may assume V_C is relatively compact in U_C.

Outside U_C, the $(1,1)$-form

$$\sqrt{-1}\Theta_h(L) + dd^c\psi_C = \sqrt{-1}\Theta_h(L) + dd^c\varphi_s$$

has n positive eigenvalues. On the other hand, inside V_C the $(1,1)$-form

$$\sqrt{-1}\Theta_h(L) + dd^c\psi_C = \sqrt{-1}\Theta_h(L) + dd^c\varphi_{\mathbb{B}_+}$$

has $(n-q)$-positive eigenvalues. In order to investigate the positivity on $\overline{U_C} \setminus V_C$, we prepare the following lemma.

LEMMA 4.4. — *Let γ be a smooth d-closed $(1,1)$-form on X and an L^1-function φ_i (for $i = 1,2$) with $dd^c\varphi_i \geqslant \gamma$. Then the function $\max(\varphi_1, \varphi_2)$ also satisfies $dd^c \max(\varphi_1, \varphi_2) \geqslant \gamma$.*

Proof. — Notice that the conclusion is a local property. We can locally take a smooth potential function of γ since γ is a d-closed form. Thus we have $\gamma = dd^c\psi$ for some function ψ. By the assumption, $dd^c(\varphi_i - \psi)$ is a positive current. Therefore the Levi form of

$$\max(\varphi_1, \psi + \varphi_2 - \psi) = \max(\varphi_1, \varphi_2) - \psi$$

is also a positive current. It implies that $dd^c \max(\varphi_1, \varphi_2) \geqslant \gamma$. □

Since U_C is relatively compact in U, $\sqrt{-1}\Theta_h(L) + dd^c\varphi_{\mathbb{B}_+}$ is q-positive on $\overline{U_C}$. Certainly $\sqrt{-1}\Theta_h(L) + dd^c\varphi_s$ is q-positive (0-positive) on $\overline{U_C} \setminus V_C$. Therefore it follows from the lemma above that $\sqrt{-1}\Theta_h(L) + dd^c\psi_C$ is q-positive on $\overline{U_C} \setminus V_C$. The function ψ_C may not be smooth. However we can approximate it with smooth functions without the loss of the q-positivity since ψ_C is continuous. Therefore L is q-positive on X. □

When the dimension of the non-ample locus is smaller q, the assumption in Theorem 4.1 is automatically satisfied. Thus we have the following corollary.

COROLLARY 4.5. — *Assume the dimension of the non-ample locus of L is less than or equal to q. Then L is q-positive.*

Under the assumption in Corollary 4.5, L is cohomologically q-ample (cf. [13], [15, Theorem 1.6]). Corollary 4.5 claims that the q-positivity has the same property.

5. Relations with the holomorphic Morse inequality

In his paper [8], Demailly proved the converse of the holomorphic Morse inequality on a surface. This result has the similarity to the converse of the Andreotti-Grauert theorem. In this section, we explain the difference between his result and the result (Theorem 1.3) in this paper. First we recall the holomorphic Morse inequality which is closely related with the Andreotti-Grauert vanishing theorem.

DEFINITION 5.1. — *Let L be a line bundle on a compact complex manifold X. Then the asymptotic q-cohomology of L is defined to be*

$$\hat{h}^i(L) := \limsup_{m\to\infty} \frac{n!}{m^n} h^i(X, \mathcal{O}_X(L^{\otimes m})).$$

In his paper [5], Demailly gave a relation between the dimension of the asymptotic cohomology of a line bundle and certain Monge-Ampère integrals of the curvature. It is so-called Demailly's holomorphic Morse inequality. For simplicity, we assume that X is projective.

THEOREM 5.2. — ([5]). *For every holomorphic line bundle L on a projective manifold X, we has the (weak) Morse inequality*

$$\hat{h}^i(L) \leqslant \inf_{h:\text{hermitian metric on } L} \int_{X(h,i).} (\sqrt{-1}\Theta_h(L))^n (-1)^i,$$

where h runs through smooth hermitian metrics on L, and $X(h,i)$ is the set defined by

$$X(h,i) := \{x \in X \mid \sqrt{-1}\Theta_h(L) \text{ has a signature } (n-i,i) \text{ at } x.\}.$$

The holomorphic Morse inequality would be seen as an asymptotic version of the Andreotti-Grauert vanishing theorem. In his paper [7], Demailly conjectured that the inequality would actually be an equality. The conjecture has the similarity to Problem 1.2. In [8], he showed the converse of holomorphic Morse inequality holds in the following case:
(1) The case where X is projective surface.
(2) The case where X is an arbitrary projective manifold and $i = 0$.
Result (2) can be seen as a "partial" converse of the Andreotti-Grauert theorem. However, Result (2) seems not to lead to Theorem 1.3.

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Manuscrit reçu le 20 mai 2011,
révisé le 4 juin 2012,
accepté le 23 novembre 2012.

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