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## ON $G$ -SETS AND ISOSPECTRALITY

by Ori PARZANCHEVSKI (\*)

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ABSTRACT. — We study finite  $G$ -sets and their tensor product with Riemannian manifolds, and obtain results on isospectral quotients and covers. In particular, we show the following: If  $M$  is a compact connected Riemannian manifold (or orbifold) whose fundamental group has a finite non-cyclic quotient, then  $M$  has isospectral non-isometric covers.

RÉSUMÉ. — Nous étudions les  $G$ -ensembles finis et leur produit tensoriel avec des variétés Riemanniennes et obtenons certains résultats sur les quotients et revêtements isospectraux. Nous démontrons en particulier le théorème suivant : Soit  $M$  une variété (ou orbifold) Riemannienne compacte et connexe dont le groupe fondamental possède un quotient fini non cyclique. Alors  $M$  admet des revêtements isospectraux non isométriques.

### 1. Introduction

Two Riemannian manifolds are said to be *isospectral* if they have the same spectrum of the Laplace operator (see Definition 3.2). The question whether isospectral manifolds are necessarily isometric has gained popularity as “*Can one hear the shape of a drum?*” [15], and it was answered negatively for many classes of manifolds (e.g., [19, 4, 12, 5]). In 1985, Sunada described a general group-theoretic method for constructing isospectral Riemannian manifolds [24], and recently this method was presented as a special case of a more general one [20]. In this paper we explore a broader special case of the latter theory, obtaining the following, somewhat surprising, result (Corollary 4.5):

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*Keywords:* isospectrality, laplacian,  $G$ -sets, Sunada.

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Let  $G$  be a finite non-cyclic group which acts faithfully by isometries on a compact connected Riemannian manifold  $M$ . Then there exist  $r \in \mathbb{N}$  and subgroups  $H_1, \dots, H_r$  and  $K_1, \dots, K_r$  of  $G$  such that the disjoint unions  $\bigcup_{i=1}^r M/H_i$  and  $\bigcup_{i=1}^r M/K_i$  are isospectral non-isometric manifolds (or orbifolds<sup>(\*)</sup>).

The result mentioned in the abstract follows immediately (Corollary 4.6).

Throughout this paper  $M$  denotes a compact Riemannian manifold, and  $G$  a finite group which acts on it by isometries. In these settings, Sunada's theorem [24] states that if two subgroups  $H, K \leq G$  satisfy

$$\forall g \in G : \quad |[g] \cap H| = |[g] \cap K|, \quad (1.1)$$

where  $[g]$  denotes the conjugacy class of  $g$  in  $G$ , then the quotients  $M/H$  and  $M/K$  are isospectral. In fact, it is not harder to show (Corollary 3.3) that if two collections  $H_1, \dots, H_r$  and  $K_1, \dots, K_r$  of subgroups of  $G$  satisfy

$$\forall g \in G : \quad \sum_{i=1}^r \frac{|[g] \cap H_i|}{|H_i|} = \sum_{i=1}^r \frac{|[g] \cap K_i|}{|K_i|} \quad (1.2)$$

then  $\bigcup M/H_i$  and  $\bigcup M/K_i$  are isospectral<sup>(†)</sup>. We shall see, however, that in contrast with Sunada pairs ( $H, K$  satisfying (1.1)), collections satisfying (1.2) are rather abundant. In fact, we will show that every finite non-cyclic group  $G$  has such collections, and furthermore, that some of them (which we denote *unbalanced*, see Definition 3.4) necessarily yield non-isometric quotients.

### 1.1. Example

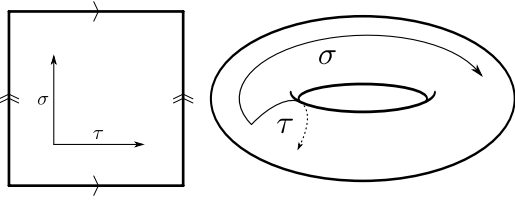
Let  $T$  be the torus  $\mathbb{R}^2/\mathbb{Z}^2$ . Let  $G = \{e, \sigma, \tau, \sigma\tau\}$  be the non-cyclic group of size four (i.e.  $G \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ ), and let  $\sigma, \tau \in G$  act on  $T$  by two perpendicular rotations:  $\sigma \cdot (x, y) = (x, y + \frac{1}{2})$  and  $\tau \cdot (x, y) = (x + \frac{1}{2}, y)$  (Figure 1.1).

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(\*) If  $G$  does not act freely on  $M$  (i.e. some  $g \in G \setminus \{e\}$  acts on  $M$  with fixed points), then  $\bigcup M/H_i$  and  $\bigcup M/K_i$  are in general orbifolds. A reader not interested in orbifolds can assume that all spaces in the paper are manifolds, at the cost of limiting the discussion to free actions.

(†) In this paper  $\bigcup$  always stands for disjoint union.

Figure 1.1. Two views of an action of  $G = \{e, \sigma, \tau, \sigma\tau\} \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$  on the torus  $T$ .



Now, the subgroups

$$\begin{aligned}
 H_1 &= \{e, \sigma\} & K_1 &= \{e\} \\
 H_2 &= \{e, \tau\} & K_2 &= K_3 = G \\
 H_3 &= \{e, \sigma\tau\}
 \end{aligned}
 \tag{1.3}$$

satisfy (1.2) (since  $G$  is abelian, (1.2) becomes

$$\forall g \in G : \sum_{i: g \in H_i} \frac{1}{|H_i|} = \sum_{i: g \in K_i} \frac{1}{|K_i|},$$

which is easy to verify). Thus, the unions of tori

$$\bigcup T/H_i = T/\langle\sigma\rangle \cup T/\langle\tau\rangle \cup T/\langle\sigma\tau\rangle \quad \text{and} \quad \bigcup T/K_i = T \cup T/G \cup T/G$$

are isospectral (Figure 1.2).

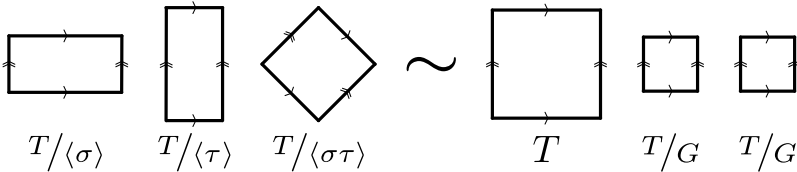


Figure 1.2. An isospectral pair consisting of quotients of the torus  $T$  (Figure 1.1) by the subgroups of  $G$  described in (1.3).

This isospectral pair, which we shall return to in §4.2.1, was immortalized in the words of Peter Doyle [9]:

Two one-by-ones and a two-by-two,  
Two two-by-ones and a roo-by-roo.

This paper is organized as follows. Section 2 describes the elements we shall need from the theory of  $G$ -sets: their classification, linear equivalence, and tensor product. Section 3 explains why tensoring a manifold with linearly equivalent  $G$ -sets gives isospectral manifolds, and defines the notion of unbalanced  $G$ -sets, which yield isospectral manifolds which are also non-isometric. At this point the focus turns to the totality of isospectral pairs arising from a single action, and it is shown that it possesses a natural

structure of a lattice. Section 4 is devoted to the proof that every finite non-cyclic group admits an unbalanced pair, and various isospectral pairs are encountered along the way. Section 5 demonstrates a detailed computation of (generators for) the lattice of isospectral pairs arising from the symmetries of the regular hexagon. Finally, Section 6 hints at possible generalizations of the results presented in this paper.

## 2. $G$ -sets

To explain where the conditions (1.1) and (1.2) come from, we invoke the theory of  $G$ -sets. We start by recalling the basic notions and facts.

### 2.1. $G$ -sets and their classification

For a group  $G$ , a (left)  $G$ -set  $X$  is a set equipped with a (left) action of  $G$ , i.e. a multiplication rule  $G \times X \rightarrow X$ . Such an action partitions  $X$  into *orbits*, the subsets of the form  $Gx = \{gx \mid g \in G\}$  for  $x \in X$ . A  $G$ -set with one orbit is said to be *transitive*, and every  $G$ -set decomposes uniquely as a disjoint union of transitive ones, its orbits. For every subgroup  $H$  of  $G$ , the set of left cosets  $G/H$  is a transitive (left)  $G$ -set.

We denote by  $\text{Hom}_G(X, Y)$  the set of  $G$ -set homomorphisms from  $X$  to  $Y$ , which are the functions  $f : X \rightarrow Y$  which commute with the actions, i.e. satisfy  $f(gx) = gf(x)$  for all  $g \in G$ ,  $x \in X$ . An isomorphism is, as usual, an invertible homomorphism.

Every transitive  $G$ -set is isomorphic to  $G/H$ , for some subgroup  $H$  of  $G$ , and  $G/H$  and  $G/K$  are isomorphic if and only if  $H$  and  $K$  are conjugate subgroups of  $G$ . More generally, every  $G$ -set is isomorphic to  $\bigcup_{i \in I} G/H_i$  for some collection (possibly with repetitions) of subgroups  $H_i$  ( $i \in I$ ) in  $G$ , and these are determined uniquely up to order and conjugacy. Namely,  $X = \bigcup G/H_i$  and  $Y = \bigcup G/K_i$  are isomorphic if and only if after some reordering  $H_i$  is conjugate to  $K_i$  for every  $i$ .

A *right*  $G$ -set is a set equipped with a right action of  $G$ , i.e. a multiplication rule  $X \times G \rightarrow X$  (satisfying  $x(gg') = (xg)g'$ ). The classification of right  $G$ -sets by right cosets is analogous to that of left  $G$ -sets by left ones.

### 2.2. Linearly equivalent $G$ -sets

Henceforth  $G$  is a finite group, and all  $G$ -sets are finite, so that every  $G$ -set is isomorphic to a finite disjoint union of the form  $\bigcup G/H_i$ . For a  $G$ -set

$X, \mathbb{C}[X]$  denotes the  $\mathbb{C}G$ -module (i.e. complex representation of  $G$ ) having  $X$  as a basis, with  $G$  acting on  $\mathbb{C}[X]$  by the linear extension of its action on  $X$ , i.e.  $g \sum a_i x_i = \sum a_i g x_i$  ( $g \in G, a_i \in \mathbb{C}, x_i \in X$ ).

If  $X \cong Y$  (as  $G$ -sets), then  $\mathbb{C}[X] \cong \mathbb{C}[Y]$  (as  $\mathbb{C}G$ -modules), but not vice versa. In fact, this is precisely where (1.1) and (1.2) come from:

PROPOSITION 2.1. — For two (finite)  $G$ -sets  $X, Y$  the following are equivalent:

- (1)  $\mathbb{C}[X] \cong \mathbb{C}[Y]$  as complex representations of  $G$ .
- (2) Every  $g \in G$  fixes the same number of elements in  $X$  and in  $Y$ .
- (3)  $X \cong \bigcup G/H_i$  and  $Y \cong \bigcup G/K_i$  for  $H_i, K_i \leq G$  satisfying (1.2).

Proof. — The character of  $\mathbb{C}[X]$  is  $\chi_{\mathbb{C}[X]}(g) = |\text{fix}_X(g)|$ , hence by character theory (1) is equivalent to (2). It is a simple exercise to show that  $|\text{fix}_{G/H}(g)| = \frac{|[g] \cap H| |C_G(g)|}{|H|}$ , so that for  $H_i$  such that  $X \cong \bigcup G/H_i$  we obtain

$$|\text{fix}_X(g)| = \sum_i |\text{fix}_{G/H_i}(g)| = |C_G(g)| \cdot \sum_i \frac{|[g] \cap H_i|}{|H_i|},$$

showing that (2) is equivalent to (3). □

DEFINITION 2.2. —  $G$ -sets  $X$  and  $Y$  as in Proposition 2.1 are said to be linearly equivalent.

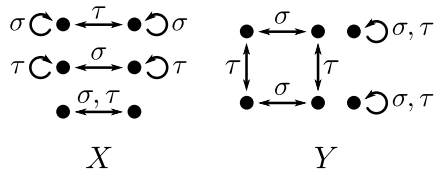
Remark. — In the literature one encounters also the terms *arithmetically equivalent*, *almost equivalent*, *Gassman pair*, or *Sunada pair*. Also, sometimes the “trivial case”, namely when  $X \cong Y$  as  $G$ -sets, is excluded.

### 2.2.1. Back to the example

In (1.3) we presented subgroups  $H_i, K_i$  of  $G = \{e, \sigma, \tau, \sigma\tau\} \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ , which satisfied condition (1.2). Figure 2.1 shows the corresponding  $G$ -sets  $X = \bigcup G/H_i$  and  $Y = \bigcup G/K_i$ , and one indeed sees that

$$|\text{fix}_X(g)| = |\text{fix}_Y(g)| = \begin{cases} 6 & g = e \\ 2 & g = \sigma, \tau, \sigma\tau \end{cases}$$

Figure 2.1.  $X$  and  $Y$  are linearly equivalent  $G$ -sets for  $G = \{e, \sigma, \tau, \sigma\tau\}$ , corresponding to the subgroups in (1.3).



We note that  $X$  and  $Y$  are not isomorphic as  $G$ -sets, as the sizes of their orbits are different:  $X$  has three orbits of size two, whereas  $Y$  has one orbit of size four and two orbits of size one.

### 2.2.2. The transitive case - Gassman-Sunada pairs

When restricting to transitive  $G$ -sets,  $X$  and  $Y$  are linearly equivalent exactly when  $X \cong G/H$ ,  $Y \cong G/K$  for  $H, K \leq G$  satisfying the Sunada condition (1.1). In the literature  $H, K$  are known as *almost conjugate*, *locally conjugate*, *arithmetically equivalent*, *linearly equivalent*, *Gassman pair*, or *Sunada pair*, and again one usually excludes the trivial case, which is when  $H$  and  $K$  are conjugate. For a group to have a Sunada pair its order must be a product of at least five primes [8], but there exist such  $n$  (the smallest being 80), for which no group of size  $n$  has one. The smallest group which admits a Sunada pair is  $\mathbb{Z}/8\mathbb{Z} \rtimes \text{Aut}(\mathbb{Z}/8\mathbb{Z})$  (of size 32).

## 2.3. Tensor product of $G$ -sets

The theory of  $G$ -sets is parallel in many aspects to that of  $R$ -modules (where  $R$  stands for a non-commutative ring). This section describes in some details the  $G$ -set analogue of the tensor product of modules. Except for Definition 2.3, and the universal property (2.1), this section may be skipped by abstract nonsense haters.

If  $M$  is a right  $R$ -module, for every abelian group  $A$  the group of homomorphisms  $\text{Hom}_{Ab}(M, A)$  has a structure of a (left)  $R$ -module, by  $(rf)(m) = f(mr)$ . In fact,  $\text{Hom}_{Ab}(M, \_)$  is a functor from  $Ab$  to  $Rmod$ , the category of left  $R$ -modules. This functor has a celebrated left adjoint, the tensor product  $M \otimes_R \_ : Rmod \rightarrow Ab$ . This means that for every  $R$ -module  $N$  there is an isomorphism

$$\text{Hom}_{Ab}(M \otimes_R N, A) \cong \text{Hom}_R(N, \text{Hom}_{Ab}(M, A))$$

which is natural in  $N$  and  $A$ .

The analogue for  $G$ -sets is this: If  $X$  is a right  $G$ -set, then for every set  $S$  the set  $\text{Hom}_{Set}(X, S)$  has a structure of a (left)  $G$ -set, by  $(gf)(x) = f(xg)$ . Here  $\text{Hom}_{Set}(X, \_)$  is a functor from  $Set$  to  $Gset$  (the category of left  $G$ -sets), and again it has a left adjoint:

**DEFINITION 2.3.** — *The tensor product over  $G$  of a right  $G$ -set  $X$  and a left  $G$ -set  $Y$ , denoted  $X \times_G Y$ , is the set  $X \times Y / (xg, y) \sim (x, gy)$ , i.e. the quotient set of the Cartesian product  $X \times Y$  by the relations  $(xg, y) \sim (x, gy)$  (for all  $x \in X$ ,  $g \in G$ ,  $y \in Y$ ).*

The functor  $X \times_G \_ : Gset \rightarrow Set$  is indeed the left adjoint of  $Hom_{Set}(X, \_)$ : For every  $G$ -set  $Y$  there is an isomorphism (natural in  $Y$  and  $S$ )

$$Hom_{Set}(X \times_G Y, S) \cong Hom_G(Y, Hom_{Set}(X, S)).$$

As it is custom to write  $B^A$  for  $Hom_{Set}(A, B)$ , this can be written as

$$S^{X \times_G Y} \cong Hom_G(Y, S^X) \tag{2.1}$$

which for  $G = 1$  is the familiar isomorphism of sets  $S^{X \times Y} \cong (S^X)^Y$ .

The tensor product of  $G$ -sets behaves much like that of modules, e.g., there are natural isomorphisms as follows:

- Distributivity:  $(\bigcup X_i) \times_G Y \cong \bigcup (X_i \times_G Y)$ .
- Associativity:  $(X \times_G Y) \times_H Z \cong X \times_G (Y \times_H Z)$  (where  $Y$  is a  $(G, H)$ -biset, i.e.  $(gy)h = g(yh)$  holds for all  $g \in G, y \in Y, h \in H$ ).
- Neutral element:  $G \times_G X \cong X$ .
- Extension of scalars: if  $H \leq G$ ,  $G$  is a  $(G, H)$ -biset. For an  $H$ -set  $X$ , this gives  $G \times_H X$  a  $G$ -set structure (by  $g'(g, x) = (g'g, x)$ ). This construction is adjoint to the restriction of scalars: for any  $G$ -set  $Y$  one has

$$Hom_G(G \times_H X, Y) \cong Hom_H(X, Y). \tag{2.2}$$

*Remark.* — A point in which groups and rings differ is the following: A left  $G$ -set can be regarded as a right one, by defining the right action to be  $xg = g^{-1}x$ . Thus, we shall allow ourselves to regard left  $G$ -sets as a right ones, and vice versa<sup>(\*)</sup>. Going back to Definition 2.3, if we choose to regard  $X$  as a left  $G$ -set, we obtain

$$\begin{aligned} X \times_G Y &= \frac{X \times Y}{(xg, y) \sim (x, gy)} = \frac{X \times Y}{(g^{-1}x, y) \sim (x, gy)} = \frac{X \times Y}{(x, y) \sim (gx, gy)} \\ &= X \times Y / G \end{aligned}$$

i.e. the tensor product is the orbit set of the normal (Cartesian) product of the left  $G$ -sets  $X$  and  $Y$ . A word of caution: the process of turning a left  $G$ -set into a right one does not give it, in general, a  $(G, G)$ -biset structure.

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(\*) For rings, a left  $R$ -module can only be regarded as a right  $R^{opp}$ -module, and in general  $R \not\cong R^{opp}$ . In groups,  $G \cong G^{opp}$  canonically by the inverse map.



### 3. Action and spectrum

#### 3.1. Tensor product of $G$ -manifolds

Assume we have an action of  $G$  by isometries on a Riemannian manifold  $M$  and on a finite  $G$ -set  $X$ . Our purpose is to study  $M \times_G X$ , which has a Riemannian orbifold structure as a quotient of  $M \times X$  (where  $X$  is given the discrete topology)<sup>(\*)</sup>. In §1 we discussed unions of the form  $\bigcup M/H_i$  for subgroups  $H_i \leq G$ , and this is still our object of study: we can choose subgroups  $H_i$  of  $G$  such that  $X \cong \bigcup G/H_i$ , and for any such choice we have an isometry  $M \times_G X \cong \bigcup M/H_i$ . This can be verified directly, or by the tensor properties:

$$\begin{aligned} M \times_G X &\cong M \times_G \left( \bigcup^{G/H_i} \right) \cong \bigcup (M \times_G G/H_i) \cong \bigcup (M \times_G (G \times_{H_i} \mathbf{1})) \\ &\cong \bigcup ((M \times_G G) \times_{H_i} \mathbf{1}) \cong \bigcup (M \times_{H_i} \mathbf{1}) \cong \bigcup M/H_i \end{aligned}$$

where  $\mathbf{1}$  denotes a one-element set. In this light, the tensor product generalizes the notion of quotients, since quotients by subgroups of  $G$  correspond to tensoring with transitive  $G$ -sets:  $M/H \cong M \times_G G/H$ . The advantage of studying  $M \times_G X$  rather than  $\bigcup M/H_i$  is that the former is free of choices, and thus more suitable for functorial constructions, and yields more elegant proofs. On the other hand,  $\bigcup M/H_i$  is much more familiar, and the reader is encouraged to envision  $M \times_G X$  as a union of quotients of  $M$ .

The next theorem, which describes the space of functions on  $M \times_G X$ , is the heart of our isospectrality technique.

**THEOREM 3.1.** — *If a finite group  $G$  acts by isometries on a Riemannian manifold  $M$  then for every finite  $G$ -set  $X$  there is an isomorphism*

$$L^2(M \times_G X) \cong \text{Hom}_{\mathbb{C}G}(\mathbb{C}[X], L^2(M))$$

(where  $L^2(M)$  is a representation of  $G$  by  $(gf)(m) = f(g^{-1}m)$ .)

*Remark.* — In the language of [1, 20], this means that  $M \times_G X$  is an  $M/\mathbb{C}[X]$ -manifold, and since  $M \times_G X \cong \bigcup M/H_i$ , this is implied in [1, §9.3]. However, the perspective of tensor product gives a direct proof.

*Proof.* — We have isomorphisms of vector spaces:

$$\mathbb{C}^{M \times_G X} \cong \text{Hom}_G(X, \mathbb{C}^M) \cong \text{Hom}_{\mathbb{C}G}(\mathbb{C}[X], \mathbb{C}^M). \tag{3.1}$$

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<sup>(\*)</sup> More generally, if  $M$  and  $M'$  are  $G$ -manifolds,  $M \times_G M'$  is an orbifold (manifold, if  $G$  acts freely on  $M \times M'$ ), but in this paper we shall only consider the tensor product of a  $G$ -manifold and a finite  $G$ -set (which can be regarded as a compact manifold of dimension 0).

The left one is by adjointness of tensor and hom (2.1), and it is given explicitly by sending  $f \in \mathbb{C}^{M \times_G X}$  to  $F \in \text{Hom}_G(X, \mathbb{C}^M)$  defined by  $F(x)(m) = f(m, x)$ . The next isomorphism is by adjointness of the free construction  $X \mapsto \mathbb{C}[X]$  and the forgetful functor  $\mathbb{C}G\text{mod} \rightarrow G\text{set}$ , i.e.

$$\text{Hom}_G(X, \_) \cong \text{Hom}_{\mathbb{C}G}(\mathbb{C}[X], \_), \tag{3.2}$$

and is given explicitly by linear extension, namely, defining  $F(\sum a_i x_i) = \sum a_i F(x_i)$ . The correspondence of the  $L^2$  conditions then follows from the finiteness of  $G$  and  $X$ , and the fact that  $\int_{M \times X} |f|^2 = \sum_{x \in X} \int_M |f(\cdot, x)|^2$ . □

**DEFINITION 3.2.** — *The spectrum of a Riemannian manifold  $M$  is the function  $\text{Spec}_M : \mathbb{R} \rightarrow \mathbb{N}$  which prescribes to every number its multiplicity as an eigenvalue of the Laplace operator on  $M$ , i.e.  $\text{Spec}_M(\lambda) = \dim L^2_\lambda(M)$  where  $L^2_\lambda(M) = \{f \in L^2(M) \mid \Delta f = \lambda f\}$ .*

**COROLLARY 3.3.** — *If  $G$  acts on  $M$ , and  $X$  and  $Y$  are linearly equivalent  $G$ -sets, then  $M \times_G X$  and  $M \times_G Y$  are isospectral.*

*Remark.* — For transitive  $X$  and  $Y$ , this is equivalent to Sunada’s theorem.

*Proof.* — By Theorem 3.1, we have  $L^2(M \times_G X) \cong L^2(M \times_G Y)$ , but we must verify that this isomorphism respects the Laplace operator. If  $y \mapsto \sum_{x \in X} a_{y,x} x$  is a  $\mathbb{C}G$ -module isomorphism from  $\mathbb{C}[Y]$  to  $\mathbb{C}[X]$ , then  $\mathcal{T} : L^2(M \times_G X) \xrightarrow{\cong} L^2(M \times_G Y)$  is given explicitly by  $(\mathcal{T}f)(m, y) = \sum_{x \in X} a_{y,x} f(m, x)$  ( $\mathcal{T}$  is a *transplantation* map, see [4, 2, 5, 6]). This isomorphism commutes with the Laplace operators on their domains of definition, hence inducing isomorphism of eigenspaces, and in particular equality of spectra. Alternatively, one can replace  $L^2$  throughout Theorem 3.1 with  $L^2_\lambda$ , obtaining directly  $L^2_\lambda(M \times_G X) \cong \text{Hom}_{\mathbb{C}G}(\mathbb{C}[X], L^2_\lambda(M))$ , and thus  $L^2_\lambda(M \times_G X) \cong L^2_\lambda(M \times_G Y)$ . □

The theorem and corollary above give us isospectral manifolds, but do not tell us whether they are isometric or not. First of all, if  $X$  and  $Y$  are isomorphic as  $G$ -sets then  $M \times_G X$  and  $M \times_G Y$  are certainly isometric. However, this may happen also for non-isomorphic  $G$ -sets<sup>(\*)</sup>. The next section deals with this inconvenience.

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<sup>(\*)</sup>For example, if  $H$  and  $K$  are isomorphic subgroups of  $G$ , and the action of  $G$  on  $M$  can be extended to an action of some supergroup  $\widehat{G}$  in which  $H$  and  $K$  are conjugate, then  $M/H$  and  $M/K$  are also isometric.

### 3.2. Unbalanced pairs

In §2.2.1 we concluded that the  $G$ -sets  $X$  and  $Y$  in Figure 2.1 were non-isomorphic by pointing out differences in the sizes of their orbits. This property is stronger than just being non-isomorphic, and we give it a name.

**DEFINITION 3.4.** — *For a finite group  $G$ , a pair of finite  $G$ -sets  $X, Y$  is an unbalanced pair if they are linearly equivalent (i.e.  $\mathbb{C}[X] \cong \mathbb{C}[Y]$  as  $\mathbb{C}G$ -modules), and if in addition they differ in the sizes of their orbits, namely, for some  $n$  the number of orbits of size  $n$  in  $X$  and the number of such orbits in  $Y$  are different.*

**Remark 3.5.** — Since the size of a  $G$ -set  $X$  equals  $\dim \mathbb{C}[X]$ , and the number of orbits in  $X$  equals  $\dim \left( \mathbb{C}[X]^G \right)^{(*)}$ , linearly equivalent  $G$ -sets necessarily have the same size and number of orbits. Thus, there are no unbalanced pairs in which one of the sets is transitive, and in particular there are no unbalanced Sunada pairs.

**THEOREM 3.6.** — *If  $X, Y$  is an unbalanced pair of  $G$ -sets, then for any faithful action of  $G$  by isometries on a compact connected manifold  $M$ , the manifolds (or orbifolds)  $M \times_G X$  and  $M \times_G Y$  are isospectral and non-isometric.*

*Proof.* — Isospectrality was obtained in Corollary 3.3. To show that  $M \times_G X$  and  $M \times_G Y$  cannot be isometric, we choose  $H_i$  such that  $X \cong \bigcup G/H_i$ , and observe that

- Since  $M$  is connected,  $\{M/H_i\}$  form the connected components of  $M \times_G X$ .
- Since  $G$  acts faithfully and  $M$  is connected,  $\text{vol}^{M/H_i} = \frac{\text{vol } M}{|H_i|}$ .

Thus, the sizes of orbits in  $X$  correspond to the volumes of connected components in  $M \times_G X$  <sup>(†)</sup>. Therefore, if  $X$  and  $Y$  form an unbalanced pair then  $M \times_G X$  and  $M \times_G Y$  differ in the volumes of their connected components. To be precise, if  $X$  and  $Y$  have different numbers of orbits of size  $n$ , then  $M \times_G X$  and  $M \times_G Y$  have different numbers of connected components of volume  $\frac{n \cdot \text{vol } M}{|G|}$ .  $\square$

---

(\*)  $V^G$  denotes the  $G$ -invariant part of a representation  $V$ :  $V^G = \{v \in V \mid gv = v \forall g \in G\}$ .

(†) This correspondence between sizes of orbits and volumes of components is apparent in Figures 2.1 and 1.2.

### 3.3. The Burnside ring and the lattice of isospectral quotients

A nice point of view is attained from  $\Omega(G)$ , the Burnside ring of the group  $G$ . Its elements are formal differences of isomorphism classes of finite  $G$ -sets, namely  $X - Y$  where  $X$  and  $Y$  are finite  $G$ -sets, with  $X - Y = X' - Y'$  whenever  $X \cup Y' \cong X' \cup Y$ . The operations in  $\Omega(G)$  are disjoint union and Cartesian product (extended to formal differences by distributivity). If we fix representatives  $H_1, \dots, H_r$  for the conjugacy classes of subgroups in  $G$ , the classification of  $G$ -sets (see §2.1) tells us that  $\Omega(G) = \{\sum_{i=1}^r n_i \cdot G/H_i \mid n_i \in \mathbb{Z}\}$ , so that as an abelian group  $\Omega(G)^+ \cong \mathbb{Z}^r$  with  $\{G/H_i\}_{i=1}^r$  being a basis.

Now, instead of looking at a pair of  $G$ -sets  $(X, Y)$ , we look at the element  $X - Y$  in  $\Omega(G)$ . First, we note that some information is lost: For any  $G$ -set  $Z$ , the pair  $(X, Y)$  and the pair  $(X' = X \cup Z, Y' = Y \cup Z)$  both correspond to the same element in  $\Omega(G)$ , i.e.  $X - Y = X' - Y'$ . Second, we notice this is in fact desirable. In order to produce elegant isospectral pairs, one would like to “cancel out” isometric connected components shared by two isospectral manifolds (as in [6]), and the pair  $M \times_G X', M \times_G Y'$  is just the pair  $M \times_G X, M \times_G Y$  with each manifold added  $M \times_G Z$ .

Thus, we would like to look at *reduced pairs*, pairs of  $G$ -sets  $X, Y$  which share no isomorphic sub- $G$ -sets (equivalently, no isomorphic orbits). The map  $(X, Y) \mapsto X - Y$  gives a correspondence between reduced pairs and the elements of  $\Omega(G)^{(*)}$ . Since  $X \cong Y$  if and only if  $X - Y = 0$ , nonzero elements in  $\Omega(G)$  correspond to reduced pairs of non-isomorphic  $G$ -sets, and 0 corresponds to the (reduced) pair  $(\emptyset, \emptyset)$ .

A second ring of interest is  $R(G)$ , the representation ring of  $G$ . Its elements are formal differences of isomorphism classes of complex representations of  $G$ , with the operations being direct sum and tensor product.  $R(G)$  also denotes the ring of virtual characters of  $G$ , which is isomorphic to the representation ring (see e.g. [21, §9.1]). There is a ring homomorphism from  $\Omega(G)$  into  $R(G)$ , given by  $X \mapsto \mathbb{C}[X]$  (or  $X \mapsto \chi_{\mathbb{C}[X]}$ , considering  $R(G)$  as the character ring). We denote the kernel of this homomorphism by  $\mathcal{L}(G)$ , and say that its elements are *linearly trivial*. The formal difference  $X - Y$  is in  $\mathcal{L}(G)$  if and only if  $\mathbb{C}[X] \cong \mathbb{C}[Y]$ , so that we have a correspondence between linearly trivial elements in  $\Omega(G)$  and reduced pairs of linearly equivalent  $G$ -sets.

---

(\*) Just like the map  $(x, y) \mapsto \frac{x}{y}$  gives a correspondence between reduced pairs of positive integers  $(x, y \in \mathbb{N}$  such that  $\gcd(x, y) = 1$ ), and positive rationals.

Since  $\mathcal{L}(G)$ , the ideal of linearly trivial elements, is a subgroup of the free abelian group  $\Omega(G)^+ \cong \mathbb{Z}^r$ , it is also free abelian:  $\mathcal{L}(G) \cong \mathbb{Z}^m$  for some  $m \leq r$ . This means that we can find a  $\mathbb{Z}$ -basis for  $\mathcal{L}(G)$  (we demonstrate how to compute such a basis in §5). This gives a lattice of linearly equivalent reduced pairs, as follows: if  $\{X_i - Y_i\}_{i=1..m}$  is a basis for  $\mathcal{L}(G)$ , and we define for  $\bar{n} = (n_1, \dots, n_m) \in \mathbb{Z}^m$

$$X_{\bar{n}} = \left( \bigcup_{i: n_i > 0} n_i X_i \right) \cup \left( \bigcup_{i: n_i < 0} |n_i| Y_i \right)$$

$$Y_{\bar{n}} = \left( \bigcup_{i: n_i < 0} |n_i| X_i \right) \cup \left( \bigcup_{i: n_i > 0} n_i Y_i \right)$$

then every reduced pair of linearly equivalent  $G$ -sets  $(X, Y)$  is obtained by canceling out common factors in  $(X_{\bar{n}}, Y_{\bar{n}})$ , for a unique  $\bar{n} \in \mathbb{Z}^m$ .

Given an action of  $G$  on a manifold  $M$ , we associate with every  $G$ -set  $X$  a manifold, namely  $M \times_G X$ . The lattice of linearly equivalent pairs then maps to a lattice of isospectral pairs (see the example in §5). For a general manifold  $M$ , this might be only a sublattice of the lattice of isospectral quotients, which can be described as follows: We pull the spectrum function backwards to  $\Omega(G)$ , defining  $\text{Spec}_{X-Y} = \text{Spec}_{M \times_G X} - \text{Spec}_{M \times_G Y}$  (so that we have  $\text{Spec} : \Omega(G) \rightarrow \mathbb{Z}^{\mathbb{R}}$ ). Isospectral pairs of the form  $(M \times_G X, M \times_G Y)$  are exactly those for which  $X - Y \in \ker \text{Spec}$ , and Corollary 3.3 states that this kernel (for any  $M$ ) contains  $\mathcal{L}(G)$ .

### 4. Construction of unbalanced pairs

Our objective in this section is to find unbalanced pairs. That is, given a group  $G$ , to find two  $G$ -sets  $X, Y$  which differ in the number of orbits of some size, and such that  $\mathbb{C}[X] \cong \mathbb{C}[Y]$  as  $\mathbb{C}G$ -modules. We shall do so by “balancing” unions of transitive  $G$ -sets, which correspond to coset spaces of the form  $G/H$ . For every subgroup  $H \leq G$  we denote by  $\mathcal{S}_H$  the function

$$\mathcal{S}_H(g) = \chi_{\mathbb{C}[G/H]}(g) = |\text{fix}_{G/H}(g)| = \frac{|[g] \cap H| |C_G(g)|}{|H|} \tag{4.1}$$

$\mathbb{C}[G/H]$  is sometimes called the *quasiregular representation* of  $G$  on  $H$ , and  $\mathcal{S}_H$  is thus the *quasiregular character*. It also bears the names  $\mathbf{1}_H^G$ ,  $\mathbf{1}^G_H$ , or  $\text{Ind}_H^G \mathbf{1}$ , being the induction of the trivial character of  $H$  to  $G$ . Lastly, it is the image of  $G/H$  under the map  $\Omega(G) \rightarrow R(G)$ , when the latter is regarded as the ring of virtual characters of  $G$ .

In light of Proposition 2.1, we shall seek  $H_i, K_i$  such that  $\sum_i \mathcal{S}_{H_i} = \sum_i \mathcal{S}_{K_i}$ , and then check that the obtained linearly equivalent pair is unbalanced. We use a few easy computations:

(1) For the trivial subgroup  $1 \leq G$ , we have

$$\mathcal{S}_1(g) = \begin{cases} |G| & g = e \\ 0 & g \neq e \end{cases} \tag{4.2}$$

(2) For  $H = G$ ,

$$\mathcal{S}_G \equiv 1 \tag{4.3}$$

(3) For any  $H$ ,

$$\mathcal{S}_H(e) = [G : H] \tag{4.4}$$

(4) For  $G$  abelian  $[g] = \{g\}$  and  $C_G(g) = G$ , so that  $\mathcal{S}_H = [G : H] \cdot \mathbf{1}_H$ , i.e.

$$\mathcal{S}_H(g) = \begin{cases} [G : H] & g \in H \\ 0 & g \notin H \end{cases} \tag{4.5}$$

### 4.1. Cyclic groups

Finite cyclic groups have no unbalanced pairs. This follows from the following:

PROPOSITION 4.1. — *If  $G$  is finite cyclic, linearly equivalent  $G$ -sets are isomorphic.*

*Proof.* — Let  $G = \mathbb{Z}/n\mathbb{Z}$ , and  $D = \{d \mid d > 0, d \mid n\}$ . The subgroups of  $G$  are  $H_d = \langle d \rangle$  for  $d \in D$ , and by (4.5)  $\mathcal{S}_{H_d} = \frac{n}{d} \cdot \mathbf{1}_{H_d}$ . A non-trivial pair of linearly equivalent  $G$ -sets corresponds to two different  $\mathbb{N}$ -combinations of  $\{\mathcal{S}_{H_d}\}_{d \in D}$  that agree as functions. Finding such a pair is equivalent to finding a nonzero  $\mathbb{Z}$ -combination of  $\{\mathcal{S}_{H_d}\}_{d \in D}$  which vanishes. However, the matrix  $(\mathcal{S}_{H_d}(d'))_{d, d' \in D}$  is upper triangular with non-vanishing diagonal, which means that  $\left\{ \mathcal{S}_{H_d} \Big|_D \right\}_{d \in D}$  are linearly independent over  $\mathbb{Q}$ , hence so are  $\{\mathcal{S}_{H_d}\}_{d \in D}$ . □

### 4.2. $G = \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$

Here we generalize the pair which appeared in Sections 1.1 and 2.2.1. Let  $p$  be a prime.  $G = \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$  has  $p + 1$  subgroups of size (and index)

$p$ :  $H_\lambda = \left\{ (x, y) \mid \frac{x}{y} = \lambda \right\}$ , where  $\lambda \in P^1(\mathbb{F}_p) = \{0, 1, \dots, p-1, \infty\}$ . Every non-identity element in  $G$  appears in exactly one of these, and we obtain by (4.4) and (4.5)

$$\sum_{\lambda \in P^1(\mathbb{F}_p)} \mathcal{S}_{H_\lambda}(g) = \begin{cases} p(p+1) & g = e \\ p & g \neq e \end{cases}.$$

Consulting (4.2) and (4.3), we find that this is the same as  $p \cdot \mathcal{S}_G + \mathcal{S}_1$ , so there is linear equivalence between

$$X = \bigcup_{\lambda \in P^1(\mathbb{F}_p)} G/H_\lambda \quad \text{and} \quad Y = \underbrace{\mathbf{1} \cup \dots \cup \mathbf{1}}_p \cup G, \tag{4.6}$$

where  $\mathbf{1}$  denotes the  $G$ -set with one element (corresponding to  $G/G$ ). Obviously, this is an unbalanced pair ( $X$  has  $p+1$  orbits of size  $p$ , and  $Y$  has one orbit of size  $p^2$  and  $p$  orbits with a single element). Figure 2.1 shows  $X, Y$  for  $p=2$  (by their Schreier graphs with respect to the standard basis of  $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ ).

### 4.2.1. Application - Hecke pairs

Let

$$G = \langle \sigma, \tau \mid \sigma^p = \tau^p = 1, \sigma\tau = \tau\sigma \rangle \cong \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$$

act on the torus  $T = \mathbb{R}^2/\mathbb{Z}^2$  by the rotations  $\sigma \cdot (x, y) = \left(x, y + \frac{1}{p}\right)$  and  $\tau \cdot (x, y) = \left(x + \frac{1}{p}, y\right)$ . From the unbalanced pair (4.6) one obtains the isospectral pair  $T \times_G X$  and  $T \times_G Y$ , each a union of  $p+1$  tori. These examples were constructed using different techniques by Doyle and Rossetti, who baptized them ‘‘Hecke pairs’’ [9]. The cases  $p=2, 3, 5$  are illustrated in Figure 4.1. One can verify that the analogue pair for  $p=4$ , for example, is not isospectral - the reason is that unlike in the prime case the subgroups

$$H_\lambda = \begin{cases} \{(x, \lambda x) \mid x \in \mathbb{Z}/4\mathbb{Z}\} & \lambda = 0..3 \\ \{(0, x) \mid x \in \mathbb{Z}/4\mathbb{Z}\} & \lambda = \infty \end{cases}$$

do not cover  $(\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}) \setminus \{0\}$  evenly.

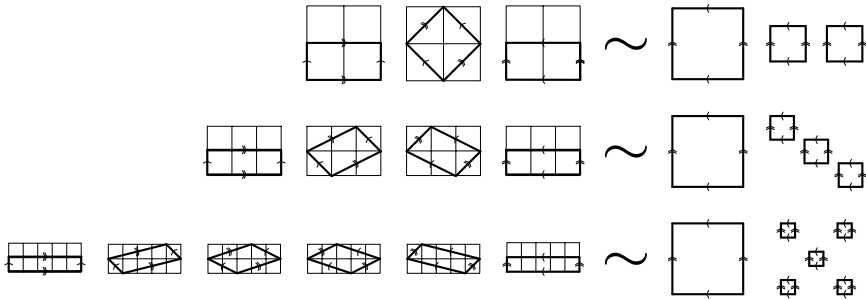


Figure 4.1. Isospectral pairs consisting of unions of tori, obtained as the tensor product over  $G = \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$  of the torus  $\mathbb{R}^2/\mathbb{Z}^2$  and the  $G$ -sets in (4.6), for  $p = 2, 3, 5$ . Grids are drawn to clarify the sizes.

Remark. — Since the spectrum of a flat torus is represented by a quadratic form, isospectrality between flat tori can be interpreted as equality in the representation numbers of forms<sup>(\*)</sup>. For example, isospectrality in the case  $p = 2$  (Figure 4.1, top) asserts that together the quadratic forms  $4m^2 + n^2$ ,  $2m^2 + 2n^2$  and  $4m^2 + n^2$  represent (over the integers) every value the same number of times as do  $m^2 + n^2$ ,  $4m^2 + 4n^2$ , and  $4m^2 + 4n^2$  together.

4.3.  $G = \mathbb{Z}/q\mathbb{Z} \rtimes \mathbb{Z}/p\mathbb{Z}$

Now let  $G$  be the non-abelian group of size  $pq$ , where  $p$  and  $q$  are primes such that  $q \equiv 1 \pmod{p}$ .  $G$  has one subgroup  $Q$  of size  $q$ , and  $q$  subgroups  $P_1, P_2, \dots, P_q$  of size  $p$ . Since  $Q$  is normal we have

$$[g] \cap Q = \begin{cases} [g] & g \in Q \\ \emptyset & g \notin Q \end{cases} \Rightarrow \mathcal{S}_Q(g) = \begin{cases} p & g \in Q \\ 0 & g \notin Q \end{cases} .$$

Every non-identity element of  $G$  generates its entire centralizer, for otherwise it would be in the center. Thus for  $g \neq e$

$$\begin{aligned} \sum_{i=1}^q \mathcal{S}_{P_i}(g) &= \frac{|C_G(g)|}{p} \sum_{i=1}^q |[g] \cap P_i| = \frac{|C_G(g)|}{p} \cdot |[g] \cap (G \setminus Q)| \\ &= \begin{cases} 0 & g \in Q \setminus \{e\} \\ q & g \notin Q \end{cases} \end{aligned}$$

(\*) This insight (in the opposite direction) led Milnor to the first construction of isospectral manifolds [19].



but since  $P_i$  are all conjugate we have  $\mathcal{S}_{P_i} = \mathcal{S}_{P_1}$  for all  $i$ . Denoting  $P = P_1$ , we have by the above and (4.4)

$$\mathcal{S}_P(g) = \begin{cases} q & g = e \\ 0 & g \in Q \setminus e \\ 1 & g \notin Q \end{cases}$$

and we find that

$$(p \cdot \mathcal{S}_P + \mathcal{S}_Q)(g) = (p \cdot \mathcal{S}_G + \mathcal{S}_1)(g) = \begin{cases} pq + p & g = e \\ p & g \neq e \end{cases}$$

which gives us the unbalanced pair

$$X = \underbrace{G/P \cup \dots \cup G/P}_p \cup G/Q \quad \text{and} \quad Y = \underbrace{\mathbf{1} \cup \dots \cup \mathbf{1}}_p \cup G.$$

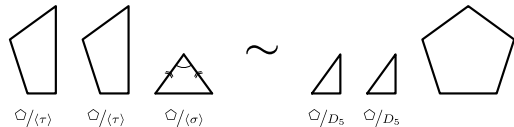
This pair was discovered and used for constructing isospectral surfaces by Hillairet [14].

### 4.3.1. Example - dihedral groups

A nice family of groups of the form  $\mathbb{Z}/q\mathbb{Z} \rtimes \mathbb{Z}/p\mathbb{Z}$  is formed by the dihedral groups of order  $2q$ , where  $q$  is an odd prime.  $D_q = \langle \sigma, \tau \mid \sigma^q, \tau^2, (\sigma\tau)^2 \rangle$  acts by symmetries on the regular  $q$ -gon (say, with Neumann boundary conditions). In this case, the unbalanced pair we obtained above is  $X = D_q/\langle \tau \rangle \cup D_q/\langle \tau \rangle \cup D_q/\langle \sigma \rangle$ ,  $Y = \mathbf{1} \cup \mathbf{1} \cup D_q$ , which gives for every  $q$  an isospectral pair consisting of six orbifolds, five of which are planar domains with Neumann boundary conditions, and the sixth (the quotient by  $\langle \sigma \rangle$ ) a  $\frac{2\pi}{q}$ -cone. Figure 4.2 shows the case  $q = 5$ .

Let us remark that similar pairs with different boundary conditions can be constructed by observing other representations of  $D_q$  and its subgroups - see [1, §9.3] for an example.

Figure 4.2. An isospectral pair obtained from the action of  $D_5 \cong \mathbb{Z}/5\mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z}$  on a regular pentagon. All boundary conditions are Neumann.



### 4.4. Non-cyclic groups

A group  $H$  is said to be *involved* in a group  $G$  if there exist some  $L \trianglelefteq K \trianglelefteq G$  such that  $K/L \cong H$ .

**PROPOSITION 4.2.** — *If a group  $H$  which has an unbalanced pair is involved in  $G$ , then  $G$  has an unbalanced pair.*

*Proof.* — It is enough to assume that  $H$  is either a subgroup or a quotient of  $G$ . Assume first that  $H \leq G$ . If  $X, Y$  is an unbalanced pair of  $H$ -sets, the induced  $G$ -sets  $G \times_H X$  and  $G \times_H Y$  (see §2.3) form an unbalanced pair as well:

- They are linearly equivalent: we have natural isomorphisms

$$\begin{aligned} \text{Hom}_{\mathbb{C}G}(\mathbb{C}[G \times_H X], \_) &\cong \text{Hom}_G(G \times_H X, \_) \\ &\cong \text{Hom}_H(X, \_) \cong \text{Hom}_{\mathbb{C}H}(\mathbb{C}[X], \_) \end{aligned}$$

where the first and last isomorphisms are by (3.2), and the middle one is by (2.2). Since  $\mathbb{C}[X] \cong \mathbb{C}[Y]$  as  $\mathbb{C}H$ -modules, we obtain that  $\mathbb{C}[G \times_H X] \cong \mathbb{C}[G \times_H Y]$  as  $\mathbb{C}G$ -modules.

- The sizes of orbits in  $G \times_H X$  are the sizes of orbits in  $X$  multiplied by  $[G : H]$ , since if  $X \cong \bigcup H/H_i$  is a decomposition of  $X$  into  $H$ -orbits then

$$\begin{aligned} G \times_H X &\cong G \times_H \left( \bigcup H/H_i \right) \cong \bigcup G \times_H H/H_i \\ &\cong \bigcup G \times_H \times_H \times_{H_i} \mathbf{1} \cong \bigcup G \times_{H_i} \mathbf{1} \cong \bigcup G/H_i \end{aligned}$$

is a decomposition of  $G \times_H X$  into  $G$ -orbits.

Assume now that  $\pi : G \twoheadrightarrow H$  is an epimorphism. An  $H$ -set  $X$  has a  $G$ -set structure by  $gx = \pi(g)x$ , and an unbalanced pair of  $H$ -sets  $X, Y$  is also an unbalanced pair of  $G$ -sets: since  $G$  realizes the same permutations in  $\text{Sym}(X)$  as does  $H$ , a linear  $H$ -equivariant isomorphism  $\mathbb{C}[X] \cong \mathbb{C}[Y]$  is also  $G$ -equivariant, and the orbits in  $X$  as a  $G$ -set and as an  $H$ -set are the same. □

*Remark.* — If  $G$  acts on a manifold  $M$ , and  $X$  is an  $H$ -set for some  $H \leq G$ , then we have

$$M \times_G (G \times_H X) \cong (M \times_G G) \times_H X \cong M \times_H X$$

i.e. the induced  $G$ -set gives the same manifold as does the original  $H$ -set.

**THEOREM 4.3.** — *Every non-cyclic finite group has an unbalanced pair.*

*Proof.* — Let  $G$  be a non-cyclic finite group. Assume first that some  $p$ -Sylow group  $P \leq G$  is not cyclic (in particular this is the case if  $G$  is abelian). Let  $\Phi(P)$  be the Frattini subgroup of  $P$ , which is the intersection of all of its maximal proper subgroups. For any  $p$ -group  $P$  the quotient  $P/\Phi(P)$  is an elementary  $p$ -group of the same rank as  $P$ , so that if  $P$  is non-cyclic then  $P/\Phi(P)$  must contain  $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$ . Therefore,  $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$  is involved in  $G$  and we are done by Proposition 4.2 and §4.2.

Zassenhaus classified the finite groups whose Sylow subgroups are all cyclic [13, Thm. 9.4.3]. They are of the form

$$G_{m,n,r} = \langle a, b \mid a^m = b^n = e, b^{-1}ab = a^r \rangle = \mathbb{Z}/m\mathbb{Z} \rtimes_{\vartheta_r} \mathbb{Z}/n\mathbb{Z}$$

for  $m, n, r$  satisfying  $(m, n(r-1)) = 1$  (here  $\vartheta_r(1)(1) = r$ , and  $r^n \equiv 1 \pmod{m}$ ) is implied to make  $\vartheta_r : \mathbb{Z}/n\mathbb{Z} \rightarrow \text{Aut}(\mathbb{Z}/m\mathbb{Z}) \cong (\mathbb{Z}/m\mathbb{Z})^\times$  a homomorphism). Since  $0 \times \ker \vartheta_r \leq Z(G)$ , and the quotient  $G/Z(G)$  is never cyclic for nonabelian  $G$ , we can assume (by Proposition 4.2) that  $\vartheta_r$  is injective. We can also assume that  $n$  is prime, for otherwise for any nontrivial factor  $k$  of  $n$  we have a proper subgroup  $\langle a, b^k \rangle = \mathbb{Z}/m\mathbb{Z} \rtimes_{\vartheta_{r,k}} \mathbb{Z}/\frac{n}{k}\mathbb{Z}$  which is non-cyclic by the injectivity of  $\vartheta_r$ . We can further assume that  $m$  is prime. Otherwise, pick some prime  $q$  dividing  $m$ , and consider  $\langle a^{m/q}, b \rangle$ : it is cyclic only if  $\vartheta_r$  fixes  $a^{m/q}$ , i.e.  $a^{r^{m/q}} = a^{m/q}$ , so that  $m \mid \frac{m}{q}(r-1)$ , which is impossible since  $(m, n(r-1)) = 1$ . Thus, by §4.3 we are done.  $\square$

Since unbalanced  $G$ -sets are in particular non-isomorphic, this together with Proposition 4.1 give the following:

**COROLLARY 4.4.** — *For a finite group  $G$ , the map  $\Omega(G) \rightarrow R(G)$  which takes a  $G$ -set  $X$  to the representation  $\mathbb{C}[X]$  is injective iff  $G$  is cyclic.*

Theorems 3.6 and 4.3 together imply the results announced in §1:

**COROLLARY 4.5.** — *If a finite non-cyclic group  $G$  acts faithfully by isometries on a compact connected Riemannian manifold  $M$ , then there exist  $G$ -sets  $X, Y$  such that  $M \times_G X$  and  $M \times_G Y$  are isospectral and non-isometric.*

From this follows:

**COROLLARY 4.6.** — *If  $M$  is a compact connected Riemannian manifold (or orbifold) such that  $\pi_1(M)$  has a finite non-cyclic quotient, then  $M$  has isospectral non-isometric covers.*

*Proof.* — Let  $\widetilde{M}$  be the universal cover of  $M$ , and  $N$  a normal subgroup in  $\pi_1(M)$  such that  $G = \pi_1(M)/N$  is finite non-cyclic.  $\widehat{M} = \widetilde{M}/N$  is a finite cover of  $M$  and thus compact, and  $G$  acts on it faithfully, with  $\widehat{M}/G = M$ . By the previous corollary there exist isospectral non-isometric unions of quotients of  $\widehat{M}$  by subgroups of  $G$ , and these are covers of  $M$ .  $\square$

### 5. Computation

Here we show how to compute, using GAP [10], a basis for  $\mathcal{L}(G)$ , the ideal of linearly trivial elements in the Burnside ring  $\Omega(G)$ , which correspond to reduced pairs of linearly equivalent  $G$ -sets. We then consider an action of  $G$  and compute the isospectral pairs which correspond to this basis and action.

Let  $G = D_6$  (see §4.3.1), and let  $\{H_i\}$  be a set of representatives for the conjugacy classes of subgroups of  $G$  (so that  $\{G/H_i\}$  is a  $\mathbb{Z}$ -basis of  $\Omega(G)$ ). In the example which follows we compute the corresponding quasiregular characters  $c_i = \mathcal{S}_{H_i}$ , which are the images of this basis under the map  $\Omega(G) \rightarrow R(G)$ . We then compute a basis for  $\mathcal{L}(G)$ , the kernel of this map, and apply the LLL algorithm to this basis in order to possibly obtain a sparser one.

```
gap > G := DihedralGroup(12);;
gap > H := List(ConjugacyClassesSubgroups(G), Representative);;
gap > c := List(H, h -> List(PermutationCharacter(G, h)));;
gap > LLLReducedBasis(NullspaceIntMat(c)).basis;
      [[0, 1, 0, -1, 0, 0, -1, 0, 1, 0], [1, -1, 0, -1, 0, 0, 0, -1, 0, 2],
      [-1, 1, 0, 1, 1, 0, -1, 0, -1, 0], [-1, 1, 1, 1, 0, -2, 0, 0, 0, 0]]
```

For example, the first element in the basis we obtained tells us that  $G/H_2 - G/H_4 - G/H_7 + G/H_9$  vanishes in  $R(G)$ , so that  $G/H_2 \cup G/H_9$  is linearly equivalent to  $G/H_4 \cup G/H_7$ . One has to explore the output of `ConjugacyClassesSubgroups(G)` to find out which subgroups these exactly are, or alternatively, to construct  $H_i$  oneself (in this case, for example,  $H_2$  belongs to the conjugacy class of  $\langle \tau \rangle$ ). The first line in Table 5.1 presents representatives for the classes returned by `ConjugacyClassesSubgroups(G)`, and the bottom four lines of the table show the basis that was calculated for  $\mathcal{L}(D_6)$  above. One may check that pairs II, III and IV are unbalanced.

$H_i$	$\langle 1 \rangle$	$\langle \tau \rangle$	$\langle \sigma^3 \rangle$	$\langle \tau\sigma \rangle$	$\langle \sigma^2 \rangle$	$\langle \tau, \tau\sigma^3 \rangle$	$\langle \tau, \tau\sigma^2 \rangle$	$\langle \sigma \rangle$	$\langle \tau\sigma, \tau\sigma^3 \rangle$	$\langle \tau, \tau\sigma \rangle$
$\circlearrowleft/H_i$										
I	0	1	0	-1	0	0	-1	0	1	0
II	1	-1	0	-1	0	0	0	-1	0	2
III	-1	1	0	1	1	0	-1	0	-1	0
VI	-1	1	1	1	0	-2	0	0	0	0

Table 5.1. Representatives for the conjugacy classes of subgroups in  $D_6$ , displayed with the corresponding quotients of the hexagon, and a basis for  $\mathcal{L}(D_6) = \ker(\Omega(D_6) \rightarrow R(D_6))$ .

Given an action of  $G$  on a manifold  $M$ , every difference of  $G$ -sets  $X - Y \in \mathcal{L}(G)$  gives rise to an isospectral pair, namely  $M \times_G X, M \times_G Y$ . We consider the standard action of  $D_6$  on the regular hexagon, which we denote by  $\circlearrowleft$ . The second line in Table 5.1 shows the quotients  $\circlearrowleft/H_i$  corresponding to the subgroups  $H_i \leq D_6$  in the topmost line, and we see that in this case there are no isometric quotients arising from non-isomorphic  $G$ -sets. The isospectral pairs corresponding to the basis we obtained for  $\mathcal{L}(D_6)$  are shown in Table 5.2.

I			$\sim$			
II			$\sim$			
III			$\sim$			
IV			$\sim$			
I - III			$\sim$			

Table 5.2. The isospectral pairs corresponding to the basis for  $\mathcal{L}(D_6)$  described in Table 5.1, and an example of an element obtained as a combination of these.

All isospectral pairs which arise from linear equivalences between  $D_6$ -sets are spanned by these four, as explained in §3.3. The bottom line in Table 5.2 demonstrates such a pair (corresponding to the element I – III). We remark that the pair corresponding to I is a hexagonal analogue of Chapman’s “two piece band” [6] - such analogues exist for every  $n$  (but for odd  $n$  the isospectral pair obtained is also isometric).

## 6. Generalizations

The isospectrality technique this paper describes (and thus Sunada’s technique as well) has actually little to do with spectral geometry, since no property of the Laplace operator is used apart from being linear and commuting with isometries. For any linear operator  $F$  (on function spaces or other bundles, over manifolds or general spaces), these methods produce  $F$ -isospectral objects, given an action of a group which commutes with  $F$ .

However, it seems that in much more general settings, when a group action is studied, Sunada pairs are worth looking at. The most famous examples are Galois theory, giving Gassmann’s construction of arithmetically equivalent number fields [11], and Riemannian coverings, giving Sunada’s isospectral construction; but Sunada pairs were also studied in the context of Lie groups [7], ergodic systems [17], dessin d’enfants [18], the spectrum of discrete graphs [3] and metric ones [22], the Ihara zeta function of graphs [23], and the Witten zeta function of a Lie group [16].

Sunada pairs in  $G$  correspond to linearly equivalent transitive  $G$ -sets, and we have seen that in the context of Riemannian coverings Sunada’s technique generalizes to non-transitive  $G$ -sets as well. We achieved this by considering the quotient  $M/H$  as the tensor product with the transitive  $G$ -set  $G/H$ , i.e., by noting that  $M/H \cong M \times_G G/H$ , and then studying  $M \times_G X$  for a general  $G$ -set  $X^{(*)}$ . It is natural to ask whether other applications of Sunada pairs can be generalized in an analogous way. Of particular interest are unbalanced pairs, which do not exist in the transitive case (see Remark 3.5). In the settings of Riemannian manifolds they allowed us to deduce non-isometry, and one may hope that they play interesting roles in other situations.

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(\*) Alternatively, for  $X = \bigcup G/H_i$  we studied the disjoint union of quotients  $\bigcup M/H_i$ .

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