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A MAXIMAL REGULAR BOUNDARY FOR SOLUTIONS OF ELLIPTIC DIFFERENTIAL EQUATIONS

by Peter A. LOEB [1] and Bertram WALSH [2]

The object of this paper is the study of the behavior of solutions of elliptic differential equations at ideal boundaries of their domains. We employ the axiomatic approach of Brelot [1], i.e. we consider a sheaf $\mathcal{H}$ of real-valued functions with open domains contained in a locally compact, non-compact, connected and locally connected Hausdorff space W, with the functions satisfying certain axioms. Specifically, by a harmonic class of functions on W we mean a class $\mathcal{H}$ of real-valued continuous functions with open domains. For each open $\Omega \subseteq W$, $\mathcal{H}_\Omega$ denotes the set of functions in $\mathcal{H}$ with domains equal to Ω ; it is assumed that $\mathcal{H}_\Omega$ is a real vector space. The three axioms of Brelot which $\mathcal{H}$ is assumed to satisfy are (1) a function is in $\mathcal{H}$ if and only if it is locally in $\mathcal{H}$; (2) there is a base for the topology of W which consists of regions regular for $\mathcal{H}$, i.e. connected open sets ω such that any continuous function f on $\partial\omega$ has a unique continuous extension in $\mathcal{H}_\omega$ which is nonnegative if f is nonnegative ; (3) the upper envelope of any increasing sequence of functions in $\mathcal{H}_\Omega$ where Ω is a region (i.e. open and connected) is either $+\infty$ or an element of $\mathcal{H}_\Omega$. In addition, we shall often assume that : (4) 1 is $\mathcal{H}$-superharmonic in W.

In the first section below, we shall establish a criterion for a point on an arbitrary ideal boundary to be a regular point with respect to the Dirichlet problem for that boundary. In the second section we shall define compactifying boundaries which contain compact subsets $\Gamma_{\mathfrak{S}}$ such that sublattices $\mathfrak{S}$ of the Banach lattice $\mathcal{B}\mathcal{H}_W$ of bounded functions in $\mathcal{H}_W$ are isometrically isomorphic to the space of continuous functions on $\Gamma_{\mathfrak{S}}$. This yields a simplified version of the boundary theories of Wiener and Royden. The third section will consider two harmonic

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classes $\mathcal{H}$ and $\mathcal{K}$ where the positive functions of $\mathcal{H}$, restricted to the complement of a fixed compact set A, are superharmonic with respect to $\mathcal{K}$. In particular, we shall reëxamine the isometric isomorphism (established in [5]) between $\mathcal{B}\mathcal{H}_W$ and $\mathcal{B}\mathcal{K}_W$ in terms of boundary correspondences, i.e. the isomorphisms between $\mathcal{B}\mathcal{H}_W$ and $\mathcal{C}_R(\Gamma_{\mathcal{H}})$ and between $\mathcal{B}\mathcal{K}_W$ and $\mathcal{C}_R(\Gamma_{\mathcal{K}})$.

The terminology and notation which we shall employ here are the same as in [5] with a few exceptions : script rather than German letters ($\mathcal{H}$ and $\mathcal{K}$) refer to harmonic classes. Instead of $\lim_{x\in A,\, x\to x_0} f(x)$, $\liminf_{x\in A,\, x\to x_0} f(x)$ and $\limsup_{x\in A,\, x\to x_0} f(x)$ we shall simply write $\lim_A f(x_0)$, $\liminf_A f(x_0)$ and $\limsup_A f(x_0)$ respectively. In order that a harmonic class $\mathcal{H}$ be called hyperbolic or parabolic on W, 1 must be superharmonic. Therefore, if $\mathcal{H}$ is described as a hyperbolic or parabolic class it is understood that 1 is $\mathcal{H}$-superharmonic. Note that we use bold face to denote constant functions.

The functions H(W), H(W − A), H(∂A , W − A) and H(∂W , W − A), which are defined in Section 5 of [5], will be used throughout this paper. We shall frequently make use of the fact that if $v \in \overline{\mathcal{H}}$ and $u \in \underline{\mathcal{H}}$ (where $\overline{\mathcal{H}}$ and $\underline{\mathcal{H}}$ are the superharmonic and subharmonic classes associated with $\mathcal{H}$ respectively) and $v \geqslant u$, then there is a function $h \in \mathcal{H}$ such that $v \geqslant h \geqslant u$. The following fact will also be needed :

PROPOSITION 0.1. – *Let* $\mathcal{H}$ *be a harmonic class on* W *with* $\mathbf{1} \in \overline{\mathcal{H}}_W$, *and let* $\mathbf{H}(u)$ *denote the least* $\mathcal{H}$*-harmonic majorant of an upper-bounded function* $u \in \underline{\mathcal{H}}$. *Then for every pair of upper-bounded functions* u_1 *and* u_2 *in* $\underline{\mathcal{H}}$, *we have* $\mathbf{H}(u_1 + u_2) = \mathbf{H}(u_1) + \mathbf{H}(u_2)$, *and for every upper-bounded* $u \in \underline{\mathcal{H}}$ *and nonnegative* $\alpha \in \mathbf{R}$,

$$\mathbf{H}(\alpha \,.\, u) = \alpha \,.\, \mathbf{H}(u).$$

Proof. – Since

$$\mathbf{H}(u_1 + u_2) \geqslant u_1 + u_2 \,,\; \mathbf{H}(u_1 + u_2) - u_1 \geqslant \mathbf{H}(u_2) \geqslant u_2 \,,$$

so
$$\mathbf{H}(u_1 + u_2) - \mathbf{H}(u_2) \geqslant \mathbf{H}(u_1) \geqslant u_1 \,.$$

Thus
$$\mathbf{H}(u_1 + u_2) \geqslant \mathbf{H}(u_1) + \mathbf{H}(u_2) \,.$$

The rest is clear. ∎

Finally, we recall that a compact set $A \subseteq W$ is *outer-regular* if it is nonempty and if there exists a barrier for $W \sim A$ at each point of ∂A. Although $W \sim A$ need not be connected, it is possible to establish the following fact :

THEOREM 0.2. – *Given a harmonic class* $\mathcal{H}$ *on* W, *let* C *be an outer-regular compact subset of* W *and* Ω *an open neighborhood of* C. *Then there is an outer-regular compact set* A *with* $C \subseteq A \subset \Omega$ *such that* $W \sim A$ *has only a finite number of components.*

Proof. – One may assume that $\bar{\Omega}$ is a compact subset of W. Let $\{U_a\}$ be the components of $W \sim C$. Since W is locally connected, each U_a is open, and since W is connected, each $\partial U_a = \bar{U}_a \sim U_a$ is a nonempty subset of ∂C. Now the collection $\{\Omega\} \cup \{U_a\}_a$ is an open covering of $\bar{\Omega}$, and thus there is a finite collection $\{U_1, \ldots, U_n\}$ of components of $W \sim C$ which covers $\partial\Omega = \bar{\Omega} \sim \Omega$ and must therefore cover $W \sim \Omega$. Setting $A = W \sim \bigcup_{i=1}^{n} U_i$, we see that A is compact, since it is a subset of Ω. Every point x in ∂A belongs to some ∂U_i and thus to ∂C, so there is a barrier for $W \sim C$ at x ; that barrier is clearly also a barrier for $W \sim A$ at x. ∎

Section 1.

In this section we shall discuss the Dirichlet problem for open subsets of W with respect to their boundaries in a Hausdorff space $\bar{W}$ which properly contains W as a dense (and therefore open) subspace. We shall fix $\bar{W}$, and in the discussion below $\bar{\Omega}$ will mean the closure of Ω in $\bar{W}$ and $\partial\Omega$ will mean $\bar{\Omega} \sim \Omega$.

Let $\mathcal{H}$ be a harmonic class on W. For a given open subset Ω of W, $\overline{\mathcal{H}}^b_\Omega$ will denote the set of lower-bounded functions in $\overline{\mathcal{H}}_\Omega$, and $\underline{\mathcal{H}}^b_\Omega$ the set of upper-bounded functions in $\underline{\mathcal{H}}_\Omega$. A function $v \in \overline{\mathcal{H}}_\Omega$ will be said to be *nonnegative at* $\partial\Omega$, or *positive at* $\partial\Omega$ respectively, if $\inf_{x_0 \in \partial\Omega} (\lim \inf_\Omega v(x_0))$ is nonnegative or positive respectively. With this nomenclature a definition due to Brelot [1, p. 98] takes the following form :

DEFINITION. – *Let* Ω *be an open subset of* W. *We say that* $\partial\Omega$ *is associated with* $\overline{\mathcal{H}}^b_\Omega$ *if every* $v \in \overline{\mathcal{H}}^b_\Omega$ *which is nonnegative at* $\partial\Omega$ *is nonnegative throughout* Ω.

It is a consequence of a result of Constantinescu and Cornea [3, Thm. 4.1, p. 32] that $\partial\Omega$ is associated with $\overline{\mathcal{H}}^b_\Omega$ whenever $\bar{\Omega}$ is compact and there exists a function $V \in \overline{\mathcal{H}}_\Omega$ with $\inf_\Omega V(x) > 0$. The following proposition shows that the property of having a boundary associated with $\overline{\mathcal{H}}^b_W$ is inherited by open subsets of W, and also establishes a partial converse for open sets with compact complements.

PROPOSITION 1.1. – *If* Ω *is an open subset of* W *and* $\Gamma = \bar{W} \sim W$ *is associated with* $\overline{\mathcal{H}}^b_W$, *then* $\partial\Omega$ *is associated with* $\overline{\mathcal{H}}^b_\Omega$. *Conversely, if* $1 \in \overline{\mathcal{H}}_W$ *and* $W \sim \Omega$ *is compact, then* Γ *is associated with* $\overline{\mathcal{H}}^b_W$ *whenever* $\partial\Omega$ *is associated with* $\overline{\mathcal{H}}^b_\Omega$.

Proof. – Suppose that Γ is associated with $\overline{\mathcal{H}}^b_W$, and let v be a function in $\overline{\mathcal{H}}^b_\Omega$ which is nonnegative at $\partial\Omega$. If one defines a function V by setting $V = 0$ in $W \sim \Omega$ and $V = v \wedge 0$ in Ω, then $V \in \overline{\mathcal{H}}_W$ and so $V \geqslant 0$, whence $v \geqslant 0$ also.

On the other hand, assume that $W \sim \Omega$ is compact, $1 \in \overline{\mathcal{H}}_W$, and $\partial\Omega$ is associated with $\overline{\mathcal{H}}^b_\Omega$. Let v be a function in $\overline{\mathcal{H}}^b_W$ which is nonnegative at Γ. Let $\alpha = \min\left(0, \inf_{x \in W \sim \Omega} v(x)\right)$; then $v - \alpha$ is nonnegative at $\partial\Omega$, whence $v \geqslant \alpha$ in Ω, and so $v \geqslant \alpha$ in W. But since $\inf_{x \in W \sim \Omega} v(x)$ is attained at some point in the compact set $W \sim \Omega$ and v cannot take a nonpositive minimum in W, α must be zero and so $v \geqslant 0$ in W. ∎

Motivated by the preceding proposition, we assume for the rest of this section that Γ is associated with $\overline{\mathcal{H}}^b_W$, and we consider a fixed open $\Omega \subseteq W$. Assume that there is a function $V \in \overline{\mathcal{H}}_\Omega$ with $\inf_\Omega V > 0$. Given a bounded real-valued function f on $\partial\Omega$, let $\mathcal{V}(f, \Omega)$ denote the set $\{v \in \overline{\mathcal{H}}^b_\Omega : \liminf_\Omega v(x) \geqslant f(x) \text{ for all } x \in \partial\Omega\}$. Let $\overline{H}(f, \Omega)$ or simply $\overline{H}(f)$ denote the lower envelope of the functions in $\mathcal{V}(f)$, and $\underline{H}(f, \Omega)$ or simply $\underline{H}(f)$ denote $-\overline{H}(-f)$. We call $\overline{H}(f)$ and $\underline{H}(f)$ the *upper-$\mathcal{H}$-extension of f in Ω* and the *lower-$\mathcal{H}$-extension of f in Ω* respectively. If $\overline{H}(f) = \underline{H}(f)$, we say that f *is resolutive on* $\partial\Omega$. Now $\overline{H}(f) \geqslant \underline{H}(f)$ and the functions $\overline{H}(f)$ and $\underline{H}(f)$ belong to $\mathcal{H}$ (same proof as in [5, p. 179]). Moreover, the proof of Brelot's comparison theorem ([5], Th. 3.2) is valid in this case. Thus if ω is an open subset of Ω and F is the function on $\partial\omega$ such that $F = f$ on $\partial\omega \cap \partial\Omega$ and $F = \overline{H}(f, \Omega)$ on $\partial\omega \cap \Omega$, then $\overline{H}(f, \Omega) = \overline{H}(F, \omega)$ in ω.

DEFINITION. – *A point x_0 on $\partial\Omega$ is said to be regular for Ω with respect to $\mathcal{H}$ or simply regular if for every bounded function f on $\partial\Omega$ the inequalities*

$$\liminf_{\partial\Omega} f(x_0) \leqslant \liminf_{\Omega} \underline{H}(f)(x_0) \leqslant \limsup_{\Omega} \overline{H}(f)(x_0)$$
$$\leqslant \limsup_{\partial\Omega} f(x_0) \tag{1}$$

hold. To show that x_0 is regular, it is clearly sufficient to show that $\limsup \overline{H}(f)(x_0) \leqslant \limsup f(x_0)$ *for every bounded function f on $\partial\Omega$.*

We now establish a criterion for the regularity of points on $\partial\Omega$.

DEFINITION. – *Let x_0 be a point on $\partial\Omega$. A positive function $b \in \mathcal{H}$ defined in the intersection of Ω with an open neighborhood of x_0 and for which $\lim_{\Omega} b(x_0) = 0$ is called an $\mathcal{H}$-barrier (or simply a barrier) for Ω at x_0. We say that there is a system of barriers for Ω at x_0 if there is a base Θ for the neighborhood system of x_0 such that on the intersection of Ω with each $\omega \in \Theta$ there is defined a barrier b for Ω at x_0 with* $\inf_{x_1 \in \partial(\omega \cap \Omega) \sim (\omega \cap \partial\Omega)} (\liminf_{\Omega} b(x_1)) > 0$ *(such a barrier is said to belong to Ω and ω). By an $\overline{\mathcal{H}}$-unit-barrier for Ω at x_0 we mean a function $\overline{b}_1 \in \overline{\mathcal{H}}$ defined on the intersection of Ω with a neighborhood of x_0, having the property that $\lim_{\Omega} \overline{b}_1(x_0) = 1$; an $\underline{\mathcal{H}}$-unit barrier for Ω at x_0 is a function $\underline{b}_1 \in \underline{\mathcal{H}}$ satisfying the same conditions.*

The same proof as [5, Thm. 3.3] will show that if $x_0 \in \partial\Omega \cap W$ and there exists a barrier b for Ω at x_0 and a function $V \in \overline{\mathcal{H}}^b_{\Omega}$ which is bounded in a neighborhood of x_0 and positive at $\partial\Omega$, then x_0 is a regular point for Ω. To handle the points $x_0 \in \partial\Omega \cap \Gamma$, we need the following theorem.

THEOREM 1.2. – *Let x_0 be a point on $\partial\Omega$. Assume that there is a function $V_0 \in \overline{\mathcal{H}}^b_{\Omega}$ which is bounded in a neighborhood of x_0 and positive at $\partial\Omega$. Assume further that there is a system of barriers, an $\overline{\mathcal{H}}$-unit-barrier $\overline{b}_1$ and an $\underline{\mathcal{H}}$-unit-barrier $\underline{b}_1$ for Ω at x_0. Then x_0 is a regular point for Ω.*

Proof. – Let f be a bounded function on $\partial\Omega$, and let

$$c = \limsup_{\partial\Omega} f(x_0) .$$

Let $\varepsilon > 0$ be given. If $c + \varepsilon \geqslant 0$, let $V = (c+\varepsilon) . \overline{b}_1$; if $c + \varepsilon < 0$, let $V = (c+\varepsilon) . \underline{b}_1$. Then $V \in \mathcal{H}$ and $\lim_\Omega V(x_0) = c + \varepsilon$. Let ω be an open neighborhood of x_0 and $\omega_0 = \omega \cap \Omega$. By taking ω smaller if necessary, we may assume that it has the following properties :

(i) $f < c + \dfrac{\varepsilon}{2}$ on $\partial\Omega \cap \omega$;

(ii) V is defined on ω_0 and $c + \dfrac{\varepsilon}{2} < V$ on ω_0 ;

(iii) V_0 is bounded on ω_0 ;

(iv) there is a barrier b for Ω at x_0 defined on ω_0 for which $m = \inf_{x_1 \in [\partial\omega_0 \sim (\omega \cap \partial\Omega)]} (\liminf_{\omega_0} b)(x_1) > 0.$

Let $F = f$ on $\partial\omega_0 \cap \partial\Omega$ and $F = \overline{H}(f, \Omega)$ on $\partial\omega_0 \cap \Omega$; then (iii) above implies that F is bounded. Setting $M = |c| + \sup_{x \in \partial\omega_0} |F|$, we have $\dfrac{M}{m} . b + V \in \mathcal{V}(F, \omega_0)$ and $\lim\left(\dfrac{M}{m} . b + V\right)(x_0) = c + \varepsilon$. Consequently $\limsup_{\omega_0} \overline{H}(F, \omega_0)(x_0) \leqslant c + \varepsilon$, and Brelot's comparison theorem gives $\limsup_\Omega \overline{H}(f, \Omega)(x_0) \leqslant c + \varepsilon$. Since $\varepsilon > 0$ was arbitrary, x_0 is a regular point for Ω. ∎

If $\overline{\Omega}$ is compact, then Ω is regular (in the sense that each continuous function f on $\partial\Omega$ can be continuously extended into Ω by a function $H(f, \Omega) \in \mathcal{H}$) if and only if inequality (1) is satisfied at each point $x_0 \in \partial\Omega$ (see the proof of 1.3 in [5]). When $\overline{\Omega}$ is compact and Ω is regular, the $\mathcal{H}$-extension of 1 is clearly an $\overline{\mathcal{H}}$- and an $\underline{\mathcal{H}}$-unit-barrier for Ω at each point $x_0 \in \partial\Omega$; moreover, we have the following results :

PROPOSITION 1.3. – *Let Ω be a regular open set in* W *such that $\overline{\Omega}$ is compact and each component has at least two boundary points. Then there is a system of barriers at a point $x_0 \in \partial\Omega$ if Ω has only a finite number of components or if x_0 is a* G_δ *in* $\partial\Omega$.

Proof. – Similar to Prop. 3.4 in [5]. ∎

PROPOSITION 1.4. – *Let Ω be a regular region such that $\overline{\Omega}$ is compact and $\partial\Omega$ consists of a single point x_0. If $1 \in \overline{\mathcal{H}_\Omega} \sim \mathcal{H}_\Omega$ then* $1 - H(1, \Omega)$ *is a barrier at x_0 and indeed there is a system of barriers for Ω at x_0.*

Section 2.

This section will be concerned primarily with the construction of ideal boundaries from lattices of bounded $\mathcal{H}$-harmonic functions, where $\mathcal{H}$ is a harmonic class hyperbolic on W. Let us note briefly some facts about the sets $\mathcal{B}\mathcal{H}_W$ (called $\mathcal{H}^\infty$ by L. Lumer-Naïm and others) consisting of all bounded $\mathcal{H}$-harmonic functions on W and $\mathcal{B}_0\mathcal{H}_{W \sim A}$ consisting of all bounded $\mathcal{H}$-harmonic functions on $W \sim A$ which vanish at ∂A (where A is an outer-regular compact set–cf. [5, § 8]). It is clear that these are ordered Banach spaces under the pointwise operations and order and the uniform norm. By agreeing to extend the functions in $\mathcal{B}_0\mathcal{H}_{W \sim A}$ to be identically zero on A, we can think of both $\mathcal{B}\mathcal{H}_W$ and $\mathcal{B}_0\mathcal{H}_{W \sim A}$ as spaces of functions defined on W. Moreover, assuming Axiom IV [5], as we shall henceforth, both these spaces have order units, and their norms are the order norms. H(W) is the order unit for $\mathcal{B}\mathcal{H}_W$ and $H(\partial W, W \sim A)$ is the order unit for $\mathcal{B}_0\mathcal{H}_{W \sim A}$ (cf. also [5, Prop. 5.3]). Certain vector subspaces of these spaces, containing the order unit, are vector lattices: the one-dimensional subspace consisting of multiples of the order unit is a vector lattice; so are the spaces $\mathcal{B}\mathcal{H}_W$ and $\mathcal{B}_0\mathcal{H}_{W \sim A}$ themselves, since given any f and g in $\mathcal{B}\mathcal{H}_W$ or $\mathcal{B}_0\mathcal{H}_{W \sim A}$ respectively, their pointwise supremum is bounded and subharmonic, and thus has a bounded least harmonic majorant in W or $W \sim A$ respectively, which is clearly the smallest bounded harmonic function which simultaneously majorizes f and g. (If $\mathcal{B}_0\mathcal{H}_{W \sim A}$ is under consideration, then the fact that both f and g are majorized by a suitable multiple of $H(\partial W, W \sim A)$ insures that the least harmonic majorant of $f \vee g$ vanishes at ∂A). Here, however, the lattice operations are not the pointwise ones. Since we will be interested in studying vector sublattices of $\mathcal{B}\mathcal{H}_W$ and $\mathcal{B}_0\mathcal{H}_{W \sim A}$, let us make the convention that the symbols $\vee_{\mathcal{H}}$, $\wedge_{\mathcal{H}}$, $|\ |_{\mathcal{H}}$, $\sup_{\mathcal{H}}$ and $\inf_{\mathcal{H}}$ will denote the lattice operations in those spaces, while the unmodified symbols $\vee$, $\wedge$, $|\ |$, sup and inf will denote the ordinary pointwise lattice operations.

With $\mathcal{H}$ a hyperbolic harmonic class on W, let a Banach sublattice $\mathfrak{S}$ of $\mathcal{B}\mathcal{H}_W$ or $\mathcal{B}_0\mathcal{H}_{W\sim A}$ (A an outer-regular compact set) be given. One may then form the Q-compactification [2, pp. 96-97 and 6] W* of W with $Q = \mathfrak{S}$; this is a compact Hausdorff space containing W as a dense (and therefore open) subspace, and is determined up to homeomorphism by the property that each function $f \in \mathfrak{S}$ has a (necessarily unique) continuous extension to W*, which extension we shall also denote by f, such that the family of all these extensions separates the points of $\Delta_{\mathfrak{S}} = W^* \sim W$. (If $\mathfrak{S} \subseteq \mathcal{B}_0\mathcal{H}_{W\sim A}$, recall that we agreed to extend all functions in $\mathcal{B}_0\mathcal{H}_{W\sim A}$ to be zero in A.)

Given this compactification, our object is to determine a smaller, more tractable associated border contained in $\Delta_{\mathfrak{S}}$. One such border has been considered by Constantinescu and Cornea [3, § 4, p. 30] : Define $\Gamma \subseteq \Delta_{\mathfrak{S}}$ to be the intersection of all sets $\Gamma_p = \{t : t \in \Delta_{\mathfrak{S}}, \liminf p(t) = 0\}$, where p is an $\mathcal{H}$-potential on W (if $\mathfrak{S} \subseteq \mathcal{B}\mathcal{H}_W$) or $W \sim A$ (if $\mathfrak{S} \subseteq \mathcal{B}_0\mathcal{H}_{W\sim A}$). We take a different approach which is more elementary and does not require the use of the Evans potentials of [3] ; the equivalence of our boundary $\Gamma_{\mathfrak{S}}$ to the Γ of [3] will be shown below.

For the sake of a unified treatment of the two cases $\mathfrak{S} \subseteq \mathcal{B}\mathcal{H}_W$ and $\mathfrak{S} \subseteq \mathcal{B}_0\mathcal{H}_{W\sim A}$, let e denote the order unit of the given vector lattice $\mathfrak{S}$ throughout the following. I.e., let $e = H(W)$ if $\mathfrak{S} \subseteq \mathcal{B}\mathcal{H}_W$, but $e = H(\partial W, W \sim A)$ if $\mathfrak{S} \subseteq \mathcal{B}_0\mathcal{H}_{W\sim A}$.

DEFINITION.

$$\Gamma_{\mathfrak{S}} = \{t \in \Delta_{\mathfrak{S}} : e(t) = 1\} \cap \bigcap_{f,g \in \mathfrak{S}} \{t \in \Delta_{\mathfrak{S}} : (f \wedge_{\mathcal{H}} g)(t) = (f \wedge g)(t)\}.$$

PROPOSITION 2.1. – *$\Gamma_{\mathfrak{S}}$ is associated with $\overline{\mathcal{H}}^b_W$, whence $\Gamma_{\mathfrak{S}}$ is nonempty.*

Proof. – If $\mathfrak{S} \subseteq \mathcal{B}\mathcal{H}_W$, let $q = 1$; if $\mathfrak{S} \subseteq \mathcal{B}_0\mathcal{H}_{W-A}$, let Ω be a fixed regular inner region containing A and let q be the continuous function on W* which is harmonic on $\Omega - A$, equal to 1 on $W^* - \Omega$ and equal to 0 on A. In either case, it follows from Proposition 5.3 of [5] that $q - e$ is a potential on W or $W - A$ respectively.

Now let ν be an element of $\overline{\mathcal{H}}^b_W$ or $\overline{\mathcal{H}}^b_{W\sim A}$ respectively with nonnegative limit infimum at $\Gamma_{\mathfrak{S}}$ or at $\partial A \cup \Gamma_{\mathfrak{S}}$ respectively. Given any $\varepsilon > 0$, the set $S = \{t : t \in \Delta_{\mathfrak{H}},\ \liminf_W \nu(t) \leqslant -\varepsilon\}$ will be a closed subset of $\Delta_{\mathfrak{H}} \sim \Gamma_{\mathfrak{H}}$, and since S is compact there will be a finite sum s of the form

$$s = \Sigma_i\ [(f_i \wedge g_i) - (f_i \wedge_{\mathcal{H}} g_i)] + (q - e)$$

which is positive on S. Multiplying s by a suitable positive constant β we can have βs larger than $-\inf_{x \in W} \nu(x)$ everywhere on S. Then clearly $\nu + \beta s$ will belong to $\overline{\mathcal{H}}^b_W$ or $\overline{\mathcal{H}}^b_{W \sim A}$ respectively, and we shall have $-\varepsilon \leqslant \inf\{\liminf_W (\nu + \beta s)(x) : x \in \Delta_{\mathfrak{H}}\}$. As a consequence $\nu + \beta s + \boldsymbol{\varepsilon}$ will be nonnegative at $\Delta_{\mathfrak{H}}$ or $\partial A \cup \Delta_{\mathfrak{H}}$ respectively, and thus $\nu + \beta s + \boldsymbol{\varepsilon} > 0$ on all of W or $W \sim A$. Letting $\mathbf{H}(u)$ denote the greatest $\mathcal{H}$-harmonic minorant of a given $\mathcal{H}$-superharmonic function in W or in $W \sim A$ respectively, we see that

$$\nu + \beta s + \boldsymbol{\varepsilon} \geqslant 0 \Longrightarrow \mathbf{H}(\nu + \beta s + \boldsymbol{\varepsilon}) = \mathbf{H}(\nu) + \beta \mathbf{H}(s) + \mathbf{H}(\boldsymbol{\varepsilon}) \geqslant 0$$

(by 0.1). Since clearly $\mathbf{H}(s) = 0$, this gives $\nu + \boldsymbol{\varepsilon} \geqslant \mathbf{H}(\nu) + \mathbf{H}(\boldsymbol{\varepsilon}) \geqslant 0$, so $\nu \geqslant -\boldsymbol{\varepsilon}$ for any $\boldsymbol{\varepsilon} > 0$, whence $\nu \geqslant 0$. ∎

PROPOSITION 2.2. – *Let* $M \subseteq \Delta_{\mathfrak{H}}$ *be a closed set which is a border associated with* $\overline{\mathcal{H}}^b_W$. *Then the restriction map* $f \longrightarrow f \mid M$ *of* $\mathfrak{H}$ *into* $\mathcal{C}_R(M)$ *is an isometry (not necessarily onto) preserving positivity in both directions.*

Proof. – Suppose first that $\mathfrak{H} \subseteq \mathcal{B}\mathcal{H}_W$. That $f \in \mathfrak{H}$ is nonnegative on W if it is nonnegative on M is immediate. Since $\|f\| \leqslant \alpha$ is equivalent to having the two $\mathcal{H}$-superharmonic functions $\boldsymbol{\alpha} + f$ and $\boldsymbol{\alpha} - f$ nonnegative, the isometry conclusion is no less immediate. In the case when $\mathfrak{H} \subseteq \mathcal{B}_0 \mathcal{H}_{W \sim A}$, one observes that by Prop. 1.1. above the set $M \cup \partial A$ is a border associated with $\overline{\mathcal{H}}^b_{W \sim A}$, and argues in the same way. ∎

THEOREM 2.3. – *The restriction mapping* $f \longrightarrow f \mid \Gamma_{\mathfrak{H}}$ *of* $\mathfrak{H}$ *into* $\mathcal{C}_R(\Gamma_{\mathfrak{H}})$ *is a surjective isometry sending the order unit of* $\mathfrak{H}$ *to the order unit* 1 of $\mathcal{C}_R(\Gamma_{\mathfrak{H}})$ *and preserving the lattice operations* (3).

Proof. – By the definition of $\Gamma_{\mathfrak{H}}$, the range of this restriction mapping is a sublattice of $\mathcal{C}_R(\Gamma_{\mathfrak{H}})$ containing 1, and 1 is the image of the order unit. Since $\mathfrak{H}$ separates points of $\Gamma_{\mathfrak{H}}$ and $\mathfrak{H} \mid \Gamma_{\mathfrak{H}}$ (being iso-

(3) Z. Semadeni has informed the authors that this theorem can also be deduced from results of K. Gęba's and his in *Spaces of Continuous Functions* (V), Studia Math. 19 (1960), 303-320.

metric to a Banach space by 2.2 above) is a closed subspace of $\mathcal{C}_{\mathbf{R}}(\Gamma_{\mathfrak{H}})$, our theorem is an immediate consequence of the Stone-Weierstrass theorem (in its lattice form). ∎

COROLLARY 2.4. – *$\Gamma_{\mathfrak{H}}$ is equal to the border Γ of* [3] *(see the discussion preceding* 2.1 *above), and no proper closed subset of $\Gamma_{\mathfrak{H}}$ is associated with $\overline{\mathcal{H}}^{b}_{\mathrm{W}}$.*

Proof. – Given the function q defined in the proof of 2.1, we see that $q - e$ is also a potential on W or W – A respectively ; thus $\Gamma \subseteq \Gamma_{\mathfrak{H}}$. That no proper closed subset of $\Gamma_{\mathfrak{H}}$ can be associated with $\overline{\mathcal{H}}^{b}_{\mathrm{W}}$ follows from the observation that any such subset is contained in the set of zeros of some nonzero, nonpositive function in $\mathcal{C}_{\mathbf{R}}(\Gamma_{\mathfrak{H}})$. Since Γ is associated [3], $\Gamma = \Gamma_{\mathfrak{H}}$. ∎

Examples.

1) Suppose W is a relatively compact regular region in some larger space Ω on which $\mathcal{H}$ is defined. Let $\mathfrak{H}$ consist of all $\mathcal{H}$-harmonic functions on W which have continuous extensions to ∂W ; then $\Delta_{\mathfrak{H}}$ and $\Gamma_{\mathfrak{H}}$ can be identified with ∂W.

2) Take $\mathfrak{H} = \mathcal{B}\mathcal{H}_{\mathrm{W}}$. The corresponding compactification, which we shall use in studying pairs of harmonic classes, is denoted simply by $\mathrm{W}^*_{\mathcal{H}}$; its ideal boundary and ideal border are denoted by $\Delta_{\mathcal{H}}$ and $\Gamma_{\mathcal{H}}$ respectively. We denote $\mathrm{W} \cup \Gamma_{\mathcal{H}}$ by $\overline{\mathrm{W}}_{\mathcal{H}}$.

3) Take $\mathfrak{H} = \mathcal{B}_0\mathcal{H}_{\mathrm{W}-\mathrm{A}}$ for some compact outer-regular A. We denote the compactification given by this lattice by $\mathrm{W}^*_{\mathcal{H}}(\mathrm{A})$, and its ideal boundary by $\Delta_{\mathcal{H}}(\mathrm{A})$. Its ideal border is denoted by $\Gamma_{\mathcal{H}}(\mathrm{A})$, while $\overline{\mathrm{W}}_{\mathcal{H}}(\mathrm{A})$ denotes $\mathrm{W} \cup \Gamma_{\mathcal{H}}(\mathrm{A})$.

4) The following example illustrates the fact that the compactifications $\mathrm{W}^*_{\mathcal{H}}$ and $\mathrm{W}^*_{\mathcal{H}}(\mathrm{A})$ are in general distinct. For ease in visualization, we realize the example in the plane: the space W is the union of the half-axes $]-1, +\infty[$ of the x- and the y-axes ; the harmonic class $\mathcal{H}$ is composed of the real-valued functions which are affine on the lines except possibly at their intersection, while at the intersection (the origin) the sum of their four directional derivatives in the directions outward from the origin is zero. Since functions in $\mathcal{B}\mathcal{H}_{\mathrm{W}}$ must be constant on the positive x- and y-axes, $\Delta_{\mathcal{H}}$ consists (making use of the

obvious natural imbedding) of the left and lower endpoints (– 1,0) and (0,– 1) of the two axes, together with a point at infinity which is the ideal meeting point of the positive halves of the x- and y-axes. It is quite easy to see that (– 1,0) and (0,– 1) are the only points of $\Gamma_{\mathfrak{H}}$. Now suppose we let $A = [1,2]$ in the positive half of the x-axis and consider the compactification $W^*_{\mathcal{H}}(A)$. Now the compactification process adjoins the points (– 1,0) and (0,– 1), but this time there are functions (in $\mathcal{B}_0\mathcal{H}_{W\sim A}$) which take different values on the infinite segments of the two axes. Hence in $W^*_{\mathcal{H}}(A)$, a point at ∞ on the x-axis and a distinct point at ∞ on the y-axis must both be adjoined; $W^*_{\mathcal{H}}$ and $W^*_{\mathcal{H}}(A)$ are non-homeomorphic.

Also note that in this example there are nonzero functions $h \in \mathcal{B}\mathcal{H}_W$ which vanish on open subsets of W.

Remarks.

1) For any $\mathfrak{H} \subseteq \mathcal{B}\mathcal{H}_W$ and any fixed $x_0 \in W$, one may define a positive linear functional of norm $\leqslant 1$ on $\mathfrak{H}$ by $\rho : f \longrightarrow f(x_0)$. This functional is strictly positive [5, Thm. 2.1], and via the positive isometric isomorphism between $\mathfrak{H}$ and $\mathcal{C}_R(\Gamma_{\mathfrak{H}})$ may be thought of as a functional on the latter space, i.e., a Radon measure of total mass $\leqslant 1$ on $\Gamma_{\mathfrak{H}}$. It is evident from the strict positivity of ρ that nonempty open subsets of $\Gamma_{\mathfrak{H}}$ must have strictly positive ρ-measure. It is possible to show (as was stated for $\mathfrak{H} = \mathcal{B}\mathcal{H}_W$ in [11]) that the isomorphism between $\mathfrak{H}$ and $\mathcal{C}_R(\Gamma_{\mathfrak{H}})$ can be given by a kernel with respect to ρ, when W is countable at ∞. Corresponding statements, mutatis mutandis, can be made for $\mathfrak{H} \subseteq \mathcal{B}_0\mathcal{H}_{W\sim A}$.

2) The case when $\mathfrak{H} = \mathcal{B}\mathcal{H}_W$ has certain additional features. $\mathcal{B}\mathcal{H}_W$ is a complete lattice, a fact which follows from Harnack's principle (Axiom III). The fact that the supremum of an upward-directed majorized family $\{f_a\}$ is a pointwise supremum on W also shows that the measure ρ defined above has the property that $\rho(\sup_{\mathcal{H}} f_a) = \sup \rho(f_a)$. The hypotheses of [9, Lemma 1.3] are thus ful lled for the measure ρ on the space $\Gamma_{\mathcal{H}}$, and consequently the natural "identity" map of $\mathcal{C}_R(\Gamma_{\mathcal{H}})$ into $L^\infty(\rho)$ is an isomorphism onto, preserving all operations (lattice and algebraic) and relations; in particular, every class in $L^\infty(\rho)$ has a unique continuous representative function. Moreover, $\mathcal{C}_R(\Gamma_{\mathcal{H}})$ is the dual space of $L^1(\rho)$, and the weak* topology on the unit ball of $\mathcal{C}_R(\Gamma_{\mathcal{H}}) = L^\infty(\rho)$ is identical with the topology of uniform convergence

on compacta for the unit ball of $\mathcal{BH}_W$ (where one identifies $\mathcal{BH}_W$ and $\mathcal{C}_R(\Gamma_{\mathcal{H}})$). Indeed, both these topologies are compact Hausdorff and evaluation at any point $x_1 \in W$ is easily seen to be weak* continuous.

3) Among the results of the analysis above is the construction of a compact Hausdorff space $\Gamma_{\mathcal{H}}$ with the property that $\mathcal{BH}_W$ is isometrically linear- and order-isomorphic to $\mathcal{C}_R(\Gamma_{\mathcal{H}})$, with H(W) corresponding to 1. Other constructions of such a space are possible: the Banach lattice $\mathcal{BH}_W$ is an abstract M-space with unit in the sense of Kakutani [4], H(W) being the unit. By Kakutani's representation theorem, there exists a compact Hausdorff space X for which $\mathcal{BH}_W$ is isomorphic as an abstract M-space with unit to $\mathcal{C}_R(X)$, i.e., is isometrically isomorphic as a Banach lattice with H(W) corresponding to 1. This observation, at least in the classical case of harmonic functions in plane domains, is due to Stone [4, p. 999, footnote]. Another way of constructing such a space X has been given by L. Lumer-Naïm [8] ; her method involves making $\mathcal{BH}_W$ into a Banach algebra with identity 1 (for which she assumes that $1 \in \mathcal{H}$, so $1 = H(W)$). $\mathcal{BH}_W$ is then represented as algebraically isomorphic and isometric to a continuous-function algebra $\mathcal{C}_R(X)$, with $1 = H(W)$ as the identity element; it is not difficult to show that this isomorphism also preserves positivity. Still another approach offering a construction of a compact Hausdorff space X with $\mathcal{BH}_W$ isometrically linear- and order-isomorphic to $\mathcal{C}_R(X)$ follows from the compactness of the unit wedge (i.e., the intersection of the unit ball and the positive cone) of $\mathcal{BH}_W$ in the u.c.c. topology, a fact which implies that it possesses extremal points; these extremal points can be shown to form a Boolean algebra under their natural ordering, with unit H(W), operations $\vee$ and $\wedge$, and complementation given by subtraction from H(W). (In the present context this is quite easy to see because of the isomorphisms between $\mathcal{BH}_W$, $\mathcal{C}_R(\Gamma_{\mathcal{H}})$, and $L^\infty(\rho)$.) The Stone space of that Boolean algebra can be shown to have the property that its continuous-function vector lattice is isomorphic to $\mathcal{BH}_W$. (Cf. the recent paper of Taylor [10].) In [3], Constantinescu and Cornea exhibit a positive isometry between $\mathcal{BH}_W$ and $\mathcal{C}_R(\Gamma_{\mathcal{W}})$, where $\Gamma_{\mathcal{W}}$ is the harmonic part of the Wiener boundary. Regardless of what method one uses to construct the compact Hausdorff space X, homeomorphic spaces result, since any such space is homeomorphic to the space formed by the extremal points of the unit wedge of the dual of $\mathcal{BH}_W$, equipped with the weak* topology. Indeed, if the $\mathcal{BH}_W$-compactification $W^*_{\mathcal{H}}$ of W is constructed by the method of [6], $\Gamma_{\mathcal{H}}$ is

exactly that set of extremal points. It is again possible to repeat the analysis above for the compactification obtained by using $\mathfrak{H} = \mathcal{B}_0\mathcal{H}_{W\sim A}$.

Let us now return to our original setting. We begin by showing that with one possible exception, any border $\Gamma_{\mathfrak{H}}$ meets the regularity criteria given in § 1 above.

PROPOSITION 2.5. – *Except perhaps when* $\mathfrak{H}$ *consists only of constant functions, there is an* $\overline{\mathcal{H}}$*-unit-barrier, an* $\underline{\mathcal{H}}$*-unit barrier and a system of barriers for* W *at each point of* $\Gamma_{\mathfrak{H}}$.

Proof. – If e is the order unit of $\mathfrak{H}$, then it is clear that e is simultaneously an $\overline{\mathcal{H}}$- and an $\underline{\mathcal{H}}$-unit-barrier for each point of $\Gamma_{\mathfrak{H}}$, by the definition of $\Gamma_{\mathfrak{H}}$. Supposing for the time being that $\Gamma_{\mathfrak{H}}$ has at least two points, so that there exist nonnegative continuous functions on $\Gamma_{\mathfrak{H}}$ which vanish at a given point without being identically zero, then we can get a system of barriers at a given point $t_0 \in \Gamma_{\mathfrak{H}}$. To do this, we define a class of neighborhoods of t_0 by

$$N(f, \varepsilon) = \{t \in W^*_{\mathfrak{H}} : f(t) \leqslant \varepsilon \quad \text{and} \quad e(t) \geqslant 1 - \varepsilon\}$$

for each nonnegative $f \in \mathfrak{H}$ with $f(t_0) = 0$ and each positive $\varepsilon < 1$. These sets have the finite intersection property since

$$N(f \vee_{\mathcal{H}} g , \varepsilon \wedge \delta) \subseteq N(f, \varepsilon) \cap N(g , \delta) .$$

Thus if U is an open subset of $W^*_{\mathfrak{H}}$ containing t_0 but not containing any $N(f, \varepsilon)$, there must exist some $t_1 \neq t_0$ which belongs to all $N(f, \varepsilon)'s$. But then, since $t_0 \in \Gamma_{\mathcal{H}}$ and therefore

$$h(t_0) = 0 \implies (h \vee_{\mathcal{H}} 0)(t_0) = 0 = (h \wedge_{\mathcal{H}} 0)(t_0)$$

for any $h \in \mathfrak{H}$, it follows readily (using the fact that $e(t_1) = 1 = e(t_0)$) that $\mathfrak{H}$ cannot separate t_0 from t_1 . Since $t_1 \notin W$ because no positive harmonic function can have a zero in W, t_1 must belong to $\Delta_{\mathfrak{H}}$, a conclusion which contradicts the fact that $\mathfrak{H}$ separates points of $\Delta_{\mathfrak{H}}$. Thus U contains some $N(f, \varepsilon)$.

Given $N(f, \varepsilon)$, set $b = f + (1 - e)$; we claim that b is a barrier which belongs to W and $N = N(f, \varepsilon)^0$ in the sense of the definition preceding 1.2 above. For ∂N is a subset of the set

$$\{t : f(t) \leqslant \varepsilon \quad \text{and} \quad e(t) = 1 - \varepsilon\} \cup \{t : f(t) = \varepsilon \quad \text{and} \quad e(t) \geqslant 1 - \varepsilon\} .$$

The fact that $\partial(N \cap W) \sim (N \cap \Delta) \subseteq \partial N$ shows that the infimum of b on the set $\partial(N \cap W) \sim (N \cap \partial W) = \partial(N \cap W) \sim (N \cap \Delta)$ is positive, and in fact is $\geqslant \varepsilon$.

Only the trivial case where $\Gamma_{\mathfrak{H}}$ consists of a single point remains. If $e \neq 1$, it is easy to see that the function $1 - e$ is a barrier at that point and has positive lim inf at every other point of $\Delta_{\mathfrak{H}}$. On the other hand, if $e = 1$, so $W^*_{\mathfrak{H}}$ is just the one-point compactification of W, the assertion of the theorem is false in general; for example let $\mathcal{H}$ be the affine functions on the space $W =]0, \infty[$. ∎

Remark. – Let W be an open Riemann surface. The above results show that a great deal of information about the harmonic part of the Wiener boundary can be obtained from an investigation of $W^*_{\mathcal{H}}$ and $\Gamma_{\mathcal{H}}$. Also one may consider the linear space $\mathcal{BHD}$ consisting of all bounded harmonic functions on W with finite Dirichlet integral. It is clear that if $f \in \mathcal{BHD}$ then $|f|$ has finite Dirichlet integral ($|f|$ is a Dirichlet function on W, by [2, p. 78]); by Dirichlet's principle, $|f|_{\mathcal{H}} = H(|f|) \in \mathcal{BHD}$. Letting $\mathfrak{H}$ be the uniform closure of $\mathcal{BHD}$, one may form the spaces $W^*_{\mathfrak{H}}$, $\Delta_{\mathfrak{H}}$, $\Gamma_{\mathfrak{H}}$ and $\overline{W}_{\mathfrak{H}}$ as above. Now let W^*_D denote the usual Royden compactification of W [2, p. 98]; then, since the elements of $\mathcal{BHD}$ are Dirichlet functions, there is a natural mapping $p : W^*_D \longrightarrow W^*_{\mathfrak{H}}$, which is onto, and it can be shown (either from the known regularity of Γ_D [2, p. 101] or directly) that $p \mid \Gamma_D$ is a homeomorphism onto $\Gamma_{\mathfrak{H}}$. Thus $\overline{W}_{\mathfrak{H}}$ is a 1-1 continuous image of $W \cup \Gamma_D$ under a map which is the identity on W and a homeomorphism on Γ_D. Consequently one can establish many properties of the harmonic part of the Royden compactification using $W^*_{\mathfrak{H}}$. Note that $\Gamma_{\mathfrak{H}}$ is "attached" to W with a topology which is weaker than that attaching Γ_D. Since a system of barriers for a point $t_0 \in \Gamma_{\mathfrak{H}}$ can readily be seen to be a system of barriers for $p^{-1}(t_0) \in \Gamma_D$, we have demonstrated the existence of systems of barriers at the points of the harmonic part of the Royden boundary. Similar considerations can be given for solutions of $\Delta u = Pu$ on a Riemann surface W, where $P \geqslant 0$ is an integrable density on W; one need only replace the usual Dirichlet bilinear form $D(u, v)$ by the form $E(u, v) = D(u, v) + \int uvP$.

Let us again consider the case where $\mathfrak{H} = \mathcal{BH}_W$. The definition of $\Delta_{\mathfrak{H}}$ insures that every bounded function in $\mathcal{H}_W$ has a unique continuous extension to $\Delta_{\mathcal{H}}$; we shall now show that every bounded function in $\overline{\mathcal{H}}_W$ or $\underline{\mathcal{H}}_W$ has a continuous extension to $\Gamma_{\mathcal{H}}$. The extension is described by the next theorem, where $\mathcal{B}\overline{\mathcal{H}}_W$ and $\mathcal{B}\underline{\mathcal{H}}_W$ denote, respectively, the cones of bounded $\mathcal{H}$-superharmonic functions and bounded $\mathcal{H}$-subharmonic functions on W.

THEOREM 2.6. – *For any* $v \in \mathcal{B}\overline{\mathcal{H}}_W$, *let* $I(v)$ *be the function on* $\Gamma_{\mathcal{H}}$ *defined by*

$$I(v)(t) = \liminf_W v(t) \qquad \text{for} \qquad t \in \Gamma_{\mathcal{H}} .$$

Then $I(v)$ *is continuous on* $\Gamma_{\mathcal{H}}$ *for each* $v \in \mathcal{B}\overline{\mathcal{H}}_W$, *and the mapping* $I : \mathcal{B}\overline{\mathcal{H}}_W \longrightarrow \mathcal{C}_{\mathbf{R}}(\Gamma_{\mathcal{H}})$ *is positive-homogeneous and additive.*

Proof. – Given any $v \in \mathcal{B}\overline{\mathcal{H}}_W$, let $\mathbf{H}(v)$ denote the greatest $\mathcal{H}$-harmonic minorant of v, and let $\mathbf{H}(v)$ also denote the unique continuous extension to all of $W^*_{\mathcal{H}}$. Since $v - \mathbf{H}(v)$ is a positive potential on W, $\liminf_W (v - \mathbf{H}(v)) = \mathbf{0}$ at $\Gamma_{\mathcal{H}}$, and thus

$$0 = \liminf_W (v - \mathbf{H}(v))(t) \geqslant \liminf_W v(t) + \liminf_W - \mathbf{H}(v)(t)$$

$$= I(v)(t) - \mathbf{H}(v)(t) ,$$

or $\mathbf{H}(v)(t) \geqslant I(v)(t)$; but the reverse inequality is obvious. Thus $I(v) = \mathbf{H}(v) \mid \Gamma_{\mathcal{H}}$, showing that $I(v)$ is continuous for each $v \in \mathcal{B}\overline{\mathcal{H}}_W$. The additivity (and obvious positive homogeneity) of $v \longrightarrow \mathbf{H}(v)$, proved in 0.1 above, then immediately imply the same properties for I. ∎

The mapping I can now be extended in a unique manner to the linear space of functions formed by $\mathcal{B}\overline{\mathcal{H}}_W - \mathcal{B}\overline{\mathcal{H}}_W$ by setting

$$I(v_1 - v_2) = I(v_1) - I(v_2) ,$$

where $v_1 , v_2 \in \mathcal{B}\overline{\mathcal{H}}_W$. In particular, one sees that $I(u) = \limsup_W u$ if $u \in \mathcal{B}\underline{\mathcal{H}}_W$.

Now let A be an outer-regular compact subset of W. It is known [5, Cor. 8.2] that $\mathcal{B}\mathcal{H}_W$ and $\mathcal{B}_0\mathcal{H}_{W \sim A}$ are isometrically isomorphic ; we shall now show that this isometric isomorphism can be realized as a boundary correspondence on $\Gamma_{\mathcal{H}}$, with the usual exception.

THEOREM 2.7. – *Let* $\mathcal{H}$ *be hyperbolic on* W ; *let* A *be an outer-regular compact subset of* W. *Suppose that* $\mathcal{B}\mathcal{H}_W$ *contains non-constant functions. Then each* $h \in \mathcal{B}_0\mathcal{H}_{W \sim A}$ *has a continuous extension (which* *we shall also denote by h)* to $\Gamma_{\mathcal{H}} \cup (W \sim A)$. *The mapping*

$$h \mid W \sim A \longrightarrow h \mid \Gamma_{\mathcal{H}}$$

is an isometric isomorphism, positive in both directions, of $\mathcal{B}_0\mathcal{H}_{W\sim A}$ *with* $\mathcal{C}_{\mathbf{R}}(\Gamma_{\mathcal{H}})$.

Proof. – Since we have excluded the exceptional case of 2.5 above, 2.5 and 1.2 imply that every continuous function f on $\Gamma_{\mathcal{H}}$ can be extended to a function continuous on $(W \sim A) \cup \Gamma_{\mathcal{H}} \cup \partial A$ which vanishes at ∂A ; one simply solves the Dirichlet problem with boundary values f on $\Gamma_{\mathcal{H}}$ and $\mathbf{0}$ on ∂A. Denote the solution of that Dirichlet problem by $H_0(f)$. By [5, Thm. 5.3], $H_0(1) \leqslant H(\partial W , W \sim A)$; on the other hand, since

$$\liminf_{W\sim A}[H_0(1) - H(\partial W , W \sim A)] = 1 - \limsup_W H(\partial W , W \sim A) \geqslant 0$$

everywhere on $\Gamma_{\mathcal{H}}$ while $\lim_{W\sim A}[H_0(1) - H(\partial W , W \sim A)] = \mathbf{0}$ on ∂A, the reverse inequality holds and one has $H(\partial W , W \sim A) = H_0(1)$. To prove that $\sup_{x \in W\sim A} |H_0(f)(x)| = \sup_{t \in \Gamma_{\mathcal{H}}} |f(t)|$ is entirely straightforward. To complete the proof, one need only show that every element of $\mathcal{B}_0\mathcal{H}_{W\sim A}$ has a continuous extension to $(W \sim A) \cup \Gamma_{\mathcal{H}}$.

Given a nonnegative $h \in \mathcal{B}_0\mathcal{H}_{W\sim A}$, consider the extension of h to W obtained by setting $h = \mathbf{0}$ on A. This extension (which we also denote by h) is bounded and subharmonic, hence the operator I of 2.6 above extends h continuously to $\Gamma_{\mathcal{H}}$. In fact, if $\alpha > 0$ is any real number for which $h \leqslant \boldsymbol{\alpha}$ on $W \sim A$, then

$$\begin{aligned} \boldsymbol{\alpha} &= I(\alpha \,.\, H_0(1)) = I(\alpha \,.\, H(\partial W , W \sim A)) \\ &= I[\alpha \,.\, H(\partial W , W \sim A) - h] + I(h) \\ &= \limsup_W [\alpha \,.\, H(\partial W , W \sim A) - h] + \limsup_W h \\ &= \boldsymbol{\alpha} - \liminf_W h + \limsup_W h \end{aligned}$$

or $$\liminf_W h = \limsup_W h = I(h) \quad \text{on} \quad \Gamma_{\mathcal{H}} .$$

Thus $I(h)$ extends any nonnegative h in $\mathcal{B}_0\mathcal{H}_{W\sim A}$ to be continuous on $(W \sim A) \cup \Gamma_{\mathcal{H}}$. Since $\mathcal{B}_0\mathcal{H}_{W\sim A}$ is generated by its nonnegative elements and I is linear on $\overline{\mathcal{B}\mathcal{H}}_W - \underline{\mathcal{B}\mathcal{H}}_W$, the theorem is proved. ∎

COROLLARY 2.8. – *If* $\mathcal{H}$ *is hyperbolic on* W, *then for any outer-regular compact* $A \subseteq W$ *the spaces* $\mathcal{B}\mathcal{H}_W$ *and* $\mathcal{B}_0\mathcal{H}_{W \sim A}$ *are isometrically isomorphic, with positivity preserved in both directions.*

Proof. – If $\mathcal{B}\mathcal{H}_W$ consists only of multiples of 1, then by 1.2 and 2.5, $\mathcal{B}_0\mathcal{H}_{W \sim A}$ consists of multiples of $H(\partial W, W \sim A)$, whence the corollary follows for this case. If $\mathcal{B}\mathcal{H}_W$ does not reduce to the constants, then any function in either space has a unique continuous extension to $\overline{W}_{\mathcal{H}} = W \cup \Gamma_{\mathcal{H}}$ (vanishing identically on A in the case of $\mathcal{B}_0\mathcal{H}_{W \sim A}$). The isometry between $\mathcal{B}\mathcal{H}_W$ and $\mathcal{B}_0\mathcal{H}_{W \sim A}$ is given by composing the isometry between one space and its boundary values on $\Gamma_{\mathcal{H}}$ with the inverse of the isometry between the other space and its boundary values on $\Gamma_{\mathcal{H}}$. ∎

This last corollary already indicates that $\Gamma_{\mathcal{H}}$ and $\Gamma_{\mathcal{H}}(A)$ are homeomorphic spaces, since their continuous-function lattices are isometrically linearly and lattice isomorphic [4, Thm. 7, p. 1008]. We shall now show that the borders are attached to W in the same way, i.e., that the spaces $\overline{W}_{\mathcal{H}} = W \cup \Gamma_{\mathcal{H}}$ and $\overline{W}_{\mathcal{H}}(A) = W \cup \Gamma_{\mathcal{H}}(A)$ are homeomorphic under a mapping which extends the identity mapping on W. Thus with the exception of the one trivial excluded case, we can reduce all considerations to a single space $\overline{W}_{\mathcal{H}}$, and we shall see that $\overline{W}_{\mathcal{H}}(A)$ was in fact independent of A.

THEOREM 2.9. – *Suppose* $\mathcal{H}$ *is hyperbolic and* $\mathcal{B}\mathcal{H}_W$ *does not reduce to the constants. Let* A *be an outer-regular compact set. Then there is a unique continuous mapping* $j : \overline{W}_{\mathcal{H}} \longrightarrow \overline{W}_{\mathcal{H}}(A)$ *for which* $j \mid W =$ *identity and which is a homeomorphism of the two spaces.* $j \mid \Gamma_{\mathcal{H}}$ *may be described in the following way : if* $h_0 \in \mathcal{B}_0\mathcal{H}_{W \sim A}$, *then* $h_0(j(s)) = h_0(s)$ *for all* $s \in \Gamma_{\mathcal{H}}$, *where* $h_0(s)$ *is the boundary value of* h_0 *at* $s \in \Gamma_{\mathcal{H}}$.

Proof. – Having defined j on W by the identity mapping, we extend j to $\Gamma_{\mathcal{H}}$ as follows : let $\mathfrak{F}$ be a filter in W whose limit in $\overline{W}_{\mathcal{H}}$ is $s \in \Gamma_{\mathcal{H}}$. Then for any $h_0 \in \mathcal{B}_0\mathcal{H}_{W \sim A}$, clearly $\lim_{\mathfrak{F}} h_0 = h_0(s)$; but looking at $\mathfrak{F}$ in $\overline{W}_{\mathcal{H}}(A)$ it is clear, since $\mathfrak{F}$ contains the complements of compact sets, that $\mathfrak{F}$ can have limit points on $\Delta_{\mathcal{H}}(A)$ only. Since $\lim_{\mathfrak{F}} h_0 = h_0(s)$, $\mathcal{B}_0\mathcal{H}_{W \sim A}$ cannot distinguish those limit points and $\mathfrak{F}$ must therefore be convergent to a point $j(s) \in \Delta_{\mathcal{H}}(A)$. This defines j ; reëxamining the argument shows that j must now be continuous from

$\overline{W}_{\mathcal{H}}$ to $W^*_{\mathcal{H}}(A)$. We had observed above that by [4, Thm. 7] for each point $s \in \Gamma_{\mathcal{H}}$ there had to be one and only one point $t \in \Gamma_{\mathcal{H}}(A)$ for which $h_0(s) = h_0(t)$ for every $h_0 \in \mathcal{B}_0\mathcal{H}_{W \sim A}$; since we have found such a point $j(s) \in \Delta_{\mathcal{H}}(A)$ and $\mathcal{B}_0\mathcal{H}_{W \sim A}$ separates the points of $\Delta_{\mathcal{H}}(A)$, we see that $j(s)$ is the point t corresponding to s by the Kakutani theorem. Thus j maps $\Gamma_{\mathcal{H}}$ 1-1 onto $\Gamma_{\mathcal{H}}(A)$ and $j \mid \Gamma_{\mathcal{H}}$ is already seen to be a homeomorphism.

We now have j sending $\overline{W}_{\mathcal{H}}$ 1-1 onto $\overline{W}_{\mathcal{H}}(A)$, so j^{-1} is well defined ; it remains to show that it is continuous. Since its restrictions to W and to $\Gamma_{\mathcal{H}}(A)$ are clearly continuous, it suffices to show that if $\mathfrak{F}$ is a filter in W with limit $t \in \Gamma_{\mathcal{H}}(A)$, and if $t = j(s)$, then for every $h \in \mathcal{B}\mathcal{H}_W$ one has $\lim_{\mathfrak{F}} h = h(s)$. Suppose that $f \in \mathcal{C}_R(\Gamma_{\mathcal{H}})$ is a non-negative continuous function which takes its maximum value at $s = j^{-1}(t)$, and let h and h_0 be the elements of $\mathcal{B}\mathcal{H}_W$ and $\mathcal{B}_0\mathcal{H}_{W \sim A}$ respectively which take the boundary values given by f. Since $h - h_0$ is $\mathcal{H}$-superharmonic on W and nonnegative at $\Gamma_{\mathcal{H}}$, it is nonnegative on W. Consequently $\liminf_{\mathfrak{F}} h \geqslant \liminf_{\mathfrak{F}} h_0 = \lim_{\mathfrak{F}} h_0 = f(s)$; but the inequality $f(s) \geqslant h(x)$ for all $x \in W$ is clear, and therefore $\lim_{\mathfrak{F}} h = f(s)$. Since functions f satisfying the description above generate $\mathcal{C}_R(\Gamma_{\mathcal{H}})$, we have $\lim_{\mathfrak{F}} h = h(s)$ for any $h \in \mathcal{B}\mathcal{H}_W$; consequently j^{-1} sends the filter $\mathfrak{F}$ to a filter which converges to $s = j^{-1}(t)$, and continuity is proved. ■

We may consequently cease to distinguish between $\overline{W}_{\mathcal{H}}$ and $\overline{W}_{\mathcal{H}}(A)$, $\Gamma_{\mathcal{H}}$ and $\Gamma_{\mathcal{H}}(A)$, and we shall simply delete the "(A)". The interested reader may consider the exceptional case where $\mathcal{B}\mathcal{H}_W$ consists only of constants.

Section 3.

This section will be concerned with the boundary behavior of functions in a pair $(\mathcal{H}, \mathcal{K})$ of harmonic classes where $\mathcal{H} \geqslant \mathcal{K}$ in the sense of [5, § 7 ff.]. We shall fix $\mathcal{H}$ and $\mathcal{K}$ throughout the section and assume that $1 \in \overline{\mathcal{H}_W}$ and $1 \in \overline{\mathcal{K}_W}$. We shall also fix an outer-regular compact set $A \subseteq W$ which is an excluded set for the pair $(\mathcal{H}, \mathcal{K})$ in the sense of [5, § 7], that is, a set with $\mathcal{H}^+ \mid W \sim A \subseteq \overline{\mathcal{K}}$. By 0.2 above, there is no loss of generality in assuming that $W \sim A$ has only finitely many components.

PROPOSITION 3.1. – *Let* $\overline{W}$ *be a Hausdorff space of which* W *is a proper dense subspace. If* $\partial W = \overline{W} \sim W$ *is associated with* $\overline{\mathcal{H}}^b_W$, *then* ∂W *is associated with* $\overline{\mathcal{K}}^b_W$.

Proof. – By the "converse" part of 1.1 above, it suffices to show that any $\nu \in \overline{\mathcal{K}}_{W \sim A}$ which is nonnegative at $\partial(W \sim A)$ is nonnegative. But given any such ν, the function $\nu \wedge \mathbf{0}$ is in $\overline{\mathcal{H}}_{W \sim A}$ by [5, Prop. 7.2 (iv)]. Consequently, since $\nu \wedge \mathbf{0}$ is clearly nonnegative at $\partial(W \sim A)$ and $\partial(W \sim A)$ is associated with $\overline{\mathcal{H}}^b_{W \sim A}$, we must have $\nu \wedge \mathbf{0} \geqslant \mathbf{0}$, or $\nu \geqslant \mathbf{0}$, on $W \sim A$. ∎

PROPOSITION 3.2. – *Let* $\overline{W}$ *be a Hausdorff space of which* W *is a proper dense subspace. If there are no positive* $\mathcal{H}$*-harmonic* $\mathcal{K}$*-potentials in* $W \sim A$, *then* $H(\partial W, W \sim A)$ *is the least* $\mathcal{H}$*-harmonic majorant of* $K(\partial W, W \sim A)$. *If* $H(\partial W, W \sim A)$ *is the least* $\mathcal{H}$*-harmonic majorant of* $K(\partial W, W \sim A)$ *and* ∂W *is associated with* $\overline{\mathcal{K}}^b_W$, *then* ∂W *is associated with* $\overline{\mathcal{H}}^b_W$.

Proof. – Let h_0 be the least $\mathcal{H}$-harmonic majorant of $K(\partial W, W \sim A)$. The first statement follows from the fact that $H(\partial W, W \sim A) - h_0$ is a $\mathcal{K}$-potential. To prove the second statement let V be a function in $\overline{\mathcal{H}}^b_{W \sim A}$ which is nonnegative at $\partial(W \sim A)$, and for the purpose of arriving at a contradiction, suppose that $\alpha = - \inf_{x \in W \sim A} V(x) > 0$ (see 1.1). We may replace V by $\nu = \left(\frac{1}{\alpha} V\right) \wedge \mathbf{0}$ and reduce the problem to considering a nonpositive function $\nu \in \overline{\mathcal{H}}^b_{W \sim A}$ which has limit 0 at $\partial(W \sim A)$ and whose infimum on $W \sim A$ is -1. The nonnegative function $-\nu$ is $\mathcal{H}$-subharmonic on $W \sim A$, has limit 0 at ∂A, and has supremum 1; by the characterization of $H(\partial W, W \sim A)$ of [5, Prop. 7.2 (iv)], it follows that $-\nu \leqslant H(\partial W, W \sim A)$, or $\mathbf{0} \leqslant H(\partial W, W \sim A) + \nu$. The function on the right of this inequality belongs to $\overline{\mathcal{H}}_{W \sim A}$ and is nonnegative, so it belongs to $\overline{\mathcal{K}}_{W \sim A}$; hence

$$H(\partial W, W \sim A) + \nu - K(\partial W, W \sim A) \in \overline{\mathcal{K}}_{W \sim A}$$

and is nonnegative at $\partial(W \sim A)$, because

$$H(\partial W, W \sim A) - K(\partial W, W \sim A) \geqslant \mathbf{0}.$$

Since $\partial(W \sim A)$ is associated with $\overline{\mathcal{H}}^b_{W \sim A}$

$$H(\partial W , W \sim A) + \nu - K(\partial W , W \sim A) \geqslant 0$$

holds everywhere in $W \sim A$. This can be written as

$$H(\partial W , W \sim A) + \nu \geqslant K(\partial W , W \sim A) ;$$

since the left side is $\mathcal{H}$-superharmonic and the right side $\mathcal{H}$-subharmonic, the inequality is unaltered by replacing $K(\partial W , W \sim A)$ by its least $\mathcal{H}$-harmonic majorant: $H(\partial W , W \sim A) + \nu \geqslant H(\partial W , W \sim A)$, or $\nu \geqslant 0$. This finishes the proof. ■

In order to motivate the next theorems, we consider the following example : let $W =]0 , 1[$, let $\mathcal{H}$ be the class of solutions of $u'' = 0$, and $\mathcal{K}$ be the class of solutions of $x^2 . u'' = 2u$, a basis for which is $\left\{\frac{1}{x} , x^2\right\}$. It is easily verified that $\mathcal{H} \geqslant \mathcal{K}$, that $W^*_{\mathcal{H}} = W^*_{\mathcal{K}} = [0 , 1]$, and that $\Gamma_{\mathcal{H}} = \{0 , 1\}$ but $\Gamma_{\mathcal{K}} = \{1\}$. It is immediate that $\Gamma_{\mathcal{K}}$ is associated with $\overline{\mathcal{K}}^b_W$ but not with $\overline{\mathcal{H}}^b_W$. And although $\Gamma_{\mathcal{K}}$ is regular for W and $\mathcal{K}$, and $\Gamma_{\mathcal{H}}$ is regular for W and $\mathcal{H}$, $\Gamma_{\mathcal{H}}$ is not regular for W and $\mathcal{K}$; indeed, every bounded function in $\mathcal{K}$ must vanish at 0. The difficulty in this case lies with barriers: clearly every $\mathcal{H}$-barrier is a $\mathcal{K}$-barrier, but there is no $\underline{\mathcal{K}}$-unit-barrier at the point $0 \in \Gamma_{\mathcal{H}} = \partial W$. This example, incidentally, will also prove useful in demonstrating the sharpness of 3.7 below.

The following two theorems provide a partial answer to the question " under what conditions does the regularity of W (as a dense subspace of a Hausdorff space $\overline{W}$) with respect to one of two comparable classes $\mathcal{H}$ and $\mathcal{K}$ imply its regularity with respect to the other ?"

THEOREM 3.3. – *Let $\overline{W}$ be a Hausdorff space containing* W *as a dense subspace such that* $\partial W = \overline{W} \sim W$ *is associated with* $\overline{\mathcal{H}}^b_W$ *(and thus with* $\overline{\mathcal{K}}^b_W$*, by 3.1 above). Let* $x_0 \in \partial W$. *If there is a* $\underline{\mathcal{K}}$*-unit-barrier* u *for* W *at* x_0 *and a system of* $\mathcal{K}$*-barriers for* W *at* x_0*, then there is an* $\underline{\mathcal{H}}$*-unit-barrier for* W *at* x_0 *and a system of* $\mathcal{H}$*-barriers for* W *at* x_0*, whence* x_0 *is a regular point with respect to* $\mathcal{H}$.

Proof. – It is clear that $u \vee 0$ is an $\underline{\mathcal{H}}$-unit-barrier at x_0. Since x_0 is regular with respect to $\mathcal{K}$ (by 1.2 above), there is a neighborhood

ω_0 of x_0 for which $\omega_0 \subseteq \overline{W} \sim A$ and $K(\partial W, W \sim A) > \frac{1}{2}$ in $\omega_0 \cap W$. Let b_0 be a barrier in the system of $\mathcal{K}$-barriers for W at x_0. Without loss of generality, assume that the domain of b is the intersection of a neighborhood $\omega \subseteq \omega_0$ with W and that $b_0 < \frac{1}{2}$ on $\omega \cap W$ (taking $b_0 \wedge 1/2$ if necessary). Since $b_0 - K(\partial W, W \sim A) \leqslant 0$ on ω,

$$b_0 - K(\partial W, W \sim A) \in \overline{\mathcal{H}}_{\omega \cap W}.$$

Consequently the function b defined by

$$b = b_0 - K(\partial W, W \sim A) + H(\partial W, W \sim A)$$

satisfies the conditions for an $\mathcal{H}$-barrier for W at x_0, and since $H(\partial W, W \sim A) \geqslant K(\partial W, W \sim A)$, b is positive wherever b_0 is positive ; in particular, since b_0 is positive at $\partial(\omega \cap W) - (\omega \cap \partial W)$, b is positive there also. Thus there is a system of $\mathcal{H}$-barriers for W at x_0. ∎

THEOREM 3.4. – *Let* $\overline{W}$ *be a compact Hausdorff space containing* W *as a dense subspace. Assume either that* $\partial W = \overline{W} \sim W$ *has at least two points or that* $1 \notin \mathcal{H}_W$ *and* $1 \notin \mathcal{K}_W$. *Then* W *(as a subset of* $\overline{W}$*) is regular with respect to* $\mathcal{K}$ *if it is regular with respect to* $\mathcal{H}$ *and there is a* $\underline{\mathcal{K}}$*-unit-barrier at each point of* ∂W.

Proof. – This is an immediate consequence of 1.2, 1.3, 1.4, and 3.3 above. ∎

THEOREM 3.5. – *Let* $W^*_{\mathcal{K}}(A)$ *denote the compact Hausdorff space* $W \cup \Delta_{\mathcal{K}}(A)$. *If* f *is a continuous function on* $\partial W = \Delta_{\mathcal{K}}(A)$, *then* f *is resolutive with respect to* $\mathcal{H}$ *on* ∂W. *Similarly, if* $f \in \mathcal{C}_R(\Delta_{\mathcal{K}}(A))$ *and* f_0 *denotes the extension of* f *to* $\partial(W \sim A)$ *obtained by setting* $f_0 \mid \partial W = f$ *and* $f_0 \mid \partial A = 0$, *then* f_0 *is resolutive with respect to* $\mathcal{H}$ *on* $\partial(W \sim A)$.

Proof. – We prove only the first part of the theorem ; the other half is similar. Since 1 is $\mathcal{H}$-superharmonic, we have $\overline{H}(1, W) \leqslant 1$ and $\overline{H}(1, W) \leqslant \underline{H}(1, W)$, whence 1 is $\mathcal{H}$-resolutive on ∂W. Making the convention (as we did in § 2 above) that functions in $\mathcal{B}_0\mathcal{K}_{W \sim A}$ are to be thought of as extended (continuously) to all of W by setting them equal to $\mathbf{0}$ on A, we see that if k is a nonnegative element of $\mathcal{B}_0\mathcal{K}_{W\sim A}$, then $k \in \underline{\mathcal{H}}$. Consequently $k \leqslant \underline{H}(k \mid \partial W, W)$, whence

$$\overline{H}(k \mid \partial W, W) \leqslant \underline{H}(k \mid \partial W, W)$$

and $k \mid \partial W$ is $\mathcal{H}$-resolutive. It is known [1, pp. 100 ff.] that the set of $\mathcal{H}$-resolutive continuous functions on ∂W forms a closed vector sublattice of $\mathcal{C}_R(\partial W)$, and we have shown that this sublattice contains $\mathbf{1}$ and contains $\mathcal{B}_0\mathcal{K}_{W\sim A} \mid \partial W$ (since $\mathcal{B}_0\mathcal{K}_{W\sim A}$ is generated by its nonnegative elements). Since $\mathcal{B}_0\mathcal{K}_{W\sim A} \mid \partial W$ separates points of ∂W, the lattice form of the Stone-Weierstrass theorem implies that the $\mathcal{H}$-resolutive continuous function on ∂W constitute all of $\mathcal{C}_R(\partial W)$. ■

In [5, § 8], one of the authors showed that there is an isometric isomorphism from $\mathcal{B}\mathcal{K}_W$ into $\mathcal{B}\mathcal{H}_W$ when $\mathcal{H}$ and $\mathcal{K}$ are both hyperbolic on W. We shall now reëstablish that isomorphism, realizing it by identifying functions with their boundary values on $\Gamma_{\mathcal{H}}$ and $\Gamma_{\mathcal{K}}$.

THEOREM 3.6. – *Let $\mathcal{H}$ and $\mathcal{K}$ be hyperbolic on* W *and assume that $\mathcal{B}\mathcal{K}_W$ does not reduce to multiples of* $\mathbf{1}$. *Then there are isometric isomorphisms of $\mathcal{B}\mathcal{K}_W$ and $\mathcal{B}_0\mathcal{K}_{W\sim A}$ onto subspaces of $\mathcal{B}\mathcal{H}_W$ and $\mathcal{B}_0\mathcal{H}_{W\sim A}$, given by*

$$k \longrightarrow k \mid \Gamma_{\mathcal{K}} \longrightarrow K_0(k \mid \Gamma_{\mathcal{K}}) \longrightarrow H[(K_0(k \mid \Gamma_{\mathcal{K}})) \mid \Delta_{\mathcal{K}}(A), W]$$

$$k \longrightarrow k \mid \Gamma_{\mathcal{K}} \longrightarrow K_0(k \mid \Gamma_{\mathcal{K}})$$

$$\longrightarrow H[(K_0(k \mid \Gamma_{\mathcal{K}})) \mid \Delta_{\mathcal{K}}(A) \cup \partial A, W \sim A]$$

$$k \longrightarrow k \mid \Delta_{\mathcal{K}}(A) \longrightarrow H[k \mid \Delta_{\mathcal{K}}(A), W]$$

and

$$k \longrightarrow k \mid \Delta_{\mathcal{K}}(A) \longrightarrow H[k \mid \Delta_{\mathcal{K}}(A) \cup \partial A, W \sim A],$$

where for any continuous f defined on $\Gamma_{\mathcal{K}}$, $K_0(f)$ means the unique element of $\mathcal{B}_0\mathcal{K}_{W\sim A}$ whose restriction to $\Gamma_{\mathcal{K}}$ is f. The isomorphisms of $\mathcal{B}_0\mathcal{K}_{W\sim A}$ into $\mathcal{B}\mathcal{H}_W$ and $\mathcal{B}_0\mathcal{H}_{W\sim A}$ respectively send nonnegative functions $k \in \mathcal{B}_0\mathcal{K}_{W\sim A}$ to their least $\mathcal{H}$-harmonic majorants on W *or* $W \sim A$ *respectively, and all isomorphisms preserve boundary values on $\Gamma_{\mathcal{K}}$.*

Proof. – It is clear that each of the mappings constructed above is a composition of isometries and is thus an isometry. Boundary values at $\Gamma_{\mathcal{K}}$ are preserved because the points of $\Gamma_{\mathcal{K}}$ are regular for W and $\Delta_{\mathcal{K}}(A)$ with respect to $\mathcal{H}$, by 3.4 above. (It should be observed, however, that elements of $\mathcal{B}\mathcal{H}_W$ or $\mathcal{B}_0\mathcal{H}_{W\sim A}$ are not in general determined by their boundary values at $\Gamma_{\mathcal{K}}$, since the latter may not be associated with $\overline{\mathcal{H}}^b_W$.) Finally, the mapping which sends each nonnegative k in $\mathcal{B}_0\mathcal{K}_{W\sim A}$ into the function $H(k \mid \Delta_{\mathcal{K}}(A), W)$ surely sends k to its least $\mathcal{H}$-harmonic majorant, for $\Delta_{\mathcal{K}}(A)$ compactifies W and k is continuous on $W^*_{\mathcal{K}}(A) = W \cup \Delta_{\mathcal{K}}(A)$, so any function $s \in \overline{\mathcal{H}}_W$ has $\liminf_W s \geqslant k$ on $\Delta_{\mathcal{K}}(A)$ if and only if $s \geqslant k$ on all of W. The argument for sending $k \longrightarrow H[k \mid (\Delta_{\mathcal{K}}(A) \cup \partial A), W \sim A]$ is identical.

Note that if $\mathcal{B}\mathcal{K}_W$ consists only of the constant functions, then $\mathcal{B}_0\mathcal{K}_{W\sim A} = \mathcal{B}_0\mathcal{H}_{W\sim A}$ since 1 is both $\mathcal{H}$- and $\mathcal{K}$-harmonic in $W \sim A$ (see 7.3 of [5]). This fact together with 2.8 implies that in this case there are isometric isomorphisms of $\mathcal{B}\mathcal{K}_W$ and $\mathcal{B}_0\mathcal{K}_{W\sim A}$ onto $\mathcal{B}\mathcal{H}_W$ and $\mathcal{B}_0\mathcal{H}_{W\sim A}$.

THEOREM 3.7. – *Assume that neither $\mathcal{B}\mathcal{H}_W$ nor $\mathcal{B}\mathcal{K}_W$ reduces to the constant functions. There is a continuous (necessarily unique) mapping $f : \overline{W}_{\mathcal{K}} \longrightarrow \overline{W}_{\mathcal{H}}$ which extends the identity mapping on* W *and which is determined by the following property: if $k \in \mathcal{B}_0\mathcal{K}_{W\sim A}$, then* $H(k \mid \Delta_{\mathcal{K}}(A) \cup \partial A, W \sim A)$ *takes the same value at $f(t)$ as k does at t, where $t \in \Gamma_{\mathcal{K}}$. The mapping f is* 1-1, *and thus establishes a homeomorphism between $\Gamma_{\mathcal{K}}$ and $f[\Gamma_{\mathcal{K}}] = C \subseteq \Gamma_{\mathcal{H}}$; C is open and closed in $\Gamma_{\mathcal{H}}$.*

Proof. – Each nonnegative element of $\mathcal{B}_0\mathcal{H}_{W\sim A}$ belongs to $\overline{\mathcal{K}}_{W\sim A}$, so if $K(h)$ denotes the greatest $\mathcal{K}$-harmonic minorant of a nonnegative $h \in \mathcal{B}_0\mathcal{H}_{W\sim A}$ (on the set $W \sim A$), then the map $h \longrightarrow K(h)$ is additive and positively homogeneous by 0.1 (applied to each component of $W \sim A$), and it has an additive extension to $\mathcal{B}_0\mathcal{H}_{W\sim A}$. Since $h - K(h)$ is a $\mathcal{K}$-potential on $W \sim A$ for any nonnegative $h \in \mathcal{B}_0\mathcal{H}_{W\sim A}$, one has $\liminf_{W\sim A} h = \lim K(h)$ at $\Gamma_{\mathcal{K}}$, and the right side of that equation is a continuous function (cf. 2.7 above). Furthermore, the inequality $1 \geqslant H(\partial W, W \sim A) \geqslant K(\partial W, W \sim A)$ implies that $H(\partial W, W \sim A)$ has limit 1 at each point of $\Gamma_{\mathcal{K}}$. A straightforward modification of the argument employed in the proof of 2.7 above will now show that

$$\liminf_{W\sim A} h = \mathbf{K}(h) = \limsup_{W\sim A} h$$

at each point of $\Gamma_{\mathcal{K}}$. Consequently, extending h to take the same boundary values as $\mathbf{K}(h)$ on $\Gamma_{\mathcal{K}}$ yields a continuous extension (obviously the unique such extension) of h to $\overline{W}_{\mathcal{K}} = W \cup \Gamma_{\mathcal{K}}$. Thus, as in the proof of 2.9 above, the identity mapping on W can be continuously extended to map $\overline{W}_{\mathcal{H}}$ into $W^*_{\mathcal{K}}(A)$; denote that extension by f. (Cf. the construction of $W^*_{\mathcal{K}}(A)$ given by the method of [6].) It follows readily from the facts that every $\mathcal{H}$-potential on $W \sim A$ is a $\mathcal{K}$-potential on $W \sim A$ and that $H(\partial W, W \sim A)$ has limit 1 at $\Gamma_{\mathcal{K}}$, that $f[\Gamma_{\mathcal{K}}] \subseteq \Gamma_{\mathcal{H}}$; denote $f[\Gamma_{\mathcal{K}}]$ by C. By 3.6 above, every continuous function on $\Gamma_{\mathcal{K}}$ is a set of boundary values for an appropriate element of $\mathcal{B}_0\mathcal{H}_{W\sim A}$, so the elements of $\mathcal{B}_0\mathcal{H}_{W\sim A}$ (or, more accurately, their extensions to $\Gamma_{\mathcal{K}}$) separate the points of $\Gamma_{\mathcal{K}}$; consequently f is 1-1, so C and $\Gamma_{\mathcal{K}}$ are homeomorphic.

It remains to show that C is open and closed in $\Gamma_{\mathcal{H}}$. To show this, we begin by observing that for a nonnegative $h \in \mathcal{B}_0\mathcal{H}_{W\sim A}$ to be a $\mathcal{K}$-potential it is necessary and sufficient that it takes zero boundary values on C, because then it also takes zero boundary values on $\Gamma_{\mathcal{K}}$ and thus can have no nonzero nonnegative $\mathcal{K}$-harmonic minorant. We claim that there is a largest $\mathcal{H}$-harmonic $\mathcal{K}$-potential dominated by $H(\partial W, W \sim A)$. Indeed, let h_0 be the least $\mathcal{H}$-harmonic majorant of $K(\partial W, W \sim A)$, so that the function $p = H(\partial W, W \sim A) - h_0$ is the greatest $\mathcal{H}$-harmonic minorant of $H(\partial W, W \sim A) - K(\partial W, W \sim A)$; since the latter is a $\mathcal{K}$-potential (cf. 3.2 above), so is p. Now, if h is any $\mathcal{H}$-harmonic $\mathcal{K}$-potential and $h \leqslant H(\partial W, W \sim A)$, then

$$[H(\partial W, W \sim A) - h] - K(\partial W, W \sim A)$$

is $\mathcal{K}$-superharmonic on $W \sim A$, with limit zero on ∂A ; on the other hand, since h has limit zero at $\Gamma_{\mathcal{K}}$, one has

$$0 \leqslant [H(\partial W, W \sim A) - h] - K(\partial W, W \sim A)$$

at $\Gamma_{\mathcal{K}}$. Hence (since $\partial A \cup \Gamma_{\mathcal{K}}$ is associated with $\overline{\mathcal{K}}^b_{W\sim A}$) one has $0 \leqslant H(\partial W, W \sim A) - h - K(\partial W, W \sim A)$, or

$$h \leqslant H(\partial W, W \sim A) - K(\partial W, W \sim A),$$

everywhere on $W \sim A$, whence $h \leqslant p$. Clearly p takes the value zero on C ; if on the other hand $p(t_0) < 1$ for some $t_0 \in \Gamma_{\mathcal{H}} \sim C$, then there is an $f \in \mathcal{C}_R(\Gamma_{\mathcal{H}})$ for which $f[C] = 0$, $0 \leqslant f \leqslant 1 - p$, and

$f(t_0) = 1 - p(t_0)$. Letting $h_1 \in \mathcal{B}_0\mathcal{H}_{W\sim A}$ be the function whose boundary values are those given by f, we have $p + h_1 \leqslant H(\partial W, W \sim A)$ and $p + h_1$ is a $\mathcal{K}$-potential, yielding a contradiction. Consequently $p \mid \Gamma_{\mathcal{H}}$ is the characteristic function of $\Gamma_{\mathcal{H}} \sim C$, whence C is open and closed. ■

Remarks.

1) From the definition of the mapping $f : \Gamma_{\mathcal{K}} \longrightarrow C$ (which is a homeomorphism) and the definition of the isometry from $\mathcal{B}_0\mathcal{K}_{W\sim A}$ into $\mathcal{B}_0\mathcal{H}_{W\sim A}$ in 3.6 above, it is easy to see that the isometry may now be interpreted as follows: if $k \in \mathcal{B}_0\mathcal{K}_{W\sim A}$ has the boundary-value function g on $\Gamma_{\mathcal{K}}$, then the image of k in $\mathcal{B}_0\mathcal{H}_{W\sim A}$ under the isometry has the boundary-value function which equals $g \circ f^{-1}$ on C and **0** on $\Gamma_{\mathcal{H}} \sim C$. (The choice of **0** is dictated by the fact that if k is non-negative, then its image in $\mathcal{B}_0\mathcal{H}_{W\sim A}$ is its *least* $\mathcal{H}$-harmonic majorant.) Similar interpretations are available for the other isometries of 3.6.

2) If $K(\partial W, W \sim A)$ has limit 1 at each point of C, then $K(\partial W, W \sim A)$ is a $\mathcal{K}$-unit-barrier at each point of C, whence each such point is regular with respect to $\mathcal{K}$. Should this happen, the same methods employed above will show that f is a homeomorphism of $\overline{W}_{\mathcal{K}}$ and $W \cup C$. The authors have not found a simple necessary and sufficient condition that this occur.

3) The example of two classes $\mathcal{H}$ and $\mathcal{K}$ on $W =]0, 1[$ considered in the discussion preceding 3.3 above is an example in which $C \neq \Gamma_{\mathcal{H}}$.

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