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EFFECTIVE FINITE GENERATION FOR ADJOINT RINGS

by Paolo CASCINI & De-Qi ZHANG (*)

Abstract. — We describe a bound on the degree of the generators for some adjoint rings on surfaces and threefolds.

Résumé. — Nous établissons une borne sur le degré des générateurs pour les anneaux adjoints de surfaces et de variétés algébriques de dimension 3.

1. Introduction

The aim of this paper is to provide a first step towards an effective version of the finite generation of adjoint rings.

The study of effective results in birational geometry has a long history. On the one hand, the boundedness results on the pluricanonical maps for varieties of general type, due to Hacon, McKernan, Takayama and Tsuji [10, 22], laid the foundation for a great deal of work towards an effective version of the (log)-Iitaka fibrations (*e.g.* [24, 23]). On the other hand, Kollár's effective version [15] of the Kawamata-Shokurov's base point free theorem provides an explicit bound for a multiple of a nef adjoint divisor to be base point free. Finally, many recent results on the geography of projective threefolds of general type yield, in particular, a description of the singularities which may appear on varieties of general type (*e.g.* see [5, 6]).

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The main goal of this paper is to combine together some of these results and study an effective version of the finite generation for adjoint rings. More specifically, given a Kawamata log terminal pair (X, B), the main result of [2] implies that the canonical ring $R(X, K_X + B)$ of $K_X + B$ is finitely generated (see also [21, 4]). Moreover, if A is an ample $\mathbb{Q}$-divisor and $B_1, \ldots, B_k$ are $\mathbb{Q}$-divisors such that (X, B_i) is Kawamata log terminal for all $i = 1, \ldots, k$, then the associated adjoint ring $R(X; K_X + A + B_1, \ldots, K_X + A + B_k)$ is also finitely generated (*cf.* Definition 2.8). On the other hand, the problem about the finite generation of $R(X; K_X + B_1, \ldots, K_X + B_k)$, without the assumption of B_i being big, is still open and it implies the abundance conjecture (*e.g.* see [8]). Thus, it is reasonable to ask if there exists a bound on the number of generators of the adjoint ring on a smooth projective variety which depends only on the numerical and topological invariants associated to the pairs (X, B_i) for $i = 1, \ldots, k$.

Inspired by these questions, our main result is the following:

Theorem 1.1. — *Let (X, B) be a Kawamata log terminal projective threefold such that X is smooth. Assume that B is nef and that B or $K_X + B$ is big. Let a be a positive integer such that aB is Cartier.*

Then, there exists a positive integer q, depending only on the Picard number $\rho(X)$ of X such that $R(X, qa(K_X + B))$ is generated in degree 5 (cf. Definition 2.8).

As a direct consequence, we obtain:

Corollary 1.2. — *Let X be a smooth projective threefold of general type.*

Then there exists a constant m which depends only on the Picard number $\rho(X)$ such that the stable base locus of K_X coincides with the base locus of the linear system $|mK_X|$.

In addition, we obtain a stronger version of the theorem above in the case of surfaces, generalizing some of the results in [9], which studies the problem in the case of smooth minimal surfaces of general type. It is worth to mention that the proofs of these results rely in a crucial way on the classification of Kawamata log terminal surface singularities and terminal threefold singularities.

The paper is organized as follows. In Section 2, we describe the main tools used in the paper, which are mainly based on Kollár's effective base point free theorem, Mumford's regularity theorem and the classification of surface and threefold singularities. In Section 3, we describe a bound on the index of the singularities of the minimal model of a smooth projective threefold X, which depends on the Picard number $\rho(X)$ of X. The bound is obtained as a consequence of a recent result by Chen and Hacon [7]. Finally, in Section 4, we prove the main results of the paper and we show, in many examples, that some of these results are optimal. In particular, the bound on the degree of the generators of a Kawamata log terminal pair (X,B) depends on the Picard number of the projective variety X.

Note that all the bounds obtained in the paper are easily computable, but they are far away from being sharp. For this reason, we often omit an explicit description of these bounds.

2. Preliminary results

2.1. Notation

We work over the field of complex numbers $\mathbb{C}$. We refer to [16] for the classical definitions of singularities in the Minimal Model Program. In particular, given a log pair (X,B), we denote by $a(\nu,X,B)$ the *discrepancy* of (X,B) with respect to a valuation ν. We say that a log pair (X,B) is *log smooth* if X is smooth and the support of B is a divisor with simple normal crossings.

A rational map $f\colon X \dashrightarrow Y$ between normal projective varieties X and Y is a *contraction* if the inverse map f^{-1} does not contract any divisors. The *exceptional locus* of f is the subset of X on which f is not an isomorphism.

Let $f\colon X \dashrightarrow Y$ be a proper birational contraction of normal projective varieties and let D be an $\mathbb{R}$-Cartier divisor on X such that $D_Y = f_*D$ is also $\mathbb{R}$-Cartier. Then f is *D-non-positive* (respectively *D-negative*) if for some common resolution $p\colon W \longrightarrow X$ and $q\colon W \longrightarrow Y$ which resolves the indeterminacy locus of f, we may write

$$p^*D = q^*D_Y + E$$

where $E \geqslant 0$ is q-exceptional (respectively $E \geqslant 0$ is q-exceptional and the support of E contains the strict transform of the exceptional divisor of f). In particular, if (X,B) is a Kawamata log terminal pair, $D = K_X + B$ and $D_Y = K_Y + B_Y$, then f is (K_X+B)-non-positive if and only if

$$a(F,X,B) \leqslant a(F,Y,f_*B)$$

for all the prime divisors F which are exceptional over Y and f is (K_X+B)-negative if, in addition, the strict inequality holds for all the prime divisors F in X which are exceptional over Y.

Let (X,B) be a Kawamata log terminal pair. A proper birational contraction $f\colon X \dashrightarrow Y$ of normal projective varieties is a *log terminal model* for (X,B) if f is (K_X+B)-negative, Y is $\mathbb{Q}$-factorial and K_Y+f_*B is nef. If $B=0$ then a log terminal model of (X,B) is called a *minimal model* of X. We denote by LMMP_n the classical conjectures in the Log Minimal Model program in dimension n, such as the abundance conjecture and termination of flips. In particular, LMMP_n implies that each Kawamata log terminal pair (X,B) such that K_X+B is pseudo-effective admits a log terminal model.

Definition 2.1. — *Let X be a smooth projective variety and let D be a $\mathbb{Q}$-divisor on X. We denote by $\kappa(X,D)$ the Kodaira dimension of D. For any positive integer q such that qD is Cartier and the linear system $|qD|$ is not empty, we denote by* $\mathrm{Fix}\,|qD|$, *the fixed part of the linear system $|qD|$. Thus, if $\kappa(X,D)\geqslant 0$, we may define*

$$\mathbf{Fix}(D)=\liminf_{q\to\infty}\frac{1}{q}\,\mathrm{Fix}\,|qD|,$$

where the limit is taken over all sufficiently divisible positive integers.

Remark 2.2. — Let (X,B) be a log smooth projective pair of dimension n such that $\lfloor B\rfloor=0$ and let $f\colon X\dashrightarrow Y$ be a log terminal model of K_X+B. Then, if $B_Y=f_*B$, we may write

$$K_X+B=f^*(K_Y+B_Y)+E$$

for some f-exceptional $\mathbb{Q}$-divisor $E\geqslant 0$. The negativity lemma (*e.g.* [2, Lemma 3.6.2]) implies that E does not depend on the log terminal model f. In addition, assuming LMMP_n, it is easy to check that $E=\mathbf{Fix}(K_X+B)$.

Lemma 2.3. — *Let (X,B) be a $\mathbb{Q}$-factorial projective Kawamata log terminal pair. Assume that B is nef, K_X+B is pseudo-effective and that B or K_X+B is big.*

Then there exists a sequence of steps

$$X=X_0\dashrightarrow\ldots\dashrightarrow X_k=Y$$

of the K_X-minimal model program such that the induced birational map $f\colon X\dashrightarrow Y$ is a log terminal model of (X,B).

Proof. — Let $A\geqslant 0$ be an ample $\mathbb{Q}$-divisor. For any rational number $\varepsilon>0$, since B is nef, we have that $B+\varepsilon A$ is ample. Thus, there exist a $\mathbb{Q}$-divisor $H_\varepsilon\sim_{\mathbb{Q}}B+\varepsilon A$ and $\lambda_\varepsilon>0$ such that, for any sufficiently small ε, the pair $(X,\lambda_\varepsilon H_\varepsilon)$ is Kawamata log terminal and $K_X+\lambda_\varepsilon H_\varepsilon$ is nef.

Let us consider the K_X-minimal model program of X with scaling of H_ε [2, Remark 3.10.10].

Note that if B is not big, then by assumption, K_X+B is big and therefore there exist $\delta,\eta>0$ such that $\eta(K_X+B)\sim_{\mathbb{Q}}\delta A+B'$ for some $\mathbb{Q}$-divisor $B'\geqslant 0$ such that $(X,B+B')$ is Kawamata log terminal. Let $C=B+B'+(\delta+\varepsilon(1+\eta))A$. We may assume that (X,C) is Kawamata log terminal. If $1\leqslant t\leqslant\lambda_\varepsilon$, we have

$$\begin{aligned}(1+\eta)(K_X+tH_\varepsilon) &\sim_{\mathbb{Q}} (1+\eta)(K_X+B+\varepsilon A+(t-1)H_\varepsilon)\\ &\sim_{\mathbb{Q}} K_X+B+B'+(\delta+\varepsilon(1+\eta))A+(t-1)(1+\eta)H_\varepsilon\\ &\sim_{\mathbb{Q}} K_X+C+(t-1)(1+\eta)H_\varepsilon.\end{aligned}$$

Thus, if $1\leqslant t\leqslant\lambda_\varepsilon$, a log terminal model of (X,tH_ε) is also a log terminal of $(X,C+(t-1)(1+\eta)H_\varepsilon)$ and the K_X-minimal model with scaling of H_ε coincides with the (K_X+C)-minimal model with scaling of $(1+\eta)H_\varepsilon$.

Therefore, after a finite number steps of the K_X-minimal model program, we obtain a K_X-negative map $f_\varepsilon\colon X\dashrightarrow X_\varepsilon$ such that X_ε is $\mathbb{Q}$-factorial and $K_{X_\varepsilon}+f_{\varepsilon*}H_\varepsilon$ is nef. By finiteness of models [2, Theorem E], there exists a sequence ε_i such that $\lim\varepsilon_i=0$ and X_{ε_i} is constant. In particular $K_{X_{\varepsilon_i}}+f_{\varepsilon_i*}B$ is nef. Note that since $f_{\varepsilon_i}\colon X\dashrightarrow X_{\varepsilon_i}$ is $(K_X+H_{\varepsilon_i})$-negative and A is ample, it is also (K_X+B)-negative. Thus f_{ε_i} is a log terminal model of (X,B). □

2.2. Kollár's effective base point freeness

In this section we describe some easy generalisations of Kollár's base point freeness theorem and Mumford's regularity theorem.

THEOREM 2.4. — *Let (X,B) be a projective Kawamata log terminal pair of dimension n such that K_X+B is nef and B or K_X+B is big. Let a be a positive integer such that $a(K_X+B)$ is Cartier.*

Then there exists a positive integer q depending only on n such that the linear system $|qa(K_X+B)|$ is base point free.

Proof. — If B is big, then there exists an ample $\mathbb{Q}$-divisor A and an effective $\mathbb{Q}$-divisor D such that $B\sim_{\mathbb{Q}}A+D$. Then if $\varepsilon>0$ is a sufficiently small rational number and $\Delta=(1-\varepsilon)B+\varepsilon D$, then the pair (X,Δ) is Kawamata log terminal and $B\sim_{\mathbb{Q}}\Delta+\varepsilon A$. Thus, if $M=a(K_X+B)$, then

$$M-(K_X+\Delta)\sim_{\mathbb{Q}}\varepsilon A+(a-1)(K_X+B)$$

is ample and the result follows by Kollár's effective base point freeness Theorem [15].

Thus, we may assume that $K_X + B$ is big and nef. Let $L = 2a(K_X + B)$. Then L is Cartier and $L-(K_X+B)$ is big and nef. The result follows again from [15]. □

Remark 2.5. — Using the notation of Theorem 2.4, by [15, Theorem 1.1] we can take $q = 4(n+2)!(n+1)$.

Lemma 2.6. — *Let (X, B) be a projective Kawamata log terminal surface such that K_X+B is nef. Let a be a positive integer such that $a(K_X+B)$ is Cartier.*

Then there exists a positive integer m depending only on a such that $|m(K_X + B)|$ is base point free.

Proof. — By Theorem 2.4, we may assume that $K_X + B$ is not big. If $K_X + B \sim_{\mathbb{Q}} 0$, then the results follows from [23, Theorem 3.1].

Thus, we may assume that there exists a map $f\colon X \longrightarrow C$ onto a smooth curve C such that $K_X + B = f^*D$ for some ample $\mathbb{Q}$-divisor D on C. We may assume that $D = K_C + B_C$ for some effective $\mathbb{Q}$-divisor B_C on C such that $\lfloor B_C \rfloor = 0$ (e.g. see [14]). By [19, Theorem 8.1], there exists a constant b, depending only on a such that bB_C is Cartier. Thus, the results follows. □

The next result follows closely the proof of Mumford's regularity theorem (e.g. see [17, Theorem 1.8.3]). A similar result also appeared in [9, Theorem 3.15].

Proposition 2.7. — *Let X be a normal projective variety of dimension n. Let $B_1, \dots, B_k$ be $\mathbb{Q}$-divisors on X and let $a_1, \dots, a_k$ be positive integers such that (X, B_i) is a Kawamata log terminal pair and there exist Cartier divisors $L_1, \dots, L_k$ such that $L_i \sim_{\mathbb{Q}} a_i(K_X+B_i)$ and the linear system $|L_i|$ is base point free, for $i = 1, \dots, k$. Let $G = \sum_{i=1}^k b_i L_i$ for some positive integers $b_1, \dots, b_k$ and assume that $b_\ell > n+1$.*

Then, the natural map

$$H^0(X, \mathcal{O}_X(G)) \otimes H^0(X, \mathcal{O}_X(L_\ell)) \to H^0(X, \mathcal{O}_X(G + L_\ell))$$

is surjective.

Proof. — We first assume that $\sum_{i=1}^k L_i$ is not big. Since the linear system $|\sum_{i=1}^k L_i|$ is base point free, there exists a morphism with connected fibres $f\colon X \longrightarrow Y$ onto a normal projective Y such that $\sum_{i=1}^k L_i = f^*A$ for some very ample divisor A on Y. Since $L_1, \dots, L_k$ are nef, if ξ is a curve contracted by f then $L_i \cdot \xi = 0$ for any $i = 1, \dots, k$. In particular, since $|L_i|$ is base point free, and the restriction of L_i to any fibre of f is trivial, it follows that there exist Cartier divisors $L'_1, \dots, L'_k$ such that $L_i = f^*L'_i$. By [1, Theorem 4.1], it follows that $L'_i \sim_{\mathbb{Q}} a_i(K_Y + B'_i)$ for some $\mathbb{Q}$-divisor B'_i such that (X, B'_i) is Kawamata log terminal for any $i = 1, \dots, k$. Note that the linear system $|L'_i|$ is base point free and that

$$H^0\Big(X, \mathcal{O}_X\Big(\sum_{i=1}^k c_i L_i\Big)\Big) \simeq H^0\Big(Y, \mathcal{O}_Y\Big(\sum_{i=1}^k c_i L'_i\Big)\Big)$$

for any non-negative integers $c_1, \dots, c_k$. Thus, after replacing X by Y, L_i by L'_i and B_i by B'_i, we may assume that $\sum_{i=i}^k L_i$ is big.

Let $V = H^0(X, \mathcal{O}_X(L_\ell))$ and let $\mathcal{V} = V \otimes \mathcal{O}_X$. Then

$$\mathcal{V} \otimes \mathcal{O}_X(-L_\ell) \to \mathcal{O}_X$$

is surjective. Thus, if $r = \dim V$, then the sequence

$$\begin{aligned} 0 = \wedge^{r+1}\mathcal{V} \otimes \mathcal{O}_X(-(r+1)L_\ell) \to \dots \\ \to \wedge^2\mathcal{V} \otimes \mathcal{O}_X(-2L_\ell) \to \mathcal{V} \otimes \mathcal{O}_X(-L_\ell) \to \mathcal{O}_X \to 0 \end{aligned}$$

is exact. Twisting by $\mathcal{O}_X(G + L_\ell)$ gives

$$\begin{aligned} 0 \to \wedge^r \mathcal{V} \otimes \mathcal{O}_X(G - (r-1)L_\ell) \to \dots \\ \to \wedge^2\mathcal{V} \otimes \mathcal{O}_X(G - L_\ell) \to \mathcal{V} \otimes \mathcal{O}_X(G) \to \mathcal{O}_X(G + L_\ell) \to 0. \end{aligned}$$

Since $\sum_{i=1}^k L_i$ is big, $b_\ell \geqslant n+2$ and $b_i \geqslant 1$ for $i \neq \ell$, we have that

$$\begin{aligned} &\sum_{i \neq \ell} b_i L_i + (b_\ell - j)L_\ell - (K_X + B_\ell) \\ &\qquad \sim_{\mathbb{Q}} \sum_{i \neq \ell} b_i L_i + (b_\ell - j)L_\ell - \frac{1}{a_\ell} L_\ell \\ &\qquad \sim_{\mathbb{Q}} \sum_{i=1}^k L_i + \sum_{i \neq \ell}(b_i - 1)L_i + \left(b_\ell - j - 1 - \frac{1}{a_\ell}\right) L_\ell \end{aligned}$$

is big and nef. Thus, Kawamata-Viehweg vanishing implies that

$$\begin{aligned} H^j(X, \wedge^{j+1}\mathcal{V} \otimes \mathcal{O}_X(G - jL)) &= \wedge^{j+1} V \otimes H^j(X, \mathcal{O}_X(G - jL_\ell)) \\ &= \wedge^{j+1} V \otimes H^j\Big(X, \mathcal{O}_X\Big(\sum_{i \neq \ell} b_i L_i + (b_\ell - j)L_\ell\Big)\Big) = 0 \end{aligned}$$

for any $j > 0$. Since

$$H^0(X, \mathcal{O}_X(G)) \otimes H^0(X, \mathcal{O}_X(L_\ell)) = H^0(X, \mathcal{V} \otimes \mathcal{O}_X(G)),$$

the map

$$H^0(X, \mathcal{O}_X(G)) \otimes H^0(X, \mathcal{O}_X(L_\ell)) \to H^0(X, \mathcal{O}_X(G + L_\ell))$$

is surjective and the claim follows. □

2.3. Adjoint Rings

In this section, we recall some basic notion about adjoint rings.

DEFINITION 2.8. — *Let X be a smooth projective variety and let D_1,...,D_k be Cartier divisors on X. The* adjoint ring associated to D_1,...,D_k *is*

$$R(X;D_1,\ldots,D_k)=\bigoplus_{(a_1,\ldots,a_k)\in\mathbb{Z}^k_{\geqslant 0}} H^0\Big(X,\mathcal{O}_X\Big(\sum_{i=1}^k a_iD_i\Big)\Big).$$

We say that $R=R(X;D_1,\ldots,D_k)$ is generated in degree m *if R is generated by sections of $H^0(X,\mathcal{O}_X(\sum_{i=1}^k a_iD_i))$, for $a_1,\ldots,a_k\in\{0,\ldots,m\}$.*

Remark 2.9. — The definition of adjoint rings can be easily extended to $\mathbb{Q}$-divisors $D_1,\ldots,D_k$ on a smooth projective variety X, but it will not be used in this paper in this generality (*e.g.* see [4] for more details).

Remark 2.10. — Let X be a smooth projective variety and let B be a $\mathbb{Q}$-divisor on X such that (X,B) is Kawamata log terminal. Let q be a positive integer such that qB is Cartier. Then $R=R(X,q(K_X+B))$ is generated in degree m if and only if the sections of

$$\bigoplus_{a\leqslant m} H^0(X,\mathcal{O}_X(aq(K_X+B)))$$

generate R.

PROPOSITION 2.11. — *Let X be a smooth projective variety of dimension n. Let $B_1,\ldots,B_k$ be $\mathbb{Q}$-divisors on X and let $a_1,\ldots,a_k$ be positive integers such that (X,B_i) is Kawamata log terminal and there exist Cartier divisors $L_1,\ldots,L_k$ such that $L_i\sim_{\mathbb{Q}} a(K_X+B_i)$ and the linear system $|L_i|$ is base point free, for $i=1,\ldots,k$.*

Then $R(X;L_1,\ldots,L_k)$ is generated in degree $n+2$.

Proof. — Let $G=\sum_{i=1}^k m_iL_i$ for some integers $m_1,\ldots,m_k\geqslant 0$ and assume that there exists $\ell\in\{1,\ldots,k\}$ such that $m_\ell>n+2$. Then Proposition 2.7 implies that

$$H^0(X,\mathcal{O}_X(G-L_\ell))\otimes H^0(X,\mathcal{O}_X(L_\ell))\to H^0(X,\mathcal{O}_X(G))$$

is surjective and the claim follows. □

Lemma 2.12. — *Let X be a smooth projective variety and let D be a Cartier divisor on X. Assume that $R(X,D)$ is generated in degree m and let $q = m!$.*

Then the stable base locus of D is equal to the base locus of the linear system $|qD|$. In addition, if $F = \mathbf{Fix}(D)$ (cf. Definition 2.1), then qF is a Cartier divisor.

Proof. — If x is a point contained in the base locus of the linear system $|qD|$ and $m' \leqslant m$ is a positive integer, then, since m' divides q, it follows that any section of $H^0(X, \mathcal{O}_X(m'D))$ vanishes at x. Thus, by assumption, any section of $H^0(X, \mathcal{O}_X(\ell D))$ vanishes at x, for any positive integer ℓ. In particular, x is contained in the stable base locus of D and the first claim follows. The proof of the second claim is analogous. □

2.4. Surface and threefolds singularities

In this section, we recall a few known facts about Kawamata log terminal singularities in dimension 2 and terminal singularities in dimension 3.

Lemma 2.13. — *Let (X,B) be a log smooth surface such that $\lfloor B \rfloor = 0$ and $K_X + B$ is pseudo-effective. Let $f\colon X \longrightarrow Y$ be the log terminal model of (X,B).*

Then there exist birational morphisms $g\colon X \longrightarrow Z$ and $h\colon Z \longrightarrow Y$ which factorize f and such that

(1) *g is a sequence of smooth blow-ups;*
(2) *h contracts only divisors contained in the support of g_*B.*

Proof. — Let $h\colon Z \longrightarrow Y$ be the minimal resolution of Y. Then, since X is smooth, there exists a morphism $g\colon X \longrightarrow Z$, which is a sequence of smooth blow-ups and such that $f = h \circ g$. Let $C = g_*B$. Then, h is the log terminal model of (Z,C) and since Z is the minimal resolution, it follows that h does not contract any (-1)-curve. In particular, $a(F,Y) \leqslant 0$ for any curve F contracted by h. Thus, $a(F,Y,f_*B) \leqslant 0$ and since h is $(K_Z + C)$-negative, it follows that $a(F,Z,C) < 0$. Therefore F is contained in the support of C. □

We proceed by bounding the index of a Kawamata log terminal surface with respect to the graph of its minimal resolution:

Proposition 2.14. — *Let (S,p) be the germ of a Kawamata log terminal surface and let $f\colon T \to S$ be the minimal resolution of S. Assume* *that $E_1, \dots, E_k$ are the irreducible components of the exceptional divisor of f and let $\varepsilon > 0$ be such that $a(E_i, S) \geqslant -1 + \varepsilon$ for all $i = 1, \dots, k$.*

Then there exists a constant $r = r(k, \varepsilon)$, depending only on k and ε, such that rD is Cartier for all Weil divisors D on S.

Proof. — The germ of a Kawamata log terminal surface (S, p) is given by the quotient of $\mathbb{C}^2$ by a finite subgroup G of $\mathrm{GL}(2, \mathbb{C})$, without quasi-reflections. Thus, it is enough to bound the order of G, depending on the graph associated to the minimal resolution of (S, p). It follows from [3, p. 348] that the order of G is at most $r = 120k^2/\varepsilon^3$. □

We now consider the germ (X, p) of a terminal singularity in dimension 3. The *index* of X at p is the smallest positive integer $r = r(X, p)$ such that rK_X is Cartier. In addition, it follows from the classification of terminal singularities [18], that there exists a deformation of (X, p) into a variety with $k \geqslant 1$ terminal singularities $p_1, \dots, p_k$ which are isolated cyclic quotient singularities of index $r(p_i)$. The set $\{p_1, \dots, p_k\}$ is called the *basket* $\mathcal{B}(X, p)$ of singularities of X at p [20]. The number k is called the *axial weight* $\mathrm{aw}(X, p)$ of (X, p). As in [7], we define

$$\Xi(X, p) = \sum_{i=1}^{k} r(p_i).$$

Thus, if X is a projective variety of dimension 3 and with terminal singularities, then the set $\mathrm{Sing}\, X$ of singular points of X is finite. In particular, we may define

$$\Xi(X) = \sum_{p \in \mathrm{Sing}\, X} \Xi(X, p) \qquad \text{and} \qquad \mathrm{aw}(X) = \sum_{p \in \mathrm{Sing}\, X} \mathrm{aw}(X, p).$$

Definition 2.15. — *Let X be a terminal projective variety of dimension 3. Then a w-resolution of X is a sequence*

$$Y = X_N \longrightarrow \dots \longrightarrow X_0 = X$$

such that

(1) *X_i has only terminal singularities for $i = 1, \dots, N$ and X_N has only Gorenstein singularities;*
(2) *the map $X_{i+1} \longrightarrow X_i$ is a weighted blow-up at $P_i \in X_i$ with minimal discrepancy $1/m_i$ where m_i is the index of X_i at P_i, for $i = 0, \dots, N-1$.*

The following result is due to Hayakawa.

Theorem 2.16. — *Let X be a terminal projective threefold.*
Then X admits a w-resolution.

Proof. — See [12, Theorem 6.1]. □

Given X as above, we define $\operatorname{dep}(X)$ the *depth* of X to be the minimum length of any w-resolution of X.

Proposition 2.17. — *Let X, X' be terminal projective varieties of dimension 3. Then*

(1) *If $X \dashrightarrow X'$ is a a flip, then $\operatorname{dep}(X) > \operatorname{dep}(X')$.*
(2) *If $X \longrightarrow X'$ is an extremal divisorial contraction to a curve, then $\operatorname{dep}(X) \geqslant \operatorname{dep}(X')$.*
(3) *If $X \longrightarrow X'$ is an extremal divisorial contraction to a point, then $\operatorname{dep}(X) \geqslant \operatorname{dep}(X') - 1$.*

Proof. — See [7, Proposition 2.15, 3.8, 3.9]. □

3. Bounding threefold terminal singularities

The aim of this section is to give a bound on the singularities of a minimal projective threefold depending on the topology of its resolution. More specifically, we show that we can bound the sum of the indices of all the points in all the baskets of Y by a constant which depends only on the Picard number $\rho(X)$ of X.

We first show a bound on the number of flips for the minimal model program of a smooth projective threefold X:

Lemma 3.1. — *Let X be a smooth projective threefold.*

Then both the number of divisorial contractions and the number of flips in the minimal model program of X are bounded by $\rho(X)$. In addition, if Y is the minimal model of X then the number of points of Y with index greater than one is also bounded by $\rho(X)$.

Proof. — Clearly the number of divisorial contractions is bounded by $\rho(X)$. Let

$$X = X_0 \dashrightarrow \dots \dashrightarrow X_k = Y$$

be a sequence of steps for the K_X-minimal model program of X, where $X \dashrightarrow Y$ is a minimal model of X. Let $d(X_i)$ be the Shokurov's difficulty of X_i, *i.e.*,

$$d(X_i) = \#\{\nu = \text{valuation} \mid a(\nu, X_i) < 1, \quad \nu \text{ is exceptional over } X_i\}.$$

Then, since X is smooth, we have $d(X) = 0$. In addition, if $X_i \dashrightarrow X_{i+1}$ is an extremal divisorial contraction, then

$$d(X_i) \leqslant d(X_{i-1}) + 1$$

Finally, if $X_i \dashrightarrow X_{i+1}$ is a flip, then

$$d(X_i) \leqslant d(X_{i-1}) - 1,$$

(*e.g.* see [16]) Thus, the total number of flips is bounded by the number of divisorial contractions.

Finally, for each point $p \in Y$ such that $r(Y, p) > 1$, the main result in [13] implies that there exists a weighted blow-up $f\colon W \to Y$ of minimal discrepancy, *i.e.*, if E is the exceptional divisor of f then

$$a(E, Y) = \frac{1}{r(Y, p)}.$$

Thus, it follows that the number of such points is bounded by $d(Y)$. Thus, it is also bounded by $\rho(X)$ and the claim follows. □

The following is a generalization of [7, Proposition 2.13]:

Lemma 3.2. — *Let Z be a terminal projective variety of dimension* 3. *Then,*

$$\Xi(Z) \leqslant 2\,\mathrm{dep}(Z).$$

Proof. — We proceed by induction on $\mathrm{dep}(Z)$. If $\mathrm{dep}(Z) = 0$ then Z is Gorenstein and $\Xi(Z) = 0$. Thus, the claim follows.

Assume now that $\mathrm{dep}(Z) > 0$. We claim that if $f\colon Y \longrightarrow Z$ is a weighted blow-up of minimal discrepancy at the point $p \in Z$, then

$$\Xi(Y) \geqslant \Xi(Z) - 2.$$

We first prove the Lemma, assuming the claim. Let $f\colon Y \longrightarrow Z$ be the first weighted blow-up of minimal discrepancy in a w-resolution of Z. Then, by definition we have that $\mathrm{dep}(Z) = \mathrm{dep}(Y) + 1$ and by induction, we have that $\Xi(Y) \leqslant 2\,\mathrm{dep}(Y)$. Thus

$$\Xi(Z) \leqslant \Xi(Y) + 2 \leqslant 2\,\mathrm{dep}(Y) + 2 = 2\,\mathrm{dep}(Z),$$

and the lemma follows.

The claim follows from the classification of weighted blow-ups in [11, 12]. For example, assume that (Z, p) is a point of type cA/r and $Y \longrightarrow Z$ is a weighted blow-up of minimal discrepancy. Then, by the proof of [11, Theorem 6.4], there exist positive integers a, b satisfying $a + b = kr$ with $k \leqslant \mathrm{aw}(Z)$ and such that the only points of index greater than 1 on Y are the following: a point Q_1 which is a cyclic quotient singularity of index a, a point Q_2 which is a cyclic quotient singularity of index b and, if $k < \mathrm{aw}(Z, p)$, a point Q_3 which is of type cA/r of index r and axial weight $\mathrm{aw}(Z) - k$. Thus, since $\Xi(Z, p) = r\,\mathrm{aw}(Z, p)$, it follows that

$$\Xi(Y) - \Xi(Z) = a + b + r(\mathrm{aw}(Z, p) - k) - r\,\mathrm{aw}(Z, p) = 0.$$

Similarly, if (Z,p) is a point of of type $cAx/4$ and $Y \longrightarrow Z$ is the weighted blow-up given by [11, Theorem 7.4], then there exists a positive integer k such that the only points of index greater than 1 on Y are the following: a point Q_1 which is cyclic of quotient singularity of index $2k+3$ and, if $\mathrm{aw}(Z,p) > k+1$, a point Q_2 which is of type $cD/2$ and such that $\mathrm{aw}(Y,Q_2) = \mathrm{aw}(Z,p) - k - 1$. Thus, since $\Xi(Z,p) = 2\,\mathrm{aw}(Z,p) + 2$, it follows that

$$\Xi(Y) - \Xi(Z) = 2(\mathrm{aw}(Z,p) - k - 1) + 2k + 3 - (2\,\mathrm{aw}(Z,p) + 2) = -1.$$

It is easy to check that if (Z,p) is a singularity of different type, then Hayakawa's list of weighted blow-ups of minimal discrepancy implies that the inequality is satisfied in all the cases. □

Proposition 3.3. — *Let X be a smooth projective threefold and assume that*

$$X = X_0 \dashrightarrow \ldots \dashrightarrow X_k = Y$$

is a sequence of steps for the K_X-minimal model program of X.

Then

$$\Xi(Y) \leqslant 2\rho(X).$$

In particular, the inequality holds if Y is the minimal model of X.

Proof. — Since X is smooth, we have that $\mathrm{dep}(X) = 0$. By Proposition 2.17, it follows that $\mathrm{dep}(X_i) \leqslant \mathrm{dep}(X_{i-1})$ unless $X_{i+1} \dashrightarrow X_i$ is a divisorial contraction to a point and in this case, we have $\mathrm{dep}(X_i) \leqslant \mathrm{dep}(X_{i-1}) + 1$. Thus, $\mathrm{dep}(X_i)$ is bounded by the number of divisorial contractions in the first i steps of the minimal model program of X. Thus, Lemma 3.1 implies that $\mathrm{dep}(Y) \leqslant \rho(X)$.

On the other hand, Lemma 3.2 implies that

$$\Xi(Y) \leqslant 2\,\mathrm{dep}(Y).$$

Thus, the claim follows. □

4. Effective Finite Generation

We now proceed to show a bound on the degree of the generators of the adjoint ring associated to a Kawamata log terminal surface depending on the number of components of its boundary and the corresponding coefficients.

Proposition 4.1. — *Let* $(X, B = \sum_{i=1}^{p} a_i S_i)$ *be a log smooth projective surface, where* $S_1, \dots, S_p$ *are distinct prime divisors and* $\lfloor B \rfloor = 0$*. Let* a *be a positive integer such that* aB *is Cartier.*

Then, there exists a positive integer $m = m(a,p)$*, depending only on* a *and* p*, such that* $R(X, m(K_X + B))$ *is generated in degree* 4*.*

Proof. — Let $f\colon X \longrightarrow S$ be the log terminal model of (X, B). Lemma 2.13 implies that f factorizes through the minimal resolution $h\colon S' \longrightarrow S$ of S. Let $g\colon X \longrightarrow S'$ be the induced map. Then h contracts only prime divisors which are rational curves with negative self-intersection and contained in the support of $g_* B$. Let Γ be the strict transform of such a curve on X. Then

$$\mathrm{mult}_\Gamma B \geqslant -a(\Gamma, S).$$

Thus, since $a\mathrm{mult}_\Gamma B$ is a positive integer and $\mathrm{mult}_\Gamma B < 1$, it follows that

$$a(\Gamma, S) \geqslant -1 + \frac{1}{a}.$$

By Proposition 2.14, it follows that there exists a constant $r = r(a,p)$ depending only on a and p such that rD is Cartier for any Weil divisor D on S. In particular, if $B_S = f_* B$, then $ar(K_S + B_S)$ is a Cartier divisor. Thus, by Lemma 2.6 there exists a constant $q = q(a,p)$ depending on a and p such that the linear system $|q(K_S + B_S)|$ is base point free. Let $L = q(K_S + B_S)$. Proposition 2.7 implies that if $b > 3$ then the natural map

$$H^0(S, \mathcal{O}_S(bL)) \otimes H^0(S, \mathcal{O}_S(L)) \to H^0(S, \mathcal{O}_S((b+1)L))$$

is surjective. Thus, $R(X, q(K_X + B))$ is generated in degree 4 and the claim follows. □

Remark 4.2. — Using the same notation as in Proposition 4.1, let q be the number of components of B which are contained in the stable base locus of $K_X + B$. Then, the same proof shows that, $R(X, m(K_X + B))$ is generated in degree 4 where m is a constant depending on a and q. Note that $q \leqslant \rho(X)$. Thus, we can bound m by a constant which depends on a and the Picard number $\rho(X)$ of X.

The following example shows that the degree of the generators depends on the number of components p.

Example 4.3. — Let r be a prime and let X_r be the smooth toric surface obtained by blowing-up a sequence of r infinitesimal near points of $\mathbb{P}^2$. More specifically, let $X_0 = \mathbb{P}^2$ with a torus $T = (\mathbb{C}^*)^2$ acting on it, let X_1 be the blow-up of $\mathbb{P}^2$ at a T-invariant point p and for each $i = 1, \dots, r$ let

X_i be the surface obtained by blowing-up the T-invariant point in the exceptional divisor of $f_{i-1}\colon X_{i-1} \longrightarrow X_{i-2}$ which is not contained in the strict transform of the exceptional divisor of f_{i-2}. Then X_r admits a chain of T-invariant (-2)-curves $e_1,\dots,e_{r-1}$ and a T-invariant (-1)-curve e_r, given the exceptional curve of f_r. Let $E=\sum_{i=1}^r e_i$, let $\pi\colon X_r \longrightarrow X_0$ be the induced map and let $h=\pi^*\ell$ where ℓ is the T-invariant line in X_0 not passing through p. Note that if $f\colon X_r \longrightarrow Y$ is the map obtained by contracting the curves $e_1,\dots,e_{r-1}$, then Y admits a unique singular point of type A_{r-1}. Let

$$G=\frac{1}{r}\sum_{i=1}^r ie_i.$$

Then the T-invariant curves on X_r are contained in the classes

$$e_1,\dots,e_r,h,h-rG,h-e_1.$$

It follows that if a is a sufficiently large positive integer then $ah-2rG$ is nef and therefore the linear system $|ah-2rG|$ is base point free. Thus, there exists a $\mathbb{Q}$-divisor $B_0\geqslant 0$ such that $B_0\sim_{\mathbb{Q}} ah-2rG$, and if $B=B_0+\frac{1}{2}\sum_{i=1}^r e_i$ then (X,B) is log smooth, $\lfloor B\rfloor=0$, and $2B$ is Cartier. It is easy to check that $f\colon X_r\longrightarrow Y$ is the log terminal model of (X,B) and that

$$K_X+B=f^*(K_Y+f_*B)+E$$

where

$$E=\frac{1}{2r}\sum_{i=1}^{r-1}(r-i)e_i.$$

In particular, by Remark 2.2, we have $E=\mathbf{Fix}(K_X+B)$. Thus, since r is prime, Lemma 2.12 implies that if a and m are positive integer such that a is even and $R(X,a(K_X+B))$ is generated in degree m then either $a\geqslant r$ or $m\geqslant r$.

The following example shows that Proposition 4.1 cannot be extended to the log canonical (or dlt) case.

Example 4.4. — Let X be the Hirzebruch surface $\mathbb{F}_r$ of prime degree $r\geqslant 3$, let S be the unique curve of negative self-intersection $-r$ and let H be a curve of self-intersection r. Then there exists an effective $\mathbb{Q}$-divisor $B'\sim_{\mathbb{Q}} 2H$ such that, S is not contained in the support of B' and if $B=S+B'$ then (X,B) is log smooth, $\lfloor B'\rfloor=0$ and $2B'$ is Cartier. In particular, the support of B' does not intersect S.

It is easy to check that

$$\mathbf{Fix}(K_X + B) = \frac{2}{r} S.$$

Thus, Lemma 2.12 implies that if a is an even positive integer such that $R(X, a(K_X + B))$ is generated in degree m then either $a \geqslant r$ or $m \geqslant r$.

We now proceed with the proof of Theorem 1.1:

Proof of Theorem 1.1. — We may assume that $K_X + B$ is pseudo-effective, otherwise $R(X, K_X + B)$ is trivial. Since B is big or $K_X + B$ is big, Lemma 2.3 implies that there exists a sequence of steps

$$X = X_0 \dashrightarrow \dots \dashrightarrow X_k = Y$$

of the K_X-minimal model program such that the induced birational map $f\colon X \dashrightarrow Y$ is a log terminal model of (X, B). Note that, in particular, since X is smooth, Y has terminal singularities.

By Proposition 3.3, it follows that there exists a constant r depending only on $\rho(X)$ such that rD is Cartier for any Weil divisor D on Y. In particular, if $B_Y = f_*B$, then $ra(K_Y + B_Y)$ is a Cartier divisor. By Theorem 2.4 there exists a constant q' such that, if $q = q'r$, then the linear system $|qa(K_Y + B_Y)|$ is base point free. Similarly to Proposition 4.1, Proposition 2.7 implies that $R(X, qa(K_X + B))$ is generated in degree 5, as claimed. □

We expect that a more general result holds for any Kawamata log terminal pair threefold or even more in general for adjoint rings on threefolds, as in the case of surfaces. On the other hand, the following example shows that it is not possible to bound the degree of the generators of the canonical ring $R(X, K_X)$ of a smooth projective variety X independently of $\rho(X)$, even assuming $B = 0$.

Example 4.5. — Let r be a prime number and let Y_r be a projective variety of dimension 3 such that K_{Y_r} is ample and Y_r admits a singular point of type

$$\frac{1}{r}(1, 1, r-1).$$

Let $X_r \longrightarrow Y_r$ be a resolution. Then there exists a prime divisor E on X_r which is exceptional over Y_r and such that $a(E, Y_r) = \frac{1}{r}$. In particular,

$$\operatorname{mult}_E \mathbf{Fix}(K_{X_r}) = \frac{1}{r}.$$

Thus, since r is prime, Lemma 2.12 implies that if q is a positive integer and $R(X, qK_X)$ is generated in degree m, then either $q \geqslant r$ or $m \geqslant r$.

Proof of Corollary 1.2. — It follows immediately from Theorem 1.1 and Lemma 2.12. □

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