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NON-ABELIAN p-ADIC L-FUNCTIONS AND EISENSTEIN SERIES OF UNITARY GROUPS – THE CM METHOD

by Thanasis BOUGANIS (*)

Abstract. — In this work we prove various cases of the so-called “torsion congruences” between abelian p-adic L-functions that are related to automorphic representations of definite unitary groups. These congruences play a central role in the non-commutative Iwasawa theory as it became clear in the works of Kakde, Ritter and Weiss on the non-abelian Main Conjecture for the Tate motive. We tackle these congruences for a general definite unitary group of n variables and we obtain more explicit results in the special cases of $n = 1$ and $n = 2$. In both of these cases we also explain their implications for some particular “motives”, as for example elliptic curves with complex multiplication. Finally we also discuss a new kind of congruences, which we call “average torsion congruences”

Résumé. — Dans cet article, nous démontrons divers cas particuliers de “congruences de torsion” entre les L-fonctions p-adiques abéliennes liées aux représentations automorphes de groupes unitaires définis. Ces congruences jouent un rôle central dans la théorie d’ Iwasawa non-commutative, ce qui a été mis en évidence par les résultats de Kakde, Ritter et Weiss sur la Conjecture Principale non-abélienne pour le motif de Tate. Nous nous attaquons à ces congruences pour un groupe unitaire défini général en n variables, et obtenons des résultats plus explicites dans les cas $n = 1$ et $n = 2$. Dans ces deux cas, nous expliquons aussi leur conséquences pour certains “motifs” particuliers, comme par exemple, les courbes elliptiques munies d’une multiplication complexe. Finalement, nous discutons d’un nouveau type de congruences que nous nommons “congruences de torsion modérées”.

1. Introduction

In [8, 15] a vast generalization of the Main Conjecture of the classical (abelian) Iwasawa theory to a non-abelian setting was proposed. As in the classical theory, the non-abelian Main Conjecture predicts a deep relation

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between an analytic object (a non-abelian p-adic L-function) and an algebraic object (a Selmer group or complex over a non-abelian p-adic Lie extension). However, the evidences for this non-abelian Main Conjecture are still very modest. One of the central difficulties of the theory seems to be the construction of non-abelian p-adic L-functions. Actually, the only known results in this direction are mainly restricted to the Tate motive, initially for particular totally real p-adic Lie extensions (see [19, 31, 33, 37]) and later for a large family of totally real p-adic Lie extension as it is shown by Ritter and Weiss in [37, 38] and Kakde [32].

For other motives besides the Tate motive not much is known. For elliptic curves there are some evidences for the existence of such non-abelian p-adic L-functions offered in [4, 9] and also some computational evidences offered in [10, 11]. Also, there is some recent progress, achieved in [6], for elliptic curves with complex multiplication defined over $\mathbb{Q}$ with respect to the p-adic Lie extension obtained by adjoing to $\mathbb{Q}$ the p-power torsion points of the elliptic curve.

The main aim of this work, as well as its companion work [2], is to tackle the question of the existence of non-abelian p-adic L-functions for "motives", whose classical L-functions can be studied through L-functions of automorphic representations of definite unitary groups. In this work we will prove the so called "torsion congruences" (to be explained below) for these motives. In a second part of this work [1] we use our approach to tackle also the so called Möbious-Wall congruences (as for example are described in [38]). Without going into details, we simply mention here that these results allow one to conclude, under some assumptions, the existence of the non-abelian p-adic L-function in the $K_1(\widehat{\Lambda(G)_S}[\frac{1}{p}])$. The stronger result, that the non-abelian p-adic L-function actually lies in $K_1(\Lambda(G)_S[\frac{1}{p}])$, as is conjectured in [8], needs, with the present knowledge, one to assume that the classical abelian Main Conjecture holds for all the subfields of the p-adic Lie extension that corresponds to G.

The "torsion-congruences" for motives. Let p be an odd prime number. We write F for a totally real field and F' for a totally real Galois extension with $\Gamma := \mathrm{Gal}(F'/F)$ of order p. We assume that the extension is unramified outside p. We write $G_F := \mathrm{Gal}(F(p^\infty)/F)$, where $F(p^\infty)$ is the maximal abelian extension of F unramified outside p (may be ramified at infinity). We make the similar definition for $F'(p^\infty)$. Our assumption on the ramification of F'/F implies that there exist a transfer map $\mathrm{ver}\colon G_F \to G_{F'}$, which induces also a map $\mathrm{ver}\colon \mathbb{Z}_p[[G_F]] \to \mathbb{Z}_p[[G_{F'}]]$ between the Iwasawa algebras of G_F and $G_{F'}$, both of them taken with coefficients in $\mathbb{Z}_p$. Let us now consider a motive M/F (by which we really mean the usual realizations of it and their compatibilities) defined over F such that its p-adic realization has coefficients in $\mathbb{Z}_p$. Then under some assumptions on the critical values of M and some ordinarity assumptions at p (to be made more specific later) it is conjectured that there exists an element $\mu_F \in \mathbb{Z}_p[[G_F]]$ that interpolates the critical values of M/F twisted by characters of G_F. Similarly we write $\mu_{F'}$ for the element in $\mathbb{Z}_p[[G_{F'}]]$ associated to M/F', the base change of M/F to F'. Then the so-called torsion congruences read $\mathrm{ver}(\mu_F) \equiv \mu_{F'} \mod T$, where T is the trace ideal in $\mathbb{Z}_p[[G'_F]]^\Gamma$ generated by the elements $\sum_{\gamma\in\Gamma}\alpha^\gamma$ with $\alpha \in \mathbb{Z}_p[[G'_F]]$. These congruences have been introduced for first time and proved by Ritter and Weiss [37] for M/F the Tate motive. Further, under some assumptions, the author [4] has shown them for M/F equal to the motive associated to an elliptic curve with complex multiplication. We also remark that for the Tate motive, a geometric approach to the torsion congruences through the so-called Shintani decomposition has been applied in [5]. In this work we prove these congruences for motives that their L-functions can be studied by automorphic representations of definite unitary groups.

The general setting of this work. We keep the notations already introduced above. We now write K for a totally imaginary quadratic extension of F, that is K is a CM field. On our prime number p we put the following ordinary assumption: all primes above p in F are split in K. As before we consider a totally real Galois extension F' of F of degree p that is ramified only at p. We write $K' := F'K$, a CM field with $K'^+ = F'$. Now we fix, once and for all, the embeddings $\mathrm{incl}_\infty\colon \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$ and $\mathrm{incl}_p\colon \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}_p$. Next we fix, with respect to the fixed embeddings $(\mathrm{incl}_\infty, \mathrm{incl}_p)$ an ordinary CM type Σ of K and denote this pair by (Σ, K). We recall that Σ is called ordinary (see [35]) when the following condition is satisfied: "whenever $\sigma \in \Sigma$ and $\lambda \in \Sigma^\rho$ (ρ is the complex conjugation), the p-adic valuations induced from the p-adic embeddings $\mathrm{incl}_p \circ\sigma$ and $\mathrm{incl}_p \circ\lambda$ are inequivalent". We note that the splitting condition on p implies the existence of such an ordinary CM type. We consider the induced type Σ' of Σ to K'. That is, we fix a CM type for K' such that for every $\sigma \in \Sigma'$ we have that its restriction $\sigma_{|K}$ to K lies in Σ. We write (K', Σ') for this CM type, and we remark that this is also an ordinary CM type. In addition to the splitting condition we also impose the condition that the reflex field E of (K, Σ) has the property that $E_w = \mathbb{Q}_p$, where w's are the places of E corresponding to the embeddings $E \hookrightarrow \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}_p$. For example this is the case if p does not ramify in F or if the type (K, Σ) is the lift of a type (K_0, Σ_0) where K_0

is a quadratic imaginary field, such that K/K_0 is a Galois extension and p splits in K_0.

Now we are ready to define the motives M/F that appear in this work. We would like to warn the reader that the word "motive" is used here in a very loose sense. What we really need is the existence of a p-adic measure over G_F that interpolates some special values, and of course a measure over $G_{F'}$ that is associated to the base changed M/F'. Then we can formulate the torsion congruences.

Let ψ be a Hecke character of K and assume that its infinite type is $-k\Sigma$ for some integer $k \geqslant 1$. We write $M(\psi)/F$ for the motive over F that is obtained by "Weil Restriction" to F from the rank one motive over K associated to ψ. In particular we have that $L(M(\psi)/F, s) = L(\psi, s)$ or more generally for a finite character χ of G_F we have $L(M(\psi)\otimes\chi, s) = L(\psi\tilde{\chi}, s)$, where $\tilde{\chi} = \chi \circ N_{K/F}$, the base change of χ to $G(KF(p^\infty)/K)$. Now we consider the character $\psi' := \psi \circ N_{K'/K}$, the base change of ψ from K to K'. It is a Hecke character of infinite type $-k\Sigma'$. Moreover we have that $M(\psi')/F'$ is the base change of $M(\psi)/F$ to F'.

We consider now a hermitian space (W, θ) over K, that means that W is a vector space over K, we write n for its dimension, and θ is a non-degenerate hermitian form on it. Moreover we assume that the signature of the form θ_σ on the complex vector space $W \otimes_{K,\sigma} \mathbb{C}$ is the same for every embedding $\sigma\colon K \hookrightarrow \mathbb{C}$ in Σ. In particular this implies that our hypothesis on the splitting of the primes above p in K is the usual ordinary condition; for a more general ordinarity condition the reader should see [25, p. 8]. We write $U(\theta)$ for the corresponding unitary group (see section two for the definition).

We let $U(\theta')$ be the group $\mathrm{Res}_{F'/F}\, U(W)/F'$, that is the unitary group corresponding to (W', θ) where $W' = W \otimes_K K'$. The F-rational points of $U(\theta')$ are the F'-rational points of $U(\theta)$. We consider now a motive $M(\pi)/K$ over K such that there exists an automorphic representation π of some unitary group $U(\theta)(\mathbb{A}_F)$ with the property that the L-function $L(M(\pi)/K, s)$ of $M(\pi)/K$ over K is equal to $L(\pi, s)$. As we remarked above, we use the word "motive" in a very loose sense. What we really use is the fact that we can associate some periods to the various critical values and then the conjectures of Deligne on algebraicity are meaningful. In particular we may speak of p-adic L-functions.

Let now $\mathfrak{c}$ be a large enough integral ideal of K that contains the conductor of the representation π. We will be assuming that $(\mathfrak{c}, p) = 1$. We now write π' for the base change of π to $U(\theta')$. This exists in this general setting only conjecturally by Langlands' functoriality conjectures but in the cases of interest that we are going to consider later it is known to exist. Then we have that $L(M(\pi)/K', s) = L(\pi', s) = L(M(\pi')/K', s)$.

Our aim in this work is to prove the torsion congruences for the motive $M(\pi, \psi)/F$ obtained by Weil Restriction from the motive $(\psi \otimes M(\pi))/K$, where here ψ is thought as the rank one motive over K associated to the Hecke character ψ. The L-function of M/F is by the inductive properties of the L-functions equal to $L(\pi, \psi, s)$, or more general for a character χ of G_F we have $L(M(\pi, \psi)/F, \chi, s) = L(\pi, \psi\tilde{\chi}, s)$, where $\tilde{\chi} = \chi \circ N_{K/F}$. We moreover note that when the motive $M(\pi)$ is defined over F then we have that $M(\pi, \psi)/F = M(\psi)/F \times M(\pi)/F$ by Frobenius reciprocity.

We will make the following three assumptions

(1) The p-adic realizations of $M(\pi)$ and $M(\psi)$ have $\mathbb{Z}_p$-coefficients (actually our methods should work for coefficients in $\mathbb{Z}_p^{nr}$, the ring of integers of the maximal unramified extension of $\mathbb{Q}_p$).
(2) π is an automorphic representation of a definite unitary group. The infinite type of the representation is taken to be of parallel scalar weight. We denote this weight by ℓ.
(3) Let n be the number of variables of the unitary group associated to π. Then for the weight of the character ψ and of the representation π we have the condition $k + 2\ell \geqslant n$.

Now we indicate some cases of special interest that are included in the motives that we described above.

The case $n = 1$. The main application in this case is obtained with π trivial. In this setting, our theorem proves the "torsion congruences" for elliptic curves with complex multiplication, or in general for Hilbert modular forms of CM type. Results in this direction have been also obtained in our previous work [4] on these congruences. However we stress that we do not only reobtain these results with our new methods but also improve on the assumptions that we made there. Actually we obtain the same result almost unconditionally. Finally we mention that in [4] the main ingredient was the Eisenstein measure of Katz as in [35] and is related to the automorphic theory of the group GL_2/F, that is Hilbert modular forms. In this work we use the automorphic theory of unitary groups and hence hermitian modular forms.

The case $n = 2$. Let us now discuss an application of the case $n = 2$. We consider a Hilbert cuspidal form f of F, which is assumed to be a normalized newform. For simplicity we take the infinite type to be of parallel weight two. We write N_f for its conductor. We assume that N_f is square free and relative prime to p. We now impose the following assumptions on f.

(1) f has a trivial Nebentypus.
(2) There exists a finite set S of finite places of F such that we have (i) $\mathrm{ord}_v(N_f) \neq 0$ for all $v \in S$, (ii) for $v \in S$ we have that v is inert in K and finally (iii) $\sharp S + [F\colon \mathbb{Q}]$ is even.

Let us write D/F for the totally definite quaternion algebra that we can associate to the set S, *i.e.* D is ramified at all finite places $v \in S$ and also at all infinite places. Note that our assumptions imply that there exists an embedding $K \hookrightarrow D$. If we write π' for the cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_F)$ associated to f then our assumptions imply that there exists a Jacquet-Langlands correspondence $\pi := JL(\pi')$ to $D^\times(\mathbb{A}_F)$. As we will explain later there exists an isomorphism $(D^\times \times K^\times)/F^\times \cong GU(\theta)(F)$ for some totally definite two dimensional Hermitian form (W,θ). In particular, the representation π induces an automorphic representation, by abuse of notation, π on $GU(\theta)$ and by restriction to $U(\theta)$. The scalar weight of the automorphic representation is zero (*i.e.* $\ell = 0$). Moreover it is known that $L(\pi, s) = L(BC(\pi'), s)$, where $BC(\pi')$ is the base-change of π' from $\mathrm{GL}_2(\mathbb{A}_F)$ to $\mathrm{GL}_2(\mathbb{A}_K)$. In particular we may pick $M(\pi)/F$ above to be the motive associated to the Hilbert modular form f. This explains our interest in the case $n = 2$. We also remark here that Ming-Lun Hsieh in [29] has made important progress with respect to the classical abelian Iwasawa Main Conjecture of such motives, *i.e.* $M(\pi)/F \times M(\psi)/F$.

Now we are ready to state the main theorems of this work. We start with the motive $M(\psi)/F$. The precise interpolation properties of the measure $\mu_{M(\psi)/F}$ are given in Theorem 4.1. As we explain after that Theorem (in Remark 4.2(ii)), this measure has very similar interpolation properties to the measure constructed by Katz, Hida and Tilouine. Namely this measure interpolates values of the L function associated to ψ twisted by finite Hecke characters of K. However there are some differences on some normalizing factors as well as on the Euler factors that we remove. We refer the reader to Remark 4.2(ii) for more details on this.

Theorem 1.1 (Main Theorem 1). — *Let $n = 1$. Assume that the prime p is unramified in F (but may ramify in F') and let Σ be an ordinary CM type of K. Let ψ be a Hecke character over K of infinite type $-k\Sigma$ with $k \geqslant 1$ and with values in $\mathbb{Z}_p$ (its p-adic realization). Write $M(\psi)/F$ for the motive over F introduced above. Further write $M(\psi')/F'$ for the base* *change of $M(\psi)/F$ to F'. Then the torsion congruences hold true, that is*

$$\mathrm{ver}(\mu_{M(\psi)/F}) \equiv \mu_{M(\psi')/F'} \mod T,$$

where $\mu_{M(\psi)/F}$ and $\mu_{M(\psi')/F'}$ are the p-adic measures associated to $M(\psi)/F$ and $M(\psi')/F'$.

In order to state the second main theorem of this work we need to introduce some more notation. Let π be an automorphic representation of a definite unitary group $G = U(\theta)/F$ with θ a positive definite Hermitian form of dimension two over K. We write π' for the base change of π to $G' = \mathrm{Res}_{F'/F}\, U(\theta)/F$. We write $\mathfrak{c}$ for the conductor of π and $\mathfrak{c}'$ for the ideal $\mathfrak{c}$ seen as an ideal of K'.

In this case, opposite to the case of $n = 1$, a new problem appears. Namely one has to control the base change of the automorphic representation, in principle a very hard problem. We will now make a Hypothesis (or conjecture) and we will provide also a family of hermitian forms satisfying it. Moreover later we will also state another Theorem (Theorem 1.4 on "Average Torsion Congruences") where we *do not* assume the "Hypothesis". We believe that it is very interesting to compare the two theorems (see also Remark 1.5).

We consider the canonical map $\Delta\colon G(\mathbb{A}_F) \to G'(\mathbb{A}_{F'})$ induced by the embedding $F \hookrightarrow F'$. Below we write $\mathbb{A}_{F,f}$ for the finite adeles of F and similarly $\mathbb{A}_{F',f}$ for those of F'.

Hypothesis. Assume that we can associate to π (resp. π') a $\mathbb{Q}_p$-valued modular form f_π (resp. $f_{\pi'}$) of G (resp. G'), which is an eigenform for all Hecke operators away from $\mathfrak{c}$ (resp. $\mathfrak{c}'$) such that the following conditions are satisfied:

(i) f_π (resp. $f_{\pi'}$) is $\mathbb{Z}_p$-valued on $G(\mathbb{A}_{F,f}^{(p\mathfrak{c})}) := \{x \in G(\mathbb{A}_{F,f}) \mid x_v = 1, \forall v | p\mathfrak{c}\}$ (resp. $G'(\mathbb{A}_{F',f}^{(p\mathfrak{c})}) := \{x \in G'(\mathbb{A}_{F',f}) \mid x_v = 1, \forall v | p\mathfrak{c}\}$).

(ii) We write $\Delta^*(f_{\pi'})$ for the pull pack of $f_{\pi'}$ with respect to Δ, that is $\Delta^*(f_{\pi'})(x) := f_{\pi'}(\Delta(x)), \forall x \in G(\mathbb{A}_F)$. Then for all $x \in G(\mathbb{A}_{F,f}^{(p\mathfrak{c})})$ we have $\Delta^*(f_{\pi'})(x) \equiv f_\pi(x) \mod p$.

(iii) For all $x \in G'(\mathbb{A}_{F',f})$ and $\gamma \in \mathrm{Gal}(F'/F)$ we have $f_{\pi'}(x^\gamma) = f_{\pi'}(x)$, where the action of the Galois group is the induced action on $\mathbb{A}_{F',f} = \mathbb{A}_{F,f} \otimes_F F'$ obtained by the action on F'.

In Section 7 we provide a family of examples where the above **Hypothesis** does hold. Moreover at the end of the next section, after introducing some notation, we explain what does it mean that the automorphic form is $\mathbb{Q}_p$-valued.

Theorem 1.2 (Main Theorem 2). — *Let $n = 2$ and let ψ be a Hecke character over K of infinite type $-k\Sigma$ and π an automorphic representation of parallel scalar weight ℓ of a definite unitary group $U(\theta)$ with θ a positive definite Hermitian form of dimension two over K. We take $k+2\ell \geqslant n$. Write M/F for the motive $M(\pi,\psi)/F$ as introduced above. Assume that M/F has coefficients in $\mathbb{Z}_p$ in its p-adic realization. Write $M/F' = M(\pi',\psi')/F'$ for the base change of M/F to F'. Then under the* **Hypothesis** *above we have,*

$$\left(\frac{\Omega_p(Y,\Sigma)^{\Phi_{\mathfrak{p}}}}{\Omega_p(Y,\Sigma)}\right)^{2\ell} \langle f_\pi, \check{f}_\pi\rangle \operatorname{ver}(\mu^{(f_\pi)}_{(\pi,\psi)}) \equiv \langle f_\pi, \check{f}_\pi\rangle \mu^{(f_{\pi'})}_{(\pi',\psi')} \mod T,$$

where $\mu^{(f_\pi)}_{(\pi,\psi)}$ and $\mu^{(f_{\pi'})}_{(\pi',\psi')}$ are the p-adic measures associated to M/F and M/F', and the interpolation properties are given in Theorem 4.1. Here we write $\langle f_\pi, \check{f}_\pi\rangle$ for the inner product of f_π. The factor $\left(\frac{\Omega_p(Y,\Sigma)^{\Phi_{\mathfrak{p}}}}{\Omega_p(Y,\Sigma)}\right)^{2\ell}$ is an element in $\mathbb{Z}_p^\times$ and it will be defined explicitly later. We simply mention here that $\Omega_p(Y,\Sigma)$ are some canonical p-adic periods. In particular if $\langle f_\pi, \check{f}_\pi\rangle$ has trivial valuation at p and $\ell = 0$ then the torsion congruences hold true.

We remark here that the measure $\mu^{(f_\pi)}_{(\pi,\psi)}$ (resp. $\mu^{(f_{\pi'})}_{(\pi',\psi')}$) depends not only on the automorphic representation π and the character ψ (resp. π' and ψ') but also on the choice of the form f_π (resp. $f_{\pi'}$). We explain more on this choice in Remark 4.2(iii) after Theorem 4.1.

We can state another theorem where we can avoid the factor $\left(\frac{\Omega_p(Y,\Sigma)^{\Phi_{\mathfrak{p}}}}{\Omega_p(Y,\Sigma)}\right)^{2\ell}$ even when $\ell \neq 0$. For this we need an extra condition. We need to assume that there exists a $\mathbb{Q}_p$-valued eigenform f_H of the unitary group G, of conductor that divides $\mathfrak{c}$ and of parallel weight $p\ell$ such that $f_H(x) \equiv f_\pi(x) \mod p, \quad \forall x \in G(\mathbb{A}^{(p\mathfrak{c})}_{F,f})$. Our hope is that the form f_H will exist in many cases if one can put the form f_π in a Hida family, or equivalently one can find a deformation of the Galois representation associated to the motive $M(\pi)$, with weight equal to $p\ell$. We note here that the family of examples which satisfy the **Hypothesis** and we provide later do also satisfy this extra condition.

Theorem 1.3 (Main Theorem 2 (second form)). — *Let $n = 2$ and let ψ be a Hecke character over K of infinite type $-k\Sigma$ and π an automorphic representation of parallel scalar weight ℓ of a definite unitary group $U(\theta)$ with θ a positive definite Hermitian form of dimension two over K. We take $k+2\ell \geqslant n$. Write M/F and M/F' as in the previous theorem. Then under* *the* ***Hypothesis*** *above and the existence of the form f_H we have,*

$$\langle f_\pi, \check{f}_\pi \rangle \operatorname{ver}(\mu^{(f_\pi)}_{(\pi,\psi)}) \equiv \langle f_\pi, \check{f}_\pi \rangle \mu^{(f_{\pi'})}_{(\pi',\psi')} \mod T,$$

where $\mu^{(f_\pi)}_{(\pi,\psi)}$ and $\mu^{(f_{\pi'})}_{(\pi',\psi')}$ are the p-adic measures associated to M/F and M/F' as in the previous theorem. In particular if $\langle f_\pi, \check{f}_\pi \rangle$ has trivial valuation at p then the torsion congruences hold true.

We now prepare our setting for a theorem which **does not assume the Hypothesis**. We now write $\{\pi\}_{\pi \in \mathrm{Rep}(G,\mathfrak{c})}$ for the set of automorphic representations of G of conductor that is contained in $\mathfrak{c}$ and of parallel weight ℓ. We still write $M(\psi)$ for the motive associated to a Grössencharacter as before and we assume that its p-adic realization has $\mathbb{Z}_p$ coefficients. However now we consider π with no restriction on the coefficients. To each of the motives $M(\pi,\psi)/F$ we have a p-adic measure $\mu_{\pi,\psi}$ on G_F. We now consider the p-adic measure

$$\mu_F := \sum_{\pi \in \mathrm{Rep}(G,\mathfrak{c})} \mu^{(f_\pi)}_{\pi,\psi} \in \mathbb{Z}_p[[G_F]],$$

for some f_π associated to π. The fact that the measure has coefficients in $\mathbb{Z}_p$ is due to the fact that if $\pi \in \mathrm{Rep}(G,\mathfrak{c})$ then also $\pi^\sigma \in \mathrm{Rep}(G,\mathfrak{c})$ for all $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, where here π^σ denotes the representation obtained by applying σ to the coefficients of the finite part of π. This will be explained later more formally. We now introduce also measures for $G_{F'}$. We write $\{\pi'\}_{\pi' \in \mathrm{Rep}(G',\mathfrak{c}')}$ for the set of automorphic representations of $G' = \mathrm{Res}_{F'/F}\, G$ of conductor $\mathfrak{c}' = \mathfrak{c}\mathfrak{r}$. We write $M(\psi')$ for the motive associated to a Grössencharacter $\psi' := \psi \circ N_{K'/K}$. As before to each of the motives $M(\pi',\psi')/F'$ we can assign a p-adic measure $\mu_{\pi',\psi'}$ on $G_{F'}$. Moreover we define the set $\mathrm{Rep}_{BC}(G',\mathfrak{c}')$ of automorphic representations of G' that are coming from base change from G. We define the measure

$$\mu_{BC,F'} := \sum_{\pi' \in \mathrm{Rep}_{BC}(G',\mathfrak{c}')} \mu^{(f_{\pi'})}_{\pi',\psi'} \in \mathbb{Z}_p[[G_{F'}]],$$

If one assumes the torsion congruences for each of $\pi \in \mathrm{Rep}(G,\mathfrak{c})$, that is

$$\operatorname{ver}(\mu_{\pi,\psi}) \equiv \mu_{\pi',\psi'} \mod T,$$

where π' the base change of π to F', then one can conclude the torsion congruences for the measures μ_F and $\mu_{BC,F'}$. Our last theorem indicates that something in this direction is true. We first need to modify the above defined measures. We believe that this modification is related with the fact that we use automorphic periods in our interpolation formulas. But we do not wish to say more on that at this point.

Let us write $\mathcal{B} \subset G(\mathbb{A}_{f,F})$ for a fixed finite set such that $G(\mathbb{A}_F) = \coprod_{b\in\mathcal{B}} G(F)bD(\mathfrak{c})$ and $\mathcal{B} \subset G'(\mathbb{A}_{f,F'})$ so that $G(\mathbb{A}_{F'}) = \coprod_{b'\in\mathcal{B}'} G(F')b'D'(\mathfrak{c})$. Here $D(\mathfrak{c})$ is the group define later in equation 2.1. Moreover by [40, Lemma 8.12] we may take $b_v = 1$ for all $v|p\mathfrak{c}$ and similarly for $b' \in \mathcal{B}'$. The map Δ defined above induces also map $\Delta\colon \mathcal{B} \to \mathcal{B}'$, where $\Delta(b)$ is the element $b' \in \mathcal{B}'$ so that $\Delta(b) \in G(F')b'D'(\mathfrak{c})$. We now fix an orthogonal basis $\{f_j\}$ consisting of hermitian forms of G for the congruence subgroup $D(\mathfrak{c})$ and of parallel weight ℓ which are eigenforms for all relative prime to $\mathfrak{c}$ Hecke operators. Similarly we write $\{f'_j\}$ for an orthogonal basis for hermitian forms of G' for the congruence group $D'(\mathfrak{c})$, parallel weight ℓ, which are eigenforms for all relative prime to $\mathfrak{c}$ Hecke operators. For a pair of elements $a, b \in \mathcal{B}_K$ we define the twisted measures

$$\mu_{F,(a,b)} := \sum_{f_j} \frac{\check{f}_j(a) f_j(b)}{\langle f_j, \check{f}_j\rangle} \mu^{(f_j)}_{(\pi_j,\psi)} \in \mathbb{Z}_p[[G_F]],$$

where f_j is associated to π_j for some $\pi_j \in \mathrm{Rep}(G, \mathfrak{c})$. Note here that we may have multiplicities. For $G_{F'}$ we have

$$\mu_{F',(a,b)} := \sum_{f'_j} \frac{\Delta^*(\check{f}'_j)(a)\Delta^*(f'_j)(b)}{\langle f'_j, \check{f'}_j\rangle} \mu^{(f'_j)}_{(\pi'_j,\psi')} \in \mathbb{Z}_p[[G_{F'}]],$$

The fact that this quantities lie in the corresponding Iwasawa algebras will be proved later. We can now state our third theorem. We remark that this theorem can be stated for any n and we remind the reader that this theorem *does not* assume the "Hypothesis".

Theorem 1.4 (Main Theorem 3 (Average Torsion Congruences)). — *For all $a, b \in \mathcal{B}_K$ we have*

(1) *Let ε be a $\mathbb{Z}_p$ valued locally constant function on $G_{F'}$ with $\varepsilon^\gamma = \varepsilon$ for all $\gamma \in \Gamma$. Then we have the congruences,*

$$\Big(\frac{\Omega_p(Y,\Sigma)^{\Phi_\mathfrak{p}}}{\Omega_p(Y,\Sigma)}\Big)^{2\ell} \int_{G_F} \varepsilon \circ \mathrm{ver}\ \ d\mu_{F,(a,b)} \equiv \int_{G_{F'}} \varepsilon\ d\mu_{F',(a,b)} \mod p.$$

(2) *If we assume that F'/F is unramified at p then there exists a constant $c(a,b) \in \mathbb{Z}_p$ such that*

$$c(a,b)\Big(\frac{\Omega_p(Y,\Sigma)^{\Phi_\mathfrak{p}}}{\Omega_p(Y,\Sigma)}\Big)^{2\ell} \mathrm{ver}\left(\mu_{(F,(a,b))}\right) \equiv c(a,b)\mu_{(F',(a,b))} \mod T,$$

that is the torsion congruences hold for all twisted normalized measures $c(a,b)\mu_{(F,(a,b))}$ and $c(a,b)\mu_{(F',(a,b))}$, $a, b \in \mathcal{B}_K$. The constant

$c(a,b)$, which depends also on the selected basis $\{f'_j\}$, is defined as the smallest power of p so that

$$c(a,b)\frac{\Delta^*(\check{f}'_j)(a)\Delta^*(f'_j)(b)}{\langle f'_j, f'_j\rangle}$$

is integral for all these f'_j which do not belong to a representation π'_j, which comes from base change from F.

We just remark here that as before the factor $\left(\frac{\Omega_p(Y,\Sigma)^{\Phi_\mathfrak{p}}}{\Omega_p(Y,\Sigma)}\right)^{2\ell} \in \mathbb{Z}_p^\times$ could be removed if one assumes now the existence of forms $f_{H,j}$ of parallel weight $p\ell$ with similar properties as before.

Remark 1.5. — We give to this kind of congruences the name *average torsion congruences*. We would like here to remark that these congruences seem to separate the problem of proving the torsion congruences in two steps. First one proves congruences between Siegel-type Eisenstein series (as we will see they are enough to prove the average torsion congruences) and then study the behaviour of the projection of the Siegel-type Eisenstein series to the various eigenspaces associated to the selected automorphic forms by means of the doubling method. The second step needs the understanding of the behaviour of the automorphic periods under base change, which seems to be a quite challenging problem in the theory. Another feature that makes these congruences interesting to us is that they can be proved in more general settings, like indefinite unitary groups or symplectic groups, in which cases the problem of periods could turn out to be even harder to handle.

Before we discuss the general strategy for proving the above theorem we would like to remark that the condition, that p is unramified at F is imposed because up to date the so-called q-expansion principle (in its p-integral form) is not known for the group $U(n,n)/F$ when p is ramified in F.

Strategy of the proof. This work and its continuation [2] are based in the following key idea (see also the works of Kato [33], Ritter and Weiss [37] and the authors [4]) : Special values of L functions of unitary representations can be realized with the help of the doubling method either (i) as values of hermitian Siegel-type Eisenstein series on CM points of Hermitian domains or (ii) as constant terms of hermitian Klingen-type Eisenstein series for some proper Fourier-Jacobi expansion. We explain briefly the two approaches. Approach A below is used in this work and approach B is applied in [2].

Approach A: Values of Eisenstein series on CM points. In this approach we consider Siegel-type Eisenstein series of the group $U(n,n)$ with the property that their values at particular CM points are equal to the special L-values that we want to study. The CM points are obtained from the doubling method as indicated by the embedding $U(n,0) \times U(0,n) \hookrightarrow U(n,n)$. Then we make use of the fact that the CM pairs (K,Σ) and (K',Σ') that we consider are closely related (*i.e.* the second is induced from the first) which allows us to relate the various CM points over K and K'. Then we use the diagonal embedding, induced from the embedding $K \hookrightarrow K'$, between the symmetric space of $U(n,n)_{/F}$ and that of $\mathrm{Res}_{F'/F} U(n,n)_{/F'}$ to relate the Eisenstein series over the different fields and hence also their values over the CM points. But the last is nothing else than the special values that we want to study. This is also the idea that was used in [4].

Approach B: Constant term of Fourier-Jacobi expansions. In this approach we obtain Klingen-type Eisenstein series of the group $U(n+1,1)$ with the property that the constant term of their Fourier-Jacobi expansion is related with the special values that we want to study. Then again we use the embedding $K \hookrightarrow K'$ to relate these Klingen-type Eisenstein series over the different fields and hence also to obtain a relation between their constant terms. The main difficulty here is that the Klingen-type Eisenstein series have a rather complicated Fourier-Jacobi expansion, which makes hard the direct study of the arithmetic properties of these Eisenstein series. However the Klingen-type Eisenstein series are obtained with the help of the pull-back method from Siegel-type Eisenstein series of the group $U(n+1,n+1)$ using the embedding $U(n+1,1) \times U(0,n) \hookrightarrow U(n+1,n+1)$. The Siegel-type Eisenstein series have a much better understood Fourier expansion, which turns out it suffices to study also the Klingen-type Eisenstein series.

Organization of the article. This article is organized as follows. The next section serves as an introduction to the theory of hermitian forms, that is automorphic forms associated to unitary groups both from the classical complex analytic point of view as well as the arithmetic. Needless to say that nothing in that section is new. In Section 3 we introduce the Eisenstein measure studied by Harris, Li and Skinner in [24, 25] plus some important input from the work of Ming-Lun Hsieh [30, 29]. Also in this section, up to some small modifications, there is not much new material. In the next section we construct the measures μ_F and $\mu_{F'}$ that appear in the "torsion-congruences". These measures are obtained by evaluating the Eisenstein measure of Harris, Li and Skinner at particular CM points of $U(n,n)$. This construction is implicit in the papers [24, 25] and it will appear in full details in the forthcoming work of Eischen, Harris, Li and Skinner [14]. For the needs of our work we provide here some parts of this construction restricting ourselves only in the cases of interest. The main part of this work is in Section 5 where we prove congruences between Siegel-type Eisenstein series. In Section 6 we discuss CM points. In Section 7 we use the congruences between the Eisenstein series to establish the "torsion congruences" for the various motives that we make explicit in the introduction. In Section 8 we consider the "average torsion congruences". Finally there is an appendix where we simply reformulate a result of Ritter and Weiss in [37].

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2. Automorphic forms of unitary groups and their p-adic counterparts

As we indicated in the introduction, in this section we simply recall the definition and fix the notation of the key objects (automorphic forms of unitary groups, Mumford Objects *etc.*) that we are going to use later. Our references are the two books of Shimura [40, 42] and the papers [13, 24, 25, 30, 29], where all the material of this section can be found. Actually we indicate separately, at each paragraph, the references that we closely followed while writing this section and hence the reader can find there more details if he/she wishes.

Let F be a field (local or global) of characteristic different from two and we consider a couple (K,ρ) of an F-algebra of rank two and an F-linear automorphism of K. That is, K is either (i) a quadratic extension of F and ρ is the non-trivial element of $\mathrm{Gal}(K/F)$ or (ii) $K = F \times F$ and $(x,y)^\rho = (y,x)$ for $(x,y) \in F \times F$. We will always (except when we indicate otherwise) write $\mathfrak{g}$ for the ring of integers of F and $\mathfrak{r}$ for the ring of integers of K in case (i) and $\mathfrak{r} = \mathfrak{g} \times \mathfrak{g}$ in case (ii).

Let now V be a K-module isomorphic to K^1_m and let $\varepsilon = \pm 1$. By an ε-hermitian form on V we mean an F-linear map $\phi\colon V \times V \to K$ such that, (i) $\phi(x,y)^\rho = \varepsilon\phi(y,x)$ and (ii) $\phi(ax,by) = a\phi(x,y)b^\rho$ for every $a,b \in K$. Assuming ϕ is non-degenerate we define the algebraic group $GU(\phi)/F$ over F

as the algebraic group representing the functor from F-algebras to groups:

$$GU(\phi)(R) := \big\{g \in \mathrm{GL}_{K\otimes_F R}(V \otimes_F R) \mid \phi(gx, gy) \\ = \nu(g)\phi(x, y),\, \nu(g) \in R^\times\big\},$$

for an F-algebra R. Similarly we make the definition for $U(\phi)/F$ by

$$U(\phi)(R) := \big\{g \in \mathrm{GL}_{K\otimes_F R}(V \otimes_F R) \mid \phi(gx, gy) = \phi(x, y)\big\}.$$

Complex analytic hermitian forms (see [40, p. 38–40 and 78] **and** [42, p. 30]**).** We now pick $F = \mathbb{R}$ and $K = \mathbb{C}$ above and as ρ the usual complex conjugation. We consider the pair (V, ϕ) with $V = \mathbb{C}^1_n$ and with respect the standard basis we write

$$\phi = \begin{pmatrix} 0 & 0 & -i1_r \\ 0 & \theta & 0 \\ i1_r & 0 & 0 \end{pmatrix},$$

where $\theta \in \mathrm{GL}_t(\mathbb{C})$ with $\theta^* = \theta > 0$. That is, $-i\phi$ is a skew-Hermitian form and ϕ has signature $(r + t, r)$ with $n = 2r + t$. For the moment we assume that $r > s0$.

We now describe the (unbounded) symmetric spaces attached to this unitary group as well as the operation of the unitary group on these symmetric spaces. We put

$$\mathfrak{Z}^\phi := \mathfrak{Z}^r_\theta := \left\{\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{C}^{r+t}_r \mid x \in \mathbb{C}^r_r, y \in \mathbb{C}^t_r, i(x^* - x) > y^*\theta^{-1}y)\right\}.$$

For $t = 0$, we have that $U(\phi)(\mathbb{R})$ is isomorphic to $U(n, n)(\mathbb{R})$. We write $\mathbb{H}_n$ for its symmetric space. We consider now an element $\alpha \in G^\phi := GU(\phi)(\mathbb{R})$ written as

$$\alpha = \begin{pmatrix} a & b & c \\ g & e & f \\ h & l & d \end{pmatrix},$$

with $a, d \in \mathbb{C}^r_r$ and $e \in \mathbb{C}^t_t$. Then we define an action of G^ϕ on $\mathfrak{Z} := \mathfrak{Z}^r_\theta$ by

$$\alpha \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x' \\ y' \end{pmatrix}$$

with $x' = (ax + by + c)(hx + ly + d)^{-1}$, $\quad y' = (gx + ey + f)(hx + ly + d)^{-1}$. Moreover we define the following factors of automorphy

$$\lambda\colon G^\phi \times \mathfrak{Z} \to \mathrm{GL}_{r+t}(\mathbb{C}), \quad \mu\colon G^\phi \times \mathfrak{Z} \to \mathrm{GL}_r(\mathbb{C})$$

by

$$\lambda(\alpha, z) = \begin{pmatrix} h\bar{x}^t + d & h\bar{y}^t - il\theta \\ i\theta^{-1}(g\bar{x}^t + f) & i\theta^{-1}g\bar{y}^t + \theta^{-1}e\theta \end{pmatrix}$$

and

$$\mu(\alpha,z)=hx+ly+d$$

for $z=\binom{x}{y}\in\mathfrak{Z}$. Finally we define $j(\alpha,z):=\det(\mu(\alpha,z))$.

We now consider a CM-type (K,Σ). We write $\mathbf{a}$ for the set of archimedean places of K determined by Σ and we define the set $\mathbf{b}:=\mathbf{a}\cup\rho\,\mathbf{a}$. We fix a hermitian space (V,ϕ) over K and consider the symmetric space $\mathbb{H}:=\mathbb{H}_\phi:=\prod_{v\in\mathbf{a}}\mathfrak{Z}^{\phi_v}$. The group $G_+:=G^\phi_{\mathbb{R}+}:=\prod_{v\in\mathbf{a}}G^\phi_v:=\prod_{v\in\mathbf{a}}G^\phi$ operates on $\mathbb{H}_\phi$ componentwise through the operation of each component G^ϕ_v on $\mathfrak{Z}^{\phi_v}$ described above. We write (m_v,n_v) for $v\in\mathbf{a}$ for the type of ϕ_v. For a function $f\colon\mathbb{H}\to\mathbb{C}$, an element $\alpha\in G_+$ and an element $k\in\mathbb{Z}^{\mathbf{b}}$ we define the functions $f\|_k\alpha\colon\mathbb{H}\to\mathbb{C}$ by

$$(f\|_k\alpha)(z)=j_\alpha(z)^{-k}f(\alpha z),$$

where

$$j_\alpha(z)^{-k}:=j(\alpha,z)^{-k}:=\prod_{v\in\mathbf{a}}\det\left(\mu(\alpha_v,z_v)\right)^{-k_v}\det(\lambda_v(\alpha_v,z_v))^{-k_{v\rho}}\right).$$

Further we define the function $f|_k\alpha\colon\mathbb{H}\to\mathbb{C}$ by $f|_k\alpha=f\|_k(\nu(\alpha)_{\mathbf{a}}^{-\frac{1}{2}}\alpha)$, where $\nu(\alpha)_{\mathbf{a}}=(\nu(\alpha)_v)_{v\in\mathbf{a}}$.

Definition 2.1. — *Let Γ be a congruence subgroup of G. Then a function $f\colon\mathbb{H}\to\mathbb{C}$ is called a hermitian modular form for the congruence subgroup Γ of weight $\{k_v\}$ if*

1. *f is holomorphic,*
2. *$f|_k\gamma=f$ for all $\gamma\in\Gamma$,*
3. *f is holomorphic at cusps.*

We may write $j_\alpha(z)^{-k}=\prod_{v\in\mathbf{a}}\det(\alpha_v)^{k_{v\rho}}\det(\mu(\alpha_v,z_v))^{-k_v-k_{v\rho}}$ ([42, p. 32]).

Unitary automorphic forms (see [40, p. 80]**).** We write G^ϕ for $U(\phi)/F$ and $G^\phi_{\mathbb{A}}:=G^\phi(\mathbb{A}_F):=\prod_{v\in\mathbf{h}}G^\phi(F_v)\prod_{v\in\mathbf{a}}G^\phi(\mathbb{R})$ for the adelic points of the unitary group G^ϕ. We define $C_{\mathbf{a}}:=\{\alpha\in G^\phi_{\mathbf{a}}\mid\alpha(\mathbf{i})=\mathbf{i}\}$. We say that $\mathbf{f}\colon G^\phi_{\mathbb{A}}\to\mathbb{C}$ is an (unitary) automorphic form of weight $k\in\mathbb{Z}^{\mathbf{b}}$ if there exist an open compact subgroup D of $G^\phi_{\mathbf{h}}$ such that for all $\alpha\in G^\phi(F)$ and $w\in DC_{\mathbf{a}}$ we have $\mathbf{f}(\alpha xw)=j(\alpha,\mathbf{i})^{-k}\mathbf{f}(x)$. The relation between classical hermitian forms and unitary automorphic forms is as follows. We pick $q\in G^\phi_{\mathbf{h}}$ and define $\Gamma_q:=G^\phi(F)\cap qDq^{-1}$. Then the function $f_q\colon\mathbb{H}\to\mathbb{C}$ defined by the rule

$$\mathbf{f}(qy)=(f_q\|_ky)(\mathbf{i}),\ \forall y\in G^\phi_{\mathbf{a}}$$

satisfies $f_q\|_k\gamma = f_q$ for all $\gamma \in \Gamma_q$. We call $\mathbf{f}$ a unitary automorphic form if the f_q's are hermitian modular forms for all $q \in G_{\mathbf{h}}$. We denote this space by $\mathcal{M}_k(D)$. As it is well-known is we fix a decomposition $G^{\phi}_{\mathbb{A}} = \coprod_{q\in\mathcal{B}} G^{\phi}(F)qD$ for a finite set $\mathcal{B} \subset G_{\mathbf{h}}$ then the map $\mathbf{f} \mapsto f_q$ establishes a bijection between $\mathcal{M}_k(D)$ and $\prod_{q\in\mathcal{B}} M_k(\Gamma_q)$.

Some special congruence subgroups (see [40, 42]**).** We now describe some congruences subgroups that play an important role in this work. We start with $G := GU(n,n)$ and introduce some special open compact subgroups $D \subset G_{\mathbf{h}}$. We consider two fractional ideals $\mathfrak{a}$ and $\mathfrak{b}$ of F such that $\mathfrak{ab} \subseteq \mathfrak{g}$, the ring of integers of F, and define using the notation of Shimura [42, p. 11]

$$D[\mathfrak{a},\mathfrak{b}] := \left\{ x = \begin{pmatrix} a_x & b_x \\ c_x & d_x \end{pmatrix} \in G_{\mathbf{h}} | a_x \prec \mathfrak{r}, b_x \prec \mathfrak{a}\mathfrak{r}, c_x \prec \mathfrak{b}\mathfrak{r}, d_x \prec \mathfrak{r} \right\},$$

where we recall $\mathfrak{r}$ is the ring of integers of K. One usually picks either $\mathfrak{a} = \mathfrak{b} = \mathfrak{g}$ or $\mathfrak{a} = \mathfrak{b}^{-1} = \mathfrak{d}^{-1}$ with $\mathfrak{d}$ the different ideal of F over $\mathbb{Q}$. As it is explained in [40, p. 73] we may pick the finite set $\mathcal{B} \subset G_{\mathbf{h}}$ consisting of elements of the form $\mathrm{diag}[\hat{r}, r]$ for $r \in \mathrm{GL}_n(\mathbb{A}_{K,\mathbf{h}})$. Actually we may even pick the elements r to be of the form $\left(\begin{smallmatrix} t & 0 \\ 0 & 1_{n-1} \end{smallmatrix}\right)$ with $t \in \mathbb{A}_{K,\mathbf{h}}$. We moreover remark the following computation,

$$\begin{pmatrix} \hat{r} & 0 \\ 0 & r \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \hat{r} & 0 \\ 0 & r \end{pmatrix}^{-1} = \begin{pmatrix} \hat{r}a\hat{r}^{-1} & \hat{r}br^{-1} \\ rc\hat{r}^{-1} & rdr^{-1} \end{pmatrix} = \begin{pmatrix} \hat{r}ar^{*} & \hat{r}br^{-1} \\ rcr^{*} & rdr^{-1} \end{pmatrix}.$$

Finally, for an integral ideal $\mathfrak{c}$ of $\mathfrak{g}$, we introduce the notations

$$\Gamma_0(\mathfrak{b},\mathfrak{c}) := G_1 \cap D[\mathfrak{b}^{-1}, \mathfrak{bc}], \quad \Gamma_1(\mathfrak{b},\mathfrak{c}) := \left\{\gamma \in \Gamma_0(\mathfrak{c}) \mid a_\gamma - 1_n \in \mathfrak{rc}\right\}$$

and we write $\Gamma_0(\mathfrak{c}) := \Gamma_0(\mathfrak{g},\mathfrak{c})$. Now we pick an n-dimensional hermitian space (V,θ) over K with θ positive definite. Let us write M for the $\mathfrak{g}$-maximal $\mathfrak{r}$-lattice in V and for an ideal $\mathfrak{c}$ of $\mathfrak{g}$ we now define a congruence subgroup $D(\mathfrak{c})^{\theta} \subset G^{\theta}_{\mathbf{h}}$. We first define $C := \{\alpha \in G^{\theta}_{\mathbf{h}} \mid M\alpha = M\}$, $\quad \tilde{M} := \{x \in V \mid \theta(x,M) \subset \mathfrak{d}^{-1}_{K/F}\}$ and then

$$D^{\theta}(\mathfrak{c}) := \left\{\gamma \in C \mid \tilde{M}_v(\gamma_v - 1) \subset \mathfrak{c}_v M_v,\ \forall v|\mathfrak{c}\right\}. \tag{2.1}$$

Following Shimura [40, p. 87] we define an element $\sigma \in \mathrm{GL}_n(K)_{\mathbf{h}}$ such that $M'\sigma = M$ where $M' = \sum_{i=1}^{n} \mathfrak{r}e_i$ for some fixed basis $\{e_i\}$ of V. Then if we write $\theta' := \sigma\theta\sigma$ then for every finite place of F we have that

$$\gamma \in D^{\theta}(\mathfrak{c})_v \Leftrightarrow {\theta'}^{-1}_v(\sigma\gamma\sigma^{-1} - 1)_v \prec \mathfrak{c}_v\mathfrak{d}_{K/F,v}.$$

Families of polarized abelian varieties over $\mathbb{C}$ (see [42, p. 22]). We consider the following data $\mathcal{P} := \{A, \lambda, \imath, \{t_i\}_{i=1}^r\}$ where

(1) A is a complex abelian variety of dimension d.
(2) $\lambda \colon A \to A^\vee$ is a polarization of A
(3) $\imath \colon K \hookrightarrow \mathrm{End}_{\mathbb{Q}}(A)$ a ring endomorphism, where K is a CM field such that $\imath K$ is stable under the Rosati involution $\alpha \mapsto \alpha'$ of $\mathrm{End}_{\mathbb{Q}}(A)$ determined by λ.
(4) The t_i's are points of A of finite order.

We fix an analytic coordinate system of A, that is we fix an isomorphism $\xi \colon \mathbb{C}^d/\Lambda \cong A(\mathbb{C})$, where Λ a lattice in $\mathbb{C}^d$. We define a ring injection $\Psi \colon K \hookrightarrow \mathbb{C}^d_d$ such that $\imath(a)\xi(u) = \xi(\Psi(a)u)$ for $a \in K$ and $u \in \mathbb{C}^d$. Then Ψ gives the structure of a K-vector space to the $\mathbb{Q}$ liner span $\mathbb{Q}\Lambda \subset \mathbb{C}^d$, and hence $\mathbb{Q}\Lambda$ is isomorphic to K^1_r for some r such that $2d = r[K\colon \mathbb{Q}]$. We can find an $\mathbb{R}$-linear isomorphism $q \colon (K_{\mathbf{a}})^1_r \to \mathbb{C}^d$ such that $q(ax) = \Psi(a)q(x)$ for $a \in K$ and $x \in K^1_r$. We define $L := q^{-1}(\Lambda)$.

As it is explained in Shimura [41, 42], if we write $E(\cdot,\cdot)$ for the Riemann form of A determined by the polarization λ, then there is an element $\mathcal{T} \in \mathrm{GL}_r(K)$ such that $\mathcal{T}^* = -\mathcal{T}$ and $E(q(x), q(y)) = \mathrm{Tr}_{K/\mathbb{Q}}(x\mathcal{T}y^*)$, $x, y \in K^1_r$. Defining $u_i := q^{-1}(t_i)$ we have constructed the PEL-data

$$\Omega := \left\{K, \Psi, L, \mathcal{T}, \{u_i\}_{i=1}^s\right\}.$$

Now we fix a CM-type $\Sigma := \{\tau_v\}_{v\in\mathbf{a}}$ of K and write m_v resp. n_v for the multiplicity of τ_v resp $\rho\tau_v$ in Ψ. Then we have that $m_v + r_v = r$ for all $v \in \mathbf{a}$ and we can decompose $\mathbb{C}^d$ into a direct sum $\oplus_{v\in\mathbf{a}} V_v$ so that each V_v is isomorphic to $\mathbb{C}^r$ and $\Psi(a)$ acts on V_v as $\mathrm{diag}[\bar{a}_v 1_{m_v}, a_v 1_{n_v}]$ for each $a \in K$. With this definitions of m_v and n_v we have that the hermitian form $i\mathcal{T}_v$ has signature (m_v, n_v) for every $v \in \mathbf{a}$.

Lattices and polarizations (see [25] and [30, p. 8]). Even though for this paper we need only to consider the case of unitary groups isomorphic to $U(n,n)$ we present the more general case $U(m,n)$ since we will need it in [2]. We fix two nonnegative integers $m \geqslant n \geqslant 0$. We consider a K vector space W of dimension $m-n$ endowed with a skew-Hermitian form θ. We fix a basis $\{w_1, \ldots, w_{m-n}\}$ of W such that $\theta(w_i, w_j) = a_i\delta_{i,j}$. Moreover we assume that $i\sigma(a_i) > 0$ for all $\sigma \in \Sigma$ and $\sigma_p(a_i)$ is a p-adic unit for all $\sigma_p \in \Sigma_p$. We let $I^X = \oplus_{i=1}^n Kx_i$ and $I^Y = \oplus_{i=1}^n Ky_i$ and we consider the skew-Hermitian space $(V, \theta_{m,n})$ defined by $V := I^Y \oplus W \oplus I^X$ and $\theta_{m,n} := \begin{pmatrix} & & -1_n \\ & \theta & \\ 1_n & & \end{pmatrix}$.

We now pick some particular lattices in the above defined hermitian spaces. We recall that we write $\mathfrak{r}$ for the ring of integers of K and $\mathfrak{g}$ for the ring of integers of F. We define $X := \mathfrak{r}x_1 \oplus \cdots \oplus \mathfrak{r}x_n$, an $\mathfrak{r}$ lattice in I^X and $Y := \mathfrak{d}_{K/F}^{-1}y_1 \oplus \cdots \oplus \mathfrak{d}_{K/F}^{-1}y_n$, an $\mathfrak{r}$-lattice in I^Y. We choose a $\mathfrak{r}$-lattice L in W that is $\mathfrak{g}$-maximal with respect to the Hermitian form θ (see [40, p. 26] for the definition of maximal lattices). Then we define the $\mathfrak{r}$-lattice M in V as $M := Y \oplus L \oplus X$. We now let $M_p := M \otimes_{\mathfrak{r}} \mathfrak{r}_p$. We consider the following sublattices of M_p,

$$M^{-1} := Y_{\Sigma_p} \oplus L_{\Sigma_p} \oplus Y_{\Sigma_p^\rho}, \quad M^0 := X_{\Sigma_p^\rho} \oplus L_{\Sigma_p^\rho} \oplus X_{\Sigma_p},$$

where for a set S of places of K and an $\mathfrak{r}$ ideal L we write $L_S := L \otimes_{\mathfrak{r}} \prod_{v \in S} \mathfrak{r}_v$. The sublattices (M^0, M^{-1}) have the following properties: (i) they are maximal isotropic submodules of M_p and they are dual to each other with respect to the alternating form $(\cdot,\cdot)_{m,n}$ defined by as $(v,v')_{m,n} := \mathrm{Tr}_{K/\mathbb{Q}}(\theta_{m,n}(v,v'))$ and (ii) we have that $\operatorname{rank} M^{-1}_{\Sigma_p} = \operatorname{rank} M^0_{\Sigma_p^\rho} = m$ and $\operatorname{rank} M^{-1}_{\Sigma_p^\rho} = \operatorname{rank} M^0_{\Sigma_p} = n$. Such a pair is usually called a polarization of M_p. As it is explained in [25, p. 9] the existence of such a polarization is equivalent to the ordinary condition that we have imposed on p.

Shimura varieties for unitary groups (see [30, pp. 10–11] **and** [29]**).** Let $G := GU(\theta_{m,n})/F$ and M be as above our fixed lattice. For a finite place $v \in \mathbf{h}$ we set $D_v := \{g \in G(F_v) \colon M_v g = M_v\}$ and $D := \prod_{v \in \mathbf{h}} D_v$. Our ordinary assumption allow us to identify for every v above p

$$G(F_p) \xrightarrow{\sim} \prod_{v \in \Sigma_p} \mathrm{GL}_{m+n}(F_v) \times F_v^\times$$

and the maximality assumption of the lattice M gives

$$G(\mathfrak{g}_p) \xrightarrow{\sim} \prod_{v \in \Sigma_p} \mathrm{GL}_{m+n}(\mathfrak{g}_v) \times \mathfrak{g}_v^\times.$$

That is, for every $v|p$ we have $D_v = \mathrm{GL}(M_{\Sigma_p}) \times \mathfrak{g}_v^\times \xrightarrow{\sim} \mathrm{GL}_{m+n}(\mathfrak{g}_v) \times \mathfrak{g}_v^\times$. We fix an integer N relative prime to p and define an open compact subgroup $K(N)$ with the property$K(N) \subseteq \{g \in D \colon M(g-1) \subseteq NM\}$. We define for $r \geqslant 0$ the groups

$$K^r(N) := \left\{ g \in K(N) \colon g_v \equiv \begin{pmatrix} 1_m & * \\ 0 & 1_n \end{pmatrix} \mod p^r, \ \forall v | p \right\}.$$

Let us write E for the reflex field of our fixed type (K,Φ). We write O_E for its ring of integers and we consider the ring $R := O_E \otimes \mathbb{Z}_{(p)}$. Let $\mathcal{S}$ denote a finite set of rational primes. We write $U \subseteq D$ for an open-compact subgroup of $G(\mathbb{A}_{\mathbf{h}})$. Let S be a connected, locally noetherian R-scheme and

$\bar{s}$ a geometric point of S. An S-quadruple $(A, \bar{\lambda}, \imath, \bar{\eta}^{(\mathcal{S})})$ of level U consists of the following data:

(1) A is an abelian scheme of dimension $(m+n)d$ over S, where $d = [K\colon \mathbb{Q}]$,
(2) $\bar{\lambda}$ is a class of polarizations $O_{(\mathcal{S}),+}\lambda$, where λ is a prime to $\mathcal{S}$ polarization of A over S,
(3) $\imath\colon \mathfrak{r} \hookrightarrow \mathrm{End}_S(A) \otimes_{\mathbb{Z}} \mathbb{Z}_{(\mathcal{S})}$ compatible with the Rosati involution induced by λ.
(4) $\bar{\eta}^{(U)} = U\eta^{(\mathcal{S})}$, with $\eta^{(\mathcal{S})}\colon M \otimes \hat{\mathbb{Z}}^{(\mathcal{S})} \xrightarrow{\sim} \mathcal{T}^{(\mathcal{S})}(A_{\bar{s}})$ an $\mathfrak{r}$-linear $\pi_1(S, \bar{s})$-invariant isomorphism and $\mathcal{T}^{(\mathcal{S})}(A_{\bar{s}}) := \varprojlim_{(N,\mathcal{S})=1} A[N](k(s))$
(5) We write $V^{(\mathcal{S})}(A_s) := \mathcal{T}(A_s) \otimes_{\mathbb{Z}} \mathbb{A}^{(\mathcal{S})}$. Then the numerical structure induces an isomorphism $\eta^{(\mathcal{S})}\colon M \otimes \mathbb{A}^{(\mathcal{S})} \xrightarrow{\sim} V^{(\mathcal{S})}(A_{\bar{s}})$. We obtain a skew-hermitian form e^{η} on $V^{(\mathcal{S})}(A_{\bar{s}})$ by
$$e^{\eta}(x, x') := \theta_{m,n}(\eta^{-1}(x), \eta^{-1}(x')).$$
Then, if we write e^{λ} for skew-hermitian on $V^{(\mathcal{S})}(A_{\bar{s}})$ induced by the polarization, we require that $e^{\lambda} = ue^{\eta}$ for $u \in \mathbb{A}_{\mathbf{h}}^{(\mathcal{S})}$.
(6) We have that
$$\det(X - \imath(b) |\, \mathrm{Lie}(A)) = \prod_{\sigma\in\Sigma} \big(X - (\sigma\circ c)(b)\big)^m \big(X - \sigma(b)\big)^n \in O_S[X],\ \forall b \in \mathfrak{r}.$$

We will consider mainly two situations for $\mathcal{S}$. Namely, the case where $\mathcal{S} = \emptyset$ and $\mathcal{S} = \{p\}$. In the first case, it follows by the theory of Shimura and Deligne that the functor $\mathcal{F}_U$ from the category of schemes over E to the category of sets defined as
$$\mathcal{F}_U(S) = \big\{\underline{A} = \big(A, \bar{\lambda}, \imath, \bar{\eta}\big)/S\big\} \,/ \cong$$
is representable by a quasi-projective scheme $Sh_G(U)$ defined over E. In the other case, it is know by the theory of Kottwitz that if we pick $U = K(N)$ neat and such that $U_p = D_p$, then the functor $\mathcal{F}_U^{(p)}$ from the category of schemes over R to sets
$$\mathcal{F}_U^{(p)}(S) = \Big\{\underline{A} = (A, \bar{\lambda}, \imath, \bar{\eta}^{(p)})/S\Big\} \,/ \cong$$
is represented by a quasi-projective scheme $Sh_G^{(p)}(K(N))/R$.

Algebraic hermitian modular forms (see [13, p. 193]**.** Let (V, ϕ) be a hermitian form with ϕ_σ of signature (m_σ, n_σ) for $\sigma \in \Sigma$. We fix an $\mathfrak{r}$-algebra R_0 and we consider the algebraic representation (ψ, Ψ) of

$\prod_{\sigma\in\Sigma} \mathrm{GL}_{m_\sigma} \times \mathrm{GL}_{n_\sigma}$ defined over R_0, that is we have for every R_0-algebra R a homomorphism

$$\psi_R\colon \prod_{\sigma\in\Sigma} \mathrm{GL}_{m_\sigma}(R) \times \mathrm{GL}_{n_\sigma}(R) \to \mathrm{GL}(\Psi_R),\ \ \Psi_R := \Psi \otimes_{R_0} R$$

that commutes with extensions of scalars $R \to R'$ of R_0-algebras. In this work we will be interested in scalar hermitian modular forms, which means that the ω above will always be taken of the form $\psi(x) = \det(x)^k$ for some $k = \{k_v\}_{v\in\mathbf{b}}$, where for $(a,b) = (a_\sigma, b_\sigma)_{\sigma\in\Sigma} \in \prod_{\sigma\in\Sigma} \mathrm{GL}_{m_\sigma}(R) \times \mathrm{GL}_{n_\sigma}(R)$ we write $\det((a,b))^k := \prod_{\sigma\in\Sigma}(\det(a_\sigma))^{k_\sigma} \prod_{\sigma\in\Sigma}(\det(b_\sigma)^{k_{\sigma\rho}}$, where we have identified the set $\mathbf{a}$ with the CM type Σ.

For an R_0-algebra R and each data $\underline{A}/R = \{A, \lambda, \imath, \alpha\}/R$ defined over R, we write $\underline{\omega}_{A/R} := H^0(A, \Omega^1_{A/R})$ for the invariant one forms of A/R. Then we define modules

$$\mathcal{E}^+_{\underline{A}/R} = \prod_{\sigma\in\Sigma} \mathrm{Isom}_R(R^{m_\sigma}, e_\sigma \underline{\omega}_{A/R}),\ \mathcal{E}^-_{\underline{A}/R} = \prod_{\sigma\in\Sigma} \mathrm{Isom}_R(R^{n_\sigma}, e_{\rho\sigma} \underline{\omega}_{A/R}),$$

and $\mathcal{E}_{\underline{A}/R} = \mathcal{E}^+_{\underline{A}/R} \oplus \mathcal{E}^-_{\underline{A}/R}$. Here for $\sigma \in \Sigma \coprod \Sigma^\rho$ we write $e_\sigma \in K \otimes_{\mathbb{Q}} \bar{\mathbb{Q}}$ for the corresponding orthogonal idempotent related to the decomposition $K \otimes_{\mathbb{Q}} \bar{\mathbb{Q}} = \otimes_{\sigma\in\Sigma \coprod \Sigma^\rho} \bar{\mathbb{Q}}$. We note that to give an element $\omega \in \mathcal{E}^\pm_{\underline{A}/R}$ is equivalent to fixing a basis for $\underline{\omega}^+_{A/R} := \prod_{\sigma\in\Sigma} e_\sigma \underline{\omega}_{A/R}$ and $\underline{\omega}^-_{A/R} := \prod_{\sigma\in\Sigma} e_{\rho\sigma} \underline{\omega}_{A/R}$. In particular we have that the group $\prod_{\sigma\in\Sigma} \mathrm{GL}_{m_\sigma}(R)$ (resp $\prod_{\sigma\in\Sigma} \mathrm{GL}_{n_\sigma}(R)$) acts on $\mathcal{E}^+_{\underline{A}/R}$ (resp. $\mathcal{E}^-_{\underline{A}/R}$) by

$$\alpha \cdot \omega(v) := \omega(\alpha^t v), \quad \alpha \in \prod_{\sigma\in\Sigma} \mathrm{GL}_{m_\sigma}(R), \omega \in \mathcal{E}^+_{\underline{A}/R}, v \in \prod_{\sigma\in\Sigma} R^{m_\sigma}$$

and hence the group $\prod_{\sigma\in\Sigma} \mathrm{GL}_{m_\sigma}(R) \times \mathrm{GL}_{n_\sigma}(R)$ on $\mathcal{E}_{\underline{A}/R}$.

Definition 2.2. — *A hermitian modular form of weight ρ and level α, defined over R_0 is a function f on the set of pairs $(\underline{A}, \omega)/R$ with values in Ψ/R such that the following hold:*

(1) *The element $f(\underline{A}, \omega)$ depends only on the R-isomorphism class of $(\underline{A}, \omega)$.*
(2) *The function f is compatible with base change $R \to R'$ of R_0-algebras, that is*

$$f(\underline{A} \times_R R', \omega \otimes_R R') = f(\underline{A}, \omega) \otimes_R 1 \in \Psi_{R'}$$

(3) *For each $(\underline{A}, \omega)$ over R and $\alpha \in \prod_{\sigma\in\Sigma} \mathrm{GL}_{m_\sigma}(R) \times \mathrm{GL}_{n_\sigma}(R)$ we have*

$$f(\underline{A}, \alpha\omega) = \psi(\alpha^t)^{-1} f(\underline{A}, \omega).$$

p-adic hermitian modular forms (see [24, 25, 30, 29]**).** For a natural number N and a prime number p with $(p,N)=1$ we consider the functor $\mathcal{F}^{(p)}_{K^n(N)}$ from R schemes to sets given by

$$\mathcal{F}^{(p)}_{K^n(N)}(S)=\left\{(\underline{A},j_n)=(A,\bar{\lambda},\imath,\bar{\eta}^{(p)},j_n)/S\right\}/\cong,$$

where j_n an $\mathfrak{r}$-linear embedding

$$j_n\colon M^0\otimes\mu_{p^n}\hookrightarrow A[p^n].$$

This functor is representable by a scheme $Ig_G(K^n(N))/R$ (this is what in [25, p. 26] is denoted by $Ig_{2,n}$). We now consider the strict ideal class group of F, that is $Cl^+_F(K)=F_+\setminus\mathbb{A}_{F,f}/\nu(K)$, and pick representatives of it in $\mathbb{A}^{(p)}_{F,f}$. From each such representative c we consider the functor $\mathcal{F}^{(p)}_{K^n(N),c}$ from R-schemes to sets

$$\mathcal{F}^{(p)}_{K^n(N),c}(S)=\left\{(\underline{A},j_n)=(A,\lambda,\imath,\bar{\eta}^{(p)},j_n)/S\colon \lambda \text{ is a } c\text{-polarization}\right\}/\cong.$$

This functor is representable by a scheme $Ig_{G_1}(K^n(N),c)/R$ and we have that

$$\coprod_c Ig_{G_1}(K^n(N),c)\cong Ig_G(K^n(N)).$$

As it is explained in [30, paragraph 2.5] the Igusa schemes $Ig_{G_1}(K^n(N),c)$ are associated to the unitary group $U(n,n)$ and hence the above decomposition gives us the bridge between the groups $GU(n,n)$ and $U(n,n)$, and hence the notions developed here (algebraic modular forms *etc.*) for $GU(n,n)$ can be extended also to $U(n,n)$.

We now take the ring R above to be a p-adic ring, that is $R=\varprojlim_k R/p^kR$. We write $R_k:=R/p^kR$. Now we fix a toroidal compactification $\overline{S_G(K(N))}/R$ of $S_G(K(N))/R$ and write $T_{0,k}:=\overline{S_G(K(N))}[1/E]/R_k$, for the ordinary locus modulo p^k. Here E is a lift of the Hasse invariant from R_1 to R_k (see [25, p. 30]). For a positive integer ℓ we set $T_{\ell,k}:=Ig_G(K^{\ell}(N))/R_k$. There exist finite étale maps $\pi_{\ell',\ell,k}\colon T_{\ell',k}\to T_{\ell,k}$ and we define $T_{\infty,k}:=\varprojlim_\ell T_{\ell,k}=Ig_G(K^\infty(N))/R_k$. Then $T_{\infty,k}$ is Galois over $T_{0,k}$ with Galois group isomorphic to $\mathrm{Aut}_{\mathfrak{r}_p}(M^0)$. For $\ell,k\in\mathbb{N}$ we define the spaces

$$V_{\ell,k}:=H^0(T_{\ell,k},\mathcal{O}_{T_{\ell,k}}),$$

and then $V_{\infty,k}:=\varinjlim_\ell V_{\ell,k}$ and $V:=\varprojlim_k V_{\infty,k}$. We call $V_p(G,K(N)):=V^{\mathcal{N}}$, the space of p-adic modular forms of level $K(N)$. Here $\mathcal{N}\leqslant \mathrm{GL}_{m+n}(\mathfrak{r}_p)$ is the upper-triangular unipotent radical of $\mathrm{GL}_{m+n}(\mathfrak{r}_p)$.

Algebraic and p-adic hermitian modular forms (see [24, 25, 30, 29]**).** Now we assume that we are given a p^∞ arithmetic structure of an abelian variety A with CM by $\mathfrak{r}$ of type (m,n) defined over a p-adic ring R. That is we have compatible $\mathfrak{r}$-linear embeddings

$$j_n \colon M^0 \otimes \mu_{p^n} \hookrightarrow A[p^n],$$

for all $n \geqslant 0$. That is, we assume an embedding

$$j_\infty \colon M^0 \otimes \mu_{p^\infty} \hookrightarrow A[p^\infty].$$

In turn we get an isomorphism

$$j \colon M^0 \otimes \hat{\mathbb{G}}_m \xrightarrow{\sim} \hat{A}.$$

Identifying $\mathrm{Lie}(A) = \mathrm{Lie}(\hat{A})$ we obtain an isomorphism

$$j \colon M^0 \otimes_R R \xrightarrow{\sim} \mathrm{Lie}(A),$$

which induces also the isomorphisms

$$j_+ \colon M^0_{\Sigma_p} \otimes_R R \xrightarrow{\sim} e_{\Sigma_p} \mathrm{Lie}(A), \quad j_- \colon M^0_{\Sigma_p^c} \otimes_R R \xrightarrow{\sim} e_{\Sigma_p^c} \mathrm{Lie}(A).$$

As $\underline{\omega}_{A/R} = \mathrm{Hom}(\mathrm{Lie}(A), R)$ we obtain isomorphisms

$$\omega(j)_+ \colon M^0_{\Sigma_p} \otimes_R R \xrightarrow{\sim} e_{\Sigma_p}\underline{\omega}_{A/R}, \quad \omega(j)_- \colon M^0_{\Sigma_p} \otimes_R R \xrightarrow{\sim} e_{\Sigma_p^c}\underline{\omega}_{A/R}$$

and then

$$\omega(j) := \omega(j)_+ \oplus \omega(j)_- \colon M^0 \otimes_R R \xrightarrow{\sim} \underline{\omega}_{A/R}.$$

In particular we obtained an element $\omega(j) \in \mathcal{E}_{\underline{A}/R}$. This construction allows us to consider every algebraic hermitian modular form also as a p-adic hermitian modular form. Indeed if f is a hermitian form we can consider it as a p-adic modular form by defining $f(\underline{A}, j) := f(\underline{A}, \omega(j))$.

Mumford Objects(1) **and q-expansions (see** [13, pp. 207–211]**).** The familiar q-expansion with respect some given cusp of an elliptic modular forms has an algebraic interpretation as the evaluation of the modular forms on the so-called Tate curve that corresponds to the selected cusp. Our next goal is to introduce the analogues of the Tate curve for the unitary groups $GU(n,n)$. We start by considering a Hermitian space (V,ϕ) over the CM field K and we assume that $\phi = \left(\begin{smallmatrix} 0 & -1_n \\ 1_n & 0 \end{smallmatrix}\right)$. That is, $G^\phi = GU(n,n)$. We now fix maximal isotropic spaces W and W' with $W \cong W' \cong K^n$ of ϕ and we have a decomposition $V = W \oplus W'$.

We consider the standard $\mathfrak{g}$-maximal lattice of ϕ in V defined as $\Lambda := \sum_{i=1}^n \mathfrak{r} e_i \oplus \sum_{i=1}^n \mathfrak{d}_{K/F}^{-1} f_i$ for the standard basis of (V,ϕ) *i.e.* $\phi(e_i, e_j) =$

(1) The author has been informed by Ellen Eischen that this terminology is not standard and the term has been used for first time in this context in her Ph.D thesis.

$\phi(f_i, f_j) = 0$ and $\phi(e_i, f_j) = -\delta_{i,j}$ and $\mathfrak{d}_{K/F}$ the relative different of K over F. We now define the lattices

$$L := W \cap \Lambda \text{ and } L' := W' \cap \Lambda.$$

Note the choice of the pair (L, L') is equivalent to the choice of a polarization as explained above. We write P for the stabilizer of W' in G^ϕ and N_P for its unipotent radical. Then N_P consists of matrices of the form $\left(\begin{smallmatrix} 1_n & B \\ 0 & 1_n \end{smallmatrix}\right)$, where $B \in S_n$. For a congruence subgroup Γ of G^ϕ we define $H := \Gamma \cap N_P$. Then $H = \left(\begin{smallmatrix} 1_n & M \\ 0 & 1_n \end{smallmatrix}\right)$, where M is a lattice in S_n. Writing $M^\vee$ for the dual lattice of M, *i.e.* $M^\vee := \{x \in S_n \mid \mathrm{Tr}_{F/\mathbb{Q}}(\mathrm{tr}(xM)) \subset \mathbb{Z}\}$, we define $H^\vee := \left(\begin{smallmatrix} 1_n & M^\vee \\ 0 & 1_n \end{smallmatrix}\right)$. For a ring R we define the ring of formal power series

$$R((q, H^\vee_{\geqslant 0})) := \left\{ \sum_{h \in H^\vee} a_h q^h | a_h \in R,\ a_h = 0,\ \text{if } h \ll 0 \right\}.$$

Over the ring $R((q, H^\vee_{\geqslant 0}))$ we now define a $\mathbb{Z}$-liner morphism $q\colon L \to (L') \otimes \mathbb{G}_m$ as follows:

$$L \to \mathrm{Hom}_{\mathbb{Z}}(H, L') \cong H^\vee \otimes L' \to L' \otimes \mathbb{G}_m,$$

where the first map is given by $\ell \mapsto (h \mapsto h(\ell))$ and the last one is given by $h' \mapsto q^{h'} \in \mathbb{G}_m(R((q, H^\vee_{\geqslant 0})))$. The Mumford object corresponding to the cusp (L, L') is given by the algebraification of the rigid analytic quotient

$$\mathrm{Mum}_{(L,L')}(q) := (L' \otimes \mathbb{G}_m)/q(L).$$

The PEL structure of $\mathrm{Mum}_{L,L'}(q)$ is given as follows. We have a canonical endomorphism $\imath_{\mathrm{can}}\colon \mathfrak{r} \hookrightarrow \mathrm{End}_{R((q,H^\vee_{\geqslant 0}))}(\mathrm{Mum}_{L,L'}(q))$ given by $\alpha \mapsto (\ell \mapsto \alpha \cdot \ell)$ for $\alpha \in \mathfrak{r}$ and $\ell \in L$. For the canonical polarization λ_{can} of $\mathrm{Mum}_{(L,L')}(q)$ we consider the dual abelian variety

$$\mathrm{Mum}_{(L,L')}(q)^\vee = \mathrm{Mum}_{(L'^\vee, L^\vee)}(q) = (L^\vee \otimes \mathbb{G}_m)/q(L'^\vee).$$

Then there exist an isogeny $\lambda_{\mathrm{can}}\colon \mathrm{Mum}_{(L,L')}(q) \to \mathrm{Mum}_{(L,L')}(q)^\vee$, induced by the isomorphism $L^\vee \otimes_{\mathfrak{r}} K \cong L' \otimes_{\mathfrak{r}} K$. The level structure is induced by the embedding $\alpha_N\colon L' \otimes \mu_N \hookrightarrow L' \otimes \mathbb{G}_m$. Finally we have a canonical differential ω_{can}. This is defined by dualizing the isomorphism

$$\mathrm{Lie}(\mathrm{Mum}_{(L,L')})(q) \cong \mathrm{Lie}(L' \otimes \mathbb{G}_m = L' \otimes R((q, H^\vee_{\geqslant 0})).$$

That is we obtain an isomorphism

$$\omega_{\mathrm{can}}\colon R((q, H^\vee_{\geqslant 0}))^n \cong L'^\vee \otimes R((q, H^\vee_{\geqslant 0})) \cong \omega_{\mathrm{Mum}_{(L,L')}(q)/R((q,H^\vee_{\geqslant 0}))}.$$

Cusps. Now we study the (0-genus) cusps of the group $U(n,n)$ with respect particular congruences subgroups. We will see that to each of these we can associate an arithmetic data $(\mathrm{Mum}_{L,L'}(q), \lambda_{\mathrm{can}}, \imath_{\mathrm{can}}, \omega_{\mathrm{can}})$. As above we write P for the standard parabolic of $G := GU(n,n)$ and N_P for its unipotent radical. Then a Levi part of P can be identified with $\mathrm{GL}_n(K)$ by the embedding $d \mapsto \mathrm{diag}[\hat{d}, d]$ for $d \in \mathrm{GL}_n(K)$. Then the set of cusps of G with respect the open locally compact subgroup K_0 is given by

$$C_0(K) := \mathrm{GL}_n(K) \times N_P(F) \setminus G(\mathbb{A}_{F,\mathbf{h}})/K_0.$$

It is well-known, see [40, lemma 9.8] that we can choose a decomposition $G(\mathbb{A}_{F,\mathbf{h}}) = \coprod_{j=1}^{m} G(F) g_j K_0$, and if we pick $g \in C_0(K)$ and write it as $g = \gamma g_i k$ with respect to the above decomposition then the Mumford object associated to the cusp g is given by $\mathrm{Mum}_{L_g, L'_g}(q)$ where $L_g := L g_i \cap V$ and $L'_g := L g_i \cap V$.

The complex analytic point of view (see [13, p. 209]**).** Now we would like to describe the complex points of the Mumford object $\mathrm{Mum}_{L,L'}(q)$. Recall that the associated to $GU(n,n)$ symmetric space is $\mathbb{H}_F := \mathbb{H}_{(n,n)}^{[F:\mathbb{Q}]}$ where

$$\mathbb{H}_{(n,n)} := \left\{ z \in M_n(\mathbb{C}) \mid i(z^* - z) > 0 \right\}.$$

We note that if we write S for the set of hermitian matrices over K then $\mathbb{H}_{(n,n)} = S + iS_+$ and hence also $\mathbb{H}_F = S_{\mathbf{a}} + \mathbf{i} S_{\mathbf{a}+}$. Given a $\tau \in \mathbb{H}_F$ we consider the lattice $L_\tau \in \mathbb{C}^{2n}$ generated by $L' \otimes_{\mathfrak{r}} 1 \in W' \otimes_K \mathbb{C}$ and $\tau L \otimes_{\mathfrak{r}} 1 \in W' \otimes_K \mathbb{C}$. Then using the exponential map $\mathbf{exp}$ we obtain $\mathbf{exp}(L_\tau) \subset W' \otimes_K \mathbb{C}^\times$. Then we have that if fix the indeterminate parameter q as $q = exp_{\mathbf{a}}(2\pi i \, \mathrm{tr}(\tau))$ we get

$$\mathrm{Mum}_{L,L'}(q)(\mathbb{C}) = W' \otimes_K \mathbb{C}^\times / \mathbf{exp}(L_\tau).$$

Analytic and algebraic q-expansions (see [40]**).** Let now $f \in M_k(\Gamma)$ be a hermitian modular form for a congruences group Γ of G. As it is explained in [42, p. 33] we can always find a $\mathbb{Z}$ lattice M in S such that $\left(\begin{smallmatrix} 1_n & \sigma \\ 0 & 1_n \end{smallmatrix}\right) \in \Gamma$, for all $\sigma \in M$. Then we have $f(z+\sigma) = f(z)$ for all $\sigma \in M$ and hence the hermitian modular form f has a Fourier expansion

$$f(z) = \sum_{h \in L} c(h) e_{\mathbf{a}}^{n}(hz), \tag{2.2}$$

where $L := \left\{ h \in S \mid \mathrm{Tr}_{F/\mathbb{Q}}(\mathrm{tr}(hM)) \subset \mathbb{Z} \right\}$. In particular, by Shimura [40, p. 147], for $\Gamma = \Gamma_0(\mathfrak{b}, \mathfrak{c})$ we have that $L = \mathfrak{d}^{-1}\mathfrak{b}T$, where T is the lattice defined as $\{x \in S \mid \mathrm{tr}(S(\mathfrak{r})x) \subset \mathfrak{g}\}$. Actually if we consider for an element

$g \in \mathrm{GL}_n(\mathbb{A}_{K,\mathbf{h}})$ groups of the form

$$\Gamma_g := G_1 \cap \begin{pmatrix} \hat{g} & 0 \\ 0 & g \end{pmatrix} D[\mathfrak{b}, \mathfrak{c}] \begin{pmatrix} \hat{g} & 0 \\ 0 & g \end{pmatrix}^{-1},$$

then we have that the lattice L above is now equal to $\mathfrak{d}^{-1}\mathfrak{b}gTg^*$.

The expression in 2.2 has an algebraic interpretation through the use of the Mumford objects introduced above. To this end, we consider now an hermitian modular form $f \in M_k(\Gamma_0(N)$ defined over the ring R. Then we may evaluate f at the Mumford data $(\underline{\mathrm{Mum}_{L,L'}(q)}, \omega_{\mathrm{can}})$ defined over the ring $R((q, H^\vee_{\geqslant 0}))$ to obtain

$$f\left(\underline{\mathrm{Mum}_{L,L'}(q)}, \omega_{\mathrm{can}}\right) = \sum_{h \in H^\vee} c(h) q^h. \tag{2.3}$$

When $R = \mathbb{C}$ we may pick $q := e^n_{\mathbf{a}}(z)$ and then the expression above is the same as the one in (2.2). Finally we close this section by recalling also the q-expansion for unitary automorphic forms. So we let $\phi \in \mathcal{M}_k(D)$, with $D = D[\mathfrak{b}^{-1}, \mathfrak{b}\mathfrak{c}]$. Then the following proposition is taken from [40, p. 148].

Proposition 2.3. — *For every $\sigma \in S_{\mathbb{A}}$ and $q \in \mathrm{GL}_n(\mathbb{A}_K)$ we have*

$$\phi\left(\begin{pmatrix} q & \sigma\hat{q} \\ 0 & \hat{q} \end{pmatrix}\right) = \sum_{h \in S} c(h,q) e^n_{\mathbb{A}}(\mathrm{tr}(h\sigma)),$$

with the following properties

1. *$(\det(q)^{-k})^\rho c(h,q)$ depends only on ϕ, h, $q_{\mathbf{h}}$ and $(qq^*)_{\mathbf{a}}$, where we recall that $\rho \in \mathrm{Gal}(K/F)$ denotes the complex conjugation,*
2. *$c(h,q) \neq 0$ if and only if $(q^*hq)_v \in \mathfrak{b}_v\mathfrak{d}_v^{-1}T_v$ for all finite places v,*
3. *$c(h,q)$ may be written as*

$$c(h,q) = \det(q)^{k\rho} c_0(h,q) e^n_{\mathbf{a}}(i \cdot \mathrm{tr}(q^*hq)),$$

where $c_0(h,q)$ depends only on ϕ, h and $q_{\mathbf{h}}$.

We now briefly explain the relation of this automorphic q-expansion with the complex analytic one in (2.2). We start with the remark that we may write $z \in \mathbb{H} = S_{\mathbf{a}} + \mathbf{i}S_{\mathbf{a}+}$ as $z = x + \mathbf{i}y$. We then define $q \in \mathrm{GL}_n(\mathbb{A}_K)$ by $q_h = 1_n$ and $q_{\mathbf{a}} = y^{1/2}$ so that $q_{\mathbf{a}}q^*_{\mathbf{a}} = y$. Further we pick $\sigma \in S_{\mathbb{A}}$ by $\sigma_{\mathbf{h}} = 1$ and $\sigma_{\mathbf{a}} = x$. With these choices we have that $\phi\left(\left(\begin{smallmatrix} q & \sigma\hat{q} \\ 0 & \hat{q} \end{smallmatrix}\right)\right) = \det(q)^{\rho k} f_1(z)$.

$\mathbb{Q}_p$-valued hermitian modular forms. We now return to the Hypothesis in the Introduction to address the question what does it mean a hermitian modular form of an definite unitary group to be $\mathbb{Q}_p$-valued. We fix embeddings $\imath_\infty\colon \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$ and $\imath_p\colon \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}_p}$. We also fix a finite set $\mathcal{B} \subset G(\mathbb{A}_{F,f})$ such that $G(\mathbb{A}_F) = \coprod_{q\in\mathcal{B}} G(F)qD$. We refer to [40, Lemma 9.8 (3), p. 73] for this. Then by Lemma 10.8 in [40] we have an isomorphism $\mathcal{M}_k(D) \cong \prod_q \mathcal{M}_k(\Gamma^q)$.

In the case of definite unitary groups an automorphic form is uniquely determined by its values on this finite set $\mathcal{B}$. Indeed in the Lemma 10.8 of (*loc. cit.*) the symmetric spaces in this case are just points and hence $f_q(\mathbf{i}) = \mathbf{f}(q)$ in the notation of Shimura. Further, it is explained for example in Shimura [42, p. 216, equation(26.41c)] that the space of algebraic forms on $\mathcal{M}_K(\Gamma^q)$ for definite unitary groups is up to a constant (a CM period which depends only on the weight) just $\overline{\mathbb{Q}}$-valued functions. This implies that the algebraic automorphic forms $\mathcal{M}_k(D,\overline{\mathbb{Q}})$ are also by definition (up to this constant) $\overline{\mathbb{Q}}$-valued on $q \in \mathcal{B}$. But then by [42, last paragraph of p. 230] an algebraic automorphic form is (up to this period) $\overline{\mathbb{Q}}$-valued on *every* finite adele of G. Indeed as Shimura observes for any $p = \alpha q u \in G(\mathbb{A}_{F,f}) \cap G(F)qD$ where $\alpha \in G(F)$, $u \in D$ and $q \in \mathcal{B}$ we have $\mathbf{f}(p) = f_p(\mathbf{i}) = \left(f_q\|_k\alpha^{-1}\right)(\mathbf{i})$. But in the case of definite groups $f_q\|_k\alpha^{-1} = \prod_{v\in\mathbf{a}} \det(\alpha_v)^{k_{v\rho}} f_q$. In particular $\mathbf{f}(p) \in K^c(\mathbf{f}(q))$, where K^c the Galois closure of K over $\mathbb{Q}$ in the fixed $\overline{\mathbb{Q}} \subset \mathbb{C}$. Hence the field of definition of an automorphic form is uniquely determined by the values on the finite set $\mathcal{B}$, and it is a finite extension of $\mathbb{Q}$. Moreover by the discussion in paragraph 29.4 of [42]) we can take the eigenform f_π as an algebraic automorphic form. After normalizing it by this constant and using our fixed embedding $\imath_\infty$ and $\imath_p$ we can see any algebraic values as p-adic through the composition $\imath_p \circ \imath_\infty^{-1}(\overline{\mathbb{Q}}) \subset \overline{\mathbb{Q}_p}$. So the condition of an automorphic form $\mathbf{f}$ being $\mathbb{Q}_p$-valued means that after this composition the field of definition lies in $\mathbb{Q}_p$.

3. The Eisenstein measure of Harris, Li and Skinner

In this section our goal is to present the construction of an Eisenstein measure due to Harris, Li and Skinner as in [24, 25]. As we will see the key ingredients are (i) the computation of the Fourier coefficients of Siegel type Eisenstein series by Shimura (see for example [40, 42]), (ii) the definition of particular sections at places above p as done by Harris, Li and Skinner (*loc. cit.*) and (iii) the definition of sections for other "bad" primes (not including those above p) as done by Ming-Lun Hsieh [30, 29]. Finally we also mention here the recent work of Eischen [12] generalizing various aspects of the work of Harris, Li and Skinner and working the relation of their Eisenstein measure with the theory of the p-adic differential operators developed in [13].

Siegel Eisenstein series for $U(n,n)$. We start by introducing some notation. For a pair of positive integers a and b we will denote by (a,b) the element in $\mathbb{Z}^{\mathbf{b}}$ defined by taking a at all places $v \in \mathbf{a}$ and b for the rest.

In this section we follow Shimura [40] as well as [24, 25] to define Siegel-type Eisenstein series, but with a few changes on the normalization of our Eisenstein series. We let (W,ψ) be a Hermitian space that decomposes as $(W,\psi) = (V,\phi) \oplus (H_m,\eta_m)$. We write $n := \dim(V) + m$ and we define

$$(X,\omega) := (W,\psi) \oplus (V,-\phi),$$

a hermitian space of dimension $2n$ over K. We consider the decomposition of (H_m,η_m) to its maximal isotropic spaces I and I' *i.e.* $H_m = I \oplus I'$ and hence we can write $X = W \oplus V = I' \oplus V \oplus I \oplus V$. We pick a basis of W so that $\omega = \left(\begin{smallmatrix} \psi & 0 \\ 0 & -\phi \end{smallmatrix}\right)$ and hence obtain an embedding $G^{\psi} \times G^{\phi} \hookrightarrow G^{\omega}$ by $(\beta,\gamma) \mapsto \mathrm{diag}[\beta,\gamma]$. If we write the elements of X in the form (i',v,i,u) with $i' \in I', i \in I$ and $u,v \in V$ with respect to the above decomposition of X we put $U := \{(0,v,i,v) \mid v \in V, i \in I\}$, $\quad P^{\omega}_U := \{\gamma \in G^{\omega} | U\gamma = U\}$. Then U is totally ω-isotropic and P^{ω} is a parabolic subgroup of G^{ω}. From [40, p. 7] we know that if U' is another totally ω-isotropic subspace of X with $\dim(U) = \dim(U')$ then there exists $\beta \in G^{\omega}$ such that $P^{\omega}_U = \beta P^{\omega}_{U'} \beta^{-1}$.

As it is explained in Shimura [40, p. 176], we have that $(X,\omega) \cong (H_n,\eta_n)$. In the group $G := G^{\eta_n}$ we write P for the standard Siegel parabolic given by elements $x = \left(\begin{smallmatrix} a_x & b_x \\ b_x & d_x \end{smallmatrix}\right) \in G$ with $c_x = 0_n$. Now we are ready to define the classical Siegel-type Eisenstein series attached to a Hecke character χ of infinite type $-k\Sigma$ for some integer $k > 0$, with respect to the fixed CM type (K,Σ). By that we mean that χ is a character $\chi\colon \mathbb{A}_K^{\times}/K^{\times} \to \mathbb{C}^{\times}$, such that for each selected infinite place $\sigma \in \Sigma$ we have that the local component at σ of χ is of the form $\chi_{\sigma}\colon \mathbb{C}^{\times} \to \mathbb{C}^{\times}$, $\quad z \mapsto z^k$.

Further we write $\mathfrak{c} \subseteq \mathfrak{g}$ for the conductor of χ. Moreover for our applications we are going to assume that $k_{\sigma} = k \geqslant n$ for all $\sigma \in \Sigma$. We now make the following notational assumption. Given such an adelic character χ it is known thanks to Weil that one can attach a p-adic $\chi_p\colon\ \mathrm{Gal}(K(\mathfrak{c}p^{\infty})/K) \to \mathbb{C}_p^{\times}$ such that $L(\chi,s) = L(\chi_p,s)$, where $K(\mathfrak{c}p^{\infty})$ denotes the maximal abelian extension of K of conductor dividing $\mathfrak{c}p^{\infty}$.

Remark 3.1. — We make the following remarks

(1) Notational remark: In this paper we will use the same notation χ for the adelic as well as the Galois character. The setting will make clear which realization of the character is meant.
(2) We also remark that in Shimura the condition $\mathfrak{c} \neq \mathfrak{r}$ is also assumed, as for example it is used in [40, Lemma 18.8]. However, thanks to our choices of the local sections at p (see below), the condition is not needed for our purposes (see [25, Remark 3.2.2.3]).

We now note that the Siegel parabolic P in G is given by

$$P \cong \left\{ \begin{pmatrix} \hat{A} & B \\ 0 & A \end{pmatrix} : A \in \mathrm{GL}_n(K),\ B \in S_n \right\},$$

where $\hat{A} = ({}^t\bar{A})^{-1}$ and S_n the space of $n \times n$ Hermitian matrices. Let v be a place of F. We define the modulus character $\delta_{P,v} \colon P(F_v) \to \mathbb{R}_+^\times$ as

$$\delta_{P,v}(g) = |N_{K/F} \circ \det(A(g))|_v^{-1}, \quad g \in P(F_v).$$

We write $\delta_{P,\mathbb{A}_F} := \prod_v \delta_{P,v}$ for the adelic modulus character and for $s \in \mathbb{C}$ and χ our Hecke character of K we define

$$\delta_{P,\mathbb{A}}(g,\chi,s) := \chi(\det(A(g)))^{-1}\delta_{P,\mathbb{A}}(g,s),$$

where $\delta_{P,\mathbb{A}}(g,s) := |N_{K/F} \circ \det(A(g))|_{\mathbb{A}}^{-s}, \quad g \in P(\mathbb{A})$. We define I_χ as the parabolic induction $\mathrm{Ind}_{P(\mathbb{A}_F)}^{G(\mathbb{A}_F)} \delta_{P,\mathbb{A}_F}(g,\chi,s)$, *i.e.*

$$I_\chi(s) := \{\phi\colon G(\mathbb{A}_F) \to \mathbb{C}\colon \phi(pg) = \delta_{P,\mathbb{A}_F}(p,\chi,s)\phi(g), \\ p \in P(\mathbb{A}_F), g \in G(\mathbb{A}_F)\}$$

where ϕ is $D_{\mathbf{a}} \cong U(n)_{\mathbf{a}} \times U(n)_{\mathbf{a}}$ finite, for some maximal open compact subgroup $D_{\mathbf{a}}$ of $G_{\mathbf{a}}$. Given $\phi \in I_\chi(s)$ we define

$$\mathcal{E}(g,\phi,\chi,s) := \sum_{\gamma \in P(F)\diagdown G(F)} \phi(\gamma g),$$

and $D(g,\phi,\chi,s) := \left(\prod_{i=0}^{n-i} L_{\mathfrak{c}}(2s-i,\chi_1\varepsilon^i)\right) \mathcal{E}(g,\phi,\chi,s)$. Both of them are converge for $\mathrm{Re}(s) \gg 0$. Moreover they have analytic continuation to whole complex plane. We will discuss this in more details later after we have picked our section at infinity. For this we follow [30, 29] and [24, 25].

In the Hermitian symmetric space $\mathbb{H}_n^{\mathbf{a}}$ we pick a CM point $\mathbf{i}$ and write $D_{\mathbf{a}} := \{g \in G_{\mathbf{a}}(\mathbb{R}) | g\mathbf{i} = g\} \subseteq G_{\mathbf{a}}$ for the stabilizer of it. For example we may take $\mathbf{i} := (\mathbf{i}_\sigma)_{\sigma\in\Sigma}$ defined by $\mathbf{i}_\sigma = \left(\begin{smallmatrix} i1_s & 0 \\ 0 & -\frac{1}{2}\sigma(\theta) \end{smallmatrix}\right) \in \mathbb{H}_n$. We use the group $D_{\mathbf{a}}$ to identify $G_{\mathbf{a}}(\mathbb{R})/D_{\mathbf{a}}$ with the symmetric space $\mathbb{H}_n^{\mathbf{a}}$. That is for every element $z \in \mathbb{H}_n^{\mathbf{a}}$ we find $p \in P_{\mathbf{a}}(\mathbb{R})$ such that $p(\mathbf{i}) = z$. For $g \in G(\mathbb{R})$ and $z \in \mathbb{H}_n^{\mathbf{a}}$ we define the automorphy factors attached to $D_{\mathbf{a}}$ by

$$J(g,z) := \det(c_g z + d_g) \text{ and } J'(g,z) := \det(\bar{c}_g{}^t z + \bar{d}_g) = \det(g)^{-1} J(g,z).$$

For a given pair of integers (k,ℓ) we define the local section at infinity as

$$\phi_\infty(g,s) := \det(g)^\ell J(g,\mathbf{i})^{-k-2\ell} |J(g,\mathbf{i})|^{-2(s-\ell)} \in I_{\chi,\infty}(s).$$

With this choice of section we have that our Eisenstein series is holomorphic at $s = \ell$, whenever $k + 2\ell \geqslant n$. Moreover they are of weight $(k+\ell,\ell) \in \mathbb{Z}^{\mathbf{b}}$. Indeed by [40, Theorem 19.3] and [42, Theorem 17.12]), we have

(1) the Eisenstein series $\mathcal{E}(g,\phi,\chi,\ell)$ is an automorphic form of weight $(k+\ell,\ell)$ when $k+2\ell \geqslant n$ except when $F = \mathbb{Q}$, $k+2\ell = n+1$ and the restriction χ_1 of χ to $\mathbb{Q}$ is equal to ε^{n+1} where ε the non-trivial character of $K/\mathbb{Q}$,
(2) the normalized Eisenstein series $D(g,\phi,\chi,\ell)$ is an automorphic form of weight $(k+\ell,\ell)$ for $k+2\ell \geqslant n$.

Remark 3.2. — Here one should notice that Shimura in his books [40, 42] considers always *unitary* characters, that is with values in $\mathbf{T} := \{z \in \mathbb{C}^\times \mid |z| = 1\}$. The relation to our character here is by multiplying our character by $N_{K/\mathbb{Q}}^{k/2}$, where $N_{K/\mathbb{Q}}$ the norm character. This has as result a shifting of the variable s in Shimura's books by $-\frac{k}{2}$.

Fourier expansion of automorphic forms of G. : We let $\psi\colon \mathbb{A}_F/F \to \mathbb{C}^\times$ be the non-trivial additive character with the property

$$\psi_\infty(x) = \mathbf{exp}\Big(2\pi i \sum_\sigma x_\sigma\Big).$$

Then all the additive characters of $\mathbb{A}_F/F$ can be obtained as $\psi_a(x) := \psi(ax)$ with $a \in F^\times$. For β a hermitian $n \times n$ matrix we define the character $\psi_\beta\colon U_P(F) \smallsetminus U_P(\mathbb{A}_F) \to \mathbb{C}, \quad n(b) \mapsto \psi(\mathrm{tr}(\beta b))$, where we have used the fact that there is an isomorphism $n\colon S_n(\mathbb{A}_F) \cong U_P(\mathbb{A}_F)$. Here U_P is the unipotent radical of the parabolic group P which is given by $U_P(F) := \left\{ \left(\begin{smallmatrix} 1_n & X \\ 0 & 1_n \end{smallmatrix}\right) \mid X \in S_n(F) \right\}$. The β-th Fourier coefficient of $\mathcal{E}(g,\phi,\chi,s)$ is given by

$$\mathcal{E}_\beta(\phi,\chi,s)(g) = \int_{U_P(F)\smallsetminus U_P(\mathbb{A}_F)} \mathcal{E}(ug,\phi,\chi,s)\psi_{-\beta}(u)du.$$

When β is of full rank n the Fourier coefficient equals up to a normalized factor the Whittaker function $W_\beta(g,\phi,s)=\prod_v W_{\beta,v}(g_v,\phi_v,s)$ with

$$W_{\beta,v}(g_v,\phi_v,s)=\int_{U_P(F_v)}\phi_v(wn_vg,\chi_v,s)\psi_{-\beta}dn_v$$

with $w=\left(\begin{smallmatrix}0 & -1_r\\ 1_r & 0\end{smallmatrix}\right)$.

The local sections. We pick a finite set S of places of F that include:

(i) all places v in F above p,
(ii) all places of F ramified in K/F,
(iii) all places of F that are below the conductor of χ,
(iv) all places of F such that after localization of (V,ϕ) at v we have $(V_v,\phi_v)\equiv(T_v,\theta_v)\oplus(H_{r_v},\eta_{r_v})$ with $t_v:=\dim(T_v)=2$. It is known (see ([40, Prop. 10.2(1)] that this set is finite,
(iv) finally S contains all archimedean places of F.

The spherical sections We consider the places v of F that are not in S. For such a v, we pick the section $\phi_v\in I_v(\chi,s)$ to be the normalized spherical sections for the group $D_v=D[\mathfrak{g},\mathfrak{g}]_v$, a maximal normal subgroup of $U(n,n)(F_v)$. Here normalized means that $\phi_v(1,\chi_v,s)=1$. We moreover define the lattice T in $S:=S_n$ by $T:=\{x\in S\mid \mathrm{tr}(S(\mathfrak{r})x)\subseteq\mathfrak{g}\}$. Then Shimura has the local Fourier coefficients for such v's explicitly computed for β of full rank. We summarize his result in the next proposition.

Proposition 3.3 (Shimura). — *Let ϕ_v be the spherical local section above. Let $m=m(A)\in M(F_v)$. Then $W_{\beta,v}(m,\phi_v,s)=0$ unless $\bar{A}^t\beta A\in\mathfrak{d}^{-1}_{F_v/\mathbb{Q}_v}T_v$. In this case $W_{\beta,v}(m,\phi_v,s)$ is equal to*

$$|N\circ\det A|_v^{n/2-s}\chi_v(\det A)g_{\beta,m,v}(\chi_v(\varpi_v)q_v^{-2s})\prod_{j=0}^{n-1}L_v(2s-j,\chi_{|F}\varepsilon^j_{K/F})^{-1}$$

with $g_{\beta,m,v}(X)\in\mathbb{Z}[X]$ with $g_{\beta,m,v}(0)=1$. When v is unramified in K and $\det(a\delta_v\bar{A}^t\beta A)\in\mathfrak{g}_v^\times$ then $g_{\beta,m,v}(X)=1$.

The non-spherical sections Now following [24, 25, 30, 29] we deal with the non-spherical finite places, that is the finite places v of F that belong to the set S. We are going to distinguish between those above p, we call this set S_p and the rest.

Finite places not dividing p Our choice of the local section is the one described by Ming-Lun Hsieh [30, 29]. We first recall the sections that are defined by Shimura [40, p. 149]. We define a function $\tilde{\phi}_v$ on G_v by

$\tilde{\phi}_v(x) = 0$, for $x \notin P_v D_v$ and

$$\tilde{\phi}(pw) = \chi(\det(d_p))^{-1}\chi_{\mathfrak{c}}(\det(d_w))^{-1}|\det d_p \bar{d}_p|^{-s},$$

for $p \in P_v$ and $w \in D_v = D[\mathfrak{g},\mathfrak{g}]_v$. Now we recall that we are considering a form $(W,\psi) = (V,\theta) \oplus (H_m,\eta_m)$ and we have defined an element $\Sigma_{\mathbf{h}} := \mathrm{diag}[1_m, \sigma_{\mathbf{h}}, 1_m, \hat{\sigma}_{\mathbf{h}}]$ with $\sigma_v = 1$ for $v \notin S$. Then we define $\phi_v(g) := \tilde{\phi}_v(gw'\Sigma^{-1})$, where $w' := \begin{pmatrix} & & -1_m & \\ & & & -\frac{\theta}{2} \\ 1_m & & & \\ & -\frac{\theta^{-1}}{2} & & \end{pmatrix}$. We now define $u := \begin{pmatrix} 1_m & \\ & \frac{\theta}{2}\sigma^* \end{pmatrix}$ and the lattice $L_v := \mathrm{Her}_n(F_v) \cap (uM_n(2\mathfrak{c}_v)u^*)$. Then ϕ_v is the unique section such that $\mathrm{supp}(\phi_v) = P(F_v)wU_P(L_v)$, and $\phi_v(w\left(\begin{smallmatrix} 1_n & \ell \\ 0 & 1_n \end{smallmatrix}\right) = \chi_v^{-1}(\det u)|\det(u\bar{u})|^{-s}, \ell \in L_v$, where we recall $w = \left(\begin{smallmatrix} 0 & -1_n \\ 1_n & 0 \end{smallmatrix}\right)$. Then as it is explained in [30, 29] we have that

$$\begin{aligned} &W_{\beta,v}(m(A),\phi_v,s) \\ &\quad = \mathbb{I}_{L_v^\vee}(\bar{A}^t\beta A)|\det A|_v^{n/2-s}\chi_v(\det A)\,\mathrm{vol}(L_v)\chi^{-1}(\det u)|\det(u\bar{u})|_v^{-2s}. \end{aligned}$$

The sections at infinite places The Fourier expansion has been computed in [30, 29] as well as in [24, 25]. We set $\Gamma_n(s) := \pi^{\frac{n(n-1)}{2}}\prod_{j=0}^{n-1}\Gamma(s-j)$ and

$$L_{n,\infty}(2\ell+k,s) := i^{n(2\ell+k)}2^{-n(2\ell+k-n+1)}\pi^{-n(s+2\ell+k)}\Gamma_n(s+2\ell+k).$$

Then as it is explained in [30, 29] and in [25] we have for every $\sigma \in \Sigma$ and $p_\sigma \in P(\mathbb{R})$

$$\begin{aligned} &J(p_\sigma,\mathbf{i}_\sigma)^{k+\ell}W_{\beta,\sigma}(p_\sigma,\phi_{\infty,s})|_{s=0} \\ &\qquad\qquad = L_{n,\infty}(k+2\ell,0)^{-1}\det(\sigma(\beta))^{k+2\ell-n}e^{2\pi i\,\mathrm{tr}(\sigma(\beta)z_\sigma)}, \end{aligned}$$

if $\beta \geqslant 0$ and zero otherwise, where $z_\sigma := p_\sigma(\mathbf{i}_\sigma)$.

Finite places dividing p, the sections of Harris, Li and Skinner, after [24, 25] Now we turn to the sections at finite places above p as defined by Harris, Li and Skinner. (The interested reader should also see the work of Eischen [12] for a more detailed study of these sections). We are assuming that we are given the following data: A character χ of $F_v^\times \times F_v^\times$, a partition $n = n_1+n_2+\cdots+n_\ell$ with $\ell \in \mathbb{N}$ and an ℓ-tuple $(\nu_1,\ldots,\nu_\ell)$ of characters ν_j of $F_v^\times$. Moreover we assume that ordinary condition that is all the primes v above p in F split in K.

We identify $U(n,n)(F_v)$ with $\mathrm{GL}(2n,F_v)$ and the character χ used in the parabolic induction of the Siegel-Eisenstein series with the character

$$\begin{pmatrix} A & * \\ 0 & B \end{pmatrix} \mapsto \chi_1(\det(B))^{-1}\chi_2(\det(A))|\det(AB^{-1})|^s,$$

where we write $\chi = (\chi_1, \chi_2)$ for χ as a character on $F_v \times F_v$.

For a partition $n: = n_1 + \cdots + n_\ell$ of n and write P for the corresponding parabolic subgroup. For a set of characters $\nu := (\nu_1, \dots, \nu_\ell)$ of $F^\times$ we pick an integer $t \in \mathbb{N}$ that is bigger than all the conductors of the characters μ_j, $j = 1, \dots, \ell$. Writing $\mathfrak{p}$ for the maximal ideal of $\mathfrak{g}_v$ we define the group $\Gamma(\mathfrak{p}^t)$ to be the subgroup of $\mathrm{GL}(\mathfrak{g}_v)$ consisting of matrices whose off diagonal blocks, determined by the selected partition of n, are divisible by $\mathfrak{p}^t$. Then we define the Bruhat-Schwartz function

$$\Phi_\nu(X) := \begin{cases} \nu_1(\det(X_{11})) \cdots \nu_\ell(\det(X_{\ell\ell})), & X \in \Gamma(\mathfrak{p}^t) \\ 0, & \text{otherwise.} \end{cases}$$

Now we define the ℓ-tuple $(\mu_1, \dots, \mu_\ell)$ where $\mu_j := \nu_j^{-1}\chi_2^{-1}$. Using this ℓ-tuple we define Φ_μ in the same way that we have defined Φ_ν. Then we define the function

$$\tilde{\Phi}_\mu(x) = \begin{cases} \mathrm{vol}(\Gamma(\mathfrak{p}^t))^{-1}\Phi_\mu(x), & x \in \Gamma(\mathfrak{p}^t) \\ 0, & \text{otherwise,} \end{cases}$$

where $\mathrm{vol}(\Gamma(\mathfrak{p}^t))$ is defined as in [25] p. 59. We now define the Bruhat-Schwartz function on $M(n, 2n, F_v)$ by $\Phi(x, y) = \tilde{\Phi}_\mu(x)\widehat{\Phi}_{\nu^{-1}\chi_1}(y)$ and then the section $\phi_v(h; \chi, s) := f_\Phi(h, s)$. where

$$f_\Phi(h, s) := \chi_2(\det h)|\det h|^s \int_{\mathrm{GL}_n(F_v)} \Phi((0, Z)h)\chi_2\chi_1(\det Z)|\det Z|^{2s} d^\times Z,$$

where $\widehat{\Phi}_{\nu^{-1}\chi_1}$ is the Fourier transform (normalized slightly different than in [25]) of $\Phi_{\nu^{-1}\chi_1}$ defined as

$$\widehat{\Phi}_{\nu^{-1}\chi_1}(Z) := |\mathfrak{d}_{F_v}| \int_{M_n(F_v)} \Phi_{\nu^{-1}\chi_1}(X)\psi(\mathrm{tr}({}^tXZ)).$$

Remark 3.4. — This section is slightly different from the one of Harris, Li and Skinner. This choice will be justified by our computations in the proof of Lemma 4.14 in the next section. Note that if the ν_j are unramified characters then the section does not depend on them. We remark here that similar modifications occur in the works of Eischen [12] and of Ming-Lun Hsieh [30, 29].

We now compute the local Fourier coefficient at v. By definition we have that

$$W_{\beta,v}(g, \phi_v, s) = \int_{S_n} \phi_v(w_n n(S) g)\psi_{-\beta}(S) dS,$$

where $w_n = \left(\begin{smallmatrix} 0 & -1_n \\ 1_n & 0 \end{smallmatrix}\right)$ and $n(S) := \left(\begin{smallmatrix} 1_n & S \\ 0 & 1_n \end{smallmatrix}\right)$. Note that since we are in the split case we have that $S_n = M_n(F_v)$. Hence putting also the definition of our section we obtain that $W_{\beta,v}(1,\phi_v,s)$ equals

$$= \int_{M_n(F_v)} \int_{\mathrm{GL}_n(F_v)} \Phi\left((0,Z)\begin{pmatrix} 0 & -1_n \\ 1_n & X \end{pmatrix}\right) \chi_2\chi_1(\det Z) \times |\det Z|^{2s} \psi_{-\beta}(X) d^\times Z dX$$

$$= \int_{M_n(F_v)} \int_{\mathrm{GL}_n(F_v)} \Phi(Z, ZX)\chi_2\chi_1(\det Z)|\det Z|^{2s}\psi_{-\beta}(X)d^\times Z dX.$$

But $\Phi(Z,ZX) = \tilde{\Phi}_\mu(Z)\widehat{\Phi}_{\nu^{-1}\chi_1}(ZX)$ and hence the integral above reads

$$\int_{\mathrm{GL}_n(F_v)} \tilde{\Phi}_\mu(Z)\left(\int_{M_n(F_v)} \widehat{\Phi}_{\nu^{-1}\chi_1}(ZX)\psi_{-\beta}(X)dX\right)\chi_2\chi_1 \times (\det Z)|\det Z|^{2s} d^\times Z.$$

But by the Fourier inversion formula, after setting $X \mapsto Z^{-1}X$, we have that

$$\int_{M_n(F_v)} \widehat{\Phi}_{\nu^{-1}\chi_1}(ZX)\psi_{-\beta}(X)dX = |\det Z|^{-n}\Phi_{\nu^{-1}\chi_1}({}^tZ^{-1}{}^t\beta),$$

and hence we have that $W_{\beta,v}(1,\phi_v,s)$ is equal to

$$\int_{\mathrm{GL}_n(F_v)} \tilde{\Phi}_\mu(Z)|\det Z|^{-n}\Phi_{\nu^{-1}\chi_1}({}^tZ^{-1}{}^t\beta)\chi_2\chi_1(\det Z)|\det Z|^{2s}d^\times Z.$$

By the definition of $\tilde{\Phi}_\mu$ we have that the integral is over $\Gamma(\mathfrak{p}^t)$ and by the support of $\Phi_{\nu^{-1}\chi_1}$ it is non zero only if $\det(\beta) \neq 0$. In this case, after making the change of variables $Z \mapsto Z\beta$, we get that $W_{\beta,v}(1,\phi_v,s)$ is equal to

$$\begin{aligned}
&\chi_2\chi_1(\det(\beta))|\det(\beta)|^{2s-n} \\
&\qquad \times \int_{\Gamma(\mathfrak{p}^t)} \tilde{\Phi}_\mu(Z\beta)\Phi_{\nu^{-1}\chi_1}({}^tZ^{-1})\chi_2\chi_1(\det Z)|\det Z|^{2s-n}d^\times Z \\
&\qquad = \chi_2\chi_1(\det(\beta))|\det(\beta)|^{2s-n}\Phi_\mu(\beta)\int_{\Gamma(\mathfrak{p}^t)} \tilde{\Phi}_\mu(Z) \\
&\qquad\qquad \times \Phi_{\nu^{-1}\chi_1}({}^tZ^{-1})\chi_2\chi_1(\det Z)d^\times Z \\
&\qquad = \chi_2\chi_1(\det(\beta))|\det(\beta)|^{2s-n}\Phi_\mu(\beta),
\end{aligned}$$

because of the normalization of $\tilde{\Phi}_\mu(Z)$. We summarize the computations in the following lemma

Lemma 3.5. — *With the choices as above we have*

$$W_{\beta,v}(1,\phi_v,s) = \chi_2\chi_1(\det(\beta))|\det(\beta)|^{2s-n}\Phi_\mu(\beta).$$

We note also the following lemma, a proof of which can be found in [25, p. 52].

Lemma 3.6. — *Let $m(A) = \left(\begin{smallmatrix} A & 0 \\ 0 & {}^t\bar{A}^{-1} \end{smallmatrix}\right) \in U(n,n)(F)$ for $A \in \mathrm{GL}_n(K)$. Then*

$$W_{\beta,v}(m(A)g_v, \phi_v, s) = |N \circ \det(A)|^{\frac{n}{2}-s} \chi_v(\det(A)) W_{{}^t\bar{A}\beta A, v}(g_v, \phi_v, s).$$

Normalization. We now normalize the Eisenstein series $D(g,\phi,\chi,s)$. We introduce the quantity

$$c(n,F,K) := 2^{n(n-1)[F:\,\mathbb{Q}/2]} |\delta(F)|^{-n/2} |\delta(K)|^{-n(n-1)/4}$$

Then as it is explained in [40] p. 153 we have $dx = c(n,F,K)\prod_v dx_v$. Here dx is the Haar measure on $S_{\mathbb{A}}$ normalized such that $\int_{S_{\mathbb{A}}/S} dx = 1$. For $v \in \mathbf{h}$ the dx_v are measures on S_v that are normalized to give volume 1 to the maximal compact subgroup $Ł_v := S(O_F)_v$. For $v \in \mathbf{a}$ we refer to Shimura. Then we define $C^S(n,K,s) := c(n,K^+,K) L_{n,\infty}(k+2\ell,s)^{-1}$, and normalize $E(g,\phi,\chi,s) := C^S(n,K,s)^{-1} D(g,\phi,\chi,s)$.

We now pick $g_f := m(A) \in M(\mathbb{A}_f)$, $A \in \mathrm{GL}_n(\mathbb{A}_{f,K})$ and consider the hermitian form

$$E(z, m(A), \phi, \chi) := J_{(k,\ell)}(g_\infty, \mathbf{i}) E(g,\phi,\chi,s)_{|s=\ell}$$

where $g = g_f g_\infty$ with $g_\infty \mathbf{i} = z \in \mathbb{H}_n^{\mathbf{a}}$ and $J_{(k,\ell)}(a,z) = \det(g)^\ell J(g,z)^{-k-2\ell}$. As we have seen the definition of the section ϕ depends on the characters χ and ν. Hence we may sometimes write $E(z,m(A),\nu,\chi)$ instead of $E(z,m(A),\phi,\chi)$.

As in [25] we write $m(A) = m(A^{(p)}\prod_{v|p} h_v$ where we identify h_v with elements in $\mathrm{GL}_n(F_v)$ and in particular write $h_v = \mathrm{diag}[A(h_v), B(h_v)]$. Moreover we recall that $\chi_v = (\chi_{1,v}, \chi_{2,v})$. From the remarks above we have that the global Fourier coefficient of $E(z,m(A),\phi,\chi)$ at $\beta \in S$ is given by

$$\begin{aligned} E_\beta(m(A),\phi,\chi) = {} & T^0(\beta, m(A)) N_{F/\mathbb{Q}}(\det(\beta))^{(k-n)} \chi(\det(A)) |\det(A)|^n \\ & \times \prod_{v\in\Sigma_p} \chi_{1,v}^{-1}(\det(B(h_v)))\chi_{2,v}(\det(A(h_v)))|\det(\beta)|_v^{-n} \\ & \times \Phi_\mu({}^tA(h_v)\beta B(h_v)^{-1}) \\ & \times \prod_{v\in S^{(p)}} \mathbb{I}_{L_v^\vee}({}^t\bar{A}\beta A)\,\mathrm{vol}(L_v)\chi_v^{-1}(\det u), \end{aligned}$$

where $T^0(\beta, m(A)) := \prod_{v\notin S} \mathbb{I}_{\mathfrak{d}^{-1}_{F_v/\mathbb{Q}_v} T_v}({}^t\bar{A}\beta A) g_{\beta,m(A),v}(\chi_v(\varpi_v))$. We define

$$Q(\beta, A, k, \nu) := N_{F/\mathbb{Q}}(\det(\beta))^{(k-n)} \left(\prod_{v\in\Sigma_p} |\det(\beta)|_v^{-n}\right) |\det(A)|^n \\ \times \prod_{v\notin S} \mathbb{I}_{\mathfrak{d}^{-1}_{F_v/\mathbb{Q}_v} T_v}({}^t\bar{A}\beta A) \prod_{v\in S^{(p)}} \mathbb{I}_{L_v^\vee}({}^t\bar{A}\beta A)\operatorname{vol}(L_v).$$

$$P_{\beta,m(A)} := \prod_{v\notin S} g_{\beta,m(A),v}(\varpi_v) = \sum_{(\mathfrak{a},S)=1} n_{\mathfrak{a}}(\beta, m(A))\mathfrak{a}, \quad n_{\mathfrak{a}}(\beta, m(A)) \in \mathbb{Z},$$

where $n_{\mathfrak{a}}(\beta, m(A)) = 0$ for almost all $\mathfrak{a}$ and $P^{(S)}_{\beta,m(A)}(\chi) := \chi(P^{(S)}_{\beta,m(A)}) := \sum_{(\mathfrak{a},S)=1} n_{\mathfrak{a}}(\beta)\chi^{(S)}(\mathfrak{a})$. Then

$$\begin{aligned} &E_\beta(m(A), \phi, \chi) \\ &\quad = Q(\beta, A, k, \nu)\chi(\det(A))P^{(S)}_{\beta,m(A)}(\chi) \\ &\qquad \times \left(\prod_{v\in\Sigma_p} \chi_{1,v}^{-1}(\det(B(h_v)))\chi_{2,v}(\det(A(h_v)))\Phi_\mu(A(h_v)^t\beta B(h_v)^{-1})\right) \\ &\qquad \times \left(\prod_{v\in S^{(p)}} \chi_v(\det u^{-1})\right) \\ &\quad = Q(\beta, A, k, \nu) \sum_{(\mathfrak{a},S)=1} n_{\mathfrak{a}}(\beta, m(A))\chi(\det(A))\chi^{(S)}(\mathfrak{a}) \\ &\qquad \times \left(\prod_{v\in\Sigma_p} \chi_{1,v}^{-1}(\det(B(h_v)))\chi_{2,v}(\det(A(h_v)))\Phi_\mu({}^tA(h_v)\beta B(h_v)^{-1})\right) \\ &\qquad \times \left(\prod_{v\in S^{(p)}} \chi_v(\det u^{-1})\right). \end{aligned}$$

Let now $\varepsilon := \sum_j c_j\chi_j$ be a locally constant function of G_K^{ab} written in a unique way as a finite sum of finite characters. Moreover we let ψ be a character of infinite type $k\Sigma$ as above. Then we define

$$E(z, m(A), \phi, \varepsilon\psi) := \sum_j c_j E(z, m(A), \phi, \chi_j\psi)$$

and then

$$\begin{aligned}E_\beta(m(A),\phi,\varepsilon\psi) =& Q(\beta,A,k)\sum_{(\mathfrak{a},S)=1} n_{\mathfrak{a}}(\beta,m(A))\sum_j c_j\chi_j\psi(\det(A))\\ &\times\Bigg(\prod_{v\in\Sigma_p}\chi_{1,v,j}\psi_{1,v}^{-1}(\det(B(h_v)))\chi_{2,v,j}\psi_{2,v}\\ &\qquad\times(\det(A(h_v)))\Phi_\mu({}^tA(h_v)\beta B(h_v)^{-1})\Bigg)\\ &\times\Bigg(\prod_{v\in S^{(p)}}\chi_{v,j}\psi_v(\det u^{-1})\Bigg)\chi_j^{(S)}(\mathfrak{a}).\end{aligned}$$

Moreover it follows easily from the above description that $E_\beta(m(A),\phi,\varepsilon\psi)\in\mathbb{Z}[\varepsilon\psi,\nu]$. Indeed one needs to observe that the values of Φ_μ are also given by the local characters at $v|p$.

The Eisenstein measure of Harris, Li and Skinner. In the definition of the p-adic sections we have fixed an integer $\ell\in\mathbb{N}$. We consider the group

$$T(\ell):=\prod_{v|p}(\mathfrak{g}_v^\times)^\ell\times G_{\mathfrak{c}}\cong\prod_{v\in\Sigma_p}(\mathfrak{r}_v^\times)^\ell\times G_{\mathfrak{c}}.$$

We write $X_{fin}(T(\ell))$ for the set of finite characters of $T(\ell)$. This set can be parametrized by the $(\ell+1)-$tuples $(\nu_1,\dots,\nu_\ell,\chi)$ where ν_j are characters of $\prod_{v|p}\mathfrak{g}_v^\times$ and χ is a character of $G_{\mathfrak{c}}$. Recall that to the tuple $(\nu_1,\dots,\nu_\ell,\chi)$ we have attached another tuple $(\mu_1,\dots,\mu_\ell)$. Then Harris, Li and Skinner in [25, p. 67] have obtained:

Theorem 3.7. — *There is a measure μ_{Eis}^{HLS} on $T(\ell)$ with the property that, for any $(\ell+1)$-tuple $(\nu_1,\dots,\nu_\ell,\chi)$ we have*

$$\int_{T(\ell)}(\nu_1,\dots,\nu_\ell,\chi)d\mu_{Eis}^{HLS}=E(\cdot,\phi,\chi),$$

where ϕ is the section described above. Here, by abusing notation, we write both χ for Grossencharacter and its p-adic realization as a character of $G_{\mathfrak{c}}$.

For the applications that we have in mind, we are going to keep the tuple $(\nu_1,\dots,\nu_\ell)$ fixed and vary χ. Let us explain now how we are going to fix the ℓ-tuple $(\nu_1,\dots,\nu_\ell)$ for the applications that we have in mind.

Definition 3.8 (see [24], pp. 14–15). — *Let π be an irreducible cuspidal automorphic representation of $U(V)(\mathbb{A}_F)$ with $\dim_K(V)=n$. Let v be place of F that splits in K. Then π is called of type $\nu=(\nu_1,\dots,\nu_\ell)$ at v if π_v is a principal series of $U(V)(F_v)\cong\mathrm{GL}_n(F_v)$ and if it is an eigenvector* *for $P(\mathfrak{g}_v) \subset \mathrm{GL}_n(F_v)$ with eigenvalues given by the ℓ-tuple $(\nu_1, \ldots, \nu_\ell)$. Of course here P is the parabolic of GL_n corresponding to the fixed partition $n = \sum_{j=1}^{\ell} n_j$.*

In our applications the representation π_v will be P-ordinary at v (see [24, p. 14] for a definition).

Finally we close this section with a remark pointed out by the referee. For the construction of the Eisenstein measure in the situation which we consider it was really necessary to compute the q-expansion for the constructed Eisenstein series in various cusps, namely around cusps of each connected component of the Igusa tower. The expansion around only one cusp would not have been enough. The reason is that the Igusa curve in the unitary case is known after the work of Chai and Hida not to be irreducible.

4. Construction of the p-adic measure $\mu^{HLS}_{\pi,\chi}$

In this section we are going to use the Eisenstein measure of Harris, Li and Skinner to obtain a measure that interpolates critical values (and their twists) of the L-functions that we are interested in. The path is well known, we will evaluate the Eisenstein measure at CM points and then use the doubling method (in this setting the analogue of Damerell's formula) to prove the interpolation properties. We will need to compute some zeta integrals in order to prove that our measure has the right interpolation properties. As we mentioned in the introduction a more general study of such measures is the subject of the work in preparation of Eischen, Harris, Li and Skinner [14].

Here we are going to restrict ourselves in the cases that we need for our work. We consider a motive $M(\pi)/F \otimes M(\chi)/F$ as in the introduction (here we write χ instead of ψ there and it is a Hecke character of infinity type $-k\Sigma$). The automorphic representation π of $U(n) = U(n, 0)$ is taken of parallel scalar weight $\ell > 0$ by which we mean that if we can associate to π an automorphic form f_π which is an eigenform for all "good" Hecke operators (*i.e.* away from the conductor of π) and has the property $f_\pi(gk) = f_\pi(g)\det(k)^{-\ell}$, for $k \in G^\theta(\mathbb{R})$. The reader should notice here the sign convention of the weight (*i.e.* $-\ell$). We remark here that this a convention as taking the dual of π will give a representation of weight $-\ell$ which in turn can be seen also itself as a representation of weight ℓ of the group $U(0, n)$, since $\det(g)\overline{\det(g)} = 1$ for $g \in U(n, 0)$. Moreover we write $\mathfrak{c}$ for a, relative prime to p, integral ideal of F that contains the conductor of π (*i.e.* π is spherical at all v above p) as well as the non-p part of the conductor of ψ and some other bad primes which we describe in details below.

We write $L(BC(\pi),\chi,s)$ for the L-function associated to π and χ normalized as in [22, p. 141] after replacing there s with $s-\frac{n}{2}$. Here $BC(\pi)$ is the base changed automorphic representation to $\mathrm{GL}_n(K)$. Our goal in this section is to prove the following theorem.

Theorem 4.1. — *There exists a measure $\mu^{HLS,(f_\pi)}_{\pi,\chi}$ on $G_{\mathfrak{c}}$ (the Galois group of $K(p^\infty\mathfrak{c})/K$) such that for every finite character ψ of $G_{\mathfrak{c}}$ we have*

$$\frac{1}{\Omega_p(Y,\Sigma)^{k+2\ell}}\int_{G_{\mathfrak{c}}}\psi d\mu^{HLS,(f_\pi)}_{\pi,\chi} = \frac{(2\pi i)^{n(k+2\ell)}}{\Delta\times C^S(n,K,0)}\frac{L_S(BC(\pi),\chi\psi,\ell)}{\Omega_\infty(Y,\Sigma)^{k+2\ell}}Z_S(\pi,\check{\pi},\chi\psi,f_\Phi),$$

where $\Omega_\infty(Y,\Sigma)$ (resp. $\Omega_p(Y,\Sigma)$) is the archimedean (resp. p-adic) period corresponding to the CM pair (Y,Σ) for a CM-algebra Y and they will be defined below. Here S is the finite set of primes consisting of (i) all places v in F above p, (ii) all places of F ramified in K/F (iii) all places of F that are below the conductor of χ (iv) all places of F such that after localization of (V,θ) at v we have $(V_v,\theta_v)\equiv(T_v,\xi_v)\oplus(H_{r_v},\eta_{r_v})$ with $t_v:=\dim(T_v)=2$, (it is known (see ([40, Prop. 10.2 (1)] *that this set is finite), and (v) all places v where π is ramified. Finally we take $\mathfrak{c}$ so that if $v\in S\smallsetminus\{v|p\}$ then $v|\mathfrak{c}$. Also the quantity*

$$\Delta:=(\chi\psi)_{\mathbf{h}}(\det(\sigma^*))^{-1}|2^{-n}\det(\theta)^n\det(\sigma_{\mathbf{a}})^{-2}|^{-k/2}\,\mathrm{vol}(D^\theta(\mathfrak{c}))^2,$$

will be explained below. Also we recall that $C^S(n,K,0)$ defined as the product $c(n,K^+,K)L_{n,\infty}(k+2\ell,0)^{-1}$ is equal to

$$\frac{c(n,K^+,K)}{i^{n(2\ell+k)}2^{-n(2\ell+k-n+1)}\pi^{-n(2\ell+k)}\Gamma_n(2\ell+k)},$$

where

$$c(n,K^+,K):=2^{n(n-1)[F:\mathbb{Q}]/2}|\delta(F)|^{-n/2}|\delta(K)|^{-n(n-1)/4}.$$

Moreover we have that

$$Z_S(\pi,\check{\pi},\chi\psi,f_\Phi)=Z_{S\smallsetminus\{v|p\}}(\pi,\check{\pi},\chi\psi,f_\Phi)\times\prod_{v|p}Z_v(s,\pi,\check{\pi},\chi\psi,f_\Phi),$$

and the following explicit description of the factors $Z_v(s,\pi,\check{\pi},\chi\psi,f_\Phi)$ for the places of $v\in S$.

(1) *If $n = 1$ and we take π the trivial representation (hence $\ell = 0$) then for the place $\mathfrak{p} \in \Sigma_p$ above $v|p$ we have*

$$Z_v(\pi, \check{\pi}, \chi\psi, f_\Phi) = \alpha(\chi, \psi) \times \frac{L_{\mathfrak{p}}(0, \chi\psi)}{e_{\mathfrak{p}}(0, \chi\psi) L_{\mathfrak{p}}(1, \chi^{-1}\psi^{-1})},$$

for $\alpha(\chi, \psi)$ a factor that we make explicit in lemma 4.14. For the rest of the places $v \in S$ we have

$$\prod_{v \in S \smallsetminus \{v|p\}} Z_v(\pi, \check{\pi}, \chi\psi, f_\Phi) = \mathrm{vol}(D(\mathfrak{c}))$$

(2) *If $n = 2$ and after the identification $U(F_v) \cong \mathrm{GL}_2(F_v)$ for all $v|p$ we have $\pi_v = \pi(\nu_1, \nu_2)$ with ν_1, ν_2 unramified then $Z_v(\pi, \check{\pi}, \chi\psi, f_\Phi)$ equals*

$$\alpha(\chi, \psi) \frac{L_v(\ell - 1, \nu_1^{-1}\phi_1)}{e_v(\ell - 1, \nu_1^{-1}\phi_1) L_v(2 - \ell, \nu_1\phi_1^{-1})} \times \frac{L_v(\ell, \nu_2^{-1}\phi_1)}{e_v(\ell, \nu_2^{-1}\phi_1) L_v(1 - \ell, \nu_2\phi_1^{-1})},$$

where $\phi := \chi\psi$ and $\phi = (\phi_1, \phi_2)$ corresponding to the split $v\mathfrak{r} = \mathfrak{p}\bar{\mathfrak{p}}$ with $\mathfrak{p} \in \Sigma_p$. Here, as above, the factor $\alpha(\chi, \psi)$ will be made explicit in Lemma 4.14. For the rest of the local bad integrals we only remark that one can choose the form f_π attached to π (as for example in [29]*) so that*

$$Z_{S \smallsetminus \{v|p\}}(\pi, \check{\pi}, \chi\psi, f_\Phi) = \prod_{v \in S \smallsetminus \{v|p\}} Z_v(\pi, \check{\pi}, \chi\psi, f_\Phi) = \mathrm{vol}(D(\mathfrak{c})).$$

Remark 4.2. — At this point we would like to remark the following

(1) According to [24, Proposition 4.3] one has a similar form for the local integrals $Z_v(\pi, \check{\pi}, \chi\psi, f_\Phi)$ for any n. We will compute these local integrals in the cases $n = 1, 2$ that we need in our applications. Moreover, our theorem should be a special case of the theorem 4.4 announced in [24] whose proof should appear in [14].

(2) It is interesting to compare the interpolation formula of our the measure in the theorem above in the case of $n = 1$ and the measure of Katz-Hida-Tilouine [35, 26]. The interpolation formula is almost identical. However it is interesting to remark that the index factor $[O_K^\times : O_F^\times]$ that appear in the measure of Katz-Hida-Tilouine does not appear in the formula above. Another difference is the Euler factors that we remove. In our construction we also need to remove also the Euler factors at all the primes of K that are over primes

that ramify in K/F. This is not needed in the construction of Katz-Hida-Tilouine.

(3) Our measures depend on the choice of f_π. As it will be seen in the construction, the dependency is with respect to the factor $Z_{S\smallsetminus\{v|p\}}(\pi,\check{\pi},\chi\psi,f_\Phi)$. For different choices of f_π one gets different values for this factor. As we state in the theorem for some choices of f_π we may compute quite explicitly this factor. However the rest of the interpolation values of the measure is independent of the choice of f_π and depends only on the eigenvalues of it at the "good" Hecke operators (*i.e.* away from the conductor of π).

The doubling method of Garrett, Shimura, Piateski-Shapiro and Rallis. We start with an exposition to the doubling method as was developed by Garrett, Shimura, Piateski-Shapiro and Rallis. Our references are [22, 24, 25, 17]. We write (V,θ) for an n-dimensional hermitian vector space over K and we write G^θ for the corresponding unitary group. As before we define the hermitian space $(V,-\theta)$ and the $2n$-dimensional hermitian space $(2W = W\oplus -W, \theta\oplus-\theta)$. Then we have $U(\theta\oplus-\theta) = U(2W) \cong U(n,n)$. Later we will discuss this isomorphism a little bit more explicit (see also [40, p. 176]. Fixing such an isomorphism, we have an embedding $G := G^\theta \times G^\theta \hookrightarrow U(n,n)$.

We pick a Haar measure $dg = \otimes_v dg_v$ on $G(\mathbb{A}_F)$ such that for almost places v of F we have dg_v assigns volume 1 to a fixed hyperspecial maximal compact subgroup $K_v \subset G(E_v)$. We consider an irreducible cuspidal automorphic representation (V_π,π) of G^θ and we write $(V_{\check{\pi}},\check{\pi})$ for its dual representation. We fix now decompositions $\pi \cong \bigotimes_v \pi_v$ and $\check{\pi} \cong \bigotimes_v \check{\pi}_v$. We now pick $\phi \in V_\pi$ and $\check{\phi} \in V_{\check{\pi}}$ such that

(1) The vectors ϕ and $\check{\phi}$ are pure tensors away from $S^{(p)} := S \smallsetminus \{v \mid p\}$. That is $\phi = \otimes_{v\notin S^{(p)}}\phi_v \otimes \phi_{S^{(p)}}$ and $\check{\phi} = \otimes_{v\notin S^{(p)}}\check{\phi}_v \otimes \check{\phi}_{S^{(p)}}$ with $\phi_v \in V_{\pi_v}$ and $\check{\phi}_v \in V_{\check{\pi}_v}$.
(2) They are normalized, that is for all v's that π_v is spherical we pick these so that $\langle \phi_v, \check{\phi}_v\rangle = 1$.

After fixing an embedding V_π in the space of automorphic forms then this ϕ corresponds to our f_π, that appears in the theorem.

Let χ be a Hecke character of K and consider a section $\mathcal{F} \in I_\chi(s)$. We consider the integral

$$Z(s,\phi,\check{\phi},\chi,\mathcal{F}) = \int_{G(F)\smallsetminus G(\mathbb{A}_F)} \mathcal{E}((g,g'),\mathcal{F},\chi,s)\phi(g)\check{\phi}(g')\chi(\det g')dgdg'.$$

Let us now write $\mathcal{F} = \bigotimes_v \mathcal{F}_v$ with respect to the decompositions $I_\chi \cong \bigotimes_v I_{\chi,v}$. We now state the following important results:

Theorem 4.3 (Key Identity of Piatetski-Shapiro, Rallis and Shimura).
$$Z(s,\phi,\check{\phi},\chi,\mathcal{F}) = \int_{G^\theta(\mathbb{A}_F)} \mathcal{F}((g,1))(\pi(g)\phi,\check{\phi})dg.$$

This formula implies by computations of Li in [36] the formula,

Theorem 4.4 (Li's computations of the spherical integrals).
$$d_n(s,\chi)Z(s,\phi,\check{\phi},\chi,\mathcal{F}) = \langle\phi,\check{\phi}\rangle Z_S(s,\phi,\check{\phi},\chi,\mathcal{F})L_S(BC(\pi),\chi,s).$$

Remark 4.5. — Here we remind the different way that we have normalized our Eisenstein series. The above formula is obtained by replacing s with $s-\frac{n}{2}$ in the formula of [22, p. 141].

Let us explain the various notations arising in the formula

(1) S is the finite set of places but here including the archimedean ones, which we have defined above. If we assume that ϕ is a pure tensor at a prime $v \in S$ then the local factors for v is given by
$$Z_v(s,\phi,\check{\phi},\chi,\mathcal{F}) := \int_{G^\theta(F_v)} \mathcal{F}_v((g_v,1);\chi,s)(\pi_v(g_v)\phi_v,\check{\phi}_v)dg_v.$$
(2) The inner product $\langle\phi,\check{\phi}\rangle$ is defined as
$$\langle\phi,\check{\phi}\rangle = \int_{G^\theta(F)\smallsetminus G^\theta(\mathbb{A}_F)} \phi(g)\check{\phi}(g)dg.$$
(3) The factor $d_n(s,\chi)$ is a product of Dirichlet L functions
$$d_n(s,\chi) := \prod_{i=0}^{n-1} L_S(2s-i,\chi_1\varepsilon^i_{K/F}),$$
where χ_1 is the restriction of the Hecke character from $\mathbb{A}_K^\times$ to $\mathbb{A}_F^\times$.
(4) Finally the L-function $L_S(BC(\pi),\chi,s))$ is the standard L function associated to the automorphic representation $BC(\pi)\otimes\chi$ of $\mathrm{GL}_n(K)$, where $BC(\pi)$ is the base change from G^θ to $\mathrm{Res}_{K/F} G^\theta = \mathrm{GL}_n/K$. As usual the S subscript indicate that we have removed the Euler factors at the places that are in S.

The complex analytic point of view. We consider a hermitian space (V,θ) over K with θ positive definite. By Shimura [40, p. 171] there exists a matrix $\lambda \in K_n^n$ and a p-adic unit $\kappa \in K^\times$, such that (i) $\kappa^\rho = -\kappa$, (ii) $i\kappa_v\theta_v$ has signature $(n,0)$ for all $v \in \Sigma$ and (iii) $\kappa\theta = \lambda^* - \lambda$. We then consider the embedding
$$\gamma_n\colon G_{\theta,\theta} \hookrightarrow G := U(n,n), \quad \mathrm{diag}(a,b) \mapsto S^{-1}\,\mathrm{diag}(a,b)S,$$

where $S = \left(\begin{smallmatrix} 1_n & -\lambda \\ -1_n & \lambda^* \end{smallmatrix}\right)$. Let us write M for the $\mathfrak{g}$-maximal $\mathfrak{r}$-lattice in V used to define the congruences subgroup $D^\theta \subset G^\theta_{\mathbf{h}}$. That is, for an ideal $\mathfrak{c}$ of $\mathfrak{r}$ we define

$$C := \left\{\alpha \in G^\theta_{\mathbf{h}} \mid M\alpha = M\right\}, \quad \tilde{M} := \left\{x \in V \mid \theta(x, M) \subset \mathfrak{d}^{-1}_{K/F}\right\}$$

and then

$$D^\theta(\mathfrak{c}) := \left\{\gamma \in C \mid \tilde{M}_v(\gamma_v - 1) \subset \mathfrak{c}_v M_v,\ \forall v|\mathfrak{c}\right\}.$$

Following Shimura [40, p. 87] we have defined an element $\sigma \in \mathrm{GL}_n(K)_{\mathbf{h}}$ such that $M'\sigma = M$ where $M' = \sum_{i=1}^n \mathfrak{r}e_i$ for some fixed basis $\{e_i\}$ of V. We define the element $\Sigma_{\mathbf{h}} \in G_{\mathbf{h}}$ by $\Sigma_{\mathbf{h}} := \mathrm{diag}[\sigma, \hat{\sigma}]$.

Let now D be an open compact subgroup of $G_{\mathbf{h}}$ we fix a set $\mathcal{C}$ of representatives of the double coset $G = G(F) \setminus G(\mathbb{A}_F)/DD_\infty$, where $K_\infty \cong U(n)(F_{\mathbf{a}}) \times U(n)(F_{\mathbf{a}})$, the standard compact subgroup in $GU(n,n)$. It is known [40, p. 73] that we can pick the elements in $\mathcal{C}$ in the form $\mathrm{diag}(r, \hat{r})$ with $r \in \mathrm{GL}(K)_{\mathbf{h}}$ and $r_v = 1$ for every v in a selected finite set of places v of F. We have already seen that an automorphic form ϕ of G with respect to D is equivalent to tuple of hermitian modular forms $(f_r)_{r\in\mathcal{C}}$, where we have abused the notation and wrote r for $\mathrm{diag}(r, \hat{r})$. As it is explained in [40, p. 181] there exist an element $U \in G_{\mathbf{a}}$ such that if we consider $2^{-1}i\theta \in \mathbf{i}S_{\mathbf{a},+} \subset \mathbb{H}_{\mathbf{a}}$ and define $z_{CM,\theta} := U^{-1} \cdot (2^{-1}i\theta) \in \mathbb{H}_{\mathbf{a}}$ then we have that

$$\gamma_n(a, b)(z_{CM,\theta}) = z_{CM,\theta}, \quad (a, b) \in G_{\theta,\theta}.$$

We now fix a set $\mathcal{B}$ of $G^\theta(F) \setminus G^\theta_{\mathbf{h}}/D^\theta(\mathfrak{c})$ such that for all $b \in \mathcal{B}$ we have $b_v = 1$ for all v in the set S. For an element $(b_1, b_2) \in G_{\theta,\theta,\mathbf{h}}$, where $b_1, b_2 \in \mathcal{B}$ we write

$$\gamma_n(b_1, b_2) = \alpha(b_1, b_2)\, r(b_1, b_2)\, k(b_1, b_2),$$

where $\alpha(b_1, b_2) \in G$, $r(b_1, b_2) \in \mathcal{C}$ and $k(b_1, b_2) \in D[\mathfrak{g}, \mathfrak{c}]$. By the key identity of the doubling method and recalling our definition of the Eisenstein series E where the section $\mathcal{F}$ is defined as $\mathcal{F} = \otimes_{v\nmid p}\phi_v \otimes_{v|p} \phi^Y_v$, where ϕ_v are the local sections which were defined in Section 3 and should not be confused with the local vectors of the automorphic form ϕ. However since we will use this notation again we allow this abuse. Moreover $\phi^Y_v(g) := \phi_v(gY^{-1})$ and Y is an element in $U(\mathbb{Q}_p)$ defined as follows.

$$Y := \begin{pmatrix} 1_n & -\lambda \\ -1_n & \lambda^* \end{pmatrix} \in U(\mathbb{Z}_p),$$

where the last inclusion follows from our assumption on $\det(\theta)$ being a p-adic unit. In most of the applications we will be starting with a Hermitian form θ with $\det(\theta)$ a p-adic unit and θ skew hermitian for all embeddings of Σ. Then we can simply take $k=1$ and $\lambda=-\theta/2$. For the Eisenstein series we observe,

$$E(g,\mathcal{F},\chi)=E(gY^{-1},\otimes_v\phi_v,\chi).$$

Then by the doubling method we have (note that in the doubling method we took the Eisenstein series without normalization)

$$\begin{aligned}C^S(n,K,s)\int_{G_{\theta,\theta}(F)\backslash G_{\theta,\theta}(\mathbb{A}_F)} & E(\gamma_n(g,g'),\mathcal{F},\chi)\phi(g)\check{\phi}(g')\chi^{-1}(\det g')dgdg' \\ &= \langle\phi,\check{\phi}\rangle Z_S(s-\frac{n}{2},\phi,\check{\phi},\chi,\mathcal{F})L_S(BC(\pi),\chi,s),\end{aligned}$$

where here we write for simplicity χ instead of $\chi\psi$. If we write $z_{b_1,b_2}:=\alpha(b_1,b_2)\cdot z_{CM,\theta}\in\mathbb{H}_{\mathbf{a}}$, then the integral on the left hand side can be rewritten as

$$\begin{aligned}\int_{G_{\theta,\theta}(F)\backslash G_{\theta,\theta}(\mathbb{A}_F)} & E(\gamma_n(g,g'),\mathcal{F},\chi,s)\phi(g)\check{\phi}(g')\chi(\det g')dgdg' \\ &= C(s)\int_{G_{\theta,\theta}(F)\backslash G_{\theta,\theta}(\mathbb{A}_F)/(D^\theta(\mathfrak{c})\times D^\theta(\mathfrak{c}))} E(\gamma_n(g,g'),\mathcal{F},\chi,s) \\ &\qquad\qquad\times\phi(g)\check{\phi}(g')\chi(\det g')dgdg' \\ &= C(s)\sum_{b_1,b_2\in\mathcal{B}} E(\gamma(b_1,b_2),\mathcal{F},\chi,s)\chi(\det(b_2))\phi(b_1)\check{\phi}(b_2),\end{aligned}$$

where,

$$\begin{aligned}C(s):=\chi_{\mathbf{h}}(\det(\sigma^*))^{-1}|\det(\sigma_{\mathbf{h}})|^{-s}|2^{-n}\det(\theta)^n \\ \times\det(\sigma_{\mathbf{a}})^{-2}|^{s-k/2}\operatorname{vol}(D^\theta(\mathfrak{c})\times D^\theta(\mathfrak{c})).\end{aligned}$$

Note that this plus our considerations over the doubling method are equivalent to the formula of Shimura [40, equation 22.11.3] after one also multiplies the formula by $\check{f}_b$ and takes an extra summation over b (with the notation as in Shimura formulas).

Now we explain the first equation, the second being trivial. We have to study the integrand

$$E(\gamma_n(g,g'),\mathcal{F},\chi,s)\phi(g)\check{\phi}(g')\chi(\det g'),$$

with respect to the left translation by elements of $\gamma_n(D^\theta(\mathfrak{c})\times D^\theta(\mathfrak{c}))$. We first study its behavior for the archimedean places and the we turn to the finite part. We first note that since we take the automorphic representation of scalar parallel weight ℓ, we have that $\phi(gk)=\phi(g)\det(k)^{-\ell}$ for $k\in$

$D^\theta(\mathfrak{c})_\infty$. For $\check{\phi}$ we have $\check{\phi}(g'k') = \check{\phi}(g')\det(k')^\ell$. From the infinite type of the character we have $\chi(\det(g'k') = \chi(\det(g'))\det(k')^k$. (In this setting we remind to the reader that for an element $k \in K$ and an integer n we write $\det(k)^n = \prod_{v\in\mathbf{a}}\det(k_v)^n$). From Shimura [40, p. 182, (22.2.4)] we have for $(k,k') \in D^\theta(\mathfrak{c})_\infty \times D^\theta(\mathfrak{c})_\infty$,

$$E(\gamma_n(gk, g'k'), \mathcal{F}, \chi, s) = E(\gamma_n(g,g'), \mathcal{F}, \chi, s)\det(k)^\ell\det(k')^\ell\det(k')^{-k-2\ell},$$

from which we conclude the invariance with respect the archimedean places.

Now we turn to the behavior of the integrand with respect to the finite part of $\gamma_n(D^\theta(\mathfrak{c})\times D^\theta(\mathfrak{c}))$. The first point that we will explain is why we can take here $(\mathfrak{c},p)=1$ even thought the Eisenstein series has level divisible by primes above p when the character χ is ramified there. Actually this will also explain the modification of the section used in the definition of the Eisenstein series by the matrix Y. We write an element (k,k') in $D^\theta(\mathfrak{c}) \times D^\theta(\mathfrak{c})$ as $(k,k') = (k^{(p)}, k'^{(p)})(k_p,k'_p)$ where $(k_p,k'_p) \in \otimes_{v|p}G^\theta(F_v)\times G^\theta(F_v)$. Since the level of ϕ is prime to p we have that $\phi(gk_p) = \phi(g)$ and similarly $\phi(g'k'_p) = \phi(g')$. The same holds for $\check{\phi}$. We assume that χ is ramified at some prime above p. We claim that

$$E(\gamma_n(gk_p, g'k'_p), \mathcal{F}, \chi, s)\chi(\det(g'k'_p)) = E(\gamma_n(g,g'), \mathcal{F}, \chi, s)\chi(\det(g')).$$

Indeed we have by the definition of $\mathcal{F}$ that

$$\begin{aligned} E(\gamma_n(gk_p, g'k'_p), \mathcal{F}, \chi, s) &= E(\gamma_n(gk_p, gk'_p)Y^{-1}, \otimes\phi_v, \chi, s) \\ &= E(S_p^{-1}(\gamma_n(g,g)Y^{-1})^{(p)}\operatorname{diag}(k_p,k'_p), \otimes\phi_v, \chi, s) \\ &= E(\gamma_n(g,g'), \mathcal{F}, \chi, s)\chi_p^{-1}(\det(k'_p)). \end{aligned}$$

The last equality follows from the the definition of the local sections ϕ_v at places v above p. Indeed for their construction we have identified the group $G(F_v)$ with $\mathrm{GL}_{2n}(F_v)$ by taking the projection to to the first component (corresponding to the primes Σ_p). With respect to this projection the image of $\operatorname{diag}(k_v,k'_v)$ is again $\operatorname{diag}(k_v,k'_v)$, but now as an element of $\mathrm{GL}_{2n}(F_v)$. We then have the following lemma.

Lemma 4.6. — *For the section ϕ_v for v above p we have that*

$$\phi_v(\gamma_n(g_vk_v, g'_vk'_v)) = \phi_v(g)\chi_{2,v}(\det(k'_v))\chi_{1,v}^{-1}(\det(k'_v)).$$

Proof. — We recall first the definition of the section $\phi_v(h) = \phi_v^Y(h;\chi,s) = f_{\Phi Y^{-1}}(h,s)$.

$$\phi_v(h) = \chi_{2,v}(\det hY^{-1})|\det hY{-}1|^s \int_{\mathrm{GL}_n(F_v)} \Phi((0,Z)hY^{-1}) \times \chi_{2,v}\chi_{1,v}(\det Z)|\det Z|^{2s} d^\times Z,$$

hence

$$\begin{aligned}
&\phi_v(\gamma_n(g_vk_v, g'_vk'_v)Y^{-1}) \\
&\quad = \chi_{2,v}(\det(\gamma_n(g_vk_v, g'_vk'_v))Y^{-1})|\det(\gamma_n(g_vk_v, g'_vk'_v)Y^{-1})|^s \\
&\qquad \times \int_{\mathrm{GL}_n(F_v)} \Phi((0,Z)\gamma_n(g_vk_v, g'_vk'_v)Y^{-1})\chi_{2,v}\chi_{1,v}(\det Z)|\det Z|^{2s} d^\times Z \\
&\quad = \chi_{2,v}(\det(\gamma_n(g_v, g'_v))Y^{-1})\chi_{2,v}(\det(k_vk'_v))|\det(\gamma_n(g_v, g'_v)Y^{-1})|^s \\
&\qquad \times \int_{\mathrm{GL}_n(F_v)} \Phi((Zg_vk_v, Zg'_vk'_v))\chi_{2,v}\chi_{1,v}(\det Z)|\det Z|^{2s} d^\times Z
\end{aligned}$$

By the definition of Φ we have that

$$\Phi((Zg_vk_v, Zg'_vk'_v)) = \Phi((Zg_v, Zg'_v))\chi_{2,v}^{-1}(\det(k_v))\chi_{1,v}^{-1}(\det(k'_v)).$$

Hence we have

$$\begin{aligned}
&\int_{\mathrm{GL}_n(F_v)} \Phi((Zg_vk_v, Zg'_vk'_v))\chi_{2,v}\chi_{1,v}(\det Z)|\det Z|^{2s} d^\times Z \\
&\qquad = \chi_{2,v}^{-1}(\det(k_v))\chi_{1,v}(\det(k'_v)) \int_{\mathrm{GL}_n(F_v)} \Phi((0,Z)\gamma_n(g_v, g'_v)Y^{-1}) \\
&\qquad\qquad \times \chi_{2,v}\chi_{1,v}^{-1}(\det Z)|\det Z|^{2s} d^\times Z,
\end{aligned}$$

from where we conclude the lemma. □

Using the lemma we can conclude our claim. Indeed we need only to notice that $\chi_v(\det(k'_v)) = \chi_{1,v}(\det(k'_v))\chi_{2,v}^{-1}(\det(k'_v))$ since we use the projection to the first component (*i.e.* $v \in \Sigma$ and not $\bar{v}$) in our identification of $G^\theta(F_v)$ with $\mathrm{GL}_{2n}(F_v)$, that is the first component of the map

$$G^\theta(F_v) \to \mathrm{GL}_{2n}(F_v) \times \mathrm{GL}_{2n}(F_v), \quad g \mapsto (g, {}^t\theta\ {}^tg^{-1}\ {}^t\theta^{-1}).$$

After the above argument we can now assume that χ is unramified at places above p. Our first remark then is that for $k \in D^\theta(\mathfrak{c})_{\mathbf{h}}$ we have that $\det(k) \in 1 + \mathfrak{c}$ and hence $\chi^{-1}(k) = 1$ as the character χ is taken trivial modulo $\mathfrak{c}$. Indeed from [40, p. 88] we have for every finite place v of F that

$$k_v \in D^\theta(\mathfrak{c})_v \Leftrightarrow {\theta'}_v^{-1}(\sigma k_v \sigma^{-1} - 1)_v \prec \mathfrak{c}_v\mathfrak{d}_{K/F,v}.$$

In particular since $\theta' \prec \mathfrak{d}_{K/F}^{-1}$ we conclude that for $k \in D^{\theta}(\mathfrak{c})$ we have that $\det(k) - 1 = \det(\sigma k \sigma^{-1}) - 1 \prec \mathfrak{c}$ which concludes our claim. Now we state the following facts that are taken from [40, pp. 178–179].

Lemma 4.7. — *Set $\mathfrak{b} := \kappa^{-1}\mathfrak{d}_{K/F} \cap F$. Let $\varepsilon := \Sigma_{\mathbf{h}} S^{-1} \operatorname{diag}[1, \gamma] S \Sigma_{\mathbf{h}}^{-1}$ with $\gamma \in D^{\theta}(\mathfrak{c})$. Then $\varepsilon \in D[\mathfrak{b}^{-1}, \mathfrak{b}\mathfrak{c}]$ and $d_{\varepsilon} - 1 \prec \mathfrak{c}$ for $\varepsilon = \begin{pmatrix} a_{\varepsilon} & b_{\varepsilon} \\ c_{\varepsilon} & d_{\varepsilon} \end{pmatrix}$.*

Our choice of $\mathfrak{c}$ (*i.e.* $\mathfrak{d}_{K/F,v} \neq \mathfrak{r}_v$ implies $v|\mathfrak{c}$) we have from [40, p. 177, lemma 21.4(iii)] that for $\alpha \in D^{\theta}(\mathfrak{c})$ we have that $x := \Sigma_{\mathbf{h}} \gamma_n(\alpha, 1) \Sigma_{\mathbf{h}}^{-1} \in D[\mathfrak{b}^{-1}, \mathfrak{b}\mathfrak{c}]$. Moreover by [40, equation 21.6.3, p. 179] for all finite places v of F with $v|\mathfrak{c}$ we have that

$$(\det(d_x)^{-1} \det(\sigma^*))_v \in 1 + \mathfrak{c}_v.$$

These remarks and the modular properties of the Eisenstein series and the automorphic forms ϕ and $\hat{\phi}$ allow us to conclude that for $k, k' \in D^{\theta}(\mathfrak{c})_{\mathbf{h}}$ we have

$$\begin{aligned} &E(\gamma_n(gk, g'k'), \mathcal{F}, \chi, s)\phi(gk)\check{\phi}(g'k')\chi(\det g'k') \\ &\qquad = \chi_{\mathbf{h}}(\det(\sigma^*))^{-1} E(\gamma_n(g, g'), \mathcal{F}, \chi, s)\phi(g)\check{\phi}(g')\chi(\det g'). \end{aligned}$$

Before we proceed to the proof of Theorem 4.1 we need to define the p-adic and archimedean periods. We do that next.

The archimedean and p-adic periods $\Omega_{\infty}(Y, \Sigma)$ and $\Omega_p(Y, \Sigma)$. Now we need to explain how we pick the complex and p-adic periods that will appear in the interpolation formula. Our definition of these periods is the natural extensions of that of Katz [35, p. 268] in our setting.

In general we start with (W, θ), a positive definite Hermitian space of dimension n over a CM field K with $d := [F\colon \mathbb{Q}]$, where $F := K^+$. We write G for G^{θ} and fix a maximal open compact subgroup of $G(\mathbb{A}_{F,f})$. We note that the Shimura variety

$$Sh_G(U) := G(F) \setminus G(\mathbb{A}_{F,f})/U$$

is zero dimensional and parametrizes abelian varieties with CM by the CM-algebra $Y := \bigoplus_{j=1}^{n} K$ and additional additive structure determined by the open compact subgroup U. Indeed if U is defined as the open compact subgroup of G_f that fixes an $\mathfrak{r}$ lattice L in $W \cong K^n$, then $Sh_G(U)$ is simply the set of classes of L contained in the genus of L (see [40, p. 62]). For our considerations we assume that we may pick $L = \sum_{i=1}^{n} \mathfrak{r} e_i$ with respect to the standard basis of W over K.

We may now pick (see [40, p. 65]) representatives $\{L_i\}_{i=1}^{h}$ of the classes of L such that $L_i = L \cdot g_i$ with $g_i \in G(\mathbb{A}_{F,f})$ such that the ideal of K

corresponding the the idele $\det(g_i)$ is relative prime to p. We write $X(L_i)$ for the abelian variety corresponding to the lattice L_i. We define $A := \{a \in \overline{\mathbb{Q}} \mid \mathrm{incl}_p(a) \in D_p\}$, where D_p is the ring of integers of $\mathbb{C}_p$ and $\mathrm{incl}_p \colon \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p \hookrightarrow \mathbb{C}_p$ the fixed p-adic embedding. As in Katz (*loc. cit.*), we have that $\mathrm{Lie}(X(L_i)) = \mathrm{Lie}(X(L))$ for all $1 \leqslant i \leqslant h$, where the equality is in $L \otimes_{\mathfrak{r}} A$.

Now we let $\omega(L)$ be a nowhere vanishing differential of $X(L)$ over A, that is it induces through diality an isomorphism

$$\omega(L)\colon \mathrm{Lie}(X(L)) \xrightarrow{\sim} \mathfrak{d}_K^{-1} \otimes A.$$

From the fixed isomorphisms $\mathrm{Lie}(X(L_i)) = \mathrm{Lie}(X(L))$ we obtain for each $X(L_i)$ a nowhere vanishing differential $\omega(L_i)$ by the composition of this isomorphism with $\omega(L)$, that is

$$\omega(L_i)\colon \mathrm{Lie}(X(L_i)) = \mathrm{Lie}(X(L)) \xrightarrow{\omega(L)} \mathfrak{d}_K^{-1} \otimes A.$$

Now we write $\omega_{\mathrm{trans}}(L_i)$ for the nowhere vanishing differential on the complex analytic abelian variety $X(L_i)/\mathbb{C} := X(L_i) \times_A \mathbb{C}$, obtained after a fixed embedding $\mathrm{incl}_\infty \colon A \hookrightarrow \mathbb{C}$, corresponding to the lattice $L_i \subset \mathbb{C}^{n[F:\mathbb{Q}]}$. Then as in Katz [35, p. 269] from the very definition of the $\omega(L_i)$'s we obtain the following lemma.

Lemma 4.8. — *There exists an element $\Omega_\infty(Y,\Sigma) = (\dots,\Omega_\infty(i,\sigma),\dots \dots)_{i,\sigma}$ in $(\mathbb{C}^\times)^{n|\Sigma|}$ such that for all selected L_i as above we have*

$$\omega(L_i) = \Omega_\infty(Y,\Sigma)\cdot\omega_{\mathrm{trans}}(L_i).$$

Now we can also define in the same way as in Katz the p-adic periods. Our first step is to explain how we can give a p^∞-structure to the abelian varieties $X(L_i)$. We recall that the ordinary condition implies that all the primes above p in F split in K. Then if we consider $L_p := L \otimes \mathbb{Z}_p$ the splitting condition implies a splitting $L_p = L_p(\Sigma) \oplus L_p(\rho\Sigma)$ with $L_p(\Sigma) \cong L_p(\rho\Sigma) \cong \sum_{i=1}^n \mathfrak{g} \otimes \mathbb{Z}_p$. Then the embedding $\sum_{i=1}^n \mathfrak{g}\otimes\mathbb{Z}_p \hookrightarrow L_p$ to the first component provides the needed p^∞-structure. Through the isomorphisms $X(L) \cong X(L_i)$ we define the p^∞-structure to the rest of the varieties.

Hence, after extension of scalars $\mathrm{incl}_p \colon A \hookrightarrow D_p$, we may consider the canonical differential $\omega_{\mathrm{can}}(L_i)$ associated to the p^∞ arithmetic structure of $X(L_i)$. Then as in Katz (*loc. cit.*) we have the following lemma,

Lemma 4.9. — *There exists a unit $\Omega_p(Y,\Sigma) = (\dots,\Omega_p(i,\sigma),\dots) \in (D_p^\times)^{n|\Sigma|}$ such that for the selected L_i's above we have*

$$\omega(L_i) = \Omega_p(Y,\Sigma)\cdot\omega_{\mathrm{can}}(L_i).$$

Proof. — The proof is exactly as in Katz [35, lemma 5.1.47]; one has only to remark that over A we have an isomorphism $X(L_i)[p^\infty] \cong X(L)[p^\infty]$ induced from the identification $L \otimes_{\mathfrak{r}} A = L_i \otimes_{\mathfrak{r}} A$. □

Our next goal is to relate the periods that we have associated to the abelian varieties of the definite unitary groups $U(n)/F$ to abelian varieties of the definite unitary group $U(1)/F$. We start by recalling the following theorem of Shimura [41, p. 164].

Proposition 4.10. — *Let $(A, \imath)$ be an abelian variety of type (K, Ψ) with a CM-field K and Ψ of the form $\Psi_v(a) = \mathrm{diag}[a_v 1_{r_v}, \bar{a}_v 1_{s_v}]$ (Thus $\dim(A) = d = m[K^+ : \mathbb{Q}]$ and $r_v + s_v = m$). If $\sum_{v \in \mathbf{a}} r_v s_v = 0$, then A is isogenous to the product of m copies of an abelian variety belonging to a CM-type (K, Φ) with Φ such that Ψ is equivalent to the sum of m copies of Φ.*

We now write B for an abelian variety with complex multiplication by the CM field K and type Σ. As in Katz [35] or de Shalit [39] we fix a pair $(\Omega_\infty(\Sigma), \Omega_p(\Sigma)) \in (\mathbb{C}^\Sigma, D_p^\Sigma)^\times$ of a complex and p-adic period. As with the pair $(\Omega_\infty(Y, \Sigma), \Omega_p(Y, \Sigma)$ the definition is again independent of the particular B and depends only on the (K, Σ)-type. Then we have

Lemma 4.11. — *We may pick the pairs $(\Omega_\infty(Y, \Sigma), \Omega_p(Y, \Sigma))$ and $(\Omega_\infty(\Sigma), \Omega_p(\Sigma))$ so that*

$$(\Omega_\infty(Y, \Sigma), \Omega_p(Y, \Sigma) = (\Omega_\infty(\Sigma)^n, \Omega_p(\Sigma)^n).$$

Proof. — The only thing that we need to remark, is that with notation as above we have that $X(L) = \oplus_{i=1}^n X(\mathfrak{r})$, where $X(\mathfrak{r})$ is the abelian variety with CM of type (K, Σ) associated to the lattice $\Sigma(\mathfrak{r}) \subset \mathbb{C}^{[F : \mathbb{Q}]}$. □

Notation. We will write $E^\nu_{\varepsilon\psi}(\cdot)$ for the algebraic counterpart of the Eisenstein series $E(z, m(A), \phi, \varepsilon\psi)$ defined in Section 3 as well as for its p-adic avatar.

Proof of Theorem 4.1. Now we ready to proceed to the proof of Theorem 4.1.

Proof. — We recall that we have defined the sets of representatives $\mathcal{B}$ and $\mathcal{C}$. We change now our notation and write $\mathbf{f} = \otimes_v \mathbf{f}_v$ for the normalized automorphic form associated to π. Similarly we write $\check{\mathbf{f}}$ for the associated to $\check{\pi}$. Moreover it is well know that $\mathbf{f}$ is determined by a set of data $(f(a))_{a \in \mathcal{B}}$ where $f(a) := \mathbf{f}(a)$. Moreover we have defined an embedding

$$\gamma_n : G^\theta \times G^\theta \hookrightarrow G.$$

For $a, b \in \mathcal{B}$ we write $r_{a,b}$ for the fixed representative in $\mathcal{C}$ of the element $\gamma_n(a,b)$. We introduce the following notation, given an $a \in \mathcal{B}$, an $r \in \mathcal{C}$ and a r-polarized Eisenstein series $E_\chi(\cdot)$ we denote by $E_\chi^{(a)}(\cdot)$ the Eisenstein series

$$E_\chi^{(a)}(\cdot) := E_\chi(\cdot)\chi\left(\frac{\det(r)}{\det(a)}\right).$$

For every finite character ψ of $G_{\mathfrak{c}}$ and $a \in \mathcal{B}$ we define the measures

$$\mu_{a,\chi} \colon \psi \mapsto \sum_{b \in \mathcal{B}} E_{\psi\chi}^{(a)}(\underline{A_a} \times \underline{A_b}, j_1 \times j_2)\check{f}(b),$$

and then the measure

$$\begin{aligned}\int_{G_{\mathfrak{c}}} \psi d\mu_{\pi,\chi}^{HLS} &:= \frac{1}{\langle f, \check{f}\rangle} \sum_{a \in \mathcal{B}} \mu_{a,\chi}(\psi) f(a)\\ &= \frac{1}{\langle f, \check{f}\rangle} \sum_{a,b \in \mathcal{B}} \chi\psi(\det(b)) E_{\chi\psi}(\underline{A_a} \times \underline{A_b}, j_1 \times j_2) f(a)\check{f}(b).\end{aligned}$$

The last equality follows from the fact that $\chi\psi\left(\frac{\det(r_{a,b})}{\det(a)}\right) = \chi\psi(\det(b))$. Indeed we can write $\gamma_n(a,b) = \gamma r_{a,b} k$ with $\gamma \in G(F)$ and $k \in D(\mathfrak{c})$. In particular we have that

$$\det(a)\det(b) = \det(\gamma_n(a,b)) = \det(\gamma)\det(r_{a,b})\det(k)$$

and of course $\chi\psi(\det(\gamma)\det(r_{a,b})\det(k)) = \chi\psi(\det(r_{a,b}))$. Moreover we can assume (see [40, p. 65]) that $b_v = 1$ for all finite places v of F where the representation π_v is not spherical or $v \in S$. Then we claim that this measure μ^{HLS} has the claimed interpolation property and that $(f, \check{f})\mu_{\pi,\chi}^{HLS}$ takes integral values. We use the doubling method as developed above. We start by observing that,

$$\begin{aligned}&\frac{1}{\Omega_p(Y,\Sigma)^{k+2\ell}} \sum_{a,b \in \mathcal{B}} \chi\psi(\det(b)) E_{\chi\psi}(\underline{A_a} \times \underline{A_b}, j_1 \times j_2) f(a)\check{f}(b)\\ &= \frac{(2\pi i)^{nk}}{\Omega_\infty(Y,\Sigma)^{k+2\ell}} \sum_{a,b \in \mathcal{B}} \chi\psi(\det(b)) E_{\chi\psi}(\underline{A_a} \times \underline{A_b}, \omega_\infty(A_a) \times \omega_\infty(A_b)) f(a)\check{f}(b).\end{aligned}$$

Here $\omega_\infty = (2\pi i)^n \omega_{\text{trans}}$, that is we evaluate the Eisenstein series at abelian varieties associated to our preselected lattices normalized by the factor $2\pi i$.

We obtain the equation

$$\frac{\langle f, \check{f}\rangle}{\Omega_p(Y,\Sigma)^{k+2\ell}}\int_{G_{\mathfrak{c}}}\psi d\mu^{HLS}_{\pi,\chi} = \frac{(2\pi i)^{n(k+2\ell)}}{\Omega_\infty(Y,\Sigma)^{k+2\ell}}\sum_{a\in\mathcal{B}} \times\left(\sum_{b\in\mathcal{B}}\chi\psi(\det(b))E_{\chi\psi}(\underline{A_a}\times\underline{A_b},\omega_\infty(A_a)\times\omega_\infty(A_b))\check{f}(b)\right)f(a). \tag{4.1}$$

Now we use an observation of Shimura in [40, p. 88 and 186]. Namely, the inner summation

$$g(a) := \sum_{b\in\mathcal{B}}\chi\psi(\det(b))E_{\chi\psi}(\underline{A_a}\times\underline{A_b},\omega_\infty(A_a)\times\omega_\infty(A_b))\check{f}(b)$$

corresponds to the value at $a\in\mathcal{B}$ of the adelic automorphic form $\mathbf{f}|\mathfrak{T}$ where $\mathbf{f}$ the adelic form corresponding to $(\check{f}(a))_{a\in\mathcal{B}}$ and $\mathfrak{T} := \prod_v \mathfrak{T}_v = \prod_{v\in S}\mathfrak{T}_v \times \prod_{v\notin S}\mathfrak{T}_v$, where $\mathfrak{T}_v$ for $v\notin S$ is given as in Shimura (with notation as therein but for us here normalized as for example explained in p. 168 of [40])

$$\mathbf{f}|\mathfrak{T}_v = \prod_{r=1}^{n} L_v(n-r,\chi_1\psi_1\varepsilon^r)\sum_{\tau\in D_v\diagdown\mathfrak{X}_v/D_v}(\mathbf{f}|D_v\tau D_v)\chi\psi(\nu^\sigma(\tau))N(\nu^\sigma(\tau)))^{-k}$$

and for $v\in S$ we have that

$$\mathbf{f}|\mathfrak{T}_v = \prod_{r=1}^{n} L_v(n-r,\chi_1\psi_1\varepsilon^r)\sum_{\tau\in D_v\diagdown\mathfrak{X}_v/D_v}(\mathbf{f}|D_v\tau D_v)f_{\Phi_{\chi\psi}}((\tau,1))N(\nu^\sigma(\tau)))^{-k}.$$

But we are taking $\check{\mathbf{f}}$ an eigenform and normalized. In particular $g(b)$ is equal to $\alpha f(b)$ where α is the eigenvalue of $\mathbf{f}$ with respect to the operator $\mathfrak{T}$. Then we have

$$\frac{(\mathbf{f}|\prod_{v\in S}\mathfrak{T}_v,\mathbf{f})}{(\mathbf{f},\check{\mathbf{f}})} = \frac{(\phi_S|\prod_{v\in S}\mathfrak{T}_v,\check{\phi}_S)}{(\phi_S,\check{\phi}_S)} = Z_S(\pi,\check{\pi},\chi\psi,f_{\Phi_{\chi\psi}})$$

or equivalently

$$\phi_S|\prod_{v\in S}\mathfrak{T}_v = Z_S(\pi,\check{\pi}_v,\chi\psi,f_{\Phi_{\chi\psi}})\phi_S.$$

Indeed the last equations follow directly from the definition of the integrals

$$Z_S(\pi,\check{\pi},\chi\psi,f_{\Phi_{\chi\psi}}) = \int_{G(F_S)} f_{\Phi_{\chi\psi}}((g,1)(\pi(g)\phi_S,\check{\phi}_S)dg$$

and the p-adic Cartan decomposition of $D_v\diagdown G(F_v)/D_v$. Putting the last considerations together we get that the equation 4.1 reads

$$\frac{(2\pi i)^{nk+2\ell}}{\Omega_\infty(Y,\Sigma)^{k+2\ell}}\int_X E((g,g'),\phi,\chi\psi)f(g)\check{f}(g')(\psi\chi)(\det g')dgdg',$$

where $X = (G \times G)(F) \setminus (G \times G)(\mathbb{A}_F)/D^\theta(\mathfrak{c}) \times D^\theta(\mathfrak{c}))$ and $E(x, \phi, \chi\psi)$ is our normalized Eisenstein series. By the doubling method (the reader should here note that the doubling method was without the normalizing factors $C^S(n, K, s)$). Then we have that the last expression is equal to

$$\frac{(2\pi i)^{n(k+2\ell)}(f, \check{f})}{C(0)C^S(n, K, 0)} \left(\prod_{v \in S} Z_S(\pi, \check{\pi}, \chi\psi, \phi)\right) \frac{L_S(BC(\pi), \chi\psi, \ell)}{\Omega_\infty(Y, \Sigma)^{k+2\ell}},$$

where we have used the fact that $\mathcal{B} = G(F) \setminus G(\mathbb{A}_F)/K(\mathfrak{c})$ and $G(F_\infty) = K_\infty(\mathfrak{c})$. The factor $Z_S(, \pi, \check{\pi}, \chi\psi, \phi)$ will be computed in some case below. $\square$

From its very definition it is easily seen that the measure $(f, \check{f})\mu^{HLS}_{\pi,\chi}$ is integral valued. Hence we could for example establish that the measure $\mu^{HLS}_{\pi,\chi}$ is integral valued if we knew that the quantity $(f, \check{f})$ is a p-adic unit. It is well-known (see [24, p. 2] that the p-divisibility of this quantity corresponds to congruences modulo p between forms in π and other cuspidal automorphic representations π' of $U(n)$. Hence if we assume that there are no congruences between forms of $U(n)$ we can conclude that the measure $\mu^{HLS}_{\pi,\chi}$ is p-adic integral

The local integrals for v archimedean. The local integrals for v archimedean have been computed in general by Garrett [16, Section 3, Quantitative Theorem] (see also the remarks (3) and (4) of Harris in [23]) and they are known to be elements in $K^\times$. However in our situation they are integrals over compact groups hence can be easily computed. Indeed by definition

$$Z_\infty(\ell, \phi, \check{\phi}, \chi, \mathcal{F}) = \int_{G^\theta(\mathbb{R})} \mathcal{F}(\gamma_n(g, 1))(\pi(g)\phi, \check{\phi})dg.$$

By our choice of the infinite section $\mathcal{F}$ we have that $\mathcal{F}(\gamma_n(g, 1)) = \det(g)^\ell \mathcal{F}(1) = \det(g)^\ell$. Similarly we have $(\pi(g)\phi, \check{\phi}) = \det(g)^{-\ell}(\phi, \check{\phi}) = \det(g)^{-\ell}$. Hence the integral is simply the volume of the compact group $G^\theta(\mathbb{R})$. Our measure is picked so that this volume is equal to 1 (see [22, p. 83]).

Computation of the local integrals for finite $v \in S$ not above p. In the case of $n = 2$ the integrals depend on the particular choices of ϕ_v. For some specific choices of these ϕ_v's in the case of $n = 2$ and in general for $n = 1$ the integrals have been computed by Ming-Lun Hsieh in [30, 29] where he proves:

$$Z_v(1, \phi, \check{\phi}, \chi, \mathcal{F}) = \mathrm{vol}(D^\theta(\mathfrak{c})_v).$$

Computation of the local integrals for v above p. We now compute the integrals

$$Z_p(s,\phi,\check{\phi},\chi,\mathcal{F}) = \prod_{v|p} Z_v(s,\phi_v,\check{\phi}_v,\chi_v,\mathcal{F}_v)$$

in the special case where $n=1$ or 2. We start with some general remarks with respect the Fourier transform over GL_1 and then we generalize to GL_n. Our main references are [30, 29]. We let F be a local field with ring of integers $\mathfrak{g}$ and we fix a uniformizer ϖ of $\mathfrak{g}$ and write $\mathfrak{p}=(\varpi)$. For a complex character $\chi\colon F^\times \to \mathbb{C}^\times$ we define the Bruhat-Schwartz function $\Phi_\chi(x) = \chi(x) I_{\mathfrak{g}^\times}(x)$. If we define the quantity

$$E_v(s,\chi) := \frac{L(s,\chi)}{e(s,\chi)L(1-s,\chi^{-1})},$$

where $L(s,\chi)$ the standard L factor of χ and $e(s,\chi)$ the epsilon factors of χ, then it is well known that

$$Z(s,\chi,\Phi_{\chi^{-1}}) := \int_F \chi(x)\Phi_{\chi^{-1}}(x)|x|^s d^\times x = \mathrm{vol}(\mathfrak{g})$$

and

$$Z(s,\chi,\widehat{\Phi}_\chi) := \int_F \chi(x)|x|^s \widehat{\Phi}_\chi(x) d^\times x = E_v(s,\chi),$$

where $\widehat{\Phi}_\chi$ the Fourier transform of Φ_χ defined as $\widehat{\Phi}_\chi(x) = \int_F \Phi_\chi(y)\psi(yx)dy$ for an additive character $\psi\colon F \to \mathbb{C}^\times$. Moreover it is well known that if we write $c(\chi)$ for the valuation of the conductor of χ and d for the valuation of the different of F over $\mathbb{Q}_p$ then

$$\widehat{\Phi}_\chi(x) = \begin{cases} \chi^{-1}(x) I_{c^{-1}\mathfrak{g}^\times}(x)\tau(\chi), & c(\chi)\neq 0 \\ I_{\mathfrak{g}}(x) - |\varpi| I_{\mathfrak{p}^{-1}}(x), & c(\chi)=0, \end{cases}$$

where $c := \varpi^{c(\chi)+d}$ and $\tau(\chi) = \int_{\mathfrak{g}^\times} \chi(\frac{x}{c})\psi(\frac{x}{c})d^\times x$, a Gauss sum related to the local-epsilon factor by $e(\chi,s) = |c|^s \tau(\chi)^{-1}$.

We now generalize these considerations to the case of GL_n. For a Bruhat-Schwartz function Φ of $M_n(F)$ we define its Fourier-transform $\widehat{\Phi}(X) := \int_{M_n(F)} \Phi(Y)\psi(\mathrm{tr}({}^tYX))dY$. For a partition $n\colon = n_1 + \cdots + n_\ell$ of n and a set of characters $\nu := (\nu_1,\ldots,\nu_\ell)$ of $F^\times$ we have defined the Bruhat-Schwartz function

$$\Phi_\nu(X) := \begin{cases} \nu_1(\det(X_{11}))\cdots\nu_\ell(\det(X_{\ell\ell})), & X \in \Gamma(\mathfrak{p}^t) \\ 0, & \text{otherwise.} \end{cases}$$

Now we recall the Godement-Jacquet zeta functions as introduced in [18]. So we consider an automorphic representation (π, V_π) of $\mathrm{GL}(n)(F)$, which always we take to be a principal series of the form $\pi = \pi(\nu_1, \ldots, \nu_\ell)$. We write $\omega(g) := \langle \pi(g)v, \tilde{v}\rangle$ for the matrix coefficient where $v \in V_\pi$ and $\tilde{v} \in \tilde{V}$ in the space of the contragredient representation $\tilde{\pi}$ and $\langle \cdot, \cdot \rangle$ properly normalized so that $\langle v, \tilde{v}\rangle = 1$. For a Bruhat-Schwartz function Φ of $M_n(F)$ and a character χ of $\mathrm{GL}_1(F) = F^\times$ we define the integrals

$$Z(s, \Phi, \omega, \chi) := \int_{\mathrm{GL}_n(F)} \Phi(x)\, \chi(\det(x))\, \omega(x)\, |\det(x)|^s\, d^\times x.$$

These integrals generalize the theory of Tate (in the case $n = 1$) as it is proven in [18]. It is known [18, p. 80] that if π and χ are spherical and we pick $\Phi := \mathbf{1}_{M_n(\mathfrak{g})}$ then we have that

$$Z(s, \Phi, \omega, \chi) = L(s, \pi, \chi),$$

that is the $L-$factor of π twisted by χ. Now we take $n = 2$ and $\ell = 2$. Then we have the following lemmas that generalize the case of GL_1.

Lemma 4.12. — *Consider the principal series representation $\pi = \pi(\nu_1, \nu_2)$, where ν_2 is an unramified character and define $\omega(x) = \langle \pi(x)v, \tilde{v}\rangle$ with $v \in V_\pi$ so that $\pi(x)v = \nu_1(a)v$ for $x = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma(\mathfrak{p}^t)$. Then the Godement-Jacquet integral $Z(s, \Phi, \pi, \chi)$ with $\Phi := \Phi_{\chi^{-1}\nu^{-1}}$ is equal to* $\mathrm{vol}(\Gamma(\mathfrak{p}^t))\langle v, \tilde{v}\rangle$.

Proof. — By definition we have that $Z(s, \Phi, \pi, \chi)$ equals

$$\int_{\mathrm{GL}_2(F)} \Phi(x)\, \chi(\det(x))\, \omega(x) |\det(x)|^s\, d^\times x = \int_{\Gamma(\mathfrak{p}^t)} \Phi(x)\, \chi(\det(x))\omega(x)\, |\det(x)|^s d^\times(x).$$

But for $x = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma(\mathfrak{p}^t)$ we have that $\omega(x) = \langle \pi(x)v, \tilde{v}\rangle = \langle \nu_1(a)v, \tilde{v}\rangle = \nu_1(a)\langle v, \tilde{v}\rangle$. That means we have,

$$Z(s, \Phi, \pi, \chi) = \langle v, \tilde{v}\rangle \int_{\Gamma(\mathfrak{p}^t)} \Phi(x)\, \chi(\det(x))\nu_1(a) |\det(x)|^s d^\times(x),\ x = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

But by the definition of Φ we have that $\Phi(x) = \nu_1^{-1}(a)\nu_2^{-1}(d)\chi^{-1}(ad)$ and we notice that by the choice of t, *i.e.* bigger than the conductors of ν_i and χ, we have that $\chi(\det(x)) = \chi(ad - bc) = \chi(ad)\chi(1 + \mathfrak{p}^t) = \chi(ad)$ as $a, d \in \mathfrak{g}^\times$ by the definition of $\Gamma(\mathfrak{p}^t)$. Putting these considerations together we have $Z(s, \Phi, \pi, \chi)$ is equal to

$$\langle v, \tilde{v}\rangle \int_{\Gamma(\mathfrak{p}^t)} \nu_1^{-1}(a)\nu_2^{-1}(d)\chi^{-1}(ad)\, \chi(ad)\nu_1(a) |\det(x)|^s d^\times(x),\ x = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Since ν_2 is not ramified we have that $\nu_2(\mathfrak{g}^\times) = 1$ which concludes the proof of the lemma. □

We now compute the integrals $Z(s, \widehat{\Phi}, \pi, \chi)$. As we mentioned above these integrals should be computed in full generality in [14]. In the following lemma we will compute them only in the case of interest, namely for $n = 2$. We note also here that a similar integral has been computed in [29, Lemma 6.7].

Lemma 4.13. — *With the setting as in lemma 4.12 but with $\pi = \pi(\nu_1, \nu_2)$ unramified and for $\Phi := \Phi_{\nu\chi}$ we have that the Godement-Jacquet integral $Z(s, \widehat{\Phi}, \pi, \chi)$ is given by*

$$Z(s, \widehat{\Phi}, \pi, \chi) = |\mathfrak{d}_F| \times E_v(s-1, \nu_1\chi) E_v(s, \nu_2\chi),$$

where $\mathfrak{d}_F$ is the different of F.

Proof. — We start by exploring the support of $\widehat{\Phi}$. By definition we have that $\widehat{\Phi}(X) = \int_{M_2(F)} \Phi(Y)\psi(\mathrm{tr}({}^tYX))$. Writing $X = \left(\begin{smallmatrix} x_1 & x_2 \\ x_3 & x_4 \end{smallmatrix}\right)$ and $Y = \left(\begin{smallmatrix} y_1 & y_2 \\ y_3 & y_4 \end{smallmatrix}\right)$ we have that

$$\widehat{\Phi}(X) = \int_{M_2(F)} \Phi\left(\begin{pmatrix} y_1 & y_2 \\ y_3 & y_4 \end{pmatrix}\right) \psi(x_1y_1)\psi(x_2y_2)\psi(x_3y_3)\psi(x_4y_4) dy_1 dy_2 dy_3 dy_4.$$

By the definition of $\Phi = \Phi_{\nu\chi}$ we have that

$$\widehat{\Phi}(X) = \int_{\mathcal{I}} \nu_1\chi(x_1)\nu_2\chi(x_4)\psi(x_1y_1)\psi(x_2y_2)\psi(x_3y_3)\psi(x_4y_4) dy_1 dy_2 dy_3 dy_4,$$

where $\mathcal{I} \subset M_2(F)$ the support of Φ. The above integral we can write as the product

$$\left(\int_{I_1} \nu_1\chi(x_1)\psi(x_1y_1)dy_1\right)\left(\int_{I_4} \nu_2\chi(x_4)\psi(x_4y_4)dy_4\right) \\ \times \left(\int_{I_2} \psi(x_2y_2)dy_2\right)\left(\int_{I_3} \psi(x_3y_3)dy_3\right),$$

where we have written $\mathcal{I} = \left(\begin{smallmatrix} I_1 & I_2 \\ I_3 & I_4 \end{smallmatrix}\right)$. By definition we have that $I_1 = I_4 = \mathfrak{g}^\times$ and hence we have that

$$\int_{I_1} \nu_1\chi(x_1)\psi(x_1y_1)dy_1 = \widehat{\Phi}_{\nu_1\chi}(y_1), \quad \int_{I_4} \nu_2\chi(x_4)\psi(x_4y_4)dy_4 = \widehat{\Phi}_{\nu_2\chi}(y_4).$$

For the other two integrals we have that

$$\int_{I_2} \psi(x_2y_2)dy_2 = \begin{cases} \mathrm{vol}(I_2), & x_2 I_2 \in \mathfrak{d}_F^{-1} \\ 0, & \text{otherwise.} \end{cases}$$

and similarly for $\int_{I_3}\psi(x_3y_3)dy_3$. Now we turn to the integral $Z(s,\widehat{\Phi},\pi,\chi)$. By definition we have

$$Z(s,\widehat{\Phi},\pi,\chi)=\int_{\mathrm{GL}_2(F)}\widehat{\Phi}(x)\,\chi(\det(x))\,\omega(x)|\det(x)|^s\,d^\times x.$$

By the Iwasawa decomposition $\mathrm{GL}_2(F)=B(F)K$ with $K=\mathrm{GL}_2(\mathfrak{g})$. Hence if we write $x=b_Fk$ and $b_F=\left(\begin{smallmatrix}a&y\\0&b\end{smallmatrix}\right)$ and observe that $d^\times x=|a|^{-1}dyd^\times ad^\times bdk$ then the integral above reads

$$\int_{F^\times}\int_{F^\times}\int_F\int_K\widehat{\Phi}\left(\begin{pmatrix}a&y\\0&b\end{pmatrix}k\right)\chi(ab\det(k))\,\omega\left(\begin{pmatrix}a&y\\0&b\end{pmatrix}k\right)\\ \times|ab|^s\,dy\frac{d^\times a}{|a|}d^\times bdk.$$

By definition we have that $\omega(x)=\langle\pi(x)v,\tilde{v}\rangle$ with v a normalized spherical vector. That means, as $\pi=\pi(\nu_1,\nu_2)$ that we have $\omega\left(\left(\begin{smallmatrix}a&y\\0&b\end{smallmatrix}\right)k\right)=\nu_1(a)\nu_2(b)\langle v,\tilde{v}\rangle=\nu_1(a)\nu_2(b)$. Moreover we note that by definition $\widehat{\Phi}(xk)=\widehat{\Phi}(x)\Phi(k^{-1})$. Indeed from the definition of the Fourier transform after the change of variable $y\mapsto y{}^tk^{-1}$ and noticing that $\mathrm{tr}({}^tyxk)=\mathrm{tr}(k{}^tyx)$ we have that

$$\widehat{\Phi}(xk)=\int_{M_2(F)}\Phi(y)\psi(\mathrm{tr}({}^tyxk))=\int_{M_2(F)}\Phi(y{}^tk^{-1})\psi(\mathrm{tr}({}^tyx)).$$

But now we note that by the very definition of Φ that

$$\Phi(y{}^tk^{-1})=\Phi(y)\Phi({}^tk^{-1})=\Phi(y)\Phi(k^{-1})$$

which proves our claim. Our next observation is that $\Phi(k^{-1})=\chi^{-1}(\det(k))$ since ν_1 and ν_2 are unramified characters. These considerations together give us that $Z(s,\widehat{\Phi},\pi,\chi)$ is equal to

$$=\mathrm{vol}(K)\left(\int_{F^\times}\widehat{\Phi}_{\nu_1\chi}(a)\nu_1\chi(a)|a|^{s-1}d^\times a\right)\left(\int_{F^\times}\widehat{\Phi}_{\nu_2\chi}(b)\nu_2\chi(b)|b|^sd^\times b\right)\\ \times\int_F\int_{I_2}\psi(yx)dydx.$$

But we have that

$$\int_F\int_{I_2}\psi(yx)dydx=\mathrm{vol}(I_2)\int_{\mathfrak{d}_F^{-1}I_2^{-1}}dy=\mathrm{vol}(I_2)\,\mathrm{vol}(\mathfrak{d}_F^{-1}I_2^{-1})=|\mathfrak{d}_F|.$$

□

We have defined the matrix $S=\begin{pmatrix}1_n&-\lambda\\-1_n&\lambda^*\end{pmatrix}$. We define the matrix $\omega:=\left(\begin{smallmatrix}\theta&0\\0&-\theta\end{smallmatrix}\right)$ and write G^ω for the corresponding unitary group. That is, G^ω corresponds to the hermitian space $(2V:=V\oplus V,\theta\oplus(-\theta))$. As always we write η_n for the matrix $\left(\begin{smallmatrix} 0 & -1_n \\ 1_n & 0 \end{smallmatrix}\right)$. Then as it is explained in Shimura [40, p. 176]we have $S^{-1}G^{\omega}S = G^{\eta_n}$ and if we define $P^{\omega} := \gamma \in G^{\omega} : U\gamma = U$ with $U := \{(v,v) \in 2V, v \in V\}$ then $S^{-1}P^{\omega}S = P^{\eta_n}$ with P^{η_n} the standard Siegel parabolic $\{x \in G^{\eta_n} : c_x = 0\}$. We write $\gamma_n : G^{\omega} \xrightarrow{\sim} G^{\eta_n}$, $g \mapsto S^{-1}\, g\, S$. Now we note that if we define the group $G_{\theta,\theta} := G^{\theta} \times G^{\theta}$ then we have a canonical embedding $G_{\theta,\theta} \hookrightarrow G^{\omega}$ given by $(g,g') \mapsto \mathrm{diag}(g,g')$. Then we remark that

$$\gamma_n^{-1}(P^{\eta_n}) \cap G_{\theta,\theta} = \{(g,g) \mid g \in G^{\theta}\}.$$

Lemma 4.14. — *Let v be a place over p and let $\Phi := \Phi_v$ be the Bruhat-Schwartz selected above. Then if we write $f_{\Phi}^{(S)}$ for the corresponding local section we have*

$$Z_v(s,\pi,\tilde{\pi},f_{\Phi}^{(S)})) = \int_{\mathrm{GL}_n(F_v)} f_{\Phi}^{(S)}(\gamma_n(g,1))\omega(g)d^{\times}g$$
$$= \begin{cases} \alpha(\chi,\psi,s)\dfrac{L_{\mathfrak{p}}(s-1,\chi\psi)}{e_{\mathfrak{p}}(s-1,\chi\psi)L_{\mathfrak{p}}(2-s,\chi^{-1}\psi^{-1})}, & \text{if } n=1 \\ \alpha(\chi,\psi,s)\dfrac{L_v(s-1,\nu_1^{-1}\phi_1)}{e_v(s-1,\nu_1^{-1}\phi_1)L_v(2-s,\nu_1\phi_1^{-1})} & \\ \qquad\qquad \times\dfrac{L_v(s,\nu_2^{-1}\phi_1)}{e_v(s,\nu_2^{-1}\phi_1)L_v(1-s,\nu_2\phi_1^{-1})}, & \text{if } n=2, \end{cases}$$

where the notation is as in Theorem 4.1.

Proof. — In this proof we will write F for F_v. Moreover we write (χ_1,χ_2) for the pairs $(\chi_{\mathfrak{p}}\psi_{\mathfrak{p}},\chi_{\bar{\mathfrak{p}}}\psi_{\bar{\mathfrak{p}}})$ in case $n=1$ and (ϕ_1,ϕ_2) in case $n=2$. By definition we have $f_{\Phi}^{(S)}(x) = f_{\Phi}(xS^{-1})$ with

$$f_{\Phi}(h) := \chi_2(\det h)|\det(h)|^s \int_{\mathrm{GL}_n(F)} \Phi((0,Z)h)\chi_2\chi_1(\det Z)|\det(Z)|^{2s}d^{\times}Z.$$

That is the integral $Z_p(s,\pi,\tilde{\pi},f_{\Phi}^{(S)}))$ is equal to

$$I(s) := \chi_2^{-1}(\det(S))|\det(S)|^{-s}\int_{\mathrm{GL}_n(F)}\int_{\mathrm{GL}_n(F)} \Phi((0,Z)\gamma_n$$
$$\times(\mathrm{diag}(g,1))S^{-1})\chi_2\chi_1(\det(Z))|\det(Z)|^{2s}\chi_2$$
$$\times(\det(g))|\det(g)|^s\omega(g)d^{\times}Zd^{\times}g.$$

By the definition of γ_n we have that

$$\Phi((0,Z)\gamma_n(g,1)S^{-1}) = \Phi((0,Z)S^{-1}\,\mathrm{diag}(g,1)).$$

We have that

$$S^{-1} = \begin{pmatrix} \lambda^* & \lambda \\ 1_n & 1_n \end{pmatrix}\begin{pmatrix} \kappa^{-1}\theta^{-1} & 0 \\ 0 & \kappa^{-1}\theta^{-1} \end{pmatrix}.$$

After doing the algebra we obtain that the matrix $S^{-1}\left(\begin{smallmatrix} g & 0 \\ 0 & 1_n \end{smallmatrix}\right)$ equals

$$\begin{pmatrix} \lambda^* & \lambda \\ 1_n & 1_n \end{pmatrix} \begin{pmatrix} \kappa^{-1}\theta^{-1}g & 0 \\ 0 & \kappa^{-1}\theta^{-1} \end{pmatrix} = \begin{pmatrix} \lambda^*\kappa^{-1}\theta^{-1}g & \lambda\kappa^{-1}\theta^{-1} \\ \kappa^{-1}\theta^{-1}g & \kappa^{-1}\theta^{-1} \end{pmatrix}.$$

In particular

$$\Phi\left((0,Z)\begin{pmatrix} \lambda^*\kappa^{-1}\theta^{-1}g & \lambda\kappa^{-1}\theta^{-1} \\ \kappa^{-1}\theta^{-1}g & \kappa^{-1}\theta^{-1} \end{pmatrix}\right) = \Phi(Z\kappa^{-1}\theta^{-1}g, Z\kappa^{-1}\theta^{-1}).$$

The integral now reads

$$\begin{aligned} I(s) = \chi_2^{-1}(\det(S))|\det(S)|^{-s} &\int_{\mathrm{GL}_n(F)}\int_{\mathrm{GL}_n(F)} \Phi(Z\kappa^{-1}\theta^{-1}g, Z\kappa^{-1}\theta^{-1}) \\ &\times \chi_2\chi_1(\det(Z))|\det(Z)|^{2s}\chi_2(\det g)|\det(g)|^s\omega(g)d^\times Z d^\times g. \end{aligned}$$

As in [17, p. 36], we make the change of variables $Z\kappa^{-1}\theta^{-1} \mapsto Z_2$ and $Z\kappa^{-1}\theta^{-1}g \mapsto Z_1$. That is $Z_2^{-1}Z_1 = g$ and by the translation invariance (recall that $\mathrm{GL}_n(F)$ is unimodular *i.e.* left and right Haar measures coincide) of the measures we have

$$\begin{aligned} I(s) = \chi_2^{-1}(\det(S))|\det(S)|^{-s} &\int_{\mathrm{GL}_n(F)}\int_{\mathrm{GL}_n(F)} \Phi(Z_1, Z_2)\chi_2\chi_1 \\ &\times (\det(\kappa\theta))|\det(\kappa\theta)|^{2s}\chi_2\chi_1(\det(Z_2))\chi_2(\det(Z_2)^{-1}) \\ &\times \chi_2(\det(Z_1))|\det(Z_2)|^s|\det(Z_1)|^s\omega(Z_2^{-1}Z_1)d^\times Z_1 d^\times Z_2. \end{aligned}$$

As it is explained in [17, p. 36] this integral is equal to

$$\begin{aligned} I(s) = \chi_2^{-1}&(\det(S))|\det(S)|^{-s}\chi_2\chi_1(\det(\kappa\theta))|\det(\kappa\theta)|^{2s} \\ &\times \int_{\mathrm{GL}_n(F)}\int_{\mathrm{GL}_n(F)} \Phi(Z_1, Z_2)\chi_2(\det(Z_1))\chi_1(\det(Z_2)) \\ &\times |\det(Z_2)|^s|\det(Z_1)|^s\omega(Z_2^{-1})\omega(Z_1)d^\times Z_1 d^\times Z_2 \end{aligned}$$

which in turn is equal to

$$\begin{aligned} \chi_2^{-1}&(\det(S))|\det(S)|^{-s}\chi_2\chi_1(\det(\kappa\theta))|\det(\kappa\theta)|^{2s} \\ &\times \int_{\mathrm{GL}_n(F)} \Phi_1(Z_1)\chi_2(\det(Z_1))\omega(Z_1)|\det(Z_1)|^s d^\times Z_1 \\ &\times \int_{\mathrm{GL}_n(F)} \Phi_2(Z_2)\chi_1(\det(Z_2))\omega(Z_2^{-1})|\det(Z_2)|^s d^\times Z_2. \end{aligned}$$

But $\Phi_1 = \tilde{\Phi}_{\nu^{-1}\chi_2^{-1}}$ and $\Phi_2 = \widehat{\Phi}_{\nu^{-1}\chi_1}$. These integrals we have already computed. Finally we set

$$\alpha(\chi,\psi,s) := |\mathfrak{d}_F| \times \chi_2^{-1}(\det(S))|\det(S)|^{-s}\chi_2\chi_1(\det(\kappa\theta))|\det(\kappa\theta)|^{2s}$$

and $\alpha(\chi,\psi) := \alpha(\chi,\psi,1)$. $\square$

5. Congruences between Eisenstein series

We recall briefly part of our setting. We consider a CM field K and a CM extension K' of degree p with totally real field $F = K^+$ and $F' = K'^+$. We also fix CM types (K, Σ) and (K', Σ') with Σ' be the lift of Σ. We moreover make the following assumption: The primes that ramify in F'/F are unramified in K/F. Our aim now is to study the natural embedding:

$$U(n,n)_{/F}(F) \hookrightarrow \mathrm{Res}_{F'/F}\, U(n,n)_{/F'}(F).$$

The diagonal embedding; algebraically and analytically. We start by first observing that the compatibility of the CM-types induces an embedding of the corresponding symmetric spaces. That is, we have

$$\Delta\colon \mathbb{H}_F \hookrightarrow \mathbb{H}_{F'}, \quad (z_\sigma)_{\sigma\in\mathbf{a}} \mapsto (z_{\sigma'})_{\sigma'\in\mathbf{a}'},$$

with $z_{\sigma'} := z_\sigma$ for $\sigma'|_K = \sigma$. We now consider the congruence subgroup $\Gamma_0(\mathfrak{b}, \mathfrak{c})$ of $U(n,n)_{/F}$ for an integral ideal $\mathfrak{c}$ of $\mathfrak{g}$ and a fractional ideal $\mathfrak{b}$ of F. We moreover consider the congruences subgroup $\Gamma_0(\mathfrak{b}\mathfrak{g}', \mathfrak{c}\mathfrak{g}')$ of $\mathrm{Res}_{F'/F}\, U(n,n)_{/F'}$. Then we have that the embedding

$$U(n,n)_{/F} \hookrightarrow \mathrm{Res}_{F'/F}\, U(n,n)_{/F'}$$

induces an embedding $\Gamma_0(\mathfrak{b}, \mathfrak{c}) \hookrightarrow \Gamma_0(\mathfrak{b}\mathfrak{g}', \mathfrak{c}\mathfrak{g}')$. We simplify our notation by setting $\Gamma := \Gamma_0(\mathfrak{b}, \mathfrak{c})$ and $\Gamma' := \Gamma_0(\mathfrak{b}\mathfrak{g}', \mathfrak{c}\mathfrak{g}')$. Now we easily observe that Δ induces, by pull-back, a map

$$\Delta^*\colon M_k(\Gamma') \to M_{pk}(\Gamma), \quad f \mapsto f \circ \Delta.$$

Next we study the effect of this map on the q-expansion of the hermitian modular form. Namely, if we assume that $f \in M_k(\Gamma')$ has a q-expansion of the form $f(z') = \sum_{h'\in L'} c(h') e^n_{\mathbf{a}'}(h'z')$, where we recall that $L' = \mathfrak{d}'^{-1} T'$, then the following lemma provides us the q-expansion of the form $\Delta^* f$.

Lemma 5.1. — *For $f \in M_k(\Gamma')$ as above we have*

$$\Delta^* f(z) = \sum_{h\in L} \Bigg(\sum_{h'\in L', \mathrm{Tr}_{K'/K}(h')=h} c(h') \Bigg) e^n_{\mathbf{a}}(hz).$$

Proof. — We first observe that $\mathrm{Tr}_{F'/F}\colon L' \to L$, where $\mathrm{Tr}_{F'/F}(h') := \sum_{\sigma\in \mathrm{Gal}(K'/K)} (h')^\sigma$. Indeed, we recall that $L' = \mathfrak{d}'^{-1} T'$ and $L = \mathfrak{d}^{-1} T$. To see this, consider an element $d't' \in L'$ with $d' \in \mathfrak{d}'^{-1}$ and $t' \in T'$. For an element $y \in S(\mathfrak{r})$ we compute

$$\mathrm{tr}(y\, \mathrm{Tr}_{K'/K}(d't')) = \mathrm{tr}(y \sum_\sigma (d')^\sigma (t')^\sigma)) = \sum_\sigma (d' \,\mathrm{tr}(yt'))^\sigma.$$

But $S(\mathfrak{r}) \subset S(\mathfrak{r}')$ hence we have that $g' := \mathrm{tr}(yt') \in \mathfrak{g}'$. That is we have shown $\mathrm{tr}(y\,\mathrm{Tr}_{K'/K}(d't')) = \sum_\sigma (d'g')^\sigma$ and $d'g' \in \mathfrak{d}'^{-1}$. But $\mathrm{Tr}_{K'/K}(\mathfrak{d}'^{-1}) \subset \mathfrak{d}^{-1}$ hence we have shown that $\mathrm{tr}(S(\mathfrak{r})\,\mathrm{Tr}_{K'/K}(L')) \subseteq \mathfrak{d}^{-1}$ or equivalently $\mathfrak{d}\,\mathrm{Tr}_{K'/K}(L') \subset T$ which concludes our claim.

Now we consider what happens to the h'^{th} component of f after setting $t' := \Delta(z)$. We have

$$\begin{aligned}
e^n_{\mathbf{a}'}(h'\Delta(z)) &= e^{2\pi i \sum_{\sigma'\in\Sigma'}(\sum_{i,j} h'_{i,j} z_{j,i})^{\sigma'}} \\
&= e^{2\pi i \sum_{\sigma\in\Sigma'} \sum_{i,j} {h'_{i,j}}^{\sigma'} z^{\sigma'}_{j,i}} \\
&= e^{2\pi i \sum_{\sigma\in\Sigma} \sum_{\sigma'\in\Sigma',\sigma'_{|K}=\sigma} \sum_{i,j} {h'_{i,j}}^{\sigma'} z^{\sigma'}_{j,i}} \\
&= e^{2\pi i \sum_{i,j} \sum_{\sigma\in\Sigma} z^{\sigma}_{j,i} \sum_{\sigma'\in\Sigma',\sigma'_{|K}=\sigma} {h'_{i,j}}^{\sigma'}} \\
&= e^{2\pi i \sum_{i,j} \sum_{\sigma\in\Sigma} z^{\sigma}_{j,i} (\mathrm{Tr}_{K'/K}(h'_{i,j})^{\sigma}} \\
&= e^{2\pi i \sum_{\sigma\in\Sigma} (\sum_{i,j} \mathrm{Tr}_{K'/K}(h'_{i,j}) z_{j,i})^{\sigma}} \\
&= e^n_{\mathbf{a}}(\mathrm{Tr}_{K'/K}(h')z).
\end{aligned}$$

These calculations allow us to conclude the proof of the lemma. $\square$

It is easily seen that the above considerations can be generalized to more general congruent subgroups. Namely, for an element $g \in \mathrm{GL}_n(\mathbb{A}_{K,\mathbf{h}})$ we consider groups of the form

$$\Gamma_g := G_1 \cap \begin{pmatrix} \hat{g} & 0 \\ 0 & g \end{pmatrix} D[\mathfrak{b}^{-1}, \mathfrak{b}\mathfrak{c}] \begin{pmatrix} \hat{g} & 0 \\ 0 & g \end{pmatrix}^{-1}.$$

We note that for $g = 1_n$ we have $\Gamma_{1_n} = \Gamma_0(\mathfrak{b},\mathfrak{c})$. Similarly for the same $g \in \mathrm{GL}_n(\mathbb{A}_{K,f}) \subset \mathrm{GL}_n(\mathbb{A}_{K',f})$ we define

$$\Gamma'_g := G'_1 \cap \begin{pmatrix} \hat{g} & 0 \\ 0 & g \end{pmatrix} D[\mathfrak{b}^{-1}\mathfrak{g}', \mathfrak{b}\mathfrak{c}\mathfrak{g}'] \begin{pmatrix} \hat{g} & 0 \\ 0 & g \end{pmatrix}^{-1}.$$

We now observe that the embedding $U(n,n)(\mathbb{A}_F) \hookrightarrow U(n,n)(\mathbb{A}_{F'})$ induces the embeddings $D[\mathfrak{b}^{-1},\mathfrak{b}\mathfrak{c}] \hookrightarrow D[\mathfrak{b}^{-1}\mathfrak{g}',\mathfrak{b}\mathfrak{c}\mathfrak{g}']$ and $\Gamma_g \hookrightarrow \Gamma'_g$. In particular, as before, we have that the map Δ induces as before a map $M_k(\Gamma'_g) \to M_{pk}(\Gamma_g)$.

Our next goal is to understand the above analytic considerations algebraically. We start by recalling the moduli interpretation of the Shimura varieties $\Gamma_g \backslash \mathbb{H}$. We fix a K-basis $\{e_i\}_{i=1}^{2n}$ of V so that the group $U(n,n)/F$ is represented as $U(n,n)_{/F}(F) := \{\alpha \in \mathrm{GL}_{2n}(K) \mid \alpha^*\eta_n\alpha = \eta_n\}$. We consider the $\mathfrak{g}$-maximal $\mathfrak{r}$-lattice $L := \sum_{i=1}^n \mathfrak{r}e_i \oplus \sum_{j=n+1}^{2n} \mathfrak{d}^{-1}_{K/F} e_j \subset V$. Then we note that for all $\gamma \in G_1 \cap D[\mathfrak{b}^{-1}, \mathfrak{b}]$ with $\mathfrak{b} := \mathfrak{d}_{K/F}^{-1} \cap F$ we have $L\gamma = L$. Moreover, for $g \in G(\mathbb{A}_{F,f})$ and $L_g := (L \otimes_{\mathfrak{r}} \hat{\mathfrak{r}})g^{-1} \cap V$ we have $L_g\gamma = L_g$ for all $\gamma \in G_1 \cap gD[\mathfrak{b}^{-1}, \mathfrak{b}]g^{-1}$. In particular the groups Γ_g above respect the lattices L_g. Following now Shimura [42, p. 26] we recall that the space $\Gamma_g \setminus \mathbb{H}_F$ parametrizes for every $z \in \mathbb{H}_F$ families of polarized abelian varieties $\mathcal{P}_z = (A_z, \mathcal{C}_z, i_z, \alpha_{\mathfrak{c}})$, where $A_z := (\mathbb{C}^{2n})^{\mathbf{a}}/p_z(L_g)$ with p_z defined by

$$p_z \colon (K_{\mathbf{a}})^1_{2n} \to (\mathbb{C}^{2n})^{\mathbf{a}},\ x \mapsto ([z_v\ 1_n] \cdot x_v^*, [{}^t z_v\ 1_n] \cdot {}^t x_v)_{v \in \mathbf{a}}.$$

Moreover $\mathcal{C}_z$ is the polarization of A_z defined by the Riemann form $E_z(p_z(x), p_z(y)) := \mathrm{Tr}_{K_{\mathbf{a}}/\mathbb{R}}(x\eta_n y^*)$. The map $i_z \colon K \hookrightarrow \mathrm{End}_{\mathbb{Q}}(A_z)$ is by the map $\Psi \colon K \to \mathrm{End}((\mathbb{C}^{2n})^{\mathbf{a}})$ defined for $a \in K$ as $\Psi(a) := \mathrm{diag}[\Psi_v(a)]_{v \in \mathbf{a}}$ and $\Psi_v(a) := \mathrm{diag}[\overline{a_v} 1_n,\ a_v 1_n]$. Finally the arithmetic structure α_c is induced from the embedding $\mathfrak{c}^{-1}L \hookrightarrow K^{2n}$. As is explained in Shimura (*loc. cit.* p. 26) we have that two such data $\mathcal{P}_z$ and $\mathcal{P}_w$ with $z, w \in \mathbb{H}_F$ are isomorphic if and only if there exists $\gamma \in \Gamma_g$ so that $w = \gamma z$.

Now we observe that the diagonal map $\Delta \colon \mathbb{H}_F \hookrightarrow \mathbb{H}_{F'}$ introduced above induces a map $\Gamma_g \setminus \mathbb{H}_F \to \Gamma'_g \setminus \mathbb{H}_{F'}$ by $\mathcal{P}_z \mapsto \mathcal{P}'_{\Delta(z)}$ where the structures for the group $GU(n,n)/F'$ are with respect to the $\mathfrak{r}'$-lattice $L'_g := \mathfrak{r}' \otimes_{\mathfrak{r}} L_g$. We note here the crucial assumption that the ramification of F/K and F'/F is disjoint. In particular we have that $\mathfrak{d}_{K'/F'} = \mathfrak{d}_{K/F}\mathfrak{r}'$.

We now explain briefly how the analytic considerations above can be extended to their algebraic counterparts. We consider the scheme $\mathcal{M}(\Gamma_g)/R$ over some ring R, associated to the congruence subgroup Γ_g, that represents the functor $S \mapsto (A, \lambda, i, \alpha_{\mathfrak{c}})/S$ discussed in the introduction. Then the algebraic counterpart of the map above is a map $\mathcal{M}(\Gamma_g)/R \to \mathcal{M}'(\Gamma'_g)/R$ given by $(A, \lambda, i, \alpha_{\mathfrak{c}}) \mapsto (A, \lambda, i, \alpha_{\mathfrak{c}}) \otimes_{\mathfrak{r}} \mathfrak{r}'$. When $R = \mathbb{C}$ this map is the previously defined map. In particular we see that we can define the map $\Delta^* \colon M_k(\Gamma'_g) \to M_{pk}(\Gamma_g)$ algebraically by $\Delta^* f(\underline{A}, \omega) := f(\underline{A} \otimes_{\mathfrak{r}} \mathfrak{r}', \omega \otimes_{\mathfrak{r}} \mathfrak{r}')$.

Before going further and providing also the algebraic counterpart of lemma 5.1 we remark that if write $A' := A \otimes_{\mathfrak{r}} \mathfrak{r}'$, the image of the abelian variety A, then this is isogenous to $[K' \colon K]$ many copies of A. Indeed, this follows by writing the $\mathfrak{r}$-module $\mathfrak{r}'$ as a direct sum $\mathfrak{b} \oplus \bigoplus_{i=1}^{[K' \colon K]-1} \mathfrak{r}$ for some ideal $\mathfrak{b}$ of $\mathfrak{r}$.

We now consider the Mumford object associated to the standard 0-genus cusp associated to the group Γ_g. We decompose the lattice $L = \sum_{i=1}^n \mathfrak{r}e_i + \sum_{j=n+1}^{2n} \mathfrak{d}_{K/F}^{-1} e_j$ to $L^1 := \sum_{i=1}^n \mathfrak{r}e_i$ and $L^2 := \sum_{j=n+1}^{2n} \mathfrak{d}_{K/F}^{-1} e_j$. Then we see that since we consider elements of the form $\begin{pmatrix} \hat{g} & 0 \\ 0 & g \end{pmatrix}$ for the components of $GU(n,n)$, we have that $L_g = L_g^1 \oplus L_g^2$ with $L_g^1 := L^1 g^*$ and $L^2 := L^2 g^{-1}$.

The Mumford data associated to the standard 0-cusp of the g's component is then $(\mathrm{Mum}_{L_g^1,L_g^2}(q),\lambda_{\mathrm{can}},i_{\mathrm{can}},\alpha_{\mathfrak{c}}^{\mathrm{can}})$, defined over the ring $R((q,H^\vee))$, where $H^\vee=\mathfrak{d}^{-1}gTg^*\subset S$ with $T:=\{x\in S\mid \mathrm{tr}(S(\mathfrak{r})x)\subset\mathfrak{g}\}$. We now consider the ring homomorphism $\mathrm{Tr}\colon R((q',H'^\vee))\to R((q,H^\vee))$ defined by ${q'}^{h'}\mapsto q^{\mathrm{Tr}_{K'/K}(h')}$. Then we have that

$$\begin{aligned}&(\mathrm{Mum}_{L_g^1,L_g^2}(q),\lambda_{\mathrm{can}},i_{\mathrm{can}},\alpha_{\mathfrak{c}}^{\mathrm{can}})\otimes_{\mathfrak{r}}\mathfrak{r}'\\&\qquad=(\mathrm{Mum}_{{L'}_g^1,{L'}_g^2}(q'),\lambda'_{\mathrm{can}},i'_{\mathrm{can}},{\alpha'}_{\mathfrak{c}}^{\mathrm{can}})\otimes_{R((q',H'^\vee)),\mathrm{Tr}}R((q,H^\vee)).\end{aligned}$$

In particular if $f\in M_k(\Gamma'_g)$ is an algebraic hermitian form with q-expansion given by

$$f(\mathrm{Mum}_{{L'}_g^1,{L'}_g^2}(q'),\lambda'_{\mathrm{can}},i'_{\mathrm{can}},{\alpha'}_{\mathfrak{c}}^{\mathrm{can}},\omega'_{\mathrm{can}})=\sum_{h'\in H'^\vee}c(h')q^{h'},$$

then its pull-back form $g:=\Delta^*(f)$ has q-expansion

$$\begin{aligned}&g\left(\mathrm{Mum}_{L_g^1,L_g^2}(q),\lambda_{\mathrm{can}},i_{\mathrm{can}},\alpha_{\mathfrak{c}}^{\mathrm{can}},\omega_{\mathrm{can}}\right)\\&\qquad=f\left((\mathrm{Mum}_{L_g^1,L_g^2}(q),\lambda_{\mathrm{can}},i_{\mathrm{can}},\alpha_{\mathfrak{c}}^{\mathrm{can}},\omega_{\mathrm{can}})\otimes_{\mathfrak{r}}\mathfrak{r}'\right)\\&=f\left((\mathrm{Mum}_{{L'}_g^1,{L'}_g^2}(q'),\lambda'_{\mathrm{can}},i'_{\mathrm{can}},{\alpha'}_{\mathfrak{c}}^{\mathrm{can}},\omega'_{\mathrm{can}})\otimes_{R((q',H'^\vee)),\mathrm{Tr}}R((q,H^\vee))\right)\\&\times f\left((\mathrm{Mum}_{{L'}_g^1,{L'}_g^2}(q'),\lambda'_{\mathrm{can}},i'_{\mathrm{can}},{\alpha'}_{\mathfrak{c}}^{\mathrm{can}},\omega'_{\mathrm{can}}\right)\otimes_{R((q',H'^\vee)),\mathrm{Tr}}R((q,H^\vee)).\end{aligned}$$

Again with $R=\mathbb{C}$ we have the algebraic counterpart of lemma 5.1. We summarize this discussion in the following lemma.

Lemma 5.2. — *Let $f\in M_k(\Gamma'_g,R)$ be an algebraic hermitian modular form defined over R. Then the q-expansion of $g:=\Delta^*(f)\in M_{pk}(\Gamma_g,R)$ is given by*

$$g(q):=\sum_h\left(\sum_{h'\in L',\mathrm{Tr}_{K'/K}(h')=h}c(h')\right)q^h,$$

when the q-expansion of f is given by $f(q)=\sum_{h'}c(h')q^{h'}$.

For a function $\varepsilon:=\sum_j c_j\chi_j$ of $G_{K'}^{ab}$ where χ_j are characters of the form and $\chi'_j\psi$, where ψ a fixed Hecke character of infinite type $k\Sigma$ and χ'_j finite order characters we have that $E_\beta(m(A),\phi,\varepsilon)$ equals

$$Q(\beta,A,k)\sum_{(\mathfrak{a},S)=1} n_{\mathfrak{a}}(\beta,m(A))\sum_j c_j\chi_j(\det(A))\chi_j^{(S)}(\mathfrak{a}) \times\left(\prod_{v\in S^{(p)}}\chi_{v,j}(\det u^{-1})\right) \times\left(\prod_{v\in\Sigma_p}\chi_{1,v,j}^{-1}(\det(B(h_v)))\chi_{2,v,j}(\det(A(h_v)))\Phi_\mu({}^tA(h_v)\beta B(h_v)^{-1})\right).$$

Now we assume that $\varepsilon^\gamma=\varepsilon$ for all $\gamma\in\Gamma$. Since we assume that $m(A)\in \mathrm{GL}_n(\mathbb{A}_K)$ we have that

$$\chi_j(\det(A))\Bigg(\prod_{v\in\Sigma_p}\chi_{1,v,j}^{-1}(\det(B(h_v)))\chi_{2,v,j}(\det(A(h_v)))\Phi_\mu({}^tA(h_v)\beta B(h_v)^{-1})\Bigg) \times\prod_{v\in S^{(p)}}\chi_{v,j}^{-1}(\det u)=\chi_j^\gamma(\det(A)) \left(\prod_{v\in\Sigma_p}\chi_{1,v,j}^{-\gamma}(\det(B(h_v)))\chi_{2,v,j}^{\gamma}(\det(A(h_v)))\Phi_\mu({}^tA(h_v)\beta B(h_v)^{-1})\right) \times\prod_{v\in S^{(p)}}\chi_{v,j}^{-\gamma}(\det u).$$

Claim. For the function $\varepsilon^{(S)}(\mathfrak{a},\beta)$ on $G_{K'}(\mathfrak{c}p^\infty)\times\mathrm{Her}_n(F)$ defined as

$$\sum_j c_j\chi_j(\det(A)) \times\left(\prod_{v\in\Sigma_p}\chi_{1,v,j}^{-1}(\det(B(h_v)))\chi_{2,v,j}(\det(A(h_v)))\Phi_\mu({}^tA(h_v)\beta B(h_v)^{-1})\right) \times\left(\prod_{v\in S^{(p)}}\chi_{v,j}^{-1}(\det u)\right)\chi_j^{(S)}(\mathfrak{a})$$

we have

$$\varepsilon^{(S)}(\mathfrak{a},\beta)=\varepsilon^{(S)}(\mathfrak{a}^\gamma,\beta^\gamma).$$

Indeed, we notice first that $\varepsilon^\gamma=\varepsilon$ if an only if $c_j=c_{\gamma(j)}$ where $c_{\gamma(j)}$ denotes the coefficient of the character $\chi_j^\gamma=:\chi_{\gamma(j)}$ in the sum $\sum_j c_j\chi_j$. In particular that means that we may decompose the locally constant function ε as follows

$$\varepsilon=\sum_i c_i\chi_i+\sum_k c_k\sum_{\gamma\in\Gamma}\chi_k^\gamma,$$

where for the characters χ_i that appear in the first sum we have that $\chi_i^\gamma = \chi_i$ for all $\gamma \in \Gamma$. Then the claim follows from the observation above, the definition of Φ_μ and the fact that $\nu^\gamma = \nu$.

For the applications that we have in mind we need to understand how the polynomials $g_{\beta,m,v}$ depend on β, m and v. We explain this by following closely Shimura's book [40]. We start with some definitions.

First of all we need to introduce the notion of the denominator of a matrix as is defined in [40] p18. Let $\mathfrak{r}$ be a principal ideal domain and let K denote its field of quotients. We assume that $K \neq \mathfrak{r}$. We set $E = E^n = \mathrm{GL}_n(\mathfrak{r})$. Given any matrix $X \in K_n^m$ of rank r, there exist $A \in E^m$ and $B \in E^n$ and elements $e_1, \dots, e_r \in K^\times$ such that $e_{i+1}/e_i \in \mathfrak{r}$ for all $i < r$ and

$$AXB = \begin{pmatrix} D & 0_{n-r}^r \\ 0_r^{m-r} & 0_{n-r}^{m-r} \end{pmatrix}, \quad D = \mathrm{diag}[e_1, \dots, e_r].$$

The ideals $e_i\mathfrak{r}$ are uniquely determined by X and we call them the elementary divisors of X. We call an element $X \in \mathfrak{r}_n^m$ primitive if $\mathrm{rank}(X) = min(m,n)$ and the elementary divisors are all equal to $\mathfrak{r}$. Shimura shows that for any given $x \in K_n^m$ there exist $c \in \mathfrak{r}_n^m$ and $d \in \mathfrak{r}_m^m \cap \mathrm{GL}_m(K)$ such that the matrix $[c\ d]$ is primitive and $x = d^{-1}c$ and the integral ideal $\nu_0(x) := \det(d)\mathfrak{r}$ is well-defined and called the denominator ideal of x.

Now we fix a local field F, a finite extension of $\mathbb{Q}_p$ for a prime p. We write $\mathfrak{g}$ for its ring of integers and $\mathfrak{p}$ for its maximal ideal. We put $q := [\mathfrak{g} \colon \mathfrak{p}]$. We pick an additive character $\chi \colon F \to S^1$ such that $\mathfrak{g} = \{a \in F \colon \chi(a\mathfrak{g}) = 1\}$. Following Shimura as in [40] (only for the case that we will consider) we fix symbols K,$\mathfrak{r}$,$\mathfrak{q}$,$\mathfrak{d}$,δ,ρ and ε as follows

(1) if K is a quadratic extension of F then $\mathfrak{r}$ is the integral closure of $\mathfrak{g}$ in K,$\mathfrak{q}$ its maximal ideal,$\mathfrak{d}$ the different of K relative to F,$\delta \in K$ that generates the different,ρ the non-trivial element of $\mathrm{Gal}(K/F)$ and $\varepsilon = 1$,
(2) If $K = F \times F$, then $\mathfrak{r} = \mathfrak{d} = \mathfrak{g} \times \mathfrak{g}$,$\mathfrak{q} = \mathfrak{p} \times \mathfrak{p}$,,$\delta = 1$, ρ is the automorphism of K defined as $(x,y)^\rho = (y,x)$ and $\varepsilon = 1$.

We introduce the following notations

$$S = S^n(\varepsilon) = \left\{h \in K_n^n | h^* = \varepsilon h\right\},\ S(\mathfrak{a}) := S \cap (\mathfrak{r}\mathfrak{a})_n^n,$$

where $\mathfrak{a}$ is an $\mathfrak{r}$ or $\mathfrak{g}$ ideal. Also we introduce the set of matrices

$$T = T^n(\varepsilon) = \left\{x \in S^n(\varepsilon) \mid \mathrm{tr}(S(\mathfrak{r})x) \subset \mathfrak{g})\right\}.$$

Now we extend the definition of the denominator of a matrix $x \in K_n^n$ defined above for the case where K is not a field as follows. If $K = F \times F$ then for $x = (y,z) \in F_n^n$ we define $\nu_0(x) = \nu_0(y)e + \nu_0(z)e'$ where $e = (1,0)$

and $e' = (0,1)$. Then $\nu_0(x)$ is an $\mathfrak{r}$-ideal. Shimura shows in [40] that in both cases (K being a field or not) we have that $\nu_0(\sigma) = (\mathfrak{g} \cap \nu_0(\sigma))\mathfrak{r}$ for $\sigma \in S$. We then define for $\sigma \in S$ the quantity $\nu[\sigma] := N(\mathfrak{g} \cap \nu_0(\sigma))$. Given an element $\zeta \in T^n$ we consider the formal Dirichlet series

$$\alpha_\zeta(s) := \sum_{\sigma \in S/S(\mathfrak{r})} \chi(\mathrm{tr}(\zeta\sigma))\nu[\sigma]^{-s}.$$

For $\sigma \in S$ we define the non-negative integer $k(\sigma)$ by $\nu[\sigma] = q^{k(\sigma)}$ and introducing the indeterminant t we consider the formal series

$$A_\zeta(t) = \sum_{\sigma \in S/S(\mathfrak{r})} \chi(\mathrm{tr}(\zeta\sigma))t^{k(\sigma)}$$

such that $A_\zeta(q^{-s}) = \alpha_\zeta(s)$. As Shimura explains, we have for $\gamma \in \mathrm{GL}_n(\mathfrak{r})$ that $A_{(\gamma\zeta\gamma^*)}(t) = A_\zeta(t)$. Hence we may assume that ζ is equal to 0 or equal to $\mathrm{diag}[\xi \quad 0]$ for $\xi \in T^r \cap \mathrm{GL}_r(K)$ where r the rank of ζ. The following theorem is proved in [40, p. 104].

Theorem 5.3. — *Let $\zeta \in T^n$ and let r be the rank of ζ. Suppose that $\zeta = 0$ or $\zeta = \mathrm{diag}[\xi\ 0]$ with $\xi \in T^r \cap \mathrm{GL}_r(K)$. Then $A_\zeta(t) = f_\zeta(t)g_\zeta(t)$ where $g_\zeta \in \mathbb{Z}[t]$ with $g_\zeta(0) = 1$ and f_ζ a rational function given as follows:*

$$f_\zeta(t) = \frac{\prod_{i=1}^{n}(1 - \tau^{i-1}q^{i-1}t)}{\prod_{i=1}^{n-r}(1 - \tau^{n+1}q^{n+i-1}t)},$$

where

$$\tau^i := \begin{cases} 1, & \text{if } i \text{ is even or } K = F \times F \\ -1, & \text{if } i \text{ is odd, } \mathfrak{d} = \mathfrak{r}, \text{ and } K \neq F \times F \\ 0, & \text{if } i \text{ is odd and } \mathfrak{d} \neq \mathfrak{r}. \end{cases}$$

Proposition 5.4. — *We consider the polynomial $g_{\beta,m(A),v}(t) \in \mathbb{Z}[t]$ in the K'-setting i.e. $\beta \in S^n(K')$, $A \in \mathrm{GL}_n(\mathbb{A}_{f,K'})$ and v a finite place of K'. Let now $\gamma \in \Gamma = \mathrm{Gal}(K'/K)$. Then we have $g_{\beta,m(A),v}(t) = g_{\beta^\gamma,m(A^\gamma),v^\gamma}(t)$.*

Proof. — We first note that it follows from [40, p. 156] that

$$g_{\beta,m(A),v}(t) = g_\zeta(t),$$

where $\zeta := \omega_v A_v^* \beta A_v$ with ω_v a generator of $\mathfrak{d}(F'/\mathbb{Q})_v$ and $g_\zeta(t)$ is defined as above for K'_v. We consider the following two cases,

(1) *The element γ fixes v* In this case we have to show that $g_{\zeta^\gamma}(t) = g_\zeta(t)$. Since the ranks of ζ and ζ^γ are the same we have that $f_\zeta(t) =$

$f_{\zeta^\gamma}(t)$. So in order to conclude our claim it is enough to show that $A_\zeta(t) = A_{\zeta^\gamma}(t)$. By definition

$$A_{\zeta^\gamma}(t) = \sum_{\sigma \in S/S(\mathfrak{r})} \mathbf{e}_v(d_{F_v}^{-1} \operatorname{tr}(\zeta^\gamma \sigma)) t^{k(\sigma)}.$$

This implies $A_{\zeta^\gamma}(t) = \sum_{\sigma \in S/S(\mathfrak{r})} \mathbf{e}_v((d_{F_v}^{-1})^\gamma \operatorname{tr}(\zeta \sigma^{\gamma^{-1}})) t^{k(\sigma)}$. But we have $k(\sigma) = k(\sigma^{\gamma^{-1}})$ as $\nu[\sigma^{\gamma^{-1}}] = N(\mathfrak{g} \cap \nu_0(\sigma^{-1})) = N(\mathfrak{g} \cap \nu_0(\sigma)) = \nu[\sigma]$. This means

$$A_{\zeta^\gamma}(t) = \sum_{\sigma \in S/S(\mathfrak{r})} \mathbf{e}_v((d_{F_v}^{-1})^\gamma \operatorname{tr}(\zeta \sigma^{\gamma^{-1}})) t^{k(\sigma^{\gamma^{-1}})} = A_\zeta(t),$$

which allows us to conclude the proof in this case as γ^{-1} permutes the set $S/S(\mathfrak{r})$ and $A_\zeta(t)$ is independent of the additive character $\mathbf{e}_v$ picked (*i.e.* makes no difference whether we pick $\mathbf{e}_v(d_F^{-1} \cdot)$ or $\mathbf{e}_v((d_{F_v}^{-1})^\gamma \cdot)$.

(2) *The element γ does not fix v* We fix an identification of K'_v and K'_{v^γ} and write x^γ for the image in K'_{v^γ} of an element $x \in K'_v$ with respect to this identification. We write

$$g_{\beta, m(A), v}(t) = g_{\zeta_v, v}(t)$$

with $\zeta_v := \omega_v A_v^* \beta A_v$ with ω_v a generator of $\mathfrak{d}(F'/\mathbb{Q})_v$ and $g_{\zeta_v, v}(t)$ is defined for K'_v. Similarly we have

$$g_{\beta^\gamma, m(A^\gamma), v^\gamma}(t) = g_{\zeta_{v^\gamma}, v^\gamma}(t)$$

with $\zeta_{v^\gamma} := \omega_{v^\gamma} A_v^{\gamma *} \beta^\gamma A_v^\gamma$ with ω_{v^γ} a generator of $\mathfrak{d}(F'/\mathbb{Q})_{v^\gamma}$ and $g_{\zeta_{v^\gamma}, v^\gamma}(t)$ is defined for K'_{v^γ}. We need to show that $g_{\zeta_v, v}(t) = g_{\zeta_{v^\gamma}, v^\gamma}(t)$. But the rank of ζ_v is equal to the rank of ζ_{v^γ} and hence $f_{\zeta_v, v}(t) = f_{\zeta_{v^\gamma}, v^\gamma}(t)$. So it is enough to show that $A_{\zeta_v}(t) = A_{\zeta_{v^\gamma}}(t)$, which follows from the identification of K'_v with K'_{v^γ}.

□

Now we can prove the following proposition.

Proposition 5.5. — *Let $m = m(A)$ with A in $\mathrm{GL}_n(\mathbb{A}_{K'}^{(p)}) \times \mathrm{GL}_n(\mathfrak{r}' \otimes \mathbb{Z}_p)$ be an element in the Levi component of P. Then for all $\gamma \in \Gamma$ we have,*

$$n_{\mathfrak{a}^\gamma}(\beta^\gamma, m(A^\gamma)) = n_{\mathfrak{a}}(\beta, m(A)).$$

Proof. — Let as write $\mathfrak{q}$ for the prime ideal of K' that corresponds to the place v of K'. Then we note that $n_{\mathfrak{q}^m}(\beta, m(A))$ is the m-th power coefficient of the polynomial $g_{\zeta, v}(t)$ with $\zeta := \omega_v A_v^* \beta A_v$ with ω_v a generator of $\mathfrak{d}(F'/\mathbb{Q})_v$. Moreover from its very definition we have that $n_{\mathfrak{a}}(\beta, m(A)) =$

$\prod_j n_{\mathfrak{q}_j^{m_j}}(\beta, m(A))$ for $\mathfrak{a} = \prod_j \mathfrak{q}_j^{m_j}$. But that means that we need to prove the statement for $\mathfrak{a}$ a powers of a prime ideal $\mathfrak{q}$, that is to show $n_{\mathfrak{q}^m}(\beta, m(A)) = n_{\mathfrak{q}^{\gamma m}}(\beta^\gamma, m(A^\gamma))$. But this follows directly from the previous proposition. $\square$

We need in addition to understand how the coefficients of the polynomials $g_{\beta,m(A),v}$ behave with respect to K' and K. We have the following proposition.

Proposition 5.6. — *Let $\beta \in S_K$ be positive definite and $A \in \mathrm{GL}_n(\mathbb{A}_K)$. Then we have the congruences*

$$n_{\mathfrak{q}^j}(\beta, m(A)) \equiv n'_{\mathfrak{q}'^j}(\beta, m(A)) \mod p,$$

where $\mathfrak{q}' := \mathfrak{q}O_{F'}$ for every prime ideal $\mathfrak{q}$ of F that is not in S.

Proof. — We consider the case where $\mathfrak{q}$ splits in F' and where it inerts.

$\mathfrak{q}$ splits in F'. We start with the splitting case. We write $\mathfrak{q}' = \prod_{i=1}^p \mathfrak{q}_i$ for the ideals above $\mathfrak{q}$ and v'_i for the corresponding places. Then by the considerations above (as β, A are coming from K) we have that $g_{\beta,m(A),v'_i}(t) = g_{\beta,m(A),v}(t)$. Hence in particular we conclude that

$$n_{\mathfrak{q}^j}(\beta, m(A)) = n_{\mathfrak{q}_i^j}(\beta, m(A)) \text{ for all } i.$$

But then

$$\begin{aligned} n_{\mathfrak{q}'^j}(\beta, m(A)) &= \prod_i n_{\mathfrak{q}_i^j}(\beta, m(A)) = n_{\mathfrak{q}_i^j}(\beta, m(A))^p \\ &= n_{\mathfrak{q}^j}(\beta, m(A))^p \equiv n_{\mathfrak{q}^j}(\beta, m(A)) \mod p. \end{aligned}$$

Hence we conclude the congruences in this case.

$\mathfrak{q}$ inerts in F'. As before we write v for the place of F that corresponds to $\mathfrak{q}$ and v' for the one that corresponds to $\mathfrak{q}'$, Moreover only for this proof we set $F := F_v$ and $F' := F'_{v'}$ and hence $[F' \colon F] = p$ and it is an unramified extension, since we assume that v is not a bad place. We first show that

$$f_\zeta(t) \equiv f'_\zeta(t) \mod p,$$

for $\zeta \in S_K \subset S_{K'}$ and of full rank n. We note that for the cases that we consider this is always the case as β is always a positive definite hermitian matrix. Indeed in this case we have that

$$f'_\zeta(t)=\prod_{i=1}^{n}(1-\tau^{i-1}{q'}^{i-1}t)=\prod_{i=1}^{n}(1-\tau^{i-1}q^{p(i-1)}t)$$
$$\equiv\prod_{i=1}^{n}(1-\tau^{i-1}q^{i-1}t)=f_\zeta(t)\ (p)$$

as $q'=q^p$. That is $\tilde{f}_\zeta(t)=\tilde{f}'_\zeta(t)\in\mathbb{F}_p[t]$ where tilde indicates reduction modulo p. Now we claim that also

$$A_\zeta(t)\equiv A'_\zeta(t)\mod p.$$

We note that in the case that we consider with ζ of full rank n these are polynomials in $\mathbb{Z}[t]$. Recalling the definitions we have

$$A'_\zeta(t)=\sum_{\sigma'\in S'/S'(\mathfrak{r}')}\mathbf{e}_{v'}(d_{F'}^{-1}\operatorname{tr}(\zeta\sigma'))t^{k[\sigma']}.$$

But then as $\zeta\in S_K$ and since we may pick $d_{F'}=d_F\in F\subset F'$ we have that $\mathbf{e}_{v'}(d_{F'}^{-1}\operatorname{tr}(\zeta\sigma'))$ equals

$$\mathbf{e}_{v'}(d_{F'}^{-1}\sum_{i,j}(\zeta_{ij}\sigma'_{ji}))=\mathbf{e}_{v}(d_{F}^{-1}\sum_{i,j}(\zeta_{ij}\operatorname{Tr}_{K'/K}(\sigma'_{ji})))$$
$$=\mathbf{e}_{v}(d_{F}^{-1}\operatorname{tr}(\zeta\operatorname{Tr}_{K'/K}(\sigma'))).$$

Hence we have

$$A'_\zeta(t)=\sum_{\sigma'\in S'/S'(\mathfrak{r}')}\mathbf{e}_{v}(d_{F}^{-1}\operatorname{tr}(\zeta\operatorname{Tr}_{K'/K}(\sigma')))t^{k[\sigma']}.$$

But then since $\operatorname{Tr}_{K'/K}(\sigma')=\operatorname{Tr}_{K'/K}({\sigma'}^{\gamma})$ and $k[\sigma']=k[{\sigma'}^{\gamma}]$ for $\gamma\in\operatorname{Gal}(K'/K)$ we have that

$$A'_\zeta(t)\equiv\sum_{\sigma\in S/S(\mathfrak{r})}\mathbf{e}_{v}(d_{F}^{-1}\operatorname{tr}(\zeta\operatorname{Tr}_{K'/K}(\sigma)))t^{k[\sigma]}\mod p$$

after collecting the γ orbits of order p. The last sum is equal now to

$$\sum_{\sigma\in S/S(\mathfrak{r})}\mathbf{e}_{v}(pd_{F}^{-1}\operatorname{tr}(\zeta\sigma))t^{k[\sigma]}=A_\zeta(t),$$

where the last equality follows from the fact that p is a unit in $\mathfrak{r}$ (recall that we consider places not in S and p is in S) and moreover the $A_\zeta(t)$ is independent of the character $\mathbf{e}_v$ used [40, p. 104]. That is we conclude that $\tilde{A}'_\zeta(t)=\tilde{A}_\zeta(t)$ as polynomials in $\mathbb{F}_p[t]$. Hence we obtain also that $g_\zeta(t)\equiv g'_\zeta(t)\mod p$, which concludes the proposition also in that case. □

Theorem 5.7 (Congruences of Eisenstein series of $U(n,n)$). — *Let $m = m(A)$ with A in $\mathrm{GL}_n(\mathbb{A}_K^{(p)}) \times \mathrm{GL}_n(\mathfrak{r} \otimes \mathbb{Z}_p)$ be an element in the Levi component of P. Let ε be a locally constant $\mathbb{Z}_p$-valued function on $G_{K'}(\mathfrak{c}p^\infty)$ with $\varepsilon^\gamma = \varepsilon$ for all $\gamma \in \Gamma$. Let also ψ be a Hecke character of K of infinite type $k\Sigma$ and let $\psi' := \psi \circ N_{K'/K}$. Then we have the congruences of Eisenstein series:*

$$\mathrm{Res}_K^{K'}(E'(z, m(A), \varepsilon\psi', \nu')) \equiv \mathrm{Frob}_p(E(z, m(A), \varepsilon \circ \mathrm{ver}\, \psi^p, \nu^p)) \mod p.$$

Here the Eisenstein series $E(z, m(A), \varepsilon \circ \mathrm{ver}\, \psi^p, \nu^p)$ is taken of weight $(pk, p\ell)$ when the Eisenstein series $E'(z, m(A), \varepsilon\psi', \nu')$ is of weight (k, ℓ).

Proof. — If we write the Fourier expansion of $E'_k(z, m(A), \psi, \nu')$ as

$$E(z, m(A), \varepsilon\psi', \nu') = \sum_{\beta' \in S'_+(A)} E'_{k,\beta'}(m(A), \varepsilon\psi', \nu') q^{\beta'},$$

then we have seen that

$$\mathrm{Res}_K^{K'}(E'(z, m(A), \varepsilon\psi', \nu')) = \sum_{\beta \in S(A)} \left(\sum_{\mathrm{Tr}_{K'/K}(\beta') = \beta} E'_{\beta'}(m(A), \varepsilon\psi', \nu') \right) q^\beta.$$

The group $\Gamma = \mathrm{Gal}(K'/K)$ operates on the inner sum. In particular we recall that if the write the function $\varepsilon\psi'$ as a finite sum of characters $\varepsilon\psi' = \sum_j c_j \chi_j$ then

$$E'_{\beta'}(m(A), \varepsilon\psi', \nu') = Q(\beta', A, k, \nu') \sum_{(\mathfrak{a}, S) = 1} n'_{\mathfrak{a}}(\beta', m(A)) \varepsilon\psi'^{(S)}(\mathfrak{a}, \beta'),$$

where

$$\begin{aligned} \varepsilon\psi'^{(S)}(\mathfrak{a}, \beta') = \sum_j c_j \chi_j(\det(A)) \chi_j^{(S)}(\mathfrak{a}) \left(\prod_{v \in S^{(p)}} \chi_{v,j}^{-1}(\det u) \right) \\ \times \left(\prod_{v \in \Sigma_p} \chi_{1,v,j}(\det(B(h_v))) \chi_{2,v,j}^{-1}(\det(A(h_v))) \Phi_\mu({}^tA(h_v)\beta B(h_v)^{-1}) \right). \end{aligned}$$

The group Γ operates on the pairs $(\beta', \mathfrak{a})$. From Proposition 5.6 we have that $n'_{\mathfrak{a}}(\beta', m(A)) = n'_{\mathfrak{a}^\gamma}(\beta'^\gamma, m(A))$. In particular since we assume that $\psi^\gamma = \psi$ we have seen that it implies that $\varepsilon\psi'^{(S)}(\mathfrak{a}^\gamma, \beta'^\gamma) = \varepsilon\psi'^{(S)}(\mathfrak{a}, \beta')$ and as it easily seen that $Q(\beta'^\gamma, A, k, \nu') = Q(\beta', A, k, \nu')$ we have that if the pair $(\beta', \mathfrak{a})$ is not fixed by $\gamma \in \Gamma$ then

$$\sum_{\gamma \in \Gamma} Q(\beta'^\gamma, A, k, \nu') n'_{\mathfrak{a}^\gamma}(\beta'^\gamma, m(A)) \varepsilon\psi'^{(S)}(\mathfrak{a}^\gamma, \beta'^\gamma) \equiv 0 \mod p.$$

In particular that means that modulo p we have the congruences

$$\mathrm{Res}^{K'}_{K}(E'(z,m(A),\varepsilon\psi',\nu')) \equiv \sum_{\beta\in S_+(A)}\left(Q(\beta,A,k,\nu')\sum_{(\mathfrak{a},S)=1,\mathfrak{a}\subset K} n'_{\mathfrak{a}}(\beta,m(A))\varepsilon{\psi'}^{(S)}(\mathfrak{a},\beta)\right)q^{p\beta} \mod p.$$

On the other hand we have

$$\mathrm{Frob}_p(E(z,m(A),\varepsilon\circ\mathrm{ver}\,\psi^p,\nu^p)) = \sum_{\beta\in S_+(A)} E_{pk,\beta}(m(A),\varepsilon\circ\mathrm{ver}\,\psi^p,\nu^p)q^{p\beta}.$$

Hence to conclude the congruences we have to show that

$$Q(\beta,A,k,\nu')\sum_{(\mathfrak{a},S)=1,\mathfrak{a}\subset K} n'_{\mathfrak{a}}(\beta,m(A))\varepsilon{\psi'}^{(S)}(\mathfrak{a},\beta) \equiv E_\beta(m(A),\varepsilon\circ\mathrm{ver}\,\psi^p,\nu^p).$$

We recall that (note that $\psi^p=\psi'\circ\mathrm{ver}$),

$$E_\beta(m(A),\varepsilon\circ\mathrm{ver}\,\psi^p,\nu^p) = Q(\beta,A,pk,\nu^p)\sum_{(\mathfrak{a},S)=1} n_{\mathfrak{a}}(\beta,m(A))\varepsilon{\psi'}^{(S)}(\mathrm{ver}(\mathfrak{a}),\beta).$$

But $\varepsilon{\psi'}^{(S)}(\mathrm{ver}(\mathfrak{a}),\beta) = \varepsilon{\psi'}^{(S)}(\mathfrak{a}O_{K'},\beta)$ and by Proposition 5.5 we have that $n_{\mathfrak{a}}(\beta,m(A)) = n'_{\mathfrak{a}}(\beta,m(A))$ for $\mathfrak{a}$ an ideal of K. Finally we observe that

$$Q(\beta,A,pk,\nu^p)\equiv Q(\beta,A,k,\nu') \mod p,$$

which allows us to conclude the proof of the theorem. □

Corollary 5.8. — *With the assumptions and notations as in the theorem above we have for every $a\in G^\theta(\mathbb{A}_{F,\mathbf{h}})$ that*

$$\mathrm{Res}^{K'}_{K}({E'}^{(a')}(z,m(A),\varepsilon\psi',\nu')) \equiv \mathrm{Frob}_p(E^{(a)}(z,m(A),\varepsilon\circ\mathrm{ver}\,\psi^p,\nu^p)) \mod p,$$

where $a':=\imath(a)$ under the natural embedding $\imath\colon G^\theta(\mathbb{A}_{F,\mathbf{h}})\hookrightarrow G^\theta(\mathbb{A}_{F',\mathbf{h}})$.

Proof. — The corollary follows directly from the theorem and the definition of the twisted Eisenstein series after observing that for a locally constant function ε with $\varepsilon^\gamma=\varepsilon$ also its twist $\varepsilon_y(x):=\varepsilon(xy)$ by a Γ-invariant ideal y in F is again a locally constant Γ-invariant function as we have

$$\varepsilon_y(x)^\gamma=\varepsilon_y(x^\gamma)=\varepsilon(x^\gamma y)=\varepsilon((xy)^\gamma)=\varepsilon(xy)=\varepsilon_y(x)$$

. □

Here we make a remark for the case of $\ell\neq 0$. Note that in the definition of the Eisenstein series $E^{(a)}(z,m(A),\varepsilon\circ\mathrm{ver}\,\psi,\nu)$ there is always the extra parameter ℓ. Let us write $E^{(a)}(z,m(A),\varepsilon\circ\mathrm{ver}\,\psi,\nu,\ell)$ in order to demonstrate the dependence on ℓ. Then for an $\mathbb{Z}_p$-valued locally constant function (ψ is always assumed $\mathbb{Z}_p$ valued) we have following congruences modulo p

$$\begin{aligned}E^{(a)}(z,m(A),\varepsilon\psi^p,\nu,p\ell) &\equiv E^{(a)}(z,m(A),\varepsilon\psi^p,\nu,\ell)\\ &\equiv E^{(a)}(z,m(A),\varepsilon\psi,\nu,\ell).\end{aligned}$$

This follows immediately from the q-expansion by observing that $\psi^p\equiv\psi \mod p$ and than the weight of the Eisenstein series appears only as power of the norm $N_{K/\mathbb{Q}}$ map which is $\mathbb{Z}_p$-valued and hence we have $N_{K/\mathbb{Q}}^{pm}\equiv N_{K/\mathbb{Q}}^{m} \mod p$, for any $m\in\mathbb{N}$.

A relation between archimedean and p-adic periods. For the relative setting that we consider, that is (K,Σ) and (K',Σ') as well as the CM algebras Y and Y' we have the following relation between the periods.

LEMMA 5.9. — *We have the equalities*

$$\Omega_\infty(Y',\Sigma')=\Omega_\infty(Y,\Sigma)^p,\quad \textit{and}\quad \Omega_p(Y',\Sigma')=\Omega_p(Y,\Sigma)^p$$

Proof. — We have seen in lemma 4.11 that

$$\Omega_\infty(Y',\Sigma')=\Omega_\infty(\Sigma')^n,\quad \text{and}\quad \Omega_\infty(Y,\Sigma)=\Omega_\infty(\Sigma)^n.$$

But we have that $\Omega_\infty(\Sigma')=\Omega_\infty(\Sigma)^p$, from where the equality for the archimedean periods follows. The same argument shows the equality for the p-adic periods. □

6. The theory of complex multiplication

The formalism of CM points for unitary groups and the reciprocity law. We start by recalling the notion of CM points on the symmetric space associated to the unitary group $G:=U(n,n)/F$. We will follow the books of Shimura [42, 41]. Let us write $r:=2n$. We consider the CM algebra $Y:=K_1\oplus\cdots\oplus K_t$ with CM fields K_i such that $K\subseteq K_i$ (later we just pick $K_i=K$). Let us denote by F_i the maximal real subfield of K_i and by ρ the automorphism of Y which induces the non-trivial element of $\mathrm{Gal}(K_i/F_i)$ for every i. Let us assume that we can find a K-linear ring injection $h\colon Y\to K_r^r$ such that, $h(a^\rho)=\eta_n h(a)^\star\eta_n^{-1},\quad a\in Y$. We put $Y^u=\{a\in Y\mid aa^\rho=1\}$. Then we have $h(Y^u)\subset G(F)$. But Y^u is contained in a compact subgroup of $(Y\otimes_{\mathbb{Q}}\mathbb{R})^\times$ hence the projection of $h(Y^u)$ to $G_{\mathbf{a}}$ is contained in a compact subgroup of $G_{\mathbf{a}}$ hence $h(Y^u)$ has a common fixed point in $\mathbb{H}_n$, and it can be shown that actually there is a unique one. A point of $\mathbb{H}_n$ obtained in this way is called a CM point. The case that we are mostly interested in is when $Y = K \oplus \cdots \oplus K$, r copies of K. The CM points obtained from this CM algebra correspond to an abelian varieties with multiplication by Y and of dimension exactly $[Y\colon \mathbb{Q}]$. We note also here that if $(A, \imath)$ is an abelian variety A with multiplication by Y *i.e.* $i\colon Y \hookrightarrow \mathrm{End}(A)_{\mathbb{Q}}$ and $2\dim A = [Y\colon \mathbb{Q}] = r[K\colon \mathbb{Q}]$, then A is isogenous to a product $A_1 \times \cdots \times A_r$ with $\imath_i\colon K \hookrightarrow \mathrm{End}(A_i)_{\mathbb{Q}}$ and $[K\colon \mathbb{Q}] = 2\dim A_i$.

Shimura's Reciprocity Law (for CM algebras). We consider the CM-algebra $Y = K_1 \oplus \cdots \oplus K_t$, where the K_i's are CM fields. We consider an abelian variety (A, λ) with CM by Y. As it is explained in Shimura [41, p. 129] we have that A is isogenous to $A_1 \times \cdots \times A_t$ where A_i is an abelian variety with CM by K_i and $2\dim(A_i) = [K_i\colon \mathbb{Q}]$. Let us write the type of the A_i variety as (K_i, Σ_i). Then we have that the type Ψ of A is the direct sum of the Φ_i's in the way explained in Shimura (*loc. cit.*). Let (K_i^*, Φ_i^*) be the reflex field of (K_i, Φ_i) and let K^* be the composite of the K_i^*'s. As is explained in Shimura (*loc. cit.*) we have a map $g\colon (K^*)^\times \to Y^\times$, which extends to a map $g\colon (K^*)^\times_{\mathbb{A}} \to Y^\times_{\mathbb{A}}$. In Shimura [41, p. 125 and p. 130] the following theorem is proved.

Theorem 6.1. — *Let $\mathcal{P} = (A, \lambda, \imath)$ be a structure of type $\Omega = (Y, \Psi, \mathfrak{a}, \zeta)$ and let K^* as above. Further let σ be an element of $\mathrm{Aut}(\mathbb{C}/K^*)$, and s an element of $(K^*)^\times_{\mathbb{A}}$ such that $\sigma_{|_{K^*_{ab}}} = [s, K^*]$ Then there exists an exact sequence*

$$0 \to q(g(s)^{-1}\mathfrak{a}) \to \mathbb{C}^n \xrightarrow{\xi'} A^\sigma \to 0$$

with the following properties

(1) *$\mathcal{P}^\sigma$ is of type $(Y, \Psi, g(s)^{-1}\mathfrak{a}, \zeta')$ with $\zeta' = N(s\mathfrak{r})\zeta$ with respect to ξ', where $\mathfrak{r}$ is the maximal order of K^*.*

(2) *$\xi(q(w))^\sigma = \xi'(q(g(s)^{-1}w))$, where ξ is such that*

$$0 \to q(\mathfrak{a}) \to \mathbb{C}^n \xrightarrow{\xi} A \to 0.$$

Using the Theory of Complex Multiplication. Now we explain how we can use the theory of complex multiplication to understand how Frobenius operates on values of Eisenstein series at CM points. In this section we prove the following proposition, which is just a reformulation of what is done in [34] (p. 539) in the case of quadratic imaginary fields and the group GL_2. This proposition has also been proved by Ellen Eischen in [13, section 5.2].

We first recall some of the assumptions that we have made. Recall that we consider a CM type (K,Σ) such that (i) p is unramified in F, where F the totally real field K^+, (ii) the ordinary condition is that all primes above p in F are split in K and (iii) that for $\mathfrak{p}$ in K^* above p we have that $N\mathfrak{p}=p$. We write $\Phi_{\mathfrak{p}}$ for the Frobenius element in $\mathrm{Gal}(K^*_{ab}/K^*)$ corresponding to the prime ideal $\mathfrak{p}$ of K^* through Artin's reciprocity law.

Proposition 6.2 (Reciprocity law on CM points). — *Consider the $\mathfrak{g}$-lattice $\mathfrak{U}$ of the CM algebra Y and the tuple $(\underline{X(\mathfrak{U})},\omega(\mathfrak{U}))$ defined over K^*_{ab}. Let E be a hermitian form defined over $\mathbb{Q}_{ab}$. Then we have the reciprocity law:*

$$\mathrm{Frob}_p(E)(\underline{X(\mathfrak{U})},\omega(\mathfrak{U}))=(E^{\Phi_{\mathfrak{p}}^{-1}}(\underline{X(\mathfrak{U})},\omega(\mathfrak{U})))^{\Phi_{\mathfrak{p}}}. \tag{6.1}$$

In particular if E is a hermitian form defined over K^* then we have

$$\mathrm{Frob}_p(E)(\underline{X(\mathfrak{U})},\omega(\mathfrak{U}))=(E(\underline{X(\mathfrak{U})},\omega(\mathfrak{U})))^{\Phi_{\mathfrak{p}}}. \tag{6.2}$$

Proof. — From the compatibility of hermitian modular forms with base extensions we have that

$$(E(\underline{X(\mathfrak{U})},\omega(\mathfrak{U})))^{\Phi_{\mathfrak{p}}}=E^{\Phi_{\mathfrak{p}}}((\underline{X(\mathfrak{U})},\omega(\mathfrak{U}))\otimes_{K^*_{ab},\Phi_{\mathfrak{p}}}K^*_{ab}), \tag{6.3}$$

where the tensor product is with respect to the map $\Phi_{\mathfrak{p}}: K^*_{ab}\to K^*_{ab}$, *i.e.* the base change of the tuple $(\underline{X(\mathfrak{U})},\omega(\mathfrak{U}))$ with respect to the Frobenius map. But then, from the theory of complex multiplication explained above and our assumptions on $\mathfrak{p}$ we have that

$$(\underline{X(\mathfrak{U})},\omega(\mathfrak{U}))\otimes_{K^*_{ab},\Phi_{\mathfrak{p}}}K^*_{ab}\cong(\underline{X'(\mathfrak{U})},\omega'(\mathfrak{U})), \tag{6.4}$$

where $(\underline{X'(\mathfrak{U})},\omega'(\mathfrak{U}))$ is the $g(\mathfrak{p})$-transform of $(\underline{X(\mathfrak{U})},\omega(\mathfrak{U}))$. We notice that $X'(\mathfrak{U})=X(g(\mathfrak{p})^{-1}\mathfrak{U})=X(\mathfrak{U})/H_{\mathrm{can}}$, where $H_{\mathrm{can}}:=i(M^0\otimes\mu_p)$ and i the p-numerical structure (see also [13, p. 45] or [35, p. 222]). Moreover, we have that the Mumford object $(\underline{\mathrm{Mum}(q)},\omega'_{\mathrm{can}})$ is obtained from $(\underline{\mathrm{Mum}(q)},\omega_{\mathrm{can}})$ by the map $q\mapsto q^p$ (see [13, pp. 46–47], from which we conclude the proposition. ☐

The relation of CM points with respect to the diagonal map. For this section we write G for the unitary group U^θ, with θ a definite hermitian form and G' for $\mathrm{Res}_{F'/F}U^\theta/F'$. Then for an integral ideal $\mathfrak{c}$ of $\mathfrak{g}$ we have defined the open compact subgroup $D(\mathfrak{c})\subset G(\mathbb{A}_{F,f})$ and the open compact subgroup $D(\mathfrak{c}')\subset G(\mathbb{A}_{F',f}=G'(\mathbb{A}_{F,f})$, where $\mathfrak{c}':=\mathfrak{c}\mathfrak{g}$. Then we have defined the finite sets $\mathcal{B}_K:=G(F)\smallsetminus G(\mathbb{A}_{F,f})/D(\mathfrak{c})$ and $\mathcal{B}_{K'}:=G(F')\smallsetminus G(\mathbb{A}_{F',f})/D(\mathfrak{c}')$. We write Γ for $\mathrm{Gal}(F'/F)$ and consider its action on $G(\mathbb{A}_{F',f})$. Then we note that $D(\mathfrak{c}')^\Gamma=D(\mathfrak{c})$ and also that this action induces an action of Γ on $\mathcal{B}_{K'}$. Moreover the natural inclusion $F \hookrightarrow F'$ induces a map $\imath\colon \mathcal{B}_K \to \mathcal{B}_{K'}$. We now examine the conditions under which the map $\imath\colon \mathcal{B}_K \to \mathcal{B}_{K'}^{\Gamma}$ is a bijection. The proof of the following proposition was inspired from a similar proof of Hida in [28].

Proposition 6.3. — *Assume that there exist a prime ideal $\mathfrak{q}$ of F such that*

(1) *If we write $q := \mathfrak{q} \cap \mathbb{Q}$ for the prime below $\mathfrak{q}$ in q and e for the ramification index of $\mathfrak{q}$ over q, then $\mathfrak{q}^{\nu}|\mathfrak{c}$ for some $\nu \geqslant (e+1)/(q-1)$,*
(2) *The extension F'/F is not ramified at $\mathfrak{q}$.*

Then the canonical map $\imath\colon \mathcal{B}_K \to \mathcal{B}_{K'}^{\Gamma}$ is a bijection.

Proof. — We recall that the sets $\mathcal{B}_K$ and $\mathcal{B}_{K'}$ are defined as $\mathcal{B}_K = G^{\theta}(F) \setminus G^{\theta}(\mathbb{A}_{F,f})/D(\mathfrak{c})$ and similarly $\mathcal{B}_{K'} := G^{\theta}(F') \setminus G^{\theta}(\mathbb{A}_{F',f})/D(\mathfrak{c}')$, with $\mathfrak{c}' = \mathfrak{c}\mathfrak{r}'$. The conditions above imply (see [40, p. 201, remark 2] that the groups $D(\mathfrak{c})$ are sufficiently small, that is we have for every $\alpha \in G^{\theta}(\mathbb{A}_{F,f})$ (resp. $\beta \in G^{\theta}(\mathbb{A}_{F',f})$) that $G^{\theta}(F) \cap \alpha D(\mathfrak{c})\alpha^{-1} = \{1\}$ (resp. $G^{\theta}(F') \cap \beta D(\mathfrak{c}')\beta^{-1} = \{1\}$). Now we are ready to prove the injectivity.

Assume that $\imath(x) = \imath(x')$ for $x, x' \in \mathcal{B}_K$. Then there exists $\gamma \in G^{\theta}(F')$ and $d \in D(\mathfrak{c}')$ such that $gx = x'd$. This implies, that for all $\gamma \in \Gamma(F'/F)$ that $g^{\gamma}x = x'd^{\gamma}$. In particular we conclude that $g^{\gamma-1} = xD(\mathfrak{c}')x^{-1} \cap G(F') = \{1\}$. Hence, we obtain that $g \in G(F)$ and similarly that $d \in D(\mathfrak{c}') \cap G(\mathbb{A}_{F,f}) = D(\mathfrak{c})$. Hence $x = x'$ in $\mathcal{B}_K$.

Next we prove the surjectivity of the map $\imath$. Let $x \in G(\mathbb{A}_{F',f})/D(\mathfrak{c}')$. Then for $\gamma \in \Gamma$ we define $g_{\gamma} \in G(F')$ by $g_{\gamma}x = x^{\gamma}$. Then for $\gamma_1, \gamma_2 \in \Gamma$ we have

$$g_{\gamma_1\gamma_2}x = x^{\gamma_1\gamma_2} = (x_1^{\gamma})^{\gamma_2} = (g_{\gamma_1}x)^{\gamma_2} = g_{\gamma_1}^{\gamma_2}g_{\gamma_2}x.$$

Under the conditions of the lemma we have that the stabilizer of $G(\mathbb{A}_{F',f})/D(\mathfrak{c}')$ in $G(F')$ is trivial. That is we have that $g_{\gamma_1\gamma_2} = g_{\gamma_1}^{\gamma_2}g_{\gamma_2}$. That is $\gamma \mapsto g_{\gamma}$ gives an element in $H^1(\Gamma, G(F'))$. As we will show in the next proposition, we have that $H^1(\Gamma, G(F')) = \{1\}$, *i.e.* it is trivial. Granted this, we then can find a $b \in G(F')$ so that $g_{\gamma} = b^{\gamma}/b$. That means, $b^{\gamma}b^{-1}x = x^{\gamma}$ and hence $b^{-1}x \in G(\mathbb{A}_{F,f})$. This in turn implies the surjectivity of the map $\imath$. $\square$

Proposition 6.4. — *Let $\Delta :=$ Gal(F'/F) be the Galois group of a totally real field extension and assume that $(2, |\Delta|) = 1$. Consider G, a unitary group over F (with CM field K), and write G' for the base changed to F' unitary group. Then the first non-abelian cohomology group is trivial, that is $H^1(\Delta, G'(F)) = H^1(\Delta, G(F') = 1$.*

Before we start with the proof of the above proposition we recall the following Hasse principle for unitary groups (see [20] for more details). We introduce some notation first. Let K/F be a quadratic extension of p-adic fields. As it is explained in [40, p. 30 and p. 56] for each even n there exits, up to isomorphism, exactly two n-dimensional hermitian spaces. The unitary group $U(V^+)$ corresponding to the one of them is quasi-split, it is associated to the hermitian space with maximal isotropic space of dimension $n/2$. We write $U(V^-)$ for the other one. It corresponds to the hermitian space with an anisotropic subspace of dimension 2 over K. For n odd there is only one isomorphic class of unitary groups for K/F. For a hermitian space V we define $\varepsilon(V) = \pm 1$ if $\dim_K(V)$ is even and $V \cong V^{\pm}$ and $\varepsilon(V) = 1$ if $\dim_K(V)$ is odd. Now we consider the archimidean case. We pick complex hermitian space (V, ϕ). If $\dim_{\mathbb{C}} V$ is odd we set $\varepsilon(V) = 1$. If $\dim_{\mathbb{C}} V$ is even, then if we write $U(V) \cong U(p,q)$, we set $\varepsilon(V) = (-1)^{\frac{n}{2}-p}$.

We turn now to global considerations. We consider a totally real field F and totally imaginary quadratic extension K. Then we have the following well known result (see [20]).

Theorem 6.5. — *Let n be a natural number. For every place v of F that is not split in K choose a hermitian space V_v of dimension n associated to the extension K_v/F_v such that if we write G_v for the corresponding unitary group and define $\varepsilon_v(G_v) := \varepsilon_v(V_v)$, then $\varepsilon(G_v) = 1$ for almost all v. Then, there exists a unitary group G over F such that for each place v of F $G \otimes_F F_v \cong G_v$ if and only if $\prod_v \varepsilon_v(G_v) = 1$.*

We note that the condition is trivial if n is odd. Before we start with the proof of the Proposition 6.4 we need one more lemma. That is,

Lemma 6.6. — *Let F'/F be a finite Galois extension of p-adic fields such that $(|G|, 2) = 1$ for $G := G(F'/F)$. Let (V, ϕ) be a hermitian form over K/F and write (V', ϕ') for the base-changed hermitian form to F'. Then we have $\varepsilon(V) = \varepsilon(V')$.*

Proof. — The statement is clear if n is odd. So we are left with the situation where n is even. Now we can reduce everything to the case $n = 2$. Indeed, by definition, $\varepsilon(V) = -1$ if V has an anisotropic space of dimension two and $\varepsilon(V) = 1$ if there is none. That means in order to prove the lemma, we need to show that in our situation a (two-dimensional) anisotropic hermitian (V, ϕ) over F remains anisotropic after base change to F'. But we can study this question by study the same question for quaternion algebras (see [40, pp. 24–25], that is, if we write B/F for the corresponding to V division algebra (since V is anisotropic), then the base changed quaternion algebra $B' := B \otimes_F F'$ is a division algebra. But we know that F' will split B, that is B' is not a division algebra if and only if there is an F-algebra A that is similar to B, contains F' and $[A\colon F'] = [F'\colon F]^2$. But that means that $A \cong M_m(B)$ for some m, and hence $[A\colon F] = 4m$. But $[A\colon F] = [A\colon F'][F'\colon F] = [F'\colon F]^3$. Since we assume that $([F'\colon F], 2) = 1$ we conclude the proof of the lemma. □

Proof of Proposition 6.4. — Let us write $(V, \phi)/K$ for the hermitian space over K that correspond to the group $G(F)$. Then the space $(V' := V \otimes_K K', \phi' := \phi_K \otimes K')$ correspond to the group $G(F')$. Then we know that the group $H^1(\Gamma, G(F'))$ classifies classes over K of hermitian forms $(W, \theta)/K$ that become isomorphic to (V', ϕ') over K'. Since the signature at the archimedean places is determined by ϕ' we know that also the signatures of the forms θ at infinite is fixed. So there is only freedom at the finite places. If n is odd there is nothing more to prove. If n is even, then we can use the previous lemma to establish that $\varepsilon_{v'}(V')$ determines $\varepsilon_v(W)$ for every v under v'. Hence there is only one class that can be base changed to (V', ϕ') and hence we conclude the proof of the proposition. □

7. Proof of the "Torsion-Congruences": The CM method

We are now ready to prove the main result of this work, namely the "torsion congruences" for the motives that we described in the introduction.

Explicit Results I: the case $n = 1$. Let E be an elliptic curve defined over $\mathbb{Q}$ with CM by the ring of integers $\mathfrak{R}_0$ of a quadratic imaginary field K_0. We fix an isomorphism $\mathfrak{R}_0 \cong \mathrm{End}(E)$ and we write Σ_0 for the implicit *CM* type of E. Let us write ψ_{K_0} for the Grössencharacter attached to E. That is, ψ_{K_0} is a Hecke character of K_0 of (ideal) type $(1, 0)$ with respect to the CM type Σ_0 and satisfies $L(E, s) = L(\psi_{K_0}, s)$. We fix an odd prime p where the elliptic curve has good ordinary reduction. We fix an embedding $\bar{\mathbb{Q}} \hookrightarrow \bar{\mathbb{Q}}_p$ and, using the selected CM type, we fix an embedding $K_0 \hookrightarrow \bar{\mathbb{Q}}$. The ordinary assumption implies that p splits in K_0, say to $\mathfrak{p}$ and $\bar{\mathfrak{p}}$ where we write $\mathfrak{p}$ for the prime ideal that corresponds to the p-adic embedding $K_0 \hookrightarrow \bar{\mathbb{Q}} \hookrightarrow \bar{\mathbb{Q}}_p$. We write N_E for the conductor of E and $\mathfrak{f}$ for the conductor of ψ_{K_0}.

We use the setting of the introduction. That is we consider a totally real field extension F'/F of degree p, unramified outside p. We let K (resp K') be the CM-field FK_0 (resp. $F'K_0 = F'K$) and recall that we write Γ for the Galois group $\mathrm{Gal}(F'/F) \cong \mathrm{Gal}(K'/K)$. We now consider the base changed elliptic curves E/F over F and E/F' over F'. We note that the above setting gives the following equalities between the L functions,

$$L(E/F,s) = L(\psi_K, s), \quad L(E/F',s) = L(\psi_{K'}, s) \tag{7.1}$$

where $\psi_K := \psi_{K_0} \circ N_{K/K_0}$ and $\psi_{K'} := \psi_K \circ N_{K'/K} = \psi_{K_0} \circ N_{K'/K_0}$, that is the base-changed characters of ψ_{K_0} to K and K'.

We write G_F for the Galois group $G(F(p^\infty)/F)$ and $G_{F'}{:=}G(F'(p^\infty)/F')$ for the analogue for F'. As we have remarked in the introduction our setting induces a transfer map $\mathrm{ver}\colon G_F \to G_{F'}$. Moreover we have an action of $\Gamma = \mathrm{Gal}(F'/F)$ on $G_{F'}$ by conjugation. We now define measures $\mu_{E/F}$ of G_F and $\mu_{E/F'}$ of $G_{F'}$ that interpolate the critical value at $s = 1$ of the elliptic curve E/F and E/F' respectively twisted by finite order characters of conductor dividing p^∞. We let $\mu_{E/F} := \mu_\psi^{HLS}$ and $\mu_{E/F'} := \mu_{\psi'}^{HLS}$, where we have taken the measure $\mu_{\pi,\chi}^{HLS}$ constructed above in the case $n = 1$, π trivial and $\chi = \psi$.

Theorem 7.1. — *We have the congruences*

$$\int_{G_F} \varepsilon \circ \mathrm{ver} \quad d\mu_{E/F} \equiv \int_{G_{F'}} \varepsilon \quad d\mu_{E/F'} \mod p\mathbb{Z}_p, \tag{7.2}$$

for all ε locally constant $\mathbb{Z}_p$-valued functions on $G_{F'}$ such that $\varepsilon^\gamma = \varepsilon$ for all $\gamma \in \Gamma$, where $\varepsilon^\gamma(g) := \varepsilon(\tilde{\gamma} g \tilde{\gamma}^{-1})$ for all $g \in G_{F'}$ and for some lift $\tilde{\gamma} \in \mathrm{Gal}(F'(p^\infty)/F))$ of γ.

Proof. — The proof of this theorem is exactly the same as the proof of Theorem 7.4 that we prove below for the case $n = 2$. One simply needs to set there $f = 1$ and π the trivial representation. The rest of the proof is identical, so we defer the proof for the next section. □

As it is explained in appendix in Theorem 9.1 a remark of Ritter and Weiss allow us to conclude from Theorem 7.1 the following theorem

Theorem 7.2 (Torsion Congruences for CM Elliptic Curves). — *With notation as in the introduction we have,*

$$\mathrm{ver}(\mu_{E/F}) \equiv \mu_{E/F'} \mod T,$$

where T is the trace ideal. That is, the torsion-congruences hold.

We note here that the important improvement in comparison to the previous result in [2] is that we do not need to make any assumption on the relation between the various class groups of F, F', K and K'.

Explicit Results II: the case $n = 2$. In this section we explain our results in the case of $n = 2$. We start by providing a family of examples where the **Hypothesis** of the introduction holds.

A family of examples for the Hypothesis We take K' and K Galois over $\mathbb{Q}$. We start from a Grössencharacter ϕ of K of conductor $\mathfrak{c} \subset \mathfrak{g}$. We moreover take $\mathfrak{c}$ relative prime to the $\mathfrak{d}_{K/F}$, the relative different of K over F, and both $\mathfrak{c}$ and $\mathfrak{d}_{K/F}$ prime to p. This character induces a character on $U(1)/F$ since its F-points is nothing else than $\{x \in K^\times \mid xx^\rho = 1\}$. We keep writing ϕ for this character of $U(1)$ (actually this is nothing else than a hermitian form for $U(1)$). We also write ϕ' for the character of $U(1)/F'$ obtained by $\phi \circ N_{K'/K}$, *i.e.* the base-change of ϕ. To the character ϕ (resp. ϕ') we now explain how we can associate a hermitian modular form f_ϕ (resp. $f_{\phi'}$) of $U(\theta)/F$ (resp $U(\theta)/F'$). We first observe that taking determinants we get maps $\det\colon U(\theta)/F \to U(1)/F$ and $\det\colon U(\theta)/F' \to U(1)/F'$. We define the functions $f_\phi := \phi \circ \det$ and $f_{\phi'} := \phi' \circ \det$. We now show that these are hermitian modular forms. We do this for f_ϕ, and similarly can be done for $f_{\phi'}$. Recall that we write G for $U(\theta)/F$. For $\alpha \in G(F)$, $x \in G(\mathbb{A}_F)$ and $w \in D$ we have $f_\phi(\alpha x w) = \phi \circ \det(\alpha x w)$ and so

$$f_\phi(\alpha x w) = \phi(\det(\alpha))\phi(\det(x))\phi(\det(w)) = f_\phi(x)\phi_{\mathbf{h}}(\det(w_{\mathbf{h}})\phi_\infty(\det(w)_\infty)$$

From this we see that if the character ϕ is of infinite type $-k\Sigma + \sum_\sigma \lambda(\sigma)(\sigma - \sigma^\rho)$ then we have that the weight of f_ϕ is $(k - 2\lambda(\sigma))_{\sigma\rho}$. Indeed since the character ϕ is taken of infinite type $-k\Sigma + \sum_\sigma \lambda(\sigma)(\sigma - \sigma^\rho)$ we have for a place $\sigma \in \Sigma$ that $\phi_\sigma(\det(w)_\sigma) = \det(w)_\sigma^k \left(\frac{\det(w)_\sigma^\rho}{\det(w)_\sigma}\right)^{\lambda(\sigma)}$. But $\det(w) \in U(1)$ hence $\det(w)_\sigma^\rho = \det(w)_\sigma^{-1}$ and hence $\psi_\sigma(\det(w)_\sigma) = \det(w)_\sigma^{k-2\lambda(\sigma)}$. Moreover if we take $\lambda(\sigma) = \lambda(\sigma')$ for all σ, σ' we get an automorphic form of parallel weight.

Now we observe that $\phi_{\mathbf{h}}(\det(w)_{\mathbf{h}}) = 1$. Indeed since we are taking the character ϕ of conductor $\mathfrak{c}$ prime to $\mathfrak{d}_{K/F}$ this follows from [40, Lemma 24.8 (3) p. 206] (see pp. 203 and 205 of (*loc. cit.*) for the definition of $U^\mathfrak{c}$ and $W^\mathfrak{c}$). Indeed this implies that we have $\det(D(\mathfrak{c}) \subset \{x \in U(1)(\mathbb{A}_F) \cap \prod_{v \in \mathbf{h}} \mathfrak{r}^\times \mid x - 1 \in \mathfrak{c}_v,\ \forall v | \mathfrak{c}_v\}$ and hence $\phi_{\mathbf{h}}(\det(w)_{\mathbf{h}}) = 1$. In particular f_ϕ is of trivial nebentypus.

We now observe that we can easily find examples where f_ϕ and $f_{\phi'}$ are $\mathbb{Z}_p$-valued (after $\imath_\infty^{-1} \circ \imath_p$ as we explained in the introduction) on finite adeles relative prime to $\mathfrak{c}p$. As an example we may take the values of the character ϕ on finite ideles to be in K_0 for a quadratic imaginary field and take p split in K_0, then the values of ϕ are in $\mathbb{Z}_p^\times$ for ideles away from the conductor and p. Such a Hecke character we may obtain from elliptic curves with CM by K_0 and taking the CM type (K,Σ) to be the inflation of the CM type (K_0,σ_0) with σ_0 the selected embedding of K_0 in $\overline{\mathbb{Q}}$. Moreover we observe that the form f_ϕ is a Hecke eigenform. Indeed for a Hecke operator $D\tau D=\bigsqcup_{y\in Y} Dy$ we have $(f_\phi|D\tau D)(x)=\sum_y f_\phi(xy^{-1})=\left(\sum_y \psi(\det(y^{-1}))\right) f_\phi(x)$. Actually we have the following for the standard L-function attached to f_ϕ.

$$L(f_\phi,s)=\prod_{\mathfrak{q}\nmid\mathfrak{c}}\left(\left(1-\tilde{\phi}(\mathfrak{q})N(\mathfrak{q})^{-s}\right)\left(1-\tilde{\phi}(\mathfrak{q})N(\mathfrak{q})^{1-s}\right)\right)^{-1},$$

where $\mathfrak{q}$ runs over the integral ideals of K prime to $\mathfrak{c}$ and $\tilde{\phi}$ is the character $\phi(x)/\phi(x^\rho)$ where ρ the non-trivial element of $\mathrm{Gal}(K/F)$. For this equation we refer to [22, p. 150 (3.5.1)]. There we take π to be the trivial representation. Then we use [40, Lemma 20.11] to conclude the above equation. The fact that Harris is working with $GU(V)$ and we are working with $U(V)$ makes no difference when it comes to the L-functions, since they are always defined by restricting the automorphic representation to $U(V)$ (see for example [22] just before equation (3.5.1) in p. 150).

Similarly we have that

$$L(f_{\phi'},s)=\prod_{\mathfrak{q}'\nmid\mathfrak{c}}\left(\left(1-\tilde{\phi}'(\mathfrak{q})N(\mathfrak{q})^{-s}\right)\left(1-\tilde{\phi}'(\mathfrak{q})N(\mathfrak{q})^{1-s}\right)\right)^{-1},$$

In particular we see that $f_{\phi'}$ is the base change of f_ϕ since we have $L(f_{\phi'},s)=\prod_\chi L(f_\phi,\chi,s)$, where χ runs over the characters of $\mathrm{Gal}(F'/F)$ and

$$L(f_\phi,\chi,s)=\prod_{\mathfrak{q}\nmid\mathfrak{c}}\left(\left(1-\tilde{\phi}(\mathfrak{q})\chi(\mathfrak{q})N(\mathfrak{q})^{-s}\right)\left(1-\tilde{\phi}(\mathfrak{q})\chi(\mathfrak{q})N(\mathfrak{q})^{1-s}\right)\right)^{-1}.$$

Now the conditions (i) (ii) and (iii) which we stated in the **Hypothesis** it is easy to see that they hold. We have already discuss (i) above. For the second one we only need to observe that since ϕ' is simply $\phi\circ N_{K'/K}$ its restriction to K is simply ϕ^p, and since we are taking the values of ϕ in $\mathbb{Z}_p$ we obtain $\phi^p\equiv\phi \mod p$ on finite ideles away from $\mathfrak{c}p$. In particular $\Delta^*(f_{\phi'})(x)=f_{\phi^p}(x)=f_\phi^p(x)\equiv f_\phi(x) \mod p$ for every $x\in G(\mathbb{A}_{F,f}^{(p\mathfrak{c})})$. The third property is also straightforward since it holds for ϕ' which is nothing else than the base change of ϕ. We also note that the family of examples that we have constructed satisfy also the existence of the form f_H in the introduction. Indeed the form f_{ϕ^p} is such a form.

We now explain in details the relation between automorphic forms of quaternion algebras and definite unitary groups of two variables, which provides an interesting family of applications of our theorem.

Quaternion algebras and unitary groups in two variables. In this section we follow closely the exposition in Harris [21]. Let D be a quaternion algebra over a totally real field F and assume that there is an embedding $i\colon K \hookrightarrow D$ for a CM field K with $K^+ = F$. We consider the algebraic group over F

$$GU_K(D) := (H_K \times D^\times)/H_F,$$

where $H_K = \mathrm{Res}_{K/\mathbb{Q}} G_m/K$ and $H_F = \mathrm{Res}_{F/\mathbb{Q}} G_m/F$ and H_F is embedded diagonally into $H_K \times D^\times$. Next we will identify the group $GU_K(D)$ with a unitary group as its notation suggests.

Let us write $\iota\colon D \to D$ for the main involution of D, that is the reduced norm and trace are related by $N_D(d) = \mathrm{Tr}_D(d \cdot d^\iota)$. We can then obtain the non-degenerate inner form $(x,y)_D := \mathrm{Tr}_D(x \cdot y^\iota)$ on the four dimensional F-vector space D and define the orthogonal group $GO(D)$ as

$$GO(D) = \{g \in \mathrm{GL}_E(D) \mid (gx, gy)_D = \nu(g)(x,y)_D\},$$

for some homomorphism $\nu\colon GO(D) \to H_F$. Further there is a map $\rho\colon D^\times \times D^\times \to GO(D)$ defined by $\rho(d_1,d_2)(x) := d_1 x d_2^{-1},\ x \in D$.

The map ρ has kernel H_F embedded diagonally in $D^\times \times D^\times$ and $\nu(\rho(d_1,d_2)) = N_D(d_1 \cdot d_2^{-1})$. Let now consider K as above i.e there is an embedding $i\colon K \hookrightarrow D$ and D splits over K. We define the group $GU_K(D)$ as the subgroup of K-linear elements of $GO(D)$. Actually the group $GU_K(D)$ is a the group of unitary similitudes of the a hermitian form $(\cdot,\cdot)_{D,K}$ characterized uniquely from the properties that $(\cdot,\cdot)_D = \mathrm{Tr}_{K/F}(\cdot,\cdot)_{D,K}$ and $(x,y)_{D,K} = x \cdot \bar{y}$ for $x, y \in K$. Now the relation with our previous considerations is that the map ρ restricted to $H_K \times D^\times$ has image in $\mathrm{GL}_K(D) \cap GO(D)$ and induces an isomorphism $(H_K \times D^\times)/H_F \cong GU_K(D)$. Finally when the quaternion algebra is unramified at all infinite places then the hermitian form $(\cdot,\cdot)_{D,K}$ is positive definite.

Now we recall a setting that we are interested in which gives interesting applications. We consider a Hilbert cuspidal form f of F, which is a newform. We take the parallel weight of this to be ℓ. We write N_f for its conductor. We assume that N_f is square free and relative prime to p. We now impose the following assumptions on f,

(1) f has a trivial Nebentypus.
(2) There exists a finite set S of finite places of F such that we have (i) $\mathrm{ord}_v(N_f) \neq 0$ for all $v \in S$, (ii) for $v \in S$ we have that v is inert in K and finally (iii) $\sharp S + [F\colon \mathbb{Q}]$ is even.

Let us write D/F for the totally definite quaternion algebra that we can associate to the set S, *i.e.* D is ramified at all finite places $v \in S$ and also at all infinite places. Note that our assumptions imply that there exist an embedding $K \hookrightarrow D$. If we write π_f for the cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_F)$ associated to f then our assumptions imply that there exist a Jacquet-Langlands correspondence $\pi := JL(\pi_f)$ to $D^\times(\mathbb{A}_F)$. As we explained above there exists an isomorphism

$$(D^\times \times K^\times)/F^\times \cong GU(\theta)(F),$$

for some totally definite two dimensional Hermitian form (W,θ). In particular, since the representation π is taken of trivial central character we can consider the representation $\mathbf{1}\times\pi$ of $(D^\times_{\mathbb{A}_F}\times\mathbb{A}_K^\times)/\mathbb{A}_F^\times$, where $\mathbf{1}$ the trivial representation (character) on $\mathbb{A}_K^\times$. In particular π induces an automorphic representation, by abuse of notation we denote it again with π, on $GU(\theta)$ and by restriction to $U(\theta)$. Moreover it is known that $L(\pi,s) = L(BC(\pi'),s)$, where $BC(\pi_f)$ is the base-change of π_f from $\mathrm{GL}_2(\mathbb{A}_F)$ to $\mathrm{GL}_2(\mathbb{A}_K)$.

We now turn to the proof of the main theorem. We start by recalling it. We assume the Hypothesis and we write $f := f_\pi$ and $f' := f_{\pi'}$ for the automorphic forms assumed by the hypothesis.

Theorem 7.3. — *Beside the* ***Hypothesis*** *of the introduction and the existence of the form f_H in the case of $\ell \neq 0$ (but see also the introduction for what can be proved if we do not assume the existence of such an f_H) we assume the following conditions are met.*

1. *The p-adic realizations of $M(\pi)$ and $M(\psi)$ have $\mathbb{Z}_p$-coefficients.*
2. *The prime p is unramified in F (but may ramify in F').*
3. *If we write K^* for the reflex field of (K,Σ) (note that this is also the reflex field of (K',Σ')), then for the primes $\mathfrak{p}$ above p in K^* we have $N_{K^*/\mathbb{Q}}(\mathfrak{p}) = p$.*
4. *If we write $\langle f,\check{f}\rangle$ for the standard normalized Peterson inner product of π, then $\langle f,\check{f}\rangle$ has trivial valuation at p.*

Then we have that the torsion congruences hold true for the motive $M(\psi,\pi)/F$, where ψ a Hecke character of K infinity type $-k\Sigma$ with $k + 2\ell \geqslant 2$.

We now construct the measures $\mu_F = \mu^{(f)}_{(\pi,\psi)}$ (resp. $\mu_{F'} := \mu^{(f')}_{(\pi',\psi')}$) on $G := G(F(p^\infty)/F)$ (resp. $G' := G(F'(p^\infty)/F')$) that appear in the theorem. We consider the CM algebras $Y = K \oplus K$ and $Y' = K' \oplus K'$. Then we define the measures by

$$\int_G \chi\, \mu_F := \frac{\int_{\mathrm{Gal}(K(p^\infty)/K)} \tilde{\chi} \mu^{HLS,(f)}_{\pi,\psi}}{\Omega_p(Y,\Sigma)^{k+2\ell}},$$
$$\int_{G'} \chi \mu_F := \frac{\int_{\mathrm{Gal}(K'(p^\infty)/K')} \tilde{\chi}'\, \mu^{HLS,(f')}_{\pi',\psi'}}{\Omega_p(Y',\Sigma')^{k+2\ell}},$$

where $\tilde{\chi}$ (resp. $\tilde{\chi}'$) is the base change of χ (resp. χ' from F (resp. F') to K (resp. K').

We will prove the following theorem. As we explain in the Appendix (see Theorem 9.1) the following theorem implies Theorem 7.3. The theorem below is of course under the same conditions as the theorem above.

Theorem 7.4. — *Assume that π_f and ψ have coefficients in $\mathbb{Q}_p$. Let ε be a locally constant $\mathbb{Z}_p$-valued function of $G_{F'}$ with $\varepsilon^\gamma = \varepsilon$, for $\gamma \in \mathrm{Gal}(F'/F)$. Then we have the congruences*

$$\langle f, \check{f}\rangle \int_{G'} \varepsilon(g)\ \mathrm{ver}(\mu_F)(g) \equiv \langle f, \check{f}\rangle \int_{G'} \varepsilon(g) \mu_{F'}(g) \mod p.$$

In particular we have that if $\langle f, \check{f}\rangle$ is a p-adic unit then the torsion congruences hold for $M(\pi)/F \otimes M(\psi)/F$ and the extension F'/F.

From the construction of the $\mu^{HLS}_{\pi,\psi}$ (resp. $\mu^{HLS}_{\pi',\psi'}$ we know that for a locally constant function ϕ of G (resp. ϕ' of G') we have

$$\int_G \phi\, \mu_F = \sum_{a,b \in \mathcal{B}_K} E^{(a),\nu}_{\phi\psi}(\underline{A_a} \times \underline{A_b}, j_1 \times j_2) \check{f}(b) f(a),$$
$$\left(\text{resp.} \quad \int_{G'} \phi'\, \mu_{F'} = \sum_{a',b' \in \mathcal{B}_{K'}} E^{(a'),\nu'}_{\phi'\psi'}(\underline{A_{a'}} \times \underline{A_{b'}}, j_1 \times j_2) \check{f}'(b') f'(a')\right).$$

Lemma 7.5. — *Let $b' = \imath(b) \in \mathcal{B}_{K'}$ and $a' = \imath(a) \in \mathcal{B}_{K'}$ for $a, b \in \mathcal{B}_K$. Then we have that*

$$\mathrm{Frob}_p(E^{(a),\nu}_{\phi\psi^p})(\underline{A_a} \times \underline{A_b}, j_1 \times j_2) \equiv E^{(a'),\nu'}_{\phi'\psi'}(\underline{A_{a'}} \times \underline{A_{b'}}, j_1' \times j_2') \mod p,$$

for ϕ' a locally constant $\mathbb{Z}_p$-valued function on G' such that ${\phi'}^\gamma = \phi'$ and $\phi := \phi' \circ \mathrm{ver}$.

Proof. — By Theorem 5.7 (see also the congruences at the end of Section 5) we have

$$\mathrm{Frob}_p(E^{(a),\nu^p}_{\phi\psi^p}) \equiv \mathrm{Res}^{K'}_K\, E^{(a'),\nu'}_{\phi'\psi'} \mod p.$$

Then the lemma follows by observing that

$$\mathrm{Res}^{K'}_K\, E^{(a'),\nu'}_{\phi'\psi'}(\underline{A_a} \times \underline{A_b}, j_1 \times j_2) = E^{(a'),\nu'}_{\phi'\psi'}(\underline{A_{a'}} \times \underline{A_{b'}}, j_1' \times j_2').$$

□

Corollary 7.6. — *Keep the notation of ϕ and ϕ' as before. Then we have the congruences*

$$\left(\sum_{a,b\in\mathcal{B}_K} E^{(a),\nu^p}_{\phi\psi^p}(\underline{A_a}\times\underline{A_b},j_1\times j_2)\check{f}(b)f(a)\right)^{\Phi_\mathfrak{p}} \equiv \sum_{a',b'\in\imath(\mathcal{B}_K)} E^{(a'),\nu'}_{\phi'\psi'}(\underline{A_{a'}}\times\underline{A_{b'}},j_1\times j_2)\check{f}'(b')f'(a') \mod p,$$

where $\Phi_\mathfrak{p}$ is an in Theorem 6.2.

Proof. — From the theory of complex multiplication (Theorem 6.2) and the assumptions on ϕ and f we have that

$$\sum_{a,b\in\mathcal{B}_K} \mathrm{Frob}_p(E^{(a),\nu^p}_{\phi\psi^p})(\underline{A_a}\times\underline{A_b},j_1\times j_2)\check{f}(b)f(a) = \left(\sum_{a,b\in\mathcal{B}_K} E^{(a),\nu^p}_{\phi\psi^p}(\underline{A_a}\times\underline{A_b},j_1\times j_2)\check{f}(b)f(a)\right)^{\Phi_\mathfrak{p}}.$$

The **Hypothesis** implies that $f'(\imath(a))\equiv f(a) \mod p$. Then the corollary follows from the lemma above. $\square$

Proposition 7.7. — *Let ε be a locally constant function such that $\varepsilon^\gamma=\varepsilon$ for all $\gamma\in\Gamma$. Then for all $a',b'\in\mathcal{B}_{K'}$ we have that*

$$E^{(a'),\nu'}_{\varepsilon\psi'}(\underline{A_{a'}}\times\underline{A_{b'}},j'_1\times j'_2) = E^{(a')^\gamma,\nu'}_{\varepsilon\psi'}(\underline{A^\gamma_{b'}}\times\underline{A^\gamma_{a'}},j'_1\times j'_2),$$

where $A^\gamma_{a'} := A_{a'^\gamma}$ and similarly $A^\gamma_{b'} := A_{b'^\gamma}$.

Proof. — We write the locally constant function ε as a sum of finite characters. That is, $\varepsilon=\sum_j c_j\chi_j$ with $c_j\in\mathbb{Q}(\varepsilon,\chi_j)$. Now the fact that $\varepsilon^\gamma=\varepsilon$ implies that this sum is of the form $\varepsilon=\sum_i c_i\chi_i+\sum_k c_k\sum_{\gamma\in\Gamma}\chi_k^\gamma$, where for the first sum we have $\chi_i^\gamma=\chi_i$, that is χ_i comes from base change from K. From the definition of the Eisenstein series we have

$$E^{\nu'}_{\varepsilon\psi}(\underline{A^\gamma_{a'}}\times\underline{A^\gamma_{b'}},j'_1\times j'_2) = \left(\frac{\Omega_p(Y,\Sigma)}{\Omega_\infty(Y,\Sigma)}\right)^{k+2\ell} E((a'^\gamma,b'^\gamma),\varepsilon\psi'),$$

where ϕ is the sections that we have constructed in Section 3. But we know that

$$E(x,\phi,\varepsilon\psi)=\sum_{\alpha\in A}\mu_{\varepsilon\psi}(\alpha x)\varepsilon(\alpha x)^{-s}|_{s=0}, \quad A:=P\setminus G.$$

The invariance of ε with respect to the action of Γ implies the invariance of $\mu_{\varepsilon\psi'}$ with respect to the action of Γ. Indeed we recall that for a character χ we have defined $\mu^{(\chi)}:=\prod_{v\in\mathbf{h}\cup\mathbf{a}}\mu_v^{(\chi_v)}$ supported on $P(\mathbb{A}_F)D(\mathfrak{c})\cap$

$P(\mathbb{A}_F)w_nP(\mathbb{A}_F)$, where for $x = pw \in P(\mathbb{A}_F)D(\mathfrak{c}) \cap P(\mathbb{A}_F)w_nP(\mathbb{A}_F)$

$$\mu_v^{(\chi_v)}(x_v) = \begin{cases} \chi_v(\lambda_0(p_v)), & v \in \mathbf{h} \text{ and } v \nmid \mathfrak{c} \\ \chi_v(\lambda_0(p_v))\chi_v(\lambda_0(w_v)), & v \in \mathbf{h} \text{ and } v|\mathfrak{c} \\ \chi_v(\lambda_0(p_v))j^k_{w_v}(\mathbf{i})^{-1}, & v \in \mathbf{a} \\ f_{\Phi_\chi}(x_v), & v|p. \end{cases}$$

Hence

$$\mu_{\varepsilon\psi'}(x) = \sum_j c_j \mu^{(\chi_j)\psi'}(x) = \sum_i c_i \mu^{(\chi_i)\psi'}(x) + \sum_k c_k \sum_{\gamma\in\Gamma} \mu^{(\chi_k^\gamma)\psi'}(x)$$

(note that ${\psi'}^\gamma{=}\psi'$). It is clear that the "trace" part, $\sum_k c_k \sum_{\gamma\in\Gamma} \mu^{(\chi_k^\gamma)\psi'}(x)$, is invariant under the operation of Γ. But also the other part is invariant thanks to the fact that the characters χ_i (as well as ψ') that appear there are of the form $\tilde{\chi}_i \circ N_{K'/K}$, with $\tilde{\chi}$ a finite order character of G_K. The only thing that needs to be also remarked is that for $v|p$ that is inert in F' we have that $f_{\Phi_{\chi_{i,v}}}(x_v^\gamma) = f_{\Phi_{\chi_{i,v}}}(x_v)$, but this follows easily from the definition of the section and the invariance of χ_i. Indeed we recall that if we write $\chi_v = (\chi_1, \chi_2)$

$$f_{\Phi_{\chi_v}}(x) = \chi_2(\det(x))|\det(x)|^s \int_{\mathrm{GL}_{2n}(F'_v)} \times \Phi_{\chi_v}((0,Z)x)\chi_1\chi_2(\det(Z))|\det(Z)|^{2s} d^\times Z,$$

where we recall that $\Phi_{\chi_v}(X,Y) = \tilde{\Phi}_{\nu^{-1}\chi_2^{-1}}(X)\widehat{\Phi_{\nu^{-1}\chi_1}}(Y)$. It is easy to see that the functions $\tilde{\Phi}_{\nu^{-1}\chi_2^{-1}}(X)$ and $\Phi_{\nu^{-1}\chi_1}(Y)$ are invariant with respect to Γ and hence also

$$\widehat{\Phi_{\nu^{-1}\chi_1}}(Y) = \int_{M_n(F'_v)} \Phi_{\nu^{-1}\chi_1}(X)\psi({}^tXY)dX,$$

since we have (note that $\psi^\gamma = \psi$)

$$\begin{aligned}\widehat{\Phi_{\nu^{-1}\chi_1}}(Y^\gamma) &= \int_{M_n(F'_v)} \Phi_{\nu^{-1}\chi_1}(X)\psi({}^tXY^\gamma)dX \\ &= \int_{M_n(F'_v)} \Phi_{\nu^{-1}\chi_1}(X^{\gamma^{-1}})\psi({}^tX^{\gamma^{-1}}Y)dX = \widehat{\Phi_{\nu^{-1}\chi_1}}(Y).\end{aligned}$$

From these observations we conclude that $f_{\Phi_{\chi_{i,v}}}(x_v^\gamma) = f_{\Phi_{\chi_{i,v}}}(x_v)$. Now we also remark that the invariance of ε follows from its very definition (see [40, p. 95]). Hence we have that

$$E(x^\gamma, \varepsilon\psi') = \sum_{\alpha\in A} \mu_{\varepsilon\psi'}(\alpha x^\gamma)\varepsilon(\alpha x^\gamma)^{-s} = \sum_{\alpha\in A} \mu_{\varepsilon\psi'}(\alpha^{\gamma^{-1}}x)\varepsilon(\alpha^{\gamma^{-1}}x)^{-s}.$$

But since γ induces an automorphism of A we have that the last summation is equal to $E(x,\varepsilon\psi')$. That is, we conclude that $E((a'^{\gamma},b'^{\gamma}),\varepsilon\psi') = E((a',b'),\varepsilon\psi')$ and hence also the proposition. $\square$

An immediate corollary of this proposition is,

Corollary 7.8. — *Assume that ϕ' is a locally constant function with $\phi'^{\gamma}=\phi$. Then we have that*

$$E^{(a'),\nu'}_{\phi'\psi'}(\underline{A_{a'}}\times\underline{A_{b'}},j'_1\times j'_2) = E^{(a'^{\gamma}),\nu'}_{\phi'\psi'}(\underline{A^{\gamma}_{a'}}\times\underline{A^{\gamma}_{b'}},j'_1\times j'_2),$$

where we also note that $A^{\gamma}_{a'}=A_{a'^{\gamma}}$ and similarly $A^{\gamma}_{b'}=A_{b'^{\gamma}}$. In particular, since the ***Hypothesis*** *implies that $f'(a'^{\gamma})=f'(a')$ and $\check{f}'(b'^{\gamma})=\check{f}'(b')$ we have that*

$$E^{(a'),\nu'}_{\phi'\psi'}(\underline{A_{a'}}\times\underline{A_{b'}},j'_1\times j'_2)\check{f}'(b')f'(a') = E^{(a'^{\gamma}),\nu'}_{\phi'\psi'}(\underline{A^{\gamma}_{a'}}\times\underline{A^{\gamma}_{b'}},j'_1\times j'_2)\check{f}'(b'^{\gamma})f'(a'^{\gamma}).$$

Lemma 7.9. — *We have the congruences*

$$\langle f,\check{f}\rangle\frac{\int_G\phi\,\mu_F}{\Omega_p(Y,\Sigma)^{k+2\ell}} \equiv \langle f,\check{f}\rangle\frac{\sum_{a,b\in\mathcal{B}_K}E^{(a),\nu}_{\phi\psi^p}(\underline{A_a}\times\underline{A_b},j_1\times j_2)\check{f}(b)f(a)}{\Omega_p(Y,\Sigma)^{pk+2\ell}}\times\frac{(\Omega_p(Y,\Sigma)^k)^{\Phi_{\mathfrak{p}}}}{\Omega_p(Y,\Sigma)^k}\mod p.$$

Proof. — Since $\psi^p\equiv\psi\mod p$ and $\nu^p\equiv\nu\mod p$ (both take values in $\mathbb{Z}_p^{\times}$) we have that

$$\langle f,\check{f}\rangle\sum_{a,b\in\mathcal{B}_K}E^{(a),\nu^p}_{\phi\psi^p}(\underline{A_a}\times\underline{A_b},j_1\times j_2)\check{f}(b)f(a) \equiv \langle f,\check{f}\rangle\sum_{a,b\in\mathcal{B}_K}E^{(a),\nu}_{\phi\psi}(\underline{A_a}\times\underline{A_b},j_1\times j_2)\check{f}(b)f(a)\mod\mathfrak{m},$$

where $\mathfrak{m}$ is the maximal ideal in J_∞, which of course contains p. Dividing by the unit $\Omega_p(Y,\Sigma)^k$ and observing that $\frac{(\Omega_p(Y,\Sigma)^k)^{\Phi_{\mathfrak{p}}}}{\Omega_p(Y,\Sigma)^k}\equiv\frac{(\Omega_p(Y,\Sigma)^k)^p}{\Omega_p(Y,\Sigma)^k}\mod\mathfrak{m}$ we get the congruences

$$\frac{\langle f,\check{f}\rangle}{\Omega_p(Y,\Sigma)^{k+2\ell}}\sum_{a,b\in\mathcal{B}_K}E^{(a)\nu}_{\phi\psi}(\underline{A_a}\times\underline{A_b},j_1\times j_2)\check{f}(b)f(a) \equiv \frac{\langle f,\check{f}\rangle}{(\Omega_p(Y,\Sigma)^{pk+2\ell})}\sum_{a,b\in\mathcal{B}_K}E^{(a),\nu}_{\phi\psi^p}(\underline{A_a}\times\underline{A_b},j_1\times j_2)\check{f}(b)f(a) \times\frac{(\Omega_p(Y,\Sigma)^k)^{\Phi_{\mathfrak{p}}}}{\Omega_p(Y,\Sigma)^k}\mod\mathfrak{m}.$$

Since both sides belong to $\mathbb{Z}_p$ we have that the congruences are modulo $\mathfrak{m}\cap\mathbb{Z}_p = p$. $\square$

Theorem 7.10. — *For ϕ' a locally constant $\mathbb{Z}_p$-valued function on G' such that ${\phi'}^{\gamma} = \phi'$ we have that*

$$\Big(\frac{\Omega_p(Y,\Sigma)^{\Phi_\mathfrak{p}}}{\Omega_p(Y,\Sigma)}\Big)^{2\ell}\langle f,\check{f}\rangle\frac{\int_G \phi\,\mu_F}{\Omega_p(Y,\Sigma)^{k+2\ell}} \equiv \langle f,\check{f}\rangle\frac{\int_{G'}\phi'\,\mu_{F'}}{(\Omega_p(Y,\Sigma)^{k+2\ell})^p} \mod p,$$

where $\phi := \phi'\circ\mathrm{ver}$. In particular if $\langle f,\check{f}\rangle\in\mathbb{Z}_p^\times$ then we have that

$$\Big(\frac{\Omega_p(Y,\Sigma)^{\Phi_\mathfrak{p}}}{\Omega_p(Y,\Sigma)}\Big)^{2\ell}\frac{\int_G \phi\,\mu_F}{\Omega_p(Y,\Sigma)^{k+2\ell}} \equiv \frac{\int_{G'}\phi'\,\mu_{F'}}{\Omega_p(Y,\Sigma)^{p(k+2\ell)}} \mod p.$$

Proof. — The fact that $\imath\colon\mathcal{B}_K\hookrightarrow\mathcal{B}_{K'}^{\Gamma}$ is a bijection and corollary 7.8 imply that

$$\langle f',\check{f}'\rangle\int_{G'}\phi'\,\mu_{F'} \equiv \Bigg(\langle f,\check{f}\rangle\sum_{a,b\in\mathcal{B}_K}E^{(a),\nu^p}_{\phi\psi^p}(\underline{A_a}\times\underline{A_b},j_1\times j_2)\check{f}(b)f(a)\Bigg)^{\Phi_\mathfrak{p}} \mod p,$$

where we have used the fact that $\langle f,\check{f}\rangle\in\mathbb{Z}_p$ and under our assumptions $\langle f,\check{f}\rangle\equiv\langle f',\check{f}'\rangle \mod p$. Dividing by the unit $\Omega_p(Y,\Sigma)^{p(k+2\ell)}$ and recall that by lemma 5.9 we have $\Omega_p(Y,\Sigma)^p = \Omega_p(Y',\Sigma')$ we get

$$\begin{aligned}&\frac{1}{\Omega_p(Y,\Sigma)^{p(k+2\ell)}}\\&\times\Bigg(\Omega_p(Y,\Sigma)^{pk+2\ell}\frac{\langle f,\check{f}\rangle\sum_{a,b\in\mathcal{B}_K}E^{(a),\nu^p}_{\phi\psi^p}(\underline{A_a}\times\underline{A_b},j_1\times j_2)\check{f}(b)f(a)}{\Omega_p(Y,\Sigma)^{pk+2\ell}}\Bigg)^{\Phi_\mathfrak{p}}\\&\qquad\equiv\langle f',\check{f}'\rangle\frac{\int_{G'}\phi'\,\mu_{F'}}{\Omega_p(Y,\Sigma)^{p(k+2\ell)}} \mod p\end{aligned}$$

But

$$\frac{\langle f,\check{f}\rangle\sum_{a,b\in\mathcal{B}_K}E^{(a),\nu^p}_{\phi\psi^p}(\underline{A_a}\times\underline{A_b},j_1\times j_2)\check{f}(b)f(a)}{\Omega_p(Y,\Sigma)^{pk+2\ell}}\in\mathbb{Z}_p.$$

Hence

$$\begin{aligned}\frac{\langle f',\check{f}'\rangle\int_{G'}\phi'\,\mu_{F'}}{\Omega_p(Y,\Sigma)^{p(k+2\ell)}} &\equiv \Big(\frac{(\Omega_p(Y,\Sigma)^{pk+2\ell})^{\Phi_\mathfrak{p}}}{\Omega_p(Y,\Sigma)^{p(k+2\ell)}}\Big)\\&\qquad\times\frac{\langle f,\check{f}\rangle\sum_{a,b\in\mathcal{B}_K}E^{(a),\nu^p}_{\phi\psi^p}(\underline{A_a}\times\underline{A_b},j_1\times j_2)\check{f}(b)f(a)}{\Omega_p(Y,\Sigma)^{pk+2\ell}} \ (p)\end{aligned}$$

But by our assumption on the reflex field K^* and the values of ψ we have $\left(\frac{\Omega_p(Y,\Sigma)^{\Phi_\mathfrak{p}}}{\Omega_p(Y,\Sigma)}\right)^p \equiv \frac{\Omega_p(Y,\Sigma)^{\Phi_\mathfrak{p}}}{\Omega_p(Y,\Sigma)} \mod p$ (see [39, p. 66]) hence we obtain

$$\frac{\langle f', \check{f}' \rangle \int_{G'} \phi' \, \mu_{F'}}{\Omega_p(Y,\Sigma)^{p(k+2\ell)}} \equiv \frac{(\Omega_p(Y,\Sigma)^{k+2\ell})^{\Phi_\mathfrak{p}}}{\Omega_p(Y,\Sigma)^{k+2\ell}} \times \frac{\langle f, \check{f} \rangle \sum_{a,b \in \mathcal{B}_K} E^{(a),\nu^p}_{\phi\psi^p}(\underline{A_a} \times \underline{A_b}, j_1 \times j_2) \check{f}(b) f(a)}{\Omega_p(Y,\Sigma)^{pk+2\ell}} \mod p.$$

But we have already shown that

$$\frac{(\Omega_p(Y,\Sigma)^k)^{\Phi_\mathfrak{p}}}{\Omega_p(Y,\Sigma)^k} \frac{\langle f, \check{f} \rangle \sum_{a,b \in \mathcal{B}_K} E^{(a),\nu^p}_{\phi\psi^p}(\underline{A_a} \times \underline{A_b}, j_1 \times j_2) \check{f}(b) f(a)}{\Omega_p(Y,\Sigma)^{pk+2\ell}} \equiv \langle f, \check{f} \rangle \frac{\int_G \phi \, \mu_F}{\Omega_p(Y,\Sigma)^{k+2\ell}} \mod p,$$

which concludes the proof of the theorem. □

Using this last theorem and the Theorem 9.1 of the Appendix I we conclude the torsion congruences of theorems 7.2 and 7.3.

The congruences using the assumption on the existence of the form f_H. Now we turn to prove the theorem assuming the existence of the the form f_H. We now write $E^{(a),\nu^p}_{\phi\psi^p}(\underline{A_a} \times \underline{A_b}, j_1 \times j_2)$ for the Eisenstein series as above but now defined with $p\ell$ instead of ℓ as the extra data. Notice that the congruences between the Eisenstein series still hold, as we have already indicated at the end of Section 5. Then as in the proof above we have the lemma; the reader should note here the difference on the powers of the periods.

Lemma 7.11. — *We have the congruences*

$$\langle f, \check{f} \rangle \frac{\int_G \phi \, \mu_F}{\Omega_p(Y,\Sigma)^{k+2\ell}} \equiv \langle f_H, \check{f}_H \rangle \frac{\sum_{a,b \in \mathcal{B}_K} E^{(a),\nu}_{\phi\psi^p}(\underline{A_a} \times \underline{A_b}, j_1 \times j_2) \check{f}_H(b) f_H(a)}{\Omega_p(Y,\Sigma)^{p(k+2\ell)}} \times \frac{(\Omega_p(Y,\Sigma)^{k+2\ell})^{\Phi_\mathfrak{p}}}{\Omega_p(Y,\Sigma)^{k+2\ell}} \mod p.$$

Proof. — Since $\psi^p \equiv \psi \mod p$, $\nu^p \equiv \nu \mod p$ and $f_H \equiv f \mod p$ (all take values in $\mathbb{Z}_p^\times$) we have that

$$\begin{aligned}\langle f, \check{f}\rangle \sum_{a,b\in\mathcal{B}_K} & E^{(a),\nu^p}_{\phi\psi^p}(\underline{A_a} \times \underline{A_b}, j_1 \times j_2)\check{f}(b)f(a) \\ &\equiv \langle f_H, \check{f}_H\rangle \sum_{a,b\in\mathcal{B}_K} E^{(a),\nu}_{\phi\psi}(\underline{A_a} \times \underline{A_b}, j_1 \times j_2)\check{f}_H(b)f_H(a) \mod \mathfrak{m},\end{aligned}$$

where $\mathfrak{m}$ is the maximal ideal in J_∞, which of course contains p. Dividing by the unit $\Omega_p(Y,\Sigma)^{k+2\ell}$ and observing that $\frac{(\Omega_p(Y,\Sigma)^{k+2\ell})^{\Phi_\mathfrak{p}}}{\Omega_p(Y,\Sigma)^{k+2\ell}} \equiv \frac{(\Omega_p(Y,\Sigma)^{k+2\ell})^{p}}{\Omega_p(Y,\Sigma)^{k+2\ell}}$ mod $\mathfrak{m}$ we get the congruences

$$\begin{aligned}\frac{\langle f, \check{f}\rangle}{\Omega_p(Y,\Sigma)^{k+2\ell}} \sum_{a,b\in\mathcal{B}_K} & E^{(a)\nu}_{\phi\psi}(\underline{A_a} \times \underline{A_b}, j_1 \times j_2)\check{f}(b)f(a) \\ &\equiv \frac{\langle f_H, \check{f}_H\rangle}{(\Omega_p(Y,\Sigma)^{k+2\ell})^p} \sum_{a,b\in\mathcal{B}_K} E^{(a),\nu}_{\phi\psi^p}(\underline{A_a} \times \underline{A_b}, j_1 \times j_2)\check{f}_H(b)f_H(a) \\ &\qquad\qquad \times \frac{(\Omega_p(Y,\Sigma)^{k+2\ell})^{\Phi_\mathfrak{p}}}{\Omega_p(Y,\Sigma)^{k+2\ell}} \mod \mathfrak{m}.\end{aligned}$$

Since both sides belong to $\mathbb{Z}_p$ we have that the congruences are modulo $\mathfrak{m} \cap \mathbb{Z}_p = p$. □

Theorem 7.12. — *For ϕ' a locally constant $\mathbb{Z}_p$-valued function on G' such that ${\phi'}^\gamma = \phi'$ we have that*

$$\langle f, \check{f}\rangle \frac{\int_G \phi\,\mu_F}{\Omega_p(Y,\Sigma)^{k+2\ell}} \equiv \langle f, \check{f}\rangle \frac{\int_{G'} \phi'\,\mu_{F'}}{(\Omega_p(Y,\Sigma)^{k+2\ell})^p} \mod p,$$

where $\phi := \phi' \circ \mathrm{ver}$. In particular if $\langle f, \check{f}\rangle \in \mathbb{Z}_p^\times$ then we have that

$$\frac{\int_G \phi\,\mu_F}{\Omega_p(Y,\Sigma)^{k+2\ell}} \equiv \frac{\int_{G'} \phi'\,\mu_{F'}}{\Omega_p(Y,\Sigma)^{p(k+2\ell)}} \mod p.$$

Proof. — As before we have the congruences modulo p

$$\langle f', \check{f}'\rangle \int_{G'} \phi'\,\mu_{F'} \equiv \left(\langle f_H, \check{f}_H\rangle \sum_{a,b\in\mathcal{B}_K} E^{(a),\nu^p}_{\phi\psi^p}(\underline{A_a}\times\underline{A_b}, j_1\times j_2)\check{f}_H(b)f_H(a)\right)^{\Phi_\mathfrak{p}},$$

where we have used the fact that $\langle f, \check{f}\rangle \in \mathbb{Z}_p$ and under our assumptions $\langle f_H, \check{f}_H\rangle \equiv \langle f', \check{f}'\rangle \mod p$. Dividing by the unit $\Omega_p(Y,\Sigma)^{p(k+2\ell)}$ and recall that by lemma 5.9 we have $\Omega_p(Y,\Sigma)^p = \Omega_p(Y',\Sigma')$ we get the congruences modulo p,

$$\langle f', \check{f}'\rangle \frac{\int_{G'} \phi' \, \mu_{F'}}{\Omega_p(Y,\Sigma)^{p(k+2\ell)}}$$
$$\equiv \frac{\left(\Omega_p(Y,\Sigma)^{p(k+2\ell)} \frac{\langle f_H, \check{f}_H\rangle \sum_{a,b\in\mathcal{B}_K} E^{(a),\nu^p}_{\phi\psi^p}(\underline{A_a}\times\underline{A_b}, j_1\times j_2)\check{f}_H(b) f_H(a)}{\Omega_p(Y,\Sigma)^{p(k+2\ell)}}\right)^{\Phi_{\mathfrak{p}}}}{\Omega_p(Y,\Sigma)^{p(k+2\ell)}}.$$

But

$$\frac{\langle f_H, \check{f}_H\rangle \sum_{a,b\in\mathcal{B}_K} E^{(a),\nu^p}_{\phi\psi^p}(\underline{A_a}\times\underline{A_b}, j_1\times j_2)\check{f}_H(b) f_H(a)}{\Omega_p(Y,\Sigma)^{p(k+2\ell)}} \in \mathbb{Z}_p.$$

Hence

$$\frac{\langle f', \check{f}'\rangle \int_{G'} \phi' \, \mu_{F'}}{\Omega_p(Y,\Sigma)^{p(k+2\ell)}} \equiv \left(\frac{(\Omega_p(Y,\Sigma)^{k+2\ell})^{\Phi_{\mathfrak{p}}}}{\Omega_p(Y,\Sigma)^{k+2\ell}}\right)^p$$
$$\times \frac{\langle f_H, \check{f}_H\rangle \sum_{a,b\in\mathcal{B}_K} E^{(a),\nu^p}_{\phi\psi^p}(\underline{A_a}\times\underline{A_b}, j_1\times j_2)\check{f}_H(b) f_H(a)}{\Omega_p(Y,\Sigma)^{p(k+2\ell)}} \mod p$$

But by our assumption on the reflex field K^* and the values of ψ we have $\left(\frac{\Omega_p(Y,\Sigma)^{\Phi_{\mathfrak{p}}}}{\Omega_p(Y,\Sigma)}\right)^p \equiv \frac{\Omega_p(Y,\Sigma)^{\Phi_{\mathfrak{p}}}}{\Omega_p(Y,\Sigma)} \mod p$ (see [39, p. 66]) hence we obtain

$$\frac{\langle f', \check{f}'\rangle \int_{G'} \phi' \, \mu_{F'}}{\Omega_p(Y,\Sigma)^{p(k+2\ell)}} \equiv \frac{(\Omega_p(Y,\Sigma)^{k+2\ell})^{\Phi_{\mathfrak{p}}}}{\Omega_p(Y,\Sigma)^{k+2\ell}}$$
$$\times \frac{\langle f_H, \check{f}_H\rangle \sum_{a,b\in\mathcal{B}_K} E^{(a),\nu^p}_{\phi\psi^p}(\underline{A_a}\times\underline{A_b}, j_1\times j_2)\check{f}_H(b) f_H(a)}{\Omega_p(Y,\Sigma)^{p(k+2\ell)}} \mod p.$$

But we have already shown that

$$\frac{(\Omega_p(Y,\Sigma)^{k+2\ell})^{\Phi_{\mathfrak{p}}}}{\Omega_p(Y,\Sigma)^{k+2\ell}} \times \frac{\langle f_H, \check{f}_H\rangle \sum_{a,b\in\mathcal{B}_K} E^{(a),\nu^p}_{\phi\psi^p}(\underline{A_a}\times\underline{A_b}, j_1\times j_2)\check{f}_H(b) f_H(a)}{\Omega_p(Y,\Sigma)^{p(k+2\ell)}}$$
$$\equiv \langle f, \check{f}\rangle \frac{\int_G \phi \, \mu_F}{\Omega_p(Y,\Sigma)^{k+2\ell}} \mod p,$$

which concludes the proof of the theorem. □

8. The "Average Torsion-Congruences"

We start by fixing an orthogonal basis $B = \{f_i\}$ of $\mathcal{S}_k(D(\mathfrak{c})$ where the f_i's are Hecke eigenforms for all the Hecke operators away from $\mathfrak{c}$ (see [40, Proposition 20.4 (1)]). Note that in (*loc. cit.*) the Hecke algebra is taken trivial for $v|\mathfrak{c}$ (see equation (11.10.7) in (*loc. cit.*)). This justifies our consideration of only the "good" Hecke operators. Moreover thanks to [42, paragraph 28.1] we have that we may pick this basis to be in $\mathcal{S}_k(D(\mathfrak{c}), \overline{\mathbb{Q}})$. For every element $\sigma \in G_{\mathbb{Q}_p}$ and f in the basis we define $f^\sigma(x) := (f(x))^\sigma$. We note that if f is an eigenfuction for all the "good" Hecke operators so is also f^σ, and f can be associated to an automorphic representation π then f^σ can be associated to the automorphic representation π^σ, the action being on the coefficients of the finite part π_f of π. We can pick the basis B such that $f_i^\sigma \in B$ if $f_i \in B$ for $\sigma \in G_{\mathbb{Q}_p}$. The same considerations apply for the situation of F'. We write B' for this basis. However here we will need to consider yet another action, that of the Galois group $\Gamma = \mathrm{Gal}(F'/F)$. For an element $\gamma \in \Gamma$ and $f' \in B'$ we may consider also the form ${}^\gamma f' := f' \circ \gamma$ defined by the composition of the action of Γ, induced form the action on $\mathbb{A}_{F'}$ and the function f'. We note here that if f' corresponds to an automorphic representation π' then ${}^\gamma f'$ to the representation ${}^\gamma \pi' := \pi' \circ \gamma$. We pick the basis B' so that if $f' \in B'$ then also ${}^\gamma f' \in B'$.

We introduce the notation $\Omega_p(k, Y, \Sigma) := \Omega_p(Y, \Sigma)^k$ and similarly for the other p-adic and archimedean periods. For an element $a \in \mathcal{B}_F$ we define the modular forms J_a on pairs $(\underline{A_b}, j_2)$ by

$$J_a((\underline{A_b}, j_2)) := \frac{\mathrm{Frob}_p(E^{(a),\nu^p}_{\phi\psi^p})(\underline{A_a} \times \underline{A_b}, j_1 \times j_2)}{\Omega_p(pk + 2\ell, Y, \Sigma)}$$

and hence

$$J_a((\underline{A_b}, j_2)) = \Big(\frac{\Omega_p(pk + 2\ell, Y, \Sigma)^{\Phi_\mathfrak{p}}}{\Omega_p(pk + 2\ell, Y, \Sigma)}\Big)\Big(\frac{E^{(a),\nu^p}_{\phi\psi^p}(\underline{A_a} \times \underline{A_b}, j_1 \times j_2)}{\Omega_p(pk + 2\ell, Y, \Sigma)}\Big)^{\Phi_\mathfrak{p}},$$

and similarly for an element $a' \in \mathcal{B}_{K'}$ the function

$$J'_{a'}((\underline{A_{b'}}, j')) := \frac{E^{(a'),\nu'}_{\phi'\psi'}(\underline{A_{a'}} \times \underline{A_{b'}}, j'_1 \times j'_2)}{\Omega_p(k + 2\ell, Y', \Sigma')},$$

and we also recall that $\Omega_p(Y', \Sigma') = \Omega_p(Y, \Sigma)^p$.

Proposition 8.1. — *The modular forms J_a and $J'_{a'}$ are $\mathbb{Z}_p$-valued when ϕ and ϕ' are $\mathbb{Z}_p$ valued locally constant functions. Moreover we have that*

$$J_a((\underline{A_b}, j_2)) = \Big(\frac{\Omega_p(pk + 2\ell, Y, \Sigma)^{\Phi_\mathfrak{p}}}{\Omega_p(pk + 2\ell, Y, \Sigma)}\Big)\frac{E^{(a),\nu^p}_{\phi\psi^p}(\underline{A_a} \times \underline{A_b}, j_1 \times j_2)}{\Omega_p(pk + 2\ell, Y, \Sigma)}$$

Proof. — It is enough to prove that

$$\frac{E^{(a),\nu^p}_{\phi\psi^p}(\underline{A_a} \times \underline{A_b}, j_1 \times j_2)}{\Omega_p(pk + 2\ell, Y, \Sigma)}, \quad \frac{E^{(a'),\nu'}_{\phi'\psi'}(\underline{A_{a'}} \times \underline{A_{b'}}, j'_1 \times j'_2)}{\Omega_p(k + 2\ell, Y', \Sigma')} \in \mathbb{Z}_p,$$

since by our assumptions we have that $\frac{\Omega_p(Y,\Sigma)^{\Phi_{\mathfrak{p}}}}{\Omega_p(Y,\Sigma)} \in \mathbb{Z}_p$. We prove the former, the later can be proved similarly. As we have already seen we have

$$\frac{E^{(a),\nu^p}_{\phi\psi^p}(\underline{A_a} \times \underline{A_b}, j_1 \times j_2)}{\Omega_p(pk+2\ell, Y, \Sigma)} = \frac{E^{(a),\nu^p}_{\phi\psi^p}(\underline{A_a} \times \underline{A_b}, \omega_\infty(A_a) \times \omega_\infty(A_b))}{\Omega_\infty(pk+2\ell, Y, \Sigma)}$$

Viewing $\frac{E^{(a),\nu^p}_{\phi\psi^p}(\underline{A_a} \times \underline{A_b}, \omega_\infty(A_a) \times \omega_\infty(A_b))}{\Omega_\infty(pk+2\ell, Y, \Sigma)}$ as a modular form on the variable b we have that

$$g_a(b) := \frac{E^{(a),\nu^p}_{\phi\psi^p}(\underline{A_a} \times \underline{A_b}, \omega_\infty(A_a) \times \omega_\infty(A_b))}{\Omega_\infty(pk+2\ell, Y, \Sigma)} = \sum_i c_i(a) f_i(b), \forall b \in \mathcal{B}_F,$$

where we note that $\overline{f_i(b)} = \check{f}_i(b)$. In order to conclude the proposition it is now enough to show that if for any $\sigma \in G_{\mathbb{Q}_p}$ and i an index of B we define the index $\sigma(i)$ by the equation $f_i^\sigma = f_{\sigma(i)}$, then we have that $c_i^\sigma(a) = c_{\sigma(i)}(a)$.

Indeed we first note that since the f_i's form an orthogonal basis we have that $c_i(a) = \frac{(g,f_i)}{(f_i,f_i)} = \frac{\langle g,\check{f}_i\rangle}{\langle f_i,\check{f}_i\rangle}$. But by definition we have

$$(g_a, f_i) = \sum_{b \in \mathcal{B}_F} \frac{E^{(a),\nu^p}_{\phi\psi^p}(\underline{A_a} \times \underline{A_b}, \omega_\infty(A_a) \times \omega_\infty(A_b))}{\Omega_\infty(pk+2\ell, Y, \Sigma)} \check{f}_i(b).$$

But we have already computed this quantity (see also [40, (22.11.3)], but note the difference on the normalization of the weights). Indeed if we write $\phi = \sum_\chi c_\chi \chi$ for χ finite order characters then we have that

$$(g_a, f_i) = \sum_\chi c_\chi C \times \frac{L_S(BC(\pi_i), \chi\psi^p, \ell)}{\Omega_\infty(pk+2\ell, Y, \Sigma)} Z_S(\pi_i, \check{\pi}_i, \chi\psi^p, f_\Phi) \check{f}_i(a),$$

for some constant C that does not depend on f_i. Here one should remark that for the local integrals at the places above p since we are taking $(\mathfrak{c}, p) = 1$ we have that the corresponding automorphic representations is spherical at every $v|p$. In particular we have that the section f_Φ in the definition of the Eisenstein series is independent of the various π's. Hence

$$c_i(a) = \frac{(g_a, f_i)}{(f_i, f_i)} = \frac{\check{f}_i(a)}{(f_i, f_i)} \sum_\chi c_\chi C \times \frac{L_S(BC(\pi_i), \chi\psi^p, \ell)}{\Omega_\infty(pk+2\ell, Y, \Sigma)} Z_S(\pi_i, \check{\pi}_i, \chi\psi^p, f_\Phi) \in \overline{\mathbb{Q}_p}.$$

Since we assume that ϕ is $\mathbb{Z}_p$ valued we have that $\sum_\chi c_\chi \chi = (\sum_\chi c_\chi \chi)^\sigma = \sum_\chi c_\chi^\sigma \chi^\sigma$ for all $\sigma \in G_{\mathbb{Q}_p}$. Here χ^σ denotes the character obtained from χ by applying σ on its values. Since the characters χ form a basis for the locally constant functions we have that $c_{\chi^\sigma} = c_\chi^\sigma$. Hence in order to establish our claim it is enough to show that the algebraic L-values have the wishing reciprocity law under the action of $G_{\mathbb{Q}_p}$. That is to show (note that ψ is taken $\mathbb{Z}_p$-valued)

$$\left(\frac{L_S(BC(\pi_i),\chi\psi^p,\ell)}{\Omega_\infty(pk+2\ell,Y,\Sigma)}Z_S\big(\pi_i,\check{\pi}_i,\chi\psi^p,f_\Phi\big)\right)^\sigma = \frac{L_S(BC(\pi_i^\sigma),\chi^\sigma\psi^p,\ell)}{\Omega_\infty(pk+2\ell,Y,\Sigma)}Z_S(\pi_i^\sigma,\check{\pi}_i{}^\sigma,\chi^\sigma\psi^p,f_\Phi^\sigma).$$

This is in general not the case if χ has an anticyclotomic part, due to the "wrong" local periods at the primes above p. But in our case χ is always cyclotomic and hence the above quantity has the right reciprocity properties. Indeed this follows from the results in [22, Corollary 3.5.9] after noticing that for χ cyclotomic the periods there corresponding to the character χ are the epsilon factors that appear in the formula above. Of course if χ had an anticyclotomic part, then this is not always the case (see [6] for a similar discussion). □

Now we are ready to prove the last theorem of this work. We recall it here.

Theorem 8.2 (Average Torsion Congruences). — *Let $n=2$, then for all $a,b\in\mathcal{B}_K$ we have*

(1) *Let ε be a $\mathbb{Z}_p$ valued locally constant function on $G_{F'}$ with $\varepsilon^\gamma=\varepsilon$ for all $\gamma\in\Gamma$. Then we have the congruences,*

$$\left(\frac{\Omega_p(2\ell,Y,\Sigma)^{\Phi_\mathfrak{p}}}{\Omega_p(2\ell,Y,\Sigma)}\right)\int_{G_F}\varepsilon\circ\mathrm{ver}\ d\mu_{F,(a,b)}\equiv\int_{G_{F'}}\varepsilon\ d\mu_{F',(a,b)}\mod p,$$

where it is implicitly assume in the statement above that both sides are p-adically integral.

(2) *If we assume that F'/F is unramified at p then there exists a constant $c(a,b)\in\mathbb{Z}_p$ such that*

$$c(a,b)\left(\frac{\Omega_p(Y,\Sigma)^{\Phi_\mathfrak{p}}}{\Omega_p(Y,\Sigma)}\right)^{2\ell}\mathrm{ver}\left(\mu_{(F,(a,b))}\right)\equiv c(a,b)\mu_{(F',(a,b))}\mod T,$$

that is the torsion congruences hold for all twisted normalized measures $c(a,b)\mu_{(F,(a,b))}$ and $c(a,b)\mu_{(F',(a,b))}$, $a,b\in\mathcal{B}_K$. The constant $c(a,b)$, which also depends on the choice of the basis $\{f'_j\}$, is defined as the smallest power of p so that

$$c(a,b)\frac{\Delta^*(\check{f}'_i)(a)\Delta^*(f'_i)(b)}{\langle f'_i,f'_i\rangle}$$

is integral for all these f'_i which do not belong to a representation π'_i that comes from base change from F.

Proof. — Now we fix an $a \in \mathcal{B}_F$ and define $a' = \imath(a) \in \mathcal{B}_{F'}$. Then from the congruences that we proved for the Eisenstein series and introducing again the notation $g_a(b) := J_a((\underline{A_b}, j_2))$ and $g_{a'}(b') := J'_{a'}((\underline{A_{b'}}, j'_2))$ we have $g_a(b) \equiv g'_{a'}(\imath(b)) \mod p$, for all $b \in \mathcal{B}_F$. However, after writing $\phi := \varepsilon \circ \mathrm{ver}$ we also observe that

$$g_a(b) \equiv \Big(\frac{\Omega_p(2\ell, Y, \Sigma)^{\Phi_\mathfrak{p}}}{\Omega_p(2\ell, Y, \Sigma)}\Big) \frac{E^{(a),\nu}_{\phi\psi}(\underline{A_a} \times \underline{A_b}, j_1 \times j_2)}{\Omega_p(k + 2\ell, Y, \Sigma)} \mod p$$

since

$$\Big(\frac{\Omega_p(k, Y, \Sigma)^{\Phi_\mathfrak{p}}}{\Omega_p(k, Y, \Sigma)}\Big) \frac{E^{(a),\nu^p}_{\phi\psi^p}(\underline{A_a} \times \underline{A_b}, j_1 \times j_2)}{\Omega_p(pk + 2\ell, Y, \Sigma)} \equiv \frac{E^{(a),\nu}_{\phi\psi}(\underline{A_a} \times \underline{A_b}, j_1 \times j_2)}{\Omega_p(k + 2\ell, Y, \Sigma)} \mod p,$$

as all the above values are in $\mathbb{Z}_p$ and the property of the p-adic periods with respect to Frobenius. Equivalently we can write the above equation as

$$\Big(\frac{\Omega_p(pk, Y, \Sigma)^{\Phi_\mathfrak{p}}}{\Omega_p(pk, Y, \Sigma)}\Big) \frac{E^{(a),\nu^p}_{\phi\psi^p}(\underline{A_a} \times \underline{A_b}, j_1 \times j_2)}{\Omega_p(pk + 2\ell, Y, \Sigma)} \equiv \frac{E^{(a),\nu}_{\phi\psi}(\underline{A_a} \times \underline{A_b}, j_1 \times j_2)}{\Omega_p(k + 2\ell, Y, \Sigma)} \mod p,$$

from which we conclude our claim. After writing $g_a = \sum_i c_i(a) f_i$ and $g'_{a'} = \sum_j c_j(a') f'_j$ we have that for any $b \in \mathcal{B}_K$

$$\sum_i c_i(a) f_i(b) \equiv \sum_j c'_j(a') f'_j(\imath(b)) \mod p.$$

The coefficients $c_i(a)$ and $c'_j(a')$ have been computed above. In particular, if we write $\varepsilon = \sum'_\chi c_{\chi'} \chi'$ and hence $\varepsilon \circ \mathrm{ver} = \sum_{\chi'} c_{\chi'} \chi' \circ \mathrm{ver}$ we have that

$$c_i(a) = \sum_{\chi'} c_{\chi'} \frac{\check{f}_i(a)}{(f_i, f_i)} C \frac{L_S(BC(\pi_i), \chi' \circ \mathrm{ver} \cdot \psi^p, \ell)}{\Omega_\infty(pk + 2\ell, Y, \Sigma)} Z_S(\pi_i, \check{\pi}_i, \chi' \circ \mathrm{ver} \cdot \psi^p, f_\Phi),$$

and hence

$$\sum_i c_i(a) f_i(b) = \Big(\frac{\Omega_p(2\ell, Y, \Sigma)^{\Phi_\mathfrak{p}}}{\Omega_p(2\ell, Y, \Sigma)}\Big) \int_{G_F} \varepsilon \circ \mathrm{ver} \quad d\mu_{F,(a,b)}.$$

Similarly,

$$c'_j(a') = \sum_{\chi'} c_{\chi'} \frac{\check{f}'_j(a')}{(f'_j, f'_j)} C \frac{L_S(BC(\pi'_j), \chi'\psi', \ell)}{\Omega_\infty(k + 2\ell, Y', \Sigma')} Z_S(\pi'_j, \check{\pi}'_j, \chi'\psi', f'_\Phi),$$

and hence $\sum_j c'_j(a') f'_j(\imath(b)) = \int_{G_{F'}} \varepsilon \, d\mu_{F',(a',b')}$.

Hence we conclude the first statement of the theorem. For the second statement we start by splitting the basis $\mathcal{B}_{F'}$ in two parts. We write $\mathcal{B}^{bc}_{F'}$ consisting of eigenforms that are coming from base change from F and $\mathcal{B}^{nbc}_{F'}$ for the rest. Our assumption that the extension F'/F is unramified at p implies that if $\pi' \in \mathrm{Rep}(G', \mathfrak{c})$ and ${}^\gamma\pi' = \pi'$ for all $\gamma \in \Gamma$ then we have that there exists $\pi \in \mathrm{Rep}(G,\mathfrak{c})$ such that π' is the base change of π from F to F'. Here of course the notation π'^{γ} means $\pi' \circ \gamma$. Then from the above consideration we need to show that if we define the measure

$$\mu^{nbc}_{F',(a,b)} := \sum_{f'_i \sim \pi' \in \mathrm{Rep}_{NBC}(G',\mathfrak{c}')} c(a,b)\frac{\Delta^*(\check{f}'_i)(a)\Delta^*(f'_i)(b)}{\langle f'_i, f'_i\rangle}\mu^{(f'_i)}_{(\pi',\psi')} \in \mathbb{Z}_p[[G_{F'}]].$$

Here we sum over all f'_i that are not base changed from F. Here we write $f'_i \sim \pi'_i$ to indicate that f'_i is associated to π'_i. Then we need to show that $\mu^{nbc}_{F',(a,b)} \in T$. Hence it is enough to show that for all π'_i that are not base changed and for all $\gamma \in \Gamma$ we have

$$\frac{\Delta^*(\check{f}'_i)(a)\Delta^*(f'_i)(b)}{\langle f'_i, \check{f'}_i\rangle}(\mu^{(f'_i)}_{(\pi'_i,\psi')})^{\gamma^{-1}} = \frac{\Delta^*({}^{\gamma}\check{f}'_i)(a)\Delta^*({}^{\gamma}f'_i)(b)}{\langle {}^{\gamma}f'_i, {}^{\gamma}\check{f'}_i\rangle}\mu^{({}^{\gamma}f'_i)}_{({}^{\gamma}\pi'_i,\psi')}.$$

Indeed this will imply that the sum

$$\sum_{\gamma\in\Gamma}\frac{\Delta^*(\check{f}'_i)(a)\Delta^*(f'_i)(b)}{\langle f'_i, \check{f'}_i\rangle}(\mu^{(f'_i)}_{(\pi'_i,\psi')})^{\gamma^{-1}} = \sum_{\gamma\in\Gamma}\frac{\Delta^*({}^{\gamma}\check{f}'_i)(a)\Delta^*({}^{\gamma}f'_i)(b)}{\langle {}^{\gamma}f'_i, {}^{\gamma}\check{f'}_i\rangle}\mu^{({}^{\gamma}f'_i)}_{({}^{\gamma}\pi'_i,\psi')}$$

is an element of the trace ideal T. We first observe that

$$\frac{\Delta^*({}^{\gamma}\check{f}'_i)(a)\Delta^*({}^{\gamma}f'_i)(b)}{\langle {}^{\gamma}f'_i, {}^{\gamma}\check{f'}_i\rangle} = \frac{\Delta^*(\check{f}'_i)(a)\Delta^*(f'_i)(b)}{\langle f'_i, \check{f'}_i\rangle}$$

since ${}^{\gamma}f'_i(x) := f'_i(x^{\gamma})$. Hence we need to show that $(\mu^{(f'_i)}_{(\pi',\psi')})^{\gamma^{-1}} = \mu^{({}^{\gamma}f'_i)}_{({}^{\gamma}\pi',\psi')}$. This last can be seen easily from the interpolation properties of the measures. Indeed we have

$$\begin{aligned}\frac{1}{\Omega_p(k+2\ell,Y,\Sigma)}\int_{G'_{\mathfrak{c}}}\chi d\mu^{({}^{\gamma}f'_i)}_{{}^{\gamma^{-1}}\pi'_i,\psi'}&\\ &= C\frac{L_S(BC(\pi'_i),\chi^{\gamma^{-1}}\psi',\ell)}{\Omega_\infty(k+2\ell,Y',\Sigma')}Z_S(\pi'_i,\check{\pi}_i{}',\chi^{\gamma^{-1}}\psi',f_\Phi),\end{aligned}$$

where $\chi^{\gamma^{-1}}$ as an ideal character is given by $\chi^{\gamma^{-1}}(\mathfrak{a}) = \chi(\mathfrak{a}^{\gamma^{-1}})$. In particular we see that this operation of γ simply permutes the various factors. Hence we see easily that $L_S(BC(\pi'_i),\chi^{\gamma^{-1}}\psi',\ell) = L_S(BC({}^{\gamma}\pi'_i),\chi\psi',\ell)$ and similarly for the rest of the factors. $\square$

9. Appendix

We introduce the following general setting. Let p be an odd prime number. We write F for a totally real field and F' for a totally real Galois extension with $\Gamma := \mathrm{Gal}(F'/F)$ of order p. We assume that the extension is unramified outside p. We write $G_F := \mathrm{Gal}(F(p^\infty)/F)$ where $F(p^\infty)$ is the maximal abelian extension of F unramified outside p (may be ramified at infinity). We make the similar definition for F'. Our assumption on the ramification of F'/F implies that there exist a transfer map $\mathrm{ver}\colon G_F \to G_{F'}$ which induces also a map $\mathrm{ver}\colon \mathbb{Z}_p[[G_F]] \to \mathbb{Z}_p[[G_{F'}]]$, between the Iwasawa algebras of G_F and $G_{F'}$, both of them taken with coefficients in $\mathbb{Z}_p$. Let us now consider a motive M/F (by which we really mean the usual realizations of it and their compatibilities) defined over F such that its p-adic realization has coefficients in $\mathbb{Z}_p$. Then under some assumptions on the critical values of M and some ordinarity assumptions at p (to be made more specific later) it is conjectured that there exists an element $\mu_F \in \mathbb{Z}_p[[G_F]]$ that interpolates the critical values of M/F twisted by characters of G_F. Similarly we write $\mu_{F'}$ for the element in $\mathbb{Z}_p[[G_{F'}]]$ associated to M/F', the base change of M/F to F'. Then the so-called torsion congruences read

$$\mathrm{ver}(\mu_F) \equiv \mu_{F'} \mod T,$$

where T is the trace ideal in $\mathbb{Z}_p[[G'_F]]^\Gamma$ generated by the elements $\sum_{\gamma\in\Gamma} \alpha^\gamma$ with $\alpha \in \mathbb{Z}_p[[G'_F]]$. We now sketch the proof of the following theorem following a similar proof given by Ritter and Weiss [37] for the case M is the Tate motive.

Theorem 9.1. — *A necessary and sufficient condition for the "torsion congruences" to hold is the following:*

For every locally constant $\mathbb{Z}_p$-valued function ε of $G_{F'}$ satisfying $\varepsilon^\gamma = \varepsilon$ for all $\gamma \in \Gamma$ the following congruences hold

$$\int_{G_F} \varepsilon \circ \mathrm{ver}(x)\mu_F(x) \equiv \int_{G_{F'}} \varepsilon(x)\mu_{F'}(x) \mod p\mathbb{Z}_p.$$

We start by recalling some notations of [37]. For a coset x of an open subgroup U of G_F we set

$$\delta^{(x)}(g) = \begin{cases} 1, & g \in x \\ 0, & \text{otherwise.} \end{cases}$$

We define for the given motive M the partial L-function $L(M, \delta^{(x)}, s) = \sum_j c_j L(M, \chi_j, s)$, where χ_j are finite order characters of G_F/U such that $\delta^{(x)}(g) = \sum_j c_j \chi_j(g)$ and $L(M, \chi_j, s)$ is the standard twisted L-function of M by χ_j. We call an open subgroup of G_F admissible if $\mathcal{N}_F(U) \subset 1 + p\mathbb{Z}_p$. We define $m_F(U) \geqslant 1$ by $\mathcal{N}_F(U) = 1 + p^{m_F(U)}\mathbb{Z}_p$. Here $\mathcal{N}_F\colon G_F \to \mathbb{Z}_p$ stands for the cyclotomic character. The following lemma is proved in [37].

Lemma 9.2. — *$\mathbb{Z}_p[[G_F]] = \varprojlim_U \mathbb{Z}_p[G_F/U]/p^{m_F(U)}\mathbb{Z}_p[G_F/U]$ with U running over the cofinal system of admissible open subgroups of G_F.*

We now assume that the motive is critical and satisfy the usual ordinarity assumption at p. Moreover we assume that its p-adic realization has coefficients in $\mathbb{Z}_p$. Then conjecturally there exits a measure $\mu_F \in \mathbb{Z}_p[[G_F]]$ such that for any finite order character χ of G_F we have

$$\int_{G_F} \chi(g)\mu_F(g) = L^*(M,\chi) \in \mathbb{Z}_p[\chi],$$

where $L^*(M,\chi,)$ involves the critical value $L(M,\chi,0)$ of M twisted by the finite order character χ, some archimedean periods related to M, a modification of the Euler factors above p, $L_p(M,\chi,s)$, and finally some epsilon factors above p of the corresponding representation $M_p \otimes \chi$.

In the same spirit as above, if $\delta^{(x)}$ is the characteristic function of a coset of an open subgroup U we define $L^*(M,\delta^{(x)}) := \sum_j c_j L^*(M,\chi_j)$. Then by the very definition of the element μ_F we have that its image in $\mathbb{Z}_p[G_F/U]/p^{m(U)}$ is given by $\sum_{x\in G_F/U} L^*(M,\delta^{(x)})\cdot x \mod p^{m(U)}$. We finally need the following lemma, which is the analogues to Lemma 3(2) of [37].

Lemma 9.3. — *Let y be a coset of a Γ-stable admissible open subgroup of $G_{F'}$. Then $L^*(M/F',\delta^{(y)}_{F'}) = L^*(M/F',\delta^{(y^\gamma)}_{F'})$, for all $\gamma\in\Gamma$. In particular we have that $\mu_{F'} \in \mathbb{Z}_p[[G_{F'}]]^\Gamma$.*

Proof. — The key observation is that M/F' is the base-change of M/F. Obviously it suffices to show the statement for finite order characters. That is to show $L^*(M/F',\chi) = L^*(M/F',\chi^\gamma)$, for χ a finite order character of $G_{F'}$. We recall that

$$L^*(M/F',\chi) = e_p(M,\chi)\mathcal{L}_p(M,\chi)\frac{L_{(S,p)}(M/F',\chi,0)}{\Omega_\infty(M)},$$

where $L_{(S,p)}(M/F',\chi,0)$ is the critical value at $s=0$ of the L-function $L(M/F',\chi,s)$ with the Euler factors at S and those above p removed, where S a finite Γ-invariant set of places of F'. Moreover $\mathcal{L}_p(M,\chi) := \prod_{v|p}\mathcal{L}_v(M,\chi)$ is a modification of the Euler factor at places above p and $e_p(M,\chi) := \prod_{v|p} e_v(M,\chi)$, the local epsilon factors above p. We now observe that we have that $L_{(S,p)}(M/F',\chi,s) = L_{(S,p)}(M/F',\chi^\gamma,s)$ since by the inductive properties of the L-functions we have that $L_{(S,p)}(M/F',\chi,s)$ equals

$$L_{(S,p)}(M/F,\mathrm{ind}_F^{F'}\chi,s) = L_{(S,p)}(M/F,\mathrm{ind}_F^{F'}\chi^\gamma,s) = L_{(S,p)}(M/F',\chi^\gamma,s).$$

Similarly one shows that $\mathcal{L}_p(M,\chi) = \mathcal{L}_p(M,\chi^\gamma)$ and $e_p(M,\chi) = e_p(M,\chi^\gamma)$ as the right sides of the equations are nothing more than permutations of the left sides of the equation (again the fact that M/F' is the base change of M/F is needed). □

The following lemma has been shown by Ritter and Weiss.

Lemma 9.4. — *If V is an admissible open subgroup of $G_{F'}$ and U an admissible open subgroup of G_F in* $\mathrm{ver}^{-1}(V)$*, then $m_F(U) \geqslant m_{F'}(V) - 1$*

In particular, as it is explained in [37], one can conclude from this lemma that the map $\mathrm{ver}\colon \mathbb{Z}_p[[G_F]] \to \mathbb{Z}_p[[G_{F'}]]$ induces a map

$$\varprojlim_U \mathbb{Z}_p[G_F/U]/p^{m_F(U)} \to \varprojlim_{V,\Gamma-\text{stable}} \mathbb{Z}_p[G_{F'}/V]p^{m_{F'}(V)-1}.$$

Now we are ready to prove Theorem 9.1 following the strategy of Ritter and Weiss in [37].

Proof of Theorem 9.1. — We consider the components of $\mu_{F'}$ and $\mathrm{ver}(\mu_F)$ in $\mathbb{Z}_p[G_{F'}/V]p^{m_{F'}(V)-1}$ for a Γ-stable admissible open subgroup V of $G_{F'}$. We note that $\mathrm{ver}(\mu_F)$ is the image under the transfer map of the U- component of μ_F where $U := \mathrm{ver}^{-1}(V) \subseteq G_F$ which contains $N := \ker(\mathrm{ver})$. These components are the images of

(1) $\sum_{y\in G_{F'}/V} L^*(M/F', \delta^{(y)})y$,
(2) $\sum_{x\in G_F/U} L^*(M/F, \delta^{(x)})\,\mathrm{ver}(x)$

in $(\mathbb{Z}_p[G_{F'}/V]/p^{m_{F'}(V)-1})^\Gamma$. We now show that the sums in (1) and (2) are congruent modulo $T(V)$, the image of the trace ideal in $(\mathbb{Z}_p[G_{F'}/V]/p^{m_{F'}(V)-1})^\Gamma$. We consider the following two case

y **is fixed by** Γ**.** Then $\delta_{F'}^{(y)}$ is a locally constant function as in the Theorem 9.1, hence we have $L^*(M/F', \delta_{F'}^{(y)}) \equiv L^*(M/F, \delta_F^{(y)} \circ \mathrm{ver}) \mod p$.

If $y = \mathrm{ver}(x)$ then $\delta_{F'}^{(y)} \circ \mathrm{ver} = \delta_F^{(x)}$. Then the corresponding summands in (i) and (ii) cancel out modulo $T(V)$ since $p\alpha$ is a Γ trace whenever α is Γ-invariant. If $y \notin \Im(\mathrm{ver})$ then $\delta_{F'}^{(y)} \circ \mathrm{ver} = 0$ and then again by the theorem we have $L^*(M/F', \delta_{F'}^{(y)}) \equiv 0$ modulo p, hence the corresponding summand vanishes modulo $T(V)$.

y **is not fixed by** Γ**.** Then we have by Lemma 9.3 that $L^*(M/F', \delta_{F'}^{(y)}) = L^*(M/F', \delta_{F'}^{(y^\gamma)})$, for all $\gamma \in \Gamma$. That means that the Γ orbit of y yields the sum $L^*(M/F', \delta_{F'}^{(y)}) \sum_{\gamma\in\Gamma} y^\gamma$, which is in $T(V)$. □

BIBLIOGRAPHY

[1] T. Bouganis, “Non abelian p-adic L-functions and Eisenstein series of unitary groups II; the CM-method”, in preparation.

[2] ———, “Non abelian p-adic L-functions and Eisenstein series of unitary groups; the Constant Term Method”, in preparation.

[3] ———, “Special values of L-functions and false Tate curve extensions”, *J. Lond. Math. Soc. (2)* **82** (2010), no. 3, p. 596-620, With an appendix by Vladimir Dokchitser.

[4] ———, “Non-abelian congruences between special values of L-functions of elliptic curves: the CM case”, *Int. J. Number Theory* **7** (2011), no. 7, p. 1883-1934.

[5] T. Bouganis & F. Nuccio, “Kongruenzen zwischen abelschen pseudo-Maßen und die Shintani Zerlegung”, preprint in German.

[6] T. Bouganis & O. Venjakob, “On the non-commutative main conjecture for elliptic curves with complex multiplication”, *Asian J. Math.* **14** (2010), no. 3, p. 385-416.

[7] J. Coates, “Motivic p-adic L-functions”, in *L-functions and arithmetic (Durham, 1989)*, London Math. Soc. Lecture Note Ser., vol. 153, Cambridge Univ. Press, Cambridge, 1991, p. 141-172.

[8] J. Coates, T. Fukaya, K. Kato, R. Sujatha & O. Venjakob, “The GL_2 main conjecture for elliptic curves without complex multiplication”, *Publ. Math. Inst. Hautes Études Sci.* (2005), no. 101, p. 163-208.

[9] D. Delbourgo & T. Ward, “Non-abelian congruences between L-values of elliptic curves”, *Ann. Inst. Fourier (Grenoble)* **58** (2008), no. 3, p. 1023-1055.

[10] ———, “The growth of CM periods over false Tate extensions”, *Experiment. Math.* **19** (2010), no. 2, p. 195-210.

[11] T. Dokchitser & V. Dokchitser, “Computations in non-commutative Iwasawa theory”, *Proc. Lond. Math. Soc. (3)* **94** (2007), no. 1, p. 211-272, With an appendix by J. Coates and R. Sujatha.

[12] E. E. Eischen, “A p-adic Eisenstein Measure for Unitary Groups”, 2011, Preprint arXiv:1106.3692v1, to appear in J. Reine Angew. Math.

[13] ———, “p-adic differential operators on automorphic forms on unitary groups”, *Ann. Inst. Fourier (Grenoble)* **62** (2012), no. 1, p. 177-243.

[14] E. E. Eischen, M. Harris, J.-S. Li & C. Skinner, “p-adic L-functions for unitary Shimura varieties, II”, in preparation.

[15] T. Fukaya & K. Kato, “A formulation of conjectures on p-adic zeta functions in noncommutative Iwasawa theory”, in *Proceedings of the St. Petersburg Mathematical Society. Vol. XII*, Amer. Math. Soc. Transl. Ser. 2, vol. 219, Amer. Math. Soc., Providence, RI, 2006, p. 1-85.

[16] P. Garrett, “Values of Archimedean zeta integrals for unitary groups”, in *Eisenstein series and applications*, Progr. Math., vol. 258, Birkhäuser Boston, Boston, MA, 2008, p. 125-148.

[17] S. Gelbart, I. Piatetski-Shapiro & S. Rallis, *Explicit constructions of automorphic L-functions*, Lecture Notes in Mathematics, vol. 1254, Springer-Verlag, Berlin, 1987, vi+152 pages.

[18] R. Godement & H. Jacquet, *Zeta functions of simple algebras*, Lecture Notes in Mathematics, Vol. 260, Springer-Verlag, Berlin-New York, 1972, ix+188 pages.

[19] T. Hara, “Iwasawa theory of totally real fields for certain non-commutative p-extensions”, *J. Number Theory* **130** (2010), no. 4, p. 1068-1097.

[20] M. Harris, “Unitary groups and Base Change”, notes available at `http://www.math.jussieu.fr/~harris/`.

[21] ———, "L-functions of 2×2 unitary groups and factorization of periods of Hilbert modular forms", *J. Amer. Math. Soc.* **6** (1993), no. 3, p. 637-719.

[22] ———, "L-functions and periods of polarized regular motives", *J. Reine Angew. Math.* **483** (1997), p. 75-161.

[23] ———, "A simple proof of rationality of Siegel-Weil Eisenstein series", in *Eisenstein series and applications*, Progr. Math., vol. 258, Birkhäuser Boston, Boston, MA, 2008, p. 149-185.

[24] M. Harris, J.-S. Li & C. M. Skinner, "The Rallis inner product formula and p-adic L-functions", in *Automorphic representations, L-functions and applications: progress and prospects*, Ohio State Univ. Math. Res. Inst. Publ., vol. 11, de Gruyter, Berlin, 2005, p. 225-255.

[25] ———, "p-adic L-functions for unitary Shimura varieties. I. Construction of the Eisenstein measure", *Doc. Math.* (2006), no. Extra Vol., p. 393-464 (electronic).

[26] H. Hida & J. Tilouine, "Anti-cyclotomic Katz p-adic L-functions and congruence modules", *Ann. Sci. École Norm. Sup. (4)* **26** (1993), no. 2, p. 189-259.

[27] H. Hida, *p-adic automorphic forms on Shimura varieties*, Springer Monographs in Mathematics, Springer-Verlag, New York, 2004, xii+390 pages.

[28] ———, "Serre's conjecture and base change for GL(2)", *Pure Appl. Math. Q.* **5** (2009), no. 1, p. 81-125.

[29] M.-L. Hsieh, "Ordinary p-adic Eisenstein series and p-adic L-functions for unitary groups", *Ann. Inst. Fourier (Grenoble)* **61** (2011), no. 3, p. 987-1059.

[30] ———, "Eisenstein congruence on unitary groups and Iwasawa main conjectures for CM fields", *J. Amer. Math. Soc.* **27** (2014), no. 3, p. 753-862.

[31] M. Kakde, "Proof of the main conjecture of noncommutative Iwasawa theory for totally real number fields in certain cases", *J. Algebraic Geom.* **20** (2011), no. 4, p. 631-683.

[32] ———, "From the classical to the noncommutative Iwasawa theory (for totally real number fields)", in *Non-abelian fundamental groups and Iwasawa theory*, London Math. Soc. Lecture Note Ser., vol. 393, Cambridge Univ. Press, Cambridge, 2012, p. 107-131.

[33] K. Kato, "Iwasawa theory of totally real fields for Galois extensions of Heisenberg type", preprint.

[34] N. M. Katz, "p-adic interpolation of real analytic Eisenstein series", *Ann. of Math. (2)* **104** (1976), no. 3, p. 459-571.

[35] ———, "p-adic L-functions for CM fields", *Invent. Math.* **49** (1978), no. 3, p. 199-297.

[36] J.-S. Li, "Nonvanishing theorems for the cohomology of certain arithmetic quotients", *J. Reine Angew. Math.* **428** (1992), p. 177-217.

[37] J. Ritter & A. Weiss, "Congruences between abelian pseudomeasures", *Math. Res. Lett.* **15** (2008), no. 4, p. 715-725.

[38] ———, "Congruences between abelian pseudomeasures, II", 2010, Preprint arXiv:1001.2091v1.

[39] E. de Shalit, "On monomial relations between p-adic periods", *J. Reine Angew. Math.* **374** (1987), p. 193-207.

[40] G. Shimura, *Euler products and Eisenstein series*, CBMS Regional Conference Series in Mathematics, vol. 93, Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 1997, xx+259 pages.

[41] ———, *Abelian varieties with complex multiplication and modular functions*, Princeton Mathematical Series, vol. 46, Princeton University Press, Princeton, NJ, 1998, xvi+218 pages.

[42] ———, *Arithmeticity in the theory of automorphic forms*, Mathematical Surveys and Monographs, vol. 82, American Mathematical Society, Providence, RI, 2000, x+302 pages.

[43] C. Skinner & E. Urban, "The Iwasawa main conjectures for GL_2", *Invent. Math.* **195** (2014), no. 1, p. 1-277.

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