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ON THE ANALYTICITY OF GENERALIZED EIGENFUNCTIONS (CASE OF REAL VARIABLES)

by **Eberhard GERLACH**

The present note is a direct continuation of Chapter III in our paper [1]; its purpose is to extend the results on analyticity of the generalized eigenfunctions to the case of proper functional Hilbert spaces consisting of functions which are (real-) analytic in a domain in Euclidean space. We continue to use the notation and numbering from Chap. III in [1].

Our basic tool will be the following.

PROPOSITION 4. — *Let* G *be a domain in Euclidean space* $\mathbf{R}^n$, *and* $\mathcal{B}$ *a class of functions defined everywhere in* G *and analytic there, and suppose that these form a proper functional Banach space* $\{\mathcal{B}, G\}$. *Then there exists a common domain* $\tilde{G}$ *in complex space* $\mathbf{C}^n$, *containing* G, *to which all* $f \in \mathcal{B}$ *can be extended analytically.*

Proof. — Since $\{\mathcal{B}, G\}$ is a p.f. Banach space, to every $x \in G$ there is an $L(x) \in \mathcal{B}'$ ($\mathcal{B}'$ is the continuous dual of $\mathcal{B}$) such that $f(x) = \langle f, L(x) \rangle$. This defines a function L from G into $\mathcal{B}'$ which is weakly-* real-analytic. It is well-known that Banachspace-valued functions defined on a complex domain which are weakly or weakly-* analytic are complex-analytic also in the strong topology. We shall show that L is strongly (real-) analytic; then it can be extended to a strongly analytic function $\tilde{L}$ (still into $\mathcal{B}'$) in some complex domain $\tilde{G}$ containing G. Finally each $f \in \mathcal{B}$ will be extended to an analytic function $\tilde{f}$ on $\tilde{G}$ by setting $f(z) = \langle f, \tilde{L}(z) \rangle$ for $z \in \tilde{G}$.

Recall (cf. for instance [2]) that for any function g which is analytic in the fixed domain $D \subset \mathbf{C}^1$ and for any compact $K \subset D$, there exists a finite number $M(g;\ K)$ such that for any choice of $\zeta, \zeta + \alpha, \zeta + \beta$ in K:

$$(6)\quad \left|\frac{1}{\alpha - \beta}\left\{\frac{1}{\alpha}[g(\zeta + \alpha) - g(\zeta)] - \frac{1}{\beta}[g(\zeta + \beta) - g(\zeta)]\right\}\right| \leqslant M(g;\ K).$$

The same is true if instead of D one has a fixed open interval $I \subset \mathbf{R}^1$.

We shall establish existence of the strong derivatives $\frac{\partial}{\partial x_i} L(x)$ in $\mathcal{B}'$. These derivatives exist in the weak-* topology since for each $f \in B$

$$\frac{\partial}{\partial x_i} f(x) = \lim_{h \to 0} \frac{1}{h}(f(x + \varepsilon_i h) - f(x))$$
$$= \lim_{h \to 0} \frac{1}{h}\langle f,\ L(x + \varepsilon_i h) - L(x)\rangle.$$

Let N be a compact neighborhood of x; then there are numbers $M(f;\ N)$ so that for all sufficiently small h and k

$$\left\|\left\langle f, \frac{1}{h - k}\left\{\frac{1}{h}[L(x + \varepsilon_i h) - L(x)] - \frac{1}{k} L(x + \varepsilon_i k) - L(x)\right\}\right\rangle\right\| \leqslant M(f;\ N).$$

Then by the uniform boundedness theorem, there is a constant $M(N)$ such that $\left\|\frac{1}{h - k}\{\dots\}\right\| \leqslant M(N)$. Letting h and k tend to zero, one now obtains existence of the strong derivative $\frac{\partial}{\partial x_i} L(x)$. Since all derivatives of the $f \in \mathcal{B}$ are analytic, the preceding procedure can be repeated; thus L possesses strong derivatives of all orders. It is easy to check that L and all its derivatives are strongly continuous.

The Taylor series for L will converge strongly to the values of L if $\|(\alpha\,!)^{-1} D_\alpha L(x)\|^{\frac{1}{|\alpha|}}$ is uniformly bounded on compacts $K \subset G$, with a bound independent of α. (Here the α_i are non-negative integers, $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$,

$|\alpha| = \Sigma\alpha_i$, $D_\alpha = \partial^{|\alpha|}/\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}$, $\alpha! = \alpha_1! \ldots \alpha_n!$). Since all $f \in \mathcal{B}$ are analytic,

$$|(\alpha!)^{-1}D_\alpha f(x)|^{\frac{1}{|\alpha|}} = |\langle f, (\alpha!)^{-1}D_\alpha L(x)\rangle|^{\frac{1}{|\alpha|}}$$

(for fixed f) is uniformly bounded on compacts $K \subset G$, independent of α. But for variable f, this expression is a sub-additive continuous functional on $\mathcal{B}$. By the uniform boundedness theorem then

$$\sup_{\|f\| \leqslant 1} |\langle f, (\alpha!)^{-1}D_\alpha L(x)\rangle|^{\frac{1}{|\alpha|}} = \Big(\sup_{\|f\|=1} |\langle \ldots \rangle|\Big)^{\frac{1}{|\alpha|}}$$
$$= \|(\alpha!)^{-1}D_\alpha L(x)\|^{\frac{1}{|\alpha|}}$$

is uniformly bounded on compacts $K \subset G$, independent of α. Thus L has a strongly convergent power series expansion in some neighborhood of any point $x \in G$.

For each $x \in G$, let $S(x)$ be the largest open ball in $\mathbf{C}^n$, centered at x, in which the Taylor series for L about x converges and set

$$\tilde{G} = \bigcup_{x \in G} S(x).$$

Then the series expansions yield an analytic continuation $\tilde{L}$ of L from G to $\tilde{G}$. Finally, for $f \in \mathcal{B}$, define

$$\tilde{f}(z) = \langle f, \tilde{L}(z)\rangle \quad \text{for} \quad z \in \tilde{G} \qquad \text{and} \qquad \|\tilde{f}\| = \|f\|;$$

this gives us a p.f. Banach space $\{\tilde{\mathcal{B}}, \tilde{G}\}$ which is isometrically isomorphic to $\{\mathcal{B}, G\}$. The proof of Proposition 4 is complete.

From now on, $\{\mathcal{F}, G\}$ will denote a p.f. Hilbert space consisting of analytic functions on a domain $G \subset \mathbf{R}^n$. Our aim is to extend the results of Corollary 2. III and Theorem 3. III in [1] to such spaces.

The anti-space $\overline{\mathcal{F}}$ of the Hilbert space $\mathcal{F}$ is identified with the dual $\mathcal{F}'$, and $\mathcal{F}$ itself with its continuous anti-dual $\mathcal{F}^*$ $(= \overline{\mathcal{F}}' = \overline{\mathcal{F}'})$ (1) by means of the canonical mappings J and θ:

$\mathcal{F}' = J\mathcal{F}$ *where* J *is the anti-isomorphism* $f \to Jf = (\cdot, f)$

(1) For these notations, cf. L. Schwartz [3].

and

$$\mathcal{F}^* = \theta\mathcal{F} \quad \textit{where } \theta \textit{ is the isomorphism } f \to \theta f = (f, \,.).$$

If K is the reproducing kernel of $\mathcal{F}$ then for $f \in \mathcal{F}$

$$f(x) = (f, \mathrm{K}(.\,, x)) = \langle f, \mathrm{L}(x) \rangle \quad \textit{for every } x \in \mathrm{G}$$

where $\langle \;,\; \rangle$ denotes the pairing of $\mathcal{F}$ and $\mathcal{F}'$. Thus $\mathrm{L}(x) = \mathrm{JK}_x$ and

$$\mathrm{K}(x, y) = (\mathrm{K}_y, \mathrm{K}_x) = \langle \mathrm{K}_y, \mathrm{JK}_x \rangle = \langle \mathrm{J}^{-1}\mathrm{L}(y), \mathrm{L}(x) \rangle.$$

By Proposition 4, L and F extend analytically to a complex domain $\tilde{\mathrm{G}}$; we obtain the p.f. Hilbert space $\{\tilde{\mathcal{F}}, \tilde{\mathrm{G}}\}$ with r.k. $\tilde{\mathrm{K}}$:

$$\tilde{f}(z) = \langle f, \tilde{\mathrm{L}}(z) \rangle = (f, \mathrm{J}^{-1}\tilde{\mathrm{L}}(z))$$

and

$$\tilde{\mathrm{K}}(z, w) = \langle \mathrm{J}^{-1}\tilde{\mathrm{L}}(w), \tilde{\mathrm{L}}(z) \rangle = (\tilde{\mathrm{K}}_w, \tilde{\mathrm{K}}_z).$$

Since the function $\tilde{\mathrm{L}}$ is strongly analytic from $\tilde{\mathrm{G}}$ into $\mathcal{F}'$ and $\tilde{\mathrm{K}}_z = \mathrm{J}^{-1}\tilde{\mathrm{L}}(z)$, we note that $\tilde{\mathrm{K}}(.\,, z)$ is strongly anti-analytic for $z \in \tilde{\mathrm{G}}$ (i.e. the function $\bar{z} \to \tilde{\mathrm{K}}(.\,, z)$ is strongly analytic from $\bar{\tilde{\mathrm{G}}} = \{z | \bar{z} \in \tilde{\mathrm{G}}\}$ into $\mathcal{F}'$). Let U denote the extension isomorphism $\mathrm{U} : \mathcal{F} \to \tilde{\mathcal{F}}$ constructed by Proposition 4. If $\{g_k\}$ is a complete orthonormal system in $\mathcal{F}$, then so is $\{\tilde{g}_k = \mathrm{U}g_k\}$ in $\tilde{\mathrm{F}}$ and $\tilde{\mathrm{K}}(z, w) = \sum_{k=1}^{\infty} \tilde{g}_k(z)\overline{\tilde{g}_k(w)}$ for $z, w \in \tilde{\mathrm{G}}$, i.e., $\tilde{\mathrm{K}}$ is also a « direct » continuation of K.

Corollary 2′. — *Let* G *be an arbitrary domain in* $\mathbf{R}^n$ *and* $\{\mathcal{F}, \mathrm{G}\}$ *any p.f. Hilbert space of functions (real-) analytic in* G. *Then* $\{\mathcal{F}, \mathrm{G}\}$ *is Hilbert-Schmidt expansible.*

Proof. — By Corollary 2, there is an H.S. operator T in $\tilde{\mathcal{F}}$ such that $\tilde{\mathrm{K}}_\zeta \in \mathrm{T}\tilde{\mathcal{F}}$ for all $\zeta \in \tilde{\mathrm{G}}$. Now $\mathrm{S} = \mathrm{U}^{-1}\mathrm{TU}$ is H.S. in $\mathcal{F}$, and $\mathrm{K}_\xi \in \mathrm{S}\mathcal{F}$ for all $\xi \in \mathrm{G}$.

Now let A be a selfadjoint operator in $\mathcal{F}$ with resolution of identity E(.) and spectral measure μ. Then $(f, g) = (\mathrm{U}f, \mathrm{U}g)^{\sim}$ for all $f, g \in \mathcal{F}$. The operator $\tilde{\mathrm{A}} = \mathrm{UAU}^{-1}$ is selfadjoint in $\tilde{\mathcal{F}}$ and unitarily equivalent to A; its resolution of identity is $\tilde{\mathrm{E}}(.) = \mathrm{UE}(.)\mathrm{U}^{-1}$, and μ is also a spectral measure for $\tilde{\mathrm{A}}$.

Both $\mathfrak{F}$ and $\tilde{\mathfrak{F}}$ are H.S.-expansible. Let $\tilde{\Lambda}_{\tilde{G}}$ denote the complement in $\mathbf{R}^1$ of the set of all λ for which

$$\frac{d(\tilde{E}(\lambda)\tilde{K}_w, E(\lambda)\tilde{K}_z)}{d\mu(\lambda)} = \tilde{K}(z, w; \lambda) \quad \textit{exists and is finite for all} \quad z, w \in \tilde{G}$$

(similar definition for Λ_G, without tildas). Then $\tilde{\Lambda}_{\tilde{G}} \supset \Lambda_G$ and $\mu(\tilde{\Lambda}_{\tilde{G}}) = 0$. Let $\tilde{\mathfrak{F}}_{\tilde{G}}^{(\lambda)}$ ($\mathfrak{F}_G^{(\lambda)}$) be the p.f. Hilbert space on $\tilde{G}(G)$ defined by the r.k. $\tilde{K}(.,.;\lambda)$ ($K(.,.;\lambda)$). For $\tilde{f} \in \tilde{\mathfrak{F}}$, let $\tilde{\Lambda}_{\tilde{f},(\tilde{G})}$ be the smallest set containing $\tilde{\Lambda}_{\tilde{G}}$ such that for all $\lambda \notin \Lambda_{\tilde{f},(\tilde{G})}$:

$$\left\{ \begin{array}{l} \dfrac{d(\tilde{E}(\lambda)\tilde{f}, \tilde{E}(\lambda)\tilde{K}_z)}{d\mu(\lambda)} = \tilde{f}(z; \lambda) \quad \textit{exists, is finite} \\ \textit{and} = 0 \quad \textit{whenever} \quad \tilde{K}(z, z; \lambda) = 0, \quad \textit{for all} \quad z \in \tilde{G} \end{array} \right.$$

and

$$\tilde{f}(., \lambda) \in \tilde{\mathfrak{F}}_{\tilde{G}}^{(\lambda)}, \quad \frac{d\|\tilde{E}(\lambda)\tilde{f}\|^2}{d\mu(\lambda)} \quad \textit{exists and equals} \quad \|\tilde{f}(.;\lambda)\|^2_{\tilde{\mathfrak{F}}_{\tilde{G}}^{(\lambda)}}$$

(similar definition for $\Lambda_{f,(G)}$, without tildas). The correspondence $\tilde{f} \to \tilde{f}(.;\lambda)$ defines $\tilde{P}_{\tilde{G}}^{(\lambda)}$ with domain $\tilde{\mathfrak{D}}_{\tilde{G}}^{(\lambda)} = \{\tilde{f} | \lambda \notin \tilde{\Lambda}_{\tilde{f},(\tilde{G})}\}$. For $\tilde{f} \in \tilde{\mathfrak{D}}_{\tilde{G}}^{(\lambda)}$, $f(.;\lambda)$ is just the restriction of $\tilde{f}(.;\lambda)$ to the domain G.

Theorem 3′. — *Let* A *be an arbitrary selfadjoint operator in* $\{\mathfrak{F}, G\}$ *with spectral measure* μ. *Then there is a set* Λ *on the real line,* $\mu(\Lambda) = 0$, *which is determined by Theorem* 3 (*and also Corollary* 2, *Theorem* 11. I, *and the above considerations*) *such that the generalized eignefunctions*

$$\frac{dE(\lambda)f(x)}{d\mu(\lambda)} = f(x; \lambda) \in \mathfrak{F}_G^{(\lambda)} \quad \textit{for} \quad \lambda \notin \Lambda \quad \textit{and} \quad \tilde{f} \in \tilde{\mathfrak{D}}_{\tilde{G}}^{(\lambda)}$$

are real-analytic in the whole domain G.

Proof. — According to the preceding preparations, set $Uf = \tilde{f}$. If $\lambda \notin \Lambda$ and $\tilde{f} \in \tilde{\mathfrak{D}}_{\tilde{G}}^{(\lambda)}$ then $\tilde{f}(.;\lambda)$ is analytic in $\tilde{G}$ by Theorem 3, and consequently its restriction $f(.;\lambda)$ is (real-) analytic in G.

BIBLIOGRAPHY

[1] E. Gerlach, On spectral representation for selfadjoint operators, Expansion in generalized eigenelements, *Ann. Inst. Fourier* (Grenoble), 15, fasc. 2 (1965), 537-574.

[2] E. Hille and R. S. Phillips, Functional Analysis and Semi-Groups, Second Edition, *Am. Math. Soc. Colloqu. Publ.*, Vol. 31, (1957).

[3] L. Schwartz, Sous-espaces hilbertiens d'espaces vectoriels topologiques et noyaux associés, (Noyaux reproduisants), *J. Analyse Math.*, 13 (1964), 115-256.

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