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ON THE ABSOLUTE CESARO SUMMABILITY FACTORS OF FOURIER SERIES (*)

by NIRANJAN SINGH

1.1. Definitions. — *Let* Σa_n *be a given infinite series with* S_n *as its* $n - th$ *partial sum. The series* Σa_n *is said to be absolutely summable* (C, α), *or summable* $|C, \alpha|$, *if the sequence* $\{\sigma_n^\alpha\}$ *is of bounded variation, that is*

$$\sum_n |\sigma_n^\alpha - \sigma_{n-1}^\alpha| < \infty,$$

where $\{\sigma_n^\alpha\}$ *is the* $n - th$ *Cesàro mean of order* α, $\alpha > -1$, *of the sequence* $\{S_n\}$.

If $\{t_n^\alpha\}$ be the $n - th$ Cesàro mean of order α of the sequence $\{na_n\}$, then we have the following identity [6].

$$t_n^\alpha = n(\sigma_n^\alpha - \sigma_{n-1}^\alpha).$$

For any sequence $\{u_n\}$, we write

$$\Delta u_n = u_n - u_{n+1}$$

and

$$\overset{r}{\Delta} u_n = \sum_{p=0}^{\infty} A_p^{-r-1} u_{n+p},$$

provided the series on the right converges.

If S is $a + ve$ integer, then

$$\overset{S}{\Delta}(u_n v_n) = \sum_{r=0}^{S} \binom{s}{r} \overset{r}{\Delta} u_n \overset{S-r}{\Delta} v_{n+r}$$

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By repeated partial summation, we observe that, for $k = 0, 1,...$

$$\sum_{p=0}^{q} A_{n-p}^{r-1} u_p a_p = \sum_{p=0}^{q} S_p^k \overset{k+1}{\Delta} (A_{n-p}^{r-1} u_p) + \sum_{j=0}^{k} \overset{j}{\Delta} (A_{n-q-1}^{r-1} u_{q+1}) S_q^j$$

where S_n^k denotes the $n-th$ Cesàro sum of order k of the sequence $\{S_n\}$. Hence, putting $q = n$, we get

$$(1.1.1) \qquad \sum_{p=0}^{n} A_{n-p}^{r-1} u_p a_p = \sum_{p=0}^{n} S_p^k \overset{k+1}{\Delta} (A_{n-p}^{r-1} u_p).$$

A sequence $\{\lambda_n\}$ is said to be convex, if $\overset{2}{\Delta} \lambda_n \geqslant 0$, and it is said to be hyper-convex of order h, if

$$\overset{h+2}{\Delta} \lambda_n \geqslant 0, \qquad (h = 0, 1, 2, ...).$$

By definition hyper-convexity of order zero is the same as convexity.

Let $f(t)$ be a periodic function with period 2π and integrable in the sense of Lebesgue over $(-\pi, \pi)$. Without any loss of generality we may assume that the constant term in the Fourier series of $f(t)$ is zero, that is

$$\int_{-\pi}^{\pi} f(t)\, dt = 0,$$

and

$$f(t) \sim \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) = \sum_{n=1}^{\infty} A_n(t).$$

We use the following notations:

$$\Phi(t) = \frac{1}{2} \{f(x+t) + f(x-t) - 2f(x)\},$$

$$\Phi_\alpha(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-u)^{\alpha-1} \Phi(u)\, du, \qquad \alpha > 0,$$

$$\Phi_0(t) = \Phi(t),$$

$$\varphi_\alpha(t) = \Gamma(\alpha+1) t^{-\alpha} \Phi_\alpha(t), \qquad \alpha \geqslant 0,$$

$$(F(t))_h = \frac{\partial^h F(t)}{\partial t^h},$$

$$\varepsilon(n) = (\log n)^{-\beta}, \qquad \beta \geqslant 0,$$

$$\varepsilon^{-1}(n) = \frac{1}{\varepsilon(n)}.$$

1.2. Various results on summability factors of Fourier series due to Prasad [12], Izumi and Kwata [5], Cheng [3], Pati [8] and Dikshit [4] were generalized by Pati and Sinha [11] in the form of the following theorem.

THEOREM A. — *Let* h *be an integer* $\geqslant 0$, *and let* $\{\lambda_n\}$ *be a monotonic non-increasing sequence when* $h = 0$, *and a hyper-convex sequence of order* $(h-1)$ *when* $h \geqslant 1$, *such that*

$$\text{(i)}\quad \Sigma \frac{\lambda_n}{n} < \infty, \qquad \text{(ii)}\quad \Sigma n^h \overset{h+1}{\Delta} \lambda_n < \infty.$$

If

$$\int_0^t |\varphi_h(u)|\, du = 0(t),$$

as $t \to 0$, *then* $\sum_{n=1}^{\infty} \lambda_n A_n(x)$ *is summable* $|C, h+1+\delta|$ *for every* $\delta > 0$.

Later on Ahmad [1] obtained the following theorem which includes as a special case for $\beta = 0$ the above theorem of Pati and Sinha.

THEOREM B. — *Let* $\{\lambda_n\}$ *be a sequence such that for all non-negative integral values of* h, $\overset{+1}{\Delta} \lambda_n \geqslant 0$, *and* $\Sigma \frac{\lambda_n}{n} < \infty$.
If

$$\int_0^t |\varphi_h(u)|\, du = 0 \left\{ t\varepsilon^{-1}\left(\frac{1}{t}\right)\right\},$$

as $t \to 0$, *then* $\sum_{n=1}^{\infty} \varepsilon(n+1)\lambda_n A_n(x)$ *is summable* $|C, h+1+\delta|$ *for every* $\delta > 0$.

In this paper we prove the following theorem for summability $|C, 1+h|$ by imposing suitable conditions on the sequence $\{\lambda_n\}$.

We prove the following theorem.

THEOREM. — *Let* $\{\lambda_n\}$ *be a sequence such that for non-negative integral values of* h, $\overset{h+1}{\Delta} \lambda_n \geqslant 0$, *and*

$$(1.2.1) \qquad \Sigma \frac{\lambda_n}{n} (\log n)^{\frac{1}{2}} < \infty.$$

If

$$(1.2.2)\qquad \int_0^t |\varphi_h(u)|\, du = 0\left\{t\varepsilon^{-1}\left(\frac{1}{t}\right)\right\}, \qquad t \to 0,$$

then, $\sum_1^\infty \varepsilon(n+1)\lambda_n A_n(x)$ *is summable* $|C, h+1|$.

It may be remarked that this theorem generalizes the following theorem of the author [14] which in turn, includes a theorem of Pati [10].

THEOREM C. — *Let* $\{\lambda_n\}$ *be a convex sequence such that* $\sum \frac{\lambda_n}{n} (\log n)^{\frac{1}{2}} < \infty$.

If

$$\int_0^t |\Phi(u)|\, du = 0\left\{t\varepsilon^{-1}\left(\frac{1}{t}\right)\right\},$$

as $t \to 0$, *then* $\sum_1^\infty \varepsilon(n+1)\lambda_n A_n(x)$ *is summable* $|C, 1|$.

1.3. For the proof of our theorem we require the following lemmas :

LEMMA 1 [9]. — *Let* $C_{n,\rho}^k$ *and* $S_n^k(t)$ *denote the* n — *th Cesàro-sums of order* k *corresponding to the series* $\sum_1^\infty (-1)^n n^\rho$ *and* $\sum_1^\infty (\sin nt)_{h+1} (h \geqslant 0)$, *respectively, then*

(i) $$C_{n,\rho}^k = 0(n^k) \quad k \geqslant \rho$$

(ii) $$S_n^k(t) = 0(n^{k+h+2}) \quad \left(0 < t \leqslant \frac{1}{n}\right), \qquad k \geqslant 0$$
$$= 0(n^{h+1}t^{-k-1}) + 0(n^k t^{-h-2}), \quad (n^{-1} < t \leqslant \pi) k \geqslant 0.$$

LEMMA 2 [2]. — *If* $k \geqslant -1, r \geqslant 0$, *necessary and sufficient conditions for* $\Sigma a_n \varepsilon_n$ *to be summable* $|C, r|$ *whenever*

$$S_n = a_0 + a_1 + \cdots + a_n = 0(1)(C, k)$$

are

(i) $$\Sigma n^{k-r} |\varepsilon_n| < \infty,$$

(ii) $$\Sigma n^{-1} |\varepsilon_n| < \infty,$$

(iii) $$\Sigma n^k \left|\overset{k+1}{\Delta} \varepsilon_n\right| < \infty.$$

LEMMA 3 [1]. — *Let* $R_n^k(t)$ *denote the* n — *th Cesàro sum of order* k $(0 \leqslant k < h + 1)$ *of the series* $\sum_1^\infty \varepsilon(n+1)(\sin nt)_{h+1}$ $(h \geqslant 0)$, *then*

(i) $R_n^k(t) = 0\{\varepsilon(n+1)n^{k+h+2}\}$ $\left(0 < t \leqslant \frac{1}{n}\right)$,

(ii) $R_n^k(t) = 0\{\varepsilon(n+1)n^{h+1}t^{-k-1}\}$ $(n^{-1} < t \leqslant \pi)$.

LEMMA 4 [1]. — *If* (1.2.2) *holds, then*

$$\int_{\frac{1}{n}}^{\pi} t^{-1}|\varphi_h(t)|\, dt = 0\{\varepsilon^{-1}(n+1)\log n\}.$$

LEMMA 5 [1]. — *Let* h *be a positive integer, and* $\{\lambda_n\}$ *be a sequence such that* $\overset{h}{\Delta}\lambda_n \geqslant 0$, *and* $\Sigma \frac{\lambda_n}{n} < \infty$, *then*

(*a*) $\overset{r}{\Delta}\lambda_n \downarrow 0$ $(r = 0, 1, \ldots, h-1)$.

(*b*) $$\lambda_n = \begin{cases} \sum_{m=n}^{\infty} \Delta\lambda_m & \text{for } h = 1 \\ \left(\underline{|h-1}\right)^{-1} \sum_{m=n}^{\infty} (m-n+1)(m-n+2)\ldots(m-n+h-1)\overset{h}{\Delta}\lambda_m & (h > 1) \end{cases}$$

(*c*) $\Sigma m^{r-1}\overset{r}{\Delta}\lambda_m < \infty$ $(r = 1, 2, \ldots, h-1)$.

LEMMA 6 [11]. — *Let* $\{\lambda_n\}$ *be a hyper-convex sequence of order* $(h-1)$ *when* $h \geqslant 1$, *or monotonic non-increasing when* $h = 0$, *such that*

$$\Sigma \frac{\lambda_n}{n} < \infty.$$

If

$$\Sigma n^h \overset{h+1}{\Delta}\lambda_n < \infty,$$

then

$$\Sigma \log(n+1) n^h \overset{h+1}{\Delta}\lambda_n < \infty.$$

LEMMA 7 [13]. — *If*

$$\int_0^t |\varphi_\alpha(u)|\, du = 0\left\{t\left(\log\frac{1}{t}\right)^\beta\right\},$$

then

$$\sum_{m=0}^{n} |\overset{*}{\sigma}{}_m^{\alpha}|^2 = 0\{n\,(\log n)^{2\beta+1}\} \quad \text{for} \quad \beta > -\frac{1}{2}$$

and $\alpha \geqslant 0$ *where* $\overset{*}{\sigma}{}_m^{\alpha}$ *is the* $m - th$ (C, α) *mean of the series* $\Sigma A_n(x)$.

LEMMA 8. — *We have for* $r = 0, 1, \ldots, h$

$$\overset{h+1-r}{\Delta} \{(\mu + r)\varepsilon_{\mu+r+1}\} = 0\left\{\frac{(\mu + 1)^{r-h}\varepsilon_{\mu+1}}{\log(\mu + 1)}\right\}.$$

Proof. — Since $\overset{p}{\Delta}(\mu + r) = 0$ for $p \geqslant 2$ we have

$$\begin{aligned}\overset{h+1-r}{\Delta} &\{(\mu + r)\varepsilon_{\mu+r+1}\} \\ &= \sum_{p=0}^{h+1-r} \binom{h+1-r}{p} \overset{p}{\Delta}(\mu + r) \overset{h+1-r-p}{\Delta} \varepsilon_{\mu+r+p+1} \\ &= (\mu + r) \overset{h+1-r}{\Delta} \varepsilon_{\mu+r+1} - (h + 1 - r) \overset{h-r}{\Delta} \varepsilon_{\mu+r+2} \\ &= 0\left\{\frac{(\mu + 1)^{r-h}\varepsilon_{\mu+1}}{\log(\mu + 1)}\right\}.\end{aligned}$$

1.4. *Proof of the Theorem.* — Since

$$\begin{aligned}A_n(x) &= \frac{2}{\pi}\int_0^{\pi} \Phi(t) \cos nt\, dt \\ &= \frac{2}{\pi}\left[\sum_{\rho=1}^{h} (-1)^{\rho-1}\Phi_\rho(t)\,(\cos nt)_{\rho-1}\right]_0^{\pi} \\ &+ (-1)^h \frac{2}{\pi}\int_0^{\pi} \Phi_h(t)\,(\cos nt)_h\, dt \\ &= A_{n,1}(x) + A_{n,2}(x), \quad \text{say.}\end{aligned}$$

Thus by virtue of the consistency theorem for absolute Cesàro-summability, it is sufficient for our purpose, to prove that each of the series

$$(1.4.1) \qquad \sum_{n=1}^{\infty} \varepsilon(n + 1)\lambda_n A_{n,1}(x),$$

and

$$(1.4.2) \qquad \sum_{n=1}^{\infty} \varepsilon(n + 1)\lambda_n A_{n,2}(x),$$

is summable $|C, h + 1|$.

Now since $\sin n\pi = 0$ and $\cos n\pi = (-1)^n$, for proving the summability $|C, h+1|$ of (1.4.1), it is enough to show that if ρ is an odd integer, $1 \leqslant \rho \leqslant h$,

$$\sum_{n=1}^{\infty} \varepsilon(n+1)\lambda_n(-1)^n n^{\rho-1} \qquad \text{is summable} \qquad |C, h+1|.$$

Taking the series Σa_n in lemma 2 to be $\Sigma(-1)^n n^{\rho-1}$, $r = h$, $k = h-1$, we have from lemma 1.

$$C_{n,\rho-1}^{h-1} = 0(n^{h-1}).$$

Also by taking ε_n to be $\lambda_n \varepsilon_{n+1}$ we find that conditions (i) and (ii) of lemma 2 are satisfied. Also

$$\Sigma n^{h-1} \left| \overset{h}{\Delta} \left(\frac{\lambda_n}{(\log n+1)} \beta \right) \right| = 0 \left\{ \sum_{n=1}^{\infty} \sum_{r=0}^{h} n^{r-1} \overset{r}{\Delta} \lambda_n \right\}$$
$$= 0(1),$$

by virtue of part (c) of lemma 5. Finally applying lemma 2 we find that $\Sigma\lambda_n\varepsilon_{n+1}(-1)^n n^{\rho-1}$ is summable $|C, h|$ and consequently summable $|C, h+1|$.

Also the summability $|C, h+1|$ of the series (1.4.2) is equivalent to the assertion that

$$(1.4.3) \qquad \sum_{n=1}^{\infty} \frac{1}{n} \left| \int_0^{\pi} \phi_h(t) L_n^{h+1}(t)\, dt \right| < \infty,$$

where

$$L_n^{h+1}(t) = \frac{t^h}{A_n^{h+1}} \sum_{\nu=0}^{n} A_{n-\nu}^h\, \varepsilon(\nu+1)\lambda_\nu\, (\sin \nu t)_{h+1}.$$

Proof of (1.4.3). — We have

$$\Sigma \equiv \sum_{\nu=1}^{n} A_{n-\nu}^h \varepsilon(\nu+1)\lambda_\nu\, (\sin \nu t)_{h+1}.$$

Applying the process of repeated summation we have in the notation of Lemma 3,

$$\Sigma = \sum_{\nu=1}^{n} R_\nu^h(t) \overset{h+1}{\Delta} (A_{n-\nu}^h \lambda_\nu)$$
$$= \sum_{r=0}^{h} \binom{h+1}{r} \sum_{\nu=1}^{n} A_{n-\nu}^{h-r} \overset{h+1-r}{\Delta} \lambda_{\nu+r} R_\nu^h(t)$$
$$+ \sum_{\nu=1}^{n} A_{n-\nu}^{-1} \lambda_{\nu+h+1} R_\nu^h(t)$$
$$= \Sigma_1 + \Sigma_2, \quad \text{say.}$$

Hence we need to prove that

$$\sum_{n=1}^{\infty} n^{-1}\left|\int_0^{\pi} \phi_h(t)\,\frac{t^h}{A_n^{h+1}}\,(\Sigma_1+\Sigma_2)\,dt\right| < \infty,$$

for which it is sufficient to show that

$$(1.4.4)\qquad \sum_{n=1}^{\infty} n^{-h-2}\int_0^{\pi} |\phi_h(t)|\,t^h\,|\Sigma_1|\,dt < \infty,$$

and

$$(1.4.5)\qquad \sum_{n=1}^{\infty} n^{-h-2}\left|\int_0^{\pi} \phi_h(t)t^h\,\Sigma_2\,dt\right| < \infty.$$

Proof of (1.4.4). — It suffices, for our purpose, to show that for $0 \leqslant r \leqslant h$,

$$\sum_{n=1}^{\infty} n^{-h-2}\sum_{\nu=1}^{n} A_{n-\nu}^{h-r}\overset{h+1-r}{\Delta}\lambda_{\nu+r}\int_0^{\pi} t^h|\phi_h(t)||R_\nu^h(t)|\,dt < \infty.$$

The above expression is

$$\sum_{n=1}^{\infty} n^{-h-2}\sum_{\nu=1}^{n} A_{n-\nu}^{h-r}\overset{h+1-r}{\Delta}\lambda_{\nu+r}\left(\int_0^{\frac{1}{\nu}}+\int_{\frac{1}{\nu}}^{\infty}\right)|\phi_h(t)|t^h|R_\nu^h(t)|\,dt$$
$$= \Sigma_{11}+\Sigma_{12},\quad \text{say}.$$

Now by lemma 3 and the hypothesis we have

$$\begin{aligned}
\Sigma_{11} &\leqslant K\,(^1)\sum_{n=1}^{\infty} n^{-h-2}\sum_{\nu=1}^{n} A_{n-\nu}^{h-r}\overset{h+1-r}{\Delta}\lambda_{\nu+r}(\nu^{2h+2}\varepsilon(\nu+1))\left(\frac{\nu^{-h-1}}{\varepsilon_{\nu+1}}\right),\\
&\leqslant K\sum_{n=1}^{\infty} n^{-h-2}\sum_{\nu=1}^{n} \nu^{h+1}(n+1-\nu)^{h-r}\overset{h+1-r}{\Delta}\lambda_{\nu+r}\\
&\leqslant K\sum_{\nu=1}^{\infty} \nu^{h+1}\overset{h+1-r}{\Delta}\lambda_{\nu+r}\sum_{n=\nu}^{\infty}(n+1-\nu)^{h-r}n^{-h-2}\\
&\leqslant K\sum_{\nu=1}^{\infty} \nu^{h+1}\overset{h+1-r}{\Delta}\lambda_{\nu+r}\nu^{-r-1}\\
&\leqslant K\sum_{\nu=1}^{\infty} \nu^{h-r}\overset{h-r+1}{\Delta}\lambda_{\nu+r}\leqslant K.
\end{aligned}$$

By lemma 5 and the fact that

$$\sum_{n=\nu}^{\infty}(n+1-\nu)^{h-r}n^{-h-2} = 0\left(\int_{\nu}^{\infty} x^{-h-2}(x-\nu)^{h-r}\,dx\right)$$
$$= 0(\nu^{-r-1})$$

(1) K is a constant not necess-arily the same at each occurrence.

Also by lemmas 3 and 4 we get

$$\Sigma_{12} \leqslant K \sum_{n=1}^{\infty} n^{-h-2} \sum_{\nu=1}^{n} A_{n-\nu}^{h-r} \overset{h+1-r}{\Delta} \lambda_{\nu+r} \nu^{h+1} \varepsilon(\nu+1) \int_{\frac{1}{\nu}}^{\pi} t^h |\varphi_h(t)| t^{-h-1}\, dt$$

$$\leqslant K \sum_{n=1}^{\infty} n^{-h-2} \sum_{\nu=1}^{n} A_{n-\nu}^{h-r} \overset{h+1-r}{\Delta} \lambda_{\nu+r} \nu^{h+1} \varepsilon(\nu+1) \int_{\frac{1}{\nu}}^{\pi} t^{-1} |\varphi_h(t)|\, dt,$$

$$\leqslant K \sum_{n=1}^{\infty} n^{-h-2} \sum_{\nu=1}^{n} A_{n-\nu}^{h-r} \overset{h+1-r}{\Delta} \lambda_{\nu+r} \nu^{h+1} \log(\nu+1),$$

$$\leqslant K \sum_{n=1}^{\infty} n^{-h-2} \sum_{\nu=1}^{n} (n+1-\nu)^{h-r} \overset{h+1-r}{\Delta} \lambda_{\nu+r} \nu^{h+1} \log(\nu+1),$$

$$\leqslant K \sum_{\nu=1}^{\infty} \nu^{h+1} \log(\nu+1) \overset{h+1-r}{\Delta} \lambda_{\nu+r} \sum_{n=\nu}^{\infty} (n+1-\nu)^{h-r} n^{-h-2},$$

$$\leqslant K \sum_{\nu=1}^{\infty} \log(\nu+1) \nu^{h-r} \overset{h-r+1}{\Delta} \lambda_{\nu+r},$$

$$\leqslant K,$$

by lemmas 5 and 6.

This completes the proof of (1.4.4).

Proof of (1.4.5). — Now we have to show that

$$\sum_{n=1}^{\infty} n^{-h-2} \left| \int_0^{\pi} t^h \varphi_h(t)\, \Sigma_2\, dt \right| < \infty.$$

Since

$$\Sigma_2 = \lambda_{n+h+1} R_n^h(t),$$

substituting the value of Σ_2, we find that the above expression is

$$\sum_{n=1}^{\infty} n^{-h-2} \left| \int_0^{\pi} t^h \varphi_h(t) \lambda_{n+h+1} R_n^h(t)\, dt \right|$$

$$\leqslant K \sum_{n=1}^{\infty} n^{-h-2} \lambda_{n+h+1} \left| \int_0^{\pi} \Phi_h(t) R_n^h(t)\, dt \right|$$

$$= K \sum_{n=1}^{\infty} n^{-h-2} \lambda_{n+h+1} \left| \sum_{\nu=1}^{n} A_{n-\nu}^h \varepsilon(\nu+1) . \nu . \int_0^{\pi} \Phi_h(t) (\cos \nu t)_h\, dt \right|$$

$$= K \sum_{n=1}^{\infty} n^{-h-2} \lambda_{n+h+1} \left| \sum_{\nu=1}^{n} A_{n-\nu}^h \varepsilon(\nu+1) . \nu . (-1)^h \right.$$

$$\Big\{(-1)^h \frac{2}{\pi} \int_0^\pi \Phi_h(t)\,(\cos \nu t)_h\, dt$$

$$+ \frac{2}{\pi} \Big[\sum_{\rho=1}^{h} (-1)^{\rho-1} \Phi_\rho(t)\,(\cos \nu t)_{\rho-1}\Big]_0^\pi$$

$$- \frac{2}{\pi} \Big[\sum_{\rho=1}^{h} (-1)^{\rho-1} \Phi_\rho(t)\,(\cos \nu t)_{\rho-1}\Big]_0^\pi\Big\}\Big|$$

$$\leqslant K \sum_{n=1}^{\infty} n^{-h-2} \lambda_{n+h+1} \Big|\sum_{\nu=1}^{n} A_{n-\nu}^{h} \varepsilon(\nu+1)\,.\,\nu\,.\,A_\nu(x)\Big|$$

$$+ K \sum_{n=1}^{\infty} n^{-h-2} \lambda_{n+h+1}$$

$$\Big|\sum_{\nu=1}^{n} A_{n-\nu}^{h} \varepsilon(\nu+1)\,.\,\nu\,.\Big[\sum_{\rho=1}^{h} (-1)^{\rho-1} \Phi_\rho(t)\,(\cos \nu t)_{\rho-1}\Big]_0^\pi\Big|$$

$$= I_1 + I_2, \quad \text{say.}$$

By repeated partial summation we have

$$\sum_{\nu=0}^{n} A_{n-\nu}^{h} \varepsilon(\nu+1)\,.\,\nu\,.\,A_\nu(x) = \sum_{\nu=0}^{n} \overset{*}{S}{}_\nu^h \overset{h+1}{\Delta} (A_{n-\nu}^{h} \nu\,.\,\varepsilon_{\nu+1}),$$

where $\overset{*}{S}{}_n^h$ denotes the $n-th$ Cesàro-sum of order h of the series $\Sigma A_n(x)$.

Now since

$$\overset{h+1}{\Delta} (A_{n-\nu}^{h} \nu\,.\,\varepsilon(\nu+1)) = \sum_{r=0}^{h+1} \binom{h+1}{r} \overset{r}{\Delta} (A_{n-\nu}^{h}) \overset{h+1-r}{\Delta} \{(\nu+r)\varepsilon_{\nu+r+1}\}$$

$$= \sum_{r=0}^{h+1} \binom{h+1}{r} A_{n-\nu}^{h-r} \overset{h+1-r}{\Delta} \{(\nu+r)\varepsilon_{\nu+r+1}\}$$

$$= \sum_{r=0}^{h} \binom{h+1}{r} A_{n-\nu}^{h-r} \overset{h+1-r}{\Delta} \{(\nu+r)\varepsilon_{\nu+r+1}\}$$

$$+ A_{n-\nu}^{-1} (\nu+h+1)\varepsilon_{\nu+h+2}.$$

It follows that

$$\sum_{\nu=1}^{n} A_{n-\nu}^{h} \varepsilon(\nu+1) \nu A_\nu(x)$$

$$= \sum_{r=0}^{h} \binom{h+1}{r} \sum_{\nu=0}^{n} \overset{*}{S}{}_\nu^h A_{n-\nu}^{h-r} \overset{h+1-r}{\Delta} \{(\nu+r)\varepsilon_{\nu+r+1}\}$$

$$+ \overset{*}{S}{}_n^h (n+h+1)\varepsilon_{n+h+2}.$$

Therefore

$$\mathrm{I}_1 \leqslant \mathrm{K} \sum_{n=1}^{\infty} n^{-h-2}\lambda_{n+h+1} \sum_{r=0}^{h} \binom{h+1}{r} \sum_{\nu=0}^{n} |\overset{*}{\mathrm{S}}{}_\nu^h| \mathrm{A}_{n-\nu}^{h-r} \left| \overset{n+1-r}{\Delta} \{(\nu + r)\varepsilon_{\nu+r+1}\}\right|$$

$$+ \mathrm{K}\Sigma n^{-h-2}\lambda_{n+h+1}(n + h + 1)\varepsilon_{n+h+2}|\overset{*}{\mathrm{S}}{}_n^h|$$

$$= \mathrm{I}_{11} + \mathrm{I}_{12}, \quad \text{say.}$$

Now

$$\mathrm{I}_{12} = \mathrm{K} \sum_{n=1}^{\infty} n^{-h-2}\lambda_{n+h+1}(n + h + 1)\mathrm{A}_n^h|\overset{*}{\sigma}{}_n^h|\varepsilon_{n+h+2}$$

$$= 0\left[\sum_{n=1}^{\infty} |\overset{*}{\sigma}{}_n^h|\lambda_n \frac{\varepsilon_{n+1}}{n}\right].$$

Applying Abel's transformation we have by Lemma 7.

$$\sum_{n=1}^{m} |\overset{*}{\sigma}{}_n^h| \frac{\lambda_n\varepsilon_{n+1}}{n} = \sum_{n=1}^{m-1} \Delta\left(\frac{\lambda_n\varepsilon_{n+1}}{n}\right) \sum_{\nu=0}^{n} |\overset{*}{\sigma}{}_\nu^h|$$

$$+ \frac{\lambda_m\varepsilon_{m+1}}{m} \sum_{n=0}^{m} |\overset{*}{\sigma}{}_n^h|$$

$$= 0\left[\sum_{n=1}^{m-1} \Delta\left(\frac{\lambda_n\varepsilon_{n+1}}{n}\right) n\varepsilon^{-1}(n+1)\,(\log(n+1))^{\frac{1}{2}}\right]$$

$$+ 0\left[\frac{\lambda_m\varepsilon_{m+1}}{m} \cdot m\varepsilon^{-1}(m+1)\,(\log(m+1))^{\frac{1}{2}}\right]$$

$$= 0\left[\sum_{1}^{m-1} \Delta\lambda_n\,(\log n + 1)^{\frac{1}{2}}\right]$$

$$+ 0\left[\sum_{1}^{m-1} \frac{\lambda_{n+1}}{n+1}\,(\log n + 1)^{\frac{1}{2}}\right]$$

$$= 0(1) + 0(1) = 0(1).$$

Since $\Delta\varepsilon_n = 0\left(\frac{\varepsilon_n}{n}\right)$ and $\lambda_m \log(m + 1) = 0(1)$.

Now in order to show that $\mathrm{I}_{11} = 0(1)$ it is sufficient to prove that

$$\sum_{n=1}^{\infty} n^{-h-2}\lambda_{n+h+1} \sum_{\nu=0}^{n} |\overset{*}{\mathrm{S}}{}_\nu^h| \mathrm{A}_{n-\nu}^{h-r}| \overset{h+1-r}{\Delta} \{(\nu + r)\varepsilon_{\nu+r+1}\}|$$

$$= 0(1) \qquad \text{for} \qquad r = 0, 1, \ldots, h.$$

The above expression is by lemma 8

$$\sum_{\nu=1}^{\infty} |\overset{*}{S}{}_\nu^h| \left| \overset{h+1-r}{\Delta} \{(\nu + r)\varepsilon_{\nu+r+1}\} \right| \left| \sum_{n=\nu}^{\infty} (n - \nu + 1)^{h-r} n^{-h-2} \lambda_{n+h+1} \right|$$

$$\leqslant K \sum_{\nu=1}^{\infty} |\overset{*}{S}{}_\nu^h| \lambda_{\nu+h+1} \left| \overset{h+1-r}{\Delta} \{(\nu + r)\varepsilon_{\nu+r+1}\} \right| \left| \sum_{n=\nu}^{\infty} (n - \nu + 1)^{h-r} . n^{-h-2} \right|$$

$$= 0\left(\sum_{\nu=0}^{\infty} |\overset{*}{S}{}_\nu^h| \lambda_{\nu+h+1} \frac{(\nu + 1)^{r-h} \varepsilon_{\nu+1}}{\log(\nu + 1)} \nu^{-r-1} \right),$$

$$= 0\left(\sum_{\nu=0}^{\infty} |\overset{*}{\sigma}{}_\nu^h| (\nu + 1)^{r-h} \frac{\varepsilon_{\nu+1}}{\log(\nu + 1)} \nu^{h-r-1} \lambda_{\nu+h+1} \right)$$

$$= 0\left(\sum_{\nu=0}^{\infty} |\overset{*}{\sigma}{}_\nu^h| \lambda_{\nu+h+1} \frac{\varepsilon(\nu + 1)}{\nu \log \overline{\nu + 1}} \right)$$

$$= 0(1),$$

as shown in the proof of $I_{12} = 0(1)$.

Hence

$$I_1 = 0(1).$$

Now we proceed to show that $I_2 = 0(1)$.

If ρ is an odd integer, then it is sufficient to show that

$$(1.4.6) \quad k \sum_{n=1}^{\infty} n^{-h-2} \lambda_{n+h+1} \left| \sum_{\nu=1}^{n} A_{n-\nu}^h \varepsilon_{\nu+1} (-1)^\nu \nu^\rho \right| < \infty,$$

for $1 \leqslant \rho \leqslant h$.

By repeated partial summation we have

$$\sum_{\nu=1}^{n} A_{n-\nu}^h \varepsilon_{\nu+1} (-1)^\nu . \nu^\rho = \sum_{\nu=0}^{n} C_{\nu,\rho}^h \overset{h+1}{\Delta} (A_{n-\nu}^h \varepsilon(\nu + 1)),$$

where $C_{n,\rho}^h$ is the $n - th$ Cesàro sum of order h of the series $\Sigma(-1)^\nu . \nu^\rho$.

Also

$$\overset{h+1}{\Delta} (A_{n-\nu}^h \varepsilon_{\nu+1}) = \sum_{r=0}^{h+1} \binom{h+1}{r} \overset{r}{\Delta} (A_{n-\nu}^h) \overset{h+1-r}{\Delta} \varepsilon_{\nu+r+1}$$

$$= \sum_{r=0}^{h+1} \binom{h+1}{r} A_{n-\nu}^{h-r} \overset{h+1-r}{\Delta} \varepsilon_{\nu+r+1}$$

$$= \sum_{r=0}^{h} \binom{h+1}{r} A_{n-\nu}^{h-r} \overset{h+1-r}{\Delta} \varepsilon_{\nu+r+1} + A_{n-\nu}^{-1} \varepsilon_{\nu+h+2}.$$

Therefore

$$\sum_{\nu=1}^{n} A_{n-\nu}^{h}\varepsilon_{\nu+1}(-1)^{\nu}\nu^{\rho} = \sum_{r=0}^{h}\binom{h+1}{r}\sum_{\nu=0}^{n} C_{\nu,\rho}^{h}A_{n-\nu}^{h-r}\overset{h+1-r}{\Delta}\varepsilon_{\nu+r+1}$$
$$+\sum_{\nu=0}^{n} C_{\nu,\rho}^{h}A_{n-\nu}^{-1}\varepsilon_{\nu+h+2}$$
$$=0\left(\sum_{\nu=0}^{n}\nu^{h}(n-\nu+1)^{h-r}\left|\overset{h+1-r}{\Delta}\varepsilon_{\nu+r+1}\right|\right)$$
$$+0(n^{h}\varepsilon_{n+h+2}),$$

by lemma 1.

Therefore the expression in (1.4.6) is

$$=0\left(\sum_{n=1}^{\infty} n^{-h-2}\lambda_{n+h+1}\sum_{\nu=0}^{n}\nu^{h}(n-\nu+1)^{h-r}\left|\overset{h-r+1}{\Delta}\varepsilon_{\nu+r+1}\right|\right)$$
$$+0\left(\sum_{n=1}^{\infty} n^{-h-2}\lambda_{n+h+1}n^{h}\varepsilon_{n+h+2}\right)$$
$$=0\left(\sum_{\nu=1}^{\infty}\nu^{h}\left|\overset{h-r+1}{\Delta}\varepsilon_{\nu+r+1}\right|\sum_{n=\nu}^{\infty}(n-\nu+1)^{h-r}n^{-h-2}\lambda_{n+h+1}\right)$$
$$+0\left(\sum_{n=1}^{\infty}\frac{\lambda_n}{n}\right).$$
$$=0\left(\sum_{\nu=1}^{\infty}\nu^{h}\left|\overset{h-r+1}{\Delta}\varepsilon_{\nu+r+1}\right|\lambda_{\nu+h+1}\sum_{n=\nu}^{\infty}(n-\nu+1)^{h-r}n^{-h-2}\right)+0(1)$$
$$=0\left(\sum_{\nu=1}^{\infty}\nu^{h}\left|\overset{h-r+1}{\Delta}\varepsilon_{\nu+r+1}\right|\lambda_{\nu+h+1}\cdot\nu^{-r-1}\right)+0(1)$$
$$=0\left(\sum_{\nu=1}^{\infty}\nu^{h-r-1}\frac{\varepsilon_{\nu+1}\lambda_{\nu}}{(\nu+1)^{h-r+1}}\right)+0(1)$$
$$=0\left(\sum_{\nu=1}^{\infty}\frac{\lambda_{\nu}}{\nu}\right)+0(1)$$
$$=0(1).$$

This completes the proof of the theorem.

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