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J-holomorphic discs and real analytic hypersurfaces

J-HOLOMORPHIC DISCS AND REAL ANALYTIC HYPERSURFACES

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Abstract. — We give in $\mathbb{R}^6$ a real analytic almost complex structure J, a real analytic hypersurface M and a vector v in the Levi null set at 0 of M, such that there is no germ of J-holomorphic disc γ included in M with $\gamma(0) = 0$ and $\frac{\partial \gamma}{\partial x}(0) = v$, although the Levi form of M has constant rank. Then for any hypersurface M and any complex structure J, we give sufficient conditions under which there exists such a germ of disc.

Résumé. — Nous définissons dans $\mathbb{R}^6$ une structure presque complexe réelle analytique J, une hypersurface réelle analytique M dont la forme de Levi est de rang constant et un vecteur v appartenant au noyau de la forme de Levi de M en 0 tels qu'il n'existe pas de germe de disque J-holomorphe γ inclus dans M vérifiant $\gamma(0) = 0$ et $\frac{\partial \gamma}{\partial x}(0) = v$. Nous donnons ensuite des conditions suffisantes pour qu'un tel germe de disque existe.

1. Introduction

Throughout all this paper, we denote by M a real analytic hypersurface in $\mathbb{R}^{2n}$ such that 0 belongs to M. We denote by φ a real analytic function such that $M = \{z \in \mathbb{R}^{2n}, \ \varphi(z) = 0\}$ and such that $d\varphi$ does not vanish on M. We also denote by TM the tangent vector bundle to M.

We equip $\mathbb{R}^{2n}$ with a real analytic almost complex structure J, i.e. a linear map from $T\mathbb{R}^{2n}$ into itself such that $J^2 = -Id$. Without restriction, we can always assume that $J(0) = J_0$, the standard complex structure in $\mathbb{R}^{2n} = \mathbb{C}^n$, and that J is as close as we need to J_0. We denote by $T^J M$ the J invariant part of TM, that is $T^J M = TM \cap J \, TM$.

The Levi form $\mathcal{L}_\varphi$ of M and its kernel are two objects linked to the geometry of M. The Levi form is defined for every vector field $X \in T^J M$ by
$$\mathcal{L}_\varphi(X) = d\varphi(J[X, JX]).$$
We will also consider $\hat{\mathcal{L}}_\varphi$, the polar form of $\mathcal{L}_\varphi$:
$$\hat{\mathcal{L}}_\varphi(X, Y) = \frac{1}{2}\left(d\varphi(J[X, JY] + J[Y, JX]) + id\varphi(J[X, Y] + J[JX, JY])\right).$$

Both $\mathcal{L}_\varphi$ and $\hat{\mathcal{L}}_\varphi$ do not depend on the derivatives of X and Y. Moreover, they depend on the defining function φ up to multiplication by a nonnegative function. Therefore $\ker \mathcal{L}_\varphi$ does no depend on φ and we simply denote it by $\ker \mathcal{L}$.

When J is integrable, or more simply when J is the standard structure, and when $\ker \mathcal{L}$ is a subbundle of $T^J M$, Freeman proved in [2] that there exists a complex foliation of M. However, when J is a generic almost complex structure, there is no foliation by J-holomorphic manifolds of dimension greater than or equal to 2 because such objects do not exist in general. Hence Freeman's theorem does not hold anymore in the non integrable case; we can only hope to find a J-holomorphic disc with prescribed derivative of order 1 included in M. A J-holomorphic disc is a C^1 map $\gamma : \mathbb{D} \to \mathbb{R}^{2n}$, where $\mathbb{D}$ is the unit disc of $\mathbb{C}$ equipped with the standard structure J_0, such that for all $\zeta \in \mathbb{D}$, we have as linear map:
$$d\gamma(\zeta) \circ J_0 = J(\gamma(\zeta)) \circ d\gamma(\zeta).$$

When γ is a J-holomorphic disc included in M, i.e. $\gamma(\mathbb{D}) \subset M$, the vector $\frac{\partial \gamma}{\partial x}(0)$ belongs to $(\ker \mathcal{L})_{\gamma(0)}$. When J is integrable and $\ker \mathcal{L}$ is a subbundle of $T^J M$, Freeman's result implies that the converse is true (even in the non real analytic case); that is, for any point $p \in M$ and any $v \in \ker \mathcal{L}$, there exists a J-holomorphic disc γ included in M such that $\gamma(0) = p$ and $\frac{\partial \gamma}{\partial x}(0) = v$. This result was generalized by Kruzhilin and Sukhov in [5] when J is not integrable and M is Levi flat, that is when $\ker \mathcal{L} = T^J M$. It was then natural to hope that this result was true in the real analytic case when $\ker \mathcal{L}$ is only a subbundle of $T^J M$. However, we will show that this is not the case, even when M is pseudoconvex. We will prove in Section 2 the following theorem:

THEOREM 1.1. — *There exist a real analytic almost complex structure J, a real analytic hypersurface M in $\mathbb{R}^6$ such that $\ker \mathcal{L}$ is a subbundle of dimension 1 of $T^J M$ and a vector $v \in \ker \mathcal{L}_0$ such that no germ of J-holomorphic disc γ, satisfying $\frac{\partial \gamma}{\partial x}(0) = v$, is tangent to M at 0 at order greater than 5.*

The following natural question then arises: under which condition on $v \in T_0^J M$ does there exist a (germ of) *J*-holomorphic disc γ included in M such that $\gamma(0) = 0$ and $\frac{\partial \gamma}{\partial x}(0) = v$?

In order to answer this question, we can either work in local coordinates or we can use a torsion free connection ∇ in order to have more intrinsic results. In this case, for simplicity sake, we use a multiplicative notation: when X and Y are two vector fields, we denote by $X \cdot Y$ the vector field $\nabla_X Y$. Since ∇ is torsion free, we have $[X, Y] = X \cdot Y - Y \cdot X$ for all vectors fields X and Y. We also introduce the following notations:

Notation 1.2. — If X is a vector field and p and q are two integers, we define $D_X^{p,q}$ by the formula

$$D_X^{p,q} \phi = \underbrace{JX \cdot (JX \cdots (JX}_{q} \cdot \underbrace{(X \cdot (X \ldots (X}_{p} \cdot \phi) \ldots)$$

where ϕ is a smooth function.

This notation will be often used with $q = 0$ in the whole paper and with $q \neq 0$ in the proof of Lemma 4.4 and 4.7 in Section 4.

Notation 1.3. — Let X be a vector field. We denote by $\mathbb{L}(X)$ the complex Lie algebra generated by X, i.e. the smallest complex Lie algebra which contains X and JX and is such that for Y and Z in $\mathbb{L}(X)$, $[Y, Z]$ belongs to $\mathbb{L}(X)$.

We will establish two sufficient conditions on a vector X under which there exists a *J*-holomorphic disc γ included in M such that for all $k \in \mathbb{N}^*$, $\frac{\partial^k \gamma}{\partial x^k}(0) = D_X^{k-1,0} X(0)$. The first one is given by the following theorem:

THEOREM 1.4. — *Let* $X \in T^J M$ *be a real analytic vector field in a neighborhood of* $0 \in M$ *such that* $\mathbb{L}(X)$ *is included in* $T^J M$.
Then, for all $p \in M$ *in a neighborhood of* 0, *there exists a germ of J-holomorphic disc* γ *such that*

(1) $\gamma(0) = p$,
(2) *for all* $k \in \mathbb{N}^*$, $\frac{\partial^k \gamma}{\partial x^k}(0) = D_X^{k-1,0} X(p)$.

Theorem 1.4 can be shown using Nagano's Theorem [7] in a good way. Here we don't use it but we give a simple direct proof of Theorem 1.4 without differential equations. Instead we construct directly the jet of the curve at the point p. This approach allows us to generalize Theorem 1.4 and to obtain Theorem 1.6 below, which is not possible to achieve with the classical Nagano's result.

We also point out that, unlike Freeman's Theorem [2], Theorem 1.4 makes no assumption on the rank of the Levi form $\mathcal{L}$. In particular, Theorem 1.4 can be applied to an integrable complex structure when $\ker \mathcal{L}$ is not a subbundle. In this case, we will show that for any $X \in \ker \mathcal{L}$, there exists a J-holomorphic disc γ included in M such that $\frac{\partial \gamma}{\partial x}(0) = X(0)$ (see Theorem 4.5). We will also give an example of application of our result in a non-integrable case and where the Levi form has constant rank (see Example 4.6).

In order to state our next result which requires a condition in the spirit of Freeman's theorem, we recall the following definition introduced in [1]:

DEFINITION 1.5. — *We say that a vector field X commutes at order k at a point p if for all $l \leqslant k$, all $X_1, \ldots, X_l \in \{X, JX\}$,*

$$[X_1, [\ldots, [X_{l-1}, X_l] \ldots](p) = 0.$$

We will prove the following result:

THEOREM 1.6. — *Let $\mathbb{L}$ be a subbundle of $T^J M$ such that the following property is true: if $X \in \mathbb{L}$ commutes at order k at the point 0, then for all $X_1, \ldots, X_{k+1} \in \{X, JX\}$, $[X_1, [\ldots, [X_k, X_{k+1}] \ldots]](0)$ belongs to $\mathbb{L}_0$. Then for all $X \in \mathbb{L}$, there exists a germ of J-holomorphic disc γ such that $\gamma(0) = 0$, $\frac{\partial^k \gamma}{\partial x^k}(0) = D_X^{k-1,0} X(0)$ for all $k \in \mathbb{N}^*$ and $\gamma(\mathbb{D}) \subset M$.*

This Theorem 1.6 enables us to prove, in the real analytic case and in a constructive way, Kruzhilin and Suhkov's theorem [5] (See Theorem 4.8). We will also give examples of applications of Theorem 1.6 (see Example 4.9).

In order to prove Theorem 1.4 and 1.6, we prove a generalization of the existence results of J-holomorphic discs with prescribed jets of finite order, due to Sikorav in the case of 1-jets (see [8]) and to Ivashkovich and Rosay [3] and independently to Barraud and Mazzilli [1] in the case of k-jets, for k finite. For this, a key point is to show that if $(x_k)_{k \in \mathbb{N}}$ is an infinite sequence of vectors in $\mathbb{R}^{2n}$ such that the series $\sum_k \frac{k+1}{k!} |x_k|$ converges, then there is a germ of J-holomorphic disc γ included in M such that $\frac{\partial^k \gamma}{\partial x^k}(0) = x_k$ for all $k \in \mathbb{N}$ (see Theorem 3.1 and Corollary 3.3). The general procedure to prove this is in some sense analogue to the case of finite jets but the proof is more intricate: we have to prove that some maps between two Banach spaces $\tilde{\Omega}$ and Ω are invertible in a neighborhood of the origin. In the case of k-jets, the Banach spaces considered were simply $\tilde{\Omega} = (\mathbb{R}^{2n})^k$ and $\Omega = C^{k+\alpha}(\overline{\mathbb{D}})$ for $\alpha > 0$, with the k-jet belonging to $(\mathbb{R}^{2n})^k$ and the disc to $C^{k+\alpha}(\overline{\mathbb{D}})$. The first difficulty we have to overcome is to find the appropriate Banach spaces.

Here, the k-jet becomes an infinite sequence of vectors of $\mathbb{R}^{2n}$, so we have to consider a space of sequences with an appropriate norm. This role will be played by the space $\tilde{\Omega}$ of all sequences $(x_k)_k$ of vectors in $\mathbb{R}^{2n}$ such that the norm $\|(x_k)_k\|_\sim := \sum_k \frac{k+1}{k!}|x_k|$ is finite. The space of functions will be the space Ω of real analytic functions $f = \sum_{k,l} f_{k,l} z^k \overline{z}^l$ on the disc $\mathbb{D}$ such that $\|f\| := \sum_{k,l}(1 + k + l + kl)|f_{kl}|$ is finite. Then, as in the case of k-jets, we associate to any jet a function and to any function a jet (See Theorem 3.1 for details). Showing that these maps are firstly well defined and secondly invertible, we find a J-holomorphic disc with prescribed derivatives of any order.

In Proposition 3.4 we will prove that if X is a real analytic vector field in a neighborhood of 0, then the sequence $(D_X^{k-1,0}X(0))_{k \in \mathbb{N}^*}$ belongs to $\tilde{\Omega}$. Hence there exists a germ of J-holomorphic disc γ such that $\gamma(0) = 0$ and $\frac{\partial^k \gamma}{\partial x^k}(0) = D_X^{k-1,0}X(0)$ for all $k \in \mathbb{N}^*$.

From this point, the problem of finding a J-holomorphic disc included in M with prescribed derivatives is reduced to finding a sequence of vector fields $(X_l)_l$ in $T^J M$ such that the sequence $(D_{X_l}^{k-1,0}X_l(0))_{k \in \mathbb{N}^*}$ does not depend on l and is such that X_l commutes at order l at 0. Indeed, Theorem 3.1 gives a J-holomorphic disc γ such that for all k and all l, $\frac{\partial^k \gamma}{\partial x^k}(0) = D_{X_l}^{k-1,0}X_l(0)$. Since X_l commutes at order l, Theorem 1 of [1] implies that γ is tangent to M at 0 at order $l + 1$, for all l. As both M and γ are real analytic, it follows that γ is actually included in M.

Starting from a vector field X which does not commute but which satisfies the assumptions of Theorem 1.4 or 1.6, we inductively construct such a sequence of vector fields by solving systems of linear equations (see Lemma 4.4 and Lemma 4.7).

The paper is organized as follows. In section 2, we prove Theorem 1.1. In Section 3, we show the existence of J-holomorphic disc with prescribed derivatives at any order. Finally, in Section 4, we prove Theorem 1.4 and 1.6 and give examples of applications of these theorems.

2. A counter-example to a generalization of Freeman's Theorem

In this section, in order to prove Theorem 1.1, we exhibit an almost complex structure J and a pseudoconvex hypersurface M in $\mathbb{R}^6$, both real analytic, such that there exists a vector v which belongs to the kernel of the Levi form of M at the point 0 but which is not the derivative at 0 of

some germ of J-holomorphic disc γ included in M, despite the fact that the kernel of the Levi form of M is a subbundle of $T^J M$. We will prove that in fact, there exists no J-holomorphic disc γ tangent to M at 0 at order 5 (i.e. such that $\varphi \circ \gamma(\zeta) = O(\zeta^5)$).

Proof of Theorem 1.1: Let $\varphi : \mathbb{R}^6 \to \mathbb{R}$ be the map defined by

$$\varphi(x_1, y_1, x_2, y_2, x_3, y_3) = y_1,$$

and let M be the set $M = \{z \in \mathbb{R}^6 \;\; \varphi(z) = 0\}$.
We define the six following vector fields

$$L_1 = \frac{\partial}{\partial x_1}, \qquad\qquad L_2 = \frac{\partial}{\partial y_1},$$

$$L_3 = \frac{\partial}{\partial x_2} - \frac{1}{2} y_3^2 \frac{\partial}{\partial x_1}, \qquad L_4 = \frac{\partial}{\partial y_2} + (-2y_3 x_3 + x_2) \frac{\partial}{\partial x_1},$$

$$L_5 = \frac{\partial}{\partial x_3} - y_2 y_3 \frac{\partial}{\partial x_1}, \qquad L_6 = \frac{\partial}{\partial y_3} + x_3 \frac{\partial}{\partial x_2} - \left(\frac{x_3 y_3^2}{2} + x_3 y_2 \right) \frac{\partial}{\partial x_1},$$

and the almost complex structure J they induce by setting

$$JL_1 = L_2, \qquad\qquad JL_3 = L_4, \qquad\qquad JL_5 = L_6.$$

Therefore $J(0)$ is the standard complex structure J_0 and TM is spanned over $\mathbb{R}$ by L_1, L_3, L_4, L_5 and L_6, while $T^J M$ is spanned over $\mathbb{C}$ by L_3 and L_5.
When we compute the Lie brackets of the L_i's belonging to the complex tangent bundle, we get:

$$[L_3, L_4] = L_1,$$

$$[L_3, L_5] = 0, \qquad\qquad [L_4, L_5] = y_3 \frac{\partial}{\partial x_1},$$

$$[L_3, L_6] = y_3 \frac{\partial}{\partial x_1} \qquad [L_4, L_6] = 0, \qquad\qquad [L_5, L_6] = L_3.$$

Therefore

$$\hat{\mathcal{L}}_\varphi(L_3, L_5) = 0,$$
$$\hat{\mathcal{L}}_\varphi(L_3, L_3) = d\varphi(L_2) = 1,$$
$$\hat{\mathcal{L}}_\varphi(L_5, L_5) = d\varphi(L_3) = 0,$$

which thus implies that M is pseudoconvex and that $\ker \mathcal{L} = span_{\mathbb{C}}\{L_5\}$; therefore $\ker \mathcal{L}$ is a subbundle of $T^J M$.

Now let us assume that there exists a J-holomorphic disc $\gamma : \mathbb{D} \to \mathbb{R}^{2n}$ tangent to M at order 5 and such that $\frac{\partial \gamma}{\partial x}(0) = L_5(0)$. Theorem 1 from [1] implies that there exists a vector field $X \in T^J M$ such that X commutes at order 4 at 0, i.e. such that all the Lie brackets of X and JX of length at most 4 vanish at 0, and such that $X(0) = \frac{\partial \gamma}{\partial x}(0) = L_5(0)$. We prove that such a vector field does not exist.

Let us consider $X = aL_3 + bL_4 + cL_5 + dL_6$ where a, b, c and d are real valued functions. Let us assume that X commutes at order 4 and that $X(0) = L_5(0)$. Therefore we have $a(0) = b(0) = d(0) = 0$ and $c(0) = 1$. We compute $[X, JX]$:

$$
\begin{aligned}
[X, JX] = {} & (a^2 + b^2)L_1 + (c^2 + d^2)L_3 \\
& + (-X \cdot b - JX \cdot a)L_3 + (X \cdot a - JX \cdot b)L_4 \\
& + (-X \cdot d - JX \cdot c)L_5 + (X \cdot c - JX \cdot d)L_6
\end{aligned}
$$

and since $[X, JX](0) = 0$, it follows that

$$
X \cdot b(0) + JX \cdot a(0) = -1. \tag{2.1}
$$

We also get from the computation of $[X, JX]$ that

$$
J[X, JX] \cdot \varphi = a^2 + b^2,
$$

thus

$$
X \cdot X \cdot J[X, JX] \cdot \varphi = 2((X \cdot a)^2 + (X \cdot b)^2 + aX \cdot X \cdot a + bX \cdot X \cdot b)
$$

and

$$
JX \cdot JX \cdot J[X, JX] \cdot \varphi = 2((JX \cdot a)^2 + (JX \cdot b)^2 + aJX \cdot JX \cdot a + bJX \cdot JX \cdot b).
$$

Since $a(0) = b(0) = 0$, if we prove that $X \cdot X \cdot J[X, JX] \cdot \varphi(0) = 0$ and $JX \cdot JX \cdot J[X, JX] \cdot \varphi(0) = 0$, we get $JX \cdot a(0) = 0$ and $X \cdot b(0) = 0$ which is incompatible with (2.1).

Let us compute $X \cdot X \cdot J[X, JX] \cdot \varphi(0)$. In local coordinates, we write $X = \sum_{j=1}^{2n} X_j \frac{\partial}{\partial x_j}$, $[X, JX] = \sum_{j=1}^{2n} Y_j \frac{\partial}{\partial x_j}$ and $J = (J_{i,j})_{i,j=1,\ldots,2n}$. We have

$$
X \cdot J[X, JX] \cdot \varphi
$$

$$
= \sum_{i,j,k=1}^{2n} X_i \frac{\partial J_{j,k}}{\partial x_i} Y_k \frac{\partial \varphi}{\partial x_j} + X_i J_{j,k} \frac{\partial Y_k}{\partial x_i} \frac{\partial \varphi}{\partial x_j} + X_i J_{j,k} Y_k \frac{\partial^2 \varphi}{\partial x_i \partial x_j}
$$

$$
= (X \cdot J)[X, JX] \cdot \varphi + J(X \cdot [X, JX]) \cdot \varphi + D^2 \varphi(X, J[X, JX]).
$$

and

$$X \cdot X \cdot J[X, JX] \cdot \varphi$$

$$= \sum_{i,j,k,l=1}^{2n} X_l \frac{\partial X_i}{\partial x_l} \frac{\partial J_{j,k}}{\partial x_i} Y_k \frac{\partial \varphi}{\partial x_j} + X_l X_i \frac{\partial^2 J_{j,k}}{\partial x_l \partial x_i} Y_k \frac{\partial \varphi}{\partial x_j} + X_l X_i \frac{\partial J_{j,k}}{\partial x_i} \frac{\partial Y_k}{\partial x_l} \frac{\partial \varphi}{\partial x_j}$$

$$+ X_l X_i \frac{\partial J_{j,k}}{\partial x_i} Y_k \frac{\partial^2 \varphi}{\partial x_j \partial x_l} + X_l \frac{\partial X_i}{\partial x_l} J_{j,k} \frac{\partial Y_k}{\partial x_i} \frac{\partial \varphi}{\partial x_j} + X_l X_i \frac{\partial J_{j,k}}{\partial x_l} \frac{\partial Y_k}{\partial x_i} \frac{\partial \varphi}{\partial x_j}$$

$$+ X_l X_i J_{j,k} \frac{\partial^2 Y_k}{\partial x_i \partial x_l} \frac{\partial \varphi}{\partial x_j} + X_l X_i J_{j,k} \frac{\partial Y_k}{\partial x_i} \frac{\partial^2 \varphi}{\partial x_j \partial x_l} + X_l \frac{\partial X_i}{\partial x_l} J_{j,k} Y_k \frac{\partial^2 \varphi}{\partial x_i \partial x_j}$$

$$+ X_l X_i \frac{\partial J_{j,k}}{\partial x_l} Y_k \frac{\partial^2 \varphi}{\partial x_i \partial x_j} + X_l X_i J_{j,k} \frac{\partial Y_k}{\partial x_l} \frac{\partial^2 \varphi}{\partial x_i \partial x_j} + X_l X_i J_{j,k} Y_k \frac{\partial^3 \varphi}{\partial x_i \partial x_j \partial x_l}$$

$$= (X \cdot X \cdot J)[X, JX] \cdot \varphi + 2(X \cdot J)(X \cdot [X, JX]) \cdot \varphi$$

$$+ 2D^2 \varphi(X, (X \cdot J)[X, JX]) + J(X \cdot X \cdot [X, JX]) \cdot \varphi$$

$$+ 2D^2 \varphi(J(X \cdot [X, JX]), X) + D^2 \varphi(X \cdot X, J \cdot [X, JX])$$

$$+ D^3 \varphi(X, X, J[X, JX]).$$

Since $[X, JX](0)$ vanishes at 0, the terms $(X \cdot X \cdot J)[X, JX] \cdot \varphi(0)$, $D^2 \varphi(X \cdot X, J[X, JX])$, $D^2 \varphi(X, (X \cdot J)[X, JX])$ and $D^3 \varphi(X, X, J[X, JX])(0)$ also vanish at 0.

Again $[X, JX](0) = 0$ yields $[X, [X, JX]](0) = X \cdot [X, JX](0)$. On the other hand, X commutes at 0 at order 4 so $[X, [X, JX]](0) = 0$ which gives $X \cdot [X, JX](0) = 0$. So $(X \cdot J)(X \cdot [X, JX]) \cdot \varphi(0) = 0$ and $D^2 \varphi(J(X \cdot [X, JX]), X)(0) = 0$.

Using $[X, [X, JX]](0) = 0$, we get

$$[X, [X, [X, JX]]](0) = X \cdot [X, [X, JX]](0)$$

and since X commutes at 0 at order 4, we have $X \cdot [X, [X, JX]](0) = 0$. We also have $X \cdot [X, [X, JX]] = X \cdot X \cdot [X, JX] - X \cdot [X, JX] \cdot X$ and since $[X, JX](0) = X \cdot [X, JX](0) = 0$, we conclude that $X \cdot X \cdot [X, JX](0) = 0$ and so $J(X \cdot X \cdot [X, JX]) \cdot \varphi(0) = 0$.

We finally have proved that $X \cdot X \cdot J[X, JX] \cdot \varphi(0) = 0$ and so $X \cdot b(0) = 0$. Analogously, we have $JX \cdot JX \cdot J[X, JX] \cdot \varphi(0) = 0$ which yields $JX \cdot a(0) = 0$.

Now, just notice that (2.1) and $X \cdot b(0) = JX \cdot a(0) = 0$ are incompatible. Hence there is no J-holomorphic disc γ tangent at order 5 at 0 to M such that $\frac{\partial \gamma}{\partial x}(0) = L_5(0)$.

3. Existence of J-holomorphic discs with prescribed derivatives

In this section, we show under an appropriate assumption, that there exists a J-holomorphic disc γ with prescribed derivatives of any order. More precisely, we prove the following

THEOREM 3.1. — Let J be a real analytic complex structure in a neighborhood of the origin of $\mathbb{R}^{2n}$, and let $(x_k)_k$ be a sequence of vectors of $\mathbb{R}^{2n}$ such that $\sum_{k=0}^{\infty} \frac{k+1}{k!}|x_k| \leqslant r$, where $r > 0$ is sufficiently small.
Then there exists a J-holomorphic disc $\gamma : \mathbb{D} \to \mathbb{R}^{2n}$ such that for all $k \in \mathbb{N}$, $\frac{\partial^k \gamma}{\partial x^k}(0) = x_k$.

Before proving Theorem 3.1, we rewrite the condition of J-holomorphicity of a disc when J is close to the standard structure J_0. As in [4], $\gamma : \mathbb{D} \to \mathbb{R}^{2n}$ is a J-holomorphic disc if and only if

$$\frac{\partial \gamma}{\partial \overline{\zeta}} - A_J(\gamma)\frac{\partial \overline{\gamma}}{\partial \overline{\zeta}} = 0$$

with $A_J(z) = (J_0 + J(z))^{-1}(J_0 - J(z))C$, C being the $\mathbb{R}$-linear application which corresponds to the complex conjugation on $\mathbb{R}^{2n} = \mathbb{C}^n$.
Let $T : C^\omega(\mathbb{D}) \to C^\omega(\mathbb{D})$ be the map defined by $T(u)(\zeta) := \int_{[0,\overline{\zeta}]} u(\zeta, \omega)d\omega$. The real analytic function $T(u)$ is a primitive of u with respect to $\overline{\zeta}$, that is $\frac{\partial T(u)}{\partial \overline{\zeta}} = u$. Let also $\Phi_J : C^\omega(\mathbb{D}) \to C^\omega(\mathbb{D})$ be the map defined by $\Phi_J(u) = u - T\left(A_J(u)\frac{\partial \overline{u}}{\partial \overline{\zeta}}\right)$.
Then, a real analytic function $\gamma : \mathbb{D} \to \mathbb{R}^{2n}$ is J-holomorphic if and only if $\Phi_J(\gamma)$ is holomorphic in the classical way. We can now prove Theorem 3.1.
Proof of Theorem 3.1: The principle of the proof is analogous to the case of finite sequences: we first introduce two well chosen Banach algebras: an algebra $\tilde{\Omega}$ of vector sequences, and an algebra Ω of real analytic functions on $\mathbb{D}$. We set

$$\tilde{\Omega} = \left\{(x_k)_k \in \left(\mathbb{R}^{2n}\right)^{\mathbb{N}}, \ \|(x_k)_k\|_\sim := \sum_{k=0}^{\infty} \frac{k+1}{k!}|x_k| < \infty\right\},$$

$$\Omega = \left\{f = \sum_{k,l} f_{k,l}\zeta^k\overline{\zeta}^l : \mathbb{D} \to \mathbb{R}^{2n}, \ \|f\| := \sum_{k,l=0}^{\infty}(1 + k + l + kl)|f_{k,l}| < \infty\right\}.$$

We also introduce two applications

$$\varphi_1 : \begin{cases} \tilde{\Omega} & \longrightarrow & C^\omega(\mathbb{D}) \\ (x_k)_k & \longmapsto & \sum_k \frac{x_k}{k!} \zeta^k \end{cases},$$

$$\varphi_2 : \begin{cases} \Omega & \longrightarrow & \left(\mathbb{R}^{2n}\right)^{\mathbb{N}} \\ f & \longmapsto & \left(\frac{\partial^k f}{\partial x^k}(0)\right)_k \end{cases}.$$

We will prove the following facts:

(i) φ_1 is an isometry from $\tilde{\Omega}$ into Ω,
(ii) φ_2 maps continuously Ω into $\tilde{\Omega}$,
(iii) $\Phi_J : \Omega \to \Omega$ is invertible in a neighborhood of 0,
(iv) $\varphi_2 \circ \Phi_J^{-1} \circ \varphi_1 : \tilde{\Omega} \to \Omega$ is also invertible in a neighborhood of 0.

Then, the disc $\gamma = \Phi_J^{-1} \circ \varphi_1 \circ \left(\varphi_2 \circ \Phi_J^{-1} \circ \varphi_1\right)^{-1} ((x_k)_k)$ will be the J-holomorphic disc we are looking for.

We first check fact (i): $\varphi((x_k)_k) = \sum_k \frac{x_k}{k!} \zeta^k$ and so, for all $l \neq 0$, the coefficient of $\zeta^k \overline{\zeta}^l$ in $\varphi_1((x_k)_k)$ vanishes. Therefore

$$\|\varphi_1((x_k)_k)\| = \sum_k (1+k) \frac{|x_k|}{k!} = \|(x_k)_k\|_\sim$$

and φ_1 is an isometry from $\tilde{\Omega}$ into Ω.

We now check fact (ii): writing ζ as $\zeta = x + iy$, we get for all k and l that $\zeta^k \overline{\zeta}^l = x^{k+l} + y P_{k,l}(x,y)$ where $P_{k,l}$ is a polynomial in x and y. Therefore

$$\frac{\partial^n \zeta^k \overline{\zeta}^l}{\partial x^n}\bigg|_{\zeta=0} = \begin{cases} (k+l)! & \text{if } k+l=n \\ 0 & \text{otherwise} \end{cases}.$$

So, for $f = \sum_{k,l} f_{k,l} \zeta^k \overline{\zeta}^l$, we have $\frac{\partial^n f}{\partial x^n}(0) = n! \sum_{k+l=n} f_{k,l}$ and

$$\varphi_2(f) = \left(n! \sum_{k+l=n} f_{k,l} \right)_{n \in \mathbb{N}},$$

from which it follows that

$$\|\varphi_2(f)\|_\sim \leqslant \sum_n (n+1) \sum_{k+l=n} |f_{k,l}|$$
$$\leqslant \|f\|.$$

So $\varphi_2(f)$ belongs to $\tilde{\Omega}$ and $\varphi_2 : \Omega \to \tilde{\Omega}$ is continuous.

In order to prove Fact (iii), we introduce for $f \in \Omega$ the functions $g = A_J(f)\frac{\partial \overline{f}}{\partial \overline{z}}$ and $G = T(g)$. Fact (iii) will be proved if we show that G belongs to Ω and that $\|G\| < \frac{1}{2}\|f\|$ when $\|f\|$ is small, because Φ_J will then be an

invertible perturbation of the identity in a neighborhood of $0 \in \Omega$. We will use the following lemma.

LEMMA 3.2. — *Let $\mathcal{A}$ be the Banach algebra of complex valued functions defined by*

$$\mathcal{A} = \{u(z,\bar{z}) = \sum_{k,l} u_{k,l} z^k \bar{z}^l, \ \|f\|_* := \sum_{k,l} (1+k)|f_{k,l}| < \infty\},$$

and let F be an analytic function bounded on $\{z \in \mathbb{C}^k, \ |z| < R\}$, $R > 0$. For $r > 0$, we denote by $\mathcal{A}_r$ the set $\mathcal{A}_r = \{f \in \mathcal{A}, \ \|f\|_* < r\}$. Then, for all $r \in]0, R[$, the following properties hold true:

(i) *for all $f_1, \ldots, f_k \in \mathcal{A}_r$, $F(f_1, f_2, \ldots, f_k)$ belongs to $\mathcal{A}$,*

(ii) $\Psi_F : \begin{cases} \mathcal{A}_r^k & \longrightarrow \quad \mathcal{A} \\ (f_1, \ldots, f_k) & \longmapsto \quad \Psi_F(f_1, \ldots, f_k) = F(f_1, \ldots, f_k) \end{cases}$ *is continuous,*

(iii) *for all $f_1, \ldots, f_k \in \mathcal{A}_r$, $\|\Psi_F(f_1, \ldots, f_k)\|_* \leqslant \left(\frac{R}{R-r}\right)^k \|F\|_\infty$.*

We admit this lemma for the moment and finish the proof of Theorem 3.1. We denote by R the radius of convergence of A_J, and we apply Lemma 3.2 to A_J. If f belongs to Ω, each of its components belongs to $\mathcal{A}$ and has norm smaller than $\|f\|$. Therefore, for all f belonging to $\Omega_{\frac{R}{2}} := \{g \in \Omega, \ \|g\| < \frac{R}{2}\}$, Lemma 3.2 implies that each coefficient in the matrix $A_J(f)$ belongs to $\mathcal{A}$ and have norm smaller that $\|A_J\|_\infty$. Moreover $\left\|\frac{\partial \bar{f}}{\partial \bar{z}}\right\|_* = \sum_{k,l}(1+l)k|f_{k,l}| \leqslant \|f\|$, and since $\mathcal{A}$ is a Banach algebra, we have $\|g\|_* = \left\|A_J(f)\frac{\partial \bar{f}}{\partial \bar{z}}\right\|_* \lesssim \|A_J\|_\infty \|f\|$, uniformly with respect to f and J.

Now we notice that the operator $T : \mathcal{A}^{2n} \to \Omega$ is continuous because for $u = \sum_{k,l} u_{k,l} z^k \bar{z}^l \in \mathcal{A}$ we have

$$\int_{[0,\bar{z}]} u(z,\omega) d\omega = \sum_{k,l} \frac{1}{l+1} u_{k,l} z^k \bar{z}^{l+1}$$

and

$$\sum_{k,l} \frac{k+(l+1)+1+k(l+1)}{l+1} |u_{k,l}| = \sum_{k,l} \frac{k+1}{l+1} |u_{k,l}| + (k+1)|u_{k,l}|$$

$$\leqslant 2\|u\|_*.$$

Therefore $G = T(g)$ belongs to Ω and $\|G\| \leqslant 2\|g\|_* \lesssim \|A_J\|_\infty \|f\|$. This proves that $\Phi_J(f)$ belongs to Ω for all $f \in \Omega_{\frac{R}{2}}$ and that $\Phi_J : \Omega_{\frac{R}{2}} \to \Omega$ is continuous. Moreover, provided J is close enough to the standard structure, $\|A_J\|_\infty$ is arbitrarily small and so $\|G\| \leqslant \frac{1}{2}\|f\|$ which implies

that Φ_J is a small perturbation of the identity. Thus $\Phi_J : \Omega_{\frac{R}{2}} \to \Phi_J(\Omega_{\frac{R}{2}})$ is continuously invertible.

Now Fact (iv) is an immediate consequence of the previous facts because, since $\varphi_2 \circ \varphi_1$ is the identity over $\tilde{\Omega}$, $\varphi_2 \circ \Phi_J^{-1} \circ \varphi_1$ is in fact a continuous perturbation of the identity which is arbitrarily small, provided J is close enough to the standard structure.

Now, for any sequence $(x_k)_k$ of vectors of $\mathbb{R}^{2n}$, provided $\|(x_k)_k\| < \frac{R}{2}$, $\gamma = \Phi_J^{-1} \circ \varphi_1 \circ \left(\varphi_2 \circ \Phi_J^{-1} \circ \varphi_1 \right)^{-1} ((x_k)_k)$ is a J-holomorphic disc such that $\varphi_2(\gamma) = (x_k)_k$. To conclude the proof of the theorem, we have to prove the crucial Lemma 3.2. $\square$

Proof of Lemma 3.2: The proof of this Lemma is inspired from abstract harmonic analysis and more precisely from Theorem 24D of [6]. In order to make the proof clearer, we prove the lemma for $k = 2$, that is for $F : \mathbb{C}^2 \to \mathbb{C}$, analytic over $P(0, R) = \{(z, w) \in \mathbb{C}^2, \ |z|^2 + |w|^2 < R^2\}$, $R > 0$. The case $k > 2$ is a direct generalization of the case $k = 2$.

Let $f = \sum_{k,l} f_{k,l} \zeta^k \overline{\zeta}^l$ and $g = \sum_{k,l} g_{k,l} \zeta^k \overline{\zeta}^l$ belonging to $\mathcal{A}$ be such that $\|f\|_* < r$ and $\|g\|_* < r$ for some $r \in]0, R[$. We write $F(z, w) = \sum_{k,l} F_{k,l} z^k w^l$ and $F(f, g)(\zeta) = \sum_{k,l} h_{k,l} \zeta^k \overline{\zeta}^l$. In order to determine a suitable expression of $h_{k,l}$ we introduce the Banach algebra

$$\hat{\mathcal{A}} = \left\{ (u_{k,l})_{k,l} \subset \mathbb{R}^{2n}, \ \|(u_{k,l})_{k,l}\|_{\hat{*}} := \sum_{k,l} (1+k)|u_{k,l}| < \infty \right\}$$

that we trivially identify to $\mathcal{A}$ via the application

$$\phi : \left\{ \begin{array}{ccc} \mathcal{A} & \longrightarrow & \hat{\mathcal{A}} \\ \sum_{k,l} u_{k,l} \zeta^k \overline{\zeta}^l & \longmapsto & (u_{k,l})_{k,l} \end{array} \right. .$$

We denote by $\mathfrak{e}$ the unit of $\hat{\mathcal{A}}$, $\mathfrak{e}_{k,l} = 1$ if $k = l = 0$, and 0 otherwise and we denote by $\mathbf{1}$ the constant function which equals 1 so that $\phi(\mathbf{1}) = \mathfrak{e}$. We have

$$h_{k,l} = \sum_{n,m} F_{n,m} \phi(f^n g^m)_{k,l}$$

and using Cauchy's Formula we get:

$$h_{k,l} = \sum_{n,m} \frac{1}{(2i\pi)^2} \int_{\substack{|\lambda|=R \\ |\mu|=R}} \frac{F(\lambda, \mu)}{\lambda^{n+1} \mu^{m+1}} \phi(f^n g^m)_{k,l} d\lambda d\mu.$$

Since $\|f\|_* < r$ and $\|g\|_* < r$, we have $\|f^n g^m\|_* < r^{n+m}$ so $|\phi(f^n g^m)_{k,l}| < r^{n+m}$.

Therefore if $r < R$, the series $\sum_{n,m} \frac{1}{\lambda^n \mu^m} \phi(f^n, g^m)_{k,l}$ converges normally for all λ and μ in $\mathbb{C}$ such that $|\lambda| = |\mu| = R$, and we can exchange the signs $\int$ and $\sum$ in the last expression of $h_{k,l}$ and which gives

$$h_{k,l} = \frac{1}{(2i\pi)^2} \int_{\substack{|\lambda|=R \\ |\mu|=R}} F(\lambda, \mu) \sum_{n,m} \left(\phi \left(\frac{f^n}{\lambda^{n+1}} \right) \phi \left(\frac{g^m}{\mu^{m+1}} \right) \right)_{k,l} d\lambda d\mu.$$

Since $\|f\|_* < r < |\lambda|$ and $\|g\|_* < r < |\mu|$, we have $(\lambda\mathfrak{e} - \phi(f))^{-1} = \sum_n \phi \left(\frac{f^n}{\lambda^{n+1}} \right)$ and $(\mu\mathfrak{e} - \phi(g))^{-1} = \sum_m \phi \left(\frac{g^m}{\mu^{m+1}} \right)$ which yields

$$h_{k,l} = \frac{1}{(2i\pi)^2} \int_{\substack{|\lambda|=R \\ |\mu|=R}} F(\lambda, \mu) \left((\lambda\mathfrak{e} - \phi(f))^{-1} (\mu\mathfrak{e} - \phi(g))^{-1} \right)_{k,l} d\lambda d\mu.$$

Using this expression of $h_{k,l}$, it follows that

$$\sum_{k,l} (k+1)|h_{k,l}|$$

$$\leqslant \frac{R^2}{(2\pi)^2} \int_0^{2\pi} \int_0^{2\pi} |F(Re^{i\theta}, Re^{i\varphi})|$$

$$\cdot \sum_{k,l} (k+1) \left| \left((Re^{i\theta}\mathfrak{e} - \phi(f))^{-1} (Re^{i\varphi}\mathfrak{e} - \phi(g))^{-1} \right)_{k,l} \right| d\theta d\varphi$$

$$\leqslant \frac{R^2}{(2\pi)^2} \int_0^{2\pi} \int_0^{2\pi} |F(Re^{i\theta}, Re^{i\varphi})| \left\| \left(Re^{i\theta}\mathfrak{e} - \phi(f) \right)^{-1} \left(Re^{i\varphi}\mathfrak{e} - \phi(g) \right)^{-1} \right\|_{\hat{*}} d\theta d\varphi.$$

Now, using again the identity $(\lambda\mathfrak{e} - \phi(f))^{-1} = \sum_n \phi \left(\frac{f^n}{\lambda^{n+1}} \right)$, for $\lambda = Re^{i\theta}$ we get

$$\left\| (\lambda\mathfrak{e} - \phi(f))^{-1} \right\|_{\hat{*}} \leqslant \frac{1}{R-r}.$$

The same holds true for g and so

$$\|F(f,g)\|_* = \sum_{k,l} (k+1)|h_{k,l}| \leqslant \left(\frac{R}{R-r} \right)^2 \|F\|_\infty,$$

which implies that $F(f,g)$ belongs to $\mathcal{A}$ and that (iii) of Lemma 3.2 holds true.

In order to establish the continuity of Ψ_F, let $\tilde{f}$ and $\tilde{g}$ be two functions of $\mathcal{A}$ such that $\|\tilde{f}\|_* < r$ and $\|\tilde{g}\|_* < r$ and set $F(\tilde{f}, \tilde{g}) = \sum_{k,l} \tilde{h}_{k,l} \zeta^k \overline{\zeta}^l$. We have

$$h_{k,l} - \tilde{h}_{k,l} = \frac{1}{(2i\pi)^2} \int_{\substack{|\lambda|=R \\ |\mu|=R}} F(\lambda, \mu) \left((\lambda\mathfrak{e} - \phi(f))^{-1} (\mu\mathfrak{e} - \phi(g))^{-1} \right.$$

$$\left. - (\lambda\mathfrak{e} - \phi(\tilde{f}))^{-1} (\mu\mathfrak{e} - \phi(\tilde{g}))^{-1} \right)_{k,l} d\lambda d\mu.$$

This yields

$$\|F(f,g) - F(\tilde{f}, \tilde{g})\|_*$$
$$\leqslant \frac{R^2}{4\pi^2} \int_0^{2\pi}\!\!\int_0^{2\pi} |F(Re^{i\theta}, Re^{i\varphi})| \cdot \left\| \left(Re^{i\theta}\mathfrak{e} - \phi(f)\right)^{-1}\left(Re^{i\varphi}\mathfrak{e} - \phi(g)\right)^{-1} \right.$$
$$- \left. \left(Re^{i\theta}\mathfrak{e} - \phi(\tilde{f})\right)^{-1}\left(Re^{i\varphi}\mathfrak{e} - \phi(\tilde{g})\right)^{-1}\right\|_{\hat{*}} d\theta d\varphi.$$

We have the following inequality:

$$\left\| \left(Re^{i\theta}\mathfrak{e} - \phi(f)\right)^{-1} - \left(Re^{i\varphi}\mathfrak{e} - \phi(\tilde{f})\right)^{-1}\right\|_{\hat{*}}$$
$$\leqslant \left\| \left(Re^{i\theta}\mathfrak{e} - \phi(f)\right)^{-1} \cdot \left(Re^{i\varphi}\mathfrak{e} - \phi(\tilde{f})\right)^{-1}\right\|_{\hat{*}} \cdot \|\phi(f) - \phi(\tilde{f})\|_{\hat{*}}$$
$$\leqslant \frac{1}{(R-r)^2}\|f - \tilde{f}\|_*.$$

Using this estimate of $\left\| \left(Re^{i\theta}\mathfrak{e} - \phi(f)\right)^{-1} - \left(Re^{i\theta}\mathfrak{e} - \phi(\tilde{f})\right)^{-1}\right\|_{\hat{*}}$ and the corresponding one for $\left\| \left(Re^{i\varphi}\mathfrak{e} - \phi(g)\right)^{-1} - \left(Re^{i\varphi}\mathfrak{e} - \phi(\tilde{g})\right)^{-1}\right\|_{\hat{*}}$, we get

$$\|F(f,g) - F(\tilde{f}, \tilde{g})\|_* \leqslant \frac{R^2}{(R-r)^3}\|F\|_\infty \cdot (\|f - \tilde{f}\|_* + \|g - \tilde{g}\|_*)$$

which proves that Ψ_F is continuous. $\square$

Under a simple growth condition of a sequence $(x_k)_k$, the following corollary gives the existence of a germ of J-holomorphic disc γ such that for all $k \in \mathbb{N}$, $\frac{\partial^k \gamma}{\partial x^k}(0) = x_k$.

COROLLARY 3.3. — *Let J be a real analytic complex structure in a neighborhood of the origin of $\mathbb{R}^{2n}$, and let $(x_k)_k$ be a sequence of vectors of $\mathbb{R}^{2n}$. Assume that x_0 is close enough to the origin and that there exists $R > 0$ such that for all $k \in \mathbb{N}^*$, $|x_k| \leqslant k!R^k$.*
Then there exists a germ of J-holomorphic disc γ such that for all $k \in \mathbb{N}$, $\frac{\partial^k \gamma}{\partial x^k}(0) = x_k$.

Proof: Let $r > 0$ be the constant given by Theorem 3.1 and for $\alpha > 0$ let $(\tilde{x}_k)_k$ be the sequence defined by $\tilde{x}_k = \left(\frac{\alpha}{R}\right)^k x_k$.
Then $\sum_{k=1}^\infty \frac{k+1}{k!}|\tilde{x}_k| \leqslant \sum_{k=1}^\infty (k+1)\alpha^k$, so if $|x_0| < r$ and if α is small enough, we have $\sum_{k=0}^\infty \frac{k+1}{k!}|\tilde{x}_k| < r$. Therefore we can apply Theorem 3.1 to $(\tilde{x}_k)_k$ and so there exists a J-holomorphic disc $\tilde{\gamma} : \mathbb{D} \to \mathbb{R}^{2n}$ such that $\frac{\partial^k \tilde{\gamma}}{\partial x^k}(0) = \tilde{x}_k$ for all k.
Now setting $\gamma(\zeta) = \tilde{\gamma}(\frac{R}{\alpha}\zeta)$ for $\zeta \in \mathbb{C}$ such that $|\zeta| < \frac{\alpha}{R}$, we get a germ of J-holomorphic disc such that $\frac{\partial^k \gamma}{\partial x^k}(0) = x_k$ for all k. $\square$

Given a point p in M and an analytic vector field X, we aim at finding a germ of *J*-holomorphic disc γ such that $\gamma(0) = p$ and $\frac{\partial^k \gamma}{\partial x^k}(0) = D_X^{k-1,0} X(p)$ for $k \in \mathbb{N}^*$ by applying Corollary 3.3 to the sequence $(x_k)_k$ defined by $x_0 = p$ and $x_k = D_X^{k-1,0} X(p)$, $k \in \mathbb{N}^*$. For this, we have to prove that the sequence $(D_X^{k-1,0} X(p))_k$ satisfies the assumption of this corollary. The estimates needed will be given in the next proposition, but before we state it, we need the following notation.

If X is a real analytic vector field defined in the polydisc $\mathcal{P}(0, R) = \{(x_1, y_1, \ldots, x_n, y_n) \in \mathbb{R}^{2n}, |x_i| < R, |y_i| < R, i = 1, \ldots, n\}$, we denote by $\tilde{X}$ the "polarization" of X. More precisely, if $X(z, \overline{z}) = \sum_{\nu,\mu} a_{\nu,\mu} z^\nu \overline{z}^\mu$ where ν and μ are multi-indices of $\mathbb{N}^n$, $z = x + iy$ belongs to $\mathbb{C}^n$ and $a_{\nu,\mu}$ belongs to $\mathbb{R}^{2n}$, we set $\tilde{X}(z, \zeta) = \sum_{\nu,\mu} a_{\nu,\mu} z^\nu \zeta^\mu$ which is defined for all z and ζ such that $|z_j|, |\zeta_j| < R$, $j = 1, \ldots, n,$. We then set

$$c_R(X) = \sup_{|z_j|,|\zeta_j|<R} |\tilde{X}(z, \zeta)|.$$

We can now state the estimates of $D_X^{k,0} X(0)$:

PROPOSITION 3.4. — *Let X be an analytic vector field in a neighborhood of the origin of $\mathbb{R}^{2n}$. Then for all $R > 0$ small enough and for all $k \in \mathbb{N}$*

$$|D_X^{k,0} X(0)| \leqslant 2nk! \left(\frac{8nc_R(X)}{R} \right)^k.$$

The proof of Proposition 3.4 relies on the following combinatorial Lemmata.

LEMMA 3.5. — *For all $\alpha_1, \ldots, \alpha_k \in \mathbb{N}$ such that $\alpha_1 + \ldots + \alpha_k = k$, we have*

$$\alpha_1! \ldots \alpha_k! \leqslant k!.$$

Proof: Without restriction we assume that $\alpha_1, \ldots, \alpha_r \geqslant 1$ and $\alpha_{r+1} = \ldots = \alpha_k = 0$, $r \leqslant k$ so that $\alpha_1 + \ldots + \alpha_r = k$.
The number $k!$ is the number of permutations of the set $\{1, \ldots, k\}$ whereas $\alpha_1! \ldots \alpha_r!$ is the number of permutations of $\{1, \ldots, k\}$ which leave stable each of the sets $\{1, \ldots, \alpha_1\}, \{\alpha_1 + 1, \ldots, \alpha_1 + \alpha_2\}, \ldots, \{\alpha_1 + \ldots + \alpha_{r-1} + 1, \ldots, k\}$. Therefore $\alpha_1! \ldots \alpha_k! \leqslant k!$. □

LEMMA 3.6. — *There are exactly $\binom{2k-1}{k}$ distinct k-tuples $(\alpha_1, \ldots, \alpha_k) \in \mathbb{N}^k$ such that $\alpha_1 + \ldots + \alpha_k = k$.*

Proof: To a k-tuple $(\alpha_1, \ldots, \alpha_k) \in \mathbb{N}^k$ we associate the k-tuple $(\beta_1, \ldots, \beta_k) = (\alpha_1 + 1, \ldots, \alpha_k + 1)$ so that $\beta_1 + \ldots + \beta_k = 2k$. This correspondence is a

bijection, and now we count the number of such k-tuples $(\beta_1, \ldots, \beta_k)$.
The sum $1 + 1 + \ldots + 1$ ($2k$-times) equals $2k$ and there are as many ways of writing $\beta_1 + \ldots + \beta_k = 2k$ as ways of separating the number 1 in the sum $1 + 1 + \ldots + 1 = 2k$ with $k - 1$-sticks, that is $\binom{2k-1}{k-1}$ ways. $\square$

LEMMA 3.7. — *For all $k \in \mathbb{N}$ we have*

$$\binom{2k-1}{k-1} \leqslant \frac{4^k}{\sqrt{\pi k}}.$$

Proof: This is a simple consequence of Stirling's formula. $\square$

We now prove Proposition 3.4:

Proof of Proposition 3.4: Let X be a real analytic vector field in a neighborhood of the origin in $\mathbb{R}^{2n}$. Let $R > 0$ be such that X is defined and continuous on $\overline{P(0,R)}$.

We write X in local coordinates as $X = \sum_{j=1}^{2n} X_j \frac{\partial}{\partial x_j}$. In order to establish the proposition, we need an upper bound for the quantity:

$$X_{i_1} \frac{\partial}{\partial x_{i_1}} \left(X_{i_2} \frac{\partial}{\partial x_{i_2}} \left(\cdots \left(X_{i_{k-1}} \frac{\partial X_{i_k}}{\partial x_{i_{k-1}}} \right) \cdots \right) \right) (0). \tag{3.1}$$

In the term above, we denote by j_i the cardinal of the set $\{l, \ i_l = i\}$.

In (3.1), for $i \in \{1, \ldots, 2n\}$ and for $\alpha_j^{(i)}$, $j \in \{1, \ldots, k\}$, such that $\sum_j \alpha_j^{(i)} = j_i$, the term

$$\prod_{j=1}^{k} \frac{\partial^{\alpha_j^{(1)} + \ldots + \alpha_j^{(2n)}} X_{i_j}}{\partial x_1^{\alpha_j^{(1)}} \ldots \partial x_{2n}^{\alpha_j^{(2n)}}} (0)$$

appears less than $\binom{j_1}{(\alpha_1^{(1)}, \ldots, \alpha_k^{(1)})} \cdots \binom{j_{2n}}{(\alpha_1^{(2n)}, \ldots, \alpha_k^{(2n)})}$ times.

On the other hand, from Cauchy's inequalities, we get

$$\left| \frac{\partial^{\alpha_j^{(1)} + \ldots + \alpha_j^{(2n)}} X_{i_j}}{\partial x_1^{\alpha_j^{(1)}} \ldots \partial x_{2n}^{\alpha_j^{(2n)}}} (0) \right| \leqslant \alpha_j^{(1)}! \ldots \alpha_j^{(2n)}! \frac{c_R(X)}{R^{\alpha_j^{(1)} + \ldots + \alpha_j^{(2n)}}}.$$

This yields

$$\left| X_{i_1} \frac{\partial}{\partial x_{i_1}} \left(X_{i_2} \frac{\partial}{\partial x_{i_2}} \left(\cdots \left(X_{i_{k-1}} \frac{\partial X_{i_k}}{\partial x_{i_{k-1}}} \right) \cdots \right) \right) (0) \right|$$

$$\leqslant \sum_{\substack{\alpha_1^{(1)} + \ldots + \alpha_k^{(1)} = j_1 \\ \vdots \\ \alpha_1^{(2n)} + \ldots + \alpha_k^{(2n)} = j_{2n}}} j_1! \ldots j_{2n}! \left(\frac{c_R(X)}{R} \right)^k.$$

Applying successively Lemma 3.6, Lemma 3.7 and Lemma 3.5 we get

$$\left| X_{i_1} \frac{\partial}{\partial x_{i_1}} \left(X_{i_2} \frac{\partial}{\partial x_{i_2}} \left(\cdots \left(X_{i_{k-1}} \frac{\partial X_{i_k}}{\partial x_{i_{k-1}}} \right) \cdots \right) \right)(0) \right|$$

$$\leqslant 4^{j_1 + \ldots + j_{2n}} j_1! \ldots j_{2n}! \left(\frac{c_R(X)}{R} \right)^k$$

$$\leqslant k! \left(\frac{4 c_R(X)}{R} \right)^k.$$

Now, since there are at most $(2n)^k$ terms

$$X_{i_1} \frac{\partial}{\partial x_{i_1}} \left(X_{i_2} \frac{\partial}{\partial x_{i_2}} \left(\cdots \left(X_{i_{k-1}} \frac{\partial X_{i_k}}{\partial x_{i_{k-1}}} \right) \cdots \right) \right)(0)$$

in $D_X^{k,0} X(0)$, the following inequaltiy holds true

$$|D_X^{k,0} X(0)| \leqslant k! \left(\frac{8 n c_R(X)}{R} \right)^k,$$

which proves the claim. $\square$

4. *J*-holomorphic discs and hypersurfaces

In this section, given a point $p \in M$ close enough to the origin and a vector $v \in T_p^J M$, we look for a *J*-holomorphic disc γ included in M such that $\frac{\partial \gamma}{\partial x}(p) = v$. Our strategy will be the following: we will look for a sequence of vector fields $(X_k)_{k \in \mathbb{N}^*}$ such that

(i) For all k, the vector field X_k belongs to $T^J M$,
(ii) $X_1(p) = v$,
(iii) $D_{X_k}^{l,0} X_k(p) = D_{X_1}^{l,0} X_1(p)$ for all k in $\mathbb{N}^*$ and l in $\mathbb{N}$,
(iv) X_k commutes at order k at p.

Then, applying Proposition 3.4 to X_1 and corollary 3.3 to the sequence $(x_k)_k$ defined by $x_0 = p$ and $x_k = D_{X_1}^{k-1,0} X_1(p)$ for $k \in \mathbb{N}^*$, we will get a germ of *J*-holomorphic disc γ such that $\gamma(0) = p$ and $\frac{\partial^k \gamma}{\partial x^k}(p) = D_{X_1}^{k-1,0} X_1(p)$ for all $k \in \mathbb{N}^*$, and in particular $\frac{\partial \gamma}{\partial x}(p) = v$. In order to check that γ is included in M, we will use Theorem 1 of [1]: since X_k commutes at order k at p, and since $\frac{\partial^l \gamma}{\partial x^l}(0) = D_{X_k}^{l-1,0} X_k(p)$ for all $l \leqslant k$, γ is tangent to M at p at order $k + 1$. Now, since this holds for all k, this implies that γ is tangent to M at p at any order. As M and γ are real analytic, γ is in fact included in M. In order to construct such a sequence of vector fields, we will need the following propositions.

PROPOSITION 4.1. — *Let X be any real analytic vector field in a neighborhood of the origin. Then the following assertions are equivalent :*

(i) *For all $0 \leqslant k \leqslant k_0 - 2$ and all $X_1, \ldots, X_k \in \{X, JX\}$,*

$$X_1 \cdot X_2 \ldots X_k \cdot [X, JX](0) = 0,$$

(ii) *X commutes at order k_0 at 0.*

Moreover, in that case, for all integers k and l such that $0 \leqslant l \leqslant k \leqslant k_0 - 1$ the following holds:

$$X_1 \ldots X_l \cdot [X_{l+1}, [\ldots [X_k, [X, JX] \ldots] (0) = [X_1, [X_2, \ldots [X_k, [X, JX] \ldots] (0).$$
$$(4.1)$$

Remark 4.2. — We point out that (4.1) holds true for $0 \leqslant k \leqslant k_0 - 1$ even if (i) is true only for $0 \leqslant k \leqslant k_0 - 2$.

Proof of proposition 4.1: The equivalence of (i) and (ii) was shown in [1], proposition 13. The second assertion is proved by induction on l, the case $l = 0$ being trivial.

We assume that the identity holds true for $l \leqslant k \leqslant k_0 - 2$ and we denote by Y the vector field $Y = [X_{l+2}, [\ldots, [X_k, [X, JX]] \ldots]]$. Then, since $\tilde{X}_1 \ldots \tilde{X}_l \cdot Y(0) = 0$ for all $\tilde{X}_1, \ldots \tilde{X}_l \in \{X, JX\}$, we have

$$X_1 \ldots X_l [X_{l+1}, [X_{l+2}, \ldots [X_k, [X, JX]] \ldots](0)$$
$$= X_1 \ldots X_l \cdot X_{l+1} \cdot Y(0) - X_1 \ldots X_l \cdot Y \cdot X_{l+1}(0)$$
$$= X_1 \ldots X_l \cdot X_{l+1} \cdot Y(0).$$

□

PROPOSITION 4.3. — *Let X be a vector field commuting at order k at 0, $k \geqslant 2$. Then for all $X_1, \ldots, X_k \in \{X, JX\}$, we have*

$$[X_1, [\ldots, [X_k, [X, JX]]]](0) = \underbrace{[JX, [\ldots, [JX}_{q}, \underbrace{[X, [\ldots, [X}_{p}, [X, JX] \ldots](0)$$

where $p = \#\{i, \ X_i = X\}$ and $q = \#\{i, \ X_i = JX\}$.

Proof: It suffices to show that for all $l \leqslant k - 2$ we have

$$[X_1, [\ldots, [X_l, [X, [JX, [X_{l+3}, [\ldots, [X, JX] \ldots](0)$$
$$= [X_1, [\ldots, [X_l, [JX, [X, [X_{l+3}, [\ldots, [X, JX] \ldots](0).$$

We set $Y = [X_{l+3}, [\ldots, [X, JX] \ldots]]$. From Jacobi identity we have:

$$[X, [JX, Y]] = [JX, [X, Y]] + [[X, JX], Y]. \qquad (4.2)$$

From Proposition 4.1, for all $0 \leqslant \tilde{l} \leqslant l$ and all $\tilde{X}_1, \dots, \tilde{X}_{\tilde{l}} \in \{X, JX\}$, we have $\tilde{X}_1 \dots \tilde{X}_{\tilde{l}} \cdot [X, JX](0) = 0$ which yields

$$[X_1, \dots, [X_l, [X, JX] \cdot Y] \dots](0) = 0 \tag{4.3}$$

Analogously, it follows from Proposition 4.1 that

$$[X_1, \dots, [X_l, Y \cdot [X, JX]] \dots](0) = 0$$

which with (4.3) gives

$$[X_1, \dots, [X_l, [[X, JX], Y]] \dots](0) = 0$$

and with (4.2) we finish the proof of the proposition. □

Taking our strategy into account, the proof of Theorem 1.4 will be a direct consequence of the following lemma:

LEMMA 4.4. — *Let* $X \in T^J M$ *be a real analytic vector field in a neighborhood of* $0 \in M$ *such that* $\mathbb{L}(X)$, *the Lie algebra generated by* X, *is included in* $T^J M$.
Then there exists a sequence $(X_k)_{k \in \mathbb{N}^*}$ *of vector fields in* $T^J M$ *such that*

(a) $D^{l,0}_{X_k} X_k(0) = D^{l,0}_X X(0)$ *for all* k *in* $\mathbb{N}^*$ *and* l *in* $\mathbb{N}$,
(b) *for all* $k \in \mathbb{N}^*$, X_k *commutes at order* k *at* 0.

Proof: We set $X_1 = X$ and by induction on k we construct a sequence $(X_k)_k$ of vector fields which satisfy (a) and (b) and which are such that for all k, X_k belongs to $\mathbb{L}(X)$. Assuming X_k constructed, we set

$$Y_{p,q} = \underbrace{[JX_k, [\dots, [JX_k,}_{q} \underbrace{[X_k, [\dots, [X_k}_{p} [X_k, JX_k] \dots]}$$

for all p and q such that $p + q = k - 1$, and we look for X_{k+1} as

$$X_{k+1} = X_k + \sum_{p+q=k-1} (a_{p,q} + b_{p,q}J)Y_{p,q},$$

where $a_{p,q}$ and $b_{p,q}$ are real analytic functions which vanish at order $k - 1$ at the origin. Hence X_{k+1} already belongs to $\mathbb{L}(X)$ and we have to show that $a_{p,q}$ and $b_{p,q}$ can be chosen so that (a) and (b) hold. We compute the

Lie bracket $[X_{k+1}, JX_{k+1}]$:

$$[X_{k+1}, JX_{k+1}] =$$
$$= [X_k, JX_k]$$
$$+ \sum_{p+q=k-1} \left(\left(X_k(a_{p,q}) - JX_k(b_{p,q}) \right) JY_{p,q} + \left(-X_k(b_{p,q}) - JX_k(a_{p,q}) \right) Y_{p,q} \right)$$
$$+ \sum_{p+q=k-1} (a_{p,q}^2 + b_{p,q}^2)[Y_{p,q}, JY_{p,q}]$$
$$+ \sum_{p+q=k-1} \left(a_{p,q} \left([X_k, JY_{p,q}] + [Y_{p,q}, JX_k] \right) - b_{p,q} \left([X_k, Y_{p,q}] + [JX_k, JY_{p,q}] \right) \right)$$
$$+ \sum_{\substack{p+q=k-1 \\ r+s=k-1}} \left(a_{r,s} Y_{r,s}(a_{p,q}) + b_{r,s} JY_{r,s}(a_{p,q}) - a_{r,s} JY_{r,s}(b_{p,q}) + b_{r,s} Y_{r,s}(b_{p,q}) \right) JY_{p,q}$$
$$+ \sum_{\substack{p+q=k-1 \\ r+s=k-1}} \left(-a_{r,s} Y_{r,s}(b_{p,q}) - b_{r,s} JY_{r,s}(b_{p,q}) - a_{r,s} JY_{r,s}(a_{p,q}) + b_{r,s} Y_{r,s}(a_{p,q}) \right) Y_{p,q}.$$

Since $a_{p,q}$ and $b_{p,q}$ vanish at order $k-1$ at the origin, and since X_k commutes at order k at 0, for all $X^{(1)}, \dots, X^{(k-2)} \in \{X_{k+1}, JX_{k+1}\}$ we have

$$X^{(1)} \dots X^{(k-2)} [X_{k+1}, JX_{k+1}](0) = 0.$$

It then follows from Proposition 4.1 that X_{k+1} already commutes at order k at 0. Moreover, $D^{r,0}_{X_{k+1}} X_{k+1}(0) = D^{r,0}_{X_k} X_k(0)$ for all $r \leqslant k-1$ and for all choices of functions $a_{p,q}$ and $b_{p,q}$, vanishing at order $k-1$ at 0.
We now choose these functions so that X_{k+1} commutes at order $k+1$ at the origin and so that $D^{r,0}_{X_{k+1}} X_{k+1}(0) = D^{r,0}_{X_k} X_k(0)$ for all $r \geqslant k$.

It follows from Proposition 4.1 that $Y_{r,s}(0) = D^{r,s}_{X_k}[X_k, JX_k](0)$ for all r and s such that $r + s = k - 1$. Since the functions $a_{p,q}$ and $b_{p,q}$ vanish at order $k-1$ at the origin, we have for all r and s such that $r + s = k - 1$

$$D^{r,s}_{X_{k+1}}[X_{k+1}, JX_{k+1}](0)$$
$$= Y_{r,s}(0) + \sum_{p+q=k-1} \left(\left(D^{r+1,s}_{X_k} a_{p,q}(0) - D^{r,s+1}_{X_k} b_{p,q}(0) \right) JY_{p,q}(0) \right)$$
$$+ \sum_{p+q=k-1} \left(\left(-D^{r+1,s}_{X_k} b_{p,q}(0) - D^{r,s+1}_{X_k} a_{p,q}(0) \right) Y_{p,q}(0) \right).$$

Perhaps after a change of coordinates if needed, we can assume that $X_k = \frac{\partial}{\partial x_1}$ and that $J(0)$ is still the standard structure. We therefore have $JX_k = \sum_{k=1}^n J_{2k-1,1} \frac{\partial}{\partial x_k} + J_{2k} \frac{\partial}{\partial y_k}$ and since $J(0)$ is the standard structure, it follows that $JX_k(0) = \frac{\partial}{\partial y_1}$. Now, since $a_{p,q}$ and $b_{p,q}$ vanishes at 0 at order $k-1$, neither X_k nor JX_k is differentiated in $D^{r,s}_{X_k} a_{p,q}(0)$ and $D^{r,s}_{X_k} b_{p,q}(0)$

and we therefore have

$$D_{X_{k+1}}^{r,s}[X_{k+1}, JX_{k+1}](0) = \tag{4.4}$$

$$= Y_{r,s}(0) + \sum_{p+q=k-1} \left(\left(\frac{\partial^k a_{p,q}}{\partial x_1^{r+1} \partial y_1^s}(0) - \frac{\partial^k b_{p,q}}{\partial x_1^r \partial y_1^{s+1}}(0) \right) JY_{p,q}(0) \right)$$

$$+ \sum_{p+q=k-1} \left(\left(-\frac{\partial^k b_{p,q}}{\partial x_1^{r+1} \partial y_1^s}(0) - \frac{\partial^k a_{p,q}}{\partial x_1^r \partial y_1^{s+1}}(0) \right) Y_{p,q}(0) \right).$$

We search for $a_{p,q}$ and $b_{p,q}$ as homogeneous polynomials in x_1 and y_1 of degree k without the term x_1^k such that

$$\forall (r,s) \in \mathbb{N}, \ r+s = k-1, \ \begin{cases} \frac{\partial^k a_{p,q}}{\partial x_1^{r+1} \partial y_1^s}(0) - \frac{\partial^k b_{p,q}}{\partial x_1^r \partial y_1^{s+1}}(0) = 0, \\ \frac{\partial^k b_{r,s}}{\partial x_1^{r+1} \partial y_1^s}(0) + \frac{\partial^k a_{p,q}}{\partial x_1^r \partial y_1^{s+1}}(0) = 0 \text{ if } (r,s) \neq (p,q), \\ \frac{\partial^k b_{p,q}}{\partial x_1^{r+1} \partial y_1^s}(0) + \frac{\partial^k a_{p,q}}{\partial x_1^r \partial y_1^{s+1}}(0) = 1. \end{cases}$$

With such a choice, (4.4) gives $D_{X_{k+1}}^{r,s}[X_{k+1}, JX_{k+1}] = 0$, for all r and s such that $r+s = k-1$, which together with Propositions 4.1 and 4.3 proves that X_{k+1} commutes at order $k+1$ at 0. So (b) holds true.

Now, we notice that $D_{X_k}^{r,0} a_{p,q}(0) = D_{X_k}^{r,0} b_{p,q}(0) = 0$ for all $r \in \mathbb{N}$, and all (p,q). Therefore $D_{X_{k+1}}^{r,0} X_{k+1}(0) = D_{X_k}^{r,0} X_k(0)$ for all $r \in \mathbb{N}$, and so (a) also holds true. □

We deduce from Theorem 1.4 the following generalization of Freeman's Theorem:

THEOREM 4.5. — *If J is integrable, then for all $X \in \ker \mathcal{L}$ and all p in a neighborhood of 0, there exists a J-holomorphic disc γ such that $\gamma(0) = p$ and $\frac{\partial \gamma}{\partial x}(0) = X(p)$.*

Proof: Let X be a vector field in $\ker \mathcal{L}$. We show that the Lie algebra generated by X and JX is included in $T^J M$ by proving by induction that all Lie brackets we can form with X and JX belong to $\ker \mathcal{L}$.

In the integrable case, we have the following characterization of $\ker \mathcal{L}$ (see [2]):

$$\ker \mathcal{L} = \{X \in T^J M, \ \forall Y \in T^J M, \ [X,Y] \in T^J M\}. \tag{4.5}$$

Assume that all Lie brackets of length at most k belongs to $\ker \mathcal{L}$. Let X_p and X_q be two Lie brackets of length p and q respectively, with p and q such that $p + q = k + 1$. Using (4.5), we show that $[X_p, X_q]$ and $[X_p, JX_q]$ belong to $\ker \mathcal{L}$.

Jacobi's identity gives for all $Y \in T^J M$:

$$[[X_p, JX_q], Y] = [X_p, [JX_q, Y]] + [JX_q, [Y, X_p]].$$

The vector field JX_q belong to ker $\mathcal{L}$ so (4.5) implies that $[JX_q, Y]$ belongs $T^J M$ and so, again (4.5) implies that $[X_p, [JX_q, Y]]$ belongs to $T^J M$. Analogously, $[JX_q, [Y, X_p]]$ belongs to $T^J M$ and so $[[X_p, JX_q], Y]$ is a element of $T^J M$ for all Y in $T^J M$, which with (4.5) implies that $[X_p, JX_q]$ belongs to ker $\mathcal{L}$.

Analogously, $[X_p, X_q]$ belongs to ker $\mathcal{L}$. Therefore all Lie brackets of X and JX belong to ker $\mathcal{L}$ and so to $T^J M$. Applying Theorem 1.4 ends the proof of Theorem 4.5. $\square$

Another example of situation where Theorem 1.4 can be useful is the following example:

Example 4.6. — Let $\varphi : \mathbb{R}^8 \to \mathbb{R}$ be the map defined by

$$\varphi(x_1, y_1, x_2, y_2, x_3, y_3, x_4, y_4) = y_1$$

and let M be the set $M = \{z \in \mathbb{R}^8, \ \varphi(z) = 0\}$. We define the eight following vector fields

$$L_1 = \frac{\partial}{\partial x_1}, \qquad\qquad L_2 = \frac{\partial}{\partial y_1},$$

$$L_3 = \frac{\partial}{\partial x_2} - \frac{1}{2} y_3^2 \frac{\partial}{\partial x_1}, \qquad L_4 = \frac{\partial}{\partial y_2} + (-2y_3 x_3 + x_2) \frac{\partial}{\partial x_1},$$

$$L_5 = \frac{\partial}{\partial x_3} - y_2 y_3 \frac{\partial}{\partial x_1}, \qquad L_6 = \frac{\partial}{\partial y_3} + x_3 \frac{\partial}{\partial x_2} - \left(\frac{x_3 y_3^2}{2} + x_3 y_2 \right) \frac{\partial}{\partial x_1},$$

$$L_7 = \frac{\partial}{\partial x_4} + y_4 \frac{\partial}{\partial y_4} \qquad L_8 = \frac{\partial}{\partial y_4}.$$

and the complex structure J they induce by setting

$$JL_1 = L_2, \qquad JL_3 = L_4, \qquad JL_5 = L_6, \qquad JL_7 = L_8.$$

This example is derived from the example of Section 2. Again $J(0)$ is the standard complex structure J_0 and the tangent space TM is spanned over $\mathbb{R}$ by L_1, L_3, L_4, L_5, L_6, L_7 and L_8 and $T^J M$ is spanned over $\mathbb{C}$ by L_3, L_5 and L_7.

When we compute the Lie brackets of the L_i's in the complex tangent

bundle, we get:

$$[L_3, L_4] = L_1,$$

$$[L_3, L_5] = 0, \qquad [L_4, L_5] = y_3 \frac{\partial}{\partial x_1},$$

$$[L_3, L_6] = y_3 \frac{\partial}{\partial x_1}, \quad [L_4, L_6] = 0, \qquad [L_5, L_6] = L_3,$$

$$[L_3, L_7] = 0, \qquad [L_4, L_7] = 0, \qquad [L_5, L_7] = 0, \qquad [L_6, L_7] = 0,$$

$$[L_3, L_8] = 0, \qquad [L_4, L_8] = 0, \qquad [L_5, L_8] = 0, \qquad [L_6, L_8] = 0,$$

and $[L_7, L_8] = -L_8$. Therefore

$$\hat{\mathcal{L}}_\varphi(L_3, L_5) = 0, \qquad\qquad \hat{\mathcal{L}}_\varphi(L_3, L_7) = 0,$$

$$\hat{\mathcal{L}}_\varphi(L_5, L_7) = 0, \qquad\qquad \hat{\mathcal{L}}_\varphi(L_3, JL_3) = d\varphi(L_2) = 1,$$

$$\hat{\mathcal{L}}_\varphi(L_5, JL_5) = d\varphi(L_3) = 0, \qquad \hat{\mathcal{L}}_\varphi(L_7, JL_7) = d\varphi(-L_8) = 0,$$

and so, M is pseudoconvex and $\ker \mathcal{L}$ is spanned by L_5 and L_7.

As in Section 2, there is no germ of *J*-holomorphic disc γ included in M such that $\gamma(0) = 0$ and $\frac{\partial \gamma}{\partial x}(0) = \frac{\partial}{\partial x_3}$ but Theorem 1.4 yields the existence of a germ of *J*-holomorphic disc γ included in M, such that $\gamma(0) = 0$ and $\frac{\partial \gamma}{\partial x}(0) = L_7(0)$.

We now prove Theorem 1.6 by applying our strategy with the following lemma:

LEMMA 4.7. — *Let* $\mathbb{L}$ *be a subbundle of* $T^J M$. *Assume that for all* $X \in \mathbb{L}$ *which commutes at order* k *at* 0, *and for all* $X_1, \ldots, X_{k+1} \in \{X, JX\}$, $[X_1, [\ldots, [X_k, X_{k+1}] \ldots]](0)$ *belongs to* $\mathbb{L}_0$.
Then for all $X \in \mathbb{L}$, *there exists a sequence* $(X_k)_k$ *of vector fields in* $\mathbb{L}$ *such that*
(α) $D_{X_k}^{l,0} X_k(0) = D_X^{l,0} X(0)$ *for all* k *in* $\mathbb{N}^*$ *and* l *in* $\mathbb{N}$,
(β) X_k *commutes at order* k *at* 0.

Proof: We proceed by induction on k. We set $X_1 = X$ and we assume X_k constructed. Maybe after a change of coordinates if needed, we can assume that $X_k = \frac{\partial}{\partial x_1}$ and $JX_k(0) = \frac{\partial}{\partial y_1}$. We set

$$X_{k+1} = X_k + \sum_{p=1}^{\ell} (a_p + Jb_p) L_p$$

where $L_1, \ldots, L_\ell$ is a basis of $\mathbb{L}$ and where a_p and b_p, $p = 1, \ldots, \ell$, are homogeneous polynomials of degree k in the variables x_1 and y_1, without the term x_1^k. Therefore X_{k+1} belongs to $\mathbb{L}$, $D_{X_{k+1}}^{l,0} X_{k+1}(0) = D_{X_1}^{l,0} X_1(0)$ for

all l and X_{k+1} commutes at order k at 0.

Analogously to the proof of Lemma 4.4, we now choose a_p and b_p so that X_{k+1} commutes at order $k+1$ at 0. We compute

$$[X_{k+1}, JX_{k+1}]$$

$$= [X_k, JX_k] + \sum_{p=1}^{\ell}(a_p^2 + b_p^2)[L_p, JL_p]$$

$$+ \sum_{p=1}^{\ell}\Big((X_k(a_p) - JX_k(b_p))JL_p + (-X_k(b_p) - JX_k(a_p))L_p\Big)$$

$$+ \sum_{p=1}^{\ell}\big(a_p\left([X_k, JL_p] + [L_p, JX_k]\right) - b_p\left([X_k, L_p] + [JX_k, JL_p]\right)\big)$$

$$+ \sum_{p,q=1}^{\ell}\big(a_q L_q(a_p) + b_q JL_q(a_p) - a_q JL_q(b_p) + b_q L_q(b_p)\big)JL_p$$

$$+ \sum_{p,q=1}^{\ell}\big(-a_q L_q(b_p) - b_q JL_q(b_p) - a_q JL_q(a_p) + b_q L_q(a_p)\big)L_p.$$

On the one hand, the functions a_p and b_p vanish at order $k-1$ at the origin. Therefore, for all r and s such that $r + s = k - 1$

$$D_{X_{k+1}}^{r,s}[X_{k+1}, JX_{k+1}](0)$$

$$= D_{X_k}^{r,s}[X_k, JX_k](0) + \sum_{p=1}^{\ell}\Big((D_{X_k}^{r+1,s}a_p(0) - D_{X_k}^{r,s+1}b_p(0))JL_p(0)\Big)$$

$$+ \sum_{p=1}^{\ell}\Big((-D_{X_k}^{r+1,s}b_p(0) - D_{X_k}^{r,s+1}a_p(0))L_p(0)\Big).$$

On the other hand, Proposition 4.1 implies that for all r and s such that $r + s = k - 1$

$$D_{X_k}^{r,s}[X_k, JX_k](0) = \underbrace{[JX_k, [\ldots, [JX_k}_{s}\underbrace{[X_k, \ldots [X_k}_{r}, [X_k, JX_k]]\ldots](0)$$

and so $D_{X_k}^{r,s}[X_k, JX_k](0)$ belongs to $\mathbb{L}_0$. Hence there exists $\alpha_1^{r,s}, \ldots, \alpha_\ell^{r,s} \in \mathbb{C}$ such that

$$D_{X_k}^{r,s}[X_k, JX_k](0) = \sum_{p=1}^{\ell}\alpha_p^{r,s}L_p(0)$$

and so

$$D_{X_{k+1}}^{r,s}[X_{k+1}, JX_{k+1}](0)$$

$$= \sum_{p=1}^{\ell} \alpha_p^{r,s} L_p(0) + \sum_{p=1}^{\ell} \left(\left(D_{X_k}^{r+1,s} a_p(0) - D_{X_k}^{r,s+1} b_p(0) \right) J L_p(0) \right)$$

$$+ \sum_{p=1}^{\ell} \left(\left(-D_{X_k}^{r+1,s} b_p(0) - D_{X_k}^{r,s+1} a_p(0) \right) L_p(0) \right).$$

Now we choose the functions a_p and b_p as in Lemma 4.4 such that

$$\forall (r,s) \in \mathbb{N}, \ r+s = k-1, \ \begin{cases} \dfrac{\partial^k a_p}{\partial x_1^{r+1} \partial y_1^s}(0) - \dfrac{\partial^k b_p}{\partial x_1^r \partial y_1^{s+1}}(0) & = -\mathrm{Im}(\alpha_p^{r,s}) \\ \dfrac{\partial^k b_{r,s}}{\partial x_1^{r+1} \partial y_1^s}(0) + \dfrac{\partial^k a_{p,q}}{\partial x_1^r \partial y_1^{s+1}}(0) & = \mathrm{Re}(\alpha_p^{r,s}) \end{cases} .$$

After solving this system of $2k$ equations and $2k$ variables, we get a vector field X_{k+1} such that $D_{X_{k+1}}^{r,s}[X_{k+1}, JX_{k+1}](0) = 0$ for all r and s such that $r+s = k-1$. Since X_{k+1} already commutes at order k, Propositions 4.1 and 4.3 then imply that in fact X_{k+1} commutes at order $k+1$ at 0. □

As a corollary of Theorem 1.6, we get the following Theorem already proved by Kruzhilin and Sukhov in [5] in the smooth case:

THEOREM 4.8. — If $\ker \mathcal{L} = T^J M$, then for all $p \in M$ and all $v \in T^J M$, there exists a germ of J-holomorphic disc γ such that $\gamma(0) = p$, $\frac{\partial \gamma}{\partial x}(p) = v$ and $\gamma(\mathbb{D}) \subset M$.

Proof: We prove that if $X \in T^J M$ commutes at order k at 0, then for all $X_1, \ldots, X_{k-1} \in \{X, JX\}$, $[X_1, [\ldots, [X_{k-1}, [X, JX]] \ldots](0)$ belongs to $T_0^J M$ and we apply Theorem 1.6.

On the one hand, since M is Levi flat, $[X, JX]$ belongs to $T^J M$ and $d\varphi(J[X, JX])$ vanishes identically.

On the other hand, since X commutes at order k at 0

$$X_1 \cdot X_2 \ldots X_{k-1} \cdot d\varphi(J[X, JX])(0)$$

$$= d\varphi(J(X_1 \cdot X_2 \ldots X_{k-1} \cdot [X, JX]))(0)$$

$$= d\varphi(J[X_1, [X_2, \ldots [X_{k-1}, [X, JX] \ldots])(0)$$

and so $[X_1, \ldots [X_{k-1}, [X, JX] \ldots](0)$ belongs to $JT_0 M$. Now, since X and JX belong to TM, $[X_1, \ldots [X_{k-1}, [X, JX] \ldots]$ also belongs to TM and finally $[X_1, \ldots [X_{k-1}, [X, JX] \ldots](0)$ to $T_0^J M$. □

Let us notice that the above theorem can in fact be generalized in the following sense. If $\mathbb{L}$ is a complex bundle included in $\ker \mathcal{L}$ such that

- for all $X \in \mathbb{L}$, $[X, JX]$ belongs to $\mathbb{L}$,
- there exist $\varphi_1, \ldots, \varphi_k$ such that for all $X \in T^J M$, $X \cdot \varphi_1 = \ldots = X \cdot \varphi_k = 0$ if and only if X belongs to $\mathbb{L}$,

then the previous proof shows that for any X in $\mathbb{L}$, there exists a J-holomorphic disc γ such that $\gamma(0) = p$, $\frac{\partial \gamma}{\partial x}(p) = v$ and $\gamma(\mathbb{D}) \subset M$. However, as we will see in the next example, Theorem 1.6 is more general.

Example 4.9. — Let $\varphi : \mathbb{R}^8 \to \mathbb{R}$ be the map defined by

$$\varphi(x_1, y_1, x_2, y_2, x_3, y_3, x_4, y_4) = y_1$$

and let M be the set $M = \{z \in \mathbb{R}^8, \ \varphi(z) = 0\}$. We also define the eight following vector fields

$$L_1 = \frac{\partial}{\partial x_1}, \qquad\qquad\qquad L_2 = \frac{\partial}{\partial y_1},$$

$$L_3 = \frac{\partial}{\partial x_2} - \frac{1}{2} y_3^2 \frac{\partial}{\partial x_1}, \qquad L_4 = \frac{\partial}{\partial y_2} + (-2y_3 x_3 + x_2) \frac{\partial}{\partial x_1},$$

$$L_5 = \frac{\partial}{\partial x_3} - y_2 y_3 \frac{\partial}{\partial x_1}, \qquad L_6 = \frac{\partial}{\partial y_3} + x_3 \frac{\partial}{\partial x_2} - \left(\frac{x_3 y_3^2}{2} + x_3 y_2 \right) \frac{\partial}{\partial x_1},$$

$$L_7 = \frac{\partial}{\partial x_4} + y_4 \frac{\partial}{\partial y_4} + y_1 y_4 L_5 \quad L_8 = \frac{\partial}{\partial y_4} + y_1 y_4 L_6.$$

and the complex structure J they induce by setting

$$JL_1 = L_2, \qquad JL_3 = L_4, \qquad JL_5 = L_6, \qquad JL_7 = L_8.$$

Again this example is derived from the example of Section 2. The tangent space TM is spanned over $\mathbb{R}$ by L_1, L_3, L_4, L_5, L_6, L_7 and L_8 and $T^J M$ is spanned over $\mathbb{C}$ by L_3, L_5 and L_7.

When we compute the Lie brackets of the L_i's in the complex tangent bundle, we get:

$$[L_3, L_4] = L_1,$$

$$[L_3, L_5] = 0, \qquad\qquad [L_4, L_5] = y_3 \frac{\partial}{\partial x_1},$$

$$[L_3, L_6] = y_3 \frac{\partial}{\partial x_1}, \qquad [L_4, L_6] = 0, \qquad\qquad [L_5, L_6] = L_3,$$

$$[L_3, L_7] = 0, \qquad\qquad [L_4, L_7] = y_1 y_3 y_4 \frac{\partial}{\partial x_1}, \quad [L_5, L_7] = 0,$$

$$[L_3, L_8] = y_1 y_3 y_4 \frac{\partial}{\partial x_1}, \quad [L_4, L_8] = 0, \qquad\qquad [L_5, L_8] = y_1 y_4 L_3,$$

$$[L_6, L_7] = y_1 y_4 L_3, \qquad [L_6, L_8] = 0$$

and

$$[L_7, L_8] = -y_1 L_5 + y_1^2 y_4^2 L_3 + 2y_1 y_4 L_6 - L_8.$$

Therefore

$$\hat{\mathcal{L}}_\varphi(L_3, L_5) = 0,$$
$$\hat{\mathcal{L}}_\varphi(L_3, L_7) = 0,$$
$$\hat{\mathcal{L}}_\varphi(L_5, L_7) = 0,$$
$$\hat{\mathcal{L}}_\varphi(L_3, JL_3) = d\varphi(L_2) = 1,$$
$$\hat{\mathcal{L}}_\varphi(L_5, JL_5) = d\varphi(L_3) = 0,$$
$$\hat{\mathcal{L}}_\varphi(L_7, JL_7) = d\varphi(y_1^2 y_4^2 L_4 - y_1 y_4 L_5 - y_1 L_6 - L_8) = 0,$$

and so, M is pseudoconvex and $\ker \mathcal{L}$ is spanned by L_5 and L_7. As in Section 2, there is no germ of J-holomorphic disc γ included in M such that $\gamma(0) = 0$ and $\frac{\partial \gamma}{\partial x}(0) = \frac{\partial}{\partial x_3}$. However, Theorem 1.6 can be applied to L_7 with $\mathbb{L}$ the subbundle generated by L_7. In order to do this, it suffices to check that for all $X_1, \ldots, X_k \in \{L_7, L_8\}$, the Lie bracket $[X_1, [\ldots, [X_k, [L_7, L_8] \ldots](0)$ is a linear combination of L_7 and L_8. By induction on k, we get

$$[X_1, [\ldots, [X_k, [L_7, L_8] \ldots] = y_1 L + \alpha L_8$$

where L is a vector field and α belongs to $\{-1, 0, 1\}$, both depending on the sequence $X_1, \ldots, X_k$. Therefore $[X_1, [\ldots, [X_k, [L_7, L_8] \ldots](0)$ belongs to the subbundle generated by L_7 at 0 and Theorem 1.6 yields the existence of a J-holomorphic disc γ included in M, such that $\gamma(0) = 0$ and $\frac{\partial \gamma}{\partial x}(0) = L_7(0)$.

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