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CHARACTERIZATION OF SEPARABILITY FOR *LF*-SPACES

by Giovanni VIDOSSICH

This note characterizes the separability of *LF*-spaces by five equivalent conditions, one being that all members of a given defining sequence (=*suite de définition* according to [2]) are separable. These conditions imply that the dual space must be hereditarily separable and Lindelöf for the weak* topology (= topology $\sigma(X', X)$ of [2]).

Concerning uniform spaces, we shall employ the terminology (and results) of the first two chapters of [4]. We shall denote by

$$\mathbf{F}$$

the scalar field, which is **R** or **C**; and we shall say

weak topology induced by $H \subseteq F(X, Y)$

the less fine topology on X making continuous all members of H (caution that this is a purely topological definition). Finally, an $\aleph_0$-space is — according to [5] — a regular space X where there exists a countable *pseudobase* $\mathfrak{P}$, i.e. a countable $\mathfrak{P} \subseteq \mathfrak{P}(X)$ such that for every compact $K \subseteq X$ and every open $U \subseteq X$ which contains K it follows $K \subseteq P \subseteq U$ for a suitable $P \in \mathfrak{P}$.

THEOREM. — *Let* X *be an* *LF-space and* $(E_n)_{n=1}^{\infty}$ *a defining sequence of* X. *The following statements are pairwise equivalent:*

(1) X *is separable.*

(2) X *is weakly separable.*

(3) *Every weakly* compact subset of* X' *is weakly* metrizable.*

(4) *Every equicontinuous subset of* X' *has a countable base for the weak* topology.*

(5) *Every* E_n *is separable.*

(6) X *is an* $\aleph_0$*-space.*

Proof. — $(1) \to (2)$: Clear.

$(2) \to (3)$: Let $e: X \to X''$ be the canonical map $x \longmapsto (f(x))_{f \in X'}$ and K a weakly* compact subset of X'. Then $e': x \longmapsto e(x)|_K$ is a continuous map from X equipped with the weak topology into the topological subspace $e'(X)$ of $F_p(K, \mathbf{F})$ (= product space of Card(K) copies of $\mathbf{F}$). By (2), $e'(X)$ is separable: let H be a countable dense subset of it. The weak* topology of K is exactly the weak topology induced by $e'(X) \subseteq F(K, \mathbf{F})$: by [3, p. 175, Footnote], this topology equals the weak topology induced by $H \subseteq F(K, \mathbf{F})$ and therefore it is metrizable.

$(3) \to (4)$: Because the weak* closure of an equicontinuous set is weakly* compact by [2, Th. 3].

$(4) \to (5)$: By [2, Cor. to Th. 3], there is a linear homeomorphism e from X onto a subspace $e(X)$ of $L_{\mathfrak{C}}(X', \mathbf{F})$, $\mathfrak{C}$ being a suitable cover of X' consisting of weakly* compact subsets of X' and $L_{\mathfrak{C}}(X', \mathbf{F})$ the space of weakly* continuous linear functionals on X' with the topology of uniform convergence on members of $\mathfrak{C}$. By [2, Th. 3], the members of $\mathfrak{C}$ are equicontinuous and hence weakly* metrizable by (4). By a well known theorem contained in [5, (J) and (D)], $C_u(K, \mathbf{F})$ (= uniform space made of all weakly* continuous maps $K \to \mathbf{F}$ and the uniformity of uniform convergence) is separable for all $K \in \mathfrak{C}$ and therefore the uniformity of $C_u(K, \mathbf{F})$ has a basis of *countable* uniform coverings (as follows easily, if you want, from [4, ii. 33 and ii. 9]). Consequently the uniformity π of $\prod_{K \in \mathfrak{C}} C_u(K, \mathbf{F})$ — and hence the trace of π on every subset — has a basis of *countable* uniform covers as it follows directly from the definition of product uniformity [4, Exercise ii. 2] (alternatively, this result may be deduced from [1, Prop. 3] and [4, ii. 33 and ii. 9]). It is well known that there is a uniform embedding $e^*: L_{\mathfrak{C}}(X', \mathbf{F}) \to \prod_{K \in \mathfrak{C}} C_u(K, \mathbf{F})$, the last space being equipped with the product uniformity π.

By what has been proved, the uniformity induced by π on $e^*(e(X))$ has a basis consisting of countable uniform coverings: consequently — because a linear homeomorphism is a uniform isomorphism for the (canonical) uniformities of linear topological spaces — the (canonical) uniformity of the linear topological space X has a basis of countable uniform covers, as well as its trace on every E_n. But this trace coincides with the (canonical) uniformity of the linear topological space $E_n (n \in \mathbf{Z}^+)$, hence it is metrizable and consequently separable (if $\{(U_{m,n})_{m=1}^{\infty} | n \in \mathbf{Z}^+\}$ is a countable base of countable uniform covers for the uniformity of E_n and if $x_{m,n}$ is an element of $U_{m,n}$ whenever this set is not empty, then $\{x_{m,n} | m, n \in \mathbf{Z}^+\}$ is dense in E_n).

(5) $\rightarrow$ (6): By [2, Prop. 4], every compact subset of X is contained in some E_n. This, together with the fact that X induces the original topology on each E_n, implies that $\bigcup_{n=1}^{\infty} \mathfrak{P}_n$ is a countable pseudobase for X whenever $\mathfrak{P}_n$ is for $E_n (n \in \mathbf{Z}^+)$.

(6) $\rightarrow$ (1): By [5, (D) and (E)]. ▌

We remark that the idea of countable uniform covers may be used to show directly that every metrizable subgroup of a separable topological group must be separable.

The above theorem points out some important examples of non-metrizable $\aleph_0$-spaces. [5, (J) and 10,4] imply some results on spaces of mappings between separable *LF*-spaces, of which we note only the following one.

Corollary. — *If an LF-space X is separable, then X' is weakly* hereditarily Lindelöf and separable.*

Proof. — By « (1) $\longleftrightarrow$ (6) » of the above theorem and [5, (J) and (D), (E)]. ▌

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