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ON QUANTITATIVE OPERATOR K-THEORY

by Hervé OYONO-OYONO & Guoliang YU (*)

Abstract. — In this paper, we develop a quantitative K-theory for filtered C^*-algebras. Particularly interesting examples of filtered C^*-algebras include group C^*-algebras, crossed product C^*-algebras and Roe algebras. We prove a quantitative version of the six term exact sequence and a quantitative Bott periodicity. We apply the quantitative K-theory to formulate a quantitative version of the Baum-Connes conjecture and prove that the quantitative Baum-Connes conjecture holds for a large class of groups.

Résumé. — Dans cet article, nous développons une K-théorie quantitative pour les C^*-algèbres filtrées. Parmi les exemples les plus intéressants de telles C^*-algèbres figurent les algèbres de Roe, les C^*-algèbres de groupes et les C^*-algèbres de produits croisés. Nous établissons une version quantitative de la suite exacte à six termes en K-théorie ainsi que de la périodicité de Bott. Nous formulons en utilisant la K-théorie quantitative une version quantitative de la conjecture de Baum-Connes. Nous montrons que cette conjecture de Baum-Connes quantitative est vérifiée pour une large classe de groupes.

Introduction

The receptacles of higher indices of elliptic differential operators are K-theory of C^*-algebras which encode the (large scale) geometry of the underlying spaces. The following examples are important for purpose of applications to geometry and topology.

- K-theory of group C^*-algebras is a receptacle for higher index theory of equivariant elliptic differential operators on covering spaces [1, 2, 5, 11];

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- K-theory of crossed product C^*-algebras and more generally groupoid C^*-algebras for foliations serve as receptacles for longitudinally elliptic operators [3, 4];
- the higher indices of elliptic operators on noncompact complete Riemannian manifolds live in K-theory of Roe algebras [15].

The local nature of differential operators implies that these higher indices can be defined in term of idempotents and invertible elements with finite propagation. Using homotopy invariance of the K-theory for C^*-algebras, these higher indices give rise to topological invariants.

In the context of Roe algebras, a quantitative operator K-theory was introduced to compute the higher indices of elliptic operators for noncompact spaces with finite asymptotic dimension [19]. The aim of this paper is to develop a quantitative K-theory for general C^*-algebras equipped with a filtration. The filtration structure allows us to define the concept of propagation. Examples of C^*-algebras with filtrations include group C^*-algebras, crossed product C^*-algebras and Roe algebras. The quantitative K-theory for C^*-algebras with filtrations is then defined in terms of homotopy classes of quasi-projections and quasi-unitaries with propagation and norm controls. We introduce controlled morphisms to study quantitative operator K-theory. In particular, we derive a quantitative version of the six term exact sequence. In the case of crossed product algebras, we also define a quantitative version of the Kasparov transformation compatible with Kasparov product. We end this paper by using the quantitative K-theory to formulate a quantitative version of the Baum-Connes conjecture and prove it for a large class of groups.

This paper is organized as follows: In section 1, we collect a few notations and definitions including the concept of filtered C^*-algebras. We use the concepts of almost unitary and almost projection to define a quantitative K-theory for filtered C^*-algebras and we study its elementary properties. In section 2, we introduce the notion of controlled morphism in quantitative K-theory. Section 3 is devoted to extensions of filtered C^*-algebras and to a controlled exact sequence for quantitative K-theory. In section 4, we prove a controlled version of the Bott periodicity and as a consequence, we obtain a controlled version of the six-term exact sequence in K-theory. In section 5, we apply KK-theory to study the quantitative K-theory of crossed product C^*-algebras and discuss its application to K-amenability. Finally in section 8, we formulate a quantitative Baum-Connes conjecture and prove the quantitative Baum-Connes conjecture for a large class of groups.

1. Quantitative K-theory

In this section, we introduce a notion of quantitative K-theory for C^*-algebras with a filtration. Let us fix first some notations about C^*-algebras we shall use throughout this paper.

- If B is a C^*-algebra and if $b_1,\dots,b_k$ are respectively elements of $M_{n_1}(B),\dots,M_{n_k}(B)$, we denote by $\operatorname{diag}(b_1,\dots,b_k)$ the block diagonal matrix $\begin{pmatrix} b_1 & & \\ & \ddots & \\ & & b_k \end{pmatrix}$ of $M_{n_1+\cdots+n_k}(B)$.
- If X is a locally compact space and B is a C^*-algebra, we denote by $C_0(X,B)$ the C^*-algebra of B-valued continuous functions on X vanishing at infinity. The special cases of $X=(0,1]$, $X=[0,1)$, $X=(0,1)$ and $X=[0,1]$, will be respectively denoted by CB, $B[0,1)$, SB and $B[0,1]$.
- For a separable Hilbert space $\mathcal{H}$, we denote by $\mathcal{K}(\mathcal{H})$ the C^*-algebra of compact operators on $\mathcal{H}$.
- If A and B are C^*-algebras, we will denote by $A\otimes B$ their spatial tensor product.

1.1. Filtered C^*-algebras

Definition 1.1. — *A filtered C^*-algebra A is a C^*-algebra equipped with a family $(A_r)_{r>0}$ of closed linear subspaces indexed by positive numbers such that:*

- $A_r\subset A_{r'}$ *if* $r\leqslant r'$;
- A_r *is stable by involution;*
- $A_r\cdot A_{r'}\subset A_{r+r'}$;
- *the subalgebra* $\bigcup_{r>0} A_r$ *is dense in* A.

If A is unital, we also require that the identity 1 is an element of A_r for every positive number r. The elements of A_r are said to have propagation r.

- Let A and A' be respectively C^*-algebras filtered by $(A_r)_{r>0}$ and $(A'_r)_{r>0}$. A homomorphism of C^* -algebras $\phi: A\longrightarrow A'$ is a filtered homomorphism (or a homomorphism of filtered C^*-algebras) if $\phi(A_r)\subset A'_r$ for any positive number r.
- If A is a filtered C^*-algebra and X is a locally compact space, then $C_0(X,A)$ is a C^*-algebra filtered by $(C_0(X,A_r))_{r>0}$. In particular the algebras CA, $A[0,1]$, $A[0,1)$ and SA are filtered C^*-algebras.

- If A is a non unital filtered C^*-algebra, then its unitarization $\widetilde{A}$ is filtered by $(A_r+\mathbb{C})_{r>0}$. We define for A non-unital the homomorphism
$$\rho_A : \widetilde{A} \to \mathbb{C};\ a+z \mapsto z$$
for $a \in A$ and $z \in \mathbb{C}$.

Prominent examples of filtered C^*-algebra are provided by Roe algebras associated to proper metric spaces, i.e. metric spaces such that closed balls of given radius are compact. Recall that for such a metric space (X,d), a X-module is a Hilbert space H_X together with a $*$-representation ρ_X of $C_0(X)$ in H_X (we shall write f instead of $\rho_X(f)$). If the representation is non-degenerate, the X-module is said to be non-degenerate. A X-module is called standard if no non-zero function of $C_0(X)$ acts as a compact operator on H_X.

The following concepts were introduced by Roe in his work on index theory of elliptic operators on noncompact spaces [15].

Definition 1.2. — *Let H_X be a standard non-degenerate X-module and let T be a bounded operator on H_X.*

(i) The support of T is the complement of the open subset of $X \times X$
$$\{(x,y) \in X \times X \text{ s.t. there exist } f \text{ and } g \text{ in } C_0(X) \text{ satisfying } f(x) \neq 0,\ g(y) \neq 0 \text{ and } f\cdot T\cdot g = 0\}.$$
(ii) The operator T is said to have finite propagation (in this case propagation less than r) if there exists a real r such that for any x and y in X with $d(x,y) > r$, then (x,y) is not in the support of T.
(iii) The operator T is said to be locally compact if $f\cdot T$ and $T\cdot f$ are compact for any f in $C_0(X)$. We then define $C[X]$ as the set of locally compact and finite propagation bounded operators of H_X, and for every $r>0$, we define $C[X]_r$ as the set of elements of $C[X]$ with propagation less than r.

We clearly have $C[X]_r \cdot C[X]_{r'} \subset C[X]_{r+r'}$. We can check that up to (non-canonical) isomorphism, $C[X]$ does not depend on the choice of H_X.

Definition 1.3. — *The Roe algebra $C^*(X)$ is the norm closure of $C[X]$ in the algebra $\mathcal{L}(H_X)$ of bounded operators on H_X. The Roe algebra in then filtered by $(C[X]_r)_{r>0}$.*

Although $C^*(X)$ is not canonically defined, it was proved in [9] that up to canonical isomorphisms, its K-theory does not depend on the choice of a non-degenerate standard X-module. Furthermore, $K_*(C^*(X))$ is the natural receptacle for higher indices of elliptic operators with support on X [15].

If X has bounded geometry, then the Roe algebra admits a maximal version [7] filtered by $(C[X]_r)_{r>0}$. Other important examples are reduced and maximal crossed product of a C^*-algebra by an action of a discrete group by automorphisms. These examples will be studied in detail in Section 5.

1.2. Almost projections/unitaries

Let A be a unital filtered C^*-algebra. For any positive numbers r and ε, we call

- an element u in A an ε-r-unitary if u belongs to A_r, $\|u^*\cdot u-1\|<\varepsilon$ and $\|u\cdot u^*-1\|<\varepsilon$. The set of ε-r-unitaries on A will be denoted by $\mathrm{U}^{\varepsilon,r}(A)$.
- an element p in A an ε-r-projection if p belongs to A_r, $p=p^*$ and $\|p^2-p\|<\varepsilon$. The set of ε-r-projections on A will be denoted by $\mathrm{P}^{\varepsilon,r}(A)$.

For n integer, we set $\mathrm{U}_n^{\varepsilon,r}(A)=\mathrm{U}^{\varepsilon,r}(M_n(A))$ and $\mathrm{P}_n^{\varepsilon,r}(A)=\mathrm{P}^{\varepsilon,r}(M_n(A))$.

For any unital filtered C^*-algebra A, any positive numbers ε and r and any positive integer n, we consider inclusions

$$\mathrm{P}_n^{\varepsilon,r}(A)\hookrightarrow \mathrm{P}_{n+1}^{\varepsilon,r}(A);\, p\mapsto \begin{pmatrix} p & 0\\ 0 & 0\end{pmatrix}$$

and

$$\mathrm{U}_n^{\varepsilon,r}(A)\hookrightarrow \mathrm{U}_{n+1}^{\varepsilon,r}(A);\, u\mapsto \begin{pmatrix} u & 0\\ 0 & 1\end{pmatrix}.$$

This allows us to define

$$\mathrm{U}_\infty^{\varepsilon,r}(A)=\bigcup_{n\in\mathbb{N}}\mathrm{U}_n^{\varepsilon,r}(A)$$

and

$$\mathrm{P}_\infty^{\varepsilon,r}(A)=\bigcup_{n\in\mathbb{N}}\mathrm{P}_n^{\varepsilon,r}(A).$$

Remark 1.4. — Let r and ε be positive numbers with $\varepsilon<1/4$;

(i) If p is an ε-r-projection in A, then the spectrum of p is included in $\left(\frac{1-\sqrt{1+4\varepsilon}}{2},\frac{1-\sqrt{1-4\varepsilon}}{2}\right)\cup\left(\frac{1+\sqrt{1-4\varepsilon}}{2},\frac{1+\sqrt{1+4\varepsilon}}{2}\right)$ and thus $\|p\|<1+\varepsilon$.

(ii) If u is an ε-r-unitary in A, then

$$1-\varepsilon < \|u\| < 1+\varepsilon/2,$$

$$1-\varepsilon/2 < \|u^{-1}\| < 1+\varepsilon,$$

$$\|u^* - u^{-1}\| < (1+\varepsilon)\varepsilon.$$

(iii) Let $\kappa_{0,\varepsilon} : \mathbb{R} \to \mathbb{R}$ be a continuous function such that
- $\kappa_{0,\varepsilon}(t) = 0$ if $t \leqslant \frac{1-\sqrt{1-4\varepsilon}}{2}$;
- $\kappa_{0,\varepsilon}(t) = 1$ if $t \geqslant \frac{1+\sqrt{1-4\varepsilon}}{2}$.

If p is an ε-r-projection in A, then $\kappa_{0,\varepsilon}(p)$ is a projection such that $\|p - \kappa_{0,\varepsilon}(p)\| < 2\varepsilon$ which moreover does not depends on the choice of $\kappa_{0,\varepsilon}$. From now on, we shall denote this projection by $\kappa_0(p)$.

(iv) If u is an ε-r-unitary in A, set $\kappa_1(u) = u(u^*u)^{-1/2}$. Then $\kappa_1(u)$ is a unitary such that $\|u - \kappa_1(u)\| < \varepsilon$.

(v) If p is an ε-r-projection in A and q is a projection in A such that $\|p - q\| < 1 - 2\varepsilon$, then $\kappa_0(p)$ and q are homotopic projections [18, Chapter 5].

(vi) If u and v are ε-r-unitaries in A, then uv is an $\varepsilon(2+\varepsilon)$-$2r$-unitary in A.

Definition 1.5. — *Let A be a C^*-algebra filtered by $(A_r)_{r>0}$.*

- *Let p_0 and p_1 be ε-r-projections. We say that p_0 and p_1 are homotopic ε-r-projections if there exists an ε-r-projection q in $A[0,1]$ such that $q(0) = p_0$ and $q(1) = p_1$. In this case, q is called a homotopy of ε-r-projections in A and will be denoted by $(q_t)_{t\in[0,1]}$.*
- *If A is unital, let u_0 and u_1 be ε-r-unitaries. We say that u_0 and u_1 are homotopic ε-r-unitaries if there exists an ε-r-unitary v in $A[0,1]$ such that $v(0) = u_0$ and $v(1) = u_1$. In this case, v is called a homotopy of ε-r-unitaries in A and will be denoted by $(v_t)_{t\in[0,1]}$.*

Example 1.6. — Let p be an ε-r-projection in a unital filtered C^*-algebra A. Set $c_t = \cos \pi t/2$ and $s_t = \sin \pi t/2$ for $t \in [0,1]$ and let us considerer the homotopy of projections $(h_t)_{t\in[0,1]}$ with $h_t = \begin{pmatrix} c_t^2 & c_t s_t \\ c_t s_t & s_t^2 \end{pmatrix}$ in $M_2(\mathbb{C})$ between $\operatorname{diag}(1,0)$ and $\operatorname{diag}(0,1)$. Set $(q_t)_{t\in[0,1]} = (\operatorname{diag}(p,0) + (1-p) \otimes h_t)_{t\in[0,1]}$. Since $q_t^2 - q_t = s_t^2(p^2 - p) \otimes I_2$, we see that $(q_t)_{t\in[0,1]}$ is a homotopy of ε-r-projections between $\operatorname{diag}(1,0)$ and $\operatorname{diag}(p, 1-p)$ in $M_2(A)$.

Next result will be used quite extensively throughout the paper and is fairly easy to prove.

LEMMA 1.7. — *Let A be a C^*-algebra filtered by $(A_r)_{r>0}$.*

(i) If p is an ε-r-projection in A and q is a self-adjoint element of A_r such that $\|p-q\| < \frac{\varepsilon - \|p^2-p\|}{4}$, then q is an ε-r-projection. In particular, if p is an ε-r-projection in A and if q is a self-adjoint element in A_r such that $\|p-q\|<\varepsilon$, then q is a 5ε-r-projection in A and p and q are connected by a homotopy of 5ε-r-projections.

(ii) If A is unital and if u is an ε-r-unitary and v is an element of A_r such that $\|u-v\| < \frac{\varepsilon - \|u^*u-1\|}{3}$, then v is an ε-r-unitary. In particular, if u is an ε-r-unitary and v is an element of A_r such that $\|u-v\|<\varepsilon$, then v is an 4ε-r-unitary in A and u and v are connected by a homotopy of 4ε-r-unitaries.

(iii) If p is a projection in A and q is a self-adjoint element of A_r such that $\|p-q\|<\frac{\varepsilon}{4}$, then q is an ε-r-projection.

(iv) If A is unital and if u is a unitary in A and v is an element of A_r such that $\|u-v\|<\frac{\varepsilon}{3}$, then v is an ε-r-unitary.

COROLLARY 1.8. — *Let u be an ε-r-unitary in a unital filtered C^*-algebra A, then $\operatorname{diag}(u,u^*)$ and I_2 are homotopic as 3ε-$2r$-unitaries in $M_2(A)$.*

Proof. — According to point (vi) of Remark 1.4 and with notations of Example 1.6, we see that $\left(\operatorname{diag}(1,u)\begin{pmatrix} c_t & -s_t \\ s_t & c_t\end{pmatrix}\cdot\operatorname{diag}(1,u^*)\cdot\begin{pmatrix} c_t & s_t \\ -s_t & c_t\end{pmatrix}\right)_{t\in[0,1]}$ is a homotopy of 3ε-$2r$-unitaries between $\operatorname{diag}(u,u^*)$ and $\operatorname{diag}(uu^*,1)$. Since $\|uu^*-1\|<\varepsilon$, we deduce from Lemma 1.7 that uu^* and 1 are homotopic 3ε-$2r$-unitaries. □

LEMMA 1.9. — *There exists a number $\lambda>4$ such that for any positive number ε with $\varepsilon<1/\lambda$, any positive real r, any ε-r-projection p and ε-r-unitary W in a filtered unital C^*-algebra A, the following assertions hold:*

(i) WpW^* is a $\lambda\varepsilon$-$3r$-projection of A;

(ii) $\operatorname{diag}(WpW^*,1)$ and $\operatorname{diag}(p,1)$ are homotopic $\lambda\varepsilon$-$3r$-projections.

Proof. — The first point is straightforward to check from Remark 1.4. For the second point, with notations of Example 1.6, use the homotopy of ε-r-unitaries

$$\begin{pmatrix} Wc_t^2+s_t^2 & (W-1)s_tc_t \\ (W-1)s_tc_t & Ws_t^2+c_t^2\end{pmatrix}_{t\in[0,1]} = \left(\begin{pmatrix} c_t & -s_t \\ s_t & c_t\end{pmatrix}\cdot\operatorname{diag}(W,1)\cdot\begin{pmatrix} c_t & s_t \\ -s_t & c_t\end{pmatrix}\right)_{t\in[0,1]}$$

to connect by conjugation $\operatorname{diag}(WpW^*,1)$ to $\operatorname{diag}(p,WW^*)$ and then connect to $\operatorname{diag}(p,1)$ by a ray. □

Recall that if two projections in a unital C^*-algebra are close enough in norm, then there are conjugated by a canonical unitary. To state a similar result in term of ε-r-projections and ε-r-unitaries, we will need the definition of a control pair.

Definition 1.10. — *A control pair is a pair (λ, h), where*

- $\lambda > 1$;
- $h : (0, \frac{1}{4\lambda}) \to (1, +\infty)$; $\varepsilon \mapsto h_\varepsilon$ *is a map such that there exists a non-increasing map* $g : (0, \frac{1}{4\lambda}) \to (0, +\infty)$, *with* $h \leqslant g$.

Lemma 1.11. — *There exists a control pair (λ, h) such that the following holds:*

for every positive number r, any ε in $(0, \frac{1}{4\lambda})$ and any ε-r-projections p and q of a filtered unital C^*-algebra A satisfying $\|p-q\| < 1/16$, there exists an $\lambda\varepsilon$-$h_\varepsilon r$-unitary W in A such that $\|WpW^* - q\| \leqslant \lambda\varepsilon$.

Proof. — We follow the proof of [18, Proposition 5.2.6]. If we set

$$z = (2\kappa_0(p) - 1)(2\kappa_0(q) - 1) + 1,$$

- then
$$\begin{aligned} \|z-2\| &\leqslant 2\|\kappa_0(p) - \kappa_0(q)\| \\ &\leqslant 8\varepsilon + 2\|p-q\| \end{aligned}$$
and hence z is invertible for $\varepsilon < 1/16$.
- Moreover, if we set $U = z|z^{-1}|$ and since $z\kappa_0(q) = \kappa_0(p)z$, then we have $\kappa_0(q) = U\kappa_0(p)U^*$.

Let us define $z' = (2p-1)(2q-1)+1$. Then we have $\|z - z'\| \leqslant 9\varepsilon$ and $\|z'\| \leqslant 3$. If ε is small enough, then $\|z'^*z' - 4\| \leqslant 2$ and hence the spectrum of z'^*z' is in $[2, 6]$. Let us consider the expansion in power serie $\sum_{k\in\mathbb{N}} a_k t^k$ of $t \mapsto (1+t)^{-1/2}$ on $(0,1)$ and let n_ε be the smallest integer such that $\sum_{n_\varepsilon \leqslant k} |a_k|/2^k \leqslant \varepsilon$. Let us set then $W = z'/2\sum_{k=0}^{n_\varepsilon} a_k(\frac{z'^*z'-4}{4})^k$. Then for a suitable λ (not depending on A, p, q or ε), we get that W is a $\lambda\varepsilon$-$(4n_\varepsilon + 2)r$-unitary which satisfies the required condition. □

Remark 1.12. — The order of h when ε goes to zero in Lemma 1.11 is $C\varepsilon^{-3/2}$ for some constant C.

1.3. Definition of quantitative K-theory

For a unital filtered C^*-algebra A, we define the following equivalence relations on $\mathrm{P}_\infty^{\varepsilon,r}(A) \times \mathbb{N}$ and on $\mathrm{U}_\infty^{\varepsilon,r}(A)$:

- if p and q are elements of $\mathrm{P}_\infty^{\varepsilon,r}(A)$, l and l' are positive integers, $(p,l)\sim(q,l')$ if there exists a positive integer k and an element h of $\mathrm{P}_\infty^{\varepsilon,r}(A[0,1])$ such that $h(0)=\operatorname{diag}(p,I_{k+l'})$ and $h(1)=\operatorname{diag}(q,I_{k+l})$.
- if u and v are elements of $\mathrm{U}_\infty^{\varepsilon,r}(A)$, $u\sim v$ if there exists an element h of $\mathrm{U}_\infty^{3\varepsilon,2r}(A[0,1])$ such that $h(0)=u$ and $h(1)=v$.

If p is an element of $\mathrm{P}_\infty^{\varepsilon,r}(A)$ and l is an integer, we denote by $[p,l]_{\varepsilon,r}$ the equivalence class of (p,l) modulo $\sim$ and if u is an element of $\mathrm{U}_\infty^{\varepsilon,r}(A)$ we denote by $[u]_{\varepsilon,r}$ its equivalence class modulo $\sim$.

Definition 1.13. — *Let r and ε be positive numbers with $\varepsilon<1/4$. We define:*

(i) $K_0^{\varepsilon,r}(A)=\mathrm{P}_\infty^{\varepsilon,r}(A)\times\mathbb{N}/\sim$ *for A unital and*

$$K_0^{\varepsilon,r}(A)=\{[p,l]_{\varepsilon,r}\in\mathrm{P}^{\varepsilon,r}(\tilde{A})\times\mathbb{N}/\sim\ \text{such that}\ \dim\kappa_0(\rho_A(p))=l\}$$

for A non unital.

(ii) $K_1^{\varepsilon,r}(A)=\mathrm{U}_\infty^{\varepsilon,r}(\tilde{A})/\sim$ *(with $A=\tilde{A}$ if A is already unital).*

Remark 1.14. — We shall see in Lemma 1.23 that as it is the case for K-theory, $K_*^{\varepsilon,r}(\bullet)$ can indeed be defined in a uniform way for unital and non-unital filtered C^*-algebras.

It is straightforward to check that for any unital filtered C^*-algebra A, if p is an ε-r-projection in A and u is an ε-r-unitary in A, then $\operatorname{diag}(p,0)$ and $\operatorname{diag}(0,p)$ are homotopic ε-r-projections in $M_2(A)$ and $\operatorname{diag}(u,1)$ and $\operatorname{diag}(1,u)$ are homotopic ε-r-unitaries in $M_2(A)$. Thus we obtain the following:

Lemma 1.15. — *Let A be a filtered C^*-algebra. Then $K_0^{\varepsilon,r}(A)$ and $K_1^{\varepsilon,r}(A)$ are equipped with a structure of abelian semi-group such that*

$$[p,l]_{\varepsilon,r}+[p',l']_{\varepsilon,r}=[\operatorname{diag}(p,p'),l+l']_{\varepsilon,r}$$

and

$$[u]_{\varepsilon,r}+[u']_{\varepsilon,r}=[\operatorname{diag}(u,v)]_{\varepsilon,r},$$

for any $[p,l]_{\varepsilon,r}$ and $[p',l']_{\varepsilon,r}$ in $K_0^{\varepsilon,r}(A)$ and any $[u]_{\varepsilon,r}$ and $[u']_{\varepsilon,r}$ in $K_1^{\varepsilon,r}(A)$.

According to Example 1.6, for every unital filtered C^*-algebra A, any ε-r-projection p in $M_n(A)$ and any integer l with $n\geqslant l$, we see that $[I_n-p,n-l]_{\varepsilon,r}$ is an inverse for $[p,l]_{\varepsilon,r}$. In the same way, using Corollary 1.8, we get that for any ε-r-unitary u in $M_n(A)$, then $[\operatorname{diag}(u,u^*)]_{\varepsilon,r}=[1]_{\varepsilon,r}$. Hence we get:

Lemma 1.16. — *If A is a filtered C^*-algebra, then $K_*^{\varepsilon,r}(A) = K_0^{\varepsilon,r}(A) \oplus K_1^{\varepsilon,r}(A)$ is a $\mathbb{Z}_2$-graded abelian group.*

We have for any filtered C^*-algebra A and any positive numbers r, r', ε and ε' with $\varepsilon \leqslant \varepsilon' < 1/4$ and $r \leqslant r'$ natural group homomorphisms

- $\iota_0^{\varepsilon,r} : K_0^{\varepsilon,r}(A) \longrightarrow K_0(A)$; $[p,l]_{\varepsilon,r} \mapsto [\kappa_0(p)] - [I_l]$;
- $\iota_1^{\varepsilon,r} : K_1^{\varepsilon,r}(A) \longrightarrow K_1(A)$; $[u]_{\varepsilon,r} \mapsto [u]$;
- $\iota_*^{\varepsilon,r} = \iota_0^{\varepsilon,r} \oplus \iota_1^{\varepsilon,r}$;
- $\iota_0^{\varepsilon,\varepsilon',r,r'} : K_0^{\varepsilon,r}(A) \longrightarrow K_0^{\varepsilon',r'}(A)$; $[p,l]_{\varepsilon,r} \mapsto [p,l]_{\varepsilon',r'}$;
- $\iota_1^{\varepsilon,\varepsilon',r,r'} : K_1^{\varepsilon,r}(A) \longrightarrow K_1^{\varepsilon',r'}(A)$; $[u]_{\varepsilon,r} \mapsto [u]_{\varepsilon',r'}$.
- $\iota_*^{\varepsilon,\varepsilon',r,r'} = \iota_1^{\varepsilon,\varepsilon',r,r'} \oplus \iota_1^{\varepsilon,\varepsilon',r,r'}$

If some of the indices r, r' or $\varepsilon, \varepsilon'$ are equal, we shall not repeat it in $\iota_*^{\varepsilon,\varepsilon',r,r'}$.

Remark 1.17. — Let p_0 and p_1 be two ε-r-projections in a filtered C^*-algebra such that $\kappa_0(p_0)$ and $\kappa_0(p_1)$ are homotopic projections. Then for any ε in $(0,1/4)$, this homotopy can be approximated for some r' by a ε-r'-projection. Hence, using point (iii) of Remark 1.4, there exists a homotopy $(q_t)_{t\in[0,1]}$ of ε-r' projections in A such that $\|p_0 - q_0\| < 3\varepsilon$ and $\|p_1 - q_1\| < 3\varepsilon$. We can indeed assume that $r' \geqslant r$ and thus by Lemma 1.7, we get that p_0 and p_1 are homotopic as 15ε-r'-projections. Proceeding in the same way for the odd case we eventually obtain:

there exists $\lambda > 1$ such that for any filtered C^*-algebra A, any $\varepsilon \in (0, \frac{1}{4\lambda})$ and any positive number r, the following holds:

Let x and x' be elements in $K_*^{\varepsilon,r}(A)$ such that $\iota_*^{\varepsilon,r}(x) = \iota_*^{\varepsilon,r}(x')$ in $K_*(A)$, then there exists a positive number r' with $r' > r$ such that $\iota_*^{\varepsilon,\lambda\varepsilon,r,r'}(x) = \iota_*^{\varepsilon,\lambda\varepsilon,r,r'}(x')$ in $K_*^{\lambda\varepsilon,r'}(A)$.

Lemma 1.18. — *Let p be a matrix in $M_n(\mathbb{C})$ such that $p = p^*$ and $\|p^2 - p\| < \varepsilon$ for some ε in $(0,1/4)$. Then there is a continuous path $(p_t)_{t\in[0,1]}$ in $M_n(\mathbb{C})$ such that*

- *$p_0 = p$;*
- *$p_1 = I_k$ with $k = \dim \kappa_0(p)$;*
- *$p_t^* = p_t$ and $\|p_t^2 - p_t\| < \varepsilon$ for every t in $[0,1]$.*

Proof. — The selfadjoint matrix p satisfies $\|p^2 - p\| < \varepsilon$ if and only if the eigenvalues of p satisfy the inequality

$$-\varepsilon < \lambda^2 - \lambda < \varepsilon,$$

i.e.

$$\lambda \in \left(\frac{1-\sqrt{1+4\varepsilon}}{2}, \frac{1-\sqrt{1-4\varepsilon}}{2}\right) \bigcup \left(\frac{\sqrt{1-4\varepsilon}+1}{2}, \frac{\sqrt{1+4\varepsilon}+1}{2}\right).$$

Let $\lambda_1,\ldots,\lambda_k$ be the eigenvalues of p lying in $\left(\frac{1-\sqrt{1+4\varepsilon}}{2},\frac{1-\sqrt{1-4\varepsilon}}{2}\right)$ and let $\lambda_{k+1},\ldots,\lambda_n$ be the eigenvalues of p lying in $\left(\frac{\sqrt{1-4\varepsilon}+1}{2},\frac{\sqrt{1+4\varepsilon}+1}{2}\right)$. We set for $t\in[0,1]$

- $\lambda_{i,t}=t\lambda_i$ for $i=1,\ldots,k$;
- $\lambda_{i,t}=t\lambda_i+1-t$ for $i=k+1,\ldots,n$.

Since $\lambda\mapsto\lambda^2-\lambda$ is decreasing on $\left(\frac{1-\sqrt{1+4\varepsilon}}{2},\frac{1-\sqrt{1-4\varepsilon}}{2}\right)$ and increasing on $\left(\frac{\sqrt{1-4\varepsilon}+1}{2},\frac{\sqrt{1+4\varepsilon}+1}{2}\right)$ then,

$$-\varepsilon<\lambda_{i,t}^2-\lambda_{i,t}<\varepsilon$$

for all t in $[0,1]$ and $i=1,\ldots,n$. If we set $p_t=u\cdot\operatorname{diag}(\lambda_{1,t},\ldots,\lambda_{n,t})\cdot u^*$ where u is a unitary matrix of $M_n(\mathbb{C})$ such that $p=u\cdot\operatorname{diag}(\lambda_1,\ldots,\lambda_n)\cdot u^*$, then

- $p_0=p$;
- $p_1=\kappa_0(p)$;
- $p_t^*=p_t$ and $\|p_t^2-p_t\|<\varepsilon$ for every t in $[0,1]$.

Since there is a homotopy of projections in $M_n(\mathbb{C})$ between $\kappa_0(p)$ and I_k with $k=\dim\kappa_0(p)$, we get the result. □

Let us equip $\mathbb{C}$ with the trivial filtration (i.e $\mathbb{C}_r=\mathbb{C}$ for every positive number r). As a consequence of the previous lemma, we obtain:

Corollary 1.19. — *For any positive numbers with $\varepsilon<1/4$, then*

$$K_0^{\varepsilon,r}(\mathbb{C})\to\mathbb{Z};\ [p,l]_{\varepsilon,r}\mapsto\dim\kappa_0(p)-l$$

is an isomorphism.

Lemma 1.20. — *Let u be a matrix in $M_n(\mathbb{C})$ such that $\|u^*u-I_n\|<\varepsilon$ and $\|uu^*-I_n\|<\varepsilon$ for ε in $(0,1/4)$. Then there is a continuous path $(u_t)_{t\in[0,1]}$ in $M_n(\mathbb{C})$ such that*

- *$u_0=u$;*
- *$u_1=I_n$;*
- *$\|u_t^*u_t-I_n\|<\varepsilon$ and $\|u_tu_t^*-I_n\|<\varepsilon$ for every t in $[0,1]$.*

Proof. — Since u is invertible, u^*u and uu^* have the same eigenvalues $\lambda_1,\ldots,\lambda_n$, and thus $\|u^*u-I_n\|<\varepsilon$ and $\|uu^*-I_n\|<\varepsilon$ if and only if $\lambda_i\in(1-\varepsilon,1+\varepsilon)$ for $i=1,\ldots,n$. Let us set

- $h_t=w\cdot\operatorname{diag}(\lambda_1^{-t/2},\ldots,\lambda_n^{-t/2})\cdot w^*$ where w is a unitary matrix of $M_n(\mathbb{C})$ such that $u^*u=w\cdot\operatorname{diag}(\lambda_1,\ldots,\lambda_n)\cdot w^*$;
- $v_t=u\cdot h_t$ for all $t\in[0,1]$. Then $v_t^*v_t=w\cdot\operatorname{diag}(\lambda_1^{1-t},\ldots,\lambda_n^{1-t})\cdot w^*$.

Since $|\lambda_i^{1-t}-1|<\varepsilon$ for all all $t\in[0,1]$, we get that $\|v_t^*v_t-I_n\|<\varepsilon$ and $\|v_tv_t^*-I_n\|<\varepsilon$ for every t in $[0,1]$. The matrix v_1 is unitary and the result then follows from path-connectness of $\mathrm{U}_n(\mathbb{C})$. □

As a consequence we obtain:

Corollary 1.21. — *For any positive numbers r and ε with $\varepsilon<1/4$, then we have $K_1^{\varepsilon,r}(\mathbb{C})=\{0\}$.*

1.4. Elementary properties of quantitative K-theory

Let A_1 and A_2 be two unital C^*-algebras respectively filtered by $(A_{1,r})_{r>0}$ and $(A_{2,r})_{r>0}$ and consider $A_1\oplus A_2$ filtered by $(A_{1,r}\oplus A_{2,r})_{r>0}$. Since we have identifications $\mathrm{P}_\infty^{\varepsilon,r}(A_1\oplus A_2)\cong \mathrm{P}_\infty^{\varepsilon,r}(A_1)\times \mathrm{P}_\infty^{\varepsilon,r}(A_2)$ and $\mathrm{U}_\infty^{\varepsilon,r}(A_1\oplus A_2)\cong \mathrm{U}_\infty^{\varepsilon,r}(A_1)\times \mathrm{U}_\infty^{\varepsilon,r}(A_2)$ induced by the inclusions $A_1\hookrightarrow A_1\oplus A_2$ and $A_2\hookrightarrow A_1\oplus A_2$, we see that we have isomorphisms $K_0^{\varepsilon,r}(A_1)\oplus K_0^{\varepsilon,r}(A_2)\stackrel{\sim}{\longrightarrow} K_0^{\varepsilon,r}(A_1\oplus A_2)$ and $K_1^{\varepsilon,r}(A_1)\oplus K_1^{\varepsilon,r}(A_2)\stackrel{\sim}{\longrightarrow} K_1^{\varepsilon,r}(A_1\oplus A_2)$.

Lemma 1.22. — *Let A be a filtered non unital C^*-algebra and let ε and r be positive numbers with $\varepsilon<1/4$. We have a natural splitting*

$$K_0^{\varepsilon,r}(\tilde{A})\stackrel{\cong}{\longrightarrow} K_0^{\varepsilon,r}(A)\oplus\mathbb{Z}.$$

Proof. — Viewing A as a subalgebra of $\tilde{A}$, the group homomorphisms

$$\begin{array}{ccc} K_0^{\varepsilon,r}(\tilde{A}) & \longrightarrow & K_0^{\varepsilon,r}(A)\oplus\mathbb{Z}\\ {[p,l]}_{\varepsilon,r} & \mapsto & ([p,\dim\kappa_0(\rho_A(p))]_{\varepsilon,r},\dim\kappa_0(\rho_A(p))-l)\end{array}$$

and

$$\begin{array}{ccc} K_0^{\varepsilon,r}(A)\oplus\mathbb{Z} & \longrightarrow & K_0^{\varepsilon,r}(\tilde{A})\\ ([p,l]_{\varepsilon,r},k-k') & \mapsto & \left[\begin{pmatrix} p & 0\\ 0 & I_k\end{pmatrix},l+k'\right]_{\varepsilon,r}\end{array}$$

are inverse one of the other. □

Let us set $A^+=A\oplus\mathbb{C}$ equipped with the multiplication

$$(a,x)\cdot(b,y)=(ab+xb+ya,xy)$$

for a and b in A and x and y in $\mathbb{C}$. Notice that

- A^+ is isomorphic to $A\oplus\mathbb{C}$ with the algebra structure provided by the direct sum if A is unital;
- $A^+=\tilde{A}$ if A is not unital.

Let us define also ρ_A in the unital case by $\rho_A : A^+ \to \mathbb{C}; (a,x) \mapsto x$. We know that in usual K-theory, we can equivalently define for A unital the $\mathbb{Z}_2$-graded group $K_*(A)$ as A^+ by

$$K_0(A) = \ker \rho_{A,*} : K_0(A^+) \to K_0(\mathbb{C}) \cong \mathbb{Z}$$

and

$$K_1(A) = K_1(A^+).$$

Let us check that this is also the case for our $\mathbb{Z}_2$-graded groups ${K_*}^{\varepsilon,r}(A)$. If the C^*-algebra A is filtered by $(A_r)_{r>0}$, then A^+ is filtered by $(A_r+\mathbb{C})_{r>0}$. Let us define for a unital filtered algebra A

$$K'^{\,\varepsilon,r}_0(A) = \{[p,l]_{\varepsilon,r} \in \mathrm{P}^{\varepsilon,r}(A^+)\times\mathbb{N}/\sim \ \text{such that}\ \dim \kappa_0(\rho_A(p)) = l\}$$

and

$$K'^{\,\varepsilon,r}_1(A) = \mathrm{U}^{\varepsilon,r}(A^+)/\sim.$$

Proceeding as we did in the proof of Lemma 1.22, we obtain a natural splitting

$$K_0^{\varepsilon,r}(A^+) \stackrel{\cong}{\longrightarrow} K'^{\,\varepsilon,r}_0(A)\oplus\mathbb{Z}.$$

But then, using the identification $A^+\cong A\oplus\mathbb{C}$ and in view of Lemmas 1.18 and 1.20, we get

Lemma 1.23. — *The $\mathbb{Z}_2$-graded groups $K_*^{\varepsilon,r}(A)$ and $K'^{\,\varepsilon,r}_*(A)$ are naturally isomorphic.*

This allows us to state functoriallity properties for quantitative K-theory. If $\phi : A \to B$ is a homomorphism of unital filtered C^*-algebras, then since ϕ preserve ε-r-projections and ε-r-unitaries, it obviously induces for any positive number r and any $\varepsilon \in (0,1/4)$ a group homomorphism

$$\phi_*^{\varepsilon,r} : K_*^{\varepsilon,r}(A) \longrightarrow K_*^{\varepsilon,r}(B).$$

In the non unital case, we can extend any homomorphism $\phi : A\to B$ to a homomorphism $\phi^+ : A^+\to B^+$ of unital filtered C^*-algebras and then we use Lemmas 1.22 and 1.23 to define $\phi_*^{\varepsilon,r} : K_*^{\varepsilon,r}(A) \longrightarrow K_*^{\varepsilon,r}(B)$. Hence, for any positive number r and any $\varepsilon\in(0,1/4)$, we get that $K_*^{\varepsilon,r}(\bullet)$ is a covariant additive functor from the category of filtered C^*-algebras (together with filtered homomorphisms) to the category of $\mathbb{Z}_2$-abelian groups.

Definition 1.24.

(i) *Let A and B be filtered C^*-algebras. Then two homomorphisms of filtered C^*-algebras $\psi_0 : A\to B$ and $\psi_1 : A \to B$ are homotopic if there exists a path of homomorphisms of filtered C^*-algebras $\psi_t : A\to B$ for $0\leqslant t\leqslant 1$ between ψ_0 and ψ_1 and such that $t\mapsto\psi_t$ is continuous for the pointwise norm convergence.*

(ii) A filtered C^*-algebra A is said to be contractible if the identity map and the zero map of A are homotopic.

Example 1.25. — If A is a filtered C^*-algebra A, then the cone of A

$$CA = \{f \in C([0,1], A) \text{ such that } f(0) = 0\}$$

is a contractible filtered C^*-algebra.

We have then the following obvious result:

Lemma 1.26. — *If $\phi : A \to B$ and $\phi' : A \to B$ are two homotopic homomorphisms of filtered C^*-algebras, then $\phi_*^{\varepsilon,r} = {\phi'}_*^{\varepsilon,r}$ for every positive numbers ε and r with $\varepsilon < 1/4$. In particular, if A is a contractible filtered C^*-algebra, then $K_*^{\varepsilon,r}(A) = \{0\}$ for every positive numbers ε and r with $\varepsilon < 1/4$.*

Let A be a C^*-algebra filtered by $(A_r)_{r>0}$ and let $(B_k)_{k\in\mathbb{N}}$ be an increasing sequence of C^*-subalgebras of A such that $\bigcup_{k\in\mathbb{N}} B_k$ is dense in A. Assume that $\bigcup_{r>0} B_k \cap A_r$ is dense in B_k for every integer k. Then for every integer k, the C^*-algebra B_k is filtered by $(B_k \cap A_r)_{r>0}$. If A is unital, then B_k is unital for some k, and thus we will assume without loss of generality that B_k is unital for every integer k.

Proposition 1.27. — *Let A be a unital C^*-algebra filtered by $(A_r)_{r>0}$ and let $(B_k)_{k\in\mathbb{N}}$ be an increasing sequence of C^*-subalgebras of A such that*

- *$\bigcup_{r>0}(B_k \cap A_r)$ is dense in B_k for every integer k,*
- *$\bigcup_{k\in\mathbb{N}}(B_k \cap A_r)$ is dense in A_r for every positive number r.*

Then the $\mathbb{Z}_2$-graded groups $K_*^{\varepsilon,r}(A)$ and $\lim_k K_*^{\varepsilon,r}(B_k)$ are isomorphic.

Proof. — In particular, we see that $\bigcup_{k\in\mathbb{N}} B_k$ is dense in A. Let us denote by

$$\Upsilon_{*,\varepsilon,r} : \lim_k K_*^{\varepsilon,r}(B_k) \to K_*^{\varepsilon,r}(A)$$

the homomorphism of groups induced by the family of inclusions $B_k \hookrightarrow A$ where k runs through integers. We give the proof in the even case, the odd case being analogous. Let p be an element of $\mathrm{P}_n^{\varepsilon,r}(A)$ and let $\delta = \|p^2 - p\| > 0$ and choose $\alpha < \frac{\varepsilon-\delta}{12}$. Since $\bigcup_{k\in\mathbb{N}}(B_k \cap A_r)$ is dense in A_r, there is an integer k and a selfadjoint element q of $M_n(B_k \cap A_r)$ such that $\|p - q\| < \alpha$. According to Lemma 1.18, q is a ε-r projection. Let q' be another selfadjoint element of $M_n(B_k\cap A_r)$ such that $\|p-q'\|<\alpha$. Then $\|q-q'\|<2\alpha$ and if we set $q_t=(1-t)q+tq'$ for $t\in[0,1]$, then

$$\begin{aligned}\|q_t^2-q_t\| &\leqslant \|q_t^2-q_tq\|+\|q_tq-q^2\|+\|q^2-q\|+\|q-q_t\|\\ &\leqslant \|q_t-q\|(\|q_t\|+\|q\|+1)+4\alpha+\delta\\ &\leqslant 12\alpha+\delta\\ &< \varepsilon,\end{aligned}$$

and thus q and q' are homotopic in $\mathrm{P}_n^{\varepsilon,r}(B_k)$. Therefore, for $p\in\mathrm{P}_n^{\varepsilon,r}(A)$ and q in some $M_n(B_k\cap A_r)$ satisfying $\|q-p\|<\frac{\|p^2-p\|}{12}$, we define $\Upsilon'_{0,\varepsilon,r}([p,l]_{\varepsilon,r})$ to be the image of $[q,l]_{\varepsilon,r}$ in $\lim_k K_*^{\varepsilon,r}(B_k)$. Then $\Upsilon'_{0,\varepsilon,r}$ is a group homomorphism and is an inverse for $\Upsilon_{0,\varepsilon,r}$. We proceed similarly in the odd case. □

1.5. Morita equivalence

For any unital filtered algebra A, we get an identification between $\mathrm{P}_n^{\varepsilon,r}(M_k(A))$ and $\mathrm{P}_{nk}^{\varepsilon,r}(A)$ and therefore between $\mathrm{P}_\infty^{\varepsilon,r}(M_k(A))$ and $\mathrm{P}_\infty^{\varepsilon,r}(A)$. This identification gives rise to a natural group isomorphism between $K_0^{\varepsilon,r}(A)$ and $K_0^{\varepsilon,r}(M_k(A))$, and this isomorphism is induced by the inclusion of C^*-algebras

$$\iota_A: A\hookrightarrow M_k(A);\ a\mapsto \mathrm{diag}(a,0).$$

Namely, if we set $e_{1,1}=\mathrm{diag}(1,0,\dots,0)\in M_k(\mathbb{C})$, definition of the functoriality yields

$$\iota_{A,*}^{\varepsilon,r}[p,l]_{\varepsilon,r}=[p\otimes e_{1,1}+I_l\otimes(I_k-e_{1,1}),l]_{\varepsilon,r}\in K_0^{\varepsilon,r}(M_k(A))$$

for any p in $\mathrm{P}_n^{\varepsilon,r}(A)$ and any integer l with $l\leqslant n$. We can verify that

$$(\iota_{A,*}^{\varepsilon,r})^{-1}[q,l]_{\varepsilon,r}=[q,kl]_{\varepsilon,r}$$

for any q in $\mathrm{P}_n^{\varepsilon,r}(M_k(A))$ and any integer l with $l\leqslant n$, where on the right hand side of the equality, the matrix q of $M_n(M_k(A))$ is viewed as a matrix of $M_{nk}(A)$.

In a similar way, we obtain in the odd case an identification between $\mathrm{U}_\infty^{\varepsilon,r}(M_k(A))$ and $\mathrm{U}_\infty^{\varepsilon,r}(A)$ providing a natural group isomorphism between $K_1^{\varepsilon,r}(A)$ and $K_1^{\varepsilon,r}(M_k(A))$. This isomorphism is also induced by the inclusion ι_A and we have

$$\iota_{A,*}[x]_{\varepsilon,r}=[x\otimes e_{1,1}+I_n\otimes(I_k-e_{1,1})]_{\varepsilon,r}\in K_1^{\varepsilon,r}(M_k(A))$$

for any x in $\mathrm{U}_n^{\varepsilon,r}(A)$.

Let us deal now with the non-unital case. For usual K-theory, Morita equivalence for non-unital C^*-algebra can be deduced from the unital case by using the six-term exact sequence associated to the split extension $0 \to A \to \tilde{A} \to \mathbb{C} \to 0$. But for quantitative K-theory this splitting only gives rise (in term of Section 2.1) to a controlled isomorphism (see Corollary 4.9). In order to really have a genuine isomorphism, we have to go through the tedious following computation. If B is a non-unital C^*-algebra, let us identify $M_k(\tilde{B})$ with $M_k(B) \oplus M_k(\mathbb{C})$ equipped with the product

$$(b, \lambda) \cdot (b', \lambda') = (bb' + \lambda b' + b\lambda', \lambda\lambda')$$

for b and b' in $M_k(B)$ and λ and λ' in $M_k(\mathbb{C})$. Under this identification, if A is not unital, let us check that the group homomorphism

$$\Phi_1 : K_1^{\varepsilon,r}(\tilde{A}) \to K_1^{\varepsilon,r}(\widetilde{M_k(A)});\ [(x, \lambda)]_{\varepsilon,r} \mapsto [(x \otimes e_{1,1}, \lambda]_{\varepsilon,r}$$

induced by the inclusion ι_A is invertible with inverse given by the composition

$$\Psi_1 : K_1^{\varepsilon,r}(\widetilde{M_k(A)}) \to K_1^{\varepsilon,r}(M_k(\tilde{A})) \stackrel{\cong}{\to} K_1^{\varepsilon,r}(\tilde{A}),$$

where the first homomorphism of the composition is induced by the inclusion

$$\widetilde{M_k(A)} \to M_k(\tilde{A});\ (a, z) \mapsto (a, zI_k).$$

Let (x, λ) be an element of $\mathrm{U}_n^{\varepsilon,r}(\tilde{A})$, with $x \in M_n(A)$ and $\lambda \in M_n(\mathbb{C})$. Then

$$\Psi_1 \circ \Phi_1[(x, \lambda)]_{\varepsilon,r} = [(x \otimes e_{1,1}, \lambda \otimes I_k)]_{\varepsilon,r},$$

where we use the identification $M_{nk}(\mathbb{C}) \cong M_n(\mathbb{C}) \otimes M_k(\mathbb{C})$ to see $x \otimes e_{1,1}$ and $\lambda \otimes I_k$ respectively as matrices in $M_{nk}(A)$ and $M_{nk}(\mathbb{C})$. According to Lemma 1.20, as a ε-r-unitary of $M_n(\mathbb{C})$, λ is homotopic to I_n. Hence

$$[(x \otimes e_{1,1}, \lambda \otimes I_k)]_{\varepsilon,r} = [(x \otimes e_{1,1}, \lambda \otimes e_{1,1} + I_n \otimes I_{k-1})]$$

and from this we get that $\Psi_1 \circ \Phi_1$ is induced in K-theory by the inclusion map $\tilde{A} \hookrightarrow M_k(\tilde{A});\ a \mapsto \mathrm{diag}(a, 0)$ which is the identity homomorphism (according to the unital case).

Conversely, let (y, λ) be an element in $\mathrm{U}_n^{\varepsilon,r}(\widetilde{M_k(A)})$ with

$$y \in M_n(M_k(A)) \cong M_n(A) \otimes M_k(\mathbb{C})$$

and $\lambda \in M_n(\mathbb{C})$. Then

$$\Phi_1 \circ \Psi_1[(y, \lambda)]_{\varepsilon,r} = [(y \otimes e_{1,1}, \lambda \otimes I_k)]_{\varepsilon,r},$$

where

- $y\otimes e_{1,1}$ belongs to $M_n(M_k(A))\otimes M_k(\mathbb{C}) \cong M_n(A)\otimes M_k(\mathbb{C})\otimes M_k(\mathbb{C})$ (the first two factors provide the copy of $M_n(M_k(A))$ where y lies in and $e_{1,1}$ lies in the last factor).
- $\lambda \otimes I_k$ belongs to the algebra $M_n(M_k(\mathbb{C})) \cong M_n(\mathbb{C}) \otimes M_k(\mathbb{C})$ that multiplies $M_n(A) \otimes M_k(\mathbb{C}) \otimes M_k(\mathbb{C})$ on the first two factors.

Let

$$\sigma : M_n(A) \otimes M_k(\mathbb{C}) \otimes M_k(\mathbb{C}) \to M_n(A) \otimes M_k(\mathbb{C}) \otimes M_k(\mathbb{C})$$

be the C^*-algebra homomorphism induced by the flip of $M_k(\mathbb{C}) \otimes M_k(\mathbb{C})$. This flip can be realized by conjugation of a unitary U in $M_k(\mathbb{C})\otimes M_k(\mathbb{C}) \cong M_{k^2}(\mathbb{C})$. Let $(U_t)_{t\in[0,1]}$ be a homotopy in $\mathrm{U}_{k^2}(\mathbb{C})$ between U and I_{k^2}. Let us define

$$\mathcal{A} = \{(x, z \otimes I_k);\, x \in M_n(A) \otimes M_k(\mathbb{C}) \otimes M_k(\mathbb{C}),\, z \in M_n(\mathbb{C})\} \subset M_n(\widetilde{M_k(A)}) \otimes M_k(\mathbb{C}),$$

where $z \otimes I_k$ is viewed as $z \otimes I_k \otimes I_k$ in

$$M_n(\widetilde{M_k(A)}) \otimes M_k(\mathbb{C}) \cong M_n(\mathbb{C}) \otimes \widetilde{M_k(A)} \otimes M_k(\mathbb{C}).$$

Then for any $t \in [0, 1]$,

$$\mathcal{A} \to \mathcal{A};\, (x, z \otimes I_k) \mapsto ((I_n \otimes U_t) \cdot x \cdot (I_n \otimes U_t)^{-1}, z \otimes I_k)$$

is an automorphism of C^*-algebra. Hence,

$$\big((I_n \otimes U_t) \cdot (y \otimes e_{1,1}) \cdot (I_n \otimes U_t^{-1}), \lambda \otimes I_k\big)_{t\in[0,1]}$$

is a path in $\mathrm{U}^{\varepsilon,r}_{nk}(\widetilde{M_k(A)})$ between $(y\otimes e_{1,1}, \lambda\otimes I_k)$ and $(\sigma(y\otimes e_{1,1}), \lambda\otimes I_k)$. The range of $\sigma(y \otimes e_{1,1})$ being in the range of the projection $I_n \otimes e_{1,1} \otimes I_k$, we have an orthogonal sum decomposition

$$(\sigma(y \otimes e_{1,1}), \lambda \otimes I_k) = (\sigma(y \otimes e_{1,1}), \lambda \otimes e_{1,1}) + (0, \lambda \otimes (I_k - e_{1,1}))$$

(recall that $\lambda \otimes e_{1,1}$ and $\lambda \otimes (I_k - e_{1,1})$ multiply $M_n(A) \otimes M_k(\mathbb{C}) \otimes M_k(\mathbb{C})$ on the first two factors). By Lemma 1.20, λ is homotopic to I_n in $\mathrm{U}^{\varepsilon,r}_n(\mathbb{C})$ and thus $(\sigma(y\otimes e_{1,1}), \lambda\otimes I_k)$ is homotopic to $(\sigma(y\otimes e_{1,1}), \lambda\otimes e_{1,1}) + (0, I_n\otimes (I_k - e_{1,1}))$ in $\mathrm{U}^{\varepsilon,r}_{nk}(\widetilde{M_k(A)}))$ which can be viewed as

$$\mathrm{diag}((y, \lambda), (0, I_{k(k-1)})$$

in $M_k(M_n(\widetilde{M_k(A)})$. From this we deduce that $[(y, \lambda)]_{\varepsilon,r} = [(y \otimes e_{1,1}, \lambda \otimes I_k)]_{\varepsilon,r}$ in $K^{\varepsilon,r}_1(\widetilde{M_k(A)})$.

For the even case, by an analogous computation, we can check that the group homomorphisms

$$K_0^{\varepsilon,r}(\tilde{A}) \to K_0^{\varepsilon,r}(\widetilde{M_k(A)});\ [(p,q),l)]_{\varepsilon,r} \mapsto [(p\otimes e_{1,1},q),l]_{\varepsilon,r}$$

and

$$K_0^{\varepsilon,r}(\widetilde{M_k(A)}) \to K_0^{\varepsilon,r}(\tilde{A});\ [(p,q),l)]_{\varepsilon,r} \mapsto [(p,q\otimes I_k),kl]_{\varepsilon,r},$$

respectively induce by restriction homomorphisms $\Phi_0 : K_0^{\varepsilon,r}(A) \to K_0^{\varepsilon,r}(M_k(A))$ and $\Psi_0 : K_0^{\varepsilon,r}(M_k(A)) \to K_0^{\varepsilon,r}(A)$ which are inverse of each other, where in the right hand side of the last formula, we have viewed $p \in M_n(M_k(A))$ as a matrix in $M_{nk}(A)$ and $q \otimes I_k \in M_n(\mathbb{C}) \otimes M_k(\mathbb{C})$ as a matrix in $M_{nk}(\mathbb{C})$. Since Φ_0 is induced by ι_A, we get from Lemma 1.22 that $\iota_{A,*}^{\varepsilon,r} : K_0^{\varepsilon,r}(A) \to K_0^{\varepsilon,r}(M_k(A))$ is an isomorphism.

Let A be a C^*-algebra filtered by $(A_r)_{r>0}$. Then $\mathcal{K}(\mathcal{H}) \otimes A$ is filtered by $(\mathcal{K}(\mathcal{H}) \otimes A_r)_{r>0}$ and applying Proposition 1.27 to the increasing family $(M_k(A)^+)_{k\in\mathbb{N}}$ of C^*-subalgebras of $\widetilde{\mathcal{K}(\mathcal{H}) \otimes A}$, Lemmas 1.22 and 1.23, and the discussion above, we deduce the Morita equivalence for $K_*^{\varepsilon,r}(\bullet)$.

Proposition 1.28. — *If A is a filtered algebra and $\mathcal{H}$ is a separable Hilbert space, then the homomorphism*

$$A \to \mathcal{K}(\mathcal{H}) \otimes A;\ a \mapsto \begin{pmatrix} a & & \\ & 0 & \\ & & \ddots \end{pmatrix}$$

induces a ($\mathbb{Z}_2$-graded) group isomorphism (the Morita equivalence)

$$\mathcal{M}_A^{\varepsilon,r} : K_*^{\varepsilon,r}(A) \to K_*^{\varepsilon,r}(\mathcal{K}(\mathcal{H}) \otimes A)$$

for any positive number r and any $\varepsilon \in (0,1/4)$.

1.6. Lipschitz homotopies

Definition 1.29. — *If A is a C^*-algebra and C is a positive integer, then a map $h = [0,1] \to A$ is called C-Lipschitz if for every t and s in $[0,1]$, then $\|h(t) - h(s)\| \leqslant C|t-s|$.*

Proposition 1.30. — *There exists a number C such that for any unital filtered C^*-algebra A and any positive numbers r and ε with $\varepsilon < 1/4$ then:*

(i) if p_0 and p_1 are homotopic in $\mathrm{P}_n^{\varepsilon,r}(A)$, then there exist integers k and l and a C-Lipschitz homotopy in $\mathrm{P}_{n+k+l}^{\varepsilon,r}(A)$ between $\mathrm{diag}(p_0, I_k, 0_l)$ and $\mathrm{diag}(p_1, I_k, 0_l)$.

(ii) if u_0 and u_1 are homotopic in $\mathrm{U}_n^{\varepsilon,r}(A)$ then there exist an integer k and a C-Lipschitz homotopy in $\mathrm{U}_{n+k}^{3\varepsilon,2r}(A)$ between $\mathrm{diag}(u_0, I_k)$ *and* $\mathrm{diag}(u_1, I_k)$.

Proof.

(i) Notice first that if p is an ε-r-projection in A, then the homotopy of ε-r-projections of $M_2(A)$ between $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} p & 0 \\ 0 & 1-p \end{pmatrix}$ in Example 1.6 is 2-Lipschitz.

Let $(p_t)_{t\in[0,1]}$ be a homotopy between p_0 and p_1 in $\mathrm{P}_n^{\varepsilon,r}(A)$. Set $\alpha = \inf_{t\in[0,1]} \frac{\varepsilon - \|p_t^2 - p_t\|}{4}$ and let $t_0 = 0 < t_1 < \ldots < t_k = 1$ be a partition of $[0,1]$ such that $\|p_{t_i} - p_{t_{i-1}}\| < \alpha$ for $i \in \{1,\ldots,k\}$. We construct a homotopy of ε-r-projections with the required property between $\mathrm{diag}(p_0, I_{n(k-1)}, 0)$ and $\mathrm{diag}(p_1, I_{n(k-1)}, 0)$ in $M_{n(2k-1)}(A)$ as the composition of the following homotopies.

- We can connect $\mathrm{diag}(p_{t_0}, I_{n(k-1)}, 0)$ and $\mathrm{diag}(p_{t_0}, I_n, 0, \ldots, I_n, 0)$ within $\mathrm{P}_{n(2k-1)}^{\varepsilon,r}(A)$ by a 2-Lipschitz homotopy.
- As we noticed at the beginning of the proof, we can connect $\mathrm{diag}(p_{t_0}, I_n, 0, \ldots, I_n, 0)$ and $\mathrm{diag}(p_{t_0}, I_n - p_{t_1}, p_{t_1}, \ldots, I_n - p_{t_k}, p_{t_k})$ within $\mathrm{P}_{n(2k-1)}^{\varepsilon,r}(A)$ by a 2-Lipschitz homotopy.
- The ε-r-projections $\mathrm{diag}(p_{t_0}, I_n - p_{t_1}, p_{t_1}, \ldots, I_n - p_{t_k}, p_{t_k})$ and $\mathrm{diag}(p_{t_0}, I_n - p_{t_0}, \ldots, p_{t_{k-1}}, I_n - p_{t_{k-1}}, p_{t_k})$ satisfy the norm estimate of the assumption of Lemma 1.7(i) and hence then can be connected within $\mathrm{P}_{n(2k-1)}^{\varepsilon,r}(A)$ by a ray which is clearly a 1-Lipschitz homotopy.
- Using once again the homotopy of Example 1.6, we see that $\mathrm{diag}(p_{t_0}, I_n - p_{t_0}, \ldots, p_{t_{k-1}}, I_n - p_{t_{k-1}}, p_{t_k})$ and $\mathrm{diag}(0, I_n, \ldots, 0, I_n, p_{t_k})$ are connected within $\mathrm{P}_{n(2k-1)}^{\varepsilon,r}(A)$ by a 2-Lipschitz homotopy.
- Eventually, $\mathrm{diag}(0, I_n, \ldots, 0, I_n, p_{t_k})$ and $\mathrm{diag}(p_{t_k}, I_{n(k-1)}, 0)$ are connected within $\mathrm{P}_{n(2k-1)}^{\varepsilon,r}(A)$ by a 2-Lipschitz homotopy.

(ii) Let $(u_t)_{t\in[0,1]}$ be a homotopy between u_0 and u_1 in $\mathrm{U}_n^{\varepsilon,r}(A)$. Set $\alpha = \inf_{t\in[0,1]} \frac{\varepsilon - \|u_t^* u_t - I_n\|}{3}$ and let $t_0 = 0 < t_1 < \ldots < t_k = 1$ be a partition of $[0,1]$ such that $\|u_{t_i} - u_{t_{i-1}}\| < \alpha$ for $i \in \{1,\ldots,k\}$. We construct a homotopy with the required property between $\mathrm{diag}(u_0, I_{2nk})$ and $\mathrm{diag}(u_1, I_{2nk})$ within $\mathrm{U}_{n(2k+1)}^{3\varepsilon,2r}(A)$ as the composition of the following homotopies.

- Since I_{nk} and $\mathrm{diag}(u_{t_1}^* u_{t_1}, \ldots, u_{t_k}^* u_{t_k})$ satisfy the norm estimate of the assumption of Lemma 1.7(ii), then $\mathrm{diag}(u_{t_0}, I_{nk})$

is a 3ε-$2r$-unitary that can be connected to $\operatorname{diag}(u_{t_0}, u_{t_1}^*u_{t_1}, \dots, u_{t_k}^*u_{t_k})$ in $\mathrm{U}^{3\varepsilon,2r}_{n(k+1)}(A)$ by a 1-Lipschitz homotopy.

- Proceeding as in the first point of Corollary 1.8, we see that $\operatorname{diag}(I_n, u_{t_1}^*, \dots, u_{t_k}^*, I_{nk})$ and $\operatorname{diag}(u_{t_1}^*, \dots, u_{t_k}^*, I_{n(k+1)})$ can be connected within $\mathrm{U}^{\varepsilon,r}_{n(2k+1)}(A)$ by a 2-Lipschitz homotopy and thus, in view of Remark 1.4,

$$\operatorname{diag}(u_{t_0}, u_{t_1}^*u_{t_1}, \dots, u_{t_k}^*u_{t_k}, I_{nk}) =$$
$$\operatorname{diag}(I_n, u_{t_1}^*, \dots, u_{t_k}^*, I_{nk}) \cdot \operatorname{diag}(u_{t_0}, u_{t_1}, \dots, u_{t_k}, I_{nk})$$

and

$$\operatorname{diag}(u_{t_1}^*, \dots, u_{t_k}^*, I_{n(k+1)}) \cdot \operatorname{diag}(u_{t_0}, u_{t_1}, \dots, u_{t_k}, I_{nk}) =$$
$$\operatorname{diag}(u_{t_1}^*u_{t_0}, \dots, u_{t_k}^*u_{t_{k-1}}, u_{t_k}, I_{nk})$$

can be connected within $\mathrm{U}^{3\varepsilon,2r}_{n(2k+1)}(A)$ by a 4-Lipschitz homotopy.

- Since $\|u_{t_i}^*u_{t_{i-1}} - I_n\| < \varepsilon$, we get by using once again Lemma 1.7(ii) that $\operatorname{diag}(u_{t_1}^*u_{t_0}, \dots, u_{t_k}^*u_{t_{k-1}}, u_{t_k}, I_{nk})$ and $\operatorname{diag}(I_{nk}, u_{t_k}, I_{nk})$ can be connected within $\mathrm{U}^{3\varepsilon,2r}_{n(2k+1)}(A)$ by a 1-Lipschitz homotopy.
- Eventually, $\operatorname{diag}(I_{nk}, u_{t_k}, I_{nk})$ can be connected to $\operatorname{diag}(u_{t_k}, I_{2nk})$ within $\mathrm{U}^{3\varepsilon,2r}_{(2k+1)n}(A)$ by a 2-Lipschitz homotopy.

□

Corollary 1.31. — *There exists a control pair (α_h, k_h) such that the following holds:*

For any unital filtered C^*-algebra A, any positive numbers ε and r with $\varepsilon < \frac{1}{4\alpha_h}$ and any homotopic ε-r-projections q_0 and q_1 in $\mathrm{P}^{\varepsilon,r}_n(A)$, then there is for some integers k and l an $\alpha_h\varepsilon$-$k_{h,\varepsilon}r$-unitary W in $\mathrm{U}^{\alpha_h\varepsilon,k_{h,\varepsilon}r}_{n+k+l}(A)$ such that

$$\|\operatorname{diag}(q_0, I_k, 0_l) - W\operatorname{diag}(q_1, I_k, 0_l)W^*\| < \alpha_h\varepsilon.$$

Proof. — According to Proposition 1.30, we can assume that q_0 and q_1 are connected by a C-Lipschitz homotopy $(q_t)_{t\in[0,1]}$, for some universal constant C. Let $t_0 = 0 < t_1 < \cdots < t_p = 1$ be a partition of $[0,1]$ such that $1/32C < |t_i - t_{i-1}| < 1/16C$. With notation of Lemma 1.11, pick for every integer i in $\{1, \dots, p\}$ a $\lambda\varepsilon$-l_ε-unitary W_i in A such that $\|W_iq_{t_{i-1}}W_i^* - q_{t_i}\| < \lambda\varepsilon$. If we set $W = W_p \cdots W_1$, then W is a $3^p\lambda\varepsilon$-$pl_\varepsilon r$-unitary such that $\|Wq_0W^* - q_1\| < 2^p\lambda\varepsilon$. Since $p < 2C$, we get the result. □

2. Controlled morphisms

As we shall see in Section 3, usual maps in K-theory such as boundary maps factorize through group homomorphism of quantitative K-theory groups with expansion of norm control and propagation controlled by a control pair. This motivates the notion of controlled morphisms for quantitative K-theory in this section.

Recall that a control pair is a pair (λ, h), where

- $\lambda > 1$;
- $h : (0, \frac{1}{4\lambda}) \to (1, +\infty); \varepsilon \mapsto h_\varepsilon$ is a map such that there exists a non-increasing map $g : (0, \frac{1}{4\lambda}) \to (1, +\infty)$, with $h \leqslant g$.

The set of control pairs is equipped with a partial order: $(\lambda, h) \leqslant (\lambda', h')$ if $\lambda \leqslant \lambda'$ and $h_\varepsilon \leqslant h'_\varepsilon$ for all $\varepsilon \in (0, \frac{1}{4\lambda'})$

2.1. Definition and main properties

For any filtered C^*-algebra A, let us define the families $\mathcal{K}_0(A) = (K_0^{\varepsilon,r}(A))_{0<\varepsilon<1/4,r>0}$, $\mathcal{K}_1(A) = (K_1^{\varepsilon,r}(A))_{0<\varepsilon<1/4,r>0}$ and $\mathcal{K}_*(A) = (K_*^{\varepsilon,r}(A))_{0<\varepsilon<1/4,r>0}$.

Definition 2.1. — *Let (λ, h) be a control pair, let A and B be filtered C^*-algebras, and let i, j be elements of $\{0, 1, *\}$. A (λ, h)-controlled morphism*

$$\mathcal{F} : \mathcal{K}_i(A) \to \mathcal{K}_j(B)$$

is a family $\mathcal{F} = (F^{\varepsilon,r})_{0<\varepsilon<\frac{1}{4\lambda},r>0}$ of group homomorphisms

$$F^{\varepsilon,r} : K_i^{\varepsilon,r}(A) \to K_j^{\lambda\varepsilon,h_\varepsilon r}(B)$$

such that for any positive numbers ε, ε', r and r' with $0 < \varepsilon \leqslant \varepsilon' < \frac{1}{4\lambda}$ and $h_\varepsilon r \leqslant h_{\varepsilon'} r'$, we have

$$F^{\varepsilon',r'} \circ \iota_i^{\varepsilon,\varepsilon',r,r'} = \iota_j^{\lambda\varepsilon,\lambda\varepsilon',h_\varepsilon r,h_{\varepsilon'} r'} \circ F^{\varepsilon,r}.$$

If it is not necessary to specify the control pair, we will just say that $\mathcal{F}$ is a controlled morphism.

Let A and B be filtered algebras. Then it is straightforward to check that if $\mathcal{F} : \mathcal{K}_i(A) \to \mathcal{K}_j(B)$ is a (λ, h)-controlled morphism, then there is group homomorphism $F : K_i(A) \to K_j(B)$ uniquely defined by $F \circ \iota_i^{\varepsilon,r} = \iota_j^{\lambda\varepsilon,h_\varepsilon r} \circ F^{\varepsilon,r}$. The homomorphism F will be called the (λ, h)-controlled homomorphism induced by $\mathcal{F}$. A homomorphism $F : K_i(A) \to K_j(B)$ is called (λ, h)-controlled if it is induced by a (λ, h)-controlled morphism. If we don't need to specify the control pair (λ, h), we will just say that F is a controlled homomorphism.

Example 2.2.

(i) Let A and B be C^*-algebras respectively filtered by $(A_r)_{r>0}$ and $(B_r)_{r>0}$ and let $f : A \to B$ be a homomorphism. Assume that there exists $d > 0$ such that $f(A_r) \subset B_{dr}$ for all positive r. Then f gives rise to a bunch of group homomorphisms
$$\left(f_*^{\varepsilon,r} : K_*^{\varepsilon,r}(A) \to K_*^{\varepsilon,dr}(B)\right)_{0<\varepsilon<\frac{1}{4},r>0}$$
and hence to a $(1,d)$-controlled morphism $f_* : \mathcal{K}_*(A) \to \mathcal{K}_*(B)$.

(ii) The bunch of group isomorphisms
$$(\mathcal{M}_A^{\varepsilon,r} : K_*^{\varepsilon,r}(A) \to K_*^{\varepsilon,r}(\mathcal{K}(\mathcal{H})\otimes A))_{0<\varepsilon<\frac{1}{4},r>0}$$
of Proposition 1.28 defines a $(1,1)$-controlled morphism
$$\mathcal{M}_A : \mathcal{K}_*(A) \to \mathcal{K}_*(\mathcal{K}(\mathcal{H})\otimes A)$$
and
$$\mathcal{M}_A^{-1} : \mathcal{K}_*(\mathcal{K}(\mathcal{H})\otimes A) \to \mathcal{K}_*(A)$$
inducing the Morita equivalence in K-theory.

If (λ, h) and (λ', h') are two control pairs, define
$$h * h' : (0, \frac{1}{4\lambda\lambda'}) \to (0, +\infty);\ \varepsilon \mapsto h_{\lambda'\varepsilon}h'_\varepsilon.$$
Then $(\lambda\lambda', h*h')$ is a control pair. Let A, B_1 and B_2 be filtered C^*-algebras, let i, j and l be in $\{0, 1, *\}$ and let $\mathcal{F} = (F^{\varepsilon,r})_{0<\varepsilon<\frac{1}{4\alpha_\mathcal{F}},r>0} : \mathcal{K}_i(A) \to \mathcal{K}_j(B_1)$ be a $(\alpha_\mathcal{F}, k_\mathcal{F})$-controlled morphism, let $\mathcal{G} = (G^{\varepsilon,r})_{0<\varepsilon<\frac{1}{4\alpha_\mathcal{G}},r>0} : \mathcal{K}_j(B_1) \to \mathcal{K}_l(B_2)$ be a $(\alpha_\mathcal{G}, k_\mathcal{G})$-controlled morphism. Then $\mathcal{G}\circ\mathcal{F} : \mathcal{K}_i(A) \to \mathcal{K}_l(B_2)$ is the $(\alpha_\mathcal{G}\alpha_F, k_\mathcal{G} * k_\mathcal{F})$-controlled morphism defined by the family $(\mathcal{G}^{\alpha_\mathcal{F}\varepsilon,k_{\mathcal{F},\varepsilon}r} \circ \mathcal{F}^{\varepsilon,r})_{0<\varepsilon<\frac{1}{4\alpha_\mathcal{F}\alpha_\mathcal{G}},r>0}$.

Remark 2.3. — The Morita equivalence for quantitative K-theory is natural, i.e
$$\mathcal{M}_B \circ f = (Id_{\mathcal{K}(\mathcal{H}))}\otimes f) \circ \mathcal{M}_A$$
for any homomorphism $f : A \to B$ of filtered C^*-algebras.

Notation 2.4. — Let A and B be filtered C^*-algebras, let (λ, h) be a control pair, and let $\mathcal{F} = (F^{\varepsilon,r})_{0<\varepsilon<\frac{1}{4\alpha_\mathcal{F}},r>0} : \mathcal{K}_i(A) \to \mathcal{K}_j(B)$ (resp. $\mathcal{G} = (G^{\varepsilon,r})_{0<\varepsilon<\frac{1}{4\alpha_\mathcal{G}},r>0}$) be a $(\alpha_\mathcal{F}, k_\mathcal{F})$-controlled morphism (resp. a $(\alpha_\mathcal{G}, k_\mathcal{G})$-controlled morphism). Then we write $\mathcal{F} \stackrel{(\lambda,h)}{\sim} \mathcal{G}$ if

- $(\alpha_{\mathcal{F}}, k_{\mathcal{F}}) \leqslant (\lambda, h)$ and $(\alpha_{\mathcal{G}}, k_{\mathcal{G}}) \leqslant (\lambda, h)$.
- for every ε in $(0, \frac{1}{4\lambda})$ and $r > 0$, then

$$\iota_j^{\alpha_{\mathcal{F}}\varepsilon,\lambda\varepsilon,k_{\mathcal{F},\varepsilon}r,h_\varepsilon r} \circ F^{\varepsilon,r} = \iota_j^{\alpha_{\mathcal{G}}\varepsilon,\lambda\varepsilon,k_{\mathcal{G},\varepsilon}r,h_\varepsilon r} \circ G^{\varepsilon,r}.$$

If $\mathcal{F}$ and $\mathcal{G}$ are controlled morphisms such that $\mathcal{F} \overset{(\lambda,h)}{\sim} \mathcal{G}$ for a control pair (λ, h), then $\mathcal{F}$ and $\mathcal{G}$ induce the same morphism in K-theory.

Remark 2.5. — Let $\mathcal{F} : \mathcal{K}_i(A_2) \to \mathcal{K}_j(B_1)$ (resp. $\mathcal{F}' : \mathcal{K}_i(A_2) \to \mathcal{K}_j(B_1)$) be a $(\alpha_{\mathcal{F}}, k_{\mathcal{F}})$-controlled (resp. a $(\alpha_{\mathcal{F}'}, k_{\mathcal{F}'})$-controlled) morphisms and let $\mathcal{G} : \mathcal{K}_{i'}(A_1) \to \mathcal{K}_i(A_2)$ (resp. $\mathcal{G}' : \mathcal{K}_j(B_1) \to \mathcal{K}_l(B_2)$) be a $(\alpha_{\mathcal{G}}, k_{\mathcal{G}})$-controlled (resp. a $(\alpha_{\mathcal{G}'}, k_{\mathcal{G}'})$-controlled) morphism. Assume that $\mathcal{F} \overset{(\lambda,h)}{\sim} \mathcal{F}'$ for a control pair (λ, h), then

- $\mathcal{G}' \circ \mathcal{F} \overset{(\alpha_{\mathcal{G}'}\lambda, k_{\mathcal{G}'} * h)}{\sim} \mathcal{G}' \circ \mathcal{F}'$;
- $\mathcal{F} \circ \mathcal{G} \overset{(\alpha_{\mathcal{G}}\lambda, h * k_{\mathcal{G}})}{\sim} \mathcal{F}' \circ \mathcal{G}$.

If i is an element in $\{0, 1, *\}$ and A is a filtered C^*-algebra, we denote by $\mathcal{I}d_{K_i(A)}$ the controlled morphism induced by Id_A.

Let $\mathcal{F} : \mathcal{K}_i(A_1) \to \mathcal{K}_{i'}(B_1)$, $\mathcal{F}' : \mathcal{K}_j(A_2) \to \mathcal{K}_l(B_2)$, $\mathcal{G} : \mathcal{K}_i(A_1) \to \mathcal{K}_j(A_2)$ and $\mathcal{G}' : \mathcal{K}_{i'}(B_1) \to \mathcal{K}_l(B_2)$ be controlled morphisms and let (λ, h) be a control pair. Then the diagram

$$\begin{array}{ccc}
\mathcal{K}_{i'}(B_1) & \xrightarrow{\mathcal{G}'} & \mathcal{K}_l(B_2) \\
{\scriptstyle\mathcal{F}}\big\uparrow & & \big\uparrow{\scriptstyle\mathcal{F}'} \\
\mathcal{K}_i(A_1) & \xrightarrow{\mathcal{G}} & \mathcal{K}_j(A_2)
\end{array}$$

is called (λ, h)-*commutative* (or (λ, h)-*commutes*) if $\mathcal{G}' \circ \mathcal{F} \overset{(\lambda,h)}{\sim} \mathcal{F}' \circ \mathcal{G}$.

Definition 2.6. — *Let (λ, h) be a control pair, and let $\mathcal{F} : \mathcal{K}_i(A) \to \mathcal{K}_j(B)$ be a $(\alpha_{\mathcal{F}}, k_{\mathcal{F}})$-controlled morphism with $(\alpha_{\mathcal{F}}, k_{\mathcal{F}}) \leqslant (\lambda, h)$.*

- *$\mathcal{F}$ is called left (λ, h)-invertible if there exists a controlled morphism*

  $$\mathcal{G} : \mathcal{K}_j(B) \to \mathcal{K}_i(A)$$

  such that $\mathcal{G} \circ \mathcal{F} \overset{(\lambda,h)}{\sim} \mathcal{I}d_{K_i(A)}$. The controlled morphism $\mathcal{G}$ is then called a left (λ, h)-inverse for $\mathcal{F}$. Notice that definition of $\overset{(\lambda,h)}{\sim}$ implies that $(\alpha_{\mathcal{F}}\alpha_{\mathcal{G}}, k_{\mathcal{F}} * k_{\mathcal{G}}) \leqslant (\lambda, h)$.
- *$\mathcal{F}$ is called right (λ, h)-invertible if there exists a controlled morphism*

  $$\mathcal{G} : \mathcal{K}_j(B) \to \mathcal{K}_i(A)$$

such that $\mathcal{F}\circ\mathcal{G} \stackrel{(\lambda,h)}{\sim} \mathcal{I}d_{K_i(B)}$. The controlled morphism $\mathcal{G}$ is then called a right (λ,h)-inverse for $\mathcal{F}$.

- *$\mathcal{F}$ is called (λ,h)-invertible or a (λ,h)-isomorphism if there exists a controlled morphism*

$$\mathcal{G}:\mathcal{K}_j(B)\to\mathcal{K}_i(A)$$

which is a left (λ,h)-inverse and a right (λ,h)-inverse for $\mathcal{F}$. The controlled morphism $\mathcal{G}$ is then called a (λ,h)-inverse for $\mathcal{F}$ (notice that we have in this case necessarily $(\alpha_\mathcal{G},k_\mathcal{G})\leqslant(\lambda,h)$).

We can check easily that indeed, if $\mathcal{F}$ is left (λ,h)-invertible and right (λ,h)-invertible, then there exists a control pair (λ',h') with $(\lambda,h)\leqslant(\lambda',h')$, depending only on (λ,h) such that $\mathcal{F}$ is (λ',h')-invertible.

Definition 2.7. — *Let (λ,h) be a control pair and let $\mathcal{F}:\mathcal{K}_i(A)\to\mathcal{K}_j(B)$ be a $(\alpha_\mathcal{F},k_\mathcal{F})$-controlled morphism.*

- *$\mathcal{F}$ is called (λ,h)-injective if $(\alpha_\mathcal{F},k_\mathcal{F})\leqslant(\lambda,h)$ and for any $0<\varepsilon<\frac{1}{4\lambda}$, any $r>0$ and any x in $K_i^{\varepsilon,r}(A)$, then $F^{\varepsilon,r}(x)=0$ in $K_j^{\alpha_\mathcal{F}\varepsilon,k_{\mathcal{F},\varepsilon}r}(B)$ implies that $\iota_i^{\varepsilon,\lambda\varepsilon,r,h_\varepsilon r}(x)=0$ in $K_i^{\lambda\varepsilon,h_\varepsilon r}(A)$;*
- *$\mathcal{F}$ is called (λ,h)-surjective, if for any $0<\varepsilon<\frac{1}{4\lambda\alpha_\mathcal{F}}$, any $r>0$ and any y in $K_j^{\varepsilon,r}(B)$, there exists an element x in $K_i^{\lambda\varepsilon,h_\varepsilon r}(A)$ such that $F^{\lambda\varepsilon,h_{\lambda\varepsilon}r}(x)=\iota_j^{\varepsilon,\alpha_\mathcal{F}\lambda\varepsilon,r,k_{\mathcal{F},\lambda\varepsilon}h_\varepsilon r}(y)$ in $K_j^{\alpha_\mathcal{F}\lambda\varepsilon,k_{\mathcal{F},\lambda\varepsilon}h_\varepsilon r}(B)$.*

Remark 2.8.

(i) It is straightforward to check that if $\mathcal{F}$ is left (λ,h)-invertible, then $\mathcal{F}$ is (λ,h)-injective and that if $\mathcal{F}$ is right (λ,h)-invertible, then there exists a control pair (λ',h') with $(\lambda,h)\leqslant(\lambda',h')$, depending only on (λ,h) such that $\mathcal{F}$ is (λ',h')-surjective.

(ii) On the other hand, if $\mathcal{F}$ is (λ,h)-injective and (λ,h)-surjective, then there exists a control pair (λ',h') with $(\lambda,h)\leqslant(\lambda',h')$, depending only on (λ,h) such that $\mathcal{F}$ is a (λ',h')-isomorphism.

2.2. Controlled exact sequences

Definition 2.9. — *Let (λ,h) be a control pair,*

- *Let $\mathcal{F}=(F^{\varepsilon,r})_{0<\varepsilon<\frac{1}{4\alpha_\mathcal{F}},r>0}:\mathcal{K}_i(A)\to\mathcal{K}_j(B_1)$ be a $(\alpha_\mathcal{F},k_\mathcal{F})$-controlled morphism, and let $\mathcal{G}=(G^{\varepsilon,r})_{0<\varepsilon<\frac{1}{4\alpha_\mathcal{G}},r>0}:\mathcal{K}_j(B_1)\to\mathcal{K}_l(B_2)$ be a $(\alpha_\mathcal{G},k_\mathcal{G})$-controlled morphism, where i,j and l are* *in $\{0,1,*\}$ and A, B_1 and B_2 are filtered C^*-algebras. Then the composition*

$$\mathcal{K}_i(A) \stackrel{\mathcal{F}}{\to} \mathcal{K}_j(B_1) \stackrel{\mathcal{G}}{\to} \mathcal{K}_l(B_2)$$

is said to be (λ,h)-exact at $\mathcal{K}_j(B_1)$ if $\mathcal{G}\circ\mathcal{F}=0$ and if for any $0<\varepsilon<\frac{1}{4\max\{\lambda\alpha_{\mathcal{F}},\alpha_{\mathcal{G}}\}}$, any $r>0$ and any y in $K_j^{\varepsilon,r}(B_1)$ such that $G^{\varepsilon,r}(y)=0$ in $K_j^{\alpha_{\mathcal{G}}\varepsilon,k_{\mathcal{G},\varepsilon}r}(B_2)$, there exists an element x in $K_i^{\lambda\varepsilon,h_\varepsilon r}(A)$ such that

$$F^{\lambda\varepsilon,h_{\lambda\varepsilon}r}(x)=\iota_j^{\varepsilon,\alpha_{\mathcal{F}}\lambda\varepsilon,r,k_{\mathcal{F},\lambda\varepsilon}h_\varepsilon r}(y)$$

in $K_j^{\alpha_{\mathcal{F}}\lambda\varepsilon,k_{\mathcal{F},\lambda\varepsilon}h_\varepsilon r}(B_1)$.

- *A sequence of controlled morphisms*

$$\cdots\mathcal{K}_{i_{k-1}}(A_{k-1})\stackrel{\mathcal{F}_{k-1}}{\to}\mathcal{K}_{i_k}(A_k)\stackrel{\mathcal{F}_k}{\to}\mathcal{K}_{i_{k+1}}(A_{k+1})\stackrel{\mathcal{F}_{k+1}}{\to}\mathcal{K}_{i_{k+2}}(A_{k+2})\cdots$$

is called (λ,h)-exact if for every k, the composition

$$\mathcal{K}_{i_{k-1}}(A_{k-1})\stackrel{\mathcal{F}_{k-1}}{\to}\mathcal{K}_{i_k}(A_k)\stackrel{\mathcal{F}_k}{\to}\mathcal{K}_{i_{k+1}}(A_{k+1})$$

is (λ,h)-exact at $\mathcal{K}_{i_k}(A_k)$.

3. Quantitative K-theory and extensions of filtered C^*-algebras

The aim of this section is to establish a controlled exact sequence for quantitative K-theory with respect to filtered extension of C^*-algebras i.e extension such that the ideal inherits a structure of filtered C^*-algebra. We also prove that for these extensions, the boundary maps are induced by controlled morphisms. As in K-theory, one is a map of exponential type and the other is an index type map, and the later in turn fits in a long (λ,h)-controlled exact sequence for some universal control pair (λ,h).

3.1. Extensions of filtered C^*-algebras

Let A be a C^*-algebra filtered by $(A_r)_{r>0}$ and let

$$0\to J\to A\stackrel{q}{\to}A/J\to 0$$

be an extension of C^*-algebras. For any positive number r set $J_r=J\cap A_r$ and assume that the bijective continuous linear map

$$A_r/J_r\longrightarrow(A_r+J)/J$$

induced by the inclusion $A_r \hookrightarrow A$ is indeed an isometry i.e for any positive number r and any x in A_r, then

$$\inf_{y\in J_r} \|x+y\| = \inf_{y\in J} \|x+y\|.$$

Then $q(A_r) = (A_r + J)/J$ is closed in A/J. Moreover, for any $x \in J$ and any number $\varepsilon > 0$ there exists a positive number r and an element a of A_r such that $\|x - a\| < \varepsilon$. Since $\|q(a)\| < \varepsilon$, there exists an element y in J_r such that $\|a - y\| < \varepsilon$ and thus $\|x - a\| < 2\varepsilon$. Hence J is filtered by $(A_r \cap J)_{r>0}$ and A/J is filtered by $(q(A_r))_{r>0}$.

Definition 3.1. — *Let A be a C^*-algebra filtered by $(A_r)_{r>0}$, let J be an ideal of A and set $J_r = J \cap A_r$. The extension of C^*-algebras*

$$0 \to J \to A \to A/J \to 0$$

is called a completely filtered extension of C^*-algebras if the bijective continuous linear map

$$A_r/J_r \longrightarrow (A_r + J)/J$$

induced by the inclusion $A_r \hookrightarrow A$ is a complete isometry i.e for any integer n, any positive number r and any x in $M_n(A_r)$, then

$$\inf_{y\in M_n(J_r)} \|x+y\| = \inf_{y\in M_n(J)} \|x+y\|.$$

Numerous examples of such extensions arise from the analogous in the setting of filtered C^*-algebras of semi-split extensions.

Definition 3.2. — *Let A be a C^*-algebra filtered by $(A_r)_{r>0}$ and let J be an ideal of A. The extension of C^*-algebras*

$$0 \to J \to A \to A/J \stackrel{q}{\to} 0$$

is said to be filtered and semi-split (or a semi-split extension of filtered C^*-algebras) if there exists a completely positive (complete) norm decreasing cross-section

$$s : A/J \to A$$

such that

$$s(q(A_r)) \subseteq A_r$$

for any number $r > 0$. Such a cross-section is said to be semi-split and filtered.

Lemma 3.3. — *Any semi-split extension of filtered C^*-algebra is completely filtered.*

Proof. — Let $0 \to J \to A \to A/J \stackrel{q}{\to} 0$ be a filtered and semi-split extension and let $s : A/J \to A$ be a semi-split and filtered crossed section. Let r be a positive number, let n be an integer and let x be an element of $M_n(A_r)$. Since $s(q(A_r)) \subseteq A_r$, there exists an element z in $M_n(J_r)$ such that $x + z = s(q(x))$. Then we have

$$\begin{aligned} \|x+z\| &\leqslant \|s(q(x))\| \\ &\leqslant \|q(x)\| \\ &\leqslant \inf_{y \in M_n(J)} \|x+y\|. \end{aligned}$$

We get hence that $\|x + z\| = \inf_{y\in M_n(J)} \|x + y\|$ and the extension is completely filtered . □

We have the following analogous of the lifting property for unitaries of the neutral component.

Lemma 3.4. — *There exists a control pair (α_e, k_e) such that for any completely filtered extension of C^*-algebras*

$$0 \longrightarrow J \longrightarrow A \stackrel{q}{\longrightarrow} A/J \longrightarrow 0,$$

with A unital, the following holds: for every positive numbers r and ε with $\varepsilon < \frac{1}{4\alpha_e}$ and any ε-r-unitary V homotopic to I_n in $\mathrm{U}_n^{\varepsilon,r}(A/J)$, then for some integer j, there exists a $\alpha_e\varepsilon$-$k_{e,\varepsilon}r$-unitary W homotopic to I_{n+j} in $\mathrm{U}_{n+j}^{\alpha_e\varepsilon,k_{e,\varepsilon}r}(A)$ and such that $\|q(W) - \mathrm{diag}(V, I_j)\| < \alpha_e\varepsilon$.

Proof. — According to Proposition 1.30, we can assume that V and I_n are connected by a C-Lipschitz homotopy $(V_t)_{t\in[0,1]}$, for some universal constant C. Let $t_0 = 0 < t_1 < \cdots < t_p = 1$ be a partition of $[0, 1]$ such that $1/16C < |t_i - t_{i-1}| < 1/8C$. Then we get that $\|V_{i-1} - V_i\| < 1/8$ and hence $\|V_{i-1}V_i^* - I_n\| < 1/2$. Let l_ε be the smallest integer such that $\sum_{k\geqslant l_\varepsilon+1} 2^{-k}/k < \varepsilon$ and $\sum_{k\geqslant l_\varepsilon+1} \log^k 2/k! < \varepsilon$ and let us consider the polynomial functions $P_\varepsilon(x) = \sum_{k=0}^{l_\varepsilon} x^k/k!$ and $Q_\varepsilon(x) = -\sum_{k=1}^{l_\varepsilon} x^k/k$. Since

$$|1 - z - P_\varepsilon \circ Q_\varepsilon(z)| = |\exp\circ\log(1 - z) - P_\varepsilon \circ Q_\varepsilon(z)| < 3\varepsilon$$

for every complex number z such that $|z| < 1/2$, we get then

$$\|V_{i-1}V_i^* - P_\varepsilon \circ Q_\varepsilon(I_n - V_{i-1}V_i^*)\| < 3\varepsilon. \tag{3.1}$$

For $i = 1, \ldots, p$, let Z_i be a lift for $I_n - V_{i-1}V_i^*$ in $M_n(A_{2l_\varepsilon r})$ such that $\|Z_i\| < 1/2$. Let us set for t in $[0, 1]$ and i in $\{1, \ldots, p\}$

$$W_i^t = P_\varepsilon\left(t\left(\frac{Q_\varepsilon(Z_i) - Q_\varepsilon(Z_i^*)}{2}\right)\right).$$

Since $\frac{Q_\varepsilon(Z_i)-Q_\varepsilon(Z_i^*)}{2}$ is skew-adjoint and $\|\frac{Q_\varepsilon(Z_i)-Q_\varepsilon(Z_i^*)}{2}\|<\log 2$, then $\exp\left(\frac{Q_\varepsilon(Z_i)-Q_\varepsilon(Z_i^*)}{2}\right)$ is a unitary such that

$$\left\|P_\varepsilon\left(t\left(\frac{Q_\varepsilon(Z_i)-Q_\varepsilon(Z_i^*)}{2}\right)\right)-\exp\left(t\left(\frac{Q_\varepsilon(Z_i)-Q_\varepsilon(Z_i^*)}{2}\right)\right)\right\|<\varepsilon$$

for every t in $[0,1]$ and i. Hence, according to Lemma 1.7, we get that $(W_i^t)_{t\in[0,1]}$ is a homotopy of 3ε-$2l_\varepsilon^2$-unitaries between I_n and $W_i^1=P_\varepsilon\left(\frac{Q_\varepsilon(Z_i)-Q_\varepsilon(Z_i^*)}{2}\right)$. Since $V_{i-1}V_i^*$ is close to the unitary $V_{i-1}V_i^*(V_iV_{i-1}^*\cdot V_{i-1}V_i^*)^{-1/2}$, then $q(W_i^1)$ is uniformly close (in i) to

$$\exp(\log(V_{i-1}V_i^*(V_iV_{i-1}^*V_{i-1}V_i^*)^{-1/2}))=V_{i-1}V_i^*(V_iV_{i-1}^*V_{i-1}V_i^*)^{-1/2}$$

(the logarithm is well defined since $\|V_{i-1}V_i^*(V_iV_{i-1}^*V_{i-1}V_i^*)^{-1/2}-I_n\|<1$). Therefore we get for some universal positive number α that $\|q(W_i^1)-V_{i-1}V_i^*\|<\alpha\varepsilon$. If we set now $W=W_1^1\cdots W_p^1$ and since $p\leqslant 16C$, then W satisfies the required property. □

Lemma 3.5. — *There exists a control pair (α,k) such that for any completely filtered extension of C^*-algebras*

$$0\to J\to A\to A/J\to 0$$

with A unital the following holds :

For any integer n, any ε-r-projection p in $M_n(A/J)$ and any self-adjoint lift x for p in $M_n(A_r)$ such that $\|x\|\leqslant 2$, there exists an element y_p in $M_n(J_{k_\varepsilon r})$ such that

$$\|I_n+y_p-\exp(2\imath\pi x)\|<\alpha\varepsilon/4.$$

In particular I_n+y_p is an $\alpha\varepsilon$-$k_\varepsilon r$-unitary of $M_n(J^+)$.

Proof. — Let k_ε be the smallest integer such that $\sum_{l=k_\varepsilon+1}^{+\infty}16^l/l!<\varepsilon$ and set

$$z_p=\sum_{l=0}^{k_\varepsilon}\frac{(2\imath\pi x)^l}{l!}.$$

Then z_p belongs to $M_n(A_{k_\varepsilon r})$ and we have

$$\begin{aligned}\|q(z_p)-I_n\| &\leqslant \|q(z_p-\exp(\imath\pi x))\|+\|q(\exp(\imath\pi x))-q(\exp(\imath\pi\kappa_0(p)))\|\\ &\leqslant \|z_p-\exp(\imath\pi x)\|+\|\exp(\imath\pi p)-\exp(\imath\pi\kappa_0(p))\|\\ &< \lambda\varepsilon,\end{aligned}$$

with $\lambda = 1 + 2e^{16}$. Hence there exists an element y_p in $M_n(J_{k_\varepsilon r})$ such that

$$\|I_n + y_p - z_p\| < \lambda\varepsilon$$

and we have

$$\|I_n + y_p - \exp(2\imath\pi x)\| < (2\lambda + 1)\varepsilon.$$

The end of the statement is then a consequence of Lemma 1.7. □

Remark 3.6. — With notations of the lemma,

(i) if y_p and y'_p are two elements of $M_n(J_{k_\varepsilon r})$ that satisfy the conclusion of the lemma, then according to Lemma 1.7,we see that $I_n + y_p$ and $I_n + y'_p$ are homotopic as $2\alpha\varepsilon$-$k_\varepsilon r$-unitaries of $M_n(J^+)$;

(ii) Let x and x' two self-adjoint lifts for p in $M_n(A_r)$ such that $\|x\| \leqslant 2$ and $\|x'\| \leqslant 2$. Applying the first point of the remark and the lemma to the completely filtered extension of C^*-algebras

$$0 \to J[0,1] \to A[0,1] \to A/J[0,1] \to 0$$

and to the constant ε-r-projection

$$[0,1] \to M_n(A/J); t \mapsto p$$

with lift

$$[0,1] \to M_n(A_r); t \mapsto (1-t)x + tx',$$

we get that x and x' give rise to homotopic $2\alpha\varepsilon$-$k_\varepsilon r$-unitaries of $M_n(J^+)$.

3.2. Controlled boundary maps

For any extension $0 \to J \to A \to A/J \to 0$ of C^*-algebras we denote by $\partial_{J,A} : K_*(A/J) \to K_*(J)$ the associated (odd degree) boundary map.

Proposition 3.7. — *There exists a control pair $(\alpha_{\mathcal{D}}, k_{\mathcal{D}})$ such that for any completely filtered extension of C^*-algebras*

$$0 \longrightarrow J \longrightarrow A \overset{q}{\longrightarrow} A/J \longrightarrow 0,$$

there exists a $(\alpha_{\mathcal{D}}, k_{\mathcal{D}})$-controlled morphism of odd degree

$$\mathcal{D}_{J,A} = (\partial^{\varepsilon,r}_{J,A})_{0<\varepsilon\frac{1}{4\alpha_{\mathcal{D}}},r} : \mathcal{K}_*(A/J) \to \mathcal{K}_*(J)$$

which induces in K-theory $\partial_{J,A} : K_*(A/J) \to K_*(J)$.

Proof. — Let us first prove the result when when A is unital.

(i) Let p be an element of $\mathrm{P}_n^{\varepsilon,r}(A/J)$ and let x be a self-adjoint lift for p in $M_n(A_r)$ such that $\|x\|\leqslant 2$. Then there exists a lift x_0 for $\kappa_0(p)$ in $M_n(A)$ such that $\|x-x_0\|<2\varepsilon$. Fix a control pair (α,k) as in Lemma 3.5, and let y_p in $M_n(J_r)$ be such that $\|I_n+y_p-\exp(2\imath x)\|<\alpha\varepsilon/4$. Then

- $\partial_{J,A}([\kappa_0(p)])$ is the class of $\exp(2\imath\pi x_0)$ in $K_1(J)$;
- I_n+y_p is an $\alpha\varepsilon$-$k_\varepsilon r$-unitary of $M_n(J^+)$, and according to Remark 3.6
  - any two such $\alpha\varepsilon$-$k_\varepsilon r$-unitaries are homotopic in $U_n^{2\alpha\varepsilon,k_\varepsilon r}(J^+)$;
  - any two self-adjoint lifts for p in $M_n(A_r)$ with norm at most 2 give rise to $\alpha\varepsilon$-$k_\varepsilon r$-unitaries which are homotopic in $U_n^{2\alpha\varepsilon,k_\varepsilon r}(J^+)$.
- $\|I_n+y_p-\exp(2\imath\pi x_0)\|<(\alpha/4+e^{20})\varepsilon$ and hence, if ε is small enough then I_n+y_p and $\exp(2\imath\pi x_0)$ are homotopic elements of $GL_n(J^+)$.

Applying Lemma 3.5 to $A/J[0,1]$, we see that the map

$$\mathrm{P}_n^{\varepsilon,r}(A/J)\longrightarrow \mathrm{U}_n^{2\alpha\varepsilon,k_\varepsilon r}(J^+);\ p\mapsto I_n+y_p$$

preserves homotopies and hence gives rise to a bunch of well defined group homomorphism

$$\partial_{J,A}^{\varepsilon,r}:K_0^{\varepsilon,r}(A/J)\longrightarrow K_1^{2\alpha\varepsilon,k_\varepsilon r}(J);\ [p,l]_{\varepsilon,r}\mapsto[I_n+y_p]_{2\alpha\varepsilon,k_\varepsilon r}$$

which in the even case satisfies the required properties for a controlled homomorphism.

(ii) In the odd case, we follow the route of [18, Chapter 8]. For any element u of $\mathrm{U}_n^{\varepsilon,r}(A/J)$, pick any element v in some $\mathrm{U}_j^{\varepsilon,r}(A/J)$ such that $\mathrm{diag}(u,v)$ is homotopic to I_{n+j} in $\mathrm{U}_{n+j}^{3\varepsilon,2r}(A/J)$ (we can choose in view of Lemma 1.8 $v=u^*$). According to Lemma 3.4, and up to replace v by $\mathrm{diag}(v,I_k)$ for some integer k, there exists an element w in $\mathrm{U}_{n+j}^{3\alpha_e\varepsilon,2k_{e,3\varepsilon}r}(A)$ such that $\|q(w)-\mathrm{diag}(u,v)\|\leqslant 3\alpha_e\varepsilon$. Let us set $x=w\,\mathrm{diag}(I_n,0)w^*$. Then x is an element in $\mathrm{P}_{n+j}^{6\alpha_e\varepsilon,4k_{e,3\varepsilon}r}(A)$ such that $\|q(x)-\mathrm{diag}(I_n,0)\|<9\alpha_e\varepsilon$. Let h be a self-adjoint element of $M_{n+j}(A_{4k_{e,3\varepsilon}r}\cap J)$ such that

$$\|x-\mathrm{diag}(I_n,0)-h\|<9\alpha_e\varepsilon. \tag{3.2}$$

According to Lemma 1.7, we get that $h+\mathrm{diag}(I_n,0)$ belongs to $\mathrm{P}_{n+j}^{45\alpha_e\varepsilon,4k_{e,3\varepsilon}r}(J)$ and we define then

$$\partial_{J,A}^{\varepsilon,r}([u]_{\varepsilon,r})=[h+\mathrm{diag}(I_n,0),n]_{3250\alpha_e\varepsilon,8k_{e,3\varepsilon}r}\,.$$

Using once again Lemma 1.7, we see that two choices of self-adjoint elements of $M_{n+j}(A_{4k_{e,3\varepsilon}r}\cap J)$ that satisfy equation (3.2) gives rise to the same class in $K_0^{3250\alpha_e\varepsilon,8k_{e,3\varepsilon}r}(J^+)$. Moreover, it is straightforward to check that (compare with [18, Chapter 8]).

- two choices of elements satisfying the conclusion of Lemma 3.4 relatively to $\mathrm{diag}(u,v)$ give rise to homotopic elements in $\mathrm{P}_{n+j}^{3250\alpha_e\varepsilon,8k_{e,3\varepsilon}r}(J)$ (this is a consequence of Lemma 1.7).
- Replacing u by $\mathrm{diag}(u,I_m)$ and v by $\mathrm{diag}(v,I_k)$ gives also rise to the same element of $K_0^{3250\alpha_e\varepsilon,8k_{e,3\varepsilon}r}(J)$.

Applying now Lemma 3.4 to the exact sequence

$$0\to J[0,1]\to A[0,1]\to A/J[0,1]\to 0,$$

we get that $\partial_{J,A}^{\varepsilon,r}([u]_{\varepsilon,r})$

- only depends on the class of u in $K_1^{\varepsilon,r}(A/J)$;
- does not depend on the choice of v such that $\mathrm{diag}(u,v)$ is connected to I_{n+j} in $\mathrm{U}_{n+j}^{\varepsilon,r}(A/J)$.

- Using Lemma 1.7, it is plain to check that for a suitable control pair $(\alpha_{\mathcal{D}},k_{\mathcal{D}})$, then $\mathcal{D}_{J,A}=(\partial_{J,A}^{\varepsilon,r})_{0<\varepsilon\frac{1}{4\alpha_{\mathcal{D}}},r}$ is a $(\alpha_{\mathcal{D}},k_{\mathcal{D}})$-controlled morphism inducing the (odd degree) boundary map $\partial_{J,A}:K_*(A/J)\to K_*(J)$.
- If A is not unital, use with notations of Section 1.4 the completely filtered extension

$$0\to J\to A^+\to A^+/J\to 0$$

to define $\partial_{J,A}^{\varepsilon,r}$ as the composition

$$K_1^{\varepsilon,r}(A/J)\xrightarrow{\cong}K_1^{\varepsilon,r}(A^+/J)\xrightarrow{\partial_{J,A^+}^{\varepsilon,r}}K_1^{\alpha_{\mathcal{D}}\varepsilon,k_{\mathcal{D},\varepsilon}r}(J)$$

and

$$K_0^{\varepsilon,r}(A/J)\hookrightarrow K_0^{\varepsilon,r}(A^+/J)\xrightarrow{\partial_{J,A^+}^{\varepsilon,r}}K_1^{\alpha_{\mathcal{D}}\varepsilon,k_{\mathcal{D},\varepsilon}r}(J),$$

where the left morphisms in the compositions are induced by the inclusion $A/J\hookrightarrow A^+/J$.

□

For a completely filtered extension of C^*-algebras

$$0\longrightarrow J\longrightarrow A\xrightarrow{q}A/J\longrightarrow 0,$$

we set $\mathcal{D}_{J,A}^0:\mathcal{K}_0(A/J)\to\mathcal{K}_1(J)$, for the restriction of $\mathcal{D}_{J,A}$ to $\mathcal{K}_0(A/J)$ and $\mathcal{D}_{J,A}^1:\mathcal{K}_1(A/J)\to\mathcal{K}_0(J)$, for the restriction of $\mathcal{D}_{J,A}$ to $\mathcal{K}_1(A/J)$.

Remark 3.8.

(i) Let A and B be two filtered C^*-algebras and let $\phi : A \to B$ be a filtered homomorphism. Let I and J be respectively ideals in A and B and assume that
  - $0 \to I \to A \to A/I \to 0$ and $0 \to J \to B \to B/J \to 0$ are completely filtered extensions of C^*-algebras.
  - $\phi(I) \subset J$,

  then $\mathcal{D}_{J,B} \circ \tilde{\phi}_* = \phi_* \circ \mathcal{D}_{I,A}$.

(ii) Let $0 \longrightarrow J \longrightarrow A \stackrel{q}{\longrightarrow} A/J \longrightarrow 0$ be a split extension of filtered C^*-algebras, i.e there exists a homomorphism of filtered C^*-algebras $s : A/J \to A$ such that $q \circ s = Id_{A/J}$. Then we have $\mathcal{D}_{J,A} = 0$.

For a filtered C^*-algebra A, we have defined the suspension and the cone respectively as $SA = C_0((0,1), A)$ and $CA = C_0((0,1], A)$. Then SA and CA are filtered C^*-algebras and evaluation at the value 1 gives rise to a semi-split filtered extension of C^*-algebras

$$0 \to SA \to CA \to A \to 0 \tag{3.3}$$

and in the even case, the corresponding boundary $\partial_{SA,CA} : K_0(A) \to K_1(SA)$ implements the suspension isomorphism and has the following easy description when A is unital: if p is a projection, then $\partial_{SA,CA}[p]$ is the class in $K_1(SA)$ of the path of unitaries

$$[0,1] \to U_n(A);\ t \mapsto pe^{2\imath\pi t} + 1 - p.$$

Let us show that we have an analogous description in term of almost projection. Notice that if q is an ε-r-projection in A, then

$$z_q : [0,1] \to A;\ t \mapsto qe^{2\imath\pi t} + 1 - q$$

is a 5ε-r-unitary in $\widetilde{SA}$. Using this, we can define a $(5,1)$-controlled morphism $\mathcal{Z}_{\mathcal{A}} = (Z_A^{\varepsilon,r})_{0<\varepsilon<1/20, r>0} : \mathcal{K}_0(A) \to \mathcal{K}_1(SA)$ in the following way:

- for any q in $\mathrm{P}_n^{\varepsilon,r}(A)$ and any integer k let us set

$$V_{q,k} : [0,1] \to \mathrm{U}_n^{5\varepsilon,r}(\widetilde{SA}) : t \mapsto \operatorname{diag}(e^{-2k\imath\pi t}, 1, \ldots, 1) \cdot (1 - q + qe^{2\imath\pi t});$$

- define then $Z_A^{\varepsilon,r}([q,k]_{\varepsilon,r}) = [V_{q,k}]_{5\varepsilon,r}$.

Proposition 3.9. — *There exists a control pair (λ, h) such that for any unital filtered C^*-algebra A, then $\mathcal{D}^0_{CA,SA} \stackrel{(\lambda,h)}{\sim} \mathcal{Z}_{\mathcal{A}}$.*

Proof. — Let $[q,k]_{\varepsilon,r}$ be an element of $K_0^{\varepsilon,r}(A)$, with q in $\mathrm{P}_n^{\varepsilon,r}(A)$ and k integer. We can assume without loss of generality that $n \geqslant k$. Namely, up to replace n by $2n$ and using a homotopy between $\operatorname{diag}(q,0)$ and $\operatorname{diag}(0,q)$

in $\mathrm{P}_{2n}^{\varepsilon,r}(A)$, we can indeed assume that q and $\operatorname{diag}(I_k,0)$ commute. As in the proof of Lemma 3.5, define l_ε as the smallest integer such that $\sum_{l=l_\varepsilon+1}^{\infty} 16^l/l! < \varepsilon$. Let us consider the following paths in $M_n(A)$

$$z:[0,1]\longrightarrow M_n(A); t\mapsto \sum_{l=0}^{l_\varepsilon}\frac{(2\imath\pi(tq+(1-t)\operatorname{diag}(I_k,0)))^l}{l!}$$

and

$$z':[0,1]\longrightarrow M_n(A); t\mapsto \exp(2\imath\pi\operatorname{diag}(-tI_k,0))(1-q+e^{2\imath\pi t}q).$$

Since q and I_k commutes, then

$$\exp(2\imath\pi(\operatorname{diag}(-tI_k,0)+tq)) \quad=\quad \exp(2\imath\pi\operatorname{diag}(-tI_k,0))\cdot\exp(2\imath\pi tq)$$

and hence

$$z(t)=\exp(2\imath\pi\operatorname{diag}(-tI_k,0))\exp(2\imath\pi tq) - \sum_{l=l_\varepsilon+1}^{\infty}\frac{(2\imath\pi(tq+(1-t)\operatorname{diag}(I_k,0)))^l}{l!}.$$

We get therefore

$$\begin{aligned}\|z(t)-z'(t)\| &\leqslant \varepsilon+\|qe^{2\imath\pi t}+(1-q)-\exp 2\imath\pi tq\|\\ &\leqslant \varepsilon+2\|\kappa_0(q)-q\|+\|\exp 2\imath\pi t\kappa_0(q)-\exp 2\imath\pi tq\|\\ &\leqslant \varepsilon(5+4e^{4\pi}).\end{aligned}$$

Let us set

$$y:[0,1]\longrightarrow M_n(A);\ t\mapsto z(t)-1-(1-t)\operatorname{diag}(I_k,0)\sum_{l=1}^{l_\varepsilon}\frac{(2\imath\pi)^l}{l!}-t\sum_{l=1}^{l_\varepsilon}\frac{(2\imath\pi q)^l}{l!}.$$

For some $\alpha_s\geqslant\alpha_\partial$, we get then that $1+y$ and z' are homotopic elements in $U_n^{\alpha_s\varepsilon,k_{\partial,\varepsilon}r}(\widetilde{SA})$. Using the semi-split filtered cross-section $A\to CA$; $a\mapsto [t\mapsto ta]$ for the extension of equation (3.3), we get in view of the proof of Proposition 3.7,

$$\iota_1^{\alpha_\partial\varepsilon,\alpha_s\varepsilon,k_{\partial,\varepsilon}r}\circ\partial_{SA,CA}^{\varepsilon,r}([q,k]_{\varepsilon,r})=[1+y]_{\alpha_s\varepsilon,k_{\partial,\varepsilon}r},$$

and thus we deduce

$$\iota_1^{\alpha_\partial\varepsilon,\alpha_s\varepsilon,k_{\partial,\varepsilon}r}\circ\partial_{SA,CA}^{\varepsilon,r}([q,k]_{\varepsilon,r})=[z']_{\alpha_s\varepsilon,k_{\partial,\varepsilon}r}.$$

We get the result by using a homotopy of unitaries in $M_n(\widetilde{SA})$ between

$$t\mapsto\operatorname{diag}(e^{-2k\pi t},1,\dots,1)$$

and $t\mapsto\exp(2\imath\pi\operatorname{diag}(-tI_k,I_{n-k}))$. □

The inverse of the suspension isomorphism is provided, up to Morita equivalence by the Toeplitz extension: let us consider the unilateral shift S on $\ell^2(\mathbb{N})$, i.e the operator defined on the canonical basis $(e_n)_{n\in\mathbb{N}}$ of $\ell^2(\mathbb{N})$ by $S(e_n)=e_{n+1}$ for all integer n. Then the Toeplitz algebra $\mathcal{T}$ is the C^*-subalgebra of $\mathcal{L}(\ell^2(\mathbb{N}))$ generated by S. The algebra of compact operators $\mathcal{K}(\ell^2(\mathbb{N}))$ is an ideal of $\mathcal{T}$ and we get an extension of C^*-algebras

$$0\to\mathcal{K}(\ell^2(\mathbb{N}))\to\mathcal{T}\stackrel{\rho}{\to}C(\mathbb{S}_1)\to 0,$$

called the Toeplitz extension, where $\mathbb{S}_1$ denote the unit circle. Let us define $\mathcal{T}_0=\rho^{-1}(C_0(0,1))$, where $C_0(0,1)$ is viewed as a subalgebra of $C(\mathbb{S}_1)$. We obtain then an extension of C^*-algebras

$$0\to\mathcal{K}(\ell^2(\mathbb{N}))\to\mathcal{T}_0\stackrel{\rho}{\to}C_0(0,1)\to 0.$$

For any C^*-algebra A, we can tensorize this exact sequence to obtain an extension

$$0\to\mathcal{K}(\ell^2(\mathbb{N}))\otimes A\to\mathcal{T}_0\otimes A{\to}SA\to 0$$

which is filtered and semi-split when A is a filtered C^*-algebra.

Proposition 3.10. — *There exists a control pair (λ,h) such that*

$$\mathcal{D}^1_{\mathcal{K}(\ell^2(\mathbb{N}))\otimes A,\mathcal{T}_0\otimes A}\circ\mathcal{Z}_A\stackrel{(\lambda,h)}{\sim}\mathcal{M}_A$$

for any unital filtered C^*-algebra A.

Proof. — Let q be an ε-r-projection in $M_n(A)$. We can assume indeed without loss of generality that $n=1$. The Toeplitz extension is semi-split by the section induced by the completely positive (complete) norm decreasing map $s:C(\mathbb{S}_1)\longrightarrow\mathcal{T};\ f\mapsto M_f$, where if π_0 stands for the projection $L^2(\mathbb{S}_1)\cong\ell^2(\mathbb{Z})\to l^2(\mathbb{N})$, then M_f is the composition

$$l^2(\mathbb{N})\hookrightarrow\ell^2(\mathbb{Z})\cong L^2(\mathbb{S}_1)\stackrel{f\cdot}{\to}L^2(\mathbb{S}_1)\stackrel{\pi_0}{\to}l^2(\mathbb{N}),$$

($f\cdot$ being the pointwise multiplication by f). Notice first that $\begin{pmatrix}S&1-SS^*\\0&S^*\end{pmatrix}$ is a unitary lift of $\mathbb{S}_1\to M_2(\mathbb{C});\ z\mapsto diag(z,\bar{z})$ in $M_2(\mathcal{T})$ under the homomorphism induced by $\rho:\mathcal{T}\to C(\mathbb{S}_1)$. Under the section induced by s, we see that z_q lifts to $1\otimes(1-q)+S\otimes q$, and hence

$$W=\begin{pmatrix}S&1-SS^*\\0&S^*\end{pmatrix}\otimes q+I_2\otimes(1-q)$$

is a lift in $\mathrm{U}_2^{5\varepsilon,r}(\mathcal{T}_0\otimes A)$ of $\operatorname{diag}(z_q,z_q^*)$. Since $\|q(1-q)\|<\varepsilon$, we see that $W^*\operatorname{diag}(1,0)W$ is close to

$$\begin{pmatrix}S^*&0\\1-SS^*&S\end{pmatrix}\begin{pmatrix}1&0\\0&0\end{pmatrix}\begin{pmatrix}S&1-SS^*\\0&S^*\end{pmatrix}\otimes q^2+\begin{pmatrix}1&0\\0&0\end{pmatrix}\otimes(1-q)^2.$$

Hence, $W^*\operatorname{diag}(1,0)W$ is an element of $\mathrm{P}_2^{10\varepsilon,2r}(\mathcal{T}_0\otimes A)$ which is close to $\operatorname{diag}(1,(1-SS^*)\otimes q)$. Since

$$\mathcal{M}_A([q,0]_{\varepsilon,r})=[\operatorname{diag}(0,(1-SS^*)\otimes q)]_{\varepsilon,r},$$

we get the existence of a positive real α_t such that the proposition holds. □

3.3. Long exact sequence

We follow the route of [18, Sections 6.3, 7.1 and 8.2] to state for completely filtered extensions of C^*-algebras (λ,h)-exact long exact sequences in quantitative K-theory, for some universal control pair (λ,h).

PROPOSITION 3.11. — *There exists a control pair (λ,h) such that for any completely filtered extension of C^*-algebras*

$$0\longrightarrow J\stackrel{\jmath}{\longrightarrow}A\stackrel{q}{\longrightarrow}A/J\longrightarrow 0,$$

the composition

$$\mathcal{K}_*(J)\stackrel{j_*}{\to}\mathcal{K}_*(A)\stackrel{q_*}{\to}\mathcal{K}_*(A/J)$$

is (λ,h)-exact at $\mathcal{K}_*(A)$.

Proof. — We can assume without loss of generality that A is unital. In the even case, let y be an element of $K_0^{\varepsilon,r}(A)$ such that $q_*(y)=0$ in $K_0^{\varepsilon,r}(A/J)$, let e be an ε-r-projection in $M_n(A)$ and let k be a positive integer such that $y=[e,k]_{\varepsilon,r}$. Up to stabilization, we can assume that $k\leqslant n$ and that $q(e)$ is homotopic to $p_k=\operatorname{diag}(I_k,0)$ as an ε-r-projection in $M_n(A/J)$. According to Corollary 1.31, there exists up to stabilization a $\alpha_h\varepsilon$-$k_{h,\varepsilon}r$-unitary W of $M_n(A/J)$ such that

$$\|Wq(e)W^*-p_k\|<\alpha_h\varepsilon.$$

Then $\operatorname{diag}(W,W^*)$ is homotopic to I_{2n} as a $3\alpha_h\varepsilon$-$2k_{h,\varepsilon}r$-unitary of $M_{2n}(A/J)$. Let choose as in Lemma 3.4, a control pair (α,l), an integer j and a $\alpha\varepsilon$-$l_\varepsilon r$-unitary V of $M_{2n+j}(A)$ such that

$$\|q(V)-\operatorname{diag}(W,W^*,I_{k+j})\|<\alpha\varepsilon.$$

If we set $e'=V\operatorname{diag}(e,0)V^*$, then e' is a $4\alpha\varepsilon$-$2l_\varepsilon r$-projection in $M_{2n+j}(A)$. Moreover, since

$$\|q(e')-\operatorname{diag}(I_n,0)\|<(4\alpha+\alpha_h)\varepsilon,$$

there exist an element f in $M_{2n+j}(J^+)$ such that

$$\|f-e'\|<(4\alpha+\alpha_h)\varepsilon.$$

Then, according to Lemma 1.7, f is for a suitable λ a $\lambda\varepsilon$-$2l_\varepsilon r$-projection of $M_{2n+k}(J^+)$ homotopic to e'. Then $x=[f,k]_{\lambda\varepsilon,2l_\varepsilon r}$ defines a class in $K_0^{\lambda\varepsilon,2l_\varepsilon r}(J)$. As in the proof of (ii) of Lemma 1.9 we can choose λ big enough so that $\mathrm{diag}(e',I_{2n+j})$ and $\mathrm{diag}(e,0,I_{2n+j})$ are homotopic $\lambda\varepsilon$-$2k_{h,\varepsilon}r$-projections of $M_{2n}(A)$ and hence we get the result in the even case.

For the odd case, let y be an element in $K_1^{\varepsilon,r}(A)$ such that $q_*(y)=0$ in $K_1^{\varepsilon,r}(A/J)$ and let us choose an ε-r-unitary V in some $M_n(A)$ such that $y=[V]_{\varepsilon,r}$. In view of Lemma 3.4 and up to enlarge the size of the matrix V, we can assume that $\|q(V)-q(W)\|\leqslant\alpha_e\varepsilon$ with W a $\alpha_e\varepsilon$-$k_{e,\varepsilon}r$-unitaries of $M_n(A)$ homotopic to I_n. Hence W^*V and V are homotopic $3\alpha_e\varepsilon$-$(k_{e,\varepsilon}+1)r$-unitary of $M_n(A)$. Since

$$\|q(W^*V)-I_n\|<(2\alpha_e+1)\varepsilon,$$

there exists U in $M_n(A)$ such that

- the coefficients of $U-I_n$ lie in $J_{k_{e,\varepsilon}+1}$;
- $\|U-W^*V\|<(2\alpha_e+1)\varepsilon$.

In particular, we get that U is a $\lambda\varepsilon$-$(k_\varepsilon+1)r$-unitary for some $\lambda\geqslant 1$ depending only on α_e. Hence, $x=[U]_{\lambda\varepsilon,(k_{e,\varepsilon}+1)r}$ defines a class in $K_1^{\lambda\varepsilon,(k_{e,\varepsilon}+1)r}(J)$ with the required property. □

Proposition 3.12. — *There exists a control pair (λ,h) such that for any completely filtered extension of C^*-algebras*

$$0\longrightarrow J\overset{\jmath}{\longrightarrow}A\overset{q}{\longrightarrow}A/J\longrightarrow 0,$$

the composition

$$\mathcal{K}_1(A)\overset{q_*}{\to}\mathcal{K}_1(A/J)\overset{\mathcal{D}^1_{J,A}}{\to}\mathcal{K}_0(J)$$

is (λ,h)-exact at $\mathcal{K}_1(A/J)$.

Proof. — We can assume without loss of generality that A is unital. Let y be an element of $K_1^{\varepsilon,r}(A/J)$ such that $\partial^{\varepsilon,r}_{J,A}(y)=0$ in $K_0^{\alpha_\partial\varepsilon,k_{\partial,\varepsilon}r}(A/J)$ and let U be an ε-r-unitary of $M_n(A/J)$ such that $y=[U]_{\varepsilon,r}$. With notation of Lemma 3.4, let j be an integer and W be a $3\alpha_e\varepsilon$-$2k_{e,3\varepsilon}r$-unitary in $M_{2n+j}(A)$ such that

$$\|q(W)-\mathrm{diag}(U,U^*,I_j)\|<3\alpha_e\varepsilon.$$

As in the proof of Proposition 3.7, set $x=W\,\mathrm{diag}(I_n,0)W^*$ and let h be an element in $M_{2n+j}(J_{4k_{e,3\varepsilon}r})$ such that

$$\|x-h-\mathrm{diag}(I_n,0)\|<9\alpha_e\varepsilon.$$

Since $\partial^{\varepsilon,r}_{J,A}(y)=0$, we can up to take a larger n assume that $h+\mathrm{diag}(I_n,0)$ is homotopic to $\mathrm{diag}(I_n,0)$ as a $\alpha_{\mathcal{D}}\varepsilon$-$k_{\mathcal{D},\varepsilon}r$-projection of $M_{2n+j}(\tilde{J})$. Since x

is close to $h+\mathrm{diag}(I_n,0)$, we get from Corollary 1.31 that up to take a larger j, there exists for a control pair (α,l), depending only on the control pairs (α_h,k_h) and $(\alpha_{\mathcal{D}},k_{\mathcal{D}})$ of Corollary 1.31 and Lemma 3.5, an $\alpha\varepsilon$-$l_\varepsilon r$-unitary V' in $M_{2n+j}(\widetilde{J})$ such that

$$\|W\,\mathrm{diag}(I_n,0)W^*-V'\,\mathrm{diag}(I_n,0)V'^*\|<\alpha\varepsilon.$$

Indeed up to unlarge the control pair (α,l) using (α_e,k_e), we can assume that $V=\rho_J(V')V'^*W$ is a $\alpha\varepsilon$-$l_\varepsilon r$-unitary in $M_{2n+j}(A)$ such that

$$\|q(V)-\mathrm{diag}(U,U^*,I_j)\|<\alpha\varepsilon.$$

Since for a suitable constant α' depending only on α we have

$$\|\rho_J(V')\,\mathrm{diag}(I_n,0)\rho_J(V'^*)-\mathrm{diag}(I_n,0)\|<\alpha'\varepsilon,$$

we obtain that

$$\|V\,\mathrm{diag}(I_n,0)V^*-\mathrm{diag}(I_n,0)\|<\alpha''\varepsilon$$

and

$$\|V^*\,\mathrm{diag}(I_n,0)V-\mathrm{diag}(I_n,0)\|<\alpha''\varepsilon$$

for some constant α'' depending only on α'. Hence the $n\times n$-left upper corner X of V is an $\alpha''\varepsilon$-$l_\varepsilon r$-unitary in $M_n(A)$ such that $\|q(X)-U\|<\alpha''\varepsilon$ and then we get the result. □

Proposition 3.13. — *There exists a control pair (λ,h) such that for any completely filtered extension of C^*-algebras*

$$0\longrightarrow J\overset{\jmath}{\longrightarrow}A\overset{q}{\longrightarrow}A/J\longrightarrow 0,$$

the composition

$$\mathcal{K}_1(A/J)\overset{\mathcal{D}^1_{J,A}}{\to}\mathcal{K}_0(J)\overset{\jmath_*}{\to}\mathcal{K}_0(A)$$

is (λ,h)-exact at $\mathcal{K}_0(J)$.

Proof. — It is enough to prove the result for A unital. Let y be an element of $K_0^{\varepsilon,r}(J)$ such that $\jmath_*^{\varepsilon,r}(y)=0$ in $K_0^{\varepsilon,r}(A)$, let e be an ε-r-projection in $M_n(J^+)$ and let k be a positive integer such that $y=[e,k]_{\varepsilon,r}$. If we set $p_k=\mathrm{diag}(I_k,0)$, we can indeed assume without loss of generality that $\|q(e)-p_k\|<2\varepsilon$ (where J^+ is viewed as a subalgebra of A). Up to stabilization, we can also assume that e is homotopic to p_k as an ε-r-projection in $M_n(A)$. According to Corollary 1.31, there exists up to stabilization a $\alpha_h\varepsilon$-$k_{h,\varepsilon}r$-unitary W of $M_n(A)$ such that

$$\|e-Wp_kW^*\|<\alpha_h\varepsilon.$$

Up to replace n by $2n$, W by $\operatorname{diag}(W,W^*)$ and e by $\operatorname{diag}(e,0)$, we can assume that W is homotopic to I_n as a $3\alpha_h\varepsilon$-$2k_{h,\varepsilon}r$-unitary. Since

$$\begin{aligned}\|q(W)p_kq(W^*)-p_k\| &\leqslant \|q(W)p_kq(W^*)-q(e)\|+\|q(e)-p_k\|\\ &< (2+\alpha_h)\varepsilon,\end{aligned}$$

then

$$\|q(W^*)p_kq(W)-p_k\|<(2+4\alpha_h)\varepsilon.$$

Hence for an $\alpha'>1$ depending only on α_h, the left-up $n\times n$ corner V_1 and the right bottom corner V_2 of $q(W)$ are $\alpha'\varepsilon$-$k_{e,\varepsilon}r$-unitaries of $M_n(A/J)$ such that

$$\|q(W)q(W^*)-\operatorname{diag}(V_1,V_2)\operatorname{diag}(V_1,V_2)^*\|<(\alpha_h+\alpha')\varepsilon$$

and

$$\|q(W^*)q(W)-\operatorname{diag}(V_1,V_2)^*\operatorname{diag}(V_1,V_2)\|<(\alpha_h+\alpha')\varepsilon.$$

Hence $q(W)$ is close to $\operatorname{diag}(V_1,V_2)$ and hence there is a $\lambda>1$ depending only on α_e such that as a $\lambda\varepsilon$-$k_{h,\varepsilon}r$-unitary of $M_n(A/J)$, then $\operatorname{diag}(V_1,V_2)$ is homotopic to $q(W)$ and hence to I_n. We can indeed choose λ big enough such that if we set $x=[V_1]_{\lambda\varepsilon,k_{e,\varepsilon}r}$, then

$$\begin{aligned}\partial_{J,A}^{\lambda\varepsilon,k_{e,\varepsilon}r}(x) &= [e,k]_{\lambda\alpha_\partial\varepsilon,k_{\partial,\alpha\varepsilon}k_{e,\varepsilon r}}\\ &= \iota_*^{\varepsilon,r,\lambda\varepsilon,k_{e,\varepsilon}r}(y).\end{aligned}$$

□

From Propositions 3.11, 3.12 and 3.13 we can derive the analogue of the long exact sequence in K-theory.

Theorem 3.14. — *There exists a control pair (λ,h) such that for any completely filtered extension of C^*-algebras*

$$0\longrightarrow J\stackrel{\jmath}{\longrightarrow}A\stackrel{q}{\longrightarrow}A/J\longrightarrow 0,$$

the sequence

$$\mathcal{K}_1(J)\stackrel{\jmath_*}{\longrightarrow}\mathcal{K}_1(A)\stackrel{q_*}{\longrightarrow}\mathcal{K}_1(A/J)\stackrel{\mathcal{D}_{J,A}}{\longrightarrow}\mathcal{K}_0(J)\stackrel{\jmath_*}{\longrightarrow}\mathcal{K}_0(A)\stackrel{q_*}{\longrightarrow}\mathcal{K}_0(A/J)$$

is (λ,h)-exact.

Remark 3.15. — With notation of Definition 3.1, the statement of the long exact sequence of Theorem 3.14 can be extended to the following situation: there exists a positive number C such that for any positive number r, any integer n and any x in $M_n(A_r)$, then

$$\inf_{y\in M_n(J_r)}\|x+y\|\leqslant C\inf_{y\in M_n(J)}\|x+y\|$$

i.e the bijective continuous linear map

$$M_n(A_r/J_r)\longrightarrow M_n((A_r+J)/J)$$

induced by the inclusion $A_r \hookrightarrow A$ has inverse bounded in operator norm by C. But in this case, the control pairs corresponding to the controlled boundary map and to controlled exactness depends on C.

As a consequence, using the exact sequence

$$0 \to SA \to CA \to A \to 0, \tag{3.4}$$

and in view of Lemma 1.26 and point (iii) of Remark 2.8, we deduce in the setting of quantitative K-theory the analogue of the suspension isomorphism in K-theory.

Corollary 3.16. — *Let $\mathcal{D}^1_A = \mathcal{D}^1_{SA,CA} : \mathcal{K}_1(A) \to \mathcal{K}_0(SA)$ be the controlled boundary morphism associated to the semi-split and filtered extension of equation (3.4) for a filtered C^*-algebra A.*

- *There exists a control pair (λ, h) such that for any filtered C^*-algebra A, then $\mathcal{D}^1_A$ is (λ, h)-invertible.*
- *Moreover, we can choose a (λ, h)-inverse which is natural: there exists a control pair (α_β, k_β) and for any filtered C^*-algebra A a (λ, h)-controlled morphism $\mathcal{B}^0_A = (\beta^{\varepsilon,r}_A)_{0<\varepsilon<\frac{1}{4\alpha_\beta},r>0} : \mathcal{K}_0(SA) \to \mathcal{K}_1(A)$ which is an (λ, h)-inverse for $\mathcal{D}^1_A$ and such that $\mathcal{B}^0_B \circ f_S = f \circ \mathcal{B}^0_A$ for any homomorphism $f : A \to B$ of filtered C^*-algebras, where $f_S : SA \to SB$ is the suspension of the homomorphism f.*

3.4. The mapping cones

We end this section by proving that the mapping cones construction can be performed in the framework of quantitative K-theory. Let

$$0 \to J \to A \overset{q}{\to} A/J \to 0$$

be a completely filtered extension of C^*-algebras. Let us set $A/J[0,1) = C_0([0,1), A/J)$ and define the mapping cone of q:

$$C_q = \{(x, f) \in A \oplus A/J[0,1);\ \text{such that } f(0) = q(x)\}.$$

It is straightforward to check that C_q is filtered by

$$(C_q \cap (A_r \oplus A/J[0,1)_r))_{r>0}\,.$$

Let us set

$$e_q : J \to C_q;\ x \mapsto (x, 0)$$

and

$$\phi_q : SA/J \to C_q;\ f \mapsto (0, f).$$

We have then a completely filtered extension of C^*-algebras

$$0 \to J \stackrel{e_j}{\to} C_q \stackrel{\pi_2}{\to} A/J[0,1) \to 0,$$

where π_2 is the projection on the second factor of $A \oplus A/J[0,1)$.

LEMMA 3.17. — *There exists a control pair (λ, h) such that $e_{q,*}$ is (λ, h)-invertible for any completely filtered extension of C^*-algebras $0 \to J \to A \stackrel{q}{\to} A/J \to 0$.*

Proof. — The even case is a consequence of Theorem 3.14. We deduce the odd case from the even one using Corollary 3.16. □

It is a standard fact in K-theory that the boundary of an extension of C^*-algebras $0 \to J \to A \stackrel{q}{\to} A/J \to 0$ can be obtain using the equality

$$e_{q,*} \circ \partial_{J,A} = \phi_{q,*} \circ \partial_{A/J},$$

where $\partial_{A/J} = \partial_{SA/J,CA/J}$ stands for the boundary map of the extension

$$0 \to SA/J \to CA/J \to A/J \to 0$$

(corresponding to the evaluation at 1). We have a similar result in quantitative K-theory:

LEMMA 3.18. — *With above notations, we have $e_{q,*} \circ \mathcal{D}_{J,A} = \phi_{q,*} \circ \mathcal{D}_{A/J}$, where $\mathcal{D}_{A/J}$ stands for $\mathcal{D}_{SA/J,CA/J}$.*

Proof. — We can assume without loss of generality that A is unital. Let p be an ε-r projection in $M_n(A/J)$ and let x be a self-adjoint lift for p in $M_n(A_r)$ such that $\|x\| \leqslant 2$. Using the notations of the proof of Lemma 3.5, let us define for t in $[0,1]$

- $y_t = ty_p + \sum_{l=1}^{k_\varepsilon} \dfrac{(2\imath\pi x)^l (t^l - t)}{l!}$ in A;
- $f_t : [0,1] \to A/J : \sigma \mapsto \sum_{l=1}^{k_\varepsilon} \dfrac{(2\imath\pi((1-\sigma)t+\sigma)p)^l - ((1-\sigma)t+\sigma)(2\imath\pi p)^l}{l!}$.

Since y_t is close to $\sum_{l=1}^{k_\varepsilon} \frac{(2\imath\pi t x)^l}{l!}$, then, $(1 + (y_t, f_t))_{t\in[0,1]}$ is a path of $\alpha\varepsilon$-$k_\varepsilon r$ unitary in $M_n(C_q^+)$ with $y_0 = 0$, $y_1 = y_p$ and $f_1 = 0$. Moreover, f_0 belongs to $M_n(SA/J)$ and satisfies the conclusion of Lemma 3.5 with respect to the semi-split extension of filtered C^*-algebras $0 \to SA/J \to CA/J \to A/J \to 0$ (corresponding to evaluation at 1) starting from the ε-r-projection p. Hence, following the construction of Proposition 3.7 in the even case, we obtain that $e_{q,*} \circ \mathcal{D}_{J,A}$ and $\phi_{q,*} \circ \mathcal{D}_{A/J}$ coincide on $\mathcal{K}_0(A/J)$.

Let us check now the odd case. Let u be an ε-r-unitary in $M_n(A/J)$. Pick any ε-r-unitary in some $M_j(A/J)$ such that $\operatorname{diag}(u,v)$ is homotopic to I_{n+j} in $\mathrm{U}^{3\varepsilon,2r}_{n+j}(A/J)$. According to Lemma 3.4, and up to replace v by $\operatorname{diag}(v,I_k)$ for some integer k, there exists an element w in $\mathrm{U}^{3\alpha_e\varepsilon,2k_{e,3\varepsilon}r}_{n+j}(A)$ homotopic to I_{n+j} as a $3\alpha_e\varepsilon$-$2k_{e,3\varepsilon}r$-unitary and such that $\|q(w)-\operatorname{diag}(u,v)\|\leqslant 3\alpha_e\varepsilon$. Let $(w_t)_{t\in[0,1]}$ be a path in $\mathrm{U}^{3\alpha_e\varepsilon,2k_{e,3\varepsilon}r}_{n+j}(A)$ with $w_0=I_{n+j}$ and $w_1=w$ and set $y_t=q(w_t)\operatorname{diag}(I_n,0)q(w_t^*)$. As in the proof of Proposition 3.7, we see that y_t is an element in $\mathrm{P}^{12\alpha_e\varepsilon,4k_{e,3\varepsilon}r}_{n+j}(A/J)$ such that $\|y_1-\operatorname{diag}(I_n,0)\|\leqslant 9\alpha_e\varepsilon$. Define

$$g:[0,1]\to M_{n+j}(A/J);\ t\mapsto y_t-\operatorname{diag}(I_n,0)-t(y_1-\operatorname{diag}(I_n,0)).$$

Then $g+\operatorname{diag}(I_n,0)$ is the element of $\mathrm{P}^{12\alpha_e\varepsilon,4k_{e,3\varepsilon}r}_{n+j}(S^+A/J)$ that we get from u and v when we perform the construction of Proposition 3.7 in the odd case with respect to the extension $0\to SA/J\to CA/J\to A/J\to 0$. Now, as in the proof of Proposition 3.7, let h be an element in $M_{n+j}(J_{4k_{e,3\varepsilon}r})$ such that

$$\|w\operatorname{diag}(I_n,0)w^*-h-\operatorname{diag}(I_n,0)\|<9\alpha_e\varepsilon$$

and define

$$h_t=w_t\operatorname{diag}(I_n,0)w_t^*-\operatorname{diag}(I_n,0)+t(h+\operatorname{diag}(I_n,0)-w\operatorname{diag}(I_n,0)w^*)$$

for t in $[0,1]$. Then $\operatorname{diag}(I_n,0)+h_t$ belongs to $\mathrm{P}^{12\alpha_e\varepsilon,4k_{e,3\varepsilon}r}_{n+j}(A)$ and $\operatorname{diag}(I_n,0)+h_1=diag(I_n,0)+h$ is the element of $\mathrm{P}^{12\alpha_e\varepsilon,4k_{e,3\varepsilon}r}_{n+j}(J)$ that we get from u and v when we perform the construction of Proposition 3.7 in the odd case with respect to the extension $0\to J\to A\stackrel{q}{\to} A/J\to 0$. Eventually, if we define

$$H_t:[0,1]\to M_{n+j}(A/J);\ \sigma\mapsto g_{(1-\sigma)t+\sigma},$$

then $((h_t,H_t)+\operatorname{diag}(I_n,0))_{t\in[0,1]}$ is a homotopy in $\mathrm{P}^{12\alpha_e\varepsilon,4k_{e,3\varepsilon}r}_{n+j}(C_q^+)$ between $((0,g)+\operatorname{diag}(I_n,0))$ and $((h,0)+\operatorname{diag}(I_n,0))$. Thus we obtain the result in the odd case. □

As a consequence, we get that the controlled suspension morphism is compatible with the controlled boundary maps.

Proposition 3.19. — *There exists a control pair (λ,h) such that for any completely filtered extension of C^*-algebras $0\to J\to A\to A/J\to 0$,* *the following diagrams are (λ, h)-commutative:*

$$\begin{array}{ccc} \mathcal{K}_0(A/J) & \xrightarrow{\mathcal{D}_{A/J}} & \mathcal{K}_1(SA/J) \\ {\scriptstyle \mathcal{D}_{J,A}}\big\downarrow & & \big\downarrow{\scriptstyle \mathcal{D}_{SJ,SA}} \\ \mathcal{K}_1(J) & \xrightarrow{\mathcal{D}_J} & \mathcal{K}_0(SJ) \end{array}$$

and

$$\begin{array}{ccc} \mathcal{K}_1(A/J) & \xrightarrow{\mathcal{D}_{A/J}} & \mathcal{K}_0(SA/J) \\ {\scriptstyle \mathcal{D}_{J,A}}\big\downarrow & & \big\downarrow{\scriptstyle \mathcal{D}_{SJ,SA}} \\ \mathcal{K}_0(J) & \xrightarrow{\mathcal{D}_J} & \mathcal{K}_1(SJ) \end{array}$$

where $\mathcal{D}_J$ and $\mathcal{D}_{A/J}$ stands respectively for the controlled suspension morphisms $\mathcal{D}_{SJ,CJ}$ and $\mathcal{D}_{SA/J,CA/J}$.

Proof. — Let $q_S : SA \to SA/J$ the suspension of the homomorphism $q : A \to A/J$. Applying Lemma 3.18 to the extensions $0 \to J \to A \to A/J \to 0$ and $0 \to SJ \to SA \to SA/J \to 0$ and using the naturality of controlled boundary maps mentioned in Remark 3.8, we get

$$\begin{aligned} e_{q_S,*} \circ \mathcal{D}_{SJ,SA} \circ \mathcal{D}_{A/J} &= \phi_{q_S,*} \circ \mathcal{D}_{SA/J} \circ \mathcal{D}_{A/J} \\ &= \mathcal{D}_{SC_q} \circ \phi_{q,*} \circ \mathcal{D}_{A/J} \\ &= \mathcal{D}_{SC_q} \circ e_{q,*} \circ \mathcal{D}_{J,A} \\ &= e_{q_S,*} \circ \mathcal{D}_J \circ \mathcal{D}_{J,A} \end{aligned}$$

The proposition is then a consequence of Lemma 3.17. □

Remark 3.20. — Proposition 3.19 extend to extensions that satisfy the assumptions of Remark 3.15, but with these notations, the control pairs involved in the proposition depend on the number C.

4. Controlled Bott periodicity

The aim of this section is to prove that there exists a control pair (λ, h) such that given a filtered C^*-algebra A, then Bott periodicity $K_0(A) \stackrel{\cong}{\to} K_0(S^2A)$ is induced in K-theory by a (λ, h)-isomorphism $\mathcal{K}_0(A) \to \mathcal{K}_0(S^2A)$. As an application, we use the controlled boundary morphism of Proposition 3.7 to close the controlled exact sequence of 3.14 into a six-term (λ, h)-exact sequence for some universal control pair (λ, h). This will be achieved by using the full power of KK-theory.

4.1. Tensorization in KK-theory

Let A be a C^*-algebra and let B be a C^*-algebra filtered by $(B_r)_{r>0}$. Let us define $A\otimes B_r$ as the closure in the spatial tensor product $A\otimes B$ of the algebraic tensor product of A and B_r. Then the C^*-algebra $A\otimes B$ is filtered by $(A\otimes B_r)_{r>0}$. Moreover, if J is a semi-split ideal of A, i.e $0 \to J \to A \to A/J \to 0$ is a semi-split extension of C^*algebras, then

$$0 \to J\otimes B \to A\otimes B \to A/J\otimes B \to 0$$

is a semi-split extension of filtered C^*-algebras. Recall from [11] that for C^*-algebras A_1, A_2 and D, G. Kasparov defined a tensorization map

$$\tau_D : KK_*(A_1, A_2) \to KK_*(A_1\otimes D, A_2\otimes D)$$

in the following way: let z be an element in $KK_*(A_1, A_2)$ represented by a K-cycle $(\pi, T, \mathcal{E})$, where

- $\mathcal{E}$ is a right A_2-Hilbert module;
- π is a representation of A_1 into the algebra $\mathcal{L}(\mathcal{E})$ of adjointable operators of $\mathcal{E}$;
- T is a self-adjoint operator on $\mathcal{E}$ satisfying the K-cycle conditions, i.e. $[T, \pi(a)]$, $\pi(a)(T^2 - \mathcal{I}d_{\mathcal{E}})$ are compact operators on $\mathcal{E}$ for any a in A_1.

Then $\tau_D(z) \in KK_*(A_1\otimes D, A_2\otimes D)$ is represented by the K-cycle $(\pi\otimes Id_D, T\otimes Id_D, \mathcal{E}\otimes D)$.

In what follows, we show that if A_1 and A_2 are C^*-algebras, if B is a filtered C^*-algebra and if z is an element in $KK_*(A_1, A_2)$, then the homomorphism $K_*(A_1\otimes B) \to K_*(A_2\otimes B)$ provided by left multiplication by $\tau_B(z)$ is induced by a controlled morphism. Moreover, we have some compatibility results with respect to Kasparov product. As an outcome, we obtain a controlled version of the Bott periodicity that induces in K-theory the Bott periodicity.

Proposition 4.1. — *Let A_1 and A_2 be C^*-algebras, let B be a filtered C^*-algebra and let z be an element in $KK_1(A_1, A_2)$. Then there exists an $(\alpha_{\mathcal{D}}, k_{\mathcal{D}})$-controlled morphism*

$$\mathcal{T}_B(z) = (\tau_B^{\varepsilon,r}(z))_{0<\varepsilon<\frac{1}{4\alpha_{\mathcal{D}}}, r>0} : \mathcal{K}_*(A_1\otimes B) \to \mathcal{K}_*(A_2\otimes B)$$

of degree 1 inducing in K-theory the right multiplication by $\tau_B(z)$.

Proof. — Recall that z can be indeed represented by a odd A_1-A_2-K-cycle $(\pi, T, \mathcal{H}\otimes A_2)$, where $\mathcal{H}$ is a separable Hilbert space, π is a representation of A_1 into the algebra $\mathcal{L}(\mathcal{H}\otimes A_2)$ of adjointable operators of $\mathcal{H}\otimes A_2$ and

T is a self-adjoint operator in $\mathcal{L}(\mathcal{H}\otimes A_2)$ satisfying the K-cycle conditions. Let us set $P_B = \frac{\mathcal{I}d_{\mathcal{H}\otimes A_2\otimes B}+T\otimes Id_B}{2}$, $\pi_B = \pi\otimes Id_B$ and define the C^*-algebra

$$E^{(\pi,T)} = \{(x,y) \in A_1\otimes B \bigoplus \mathcal{L}(\mathcal{H}\otimes A_2\otimes B) \\ \text{such that } P_B \cdot \pi_B(x) \cdot P_B - y \in \mathcal{K}(\mathcal{H}) \otimes A_2\otimes B\}.$$

Since P_B has no propagation, the C^*-algebra $E^{(\pi,T)}$ is filtered by $(E_r^{(\pi,T)})_{r>0}$ with

$$E_r^{(\pi,T)} = \{(x,\ P_B \cdot \pi_B(x) \cdot P_B + y);\ x \in A_1\otimes B_r \text{ and } y \in \mathcal{K}(\mathcal{H}) \otimes A_2\otimes B_r\}.$$

The extension of filtered C^*-algebras

$$0 \longrightarrow \mathcal{K}(\mathcal{H}) \otimes A_2\otimes B \longrightarrow E^{(\pi,T)} \longrightarrow A_1 \otimes B \longrightarrow 0 \tag{4.1}$$

is semi-split by the cross-section

$$s : A_1\otimes B \to E^{(\pi,T)};\ x \mapsto (x, P_B \cdot \pi_B(x) \cdot P_B).$$

Let us show that the associated controlled boundary (degree one) map

$$\mathcal{D}_{\mathcal{K}(\mathcal{H})\otimes A_2\otimes B,E^{(\pi,T)}} : \mathcal{K}_*(A_1\otimes B) \to \mathcal{K}_*(\mathcal{K}(\mathcal{H})\otimes A_2\otimes B)$$

only depends on the class z of $(\pi, T, \mathcal{H}\otimes A_2)$ in $KK_1(A_1, A_2)$. Assume that $(\pi, T, \mathcal{H}\otimes A_2[0,1])$ is a A_1-$A_2[0,1]$-K -cycle providing a homotopy between two A_1-A_2-K-cycles $(\pi_0, T_0, \mathcal{H}\otimes A_2)$ and $(\pi_1, T_1, \mathcal{H}\otimes A_2)$. For $t \in [0,1]$ we denote by

- $e_t : A_2[0,1] \to A_2$ the evaluation at t;
- $F_t \in \mathcal{L}(\mathcal{H}\otimes A_2)$ the fiber at t of an operator $F \in \mathcal{L}(\mathcal{H}\otimes A_2[0,1])$;
- $\pi_t : A_1 \to \mathcal{L}(\mathcal{H}\otimes A_2)$ the representation induced by π at the fiber t.

Then the homomorphism $E^{(\pi,T)} \to E^{(\pi_t,T_t)}$; $(x,y) \mapsto (x, y_t)$ satisfies the conditions of Remark 3.8 and thus we get that

$$(\mathcal{I}d_{\mathcal{K}(\mathcal{H})} \otimes e_t\otimes Id_B)_* \circ \mathcal{D}_{\mathcal{K}(\mathcal{H})\otimes A_1\otimes B[0,1],E^{(\pi,T)}} = \mathcal{D}_{\mathcal{K}(\mathcal{H})\otimes A_1\otimes B,E^{(\pi_t,T_t)}},$$

and according to Lemma 1.26, we deduce that

$$\mathcal{D}_{\mathcal{K}(\mathcal{H})\otimes A_1\otimes B_2,E^{(\pi_0,T_0)}} = \mathcal{D}_{\mathcal{K}(\mathcal{H})\otimes A_1\otimes B,E^{(\pi_1,T_1)}}.$$

This shows that for a A_1-A_2-K-cycle $(\pi, T, \mathcal{H}\otimes A_2)$, then $\mathcal{D}_{\mathcal{K}(\mathcal{H})\otimes A_1\otimes B,E^{(\pi,T)}}$ depends only on the class z of $(\pi, T, \mathcal{H}\otimes A_2)$ in $KK_1(A_1, A_2)$. Finally we define

$$\mathcal{T}_B(z) = (\tau_B^{\varepsilon,r}(z))_{0<\varepsilon<\frac{1}{4\alpha_{\mathcal{D}}}} \stackrel{\text{def}}{=\!=} \mathcal{M}^{-1}_{A_2\otimes B} \circ \mathcal{D}_{\mathcal{K}(\mathcal{H})\otimes A_1\otimes B,E^{(\pi,T)}},$$

where

- $(\pi, T, \mathcal{H}\otimes A_2)$ is any A_1-A_2-K-cycles representing z;

- $\mathcal{M}_{A_2\otimes B}$ is the Morita equivalence (see Example 2.2).

The result then follows from the observation that up to the Morita equivalence

$$K_*(\mathcal{K}(\mathcal{H})\otimes A_2\otimes B)\stackrel{\cong}{\to}K_*(A_2\otimes B),$$

the boundary $\partial_{\mathcal{K}(\mathcal{H})\otimes A_1\otimes B,E^{(\pi,T)}}$ corresponding to the exact sequence (4.1) is induced by right multiplication by $\tau_B(z)$. □

Remark 4.2. — Let B be a filtered C^*-algebra.

(i) For any C^*-algebras A_1 and A_2 and any elements z and z' in $KK_1(A_1,A_2)$ then

$$\mathcal{T}_B(z+z')=\mathcal{T}_B(z)+\mathcal{T}_B(z').$$

(ii) Let $0\to J\to A\to A/J\to 0$ be a semi-split extension of filtered C^*-algebras and let $[\partial_{J,A}]$ be the element of $KK_1(A/J,J)$ that implements the boundary map $\partial_{J,A}$. Then we have

$$\mathcal{T}_B([\partial_{J,A}])=\mathcal{D}_{J\otimes B,A\otimes B}.$$

(iii) For any C^*-algebras A_1, A_2 and D and any K-cycle $(\pi,T,\mathcal{H}\otimes A_2)$ for $KK_1(A_1,A_2)$, we have a natural identification between $E^{(\pi\otimes I_D,T\otimes I_D)}$ and $E^{(\pi,T)}\otimes D$. Hence, for any element z in $KK_1(A_1,A_2)$ then $\mathcal{T}_B(\tau_D(z))=\mathcal{T}_{B\otimes D}(z)$.

For a filtered C^*-algebra B and a homomorphism $f:A_1\to A_2$ of C^*-algebras, we set $f_B:A_1\otimes B\to A_2\otimes B$ for the filtered homomorphism induced by f.

Proposition 4.3. — *Let B be a filtered C^*-algebra and let A_1 and A_2 be two C^*-algebras.*

(i) For any C^*-algebra A_1', any homomorphism of C^*-algebras $f:A_1\to A_1'$ and any z in $KK_1(A_1',A_2)$, we have $\mathcal{T}_B(f^*(z))=\mathcal{T}_B(z)\circ f_{B,*}$;

(ii) For any C^*-algebra A_2', any homomorphism of C^*-algebras $g:A_2\to A_2'$ and any z in $KK_1(A_1,A_2)$, we have $\mathcal{T}_B(g_*(z))=g_{B,*}\circ\mathcal{T}_B(z)$.

Proof.

(i) Let A_1' be a filtered C^*-algebra, let $f:A_1\to A_1'$ be a homomorphism of C^*-algebras and let $(\pi,T,H\otimes A_2)$ be an odd A_1'-A_2-K-cycle. With the notations of the proof of Proposition 4.1, the homomorphism

$$f^E:E^{f^*(\pi,T)}\to E^{(\pi,T)};\ (x,y)\mapsto(f_B(x),y)$$

fits in the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{K}(\mathcal{H})\otimes A_2\otimes B & \longrightarrow & E^{f^*(\pi,T)} & \longrightarrow & A_1\otimes B & \longrightarrow & 0 \\ & & \Big\downarrow{=} & & \Big\downarrow{f^E} & & \Big\downarrow{f_B} & & \\ 0 & \longrightarrow & \mathcal{K}(\mathcal{H})\otimes A_2\otimes B & \longrightarrow & E^{(\pi,T)} & \longrightarrow & A'_1\otimes B & \longrightarrow & 0 \end{array}.$$

Thus, we get by Remark 3.8 that

$$\mathcal{T}_B(f^*(z)) = \mathcal{T}_B(z)\circ f_*$$

for all z in $KK_1(A'_1, A_2)$.

(ii) Let A'_2 be a C^*-algebra and let $g : A_2 \to A'_2$ be a homomorphism of C^*-algebras. For any element F in $\mathcal{L}(\mathcal{H}\otimes A_2)$, let us denote by

$$\tilde{F} = F\otimes_{A_2} Id_{A'_2} \in \mathcal{L}(\mathcal{H}\otimes A_2\otimes_{A_2} A'_2).$$

Notice that $\mathcal{H}\otimes A_2\otimes_{A_2} A'_2$ can be viewed as a right A'_2-Hilbert-submodule of $\mathcal{H}\otimes A'_2$ and under this identification, for any F in $\mathcal{K}(\mathcal{H})\otimes A_2$, then $\tilde{F}$ is the restriction to $\mathcal{H}\otimes A_2\otimes_{A_2} A'_2$ of the homomorphism $(Id_{\mathcal{K}(\mathcal{H})}\otimes g)(F)$. Let z be an element of $KK_1(A_1, A_2)$ represented by a K-cycle $(\pi, T, \mathcal{H}\otimes A_2)$. Consider the A_1-A_2-K-cycle $(\pi', T', \mathcal{H}'\otimes A_2)$ with $\mathcal{H}' = \mathcal{H}_1\oplus\mathcal{H}_2\oplus\mathcal{H}_3$, where $\mathcal{H}_1$, $\mathcal{H}_2$ and $\mathcal{H}_3$ are three copies of $\mathcal{H}$, $\pi' = 0\oplus 0\oplus\pi$ and $T' = Id_{\mathcal{H}_1\otimes A_2}\oplus Id_{\mathcal{H}_2\otimes A_2}\oplus T$. Then $(\pi', T', \mathcal{H}'\otimes A_2)$ is again a K-cycle representing z and $g_*(z)$ is represented by the K-cycle $(\pi'', T'', \mathcal{E})$, where

- $\mathcal{E} = (\mathcal{H}_1\otimes A'_2)\bigoplus(\mathcal{H}_2\otimes A'_2)\bigoplus(\mathcal{H}_3\otimes A_2\otimes_{A_2} A'_2)$;
- $\pi'' = 0\oplus 0\oplus\tilde{\pi}$;
- $T'' = \mathcal{I}d_{\mathcal{H}_1\otimes A'_2}\oplus\mathcal{I}d_{\mathcal{H}_2\otimes A'_2}\oplus\tilde{T}$.

Using Kasparov stabilization theorem, we get that $(\mathcal{H}_2\otimes A'_2)\bigoplus(\mathcal{H}_3\otimes A_2\otimes_{A_2} A'_2)$ is isomorphic as a right-A'_2-Hilbert module to $\mathcal{H}\otimes A'_2$ and hence, using this identification, we can represent $g_*(z)$ using a standard right-A'_2-Hilbert module, as in the proof of Proposition 4.1. Then, under the above identification $(\mathcal{H}_2\otimes A'_2)\bigoplus(\mathcal{H}_3\otimes A_2\otimes_{A_2} A'_2)\cong\mathcal{H}\otimes A'_2$,

$$\begin{array}{rcl} g_E : E^{(\pi,T)} & \to & E^{g_*(\pi,T)} \\ (x,y) & \mapsto & (x, P''_B\pi''(x)P''_B + (Id_{\mathcal{K}(\mathcal{H}')\otimes B}\otimes g)(y - P'_B\pi'(x)P'_B)) \end{array}$$

restricts to a homomorphism $\mathcal{K}(\mathcal{H}_1\oplus\mathcal{H}_2\oplus\mathcal{H}_3)\otimes A_2\otimes B\to\mathcal{K}(\mathcal{H}_1\oplus\mathcal{H})\otimes A'_2\otimes B$.

We get now a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{K}(\mathcal{H}_1\oplus\mathcal{H}_2\oplus\mathcal{H}_3)\otimes A_2\otimes B & \longrightarrow & E^{(\pi',T')} & \longrightarrow & A_1\otimes B & \longrightarrow & 0\\ & & \Big\downarrow{\scriptstyle g_E} & & \Big\downarrow{\scriptstyle g_E} & & \Big\downarrow{\scriptstyle =} & & \\ 0 & \longrightarrow & \mathcal{K}(\mathcal{H}_1\oplus\mathcal{H})\otimes A_2'\otimes B & \longrightarrow & E^{(\pi'',T'')} & \longrightarrow & A_1\otimes B & \longrightarrow & 0 \end{array}$$

Hence, we get by Remark 3.8 that

$$\mathcal{D}_{\mathcal{K}(\mathcal{H}_1\oplus\mathcal{H})\otimes A_2'\otimes B,E^{(\pi'',T'')}}=g_{E,*}\circ\mathcal{D}_{\mathcal{K}(\mathcal{H}_1\oplus\mathcal{H}_2\oplus\mathcal{H}_3)\otimes A_2\otimes B,E^{(\pi',T')}}.$$

But the restriction of g_E to the corner $\mathcal{K}(\mathcal{H}_1)\otimes A_2\otimes B$ of the C^*-algebra $\mathcal{K}(\mathcal{H}_1\oplus\mathcal{H}_2\oplus\mathcal{H}_3)\otimes A_2\otimes B$ is $Id_{\mathcal{K}(\mathcal{H}_1)}\otimes g\otimes Id_B$. Since the Morita equivalence

$$\mathcal{M}_{A_2'\otimes B}:\mathcal{K}_*(A_2'\otimes B)\stackrel{\cong}{\to}\mathcal{K}_*(\mathcal{K}(\mathcal{H}_1\oplus\mathcal{H})\otimes A_2'\otimes B)$$

can be implemented by an inclusion of $A_2'\otimes B$ in a corner of $\mathcal{K}(\mathcal{H}_1)\otimes A_2'\otimes B$, and similarly for the Morita equivalence

$$\mathcal{M}_{A_2\otimes B}:\mathcal{K}_*(A_2\otimes B)\stackrel{\cong}{\to}\mathcal{K}_*(\mathcal{K}(\mathcal{H}_1\oplus\mathcal{H}_2\oplus\mathcal{H}_3)\otimes A_2\otimes B),$$

we deduce that the two following compositions coincide:

$$\mathcal{K}_*(A_2\otimes B))\stackrel{g_{B,*}}{\longrightarrow}\mathcal{K}_*(A_2'\otimes B)\stackrel{\mathcal{M}_{A_2'\otimes B}}{\longrightarrow}\mathcal{K}_*(\mathcal{K}(\mathcal{H}_1\oplus\mathcal{H})\otimes(A_2'\otimes B))$$

and

$$\mathcal{K}_*(A_2\otimes B)\stackrel{\mathcal{M}_{A_2\otimes B}}{\longrightarrow}\mathcal{K}_*(\mathcal{K}(\mathcal{H}_1\oplus\mathcal{H}_2\oplus\mathcal{H}_3)\otimes A_2\otimes B)$$
$$\stackrel{g_{E,*}}{\longrightarrow}\mathcal{K}_*(\mathcal{K}(\mathcal{H}_1\oplus\mathcal{H})\otimes A_2'\otimes B).$$

Hence we get

$$\mathcal{T}_B(g_*(z))=g_*\circ\mathcal{T}_B(z)$$

for any z in $KK_1(A_1,A_2)$.

□

Let us now extend the definition of $\mathcal{T}_B$ to the even case. Consider for a suitable control pair $(\alpha_{\mathcal{B}},k_{\mathcal{B}})$ and any filtered C^*-algebra A the $(\alpha_{\mathcal{B}},k_{\mathcal{B}})$-controlled morphism of odd degree $\mathcal{B}_A:\mathcal{K}_*(SA)\to\mathcal{K}_*(A)$ defined by

- $\mathcal{B}_A^0$ on $\mathcal{K}_0(SA)$ as in Corollary 3.16;
- $\mathcal{M}_A^{-1}\circ\mathcal{D}_{\mathcal{K}(\ell^2(\mathbb{N}))\otimes A,\mathcal{T}_0\otimes A}$ on $\mathcal{K}_1(SA)$ using the Toeplitz extension

$$0\to\mathcal{K}(\ell^2(\mathbb{N}))\otimes A\to\mathcal{T}_0\otimes A\to SA\to 0$$

(see the discussion at the end of Section 3.2).

Then, according to Proposition 3.10 and Corollary 3.16 there exists a control pair (λ, h) such that $\mathcal{B}_A$ is a right (λ, h)-inverse for $\mathcal{D}_{SA,CA}$ for any filtered C^*-algebra A. Let us set $\alpha_\mathcal{T} = \lambda\alpha_\mathcal{B}$ and $k_\mathcal{T} = h * k_\mathcal{B}$.

Now, let B be a filtered C^*-algebra, let A_1 and A_2 be C^*-algebras, then define for any z in $KK_0(A_1, A_2)$ the $(\alpha_\mathcal{T}, k_\mathcal{T})$-controlled morphism

$$\mathcal{T}_B(z) = (\tau_B^{\varepsilon,r})_{0<\varepsilon<\frac{1}{4\alpha_\mathcal{T}},r>0} : \mathcal{K}_*(A_1 \otimes B) \to \mathcal{K}_*(A_2 \otimes B)$$

by

$$\mathcal{T}_B(z) \stackrel{\text{def}}{=\!=} \mathcal{B}_{A_2\otimes B} \circ \mathcal{T}_B(z \otimes_{A_2} [\partial_{A_2}])$$

where

- $[\partial_{A_2}] = [\partial_{SA_2,CA_2}] \in KK_1(A_2, SA_2)$ corresponds to the boundary of the exact sequence $0 \to SA_2 \to CA_2 \to A \to 0$;
- $\otimes_{A_2}$ stands for Kasparov product.

Up to compose on the left with $\iota_*^{\alpha_\mathcal{D}\varepsilon,\alpha_\mathcal{T}\varepsilon,k_\mathcal{D}r,k_\mathcal{T}r}$, we can in the odd case define $\mathcal{T}_B(\bullet)$ also as an $(\alpha_\mathcal{T}, k_\mathcal{T})$-controlled morphism.

Theorem 4.4. — *Let B be a filtered C^*-algebra, let A_1 and A_2 be C^*-algebras*

(i) For any element z in $KK_*(A_1, A_2)$, then $\mathcal{T}_B(z) : \mathcal{K}_*(A_1 \otimes B) \to \mathcal{K}_*(A_2 \otimes B)$ is a $(\alpha_\mathcal{T}, k_\mathcal{T})$-controlled morphism with same degree as z that induces in K-theory right multiplication by $\tau_B(z)$.

(ii) For any elements z and z' in $KK_*(A_1, A_2)$ then

$$\mathcal{T}_B(z + z') = \mathcal{T}_B(z) + \mathcal{T}_B(z').$$

(iii) Let A_1' be a filtered C^*-algebras and let $f : A_1 \to A_1'$ be a homomorphism of C^*-algebras, then $\mathcal{T}_B(f^*(z)) = \mathcal{T}_B(z) \circ f_{B,*}$ for all z in $KK_*(A_1', A_2)$.

(iv) Let A_2' be a C^*-algebra and let $g : A_2' \to A_2$ be a homomorphism of C^*-algebras then $\mathcal{T}_B(g_*(z)) = g_{B,*} \circ \mathcal{T}_B(z)$ for any z in $KK_*(A_1, A_2')$.

(v) $\mathcal{T}_B([Id_{A_1}]) \stackrel{(\alpha_\mathcal{T},k_\mathcal{T})}{\sim} \mathcal{I}d_{\mathcal{K}_*(A_1\otimes B)}$.

(vi) For any C^*-algebra D and any element z in $KK_*(A_1, A_2)$, we have $\mathcal{T}_B(\tau_D(z)) = \mathcal{T}_{B\otimes D}(z)$.

Proof. — Since $\mathcal{B}_{A_2\otimes B}$ is a right (λ, h)-inverse for $\mathcal{D}_{SA_2\otimes B,CA_2\otimes B}$, it induces in K-theory a right inverse (indeed an inverse) for the (degree 1) boundary map

$$\partial_{SA_2\otimes B,CA_2\otimes B} : K_*(A_2 \otimes B) \to K_*(SA_2 \otimes B).$$

But since $\mathcal{T}_B(z\otimes_{A_2}[\partial_{SA_2\,CA_2}])$ induces in K-theory right multiplication by $\tau_B(z\otimes_{A_2}[\partial_{SA_2,CA_2}])$, we eventually get that $\mathcal{T}_B(z\otimes_{A_2}[\partial_{SA_2,CA_2}])$ induced in K-theory the composition

$$K_*(A_1\otimes B) \stackrel{\otimes_{A_1\otimes B}\tau_B(z)}{\longrightarrow} K_*(A_2\otimes B) \stackrel{\partial_{SA_2\otimes B,CA_2\otimes B}}{\longrightarrow} K_*(SA_2\otimes B)$$

and hence we get the first point.

Point (ii) is a consequence of Remark 4.2. Point (iii) is a consequence of Proposition 4.3. Point (iv) is a consequence of Proposition 4.3 and of the naturality of $\mathcal{B}_\bullet$ (see Remark 3.8 and Corollary 3.16), point (v) holds by definition of $\mathcal{B}_\bullet$. Point (vi) is a consequence of point (iii) of Remark 4.2. □

We end this section by proving the compatibility of $\mathcal{T}_B$ with Kasparov product.

Theorem 4.5. — *There exists a control pair (λ, h) such that the following holds :*

let A_1, A_2 and A_3 be C^*-algebras and let B be a filtered C^*-algebra. Then for any z in $KK_*(A_1, A_2)$ and any z' in $KK_*(A_2, A_3)$, we have

$$\mathcal{T}_B(z\otimes_{A_2}z') \stackrel{(\lambda,h)}{\sim} \mathcal{T}_B(z')\circ\mathcal{T}_B(z).$$

Proof. — We first deal with the case z even. According to [12, Lemma 1.6.9], there exists a C^*-algebra A_4 and homomorphisms $\theta : A_4 \to A_1$ and $\eta : A_4 \to A_2$ such that

- the element $[\theta]$ of $KK_*(A_4, A_1)$ induced by θ is invertible.
- $z = \eta_*([\theta]^{-1})$.

Since $\theta_*([\theta]^{-1}) = [Id_{A_1}]$ in $KK_*(A_1, A_1)$, we get in view of Remark 2.5 and of points (iii), (iv) and (v) of Theorem 4.4 that

$$\mathcal{T}_B(z\otimes_{A_2}z') \stackrel{(\lambda,h)}{\sim} \mathcal{T}_B(\theta^*(z\otimes_{A_2}z'))\circ\mathcal{T}_B([\theta]^{-1}),$$

with $(\lambda, h) = (\alpha_{\mathcal{T}}^2, k_{\mathcal{T}} * k_{\mathcal{T}})$. But by bi-functoriality of KK-theory, we have $\theta^*(z\otimes_{A_2}z') = \eta^*(z')$ and then the result is a consequence of points (iii) and (iv) of Theorem 4.4. We can proceed similarly when z' is even. Let us prove now the result when z and z' are odd. Then $[\partial_{A_2}] = [\partial_{SA_2,CA_2}]$ is an invertible element in $KK_1(A_2, SA_2)$ and $z\otimes_{A_2}z' = z\otimes_{A_2}[\partial_{A_2}]\otimes_{SA_2}[\partial_{A_2}]^{-1}\otimes_{A_2}z'$ and hence using the even case, we get that

$$\mathcal{T}_B(z\otimes_{A_2}z') \stackrel{(\lambda,h)}{\sim} \mathcal{T}_B([\partial_{A_2}]^{-1}\otimes_{A_2}z')\circ\mathcal{T}_B(z\otimes_{A_2}[\partial_{A_2}]). \tag{4.2}$$

But

$$\begin{aligned} \mathcal{T}_B([\partial_{A_2}]^{-1}\otimes_{A_2}z') &= \mathcal{B}_{A_3\otimes B}\circ\mathcal{T}_B([\partial_{A_2}]^{-1}\otimes_{A_2}z'\otimes_{A_3}[\partial_{A_3}]) \\ &\stackrel{(\lambda',h')}{\sim} \mathcal{B}_{A_3\otimes B}\circ\mathcal{T}_B(z'\otimes_{A_3}[\partial_{A_3}])\circ\mathcal{T}_B([\partial_{A_2}]^{-1}) \end{aligned} \tag{4.3}$$

for some control pair (λ',h'), depending only on (λ,h) and $(\alpha_{\mathcal{T}},k_{\mathcal{T}})$, where equation (4.3) holds by the even case applied to $z'\otimes_{A_3}[\partial_{A_3}]$ and $[\partial_{A_2}]^{-1}$. Hence, for a control pair (λ'',h'')-depending only on (λ,h), we get applying the even case to $[\partial_{A_2}]^{-1}$ and $z\otimes_{A_2}[\partial_{A_2}]$ that

$$\mathcal{T}_B(z\otimes_{A_2}z') \stackrel{(\lambda'',h'')}{\sim} \mathcal{B}_{A_3\otimes B}\circ\mathcal{T}_B(z'\otimes_{A_3}[\partial_{A_3}])\circ\mathcal{T}_B(z). \tag{4.4}$$

In view of this equation, we deduce the odd case from the controlled Bott periodicity, which will be proved in the next lemma: if we set $[\partial]=[\partial_{C_0(0,1),C_0(0,1]}]\in KK_1(\mathbb{C},C_0(0,1))$, then there exists a control pair (α,k) such that $\mathcal{T}_A([\partial]^{-1})$ is an (α,k)-inverse for $\mathcal{D}_A$ for any filtered C^*-algebra A. Indeed, from this claim and since for some control pair (α',k'), the $(\alpha_{\mathcal{B}},k_{\mathcal{B}})$-controlled morphism $\mathcal{B}_A$ is for every filtered C^*-algebra A a right (α',k')-inverse for $\mathcal{T}_A([\partial])$, we get that

$$\mathcal{T}_A([\partial]^{-1}) \stackrel{(\alpha'',k'')}{\sim} \mathcal{B}_A$$

for some controlled pair (α'',k'') depending only on (α',k') and $(\alpha_{\mathcal{T}},k_{\mathcal{T}})$. Noticing by using point (vi) of Theorem 4.4, that $\mathcal{T}_{A_3\otimes B}([\partial]^{-1})=\mathcal{T}_B([\partial_{A_3}]^{-1})$, the proof of the theorem in the odd case is then by equation (4.4) a consequence of the even case applied to $[\partial_{A_3}]^{-1}$ and $z'\otimes_{A_3}[\partial_{A_3}]$ □

4.2. The controlled Bott isomorphism

We prove in this subsection a controlled version of Bott periodicity. The proof use the even case of Theorem 4.5 and is needed for the proof of the odd case. Let A be a filtered C^*-algebra, let us denote for short as before $\mathcal{D}_{SA,CA}$ by $\mathcal{D}_A$ and $[\partial_{SA,CA}]$ by $[\partial_A]$ and let us set $[\partial]=[\partial_{\mathbb{C}}]$.

Lemma 4.6. — *There exists a control pair (α,k) such that for every filtered C^*-algebra A, then $\mathcal{T}_A([\partial]^{-1})$ is an (α,k)-inverse for $\mathcal{D}_A$.*

Proof. — Consider the even element $z=[\partial]\otimes_S[\partial_S]$ of $KK_*(\mathbb{C},S^2)$, where $S=C_0(0,1)$ and $S^2=SS$. The lemma is a consequence of the following claim: there exists a control pair (λ,h) such that $\mathcal{D}_{SA}\circ\mathcal{D}_A \stackrel{(\lambda,h)}{\sim} \mathcal{T}_A(z)$ for any C^*-algebra A. Before proving the claim, let us see how it implies the lemma. Notice first that by point (ii) of Remark 4.2, we have $\mathcal{D}_A=\mathcal{T}_A([\partial])$. Since by associativity of Kasparov product $[\partial]^{-1}\otimes_{\mathbb{C}}z=[\partial_S]$, we get from Theorem 4.5 applied to the even case that there exists a control pair (λ',h') such that for any filtered C^*-algebra A, then $\mathcal{T}_A(z)\circ\mathcal{T}_A([\partial]^{-1})\circ\mathcal{D}_A \stackrel{(\lambda,h)}{\sim} \mathcal{D}_{SA}\circ\mathcal{D}_A$. Using the claim and since z is an invertible element of $KK_*(\mathbb{C}, S^2)$, we obtain from Theorem 4.5 applied to the even case that there exists a control pair (α, k) such that $\mathcal{T}_A([\partial]^{-1})$ is a left (α, k)-inverse for $\mathcal{D}_A$. Using associativity of the Kasparov product, we see that $[\partial] = z \otimes_{S^2} [\partial_S]^{-1}$. Then applying twice Theorem 4.5, on one hand to $[\partial] = z \otimes_{S^2} [\partial_S]^{-1}$ and on the other hand to $[\partial]^{-1} \otimes z = [\partial_S]$, we get that there exists a control pair (α', k') such that $\mathcal{T}_A([\partial]) \circ \mathcal{T}_A([\partial]^{-1}) \stackrel{(\alpha',k')}{\sim} \mathcal{T}_{SA}([\partial]^{-1}) \circ \mathcal{T}_{SA}([\partial])$. But according to what we have seen before, $\mathcal{T}_{SA}([\partial]^{-1}) \circ \mathcal{T}_{SA}([\partial]) \stackrel{(\alpha,k)}{\sim} \mathcal{I}d_{\mathcal{K}_*(SA)}$.

Let us now prove the claim. It is known that up to Morita equivalence, $[\partial_A]^{-1}$ is the element of $KK_1(SA, A)$ corresponding to the boundary element of the Toeplitz extension

$$0 \to \mathcal{K}(\ell^2(\mathbb{N})) \otimes A \to \mathcal{T}_0 \otimes A \to SA \to 0.$$

Let us respectively denote by $\mathcal{D}_A^0 : \mathcal{K}_0(A) \to \mathcal{K}_1(SA)$ and $\mathcal{D}_A^1 : \mathcal{K}_1(A) \to \mathcal{K}_0(SA)$ the restriction of $\mathcal{D}_A$ to $\mathcal{K}_0(A)$ and $\mathcal{K}_1(A)$. According to Proposition 3.10, there exists a control pair (λ', h') such that, on even elements

$$\mathcal{T}_A([\partial]^{-1}) \circ \mathcal{D}_A^0 \stackrel{(\lambda',h')}{\sim} \mathcal{I}d_{\mathcal{K}_0(A)}. \tag{4.5}$$

Since $[\partial_S] = [\partial]^{-1} \otimes z$, we get by left composition by $\mathcal{T}_A(z)$ in equation (4.5) and by using Theorem 4.5 in the even case that there exists a control pair (λ, h) depending only on (λ', h') and such that that $\mathcal{D}_{SA}^1 \circ \mathcal{D}_A^0 \stackrel{(\lambda,h)}{\sim} \mathcal{T}_A^0(z)$ (here $\mathcal{T}_A^0(z) : \mathcal{K}_0(A) \to \mathcal{K}_0(S^2 A)$ stands for the restriction of $\mathcal{T}_A(z)$ to $\mathcal{K}_0(A)$). For the odd case, we know from Corollary 3.16 that there exists a control pair (λ'', h'') such that $\mathcal{D}_{S^2A}^1 : \mathcal{K}_1(S^2A) \to \mathcal{K}_0(S^3A)$ is (λ'', h'')-invertible. Using the previous case, and since by associativity of the Kasparov product, we have $[\partial_A] \otimes_{SA} \tau_{SA}(z) = \tau_A(z) \otimes [\partial_{S^2A}]$, we get by applying twice Theorem 4.5 in the even case that there exists a control pair (λ''', h''') such that $\mathcal{D}_{S^2A}^1 \circ \mathcal{D}_{SA}^0 \circ \mathcal{D}_A^1 \stackrel{(\lambda''',h''')}{\sim} \mathcal{D}_{S^2A}^1 \circ \mathcal{T}_A^1(z)$, where $\mathcal{T}_A^1(z) : \mathcal{K}_1(A) \to \mathcal{K}_1(S^2A)$ is the restriction of $\mathcal{T}_A(z)$ to $\mathcal{K}_1(A)$. Since $\mathcal{D}_{S^2A}^1 : \mathcal{K}_1(S^2A) \to \mathcal{K}_0(S^3A)$ is (λ'', h'')-invertible, we get the result by Remark 2.5. □

4.3. The six term (λ, h)-exact sequence

Recall from Proposition 3.19 that there exists a control pair (λ, h) such that for any completely filtered extension of C^*-algebras $0 \to J \to A \to$

$A/J \to 0$, the following diagrams are (λ, h)-commutative:

$$\begin{array}{ccc} \mathcal{K}_0(A/J) & \xrightarrow{\mathcal{D}_{A/J}} & \mathcal{K}_1(SA/J) \\ {\scriptstyle \mathcal{D}_{J,A}}\downarrow & & \downarrow{\scriptstyle \mathcal{D}_{SJ,SA}} \\ \mathcal{K}_1(J) & \xrightarrow{\mathcal{D}_J} & \mathcal{K}_0(SJ) \end{array}$$

and

$$\begin{array}{ccc} \mathcal{K}_1(A/J) & \xrightarrow{\mathcal{D}_{A/J}} & \mathcal{K}_0(SA/J) \\ {\scriptstyle \mathcal{D}_{J,A}}\downarrow & & \downarrow{\scriptstyle \mathcal{D}_{SJ,SA}} \\ \mathcal{K}_0(J) & \xrightarrow{\mathcal{D}_J} & \mathcal{K}_1(SJ) \end{array}$$

As a consequence, by using Lemma 4.6 and Theorem 3.14, we get

THEOREM 4.7. — *There exists a control pair (λ, h) such that for any completely filtered extension of C^*-algebras*

$$0 \longrightarrow J \xrightarrow{\jmath} A \xrightarrow{q} A/J \longrightarrow 0,$$

the following six-term sequence is (λ, h)-exact

$$\begin{array}{ccccc} \mathcal{K}_0(J) & \xrightarrow{\jmath_*} & \mathcal{K}_0(A) & \xrightarrow{q_*} & \mathcal{K}_0(A/J) \\ {\scriptstyle \mathcal{D}_{J,A}}\uparrow & & & & \downarrow{\scriptstyle \mathcal{D}_{J,A}} \\ \mathcal{K}_1(A/J) & \xleftarrow{q_*} & \mathcal{K}_1(A) & \xleftarrow{\jmath_*} & \mathcal{K}_1(J) \end{array}$$

Remark 4.8. — Let us consider with notations of Section 3.4 the completely filtered extension of C^*-algebras

$$0 \to SA/J \xrightarrow{\phi_q} C_q \xrightarrow{\pi_1} A \to 0, \tag{4.6}$$

where $\pi_1 : C_q \to A$ is the projection on the first factor of C_q. Since we have a completely filtered extension of algebras $0 \to J \xrightarrow{e_j} C_q \xrightarrow{\pi_2} A/J[0,1) \to 0$, and since $A/J[0,1)$ is a contractible filtered C^*-algebra, we see in view of Theorem 4.7 that $e_{j,*} : \mathcal{K}_*(J) \to \mathcal{K}_*(C_q)$ is a controlled isomorphism. It is then plain to check that up to the controlled isomorphism $e_{j,*}$ and $\mathcal{D}_{A/J} : \mathcal{K}_*(SA/J) \to \mathcal{K}_*(A/J)$, we get from the completely filtered extension of C^*-algebras of equation (4.6) (for a possibly different control pair) the controlled six-term exact sequence of Theorem 4.7.

(i) The controlled six-term exact sequence extend to extensions that satisfy the assumptions of Remark 3.15, but with these notations, the control pairs involved in the proposition depend on the number C.

If we apply Theorem 4.7 to a filtered and split extension, we get:

Corollary 4.9. — *There exists a control pair (λ, h) such that for every split extension of filtered C^*-algebra $0 \to J \to A \to A/J \to 0$, and any filtered split cross-section $s : A/J \to A$, then*

$$\mathcal{K}_*(J) \oplus \mathcal{K}_*(A/J) \longrightarrow \mathcal{K}_*(A);\ (x, y) \mapsto \jmath_*(x) + s_*(y)$$

is (λ, h)-invertible.

5. Quantitative K-theory for crossed product C^*-algebras

In this section, we study quantitative K-theory for crossed product C^*-algebras and discuss its applications to K-amenability.

Let Γ be a finitely generated group. A Γ-C^*-algebra is a separable C^*-algebra equipped with an action of Γ by automorphisms. Recall that the convolution algebra $C_c(\Gamma, A)$ of finitely supported A-valued functions on Γ admits two canonical C^*-completions, the reduced crossed product $A \rtimes_{red} \Gamma$ and the maximal crossed product $A \rtimes_{max} \Gamma$. Moreover, there is a canonical epimorphism $\lambda_{\Gamma,A} : A \rtimes_{max} \Gamma \to A \rtimes_{red} \Gamma$ which is the identity on $C_c(\Gamma, A)$.

5.1. Lengths and propagation

Recall that a length on Γ is a map $\ell : \Gamma \to \mathbb{R}^+$ such that

- $\ell(\gamma) = 0$ if and only if γ is the identity element e of Γ;
- $\ell(\gamma\gamma') \leqslant \ell(\gamma) + \ell(\gamma')$ for all element γ and γ' of Γ.
- $\ell(\gamma) = \ell(\gamma^{-1})$.

In what follows, we will assume that ℓ is a word length arising from a finite generating symmetric set S, i.e $\ell(\gamma) = \inf\{d \text{ such that } \gamma = \gamma_1 \cdots \gamma_d \text{ with } \gamma_1, \ldots, \gamma_d \text{ in } S\}$. Let us denote by $B(e, r)$ the ball centered at the neutral element of Γ with radius r, i.e $B(e, r) = \{\gamma \in \Gamma \text{ such that } \ell(\gamma) \leqslant r\}$. For any positive number r, we set

$$(A \rtimes_{red} \Gamma)_r \stackrel{\text{def}}{=\!=} \{f \in C_c(\Gamma, A) \text{ with support in } B(e, r)\}.$$

Then the C^*-algebra $A \rtimes_{red} \Gamma$ is filtered by $((A \rtimes_{red} \Gamma)_r)_{r>0}$. In the same way, setting $(A \rtimes_{max} \Gamma)_r \stackrel{\text{def}}{=\!=} \{f \in C_c(\Gamma, A) \text{ with support in } B(e, r)\}$, then the C^*-algebra $A \rtimes_{max} \Gamma$ is filtered by $((A \rtimes_{max} \Gamma)_r)_{r>0}$ (notice that as sets, $(A \rtimes_{red} \Gamma)_r = (A \rtimes_{max} \Gamma)_r$). It is straightforward to check that two word lengths give rise for $A\rtimes_{red}\Gamma$ (resp. for $A\rtimes_{max}\Gamma$) to quantitative K-theories related by a $(1,c)$-controlled isomorphism for a constant c.

For a homomorphism $f : A \to B$ of Γ-C^*-algebras, we denote respectively by $f_{\Gamma,red} : A\rtimes_{red}\Gamma \to B\rtimes_{red}\Gamma$ and $f_{\Gamma,max} : A\rtimes_{max}\Gamma \to B\rtimes_{max}\Gamma$ the homomorphisms respectively induced by f on the reduced and on the maximal crossed product.

For any semi-split extension of Γ-C^*-algebras $0 \longrightarrow J \stackrel{\jmath}{\longrightarrow} A \stackrel{q}{\longrightarrow} A/J \longrightarrow 0$, we have semi-split extensions of filtered C^*-algebras

$$0 \longrightarrow J\rtimes_{red}\Gamma \stackrel{\jmath_{\Gamma,red}}{\longrightarrow} A\rtimes_{red}\Gamma \stackrel{q_{\Gamma,red}}{\longrightarrow} A/J\rtimes_{red}\Gamma \longrightarrow 0$$

and

$$0 \longrightarrow J\rtimes_{max}\Gamma \stackrel{\jmath_{\Gamma,max}}{\longrightarrow} A\rtimes_{max}\Gamma \stackrel{q_{\Gamma,max}}{\longrightarrow} A/J\rtimes_{max}\Gamma \longrightarrow 0$$

and hence, by Theorem 4.7, we get:

Proposition 5.1. — *There exists a control pair (λ,h) such that for any semi-split extension of Γ-C^*-algebras*

$$0 \longrightarrow J \stackrel{\jmath}{\longrightarrow} A \stackrel{q}{\longrightarrow} A/J \longrightarrow 0,$$

the following six-term sequences are (λ,h)-exact

$$\begin{array}{ccccc} \mathcal{K}_0(J\rtimes_{red}\Gamma) & \xrightarrow{\jmath_{\Gamma,red,*}} & \mathcal{K}_0(A\rtimes_{red}\Gamma) & \xrightarrow{q_{\Gamma,red,*}} & \mathcal{K}_0(A/J\rtimes_{red}\Gamma) \\ {\scriptstyle \mathcal{D}_{J\rtimes_{red}\Gamma,A\rtimes_{red}\Gamma}}\uparrow & & & & \downarrow{\scriptstyle \mathcal{D}_{J\rtimes_{red}\Gamma,A\rtimes_{red}\Gamma}} \\ \mathcal{K}_1(A/J\rtimes_{red}\Gamma) & \xleftarrow{q_{\Gamma,red,*}} & \mathcal{K}_1(A\rtimes_{red}\Gamma) & \xleftarrow{\jmath_{\Gamma,red,*}} & \mathcal{K}_1(J\rtimes_{red}\Gamma) \end{array}$$

and

$$\begin{array}{ccccc} \mathcal{K}_0(J\rtimes_{max}\Gamma) & \xrightarrow{\jmath_{\Gamma,max,*}} & \mathcal{K}_0(A\rtimes_{max}\Gamma) & \xrightarrow{q_{\Gamma,max,*}} & \mathcal{K}_0(A/J\rtimes_{max}\Gamma) \\ {\scriptstyle \mathcal{D}_{J\rtimes_{red}\Gamma,A\rtimes_{max}\Gamma}}\uparrow & & & & \downarrow{\scriptstyle \mathcal{D}_{J\rtimes_{max}\Gamma,A\rtimes_{max}\Gamma}} \\ \mathcal{K}_1(A/J\rtimes_{max}\Gamma) & \xleftarrow{q_{\Gamma,max,*}} & \mathcal{K}_1(A\rtimes_{max}\Gamma) & \xleftarrow{\jmath_{\Gamma,max,*}} & \mathcal{K}_1(J\rtimes_{max}\Gamma) \end{array}$$

5.2. Kasparov transformation

In this subsection we see how a slight modification of the argument used in Section 4.1 allowed to define a controlled version of the Kasparov transformation compatible with Kasparov product.

Notice first that every element z of $KK^{\Gamma}_*(A,B)$ can be represented by a K-cycle, $(\pi,T,\mathcal{H}\otimes B)$, where

- $\mathcal{H}$ is a separable Hilbert space;

- the right Hilbert B-module $\mathcal{H}\otimes B$ is acted upon by Γ;
- π is an equivariant representation of A in the algebra $\mathcal{L}(\mathcal{H}\otimes B)$ of adjointable operators on $\mathcal{H}\otimes B$;
- T is a self-adjoint operator on $\mathcal{H}\otimes B$ satisfying the K-cycle conditions, i.e. $[T,\pi(a)]$, $\pi(a)(T^2-\mathcal{I}d_{\mathcal{H}\otimes B})$ and $\pi(a)(\gamma(T)-T)$ belongs to $\mathcal{K}(\mathcal{H})\otimes B$, for every a in A and $\gamma\in\Gamma$.

Let $T_\Gamma=T\otimes_B\mathcal{I}d_{B\rtimes_{red}\Gamma}$ be the adjointable element of $(\mathcal{H}\otimes B)\otimes_B B\rtimes_{red}\Gamma\cong\mathcal{H}\otimes B\rtimes_{red}\Gamma$ induced by T and let π_Γ be the representation of $A\rtimes_{red}\Gamma$ in the algebra $\mathcal{L}(\mathcal{H}\otimes B\rtimes_{red}\Gamma)$ of adjointable operators of $\mathcal{H}\otimes B\rtimes_{red}\Gamma$ induced by π. Then $(\pi_\Gamma,T_\Gamma,\mathcal{H}\otimes B\rtimes_{red}\Gamma)$ is a $A\rtimes_{red}\Gamma$-$B\rtimes_{red}\Gamma$-K-cycle and the Kasparov transform of z is the class $J_\Gamma^{red}(z)$ of this K-cycle in $KK_*(A\rtimes_{red}\Gamma,B\rtimes_{red}\Gamma)$ [11]. In the odd case, let us set $P=\frac{\mathcal{I}d_{\mathcal{H}\otimes B}+T}{2}$. Then P induces an adjointable operator $P_\Gamma=P\otimes_B\mathcal{I}d_{B\rtimes_{red}\Gamma}$ of $(\mathcal{H}\otimes B)\otimes_B B\rtimes_{red}\Gamma\cong\mathcal{H}\otimes B\rtimes_{red}\Gamma$. Let us define

$$E^{(\pi,T)}=\{(x,y)\in A\rtimes_{red}\Gamma\oplus\mathcal{L}(\mathcal{H}\otimes B\rtimes_{red}\Gamma)\\ \text{such that } P_\Gamma\cdot\pi_\Gamma(x)\cdot P_\Gamma-y\in\mathcal{K}(\mathcal{H})\otimes B\rtimes_{red}\Gamma\}.$$

Since P_Γ has no propagation, the C^*-algebra $E^{(\pi,T)}$ is filtered by $(E_r^{(\pi,T)})_{r>0}$ with

$$E_r^{(\pi,T)}=\{(x,\,P_\Gamma\cdot\pi_\Gamma(x)\cdot P_\Gamma+y);\ x\in(A\rtimes_{red}\Gamma)_r\text{ and }y\in\mathcal{K}(\mathcal{H})\otimes(B\rtimes_{red}\Gamma)_r\}.$$

The extension of C^*-algebras

$$0\longrightarrow\mathcal{K}(\mathcal{H})\otimes B\rtimes_{red}\Gamma\longrightarrow E^{(\pi,T)}\longrightarrow A\rtimes_{red}\Gamma\longrightarrow 0$$

is filtered semi-split by the cross-section

$$s:A\rtimes_{red}\Gamma\to E^{(\pi,T)};\ x\mapsto(x,P_\Gamma\cdot\pi_\Gamma(x)\cdot P_\Gamma).$$

Let us show that $\mathcal{D}_{\mathcal{K}(\mathcal{H})\otimes B\rtimes_{red}\Gamma,E^{(\pi,T)}}$ only depends on the class of $(\pi,T,\mathcal{H}\otimes B)$ in $KK_1^\Gamma(A,B)$. Assume that $(\pi,T,\mathcal{H}\otimes B[0,1])$ is a Γ-equivariant A-$B[0,1]$-K-cycle providing a homotopy between two Γ-equivariant A-B-K-cycles $(\pi_0,T_0,\mathcal{H}\otimes B)$ and $(\pi_1,T_1,\mathcal{H}\otimes B)$. For $t\in[0,1]$ we denote by

- $e_t:B[0,1]\rtimes_{red}\Gamma\to B\rtimes_{red}\Gamma$ the evaluation at t;
- $F_t\in\mathcal{L}(\mathcal{H}\otimes B\rtimes_{red}\Gamma)$ the fiber at t of an operator $F\in\mathcal{L}(\mathcal{H}\otimes B[0,1]\rtimes_{red}\Gamma)$;
- $\pi_{\Gamma,t}$ the representation of $A\rtimes_{red}\Gamma$ induced by π_Γ at the fiber t.

Then the homomorphism $E^{(\pi,T)}\to E^{(\pi_t,T_t)}$; $(x,y)\mapsto(x,y_t)$ satisfies the conditions of Remark 3.8 and thus we get that

$$(\mathcal{I}d_{\mathcal{K}(\mathcal{H})}\otimes e_t)_*\circ\mathcal{D}_{\mathcal{K}(\mathcal{H})\otimes B[0,1]\rtimes_{red}\Gamma,E^{(\pi,T)}}=\mathcal{D}_{\mathcal{K}(\mathcal{H})\otimes B\rtimes_{red}\Gamma,E^{(\pi_t,T_t)}}.$$

According to Lemma 1.26, we deduce that

$$\mathcal{D}_{\mathcal{K}(\mathcal{H})\otimes B\rtimes_{red}\Gamma,E^{(\pi_0,T_0)}}=\mathcal{D}_{\mathcal{K}(\mathcal{H})\otimes B\rtimes_{red}\Gamma,E^{(\pi_1,T_1)}}.$$

This shows that for a Γ-equivariant A-B-K-cycles $(\pi,T,\mathcal{H}\otimes B)$, then $\mathcal{D}_{\mathcal{K}(\mathcal{H})\otimes B\rtimes_{red}\Gamma,E^{(\pi,T)}}$ depends only on the class z of $(\pi,T,\mathcal{H}\otimes B)$ in $KK_1^\Gamma(A,B)$. Eventually, if we define

$$\mathcal{J}_\Gamma^{red}(z)=\mathcal{M}^{-1}_{B\rtimes_{red}\Gamma}\circ\mathcal{D}_{\mathcal{K}(\mathcal{H})\otimes B\rtimes_{red}\Gamma,E^{(\pi,T)}},$$

where

- $(\pi,T,\mathcal{H}\otimes B)$ is any Γ-equivariant A-B-K-cycles representing z;
- $\mathcal{M}_{B\rtimes_{red}\Gamma}$ is the Morita equivalence (see Example 2.2).

we get as in Section 4.1

Proposition 5.2. — *Let A and B be Γ-C^*-algebras. Then for any element z of $KK_1^\Gamma(A,B)$, there is a odd degree $(\alpha_\mathcal{D},k_\mathcal{D})$-controlled morphism*

$$\mathcal{J}_\Gamma^{red}(z)=(J_\Gamma^{red,\varepsilon,r}(z))_{0<\varepsilon<\frac{1}{4\alpha_\mathcal{D}},r>0}:\mathcal{K}_*(A\rtimes_{red}\Gamma)\to\mathcal{K}_*(B\rtimes_{red}\Gamma)$$

such that

(i) $\mathcal{J}_\Gamma^{red}(x)$ induces in K-theory the right multiplication by $J_\Gamma^{red}(z)$;

(ii) $\mathcal{J}_\Gamma^{red}$ is additive, i.e

$$\mathcal{J}_\Gamma^{red}(z+z')=\mathcal{J}_\Gamma^{red}(z)+\mathcal{J}_\Gamma^{red}(z').$$

(iii) Let A' be a Γ-C^*-algebra and let $f:A\to A'$ be a homomorphism Γ-C^*-algebras, then

$$\mathcal{J}_\Gamma^{red}(f^*(z))=\mathcal{J}_\Gamma^{red}(z)\circ f_{\Gamma,red,*}$$

for any z in $KK_1^\Gamma(A',B)$.

(iv) Let B' be a Γ-C^*-algebra and let $g:B\to B'$ be a homomorphism of Γ-C^*-algebras, then

$$\mathcal{J}_\Gamma^{red}(g_*(z))=g_{\Gamma,red,*}\circ\mathcal{J}_\Gamma^{red}(z)$$

for any z in $KK_1^\Gamma(A,B)$.

(v) If

$$0\to J\to A\to A/J\to 0$$

is a semi-split exact sequence of Γ-C^*-algebras, let $[\partial_{J,A}]$ be the element of $KK_1^\Gamma(A/J,J)$ that implements the boundary map $\partial_{J,A}$. Then we have

$$\mathcal{J}_\Gamma^{red}([\partial_{J,A}])=\mathcal{D}_{J\rtimes_{red}\Gamma,A\rtimes_{red}\Gamma}.$$

We can now define $\mathcal{J}_\Gamma^{red}$ for even element in the following way. Set $\alpha_\mathcal{J}=\alpha_\mathcal{T}\alpha_\mathcal{D}$ and $k_\mathcal{J}=k_\mathcal{T}*k_\mathcal{D}$. If A and B are Γ-C^*-algebra and if z is an element in $KK_0^\Gamma(A,B)$, then we set with notation of Section 4.1

$$\mathcal{J}_\Gamma^{red}(z)=(J_\Gamma^{red,\varepsilon,r}(z))_{0<\varepsilon<\frac{1}{4\alpha_\mathcal{T}},r}\overset{\text{def}}{=\!=}\mathcal{T}_{B\rtimes_{red}\Gamma}([\partial]^{-1})\circ\mathcal{J}_\Gamma^{red}(z\otimes_B[\partial_{SB}]).$$

According to Lemma 4.6, there exists a control pair (λ,h) such that for any Γ-C^*-algebra A, then $\mathcal{J}_\Gamma^{red}([Id_A])\overset{(\lambda,h)}{\sim}\mathcal{I}d_{\mathcal{K}_*(A\rtimes_{red}\Gamma)}$. Up to compose with $\iota_*^{\alpha_\mathcal{D}\varepsilon,\alpha_\mathcal{J}\varepsilon,k_{\mathcal{D},\varepsilon}r,k_{\mathcal{J},\varepsilon}r}$, we can assume indeed that $\mathcal{J}_\Gamma^{red}(\bullet)$ is also, in the odd case a $(\alpha_\mathcal{J},k_\mathcal{J})$-controlled morphism. As for Theorem 4.4, we get.

Theorem 5.3. — *Let A and B be Γ-C^*-algebras.*

(i) For any element z of $KK_*^\Gamma(A,B)$, then

$$\mathcal{J}_\Gamma^{red}(z):\mathcal{K}_*(A\rtimes_{red}\Gamma)\to\mathcal{K}_*(B\rtimes_{red}\Gamma)$$

is a $(\alpha_\mathcal{J},k_\mathcal{J})$-controlled morphism of same degree as z that induces in K-theory right multiplication by $J_\Gamma^{red}(z)$.

(ii) For any z and z' in $KK_*^\Gamma(A,B)$, then

$$\mathcal{J}_\Gamma^{red}(z+z')=\mathcal{J}_\Gamma^{red}(z)+\mathcal{J}_\Gamma^{red}(z').$$

(iii) For any Γ-C^*-algebra A', any homomorphism $f:A\to A'$ of Γ-C^*-algebras and any z in $KK_*^\Gamma(A',B)$, then $\mathcal{J}_\Gamma^{red}(f^*(z))=\mathcal{J}_\Gamma^{red}(z)\circ f_{\Gamma,*}$.

(iv) For any Γ-C^*-algebra B', any homomorphism $g:B\to B'$ of Γ-C^*-algebras and any z in $KK_*^\Gamma(A,B)$, then $\mathcal{J}_\Gamma^{red}(g_*(z))=g_{\Gamma,*}\circ\mathcal{J}_\Gamma^{red}(z)$.

Using the same argument as in the proof of Theorem 4.5, we see that $\mathcal{J}_\Gamma^{red}$ is compatible with Kasparov products.

Theorem 5.4. — *There exists a control pair (λ,h) such that the following holds: for every Γ-C^*-algebras A, B and D, any elements z in $KK_*^\Gamma(A,B)$ and z' in $KK_*^\Gamma(B,D)$, then*

$$\mathcal{J}_\Gamma^{red}(z\otimes_B z')\overset{(\lambda,h)}{\sim}\mathcal{J}_\Gamma^{red}(z')\circ\mathcal{J}_\Gamma^{red}(z).$$

We can perform a similar construction for maximal cross products.

Theorem 5.5. — *Let A and B be Γ-C^*-algebras.*

(i) For any element z of $KK_*^\Gamma(A,B)$, there exists a $(\alpha_\mathcal{J},k_\mathcal{J})$-controlled morphism

$$\mathcal{J}_\Gamma^{max}(z)=(J_\Gamma^{max,\varepsilon,r}(z))_{0<\varepsilon<\frac{1}{4\alpha_\mathcal{J}},r}:\mathcal{K}_*(A\rtimes_{max}\Gamma)\to\mathcal{K}_*(B\rtimes_{max}\Gamma)$$

with same degree as z that induces in K-theory right multiplication by $J_\Gamma^{max}(z)$ and such that $\lambda_{\Gamma,B,*}\circ\mathcal{J}_\Gamma^{max}(z)=\mathcal{J}_\Gamma^{red}(z)\circ\lambda_{\Gamma,A,*}$.

(ii) For any z and z' in $KK_*^\Gamma(A,B)$, then

$$\mathcal{J}_\Gamma^{max}(z+z')=\mathcal{J}_\Gamma^{max}(z)+\mathcal{J}_\Gamma^{max}(z').$$

(iii) For any Γ-C^*-algebra A', any homomorphism $f:A\to A'$ of Γ-C^*-algebras and any z in $KK_*^\Gamma(A',B)$, then $\mathcal{J}_\Gamma^{max}(f^*(z))=\mathcal{J}_\Gamma^{max}(z)\circ f_{\Gamma,max,*}$.

(iv) For any Γ-C^*-algebra B', any homomorphism $g:B\to B'$ of Γ-C^*-algebras and any z in $KK_*^\Gamma(A,B)$, then $\mathcal{J}_\Gamma^{max}(g_*(z))=g_{\Gamma,max,*}\circ\mathcal{J}_\Gamma^{max}(z)$.

Moreover, there exists a controlled pair (λ,h) such that,

- *for any Γ-C^* algebra A, then $\mathcal{J}_\Gamma^{max}([Id_A])\stackrel{(\lambda,h)}{\sim}\mathcal{I}d_{\mathcal{K}_*(A\rtimes_{max}\Gamma)}$;*
- *For any semi-split extension of Γ algebras $0\to J\to A\to A/J\to 0$, then $\mathcal{J}_\Gamma^{max}([\partial_{J,A}])\stackrel{(\lambda,h)}{\sim}\mathcal{D}_{J,A}$.*

Theorem 5.6. — *There exists a control pair (λ,h) such that the following holds: for every Γ-C^*-algebras A, B and D, any elements z in $KK_*^\Gamma(A,B)$ and z' in $KK_*^\Gamma(B,D)$, then*

$$\mathcal{J}_\Gamma^{max}(z\otimes_B z')\stackrel{(\lambda,h)}{\sim}\mathcal{J}_\Gamma^{max}(z')\circ\mathcal{J}_\Gamma^{max}(z).$$

5.3. Application to K-amenability

The original definition of K-amenability is due to J. Cuntz [6]. For our purpose, it is more convenient to use the equivalent definition given by P. Julg and A. Valette in [10]. If Γ is a discrete group, let us denote by 1_Γ the class in $KK_0^\Gamma(\mathbb{C},\mathbb{C})$ of the K-cycle $(Id_\mathbb{C},0,\mathbb{C})$, where $\mathbb{C}$ is provided with the trivial action on Γ.

Definition 5.7. — *Let Γ be a discrete group. Then Γ is K-amenable if 1_Γ can be represented by a K-cycle such that the action of Γ on the underlying Hilbert space is weakly contained in the regular representation.*

(The previous definition indeed also makes sense for locally compact groups.)

Example 5.8. — Amenable groups are obviously K-amenable. Typical example on non-amenable K-amenable groups are free groups [6]. More generally, J. L. Tu proved in [17] that group which satisfies the strong Baum-Connes conjecture (i.e with $\gamma=1$) are K-amenable. Examples of such group are groups with the Haagerup property [8] and fundamental groups of compact and oriented 3-manifolds [13].

For a Γ-C^*-algebra B and an element T of $\mathcal{L}(\mathcal{H}\otimes B)$, where $\mathcal{H}$ is a separable Hilbert space, let us set $T_{\Gamma,max} = T\otimes_B Id_{B\rtimes_{max}\Gamma}$ and $T_{\Gamma,red} = T\otimes_B Id_{B\rtimes_{red}\Gamma}$. If A is a Γ-C^*-algebra and $\pi : A \to \mathcal{L}(\mathcal{H}\otimes B)$ is a Γ-equivariant representation, let $\pi_{\Gamma,red} : A\rtimes_{red}\Gamma \to \mathcal{L}(\mathcal{H}\otimes B\rtimes_{red}\Gamma)$ and $\pi_{\Gamma,max} : A\rtimes_{max}\Gamma \to \mathcal{L}(\mathcal{H}\otimes B\rtimes_{max}\Gamma)$ be respectively the reduced and the maximal representation induced by π. Then, we have the following (compare with the proof of [10, proposition 3.4]).

Proposition 5.9. — *Let Γ be a K-amenable discrete group and let A and B be Γ-C^*-algebras. Then any elements of $KK_*^\Gamma(A,B)$ can be represented by a K-cycle $(\pi, T, \mathcal{H}\otimes B)$ such that the homomorphism $\pi_{\Gamma,max} : A\rtimes_{max}\Gamma \to \mathcal{L}(\mathcal{H}\otimes B\rtimes_{max}\Gamma)$ factorises through the homomorphism $\lambda_{\Gamma,A} : A\rtimes_{max}\Gamma \to A\rtimes_{red}\Gamma$, i.e there exists a homomorphism*

$$\pi_{\Gamma,red,max} : A\rtimes_{red}\Gamma \to \mathcal{L}(\mathcal{H}\otimes B\rtimes_{max}\Gamma)$$

such that

$$\pi_{\Gamma,max} = \pi_{\Gamma,red,max} \circ \lambda_{\Gamma,A}.$$

As a consequence, for any Γ-C^*-algebra A, then

$$\lambda_{\Gamma,A,*} : K_*(A\rtimes_{max}\Gamma) \to K_*(A\rtimes_{red}\Gamma)$$

is an isomorphism [6].

We have the following analogous result for quantitative K-theory.

Theorem 5.10. — *There exists a control pair (λ, h) such that*

$$\lambda_{\Gamma,A,*} : \mathcal{K}_*(A\rtimes_{max}\Gamma) \to \mathcal{K}_*(A\rtimes_{red}\Gamma)$$

is a (λ, h)-isomorphism for every Γ-C^*-algebra A.

Proof. — Let $(\pi, T, \mathcal{H}\otimes SA)$ be a Γ-equivariant K-cycle as in Proposition 5.9 representing the element $[\partial_A]$ of $KK_1^\Gamma(A, SA)$ corresponding to the extension

$$0 \to SA \to CA \to A \to 0.$$

Let then choose $\pi_{\Gamma,A,red,max} : A\rtimes_{red}\Gamma \to \mathcal{L}(\mathcal{H}\otimes B\rtimes_{max}\Gamma)$ such that $\pi_{\Gamma,max} = \pi_{\Gamma,red,max} \circ \lambda_{\Gamma,A}$. Let us set $P = \frac{T+Id_{\mathcal{H}\otimes SA}}{2}$ and then define

$$\begin{aligned} E_{red}^{(\pi,T)} = \{(x,y) \in A\rtimes_{red}\Gamma \oplus \mathcal{L}(\mathcal{H} \otimes SA\rtimes_{red}\Gamma) \text{ such that} \\ P_{\Gamma,red} \cdot \pi_{\Gamma,red}(x) \cdot P_{\Gamma,red} - y \in \mathcal{K}(\mathcal{H}) \otimes SA\rtimes_{red}\Gamma\}, \end{aligned}$$

$$E_{max}^{(\pi,T)} = \{(x,y) \in A\rtimes_{max}\Gamma \oplus \mathcal{L}(\mathcal{H} \otimes SA\rtimes_{max}\Gamma) \text{ such that } \\ P_{\Gamma,max} \cdot \pi_{\Gamma,max}(x) \cdot P_{\Gamma,max} - y \in \mathcal{K}(\mathcal{H}) \otimes SA\rtimes_{max}\Gamma\}$$

and

$$E_{red,max}^{(\pi,T)} = \{(x,y) \in A\rtimes_{red}\Gamma \oplus \mathcal{L}(\mathcal{H} \otimes SA\rtimes_{max}\Gamma) \text{ such that } \\ P_{\Gamma,max} \cdot \pi_{\Gamma,red,max}(x) \cdot P_{\Gamma,max} - y \in \mathcal{K}(\mathcal{H}) \otimes A\rtimes_{max}\Gamma\}$$

Then $E_{red}^{(\pi,T)}$, $E_{max}^{(\pi,T)}$ and $E_{red,max}^{(\pi,T)}$ are respectively filtered by

$$\{(x,\, P_{\Gamma,red} \cdot \pi_{\Gamma,red}(x) \cdot P_{\Gamma,red} + y); \\ x \in A\rtimes_{red}\Gamma_r \text{ and } y \in \mathcal{K}(\mathcal{H}) \otimes SA\rtimes_{red}\Gamma_r\},$$

$$\{(x,\, P_{\Gamma,max} \cdot \pi_{\Gamma,max}(x) \cdot P_{\Gamma,max} + y); \\ x \in A\rtimes_{max}\Gamma_r \text{ and } y \in \mathcal{K}(\mathcal{H}) \otimes SA\rtimes_{max}\Gamma_r\}$$

and

$$\{(x,\, P_{\Gamma,max} \cdot \pi_{\Gamma,red,max}(x) \cdot P_{\Gamma,max} + y); \\ x \in A\rtimes_{red}\Gamma_r \text{ and } y \in \mathcal{K}(\mathcal{H}) \otimes SA\rtimes_{max}\Gamma_r\}.$$

Moreover, the extension of C^*-algebras

$$0 \longrightarrow \mathcal{K}(\mathcal{H}) \otimes SA\rtimes_{red}\Gamma \longrightarrow E_{red}^{(\pi,T)} \longrightarrow A\rtimes_{red}\Gamma \longrightarrow 0,$$

$$0 \longrightarrow \mathcal{K}(\mathcal{H}) \otimes SA\rtimes_{max}\Gamma \longrightarrow E_{max}^{(\pi,T)} \longrightarrow A\rtimes_{max}\Gamma \longrightarrow 0$$

and

$$0 \longrightarrow \mathcal{K}(\mathcal{H}) \otimes SA\rtimes_{max}\Gamma \longrightarrow E_{red,max}^{(\pi,T)} \longrightarrow A\rtimes_{red}\Gamma \longrightarrow 0$$

provided by the projection on the first factor are respectively semi-split by the filtered cross-sections

$$s_{red} : A\rtimes_{red}\Gamma \to E_{red}^{(\pi,T)};\, x \mapsto (x, P_{\Gamma,red} \cdot \pi_{\Gamma,red}(x) \cdot P_{\Gamma,red}),$$

$$s_{max} : A\rtimes_{max}\Gamma \to E_{max}^{(\pi,T)};\, x \mapsto (x, P_{\Gamma,max} \cdot \pi_{\Gamma,max}(x) \cdot P_{\Gamma,max})$$

and

$$s_{red,max} : A\rtimes_{red}\Gamma \to E_{max}^{(\pi,T)};\, x \mapsto (x, P_{\Gamma,max} \cdot \pi_{\Gamma,red,max}(x) \cdot P_{\Gamma,max}).$$

Let us set

$$f_1 : E_{max}^{(\pi,T)} \to E_{red,max}^{(\pi,T)} : (x,y) \mapsto (\lambda_{\Gamma,A,*}(x), y)$$

and

$$f_2 : E_{red,max}^{(\pi,T)} \to E_{red}^{(\pi,T)} : (x,y) \mapsto (x, y \otimes_{A\rtimes_{max}\Gamma} Id_{A\rtimes_{red}\Gamma}).$$

Then the three above extensions fit in a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{K}(\mathcal{H})\otimes SA\rtimes_{max}\Gamma & \longrightarrow & E^{(\pi,T)}_{max} & \longrightarrow & A\rtimes_{max}\Gamma & \longrightarrow & 0\\
 & & \downarrow{=} & & \downarrow{f_1} & & \downarrow{\lambda_{\Gamma,A}} & & \\
0 & \longrightarrow & \mathcal{K}(\mathcal{H})\otimes SA\rtimes_{max}\Gamma & \longrightarrow & E^{(\pi,T)}_{red,max} & \longrightarrow & A\rtimes_{red}\Gamma & \longrightarrow & 0\\
 & & \downarrow{\lambda_{\Gamma,\mathcal{K}(\mathcal{H})\otimes SA}} & & \downarrow{f_2} & & \downarrow{=} & & \\
0 & \longrightarrow & \mathcal{K}(\mathcal{H})\otimes SA\rtimes_{red}\Gamma & \longrightarrow & E^{(\pi,T)}_{red} & \longrightarrow & A\rtimes_{red}\Gamma & \longrightarrow & 0
\end{array}$$

which satisfy the conditions of Remark 3.8. Hence we deduce

$$\mathcal{D}_{\mathcal{K}(\mathcal{H})\otimes SA\rtimes_{max}\Gamma,E^{(\pi,T)}_{red,max}}\circ\lambda_{A,\Gamma,*}=\mathcal{D}_{\mathcal{K}(\mathcal{H})\otimes SA\rtimes_{max}\Gamma,E^{(\pi,T)}_{max}} \tag{5.1}$$

and

$$\lambda_{\mathcal{K}(\mathcal{H})\otimes SA,\Gamma,*}\circ\mathcal{D}_{\mathcal{K}(\mathcal{H})\otimes SA\rtimes_{max}\Gamma,E^{(\pi,T)}_{red,max}}=\mathcal{D}_{\mathcal{K}(\mathcal{H})\otimes SA\rtimes_{red}\Gamma,E^{(\pi,T)}_{red}} \tag{5.2}$$

Let us set then

$$\mathcal{D}'_A=\mathcal{M}^{-1}_{SA\rtimes_{max}\Gamma}\circ\mathcal{D}_{SA\rtimes_{max}\Gamma,E^{(\pi,T)}_{red,max}}:\mathcal{K}_*(A\rtimes_{red}\Gamma)\to\mathcal{K}_*(SA\rtimes_{max}\Gamma).$$

Since we have by definition of the quantitative Kasparov transformation the equalities

$$\mathcal{J}^{red}_\Gamma([\partial_A])=\mathcal{M}^{-1}_{SA\rtimes_{red}\Gamma}\circ\mathcal{D}_{SA\rtimes_{red}\Gamma,E^{(\pi,T)}_{red}}$$

and

$$\mathcal{J}^{max}_\Gamma([\partial_A])=\mathcal{M}^{-1}_{SA\rtimes_{max}\Gamma}\circ\mathcal{D}_{SA\rtimes_{max}\Gamma,E^{(\pi,T)}_{max}},$$

we deduce by using equations (5.1) and (5.2), Theorems 5.3, 5.4, 5.5 and 5.6 and naturality of Morita equivalence, that there exists a control pair (λ,h) such that $\mathcal{J}^{max}_\Gamma([\partial_A]^{-1})\circ\mathcal{D}'_A$ is a (α,h)-inverse for $\lambda_{\Gamma,A,*}$. □

6. The quantitative Baum-Connes conjecture

In this section, we formulate a quantitative version for the Baum-Connes conjecture and we prove it for a large class of groups.

6.1. The quantitative assembly maps

Let Γ be a finitely generated group equipped with a lenght ℓ arising from a finite and symmetric generating set. Recall that for any positive number d, then the d-Rips complex $P_d(\Gamma)$ is the set of finitely supported probability measures on Γ with support of diameter less than d for the distance induced by ℓ. We equip $P_d(\Gamma)$ with the distance induced by the norm $\|h\| = \sup\{\|h(\gamma)\|; \gamma \in \Gamma\}$ for $h \in C_0(\Gamma, \mathbb{C})$. Since ℓ is a proper function, i.e. $B(e,r)$ is finite for every positive number r, we see that $P_d(\Gamma)$ is a finite dimension and locally finite simplicial complexe and the action of Γ by left translations is simplicial, proper and cocompact.

Notice that any x in $P_d(\Gamma)$ can be written down in a unique way as a finite convex combination

$$x = \sum_{\gamma\in\Gamma} \lambda_\gamma(x)\delta_\gamma,$$

where δ_γ is the Dirac probability measure at γ in Γ. The functions

$$\lambda_\gamma : P_d(\Gamma) \to [0,1]$$

are continuous and $\gamma(\lambda_{\gamma'}) = \lambda_{\gamma\gamma'}$ for all γ and γ' in Γ. The function

$$e_{\Gamma,d} : \Gamma \to C_0(P_d(\Gamma));\ \gamma \mapsto \lambda_e^{1/2}\lambda_\gamma^{1/2}$$

is a projection of $C_0(P_d(\Gamma))\rtimes_{red}\Gamma$ with propagation less than d. Let us set then $r_{d,\varepsilon} = k_{\mathcal{J},\varepsilon/\alpha_J}d$. Recall that $k_{\mathcal{J}}$ can be chosen non increasing and in this case, $r_{d,\varepsilon}$ is non decreasing in d and non increasing in ε.

Definition 6.1. — *For any Γ-C^*-algebra A and any positive numbers ε, r and d with $\varepsilon < 1/4$ and $r \geqslant r_{d,\varepsilon}$, we define the quantitative assembly map*

$$\mu_{\Gamma,A,*}^{\varepsilon,r,d} : KK_*^\Gamma(C_0(P_d(\Gamma)), A) \to K_*^{\varepsilon,r}(A \rtimes_{red} \Gamma)$$
$$z \mapsto \left(J_\Gamma^{red,\frac{\varepsilon}{\alpha_J},\frac{r}{k_{J,\varepsilon/\alpha_J}}}(z)\right)\left([e_{\Gamma,d},0]_{\frac{\varepsilon}{\alpha_J},\frac{r}{k_{J,\varepsilon/\alpha_J}}}\right).$$

Then according to Theorem 5.3, the map $\mu_{\Gamma,A}^{\varepsilon,r,d}$ is a homomorphism of $\mathbb{Z}_2$-graded groups. For any positive numbers d and d' such that $d \leqslant d'$, we denote by $q_{d,d'} : C_0(P_{d'}(\Gamma)) \to C_0(P_d(\Gamma))$ the homomorphism induced by the restriction from $P_{d'}(\Gamma)$ to $P_d(\Gamma)$. It is straightforward to check that if d, d' and r are positive numbers such that $d \leqslant d'$ and $r \geqslant r_{d',\varepsilon}$, then $\mu_{\Gamma,A}^{\varepsilon,r,d} = \mu_{\Gamma,A}^{\varepsilon,r,d'} \circ q_{d,d',*}$. Moreover, for every positive numbers ε, ε', d, r and

r' such that $\varepsilon \leqslant \varepsilon' \leqslant 1/4$, $r_{d,\varepsilon} \leqslant r$, $r_{d,\varepsilon'} \leqslant r'$, and $r < r'$, we get by definition of a controlled morphism that

$$\iota_*^{\varepsilon,\varepsilon',r,r'} \circ \mu_{\Gamma,A,*}^{\varepsilon,r,d} = \mu_{\Gamma,A,*}^{\varepsilon',r',d}. \tag{6.1}$$

Furthermore, the quantitative assembly maps are natural in the Γ-C^*-algebra, i.e. if A and B are Γ-C^*-algebras and if $\phi : A \to B$ is a Γ-equivariant homomorphism, then

$$\phi_{\Gamma,red,*,\varepsilon,r} \circ \mu_{\Gamma,A,*}^{\varepsilon,r,d} = \mu_{\Gamma,B,*}^{\varepsilon,r,d} \circ \phi_*$$

for every positive numbers r and ε with $r \geqslant r_{d,\varepsilon}$ and $\varepsilon < 1/4$. These quantitative assembly maps are related to the usual assembly maps in the following way: recall from [2] that there is a bunch of assembly maps with coefficients in a Γ-C^*-algebra A defined by

$$\begin{array}{rcl} \mu_{\Gamma,A,*}^d : KK_*^\Gamma(C_0(P_d(\Gamma)), A) & \to & K_*(A \rtimes_{red} \Gamma) \\ z & \mapsto & [e_{\Gamma,d}] \otimes_{C_0(P_d(\Gamma))\rtimes\Gamma} J_\Gamma(z). \end{array}$$

For every positive numbers r and ε with $r \geqslant r_{d,\varepsilon}$ and $\varepsilon < 1/4$, we have

$$\iota_*^{\varepsilon,r} \circ \mu_{\Gamma,A,*}^{\varepsilon,r,d} = \mu_{\Gamma,A,*}^d. \tag{6.2}$$

Recall that since $\mu_{\Gamma,A,*}^{d'} \circ q_{d,d',*} = \mu_{\Gamma,A,*}^d$ for all positive numbers d and d' with $d \leqslant d'$, the family of assembly maps $(\mu_{\Gamma,A}^d)_{d>0}$ gives rise to a homomorphism

$$\mu_{\Gamma,A,*} : \lim_{d>0} KK_*^\Gamma(C_0(P_d(\Gamma)), A) \longrightarrow K_*(A \rtimes_{red} \Gamma)$$

called the Baum-Connes assembly map.

6.2. Quantitative statements

Let us consider for a Γ-C^*-algebra A and positive numbers $d, d', r, r', \varepsilon$ and ε' with $d \leqslant d'$, $\varepsilon' \leqslant \varepsilon < 1/4$, $r_{d,\varepsilon} \leqslant r$ and $r' \leqslant r$ the following statements:

$QI_{\Gamma,A,*}(d, d', r, \varepsilon)$: for any element x in $KK_*^\Gamma(C_0(P_d(\Gamma)), A)$, then $\mu_{\Gamma,A,*}^{\varepsilon,r,d}(x) = 0$ in $K_*^{\varepsilon,r}(A \rtimes_{red} \Gamma)$ implies that $q_{d,d'}^*(x) = 0$ in $KK_*^\Gamma(C_0(P_{d'}(\Gamma)), A)$.

$QS_{\Gamma,A,*}(d, r, r', \varepsilon, \varepsilon')$: for every y in $K_*^{\varepsilon',r'}(A \rtimes_{red} \Gamma)$, there exists an element x in $KK_*^\Gamma(C_0(P_d(\Gamma)), A)$ such that

$$\mu_{\Gamma,A,*}^{\varepsilon,r,d}(x) = \iota_*^{\varepsilon',\varepsilon,r',r}(y).$$

Using equation (6.2) and Remark 1.17 we get

Proposition 6.2. — *Assume that for all positive number d there exists a positive number ε with $\varepsilon < 1/4$ for which the following holds:*

For any positive number r with $r \geqslant r_{d,\varepsilon}$, there exists a positive number d' with $d' \geqslant d$ such that $QI_{\Gamma,A}(d,d',r,\varepsilon)$ is satisfied.

Then $\mu_{\Gamma,A,*}$ is one-to-one.

We can also easily prove the following:

Proposition 6.3. — *Assume that there exists a positive number ε' with $\varepsilon' < 1/4$ such that the following holds:*

For any positive number r' , there exist positive numbers ε, d and r with $\varepsilon' \leqslant \varepsilon < 1/4$, $r_{d,\varepsilon} \leqslant r$ and $r' \leqslant r$ such that $QS_{\Gamma,A}(d,r,r',\varepsilon,\varepsilon')$ is true.

Then $\mu_{\Gamma,A,*}$ is onto.

The following results provide numerous examples of finitely generated groups that satisfy the quantitative statements.

Theorem 6.4. — *Let A be a Γ-C^*-algebra. Then the following assertions are equivalent:*

(i) *$\mu_{\Gamma,\ell^\infty(\mathbb{N},\mathcal{K}(\mathcal{H})\otimes A),*}$ is one-to-one,*
(ii) *For any positive numbers d, ε and r with $\varepsilon < 1/4$ and $r \geqslant r_{d,\varepsilon}$, there exists a positive number d' with $d' \geqslant d$ for which $QI_{\Gamma,A}(d,d',r,\varepsilon)$ is satisfied.*

Proof. — Assume that condition (ii) holds.

Let x be an element in some $KK_*^\Gamma(C_0(P_d(\Gamma)),\ell^\infty(\mathbb{N},\mathcal{K}(\mathcal{H})\otimes A))$ such that

$$\mu^d_{\Gamma,\ell^\infty(\mathbb{N},\mathcal{K}(\mathcal{H})\otimes A),*}(x)=0.$$

Using equation (6.2), we get that $\iota_*^{\varepsilon',r'}(\mu^{\varepsilon',r',d}_{\Gamma,A,*}(x))=0$ for any ε' in $(0,1/4)$ and $r' \geqslant r_{d,\varepsilon'}$ and hence, by Remark 1.17, we can find ε and $r > r_{d,\varepsilon}$ such that $\mu^{\varepsilon,r,d}_{\Gamma,\ell^\infty(\mathbb{N},\mathcal{K}(\mathcal{H})\otimes A),*}(x)=0$. Recall from [14, Proposition 3.4] that we have an isomorphism

$$KK_0^\Gamma(C_0(P_d(\Gamma)),\ell^\infty(\mathbb{N},\mathcal{K}(\mathcal{H})\otimes A)) \stackrel{\cong}{\longrightarrow} KK_0^\Gamma(C_0(P_d(\Gamma)),A)^{\mathbb{N}} \tag{6.3}$$

induced on the j th factor and up to the Morita equivalence

$$KK_0^\Gamma(C_0(P_d(\Gamma)),A)\cong KK_0^\Gamma(C_0(P_d(\Gamma)),\mathcal{K}(\mathcal{H})\otimes A)$$

by the j th projection $\ell^\infty(\mathbb{N},\mathcal{K}(\mathcal{H})\otimes A)\to\mathcal{K}(\mathcal{H})\otimes A$. Let $(x_i)_{i\in\mathbb{N}}$ be the element of $KK_0^\Gamma(C_0(P_d(\Gamma)),A)^{\mathbb{N}}$ corresponding to x under this identification and let $d' \geqslant d$ be a number such that $QI_{\Gamma,A}(d,d',r,\varepsilon)$ holds. Naturality of the quantitative assembly maps implies that $\mu^{\varepsilon,r,d}_{\Gamma,A,*}(x_i)=0$ and hence that $q_{d,d',*}(x_i)=0$ in $KK^\Gamma_*(C_0(P_{d'}(\Gamma)),A)$ for every integer i. Using once again the isomorphism of equation (6.3), we get that $q_{d,d',*}(x)=0$ in $KK^\Gamma_*(C_0(P_{d'}(\Gamma)),\ell^\infty(\mathbb{N},\mathcal{K}(\mathcal{H})\otimes A)$ and hence $\mu_{\Gamma,\ell^\infty(\mathbb{N},\mathcal{K}(\mathcal{H})\otimes A),*}$ is one-to-one.

Let us prove the converse in the even case, the odd case being similar. Assume that there exists positive numbers d, ε and r with $\varepsilon<1/4$ and $r\geqslant r_{d,\varepsilon}$ and such that for all $d'\geqslant d$, the condition $QI_{\Gamma,A}(d,d',r,\varepsilon)$ does not hold. Let us prove that $\mu_{\Gamma,\ell^\infty(\mathbb{N},\mathcal{K}(\mathcal{H})\otimes A),*}$ is not one-to-one. Let $(d_i)_{i\in\mathbb{N}}$ be an increasing and unbounded sequence of positive numbers such that $d_i\geqslant d$ for all integer i. For all integer i, let x_i be an element in $KK^\Gamma_0(C_0(P_d(\Gamma)),A)$ such that $\mu^{\varepsilon,r,d}_{\Gamma,A,*}(x_i)=0$ in $K_0(A\rtimes_{red}\Gamma)$ and $q_{d,d_i,*}(x_i)\neq 0$ in $KK^\Gamma_0(C_0(P_{d_i}(\Gamma)),A)$. Let x be the element of $KK^\Gamma_0(C_0(P_d(\Gamma)),\ell^\infty(\mathbb{N},\mathcal{K}(\mathcal{H})\otimes A))$ corresponding to $(x_i)_{i\in\mathbb{N}}$ under the identification of equation (6.3). Let $(p_i)_{i\in\mathbb{N}}$ be a family of ε-r-projections, with p_i in some $M_{l_i}(\widetilde{A\rtimes_{red}}\Gamma)$ and n an integer such that

$$\mu^{\varepsilon,r,d}_{\Gamma,\ell^\infty(\mathbb{N},\mathcal{K}(\mathcal{H})\otimes A),*}(x)=[(p_i)_{i\in\mathbb{N}},n]_{\varepsilon,r}$$

in $K^{\varepsilon,r}_0(\ell^\infty(\mathbb{N},\mathcal{K}(\mathcal{H})\otimes A)\rtimes_{red}\Gamma)$. By naturality of $\mu^{\varepsilon,r,d}_{\Gamma,\bullet,*}$, we get that $[p_i,n]_{\varepsilon,r}=0$ in $K^{\varepsilon,r}_0(A\rtimes_{red}\Gamma)$ for all integer i. We see by using Proposition 1.30 that then $\iota^{\varepsilon,r}_*([(p_i)_{i\in\mathbb{N}},n])=0$ in $K_0(\ell^\infty(\mathbb{N},\mathcal{K}(\mathcal{H})\otimes A)\rtimes_{red}\Gamma)$. We eventually obtain that $\mu^d_{\Gamma,A}(x)=\iota^{\varepsilon,r}_*\circ\mu^{\varepsilon,r,d}_{\Gamma,A}(x)=0$. Since $q_{d,d_i,*}(x)\neq 0$ for every integer i, we get that $\mu_{\Gamma,\ell^\infty(\mathbb{N},\mathcal{K}(\mathcal{H})\otimes A),*}$ is not one-to-one. □

Theorem 6.5. — *There exists $\lambda>1$ such that for any Γ-C^*-algebra, the following assertions are equivalent:*

(i) $\mu_{\Gamma,\ell^\infty(\mathbb{N},\mathcal{K}(\mathcal{H})\otimes A),*}$ *is onto;*
(ii) *For any positive numbers ε and r' with $\varepsilon<\frac{1}{4\lambda}$, there exist positive numbers d and r with $r_{d,\varepsilon}\leqslant r$ and $r'\leqslant r$ for which $QS_{\Gamma,A}(d,r,r',\lambda\varepsilon,\varepsilon)$ is satisfied.*

Proof. — Choose λ as in Remark 1.17. Assume that condition (ii) holds. Let z be an element in $K_*(\ell^\infty(\mathbb{N},\mathcal{K}(\mathcal{H})\otimes A)\rtimes_{red}\Gamma)$ and let y be an element in $K^{\varepsilon,r'}_*(\ell^\infty(\mathbb{N},\mathcal{K}(\mathcal{H})\otimes A)\rtimes_{red}\Gamma)$ such that $\iota^{\varepsilon,r'}_*(y)=z$, with $0<\varepsilon<\frac{1}{4\lambda}$ and $r'>0$. Let y_i be the image of y under the composition

$$K^{\varepsilon,r'}_*(\ell^\infty(\mathbb{N},\mathcal{K}(\mathcal{H})\otimes A)\rtimes_{red}\Gamma)\to K^{\varepsilon,r'}_*(\mathcal{K}(\mathcal{H})\otimes A\rtimes_{red}\Gamma)\stackrel{\cong}{\to}K^{\varepsilon,r'}_*(A\rtimes_{red}\Gamma), \tag{6.4}$$

where the first map is induced by the evaluation $\ell^\infty(\mathbb{N},\mathcal{K}(\mathcal{H})\otimes A)\longrightarrow\mathcal{K}(\mathcal{H})\otimes A$ at i and the second map is the Morita equivalence of Proposition 1.28. Let d and r be numbers with $r\geqslant r'$ and $r\geqslant r_{d,\varepsilon}$ and such that $QS_{\Gamma,A}(d,r,r',\lambda\varepsilon,\varepsilon)$ holds. Then for any integer i, there exists a x_i in

$KK_*^\Gamma(C_0(P_d(\Gamma)),A)$ such that $\mu_{\Gamma,A,*}^{\lambda\varepsilon,r,d}(x_i)=\iota_*^{\varepsilon,\lambda\varepsilon,r',r}(y_i)$ in $K_*^{\varepsilon,r}(A\rtimes_{red}\Gamma)$. Let

$$x\in KK_*^\Gamma(C_0(P_d(\Gamma)),\ell^\infty(\mathbb{N},\mathcal{K}(\mathcal{H})\otimes A))$$

be the element corresponding to $(x_i)_{i\in\mathbb{N}}$ under the identification of equation (6.3). By naturality of the quantitative assembly maps, we get according to Proposition 1.30 that

$$\mu_{\Gamma,\ell^\infty(\mathbb{N},\mathcal{K}(\mathcal{H})\otimes A)),*}^{\lambda\varepsilon,r,d}(x)=\iota_*^{\varepsilon,\lambda\varepsilon,r',r}(y)$$

in $K_*^{\varepsilon,r}(\ell^\infty(\mathbb{N},\mathcal{K}(\mathcal{H})\otimes A)\rtimes_{red}\Gamma)$. We have hence

$$\mu_{\Gamma,\ell^\infty(\mathbb{N},\mathcal{K}(\mathcal{H})\otimes A)),*}^{d}(x)=\iota_*^{\varepsilon,r'}(y)=z,$$

and therefore $\mu_{\Gamma,\ell^\infty(\mathbb{N},\mathcal{K}(\mathcal{H})\otimes A),*}$ is onto.

Let us prove the converse in the even case, the odd case being similar. Assume that there exist positive numbers ε and r' with $\varepsilon<\frac{1}{4\lambda}$ such that for all positive numbers r and d with $r\geqslant r'$ and $r\geqslant r_{d,\varepsilon}$, then $QS_{\Gamma,A}(d,r,r',\lambda\varepsilon,\varepsilon)$ does not hold. Let us prove then that $\mu_{\Gamma,\ell^\infty(\mathbb{N},\mathcal{K}(\mathcal{H})\otimes A),*}$ is not onto. Let $(d_i)_{i\in\mathbb{N}}$ and $(r_i)_{i\in\mathbb{N}}$ be increasing and unbounded sequences of positive numbers such that $r_i\geqslant r_{d_i,\lambda\varepsilon}$ and $r_i\geqslant r'$. Let y_i be an element in $K_0^{\varepsilon,r'}(A\rtimes_{red}\Gamma)$ such that $\iota_*^{\varepsilon,\lambda\varepsilon,r',r_i}(y_i)$ is not in the range of $\mu_{\Gamma,A,*}^{\lambda\varepsilon,r_i,d_i}$. There exists an element y in $K_0^{\varepsilon,r'}(\ell^\infty(\mathbb{N},\mathcal{K}(\mathcal{H})\otimes A)\rtimes_{red}\Gamma)$ such that for every integer i, the image of y under the composition of equation (6.4) is y_i. Assume that for some d', there is an x in $KK_0^\Gamma(C_0(P_{d'}(\Gamma)),\ell^\infty(\mathbb{N},\mathcal{K}(\mathcal{H})\otimes A))$ such that $\iota_*^{\varepsilon,r'}(y)=\mu_{\Gamma,\ell^\infty(\mathbb{N},\mathcal{K}(\mathcal{H})\otimes A),*}^{d'}(x)$. Using Remark 1.17, we see that there exists a positive number r with $r'\leqslant r$ and $r_{d',\lambda\varepsilon}\leqslant r$ and such that

$$\iota_*^{\varepsilon,\lambda\varepsilon,r',r}\circ\mu_{\Gamma,\ell^\infty(\mathbb{N},\mathcal{K}(\mathcal{H})\otimes A),*}^{\varepsilon,r',d'}(x)=\iota_*^{\varepsilon,\lambda\varepsilon,r',r}(y).$$

But then, if we choose i such that $r_i\geqslant r$ and $d_i\geqslant d'$ we get by using naturality of the assembly map and equation (6.1) that $\iota_*^{\varepsilon,\lambda\varepsilon,r',r_i}(y_i)$ belongs to the image of $\mu_{\Gamma,A,*}^{\lambda\varepsilon,r_i,d_i}$, which contradicts our assumption. □

Replacing in the proof of (ii) implies (i) of Theorems 6.4 and 6.5 the algebra $\ell^\infty(\mathbb{N},\mathcal{K}(\mathcal{H})\otimes A)$ by $\prod_{i\in\mathbb{N}}(\mathcal{K}(\mathcal{H})\otimes A_i)$ for a family $(A_i)_{i\in\mathbb{N}}$ of Γ-C^*-algebras, we can prove the following result.

Theorem 6.6. — *Let Γ be a discrete group.*

(i) Assume that for any Γ-C^*-algebra A, the assembly map $\mu_{\Gamma,A,*}$ is one-to-one. Then for any positive numbers d, ε and r with $\varepsilon<1/4$ and $r\geqslant r_{d,\varepsilon}$, there exists a positive number d' with $d'\geqslant d$ such that $QI_{\Gamma,A}(d,d',r,\varepsilon)$ is satisfied for every Γ-C^*-algebra A;

(ii) *Assume that for any Γ-C^*-algebra A, the assembly map $\mu_{\Gamma,A,*}$ is onto. Then for some $\lambda > 1$ and for any positive numbers ε and r' with $\varepsilon < \frac{1}{4\lambda}$, there exist positive numbers d and r with $r_{d,\varepsilon} \leqslant r$ and $r' \leqslant r$ such that $QS_{\Gamma,A}(d,r,r',\lambda\varepsilon,\varepsilon)$ is satisfied for every Γ-C^*-algebra A.*

In particular, if Γ satisfies the Baum-Connes conjecture with coefficients, then Γ satisfies points (i) *and* (ii) *above.*

Recall from [16, 20] that if Γ coarsely embeds in a Hilbert space, then $\mu_{\Gamma,A,*}$ is one-to-one for every Γ-C^*-algebra A. Hence we get:

Corollary 6.7. — *If Γ coarsely embeds in a Hilbert space, then for any positive numbers d, ε and r with $\varepsilon < 1/4$ and $r \geqslant r_{d,\varepsilon}$, there exists a positive number d' with $d' \geqslant d$ such that $QI_{\Gamma,A}(d,d',r,\varepsilon)$ is satisfied for every Γ-C^*-algebra A;*

The quantitative assembly maps admit maximal versions defined with notations of Definition 6.1 for any Γ-C^*-algebra A and any positive number ε, r and d with $\varepsilon < 1/4$ and $r \geqslant r_{d,\varepsilon}$, as

$$\begin{aligned}\mu^{\varepsilon,r,d}_{\Gamma,A,max,*} : KK^\Gamma_*(C_0(P_d(\Gamma)),A) &\to K^{\varepsilon,r}_*(A\rtimes_{max}\Gamma)\\ z &\mapsto \left(J_\Gamma^{max,\frac{\varepsilon}{\alpha_J},\frac{r}{k_{J,\varepsilon/\alpha_J}}}(z)\right)\left([e_{\Gamma,d},0]_{\frac{\varepsilon}{\alpha_J},\frac{r}{k_{J,\varepsilon/\alpha_J}}}\right).\end{aligned}$$

As in the reduced case, we have using the same notations

- for any positive number d and d' such that $d \leqslant d'$, then
$$\mu^{\varepsilon,r,d}_{\Gamma,A,max,*} = \mu^{\varepsilon,r,d'}_{\Gamma,A,max,*} \circ q_{d,d',*}.$$
- for every positive numbers ε, ε', d, r and r' such that $\varepsilon \leqslant \varepsilon' \leqslant 1/4$, $r_{d,\varepsilon} \leqslant r$, $r_{d,\varepsilon'} \leqslant r'$, and $r < r'$, then
$$\iota^{\varepsilon,\varepsilon',r,r'}_* \circ \mu^{\varepsilon,r,d}_{\Gamma,A,max,*} = \mu^{\varepsilon',r',d}_{\Gamma,A,max,*}.$$
- the maximal quantitative assembly maps are natural in the Γ-C^*-algebras.

Moreover, by Theorem 5.5(i), the maximal quantitative assembly maps are compatible with the reduced ones, i.e $\mu^{\varepsilon,r,d}_{\Gamma,A,*} = \lambda^{\varepsilon,r}_{\Gamma,A,*} \circ \mu^{\varepsilon,r,d}_{\Gamma,A,max,*}$. The surjectivity of the Baum-Connes assembly map $\mu_{\Gamma,A,*}$ implies that the map

$$\lambda_{\Gamma,A,*} : K_*(A\rtimes_{max}\Gamma) \to K_*(A\rtimes_{red}\Gamma)$$

is onto. We have a similar statement in the setting of quantitative K-theory.

Theorem 6.8. — *There exists $\lambda > 1$ such the following holds : let Γ be a finitely generated discrete group and assume that for any Γ-C^*-algebra A, the assembly map $\mu_{\Gamma,A,*}$ is onto. Then for any positive numbers ε and r, with $\varepsilon < \frac{1}{4\lambda}$, there exists a positive number r' with $r' \geqslant r$ such that*

- *for any Γ-C^*-algebra A;*
- *for any x in $K_*^{\varepsilon,r}(A \rtimes_{red} \Gamma)$,*

there exists y in $K_*^{\lambda\varepsilon,r'}(A \rtimes_{max} \Gamma)$ such that $\lambda_{\Gamma,A,*}^{\lambda\varepsilon,r'}(y) = \iota_*^{\varepsilon,\lambda\varepsilon,r,r'}(x)$.

7. Further comments

The definition of quantitative K-theory can be extended to the framework of filtered Banach algebras, i.e. Banach algebra A equipped with a family $(A_r)_{r>0}$ of linear closed subspaces indexed by positive numbers such that:

- $A_r \subset A_{r'}$ if $r \leqslant r'$;
- $A_r \cdot A_{r'} \subset A_{r+r'}$;
- the subalgebra $\bigcup_{r>0} A_r$ is dense in A.

Since we no more have an involution, we need to introduce instead a norm control for almost idempotents. Let ε be in $(0, 1/4)$ and let r and N be positive numbers. An element e of A is an ε-r-N-idempotent if

- e is in A_r;
- $\|e^2 - e\| < \varepsilon$;
- $\|e\| < N$;

Similarly, if A is a unital, an element x in A is called ε-r-N-invertible if

- x is in A_r;
- $\|x\| < N$;
- there exists an element y in A_r such that $\|y\| < N$, $\|xy - 1\| < \varepsilon$ and $\|yx - 1\| < \varepsilon$.

Quantitative K-theory can then be defined in the setting of ε-r-N-idempotents and of ε-r-N-invertibles. We obtain in this way a bunch of abelian groups $(K_*^{\varepsilon,r,N}(A))_{\varepsilon\in(0,1/4),r>,N>1}$. Let us set for a fixed $N > 1$

$$\mathcal{K}_*^N(A) = (K_*^{\varepsilon,r,N}(A))_{\varepsilon\in(0,1/4),r>0}.$$

If A is a filtered C^*-algebra and e an ε-r-N-idempotent in A, then there is an obvious $(1,1)$-controlled morphism $\mathcal{K}_0(A) \to \mathcal{K}_0^N(A)$. Approximating $((2e^* - 1)(2e - 1) + 1)^{1/2} e((2e^* - 1)(e - 1) + 1)^{-1/2}$ by using a power serie (compare with the proof of Lemma 1.11), we get that for every $N > 1$, there exists a control pair (λ_N, h_N) such that $\mathcal{K}_0(A) \to \mathcal{K}_0^N(A)$ is a (λ_N, h_N)-controlled isomorphism. Using the polar decomposition, we have a similar statement in the odd case.

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